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Yang-Mills Sources in Biconformal Double Field Theory

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YANG-MILLS SOURCES IN BICONFORMAL DOUBLE FIELD

THEORY

by

Davis W. Muhwezi

A thesis submitted in partial fulfillment
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of

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ABSTRACT

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Generically, biconformal gauging of the conformal group has been shown to reproduce scale- covariant general relativity on the co-tangent bundle in any dimension. We generalize this result to include Yang-Mills matter sources formulated as $SU(N)$ gauge theories on the full $2n$-dimensional biconformal space. We show that the coupling of the sources to gravity does not stop the reduction of effective dimension $2n \rightarrow n$ of the gravity theory, and instead forces the Yang-Mills source to reduce to ordinary n-dimensional Yang-Mills theory on the gravitating cotangent bundle, with the usual Yang-Mills energy tensor as gravitational source.

(182 pages)

PUBLIC ABSTRACT

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THEORY

Davis W. Muhwezi

There is a robust and unifying approach to unraveling the roiling mysteries of the universe. Our most compelling accounts of physical reality at present rest on symmetry arguments that are conspicuously geometrical!

105 years ago, Albert Einstein derived gravity from Riemannian geometry. In the general theory of relativity, the world of our experience is a pseudo-Riemannian manifold whose curvature represents the gravitational field. Encoded in the Einstein field equation is how matter sources (energy-momentum tensor) couple to gravity (spacetime curvature). Schematically, the Einstein equation exhibits a more general structure:

$$Curvature\ of\ Spacetime\ =\ Material\ Sources$$

On one side of the equation are mathematical expressions that characterize in detail the curved shape of space and time. On the source side we write an expression collecting the energy and momentum densities of the different matter fields that move about in the world. Due to a profound theorem by Emmy Noether, these sources are built of conserved "currents" corresponding to matter sources that are described by certain symmetries.

In this thesis, we confront and resolve the question of how Yang-Mills matter fields couple to a scale-invariant model of gravity, called biconformal gravity. Biconformal gravity arises from a construction that imposes, in addition to local Lorentz symmetry, local dilatational symmetry. The need of the latter symmetry in a physical theory is nothing more inscrutable than the realization that the laws of physics should not change

when we change units (i.e. meters, miles, kilograms) of our experimental measurements.

<div align="center">v</div>

Assuming vanishing torsion, biconformal gravity not only reduces to scale invariant general relativity, but time emerges as part of the physical theory. Somewhat remarkably, we have shown that biconformal gravity also requires the sources to take the expected form.

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I would like to thank the incomparable Jim Wheeler for teaching me physics. He found me in my crib, as it were, long before I found myself there, and over countless hours of calculations and discussions, he gradually molded me into a better student of science.

I also thank the USU Physics Department for the many teaching and research opportunities I have received.

Davis W. Muhwezi

CONTENTS

I. INTRODUCTION

A profound aspiration of physicists since Galileo, has been to answer the question of *how matter behaves.* Bertrand Russell once quipped [1], "The question 'What is matter?' is of the kind that is . . . answered in vast books of incredible obscurity." To optimize our understanding of the world, physics takes the view that matter, whatever it is, interacts! The three fundamental interactions known to exist are: gravity, electroweak, and the strong force. In mathematical terms, these interactions are described as "fields". Prior to the introduction of quantum mechanics, it was believed that the world was composed of two distinct entities, namely, *particles* and *fields.* Particles, on this view, were composed of *matter* and gave rise to external fields which were composed of *energy.*

With the advent of quantum mechanics and special relativity, *quantum fields* have been understood as the essential reality. The electromagnetic force, for instance, is treated as a field whose Fourier components are quantized as a collection of harmonic oscillators, giving rise to creation and annihilation operators for photons. In this light (pun intended), a photon is the messenger of the electromagnetic interaction. The photon, like all *force carriers* that mediate fundamental interactions, is given the name *gauge boson.*

In the late 1960s, Sydney Coleman and his student Jeffrey Mandula proved a fascinating *no-go-theorem* [[2]]. They showed that the spacetime symmetries invoked in our description of gravity (Poincaré symmetry) cannot be combined with the symmetries of quantum field theory in any but a trivial way. This was understood to mean that the project of describing all known particles of different masses and spins in terms of a single, overriding symmetry had come to an end, and with it, Einstein's dream of a *grand unified theory.* It was not until 1973 that Wess , Zumino, and others [[3]] found a novel way to bypass the Coleman-Mandula theorem. A new

symmetry, dubbed *supersymmetry*, blurred the distinction between fermionic fields (matter) and bosonic fields (force), as they both appear in the theory as part of a single supermultiplet. Although supersymmetric models predict exotic partners to the known fundamental particles, there has yet to be any experimental confirmation of this beautiful idea.

Over the course of the 20$^{\text{th}}$ century, a novel insight into the relationship between fundamental interactions and the geometric structures of spacetime was born. To understand the broad outlines of this relationship, let us recall a colorful phrase found in Misner, Thorn, and Wheeler's 1973 tome , *Gravitation* [4]

Space tells matter how to move

Matter tells space how to curve

This summarizes the central pair of axioms of general relativity: the Einstein equation, which specifies the way in which matter determines the curvature of spacetime, and the geodesic equation that fixes the motion of matter in that curved arena. Einstein's equation describing the coupling between gravitation (*spacetime curvature*) and matter (*stress-energy tensor*) takes the form

$$\boldsymbol{G} \;=\; \kappa\,\boldsymbol{T} \tag{1}$$

where $\boldsymbol{G}$ is the Einstein curvature tensor that is constructed entirely from the Riemann curvature of the spacetime metric. On the right-hand side of equation [1], $\boldsymbol{T}$ is a symmetric tensor with zero divergence. In the Lagrangian formalism, the diffeomorphism invariance of the Einstein-Hilbert action guarantees that $\boldsymbol{T}$ is always divergenceless, a feature that enforces the conservation of energy-momentum. In other words, $\boldsymbol{T}$ is the frame-independent geometric object that must act as the source for gravity.

The foregoing discussion conjures a picture of the stars of our night sky as wanderers through spacetime, each helping to contribute to the spacetime curvature that is felt by all others. In this sense, the *gravitational field* is spacetime itself.

With the benefit of hindsight, it can be stated that the most revealing formulations of general relativity are those that make use of an algorithm that goes by the name of *gauge theory*. The fundamental premise of gauge formulations is that physically relevant degrees of freedom of a system of interest are those that are invariant under a transformation of the variables in the theory. The gauge idea had a near-death experience even before it had been fully developed. In an attempt to unify general relativity with electromagnetism, Hermann Weyl, the first to apply the gauge idea to a physical theory, mistakenly identified the electromagnetic potential with the dilatational gauge field that allows scale transformations to be local. With Weyl's realization that the local scale symmetry was in fact a phase change that is associated with quantum mechanical wavefunctions, a $U(1)$ gauge theory that describes the electromagnetic interaction was formalized [[5]]. The extension of this early work during the latter half of the 20th century culminated in the most celebrated gauge theory - the *standard model* of particle physics.

A standard view among physicists at present is that all models that successfully describe nature have–at least at low energy[1]–the mathematical structure known as **gauge symmetry**. The concept of gauge symmetry is rooted in the observation that the *action functional*, from which the equations of motion of a physical system are computed, is invariant with respect to coordinate or other field transformations. In Maxwell's theory of electromagnetism for instance, the vector potential A_μ is defined only up to the addition of the gradient of a scalar function. Ironically, it is this very redundancy that makes the local invariance of the action possible. The great insight

---

[1]String theory, not in itself a gauge theory, is unique in specifying a unifying symmetry group from which the standard model gauge theories may descend.

of gauge theory is that equivalence classes of field configurations correspond to the same physical reality.

In order for a variational problem to be invariant under a given transformation, say spacetime translations, its action mustn't depend explicitly on the coordinates. Put differently, the laws of physics do not change when you translate your laboratory to a different section of spacetime. The eminent mathematician Emmy Noether's supernal contribution to physics was to prove that such symmetries (e.g. spacetime translations) inexorably lead to conservation laws. Noether's theorem states that:

> For each continuous symmetry of a system, there is a corresponding conserved quantity.

In more precise language, to every differentiable symmetry that is generated by local actions, there exists a corresponding conserved current. The propagation of these quantities in time is called a *Noether current*.

The relationship between Noether currents and gauge fields echoes the dual dictum of general relativity. The global symmetries that give rise to Noether currents tell us what symmetries to make local by introducing equivalence classes of gauge potentials. These potentials give rise to new fields. These new fields are responsible for interactions between the particles of the original symmetric system, while the modified Noether currents of those original particles provide the sources for the new fields. Therefore, we may say,

> Matter gives the source for gauge fields
>
> Gauge fields tell matter fields how to move

In this thesis, we deal with a gauge theory of gravity, known as *biconformal gravity*, a model that promises to give deeper insights into the geometry of spacetime, the emergence of time, and the electroweak interaction. Biconformal gravity has been

shown to give rise to a $2n$-dimensional Kähler manifold. The curvatures in this model, in spite of their initial dependence on all $2n$ independent coordinates, reduce to the usual Riemannian curvature tensor in n-dimensions. Biconformal gravity generically reproduces the physics of vacuum general relativity [6].

We address ourselves in this thesis to the question of matter sources for biconformal gravity. Is it possible to show that, like gravity, a fully $2n$-dimensional Yang-Mills matter Lagrangian density leads to the usual Yang-Mills equations and Yang-Mills gravity sources?

In an attempt to couple biconformal gravity to Yang-Mills matter fields, we have found that the usual Yang-Mills type Lagrangian density does not lead to the correct Yang-Mills source for gravity. Indeed, the usual action functional thwarts the usual approach to gravitational solutions.

A surprising ambiguity arises from the fact that the theory has a number of invariant tensors effectively obscuring the correct way to define the Hodge dual operation as well as the inner product. This ambiguity thereby makes it possible to construct a number of "allowed" Lagrangian densities from the curvatures that arise in the theory, a problem that does not arise in general relativity.

An additional ambiguity arises from the presence of two independent non-degenerate, symmetric tensors–the Kähler form and the Killing form. We have been able to show that the correct matter couplings can only arise with the use of the Killing form as the biconformal metric.

A striking finding in dealing with the these difficulties is the introduction of a "twist" in the action for the Yang-Mills source fields. Assuming only vanishing torsion, we have shown that the n-dimensional gravitational field equations are sourced by a symmetric, divergence-free tensor on the co-tangent bundle of spacetime.

Our work thus far has explored in great detail how the field equations of a Yang-

Mills gauge theory that is formulated on a $2n$-dimensional space reduces to the usual Yang-Mills sources for general relativity on an n-dimensional sub-manifold.

II. SPACETIME

A starting point of Einstein's general theory of relativity is a description of relationships among **events**.

The "grand arena" in which everything happens, **spacetime**, is modeled as a collection of all events. Objects in the world are 4-dimensional bundles of events represented by a pair $(\mathcal{M}, \mathbf{g})$ where $\mathcal{M}$ is a set of points equipped with with a well-defined topology and $\mathbf{g}$ is a Lorentzian metric.

Definition: An **event** is a point that is characterized by the spatial coordinates $(x^i, i = 1, 2, 3)$ and the time t, denoted by $\{x^\mu, \mu = 0, 1, 2, 3\}$ where

$$x^{\mu=0} = ct \tag{2}$$

$$x^{\mu=i} = x^i \tag{3}$$

To invoke the potent tools of calculus, we consider an n-dimensional differentiable manifold $\mathcal{M}^n$, a space that is locally Euclidean. According to Galileo and Newton, whilst one *inertial observer* may choose a favorite set of coordinate labels in spacetime (t, x, y, z), a second inertial observer moving at velocity $\mathbf{v}$ relative to the first observer will label the point with different coordinates, (t, x', y', z'). Different inertial observers can thereby only agree on an **equivalence class** of coordinates related by the transformation

$$x' = x - vt$$
$$y' = y$$
$$z' = z$$
$$t' = t$$

This feature of *Galilean relativity* highlights a crucial point: a model for spacetime is defined only up to an *equivalence class.* In general relativity, spacetime $(\mathcal{M}, \mathbf{g})$ is represented by an equivalent *i.e.* diffeomorphic invariant class of ordered quadruples of the form (x^1, x^2, x^3, x^4) where $\mathcal{M}$ is a 4-dimensional Hausdorff manifold and $\mathbf{g}$ is a Lorentzian metric on $\mathcal{M}$.

Definition: An **_inertial frame_** is one in which Newton's first law holds.

The history of our lives, or any physical object in the universe, tracks a **_worldline_** of events described by the equations $x(t)$, $y(t)$, $z(t)$. These equations can be summarized by a single vector equation $\mathbf{x}(t)$. The Galilean group of transformations consists in all spacetime transformations from one inertial frame to another of the form

$$
\begin{aligned}
\mathbf{x}'(t) &= \mathbf{R}\,\mathbf{x}(t) + \mathbf{v}_0 t + \mathbf{x}_0 \\
\mathbf{v}'(t) &= \mathbf{R}\,\mathbf{v}(t) + \mathbf{v}_0 \\
t' &= t + t_0
\end{aligned}
$$

where $(\mathbf{x}_0, \mathbf{v}_0, t_0)$ are the initial conditions and $\mathbf{R}$ is a rotation matrix. Since $\mathbf{v}_0$ is constant, it follows that the magnitude of the acceleration is constant

$$
\mathbf{a}'(t) = \mathbf{R}\,\mathbf{a}(t)
$$

In components, Newton's second law for a system of n-particles reads

$$
m_i \frac{\mathrm{d}^2 \overrightarrow{\mathbf{x}}_i}{\mathrm{d}t^2} = -\sum_{k=1}^{n} \frac{\partial U}{\partial x_{ik}} \hat{\mathbf{x}}_{ik} \tag{4}
$$

where U is the potential and the inter-particle distances, x_{jk} are given by

$$x_{jk} \;=\; |\mathbf{x}_j - \mathbf{x}_k| \;=\; \sqrt{\sum_{a=1}^{3} \left(x_j^a - x_k^a\right)^2}$$

and

$$\hat{\mathbf{x}}_{ik} \;=\; \frac{\mathbf{x}_i - \mathbf{x}_k}{x_{ik}}$$

Since $\mathbf{a}' = \mathbf{a}$, we see that the laws of mechanics are the same for all observers moving at constant velocity. Rotations, translations, and boosts from one coordinate system to another form the ***Galilean group***. Since each transformation of the Galilean group corresponds to only one constant, $\mathbf{v}$, the Galilean group is said to be a 1-***parameter group*** (more about this later).

How do electromagnetic waves propagate through the vacuum of space? Is there anything in this vast universe that is absolutely at rest? With the hope of answering these questions, it was conjectured that space is filled with an invisible jelly-like medium , called the *aether*, with respect to which the speed of objects was to be measured. An object that was at rest with respect to the aether was therefore said to be in a state of *absolute rest*.

In 1900 a brilliant student of Charles Hermite, Henri Poincaré wrote[7]:

> Our aether, does it really exist? I do not believe that more precise mea-
> surements could ever reveal anything more than *relative* displacements.

Five years later, a doctoral student at the University of Zurich boldly challenged the very existence of the aether. Albert Einstein's special theory of relativity was founded on two postulates:

1. All physical laws are the same for uniformly moving observers

2. The speed of light in vacuum is the same in all inertial frames of reference, and is independent of the motion of the source.

Before we explore the consequences of these postulates, let us briefly revisit a few crucial ideas whose roots run deep in the works of two giants of human thought: Carl Friedrich Gauss, and Georg Friedrich Bernhard Riemann.

Metric

Gauss once asked himself how a small creature who did not have global information (God's eye-view) about a surface $\mathcal{M}$ on which they were confined, could nonetheless perceive the geometry of their world. He resolved this puzzle by showing that the small creature could fruitfully study infinitesimally nearby points $p \in \mathcal{M}$.

Gauss introduced to modern mathematics the concept of *curvilinear coordinates* (u, v), by way of a question- Is it possible to define the length of a curve that starts at $p_1 \in \mathcal{M}$ and terminates at $p_2 \in \mathcal{M}$ in terms of completely local data? To answer this question, he put coordinates (u, v) at p_1 and $(u + \mathrm{d}u, v + \mathrm{d}v)$ at p_2 . In order to apply Pythagoras' theorem, Gauss observed that a very small region around any point p on the surface could be approximated by a tangent plane, $T_p\mathcal{M}$ at that point such that the line element takes the form:

$$\mathrm{d}s^2 \;=\; \mathrm{d}u^2 + \mathrm{d}v^2 \tag{5}$$

The points $\mathrm{d}x$ and $\mathrm{d}y$ are transformed linearly

$$\begin{pmatrix} \mathrm{d}x \\ \mathrm{d}y \end{pmatrix} = \begin{pmatrix} a\,(u,v) & b\,(u,v) \\ c\,(u,v) & d\,(u,v) \end{pmatrix} \begin{pmatrix} \mathrm{d}u \\ \mathrm{d}v \end{pmatrix}$$

The Gaussian line element was first written in 1828 (when Riemann was 2 years old):

$$ds^2 \;=\; F\left(u,v\right)\,du^2 + G\left(u,v\right)\,dv^2 + H\left(u,v\right)\,du\,dv \tag{6}$$

where

$$F \;=\; a^2 + c^2$$
$$G \;=\; b^2 + d^2$$
$$H \;=\; 2\,a\,b + 2\,c\,d$$

To compute the distance between arbitrarily close *events* in Euclidean 4-space, compute

$$ds^2 \;=\; \delta_{\mu\nu}\,dx^\mu\,dx^\nu \;=\; \left(dx^0\right)^2 + \left(dx^1\right)^2 + \left(dx^2\right)^2 + \left(dx^3\right)^2$$

where $\delta_{\mu\nu}$ is the Euclidean *metric tensor* :

$$\delta_{\mu\nu} \;=\; \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Notice that a restriction of the metric to 3-dimensions yields Pythagoras' formula.

Riemann, a student of Gauss, reformulated the idea of a line element as generic symmetric quadratic form on an n -dimensional *manifold*. Consider an n-dimensional C^k Riemannian manifold $\mathcal{M}$ with metric

$$ds^2 = g_{\alpha\beta}dX^\alpha dX^\beta \tag{7}$$

where $\alpha, \beta, \ldots = 1, 2, \ldots n$ and the metric tensor $\mathbf{g}$ at a point $p \in \mathcal{M}$ is a symmetric second rank tensor that is used to define the inner product between two *vectors.*

Vector Fields

Consider a curve in $\mathcal{M}$ that is described in local coordinates $\{x^\mu\}$ by

$$x^\mu = x^\mu(\lambda) \quad \lambda \in [a, b]$$

The velocity vector in these coordinates is

$$\mathrm{v}^\mu \equiv \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} = \left(\frac{\mathrm{d}x^1}{\mathrm{d}\lambda}, \cdots, \frac{\mathrm{d}x^n}{\mathrm{d}\lambda} \right)$$

In general relativity, the invariant parameter on the worldline of a particle is chosen to be its proper time , τ

$$\mathrm{u}^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}$$

The 4-**velocity** expressed as

$$\mathbf{u} = \mathrm{u}^\mu \hat{\mathbf{e}}_\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} \hat{\mathbf{e}}_\mu \tag{8}$$

where $\hat{\mathbf{e}}_\mu$ are coordinate basis vectors which are defined as follows

$$\hat{\mathbf{e}}_\mu = \frac{\partial}{\partial x^\mu}$$

A ***vector*** can therefore be defined as a *directional derivative operator* at a point $p \in \mathcal{M}$ along a curve $C\left(\lambda\right)$

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} \quad = \quad \frac{\mathrm{d}x^{\mu}}{\mathrm{d}\lambda}\frac{\partial}{\partial x^{\mu}} \tag{9}$$

The tangent vector ***field*** $\frac{\mathrm{d}}{\mathrm{d}\lambda}$ changes with the parameter λ. So a vector tangent at p_0 is obtained by evaluating $\frac{\mathrm{d}}{\mathrm{d}\lambda}\big|_p$.

Vectors that are defined at a point p lie in a *tangent space* to $\mathcal{M}$, and thereby form a vector space denoted by $T_p\mathcal{M}$. The disjoint union of all the tangent spaces on $\mathcal{M}$ are called the ***tangent bundle*** of $\mathcal{M}$:

$$T\mathcal{M} \quad = \quad \cup_{p\in\mathcal{M}} T_p\mathcal{M}$$

We can use a very special metric to define an inner product between pairs of basis vectors as follows

$$g_{\mu\nu} \quad = \quad \left\langle \frac{\partial}{\partial x^{\mu}}, \frac{\partial}{\partial x^{\nu}} \right\rangle \tag{10}$$

The *inverse metric* is defined by the relation

$$g^{\mu\alpha}g_{\alpha\nu} \quad = \quad \delta^{\mu}{}_{\nu} \tag{11}$$

Since the metric tensor "maps" two vectors into a scalar, it can also be represented as a product on two 1-*forms*.

Linear Forms

Since $T_p\mathcal{M}$ is a vector space, we can define a *dual* vector space , called the *cotangent space, $T_p^*\mathcal{M}$, i.e.* the vector space of all linear maps $\boldsymbol{\omega}$ along curves through $p \in \mathcal{M}$.

According to the **fundamental theorem of calculus**, the integral of a function f over an interval $[a, b]$ is computed by finding an *antiderivative F* of F :

$$\int_a^b f(x)\, \mathrm{d}x \;=\; F(b) - F(a) \tag{12}$$

In the language of differential forms, $f(x)\, \mathrm{d}x$ is a 1-**form**, the exterior derivative of a 0-form $\mathbf{d}F$. Equation [12] generalizes to any p-form as **Stokes theorem** which reads

$$\int_{\partial\Omega} \boldsymbol{\omega} \;=\; \int_{\Omega} \mathbf{d}\boldsymbol{\omega} \tag{13}$$

In components $\boldsymbol{\omega}$ can be written as follows

$$\boldsymbol{\omega} \;=\; \omega_a\, \mathbf{e}^a \tag{14}$$

where $\{\mathbf{e}^a\}$ is a basis for $T_p^*\mathcal{M}$. Linearity means

$$\boldsymbol{\omega}\,(a\,\mathbf{v} + b\,\mathbf{u}) \;=\; a\,\boldsymbol{\omega}\,(\mathbf{v}) + b\,\boldsymbol{\omega}\,(\mathbf{u})$$

The disjoint union of all the *cotangent spaces* on $\mathcal{M}$ are called the **cotangent bundle** of $\mathcal{M}$

$$T^*\mathcal{M} \;=\; \cup_{p\in\mathcal{M}}\, T_p^*\mathcal{M}$$

A p-form is obtained by by taking the antisymmetric product given by $\wedge_k T_p\mathcal{M}$ where

the **wedge** symbol $\wedge$ denotes the antisymmetric tensor product. In local coordinates, a p-form is given by

$$\boldsymbol{\omega} \;=\; \frac{1}{p!}\,\omega_{\alpha_1\alpha_2\ldots\alpha_p}\mathrm{d}x^{\alpha_1}\wedge\mathrm{d}x^{\alpha_2}\cdots\wedge\mathrm{d}x^{\alpha_p} \tag{15}$$

The exterior derivative of a p-form is a $(p+1)$-form

$$\mathbf{d}\boldsymbol{\omega} \;=\; \partial_\beta\omega_{\alpha_1\alpha_2\ldots\alpha_p}\mathrm{d}x^\beta\wedge\mathrm{d}x^{\alpha_1}\wedge\mathrm{d}x^{\alpha_2}\cdots\wedge\mathrm{d}x^{\alpha_p} \tag{16}$$

Consider the wedge product of two differential forms:

$$\boldsymbol{\omega} \;=\; A_{i_1i_2\ldots i_p}\mathbf{d}x^{i_1}\wedge\cdots\cdot\wedge\mathbf{d}x^{i_p}$$

$$\boldsymbol{\eta} \;=\; B_{k_1k_2\ldots k_q}\mathbf{d}x^{k_1}\wedge\cdots\cdot\wedge\mathbf{d}x^{k_p}$$

So

$$
\begin{aligned}
\mathbf{d}\left(\boldsymbol{\omega}\wedge\boldsymbol{\eta}\right) \;=\;& \mathbf{d}\left(A_{i_1i_2\ldots i_p}B_{k_1k_2\ldots k_q}\right)\mathbf{d}x^{i_1}\wedge\cdots\cdot\wedge\mathbf{d}x^{i_p}\wedge\mathbf{d}x^{k_1}\wedge\cdots\cdot\wedge\mathbf{d}x^{k_p}\\
\;=\;& \left[\left(\mathbf{d}A_{i_1i_2\ldots i_p}\right)B_{k_1k_2\ldots k_q}+A_{i_1i_2\ldots i_p}\left(\mathbf{d}B_{k_1k_2\ldots k_q}\right)\right]\\
& \times\mathbf{d}x^{i_1}\wedge\cdots\cdot\wedge\mathbf{d}x^{i_p}\wedge\mathbf{d}x^{k_1}\wedge\cdots\cdot\wedge\mathbf{d}x^{k_p}\\
\;=\;& \left\{\mathbf{d}A_{i_1i_2\ldots i_p}\mathbf{d}x^{i_1}\wedge\cdots\cdot\wedge\mathbf{d}x^{i_p}\right\}\wedge\left[B_{k_1k_2\ldots k_q}\mathbf{d}x^{k_1}\wedge\cdots\cdot\wedge\mathbf{d}x^{k_p}\right]\\
& +(-1)^q\left\{A_{i_1i_2\ldots i_p}\mathbf{d}x^{i_1}\wedge\cdots\cdot\wedge\mathbf{d}x^{i_p}\right\}\wedge\left[\mathbf{d}B_{k_1k_2\ldots k_q}\wedge\mathbf{d}x^{k_1}\wedge\cdots\cdot\wedge\mathbf{d}x^{k_p}\right]\\
\;=\;& \mathbf{d}\boldsymbol{\omega}\wedge\boldsymbol{\eta}+(-1)^q\left(\boldsymbol{\omega}\wedge\mathbf{d}\boldsymbol{\eta}\right)
\end{aligned}
$$

Given a basis $\{\mathbf{e}^a\}$, there exists a unique basis $\{\mathbf{e}_a\}$ for $T_p\mathcal{M}$ satisfying

$$\langle\mathbf{e}^a,\mathbf{e}_b\rangle \;=\; \delta^a_b \tag{17}$$

A *duality* exists been vectors and forms and is defined by

$$
\begin{aligned}
\langle \boldsymbol{\omega}, \mathbf{v} \rangle &= \left\langle \omega_a \mathbf{e}^a, \mathrm{v}^b \mathbf{e}_b \right\rangle \\
&= \omega_a \mathrm{v}^b \left\langle \mathbf{e}^a, \mathbf{e}_b \right\rangle \\
&= \omega_a \mathrm{v}^b \delta^a_b \\
&= \omega_a \mathrm{v}^a
\end{aligned}
$$

Since

$$
\langle \boldsymbol{\omega}, \mathbf{v} \rangle = \omega_a \mathrm{v}^a \tag{18}
$$

We can expand the bases and invoke *linearity* to give

$$
\begin{aligned}
\omega_a \mathrm{v}^a &= \langle \boldsymbol{\omega}, \mathbf{v} \rangle \\
\Rightarrow \omega_a \, \mathrm{v}^b \, \delta^a_b &= \omega_a \, \mathrm{v}^b \left\langle \mathbf{d}x^a, \frac{\partial}{\partial x^b} \right\rangle
\end{aligned}
$$

The last line holds for all $\boldsymbol{\omega}$ and $\mathbf{v}$. Therefore the duality of the basis sets is expressed as

$$
\delta^a_b = \left\langle \mathbf{d}x^a, \frac{\partial}{\partial x^b} \right\rangle \tag{19}
$$

Invariant Interval

In **general relativity**, the invariant interval is given by

$$
\mathrm{d}s^2 = g_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu
$$

The metric tensor $g_{\mu\nu}$ is the basic field that is required to describe the gravitational interaction. The laws that govern the dynamics of a system should be intrinsic and should not depend on the set of variables chosen to describe it. To implement this idea, the **action functional** is required to be invariant under arbitrary coordinate transformations *i.e.* $x \longrightarrow x'(x)$. With the appropriate metric transformation, we have

$$
\begin{aligned}
\mathrm{d}s^2 &= g'_{\alpha\beta}\mathrm{d}x'^{\alpha}\mathrm{d}x'^{\beta} \\
&= g'_{\alpha\beta}\frac{\partial x'^{\alpha}}{\partial x^{\mu}}\frac{\partial x'^{\beta}}{\partial x^{\nu}}\mathrm{d}x^{\mu}\mathrm{d}x^{\nu} \\
&= g_{\mu\nu}\mathrm{d}x^{\mu}\mathrm{d}x^{\nu}
\end{aligned}
$$

The action is obtained by taking the integral of the Lagrangian density over spacetime.

The factor $\sqrt{-g}$ must be included in the integration measure because of the transformation properties of the *volume element* (see section 2.3.1)

To see why $\sqrt{-g}$ must be included from another view-point, we use the divergence formula

$$
D_{\alpha}\mathrm{v}^{\beta} = \frac{1}{\sqrt{-g}}\partial_{\alpha}\left(\sqrt{-g}\,\mathrm{v}^{\beta}\right)
$$

The covariant form of Gauss' theorem shows that the invariant volume element is given by $\sqrt{|g|}d^4x$:

$$
\int d^4x\,\sqrt{-g}\,D_{\alpha}\mathrm{v}^{\beta} = \int d^4x\,\partial_{\alpha}\left(\sqrt{-g}\,\mathrm{v}^{\beta}\right) = 0
$$

The Levi-Civita tensor in arbitrary coordinates

The totally antisymmetric symbol in n-dimensions is

$$\varepsilon_{i_1 i_2 \ldots i_n}$$

Under a diffeomorphism $\varepsilon_{i_1 i_2 \ldots i_n}$ transforms as

$$\varepsilon_{i_1 i_2 \ldots i_n} \frac{\partial x^{i_i}}{\partial y^{j_1}} \cdots \frac{\partial x^{i_n}}{\partial y^{j_n}} = \mathrm{J} \, \varepsilon_{j_1 j_2 \ldots j_n}$$

If we think of $\varepsilon_{i_1 i_2 \ldots i_n}$ as an n-form,

$$\boldsymbol{\Phi} = \varepsilon_{i_1 i_2 \ldots i_n} \mathbf{d}x^{i_1} \wedge \cdots \wedge \mathbf{d}x^{i_n} = \varepsilon'_{i_1 i_2 \ldots i_n} \frac{\partial x^{i_i}}{\partial y^{j_1}} \mathbf{d}y^{j_1} \wedge \cdots \wedge \frac{\partial x^{i_n}}{\partial y^{j_n}} \mathbf{d}y^{j_n}$$

so that to make the volume form invariant, we invert the Jacobian matrices to find

$$
\begin{aligned}
\varepsilon'_{j_1 \ldots j_n} &= \frac{\partial y^{i_i}}{\partial x^{j_1}} \wedge \cdots \wedge \frac{\partial y^{i_n}}{\partial x^{j_n}} \varepsilon_{i_1 \ldots i_n} \\
&= \frac{1}{J} \varepsilon_{j_1 \ldots j_n}
\end{aligned}
$$

Now, multiplying by $\sqrt{|g|}$ introduces a factor of J that cancels this one, leaving the combination

$$\sqrt{|g|} \varepsilon_{i_1 \ldots i_n}$$

to transform as a tensor, not a tensor density. This is the Levi-Civita tensor

$$\varepsilon_{i_1 i_2 \ldots i_n} \frac{\partial x^{i_i}}{\partial y^{j_1}} \cdots \frac{\partial x^{i_n}}{\partial y^{j_n}} = \mathrm{J} \, \varepsilon_{j_1 j_2 \ldots j_n}$$

where J is the Jacobian of the coordinate transformation and is given by

$$J = \det\left(\frac{\partial x^k}{\partial y^l}\right)$$

Since J is a *density* rather than a scalar, we see that $\varepsilon_{i_1 i_2 \ldots i_n}$ is not a tensor. To fix this, recall the transformation rule for the metric

$$g'_{kl} = \frac{\partial x^i}{\partial y^k}\frac{\partial x^j}{\partial y^l}g_{ij}$$

Taking the determinant on both sides:

$$\begin{aligned}
g' &= \det\left(\frac{\partial x^i}{\partial y^k}g_{ij}\frac{\partial x^j}{\partial y^l}\right)\\
&= \det\left(\frac{\partial x^i}{\partial y^k}\right)\det(g_{ij})\det\left(\frac{\partial x^j}{\partial y^l}\right)\\
&= J^2\, g
\end{aligned}$$

This allows us to define a tensorial object

$$e_{i\ldots j} = \sqrt{g}\,\varepsilon_{i\ldots j}$$

Using n-copies of the inverse metric to raise all indices on $e_{i_1 i_2 \ldots i_n}$

$$\begin{aligned}
e^{i_1 i_2 \ldots i_n} &= g^{i_1 j_1}g^{i_2 j_2}\ldots g^{i_n j_n}\left(\sqrt{g}\,\varepsilon_{j_1 j_2 \ldots j_n}\right)\\
&= \frac{\sqrt{g}}{g}\varepsilon^{i_1 i_2 \ldots i_n}\\
&= \frac{1}{\sqrt{g}}\varepsilon^{i_1 i_2 \ldots i_n}
\end{aligned}$$

For an antisymmetric rank p-tensor, F, the **Hodge dual** is defined by

$$
{}^*F \;=\; \frac{1}{p!\,(n-p)!}\,\varepsilon_{\alpha_1\cdots\alpha_{n-p}}{}^{\alpha_{n-p+1}\cdots\alpha_n}\,F_{\alpha_{n-p+1}\cdots\alpha_n}
$$

In Minkowski space, the Hodge star operation is as follows:

$$
\begin{aligned}
{}^*\left(\mathbf{d}x^\mu\right) &= \frac{1}{3!}g^{\mu\sigma}\,\varepsilon_{\sigma\nu\gamma\tau}\mathbf{d}x^\nu\wedge\mathbf{d}x^\gamma\wedge\mathbf{d}x^\tau\\
{}^*\left(\mathbf{d}x^\mu\wedge\mathbf{d}x^\nu\right) &= \frac{1}{2!}g^{\mu\sigma}\,g^{\nu\rho}\,\varepsilon_{\sigma\rho\gamma\tau}\mathbf{d}x^\gamma\wedge\mathbf{d}x^\tau\\
{}^*\left(\mathbf{d}x^\mu\wedge\mathbf{d}x^\nu\wedge\mathbf{d}x^\rho\right) &= \frac{1}{1!}g^{\mu\sigma}\,g^{\nu\delta}\,g^{\gamma\eta}\,\varepsilon_{\sigma\delta\eta\tau}\mathbf{d}x^\tau\\
{}^*\left(\mathbf{d}x^\mu\wedge\mathbf{d}x^\nu\wedge\mathbf{d}x^\rho\wedge\mathbf{d}x^\tau\right) &= g^{\mu\alpha}\,g^{\nu\beta}\,g^{\rho\delta}\,g^{\tau\gamma}\,\varepsilon_{\alpha\beta\delta\gamma}
\end{aligned}
$$

It follows from that

$$
{}^*1 \;=\; \frac{1}{4!}\,\varepsilon_{\mu\nu\alpha\beta}\,\mathbf{d}x^\mu\wedge\mathbf{d}x^\nu\wedge\mathbf{d}x^\alpha\wedge\mathbf{d}x^\beta
$$

On an n-dimensional Riemannian manifold, the above relationship generalizes as follows

$$
\begin{aligned}
{}^*\left(\mathbf{d}x^{\mu_1}\wedge\mathbf{d}x^{\mu_2}\wedge\cdots\wedge\mathbf{d}x^{\mu_p}\right) &= {}^*1\\
&= \frac{1}{(n-p)!}\sqrt{|g|}\,\varepsilon^{\mu_1\cdots\mu_p}{}_{\nu_{p+1}\cdots\nu_n}\,\mathbf{d}x^{\nu_{p+1}}\wedge\cdots\wedge\mathbf{d}x^{\nu_n}\\
&= \frac{1}{n!}\sqrt{|g|}\,\varepsilon_{\mu_1\cdots\mu_n}\,\mathbf{d}x^{\mu_1}\wedge\cdots\wedge\mathbf{d}x^{\mu_n}
\end{aligned}
$$

Notice that the Hodge star operation depends on a choice of the metric. On an n-dimensional Riemannian manifold, an n-form is the *volume form* and can be written explicitly as

$$
\mathrm{d}V \;=\; \frac{1}{n!}\sqrt{|g|}\,\varepsilon_{\mu_1\cdots\mu_p}\,\mathbf{d}x^{\mu_1}\wedge\cdots\wedge\mathbf{d}x^{\mu_p}
\tag{20}
$$

It is straightforward to prove the Poincarè lemma.

The Poincarè Lemma For any p-form $\boldsymbol{\omega}$,

$$\mathbf{d}^2\boldsymbol{\omega} \;=\; 0$$

Proof: Poincarè lemma

Consider any p-form

$$\boldsymbol{\omega} \;=\; A_{i_1 i_2 \ldots i_p}\mathbf{d}x^{i_1} \wedge \cdots \cdots \wedge \mathbf{d}x^{i_p}$$

A single exterior derivative gives

$$\mathbf{d}\boldsymbol{\omega} \;=\; \partial_j A_{i_1 i_2 \ldots i_p}\mathbf{d}x^{i_1} \wedge \cdots \cdots \wedge \mathbf{d}x^{i_p}$$
$$=\; A_{[i_1 i_2 \ldots i_p, j]}\mathbf{d}x^{j} \wedge \mathbf{d}x^{i_1} \wedge \cdots \cdots \wedge \mathbf{d}x^{i_p}$$

Therefore, applying the exterior derivative again,

$$\mathbf{d}^2\boldsymbol{\omega} \;=\; A_{[i_1 i_2 \ldots i_p, jk]}\mathbf{d}x^{k} \wedge \mathbf{d}x^{j} \wedge \mathbf{d}x^{i_1} \wedge \cdots \cdots \wedge \mathbf{d}x^{i_p}$$
$$=\; \frac{\partial^2}{\partial x^j \partial x^k}A_{i_1 i_2 \ldots i_p}\mathbf{d}x^{k} \wedge \mathbf{d}x^{j} \wedge \mathbf{d}x^{i_1} \wedge \cdots \cdots \wedge \mathbf{d}x^{i_p}$$
$$\equiv\; 0$$

The result follows because of the symmetry of mixed partials. With the definitions

1. A form $\boldsymbol{\omega}$ is said to be *closed* if $\mathbf{d}\boldsymbol{\omega} \;=\; 0$.

2. A form $\boldsymbol{\omega}$ is *exact* if there exists another form $\boldsymbol{\eta}$ such that $\boldsymbol{\omega} \;=\; \mathbf{d}\boldsymbol{\omega}$

we state an extremely useful result.

Theorem: (Converse to the Poincarè lemma) If U is a star-shaped open subset of $\mathbb{R}^n$ (or any manifold diffeomorphic to $\mathbb{R}^n$), then every *closed* differential form on U is *exact*.

This means that if the region is simple, we can immediately integrate exterior derivatives. For if

$$\mathbf{d}\boldsymbol{\omega} \;=\; 0$$

for any p-form $\boldsymbol{\omega}$ then we know there exists a $(p-1)$-form $\boldsymbol{\eta}$ such that

$$\boldsymbol{\omega} \;=\; \mathbf{d}\boldsymbol{\eta}$$

Orthonormal Bases

Locally, one can choose a set of *orthonormal vector fields* as a basis for $T_p\mathcal{M}$

$$\mathbf{e}^a \;=\; e_\mu{}^a \mathbf{d}x^\mu \tag{21}$$

The invertible matrix $e_\mu{}^a$ is determined by the condition

$$\langle \mathbf{e}^a, \mathbf{e}^b \rangle \;=\; \eta^{ab} \tag{22}$$

where η^{ab} is the Lorentz metric. Greek indices, $\mu, \nu, \cdots$ label objects in coordinate basis $\mathbf{d}x^\mu$ and Latin indices label objects in the orthonormal basis $\mathbf{e}^a$, also called the **solder form**. Using Eq.(22), we can write the metric tensor $g_{\mu\nu}$ in terms of the

coefficients of the orthonormal basis

$$
\begin{aligned}
\eta^{ab} &= \left\langle \mathbf{e}^a, \mathbf{e}^b \right\rangle \\
&= \left\langle e_\mu{}^a \mathbf{d}x^\mu, e_\nu{}^b \mathbf{d}x^\nu \right\rangle \\
&= e_\mu{}^a e_\nu{}^b \left\langle \mathbf{d}x^\mu, \mathbf{d}x^\nu \right\rangle \\
&= e_\mu{}^a e_\nu{}^b g^{\mu\nu}
\end{aligned}
$$

Inverting both sides yields

$$
e_\mu{}^a e_\nu{}^b \eta_{ab} = g_{\mu\nu}
$$

where $e_a{}^\mu$ is inverse to $e_\mu{}^a$.

Principle of Equivalence

Here is Einstein's "happiest thought": the ***inertial mass*** of a particle (in Newton's second law) is equal to the ***gravitational mass*** (in Newton's law of gravitation). Thus, the trajectory of an uncharged freely falling massive particle is independent of mass and composition.

If we are inclined to follow Einstein down this *happy* trail, we conclude that the ***principle of equivalence*** asserts that a gravitational field can *locally* be transformed away. The claim is that one can always find a neighborhood near a point $p \in \mathcal{M}$ such that the ***connection*** vanishes.

Proposition: If $\mathbf{g}$ is a Lorentzian metric on $\mathcal{M}$, for every point $p \in \mathcal{M}$ there exists a neighborhood with a coordinate system such that to order $O\left(x^2\right)$

$$
g_{\mu\nu}\left(p\right) = \eta_{\mu\nu} \tag{23}
$$

With **g** differentiable, then coordinate system can be chosen so as to guarantee that

$$\Gamma^{\alpha}_{\beta\mu}(p) \;=\; 0 \tag{24}$$

Proof: We choose an arbitrary point p . At p , there is a frame

$$\hat{\mathbf{e}}_a \;=\; e_a{}^{\mu}\frac{\partial}{\partial y^{\mu}} \tag{25}$$

where y^{μ} is a coordinate system around p. The frame satisfies

$$\mathbf{g}\left(\hat{\mathbf{e}}_a, \hat{\mathbf{e}}_b\right) \;=\; \eta_{ab}$$

So

$$g_{\mu\nu}\, e_a{}^{\mu} e_b{}^{\nu} \;=\; \eta_{ab}$$

Taking the determinants of both sides

$$\det\left(g_{\mu\nu}\right)\det\left(e_a{}^{\mu} e_b{}^{\nu}\right) \;=\; \det\left(\eta_{ab}\right)$$
$$\Rightarrow \det\left(g_{\mu\nu}\right)\det\left(e_a{}^{\mu}\right)^2 \;=\; -1$$

Therefore

$$\det\left(e_a{}^{\mu}\right) \;\neq\; 0$$

Using the new coordinate system defined in equation [25]:

$$\mathbf{g}\left(\hat{\mathbf{e}}_a, \hat{\mathbf{e}}_b\right) \;=\; e_a{}^\mu \, e_b{}^\nu \, \mathbf{g}\left(\frac{\partial}{\partial y^\mu}, \frac{\partial}{\partial y^\nu}\right) \;=\; \eta_{ab}$$

With a bit more work we can show that there exists a coordinate choice in which the first derivatives also vanish. The upshot of this is that at each point of $\mathcal{M}$, we can establish a local change of frame so as to guarantee that the metric is flat (equivalence principle).

III. LIE GROUPS

Nowadays, geometries are defined by reference to the geometry's invariance group. The geometric arena in which field theories reside is the so-called **principal $\mathcal{G}$-bundle**.

Gauge fields, on this view point, are simply **connections** on principal fiber bundles. It was Élie Cartan who first showed how these fundamental fields in the gravitation case are encoded into the structure of a connection. Cartan introduced a set of 1-**forms** that later found unity in an equation that now bears his name alongside that of Ludwig Maurer. The **Maurer-Cartan equation** lies at the heart of Lie groups, to which we now turn our attention.

Definition: A Lie group is a topological group $\mathcal{G}$ which has the structure of a **differentiable manifold**. The multiplication map is

$$
\begin{aligned}
\mathcal{G} \times \mathcal{G} \quad &\to \quad \mathcal{G} \\
(g, h) \quad &\longmapsto \quad g \cdot h
\end{aligned}
$$

The inverse map exists and is differentiable

$$
\begin{aligned}
\mathcal{G} \quad &\to \quad \mathcal{G} \\
g \quad &\to \quad g^{-1}
\end{aligned}
$$

Proposition: Let $\mathcal{G}$ and $\mathcal{H}$ be Lie groups. Then the product manifold $\mathcal{G} \times \mathcal{H}$ with the direct product structure as a group is a Lie group.

The fact a Lie group $\mathcal{G}$ is also a smooth manifold has 2 important implications:

1. Sophus Lie proved that there exists a canonical Lie algebra. He identified the

canonical Lie algebra with the ***infinitesimal generators*** of the group.

2. The Lie group $\mathcal{G}$ can smoothly act on itself by means of two canonical actions known as the ***left*** and ***right translation***.

Definition: Let $g \in \mathcal{G}$

1. ***Left translation*** by g , L_g is the map:

$$L_g : \mathcal{G} \quad \to \quad \mathcal{G}, \; h \longmapsto g \circ h$$

2. ***Right translation*** by g , R_g is the map:

$$L_g : \mathcal{G} \quad \to \quad \mathcal{G}, \; h \longmapsto h \circ g$$

Definition: A group $\mathcal{G}$ is ***Abelian*** or commutative if $g \circ h = h \circ g$ for all $g, h \in \mathcal{G}$

Definition: A group $\mathcal{G}$ that possesses two ***subgroups*** $\mathcal{H}_1$ and $\mathcal{H}_2$ is said to be ***direct product*** of $\mathcal{H}_1$ and $\mathcal{H}_2$ *i.e.* $\mathcal{G} = \mathcal{H}_1 \otimes \mathcal{H}_2$ if:

1. The two subgroups $\mathcal{H}_1$ and $\mathcal{H}_2$ have only the unit element in common;

2. The elements of $\mathcal{H}_1$ commute with those of $\mathcal{H}_2$

3. Each element g of $\mathcal{G}$ is expressible in one and only one way as $g = h_1 \cdot h_2$, in terms of the elements h_1 of $\mathcal{H}_1$ and h_2 of $\mathcal{H}_2$.

The Lie Algebra of a Group

Definition: A Lie algebra $\mathfrak{g}$ is a ***vector space*** together with a skew-symmetric bilinear map satisfying the Jacobi identity. The defining *properties* of a Lie algebra are therefore:

1. The ***Lie bracket*** $[g, h]$ is a bilinear map of $\mathfrak{g} \times \mathfrak{g}$ into $\mathfrak{g}$

2. $[g, h] = -[h, g]$ (***antisymmetric***)

3. $[g, [h, w]] + [h, [w, g]] + [w, [g, h]] = 0$ (***Jacobi Identity***)

Definition: Given a Lie group $\mathcal{G}$, the tangent space to $\mathcal{G}$ at the identity $e \in \mathcal{G}$, is called the ***Lie algebra***, $\mathfrak{g}$, of the Lie group $\mathcal{G}$ *i.e.*

$$\mathfrak{g} = T_e \mathcal{G}$$

Definition: Consider a finite n-dimensional Lie algebra $\mathfrak{g}$ with basis $\{\mathbf{e}_a\}$ where $a = 1, \cdots, n$. The *commutation* relations of $\mathfrak{g}$ in the basis $\{\mathbf{e}_a\}$ are the Lie brackets:

$$[\mathbf{e}_a, \mathbf{e}_b] = c^d{}_{ab} \mathbf{e}_d$$

where $c^d{}_{ab}$ are called the *structure constants* of the Lie algebra $\mathfrak{g}$. Because of property (2) ,

$$c^d{}_{ab} = -c^d{}_{ba}$$

Convention: For Lie groups capital Latin letters are used, $\mathcal{G} = SU(N), SO(N), \cdots$, while the corresponding Lie algebra will be denoted by calligraphic lower case Latin letters (*Fraktur*) *i.e.* $\mathfrak{g} = \mathfrak{su}(N), \mathfrak{so}(N), \cdots$

Infinitesimal Transformations

An element of a Lie algebra $\mathfrak{g}$ corresponds to an infinitesimal transformation in the Lie group $\mathcal{G}$ near the identity e. The exponential map provides the crucial link between a linear Lie group $\mathcal{G}$ and its associated Lie algebra.

Definition: The *exponential of a matrix* $X \in M(n, \mathbb{K})$ (where $\mathbb{R}$ or $\mathbb{C}$) is defined in terms of a series that converges for every X

$$\mathrm{e}^X = \sum_{k=0}^{\infty} \frac{X^k}{k!}$$

For $g \in GL(n, \mathbb{K})$,

$$g\mathrm{e}^X g^{-1} = \mathrm{e}^{gXg^{-1}}$$

To highlight the relationship between the exponential map and infinitesimal transformations, consider a differentiable map $g(\lambda) \in \mathcal{G}$ (where $\lambda \in \mathbb{R}$) that satisfies the condition

$$g(0) = e$$

The Lie algebra $\mathfrak{g}$ corresponding to the Lie group $\mathcal{G}$ is defined as follows

$$X = \lim_{\lambda \to 0} \frac{g(\lambda) - e}{\lambda}$$

For given any $X \in \mathfrak{g}$ near the identity

$$g_\varepsilon = e + \varepsilon X$$

for $0 \leq \varepsilon \ll 1$ and $g_\varepsilon : \mathbb{R} \to \mathcal{G}$. We get a 1-parameter subgroup by applying this map many times (and this is where we get the exponential map):

$$
\begin{aligned}
g\left(\lambda\right) = \lim_{\varepsilon \to 0, n \to \infty} \left(1 + \varepsilon X\right)^n &= \lim_{\varepsilon \to 0, n \to \infty} \sum_{k=0}^{\infty} \frac{n!}{k!\left(n-k\right)!} \varepsilon^k X^k 1^{n-k} \\
&= \lim_{\varepsilon \to 0, n \to \infty} \sum_{k=0}^{\infty} \frac{n\left(n-1\right)\left(n-2\right)\cdots\left(n-k+1\right)}{k!n^k} \varepsilon^k n^k X^k \\
&= \lim_{\varepsilon \to 0, n \to \infty} \sum_{k=0}^{\infty} \frac{\frac{n}{n}\left(\frac{n-1}{n}\right)\left(\frac{n-2}{n}\right)\cdots\left(\frac{n-k+1}{n}\right)}{k!} \left(\varepsilon n\right)^k X^k
\end{aligned}
$$

Then with $\lambda = \lim\left(\varepsilon n\right)$,

$$
\begin{aligned}
\lim_{\varepsilon \to 0, n \to \infty} \left(1 + \varepsilon X\right)^n &= \lim_{\varepsilon \to 0, n \to \infty} \sum_{k=0}^{\infty} \frac{1\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{k-1}{n}\right)}{k!} \lambda^k X^k \\
&= \sum_{k=0}^{\infty} \frac{1}{k!} \lambda^k X^k \\
&= e^{\lambda X}
\end{aligned}
$$

This gives a curve in $\mathcal{G}$,

$$
g\left(\lambda\right) = e^{\lambda X}
$$

Note that

$$
\left.\frac{dg}{d\lambda}\right|_{\lambda=0} = X
$$

It's also a 1-parameter Lie subgroup:

$$
e^{\lambda_1 X} e^{\lambda_2 X} = e^{\left(\lambda_1 + \lambda_2\right) X}
$$

and

$$g(0) = e$$

$$g(-\lambda) = (g(\lambda))^{-1}$$

and it's all associative.

Definition: A ***one-parameter subgroup*** of a Lie group $\mathcal{G}$ is differentiable curve

$$\gamma : \mathbb{R} \to \mathcal{G}$$

which satisfies two conditions

$$\gamma(0) = e$$

$$\gamma(t+s) = \gamma(t)\gamma(s) \qquad (26)$$

where e is the group identity and $s, t \in \mathbb{R}$

We note that one-parameter are necessarily ***Abelian*** since

$$t + s = s + t$$

Are one-parameter subgroups re-parametrization invariant? In equation [26], let

$$t \longrightarrow \sigma t$$

$$s \longrightarrow \sigma s$$

and let

$$\sigma \;=\; 1 + a$$

Then we have (using associativity)

$$
\begin{aligned}
\gamma\left(\sigma t + \sigma s\right) \;&=\; \gamma\left(t + s + at + as\right) \\
&=\; \gamma\left(t\right)\gamma\left(s\right)\gamma\left(at\right)\gamma\left(as\right)
\end{aligned}
$$

This new 1-parameter group is Abelian

$$
\begin{aligned}
\gamma\left(\sigma t + \sigma s\right) \;&=\; \gamma\left(t\right)\gamma\left(s\right)\gamma\left(at\right)\gamma\left(as\right) \\
&=\; \gamma\left(t\right)\gamma\left(at\right)\gamma\left(s\right)\gamma\left(as\right) \\
&=\; \left(\gamma\left(t\right)\gamma\left(ar\right)\right)\left(\gamma\left(s\right)\gamma\left(as\right)\right) \\
&=\; \gamma\left(t + at\right)\gamma\left(s + as\right) \\
&=\; \gamma\left(\sigma t\right)\gamma\left(\sigma s\right)
\end{aligned}
$$

This means the parameter is *not* unique, so neither is X_e.

   In summary, a one-parameter group of *infinitesimal* transformations $\gamma\left(\sigma t\right)$ satisfies

1. $\gamma\left(\sigma t\right)\gamma\left(\sigma s\right) = \gamma\left(\sigma t + \sigma s\right)$; (composition law)

2. $\gamma\left(0\right) = 1$; (existence of the identity)

3. $\gamma\left(\sigma t\right)\gamma\left(-\sigma t\right) = e$; (existence of the inverse for every transformation)

4. $\gamma\left(\sigma t\right)\left[\gamma\left(\sigma s\right)\gamma\left(\sigma m\right)\right] = \left[\left(\sigma t\right)\gamma\left(\sigma s\right)\right]\gamma\left(\sigma m\right)$; (associativity)

Theorem: Every one-parameter subgroup of a linear Lie group $\mathcal{G}$ is formed by expo-

nentiation. Furthermore, If the matrices $\boldsymbol{\gamma}\left(t\right)$ form a one-parameter subgroup of a Lie group $\mathcal{G}$, then

$$\boldsymbol{\gamma}\left(t\right) \;\; = \;\; \mathrm{e}^{t\dot{\gamma}\left(0\right)} \tag{27}$$

where

$$\dot{\gamma}\left(0\right) \;\; = \;\; \left.\frac{\mathrm{d}\boldsymbol{\gamma}\left(t\right)}{\mathrm{d}t}\right|_{t=0} \tag{28}$$

Proof: Given $\mathcal{G}$, a group of $n \times n$ matrices, then a one-parameter subgroup of $\mathcal{G}$ is all the matrices $\boldsymbol{\gamma}\left(t\right)$ such that

$$\boldsymbol{\gamma}\left(s\right) \boldsymbol{\gamma}\left(t\right) \;\; = \;\; \boldsymbol{\gamma}\left(s+t\right) \tag{29}$$

for all $-\infty < s, t < +\infty$. Notice that for $s \; = \; 0$

$$\boldsymbol{\gamma}\left(0\right) \;\; = \;\; e$$

where e is the identity of the Lie group $\mathcal{G}$. Since the one parameter subgroup is of dimension 1 , $\dot{\gamma}\left(0\right)$ exists and not identically zero. For any t in equation [29], we compute the derivative as the limit of a difference quotient:

$$\begin{aligned}
\dot{\gamma}\left(t\right) \;\; &= \;\; \lim_{s \to 0} \left[\frac{\boldsymbol{\gamma}\left(t+s\right) - \boldsymbol{\gamma}\left(t\right)}{s}\right] \\
&= \;\; \lim_{s \to 0} \left[\boldsymbol{\gamma}\left(t\right) \left(\frac{\boldsymbol{\gamma}\left(s\right) - \dot{\gamma}\left(0\right)}{s}\right)\right]
\end{aligned}$$

Thus

$$\dot{\gamma}\left(t\right) \;\; = \;\; \boldsymbol{\gamma}\left(t\right) \dot{\gamma}\left(0\right) \tag{30}$$

Equation [30] is a differential equation with a unique solution that satisfies $\boldsymbol{\gamma}(0) = \boldsymbol{e}$ given by

$$\boldsymbol{\gamma}(t) = e^{-t\dot{\gamma}(0)}$$

Now let

$$\tilde{\boldsymbol{\gamma}}(t) = \boldsymbol{\gamma}(t) e^{-t\dot{\gamma}(0)} \tag{31}$$

Taking a derivative in equation [31] with respect to t:

$$
\begin{aligned}
\dot{\tilde{\boldsymbol{\gamma}}}(t) &= \frac{\mathrm{d}\tilde{\boldsymbol{\gamma}}(t)}{\mathrm{d}t} \\
&= [\dot{\boldsymbol{\gamma}}(t) - \boldsymbol{\gamma}(t)\,\dot{\boldsymbol{\gamma}}(0)]\,e^{-t\dot{\gamma}(0)} \\
&= [\dot{\boldsymbol{\gamma}}(0) - \dot{\boldsymbol{\gamma}}(0)]\,\boldsymbol{\gamma}(t)\,e^{-t\dot{\gamma}(0)} \\
&= [\dot{\boldsymbol{\gamma}}(0) - \dot{\boldsymbol{\gamma}}(0)]\,\tilde{\boldsymbol{\gamma}}(t) \\
&= 0
\end{aligned}
$$

Consequently

$$\tilde{\boldsymbol{\gamma}}(t) = \tilde{\boldsymbol{\gamma}}(0) = \boldsymbol{e}$$

The lesson here is that if $\gamma(t) : \mathbb{R} \to GL(n, \mathbb{K})$ is a continuous group **homomorphism**, then $\boldsymbol{\gamma}(t) = e^{t\dot{\gamma}(0)}$

Definition: Let ϕ be a map from a Lie algebra $\mathfrak{g}$ onto another Lie algebra $\mathfrak{g}'$ such that

1. For all $A, B \in \mathfrak{g}$ and all α, β of the field $\mathbb{K}$,

$$\phi\left(\alpha A + \beta B\right) \;=\; \alpha \phi\left(A\right) + \beta \phi\left(B\right)$$

2. For all $A, B \in \mathfrak{g}$

$$\phi\left(\left[A, B\right]\right) \;=\; \left[\phi\left(A\right), \phi\left(B\right)\right]$$

Then the ϕ is said to be a **homomorphism** from $\mathfrak{g}$ to $\mathfrak{g}'$. In other words, ϕ is *one-to-one* and both ϕ and its inverse ϕ^{-1} are continuous.

The map ϕ is determined by its behavior in an infinitesimal neighborhood of the identity. Suppose that ϕ is a *group homomorphism*,

$$\phi : \mathcal{G} \;\rightarrow\; GL\left(n, \mathbb{K}\right)$$

then

$$\dot{\phi} : X \in \mathfrak{g} \;\rightarrow\; \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \phi\left(\mathrm{e}^{tX}\right) \;\in\; \mathfrak{gl}\left(n, \mathbb{K}\right) \;=\; M\left(n, \mathbb{K}\right)$$

satisfies

$$\phi\left(\mathrm{e}^{tX}\right) \;=\; \mathrm{e}^{\dot{\phi}(X)}$$

This follows from

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(\phi \mathrm{e}^{tX}\right) &= \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \phi\left(\mathrm{e}^{(t+s)X}\right) \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \phi\left(\mathrm{e}^{tX}\mathrm{e}^{sX}\right) \\
&= \phi\left(\mathrm{e}^{tX}\right) \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \phi\left(\mathrm{e}^{sX}\right) \\
&= \phi\left(\mathrm{e}^{tX}\right) \dot{\phi}\left(X\right)
\end{aligned}
$$

Rewrite the last line by letting $\gamma\left(t\right) = \phi\left(\mathrm{e}^{tX}\right)$,

$$
\frac{\mathrm{d}\gamma}{\mathrm{d}t} = \gamma\left(t\right) \dot{\phi}\left(X\right) \tag{32}
$$

Equation [32] has a unique solution corresponding to $\gamma\left(0\right) = e$ that is given by

$$
\gamma\left(t\right) = \mathrm{e}^{\dot{\phi}(X)}
$$

Hence

$$
\phi\left(\mathrm{e}^{tX}\right) = \mathrm{e}^{\dot{\phi}(X)} \tag{33}
$$

Furthermore, for any $g \in \mathcal{G}$,

$$
\dot{\phi}\left(gXg^{-1}\right) = \phi\left(g\right) \dot{\phi}\left(X\right) \left(\phi\left(g\right)\right)^{-1} \tag{34}
$$

This follows from

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\, \mathrm{e}^{t\dot{\phi}\left(gXg^{-1}\right)}\Big|_{t=0} &= \frac{\mathrm{d}}{\mathrm{d}t}\, \phi\left(\mathrm{e}^{t\left(gXg^{-1}\right)}\right)\Big|_{t=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\, \phi\left(g\,\mathrm{e}^{tX}\,g^{-1}\right)\big|_{t=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\, \phi\left(g\right)\phi\left(\mathrm{e}^{tX}\right)\phi\left(g^{-1}\right)\big|_{t=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\, \phi\left(g\right)\mathrm{e}^{\dot{\phi}(X)}\phi\left(g^{-1}\right)\Big|_{t=0}
\end{aligned}
$$

Hence

$$
\dot{\phi}\left(gXg^{-1}\right) = \phi\left(g\right)\dot{\phi}\left(X\right)\left(\phi\left(g\right)\right)^{-1}
$$

Finally, if (as we show below)

$$
\dot{\phi}\left([X,Y]\right) = \left[\dot{\phi}\left(X\right),\dot{\phi}\left(Y\right)\right] \tag{35}
$$

where

$$
\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\left(\mathrm{e}^{tX}\,Y\,\mathrm{e}^{-tX}\right) = [X,Y] \tag{36}
$$

and Equations [33],[34] hold, we conclude that a study of the pair (ϕ, V), called a *Lie group representation*, can be reduced to a study of the corresponding *Lie algebra representation* $\left(\dot{\phi}, V\right)$. A group representation is thought of as a homomorphism.

Before we get carried too far afield, let us prove equation [35] using the notation

of this section:

$$
\begin{aligned}
\dot{\phi}\left([X,Y]\right) &= \dot{\phi}\left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\left(\mathrm{e}^{tX}\,Y\,\mathrm{e}^{-tX}\right)\right) \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\dot{\phi}\left(\mathrm{e}^{tX}\,Y\,\mathrm{e}^{-tX}\right) \quad \text{(Linearity)} \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\phi\left(\mathrm{e}^{tX}\right)\dot{\phi}\left(Y\right)\phi\left(\mathrm{e}^{-tX}\right) \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\left(\mathrm{e}^{t\dot{\phi}(X)}\dot{\phi}\left(Y\right)\mathrm{e}^{-t\dot{\phi}(X)}\right) \\
&= \left[\dot{\phi}\left(X\right),\dot{\phi}\left(Y\right)\right]
\end{aligned}
$$

In the next section, we encounter a Lie algebra representation, $(\mathfrak{ad}_{\cdot},\mathfrak{g})$, called the *adjoint representation.*

Adjoint Representation

Definition: A *representation* of a Lie group $\mathcal{G}$ is a vector space V together with a

map

$$
\rho : \mathcal{G} \;\longrightarrow\; GL\left(V\right)
$$

Definition: The *adjoint representation* is given by the mapping

$$
\mathrm{Ad} : g \in \mathcal{G} \;\rightarrow\; \mathrm{Ad}_{\mathrm{g}} \in \mathrm{GL}\left(\mathrm{n},\mathbb{K}\right)
$$

where Ad_g acts on $A \in \mathfrak{g}$ by

$$
\mathrm{Ad}_g\left(A\right) \;=\; g\,A\,g^{-1} \in \mathfrak{g} \tag{37}
$$

To show that this is well defined map, we notice that

$$ge^{tA}g^{-1} \;=\; \sum_{k=0}^{\infty} \frac{t^k}{k_1}\left(ge^{tA}g^{-1}\right)^k \;=\; e^{t\left(gAg^{-1}\right)}$$

which means that $gAg^{-1} \in \mathfrak{g}$ if $A \in \mathfrak{g}$.

Theorem: For any elements $X, Y \in \mathfrak{g}$, $[X, Y] = (XY - YX) \in \mathfrak{g}$

Proof: If $A \in \mathfrak{g}$, then $e^{tA} \in \mathcal{G}$ .We can act on $B \in \mathfrak{g}$ by the adjoint representation

$$\mathrm{Ad}_{e^{tA}}(B) \;=\; e^{tA}Be^{-tA} \;\in \mathfrak{g}$$

A variation of λ gives a parameterized curve $\gamma(\lambda)$ in $\mathfrak{g}$. The velocity of this curve $\dot{\gamma}(t)$ is also in $\mathfrak{g}$. Thus

$$\begin{aligned}
\dot{\gamma}(t) \;&=\; \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{Ad}_{e^{tA}}(B)\right)\\
&\quad \frac{\mathrm{d}}{\mathrm{d}t}\left(e^{tA}Be^{-tA}\right)\\
&=\; \frac{\mathrm{d}}{\mathrm{d}t}\left(e^{tA}B\right)e^{-tA} + e^{tA}B\left(\frac{\mathrm{d}}{\mathrm{d}t}e^{-tA}\right)\\
&=\; Ae^{tA}Be^{-tA} - e^{tA}B\,Ae^{-tA}
\end{aligned}$$

At $t = 0$:

$$\dot{\gamma}(0) \;=\; AB - BA$$

Therefore

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\left(\mathrm{Ad}_{e^{tA}}(B)\right) \;=\; AB - BA \;=\; [A, B]$$

For each A, a **path/curve** of elements of $GL\left(n,\mathbb{R}\right)$ goes through the *identity matrix* at $t=0$ with velocity

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\left(e^{tA}\right) \;=\; A$$

This means that there exists a *unique* $A \in GL\left(n,\mathbb{R}\right)$ such that $g\left(t\right) = e^{tA}$ is a continuous map from the additive group $\mathbb{R}$ to $GL\left(n,\mathbb{R}\right)$.

For any $A, B \in \mathfrak{g}$, we compute

$$
\begin{aligned}
\mathrm{Ad}_g\left[A,B\right] \;&=\; g\left[A,B\right]g^{-1} \\
&=\; gABg^{-1} - gBAg^{-1} \\
&=\; \left(gAg^{-1}\right)\left(gBg^{-1}\right) - \left(gBg^{-1}\right)\left(gAg^{-1}\right) \\
&=\; \left[\mathrm{Ad}_g\left(A\right), \mathrm{Ad}_g\left(B\right)\right]
\end{aligned}
$$

Definition: The **adjoint Lie algebra representation** $\left(\mathfrak{ad}_{\cdot}, \mathfrak{g}\right)$ is the Lie algebra representation that is given by

$$A \in \mathfrak{g} \;\rightarrow\; \mathfrak{ad}_A$$

where $\mathfrak{ad}_A$ is a Linear map

$$\mathfrak{ad}_A\left(B\right) \;\equiv\; \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\left(\mathrm{Ad}_{e^{tA}}\left(B\right)\right) \;=\; \left[A,B\right] \tag{38}$$

Using the Jacobi identity:

$$
\begin{aligned}
\left(\left[\mathfrak{ad}_A, \mathfrak{ad}_B\right]\right)(C) &= [A,[B,C]] - [B,[A,C]] \\
&= [A,[B,C]] + [B,[C,A]] \\
&= -[C,[A,B]] \\
&= [[A,B],C] \\
&= \left(\mathfrak{ad}_{[A,B]}\right)(C)
\end{aligned}
$$

Hence $\mathfrak{ad}$. is a Lie algebra ***homomorphism***. It is worth noting that the Jacobi identity implies the existence of the adjoint representation. To see this, let $\circ$ denote a composition of linear maps . The bracket is by definition

$$
[A,B] = A \circ B - B \circ A
$$

The Lie algebra homomorphism property says that

$$
\mathfrak{ad}_{[A,B]} = \mathfrak{ad}_A \circ \mathfrak{ad}_B - \mathfrak{ad}_B \circ \mathfrak{ad}_A
$$

Let us now operate on $C \in \mathfrak{g}$

$$
\begin{aligned}
\left(\mathfrak{ad}_{[A,B]}\right)(C) &= \left(\mathfrak{ad}_A \circ \mathfrak{ad}_B\right)(C) - \left(\mathfrak{ad}_B \circ \mathfrak{ad}_A\right)(C) \\
&= [A,[B,C]] - [B,[A,C]]
\end{aligned}
$$

Using the definition given in equation [38] on the LHS:

$$
[[A,B],C] = [A,[B,C]] - [B,[A,C]]
$$

Let $\{G_a\}$ be a basis for the vector space $\mathfrak{g}$

$$\begin{aligned}
\mathfrak{ad}_{[G_a,G_b]}(G_d) &= [[G_a, G_b], G_d] \\
&= \left[c^f{}_{ab}G_f, G_d\right] \\
&= c^f{}_{ab}[G_f, G_d] \\
&= c^f{}_{ab}c^k{}_{fd}G_k
\end{aligned}$$

Because of the Jacobi identity

$$c^k{}_{f[a}c^f{}_{bd]}G_k = 0$$

Expressing the antisymmetry explicitly

$$\left(c^f{}_{bd}c^k{}_{Af} + c^f{}_{da}c^k{}_{bf} + c^f{}_{ab}c^k{}_{df}\right)G_k = 0$$

which holds for all G_k, so

$$c^f{}_{bd}c^k{}_{Af} + c^f{}_{da}c^k{}_{bf} + c^f{}_{ab}c^k{}_{df} = 0 \tag{39}$$

In the *adjoint representation*, we can write the matrix elements for $\mathfrak{ad}_{G_a}$ as follows

$$[\mathfrak{ad}_{G_b}]^a{}_d \equiv \left[\tilde{G}_b\right]^a{}_d \equiv c^a{}_{bd}$$

Since every commutator can be evaluated from a knowledge of structure constants, let us re-write the structure constants in equation [39] as matrix elements $[G_c] = [G_c]^a{}_b$

$$\left[\tilde{G}_a\right]^k{}_f \left[\tilde{G}_b\right]^f{}_d - \left[\tilde{G}_b\right]^k{}_f \left[\tilde{G}_a\right]^f{}_d = c^f{}_{ab}\left[\tilde{G}_f\right]^k{}_C \tag{40}$$

In the above equation [40], summation is in the order of normal matrix multiplication. We can therefore write

$$\tilde{G}_a\tilde{G}_b - \tilde{G}_b\tilde{G}_a \quad = \quad \left[\tilde{G}_a, \tilde{G}_b\right] \quad = \quad c^d{}_{ab}\tilde{G}_d$$

which yields the Lie bracket.

Definition: The ***Killing form*** of a Lie algebra $\mathfrak{g}$ is defined by a bilinear form:

$$\mathbf{K}\left(G_a, G_b\right) \quad = \quad \mathtt{Tr}\left(\mathfrak{ad}_{G_b} \circ \mathfrak{ad}_{G_b}\right) \ \forall\, G_a, G_b \in \mathfrak{g}$$

For a ***semisimple*** Lie algebra, the Killing form is non-degenerate *i.e.*

$$\mathbf{K}\left(G_a, G_b\right) \quad = \quad 0 \quad \forall G_b \in \mathfrak{g} \quad \Rightarrow G_a = 0$$

The Killing form K_{ab} is the metric on the Lie algebra $\mathfrak{g}$. If $\mathfrak{g}$ has a basis $\{G_a, \cdots, G_b\}$, we can write K_{ab} as follows

$$K_{ab} \quad = \quad \mathtt{Tr}\left(\mathfrak{ad}_{G_b} \circ \mathfrak{ad}_{G_b}\right) \quad = \quad c^d{}_{ac}c^c{}_{bd}$$

The inverse K^{ab} is defined by the relation

$$K_{ac}\, K^{cb} \quad = \quad \delta_a{}^b$$

Lorentz Group

In 1908 as the 80th assembly of *German Natural Scientists and Physicians* unraveled, Hermann Minkowski delivered an elegant reformulation of his former student's special theory of relativity. Standing at the podium, he said [8]

Gentlemen! The views of space and time which I wish to develop before you have sprung from the soil of experimental physics, and therein lies their strength.

In a nutshell, Minkowski argued that the fundamental invariant of special relativity is

$$dS^2 = -\left(dx^0\right)^2 + \left(dx^1\right)^2 + \left(dx^2\right)^2 + \left(dx^3\right)^2$$
$$= \eta_{\mu\nu}dx^\mu dx^\nu$$

where $\eta_{\mu\nu}$ is the Minkowski metric :

$$\eta = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

By recasting special relativity this way, Minkowski captured the basic postulates of the theory in a single equation. Recall that Einstein had postulated that the basic law of physics are invariant *w.r.t* translations in all 4 coordinates (spacetime is homogeneous). Furthermore, physical laws are tensor equations in Minkowski spacetime.

Homogeneous Lorentz transformations are continuous linear transformations $\boldsymbol{\Lambda}$ on unit coordinate vectors and coordinate components given by

$$\hat{\mathbf{e}}_\alpha \rightarrow \hat{\mathbf{e}}'_\alpha = \hat{\mathbf{e}}_\alpha \, \bar{\Lambda}^\alpha_{\ \mu}$$
$$x^\alpha \rightarrow x'^\alpha = \Lambda^\alpha_{\ \beta} \, x^\beta$$

Therefore, a vector $\mathbf{x}$ is invariant

$$
\begin{aligned}
\mathbf{x}' &= x'^{\alpha} \hat{\mathbf{e}}'_{\alpha} \\
&= \left(\Lambda^{\alpha}{}_{\beta}\, x^{\beta} \right) \left(\hat{\mathbf{e}}_{\mu}\, \bar{\Lambda}^{\mu}{}_{\alpha} \right) \\
&= x^{\beta} \hat{\mathbf{e}}_{\mu}\, \bar{\Lambda}^{\mu}{}_{\alpha} \Lambda^{\alpha}{}_{\beta} \\
&= x^{\beta} \hat{\mathbf{e}}_{\mu} \delta^{\mu}{}_{\alpha} \\
&= x^{\beta} \hat{\mathbf{e}}_{\beta} \\
&= \mathbf{x}
\end{aligned}
$$

A Lorentz transformation is any linear map

$$
\begin{aligned}
\mathbf{\Lambda} : T_p \mathcal{M} &\rightarrow T_p \mathcal{M} \\
\mathbf{v} &\mapsto \mathbf{\Lambda}\left(\mathbf{v} \right)
\end{aligned}
$$

such that for any vectors $\mathbf{v}$, $\mathbf{u}$

$$
\boldsymbol{\eta} \left(\mathbf{\Lambda}\left(\mathbf{v} \right), \mathbf{\Lambda}\left(\mathbf{u} \right) \right) = \boldsymbol{\eta}\left(\mathbf{v}, \mathbf{u} \right)
$$

where we define a scalar product $\hat{\mathbf{e}}_{\alpha} \cdot \hat{\mathbf{e}}_{\beta} \equiv \eta_{\alpha\beta}$. Then

$$
\begin{aligned}
\eta\left(\mathbf{v}, \mathbf{u} \right) &= \mathbf{v} \cdot \mathbf{u} \\
&= \left(\mathbf{v}^{\alpha} \hat{\mathbf{e}}_{\alpha} \right) \cdot \left(\mathbf{u}^{\beta} \hat{\mathbf{e}}_{\beta} \right) \\
&= \left(\hat{\mathbf{e}}_{\alpha} \cdot \hat{\mathbf{e}}_{\beta} \right) \mathbf{v}^{\alpha} \mathbf{u}^{\beta} \\
&= \eta_{\alpha\beta}\, \mathbf{v}^{\alpha} \mathbf{u}^{\beta}
\end{aligned}
$$

The components w^α of the image $\boldsymbol{\Lambda}\left(\mathbf{v}\right)$ are written as

$$
\begin{aligned}
\boldsymbol{\Lambda}\left(\mathbf{v}\right) &= \boldsymbol{\Lambda}\left(v^\alpha \hat{\mathbf{e}}_\alpha\right) \\
&= v^\alpha \boldsymbol{\Lambda}\left(\hat{\mathbf{e}}_\alpha\right) \\
&= v^\alpha \Lambda^\beta{}_\alpha \hat{\mathbf{e}}_\beta
\end{aligned}
$$

$\boldsymbol{\Lambda}$ preserves the scalar product

$$
\begin{aligned}
\hat{\mathbf{e}}_\mu \cdot \hat{\mathbf{e}}_\nu &= \eta_{\mu\nu} \\
\boldsymbol{\Lambda}\left(\hat{\mathbf{e}}_\mu\right) \cdot \boldsymbol{\Lambda}\left(\hat{\mathbf{e}}_\nu\right) &= \eta_{\mu\nu}
\end{aligned}
$$

The Lorentz invariant interval is given by

$$
\begin{aligned}
\mathrm{d}s^2 &= \eta_{\mu\nu}\,\mathrm{d}x^\mu \mathrm{d}x^\nu \\
&= \eta_{\alpha\beta}\,\Lambda^\alpha{}_\mu \Lambda^\beta{}_\nu \mathrm{d}x^\mu \mathrm{d}x^\nu
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\boldsymbol{\eta}\left(\boldsymbol{\Lambda}\left(\mathbf{v}\right), \boldsymbol{\Lambda}\left(\mathbf{u}\right)\right) &= \boldsymbol{\eta}\left(\mathbf{v}, \mathbf{u}\right) \\
\eta_{\alpha\beta}\,\Lambda^\alpha{}_\mu \Lambda^\beta{}_\nu v^\mu u^\nu &= \eta_{\mu\nu}\,v^\mu u^\nu
\end{aligned}
$$

Because u^α and v^α are independent, we conclude that

$$
\eta_{\alpha\beta}\,\Lambda^\alpha{}_\mu \Lambda^\beta{}_\nu = \eta_{\mu\nu} \tag{41}
$$

or in matrix notation

$$\boldsymbol{\eta} \;=\; \boldsymbol{\Lambda}^T \boldsymbol{\eta} \,\boldsymbol{\Lambda}$$

Groups of matrices that obey the above condition form a Lie group.

If $\boldsymbol{\Lambda}_1$ and $\boldsymbol{\Lambda}_2$ are Lorentz transformations, then their product $\boldsymbol{\Lambda}_1\boldsymbol{\Lambda}_2$ must also be a Lorentz transformation (*closure*)

$$
\begin{aligned}
\left(\boldsymbol{\Lambda}_1\boldsymbol{\Lambda}_2\right)^T \boldsymbol{\eta} \left(\boldsymbol{\Lambda}_1\boldsymbol{\Lambda}_2\right) &= \boldsymbol{\Lambda}_2^T \left(\boldsymbol{\Lambda}_1\right)^T \boldsymbol{\eta} \left(\boldsymbol{\Lambda}_1\boldsymbol{\Lambda}_2\right) \\
&= \boldsymbol{\Lambda}_2^T \left(\boldsymbol{\Lambda}_1^T \boldsymbol{\eta} \,\boldsymbol{\Lambda}_1\right) \boldsymbol{\Lambda}_2 \\
&= \boldsymbol{\Lambda}_2^T \left(\boldsymbol{\eta}\right) \boldsymbol{\Lambda}_2 \\
&= \boldsymbol{\eta}
\end{aligned}
$$

Using the Lorentz metric, we are allowed to raise and lower indices

$$\eta_{\alpha\mu} \Lambda^{\alpha}{}_{\nu} \;=\; \Lambda_{\mu\nu}$$

Then the condition labeled equation [41] becomes

$$\Lambda_{\alpha\mu} \Lambda^{\alpha}{}_{\nu} \;=\; \eta_{\mu\nu}$$

Raising μ on both sides gives

$$\bar{\Lambda}^{\alpha}{}_{\nu} \Lambda^{\nu}{}_{\beta} \;=\; \delta^{\alpha}{}_{\beta}$$

which indicates the existence of the inverse element.

Taking the *determinant* of $\boldsymbol{\eta} = \boldsymbol{\Lambda}^T \boldsymbol{\eta} \, \boldsymbol{\Lambda}$:

$$\det \boldsymbol{\eta} \;=\; (\det \boldsymbol{\Lambda})^2 \, \det \boldsymbol{\eta}$$

The group of all matrices $\boldsymbol{\Lambda}$ that satisfy the condition $\boldsymbol{\eta} = \boldsymbol{\Lambda}^T \boldsymbol{\eta} \, \boldsymbol{\Lambda}$ belong to the Lie group $O\,(3,1)$. Since $\det \boldsymbol{\eta} \neq 0$, we have

$$\det \boldsymbol{\Lambda} \;=\; \pm 1$$

Since $\det \boldsymbol{\Lambda} \neq 0$, the *inverse* matrix $\boldsymbol{\Lambda}^{-1}$ exists and takes the form

$$\left[\boldsymbol{\Lambda}^{-1}\right]^{\mu}{}_{\nu} \;=\; \eta^{\mu\alpha} \, \Lambda^{\beta}{}_{\alpha} \, \eta_{\beta\nu} \;=\; \Lambda_{\nu}{}^{\mu}$$

Let $\mu = \nu = 0$ in $\eta_{\mu\nu} = \Lambda^{\alpha}{}_{\mu} \, \eta_{\alpha\beta} \, \Lambda^{\beta}{}_{\nu}$, then

$$
\begin{aligned}
\eta_{00} \;&=\; \Lambda^{\alpha}{}_{0} \, \eta_{\alpha\beta} \, \Lambda^{\beta}{}_{0} \\
&=\; -\left(\Lambda^{0}{}_{0}\right)^2 + \left(\Lambda^{1}{}_{0}\right)^2 + \left(\Lambda^{2}{}_{0}\right)^2 + \left(\Lambda^{3}{}_{0}\right)^2
\end{aligned}
$$

or

$$\left(\Lambda^{0}{}_{0}\right)^2 \;=\; 1 + \left(\Lambda^{1}{}_{0}\right)^2 + \left(\Lambda^{2}{}_{0}\right)^2 + \left(\Lambda^{3}{}_{0}\right)^2$$

This implies that

$$\left(\Lambda^{0}{}_{0}\right)^2 \;\geq\; 1$$

Therefore

$$\Lambda^0{}_0 \geq 1$$

$$\Lambda^0{}_0 \leq -1$$

Therefore the *four* subsets of the Lorentz group $\mathcal{L}$ are

$$\mathcal{L}_+^\uparrow = \left\{ \mathbf{\Lambda} \in \mathcal{L} \mid \det \mathbf{\Lambda} = +1 \ , \ \Lambda^0{}_0 \geq 1 \right\}$$

$$\mathcal{L}_-^\uparrow = \left\{ \mathbf{\Lambda} \in \mathcal{L} \mid \det \mathbf{\Lambda} = -1 \ , \ \Lambda^0{}_0 \geq 1 \right\}$$

$$\mathcal{L}_+^\downarrow = \left\{ \mathbf{\Lambda} \in \mathcal{L} \mid \det \mathbf{\Lambda} = +1 \ , \ \Lambda^0{}_0 \leq 1 \right\}$$

$$\mathcal{L}_-^\downarrow = \left\{ \mathbf{\Lambda} \in \mathcal{L} \mid \det \mathbf{\Lambda} = -1 \ , \ \Lambda^0{}_0 \leq 1 \right\}$$

When $\Lambda^0{}_0 \geq 1$, $\mathcal{L}$ is said to be *orthochronous* and when $\det \mathbf{\Lambda} = +1$, $\mathcal{L}$ is said to be *proper*. The *proper orthochronous Lorentz group* , $\mathcal{L}_+^\uparrow$ is the most important Lorentz group since it is continuously connected to the identity. For $\mathcal{L}_+^\uparrow$

$$\det \mathbf{\Lambda} = \Lambda^0{}_\alpha \Lambda^1{}_\beta \Lambda^2{}_\mu \Lambda^3{}_\nu \, \varepsilon^{\alpha\beta\mu\nu} = -1$$

where $\varepsilon^{\alpha\beta\mu\nu}$ is the contravariant form of the Levi-Civita symbol.

Infinitesimal Lorentz Transformations

A Lorentz transformation which is close to the identity can be written as

$$\Lambda^\beta{}_\alpha = \delta^\beta{}_\alpha + \lambda \, A^\beta{}_\alpha \tag{42}$$

To obtain a condition on the matrix A we compute :

$$
\begin{aligned}
\eta_{\mu\nu} &= \Lambda^{\alpha}{}_{\mu} \eta_{\alpha\beta} \Lambda^{\beta}{}_{\nu} \\
&= \left(\delta^{\alpha}{}_{\mu} + \lambda A^{\alpha}{}_{\mu}\right) \eta_{\alpha\beta} \left(\delta^{\beta}{}_{\nu} + \lambda A^{\beta}{}_{\nu}\right) \\
&= \delta^{\alpha}{}_{\mu} \eta_{\alpha\beta} \left(\delta^{\beta}{}_{\nu} + \lambda A^{\beta}{}_{\nu}\right) + \lambda A^{\alpha}{}_{\mu} \eta_{\alpha\beta} \left(\delta^{\beta}{}_{\nu} + \lambda A^{\beta}{}_{\nu}\right) \\
&= \eta_{\mu\nu} + \lambda \eta_{\mu\beta} A^{\beta}{}_{\nu} + \lambda A^{\alpha}{}_{\mu} \eta_{\alpha\nu} + \lambda^2 A^{\alpha}{}_{\mu} \eta_{\alpha\beta} A^{\beta}{}_{\nu} \\
&= \eta_{\mu\nu} + \lambda A_{\mu\nu} + \lambda A_{\nu\mu} + \lambda^2 A^{\alpha}{}_{\mu} \eta_{\alpha\beta} A^{\beta}{}_{\nu}
\end{aligned}
$$

Drop terms that are second order in λ , we get

$$
A_{\mu\nu} = -A_{\nu\mu} \tag{43}
$$

This shows that $A_{\mu\nu}$ are antisymmetric parameters. The generators are $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ tensors,

$$
M^{\alpha}{}_{\beta} = \eta^{\alpha\mu} A_{\mu\beta}
$$

where $A_{\mu\beta} = -A_{\beta\mu}$. Using $\eta^{\alpha\mu}$ to raise 1 index changes signs for the $A_{0\beta}$ components, but not $A_{i\beta}$, making this part symmetric. It follows that boosts are hyperbolic instead of orthogonal.

$$
\begin{aligned}
A^{\mu\nu} M_{\mu\nu} &= \frac{1}{2} \left(A^{\mu\nu} M_{\mu\nu} + A^{\mu\nu} M_{\mu\nu}\right) \\
&= \frac{1}{2} \left(A^{\mu\nu} M_{\mu\nu} + A^{\nu\mu} M_{\nu\mu}\right) \\
&= \frac{1}{2} \left(A^{\mu\nu} M_{\mu\nu} - A^{\mu\nu} M_{\nu\mu}\right) \\
&= \frac{1}{2} A^{\mu\nu} \left(M_{\mu\nu} - M_{\nu\mu}\right)
\end{aligned}
$$

Therefore

$$M_{\mu\nu} = \frac{1}{2}\left(M_{\mu\nu} - M_{\nu\mu}\right)$$

$$\Rightarrow M_{\mu\nu} = -M_{\nu\mu}$$

For clarity, we adopt the following notation for the generators

$$\left[M^{\alpha\beta}\right]_{\mu\nu}$$

where the labels are $\alpha\beta$ and the components of the matrix are $\mu\nu$. A basis for antisymmetric matrices may be written as

$$\left[M^{\alpha\beta}\right]_{\mu\nu} = \delta^\alpha_\mu \delta^\beta_\nu - \delta^\beta_\mu \delta^\alpha_\nu$$

The most general antisymmetric matrix is now a linear combination of these $M^{\alpha\beta}$. So the L ie algebra is constructed from

$$\frac{1}{2} w_{\alpha\beta} M^{\alpha\beta}$$

Recall that exponentiation forms a Lie group structure whose elements are written as

$$\Lambda = e^{-\frac{1}{2}A^{\mu\nu}M_{\mu\nu}} \tag{44}$$

In the power series expansion to first order , the above relation becomes

$$\Lambda = \mathbf{1} - \frac{1}{2}A^{\mu\nu}M_{\mu\nu}$$

In components

$$\Lambda^\alpha{}_\beta = \delta^\alpha{}_\beta - \frac{1}{2} A^{\mu\nu} [M_{\mu\nu}]^\alpha{}_\beta \tag{45}$$

By comparing with $\Lambda^\alpha{}_\beta = \delta^\alpha{}_\beta + \lambda A^\alpha{}_\beta$, we conclude that

$$A^\alpha{}_\beta = -\frac{1}{2} A^{\mu\nu} [M_{\mu\nu}]^\alpha{}_\beta$$

But

$$
\begin{aligned}
[A]^\alpha{}_\beta &= \eta_{\mu\beta} A^{\alpha\mu} \\
&= \frac{1}{2} \left(\eta_{\mu\beta} A^{\alpha\mu} + \eta_{\mu\beta} A^{\alpha\mu} \right) \\
&= \frac{1}{2} \left(\delta^\alpha{}_\mu \eta_{\nu\beta} A^{\mu\nu} + \delta^\alpha{}_\nu \eta_{\mu\beta} A^{\nu\mu} \right) \\
&= \frac{1}{2} \left(\delta^\alpha{}_\mu \eta_{\nu\beta} A^{\mu\nu} - \delta^\alpha{}_\nu \eta_{\mu\beta} A^{\mu\nu} \right) \\
&= \frac{1}{2} A^{\mu\nu} \left(\delta^\alpha{}_\mu \eta_{\nu\beta} - \delta^\alpha{}_\nu \eta_{\mu\beta} \right)
\end{aligned}
$$

This leads to

$$[M_{\mu\nu}]^\alpha{}_\beta = \left(\delta^\alpha{}_\mu \eta_{\nu\beta} - \delta^\alpha{}_\nu \eta_{\mu\beta} \right)$$

The vector space that is spanned by linear combinations of the 6 generators $M_{\mu\nu}$ with real coefficients make up the Lie algebra $\mathfrak{so}(3,1)$ of the Lorentz group. The commutation relationship of the generators is

$$[M_{\alpha\beta}, M_{\mu\nu}]^\gamma{}_\varrho = [M_{\alpha\beta}]^\gamma{}_\sigma [M_{\mu\nu}]^\sigma{}_\varrho - [M_{\mu\nu}]^\gamma{}_\sigma [M_{\alpha\beta}]^\sigma{}_\varrho \tag{46}$$

In general, a Lie algebra is defined by its commutation relations.

$$[G_\alpha, G_\beta] = c^\lambda{}_{\alpha\beta} G_\lambda \tag{47}$$

where $c^c{}_{ab}$ are real constants given the name structure constants.

$$
\begin{aligned}
\frac{1}{2} c^{\sigma\rho}{}_{\alpha\beta;\mu\nu} [M_{\sigma\rho}]^\gamma{}_\varrho &= [M_{\alpha\beta}]^\gamma{}_\sigma [M_{\mu\nu}]^\sigma{}_\varrho - [M_{\mu\nu}]^\gamma{}_\lambda [M_{\alpha\beta}]^\lambda{}_\varrho \\
&= -\left(\delta^\gamma{}_\alpha \eta_{\beta\sigma} - \delta^\gamma{}_\beta \eta_{\alpha\sigma}\right)\left(\delta^\sigma{}_\mu \eta_{\nu\varrho} - \delta^\sigma{}_\nu \eta_{\mu\varrho}\right) \\
&\quad + \left(\delta^\gamma{}_\mu \eta_{\nu\lambda} - \delta^\gamma{}_\nu \eta_{\mu\lambda}\right)\left(\delta^\lambda{}_\alpha \eta_{\beta\varrho} - \delta^\lambda{}_\beta \eta_{\alpha\varrho}\right) \\
&= -\delta^\gamma{}_\alpha \left(\eta_{\beta\mu} \eta_{\nu\varrho} - \eta_{\beta\nu} \eta_{\mu\varrho}\right) + \delta^\gamma{}_\beta \left(\eta_{\alpha\mu} \eta_{\nu\varrho} - \eta_{\alpha\nu} \eta_{\mu\varrho}\right) \\
&\quad + \delta^\gamma{}_\mu \left(\eta_{\nu\alpha} \eta_{\beta\varrho} - \eta_{\nu\beta} \eta_{\alpha\varrho}\right) - \delta^\gamma{}_\nu \left(\eta_{\mu\alpha} \eta_{\beta\varrho} - \eta_{\mu\beta} \eta_{\alpha\varrho}\right) \\
&= \eta_{\beta\nu} \left(\delta^\gamma{}_\alpha \eta_{\mu\varrho} - \delta^\gamma{}_\mu \eta_{\alpha\varrho}\right) + \eta_{\alpha\mu} \left(\delta^\gamma{}_\beta \eta_{\nu\varrho} - \delta^\gamma{}_\nu \eta_{\beta\varrho}\right) \\
&\quad - \eta_{\beta\mu} \left(\delta^\gamma{}_\alpha \eta_{\nu\varrho} - \delta^\gamma{}_\nu \eta_{\alpha\varrho}\right) - \eta_{\alpha\nu} \left(\delta^\gamma{}_\beta \eta_{\mu\varrho} - \delta^\gamma{}_\mu \eta_{\beta\varrho}\right) \\
&= \eta_{\beta\nu} [M_{\alpha\mu}]^\gamma{}_\varrho + \eta_{\alpha\mu} [M_{\beta\nu}]^\gamma{}_\varrho - \eta_{\beta\mu} [M_{\alpha\nu}]^\gamma{}_\varrho - \eta_{\alpha\nu} [M_{\beta\mu}]^\gamma{}_\varrho \\
&= \eta_{\beta\mu} [M_{\alpha\nu}]^\gamma{}_\varrho + \eta_{\alpha\nu} [M_{\beta\mu}]^\gamma{}_\varrho - \eta_{\beta\nu} [M_{\alpha\mu}]^\gamma{}_\varrho - \eta_{\alpha\mu} [M_{\beta\nu}]^\gamma{}_\varrho
\end{aligned}
$$

Suppressing the ϱ, γ -indices gives the $\mathfrak{so}(p,q)$ commutation relationship

$$[M_{\alpha\beta}, M_{\mu\nu}] = \eta_{\beta\mu} M_{\alpha\nu} + \eta_{\alpha\nu} M_{\beta\mu} - \eta_{\beta\nu} M_{\alpha\mu} - \eta_{\alpha\mu} M_{\beta\nu} \tag{48}$$

To compute the structure constants we factor $M_{\sigma\rho}$:

$$[M_{\alpha\beta}, M_{\mu\nu}] = \left(\eta_{\beta\mu} \delta^\sigma{}_\alpha \delta^\rho{}_\nu + \eta_{\alpha\nu} \delta^\sigma{}_\beta \delta^\rho{}_\mu - \eta_{\beta\nu} \delta^\sigma{}_\alpha \delta^\rho{}_\mu - \eta_{\alpha\mu} \delta^\sigma{}_\beta \delta^\rho{}_\nu\right) M_{\sigma\rho}$$

Therefore the structure constants are given by

$$c^{\sigma\rho}{}_{\alpha\beta;\mu\nu} = \eta_{\beta\mu} \delta^{[\sigma}{}_\alpha \delta^{\rho]}{}_\nu + \eta_{\alpha\nu} \delta^{[\sigma}{}_\beta \delta^{\rho]}{}_\mu - \eta_{\beta\nu} \delta^{[\sigma}{}_\alpha \delta^{\rho]}{}_\mu - \eta_{\alpha\mu} \delta^{[\sigma}{}_\beta \delta^{\rho]}{}_\nu$$

We define 1-forms dual to $M^{\alpha\beta}$

$$\left\langle M^{\alpha\beta}, \boldsymbol{\omega}_{\mu\nu} \right\rangle \;=\; \delta^{\alpha}_{\ \mu}\delta^{\beta}_{\ \nu} - \delta^{\alpha}_{\ \nu}\delta^{\beta}_{\ \mu} \tag{49}$$

Using the structure constants, we arrive at the Maurer-Cartan equation for $SO\left(p, q\right)$

$$\mathbf{d}\boldsymbol{\omega}^{\alpha}_{\ \beta} \;=\; \boldsymbol{\omega}^{\mu}_{\ \beta} \wedge \boldsymbol{\omega}^{\alpha}_{\ \mu} \tag{50}$$

IV. FIBER BUNDLES

Our current formulations of field theory rely on a choice of the **structure group** $\mathcal{G}$ of a *principle fiber bundle*. The gauge concept functions as selection criteria for the connection that ultimately specifies the structural Lie group $\mathcal{G}$.

In this section, we briefly review the most relevant and basic facts underlying the theory of fiber bundles.

Definition: A fiber bundle , $(P, \pi, \mathcal{M}, \mathcal{G})$ is a geometrical structure endowed with

1. A differentiable manifold P named the **total space**

2. A differentiable manifold $\mathcal{M}$ named the **base space**

3. A differentiable manifold F named the **standard fiber**

4. A surjective map known as the **projection** $\pi : P \to \mathcal{M}$

5. A Lie group $\mathcal{G}$, named the **structure group**, which acts as a transformation group on the **standard fiber**. A fiber over the point p , F_p is given by $\pi^{-1}(x)$.

Definition: A **principal bundle** $P(\mathcal{M}, \mathcal{G})$ is a fiber-bundle where the *standard fiber* is also the *structure Lie group* $F = \mathcal{G}$ and the action of $\mathcal{G}$ on the fiber is the left (or right) multiplication

Definition: Consider a generic fiber-bundle $P : \pi \to \mathcal{M}$ with generic fibre F. We name **section of the bundle** is a rule such that associates to each point $p \in \mathcal{M}$ of the *base manifold* a point $\phi(p) \in F_p$ in the fiber above$p$, namely a map

$$\phi : \mathcal{M} \longrightarrow P$$

such that

$$\forall p \in \mathcal{M}: \qquad \phi\left(p\right) \in \pi^{-1}\left(p\right)$$

A *section* of the ***tangent bundle*** is a vector field whereas a differential 1-form $\boldsymbol{\omega}$ on a manifold $\mathcal{M}$ is a *section* of the ***cotangent bundle***. A striking realization of the mathematical structure of a cotangent bundle arises in the study of the Lie algebras of Hamiltonian systems.

In classical mechanics, the set of all possible configurations of a system, the ***configuration space*** is a group manifold $\mathcal{G}$ whose cotangent bundle describes the ***phase-space*** of a system. According to ***Liouville's Theorem***, the phase-space *distribution function* is constant along trajectories that satisfy Hamilton's equations. The distribution function is identified with a natural *volume-form* defined by a 2-form $\boldsymbol{\Omega}$ which is left-invariant under Hamiltonian flows.

Symplectic geometry

To set stage for later developments, we turn our attention to an even-dimensional differential manifold that was given the name ***symplectic*** by Hermann Weyl. The cotangent bundle, to which we have been introduced, can be thought of as a ***symplectic manifold***.

The cotangent bundle $T^*\mathcal{M}$ of a manifold $\mathcal{M}$ bears the following properties:

1. It carries a canonical 1-form $\boldsymbol{\omega}$, also known as the ***Liouville*** 1-form. In canonical coordinates, $\boldsymbol{\omega}$ is given by

$$\boldsymbol{\omega} \;=\; p_\mu \mathbf{d}q^\mu \tag{51}$$

where the functional p is called the **generalized momentum**. In the context of gauge theory, the choice of a **solder form** on a smooth principal $\mathcal{G}$-bundle is unique, or canonically determined. In this case, the solder form is the canonical 1-form $\boldsymbol{\omega}$.

2. It contains a variety of **Lagrangian sub-manifolds** that are related to the bundle structure. A sub-manifold $\mathscr{L} \subset \mathcal{M}$ is called Lagrangian if $\mathscr{L}$ is *half* the dimension of $\mathcal{M}$ and $\boldsymbol{\Omega}$ vanishes when restricted to $\mathscr{L}$.

3. Every 1-form on $\mathcal{M}$ induces a vertical vector field on $T^*\mathcal{M}$

One of the most defining results of Hamiltonian systems is that given any smooth manifold $\mathcal{M}$, the cotangent bundle $T^*\mathcal{M}$ has a natural symplectic structure.

A smooth manifold $\mathcal{M}$ with local coordinates $\{q^\alpha, \alpha = 1, , n\}$ admits canonical coordinates (q^α, p_α) for $T^*\mathcal{M}$ that describes local 1-forms in equation [51]. Under a change of coordinates

$$\mathbf{d}\tilde{q}^\mu \;\; = \;\; \frac{\partial \tilde{q}^\mu}{\partial q^\nu}\mathbf{d}q^\nu$$

The cotangent bundle has a 2-form $\boldsymbol{\Omega}$ which is **exact** because it satisfies

$$\mathbf{d}\boldsymbol{\omega} \;\; = \;\; \boldsymbol{\Omega} \tag{52}$$

By to the Poincaré Lemma, $\boldsymbol{\Omega}$ is **closed** *i.e.*

$$\mathbf{d}\boldsymbol{\Omega} \;\; = \;\; 0 \tag{53}$$

Definition: A symplectic manifold is a pair $(\mathcal{M}, \boldsymbol{\Omega})$ consisting of a **manifold** $\mathcal{M}$ and a **closed non-degenerate** 2-form $\boldsymbol{\Omega}$, called the **symplectic form** or the

symplectic structure.

Given coordinates $\{x^A, A = 1, \cdots, n\}$ on a manifold $\mathcal{M}$ where n is the dimension of $\mathcal{M}$, any 2-form can be written as

$$\mathbf{\Omega} \;\; = \;\; \frac{1}{2}\mathbf{\Omega}_{AB}\mathbf{d}x^A \wedge \mathbf{d}x^B \tag{54}$$

In *canonical* coordinates (x^α, p_β), the symplectic form can be written as

$$\mathbf{\Omega} = \mathbf{d}x^\alpha \wedge \mathbf{d}p_\alpha \tag{55}$$

$\mathbf{\Omega}$ is **non-degenerate** if

$$\det[\mathbf{\Omega}_{AB}] \;\; \neq \;\; 0 \tag{56}$$

This means that if $\mathbf{\Omega}(w, u) = 0$ for some vector u , then $u = 0$

The **Darboux theorem** states that for every point x of a symplectic manifold there exists neighborhood with coordinates (q^α, p_α) such that the symplectic form $\mathbf{\Omega}$ satisfies eqn. [55] *everywhere.* To see that $\mathbf{\Omega}$ is independent of coordinates , let us change to different coordinates $(\tilde{q}^\alpha, \tilde{p}_\alpha)$

$$\mathbf{d}\tilde{q}^\mu \wedge \mathbf{d}\tilde{p}_\mu \;\; = \;\; \frac{\partial \tilde{q}^\mu}{\partial q^\alpha}\frac{\partial \tilde{q}^\nu}{\partial q^\mu}\mathbf{d}q^\alpha \wedge \mathbf{d}p_\nu \;\; = \;\; \mathbf{\Omega}$$

In effect, a symplectic manifold is locally trivial, in contrast to Riemannian manifolds with locally invariant curvature.

The constraint on the structure of $\mathcal{M}$ in equation [53] is called an **integrability condition**. To restate Liouville's theorem, an integrable n-dimensional Hamiltonian system is integrable if it admits n first integrals that are linearly independent and in

involution.

In short, an integrable system on a symplectic manifold describes foliations by Lagrangian sub-manifolds.

Curvature on principal fiber bundles

Consider a Lie group $\mathcal{P} \times \mathcal{G}$ where $\mathcal{P}$ is the **Poincaré group** and $\mathcal{G}$ is the **gauge group**, $SU(n)$. The commutators of the Lie algebra, $\mathfrak{g}$, of $\mathcal{G}$ can be written as

$$[G_B, G_C] = c^A{}_{BC} G_A \tag{57}$$

where G_A are basis vectors of $\mathfrak{g}$, also called **generators**. To formulate a gauge theory, we associate to each generator dual 1-forms,

$$\left\langle G_B, \boldsymbol{\omega}^A \right\rangle = \delta_B^A \tag{58}$$

The dual basis satisfies the **Maurer-Cartan equations**,

$$\mathbf{d}\boldsymbol{\omega}^A = -\frac{1}{2}c^A{}_{BC}\boldsymbol{\omega}^B \wedge \boldsymbol{\omega}^C \tag{59}$$

Since the Maurer-Cartan forms $\boldsymbol{\omega}^A$ have a Lie algebra index, they describe a **connection** on a fiber bundle. We can contract the Lie algebra index with a *linear representation* of the group generators,

$$\boldsymbol{\omega} = \boldsymbol{\omega}^A[G_A]$$

where $[G_A] = [G_A]^a{}_b$ is a matrix operator on a vector representation. In the adjoint representation, the matrix elements are given by the structure constants.

$$[G_B]^A{}_C \equiv c^A{}_{BC}$$

Since all Lie algebras satisfy the **Jacobi identity** $[G_B]^A{}_C$ can be written as

$$\begin{aligned}
0 &= [G_A, [G_B, G_C]] + [G_B, [G_C, G_A]] + [G_C, [G_A, G_B]] \\
&= c^D{}_{BC} [G_A, G_D] + c^D{}_{CA} [G_B, G_D] + c^D{}_{AB} [G_C, G_D] \\
&= \left(c^D{}_{BC} c^E{}_{AD} + c^D{}_{CA} c^E{}_{BD} + c^D{}_{AB} c^E{}_{CD} \right) G_E
\end{aligned}$$

The last line holds for all G_E. We can re-arrange

$$\begin{aligned}
0 &= c^D{}_{BC} c^E{}_{AD} + c^D{}_{CA} c^E{}_{BD} + c^D{}_{AB} c^E{}_{CD} \\
&= c^E{}_{AD} c^D{}_{BC} - c^E{}_{BD} c^D{}_{AC} - c^D{}_{AB} c^E{}_{DC}
\end{aligned}$$

Hence

$$c^E{}_{AD} c^D{}_{BC} - c^E{}_{BD} c^D{}_{AC} = c^D{}_{AB} c^E{}_{DC} \tag{60}$$

Now, rewrite the structure constants as generators,

$$[G_A]^E{}_D [G_B]^D{}_C - [G_B]^E{}_D [G_A]^D{}_C = c^D{}_{AB} [G_D]^E{}_C \tag{61}$$

Observe that in equation [61], all indices are summed in the order of normal matrix multiplication, we can write

$$G_A G_B - G_B G_A = [G_A, G_B] = c^C{}_{AB} G_C \tag{62}$$

which yields the Lie bracket of generators.

Now, rewrite the Maurer-Cartan structure equations in this representation. Our Lie algebra valued 1-form is

$$\boldsymbol{\omega} \;=\; \boldsymbol{\omega}^A G_A \Rightarrow \boldsymbol{\omega}^B{}_C = \boldsymbol{\omega}^A c^B{}_{AC}$$

Contract with $c^D{}_{AE}$:

$$c^D{}_{AE}\mathbf{d}\boldsymbol{\omega}^A \;=\; \mathbf{d}\boldsymbol{\omega}^D{}_E \;=\; -\frac{1}{2}c^D{}_{AE}c^A{}_{BC}\boldsymbol{\omega}^B \wedge \boldsymbol{\omega}^C$$

Use the Jacobi identity on the right side, $0 = c^D{}_{AE}c^A{}_{BC} + c^D{}_{AB}c^A{}_{CE} + c^D{}_{AC}c^A{}_{EB}$

$$
\begin{aligned}
\mathbf{d}\boldsymbol{\omega}^D{}_E \;&=\; \frac{1}{2}c^D{}_{AE}c^A{}_{BC}\boldsymbol{\omega}^B \wedge \boldsymbol{\omega}^C \\
&=\; -\frac{1}{2}\left(-c^D{}_{AB}c^A{}_{CE} - c^D{}_{AC}c^A{}_{EB}\right)\boldsymbol{\omega}^B \wedge \boldsymbol{\omega}^C \\
&=\; \frac{1}{2}\left(c^D{}_{AB}c^A{}_{CE}\boldsymbol{\omega}^B \wedge \boldsymbol{\omega}^C + c^D{}_{AC}c^A{}_{EB}\boldsymbol{\omega}^B \wedge \boldsymbol{\omega}^C\right) \\
&=\; \frac{1}{2}\left(-c^D{}_{BA}\boldsymbol{\omega}^B \wedge c^A{}_{CE}\boldsymbol{\omega}^C - c^D{}_{CA}c^A{}_{BE}\boldsymbol{\omega}^C \wedge \boldsymbol{\omega}^B\right) \\
&=\; \frac{1}{2}\left(-\boldsymbol{\omega}^D{}_A \wedge \boldsymbol{\omega}^A{}_E - \boldsymbol{\omega}^D{}_A \wedge \boldsymbol{\omega}^A{}_E\right) \\
&=\; \boldsymbol{\omega}^A{}_E \wedge \boldsymbol{\omega}^D{}_A
\end{aligned}
$$

Therefore, any Lie algebra with a suitable adjoint representation may be written as

$$\mathbf{d}\boldsymbol{\omega}^D{}_E \;=\; \boldsymbol{\omega}^A{}_E \wedge \boldsymbol{\omega}^D{}_A \tag{63}$$

We return to the use of $\boldsymbol{\omega}^A$ as a connection. Eq.(63) lets us construct a $\mathcal{G}$-covariant derivative. For any vector in the Lie algebra,

$$v \;=\; v^A G_A$$

the components will transform under $\mathcal{G}$ as

$$\tilde{v}^A = [g]^A{}_B\, v^B \tag{64}$$

Taking the exterior derivative of equation [64] gives

$$\mathbf{d}\tilde{v}^A = \mathbf{d}\,[g]^A{}_B\, v^B + [g]^A{}_B\, \mathbf{d}v^B$$

A connection is required to keep the derivative tensorial (i.e., linear and homogeneous). Define

$$\mathbf{D}v^A = \mathbf{d}v^A + \boldsymbol{\omega}^A{}_B v^B \tag{65}$$

Then covariance requires

$$\tilde{\mathbf{D}}\tilde{v}^A = [g]^A{}_B\, \mathbf{D}v^B \tag{66}$$

Expanding each side,

$$\mathbf{d}\left([g]^A{}_B\, v^B\right) + \tilde{\boldsymbol{\omega}}^A{}_B\left([g]^B{}_C\, v^C\right) = [g]^A{}_B\left(\mathbf{d}v^B + \boldsymbol{\omega}^B{}_C v^C\right)$$

$$\mathbf{d}\,[g]^A{}_B\, v^B + [g]^A{}_B\, \mathbf{d}v^B + \tilde{\boldsymbol{\omega}}^A{}_B\,[g]^B{}_C\, v^C = [g]^A{}_B\, \mathbf{d}v^B + [g]^A{}_B\, \boldsymbol{\omega}^B{}_C v^C$$

$$\tilde{\boldsymbol{\omega}}^A{}_B\,[g]^B{}_C\, v^C = [g]^A{}_B\, \boldsymbol{\omega}^B{}_C v^C - \mathbf{d}\,[g]^A{}_B\, v^B$$

This must hold for every vector in the Lie algebra, so

$$\tilde{\boldsymbol{\omega}}^A{}_B\,[g]^B{}_C = [g]^A{}_B\, \boldsymbol{\omega}^B{}_C - \mathbf{d}\,[g]^A{}_C$$

and finally, inverting the group element and adjusting indices,

$$\tilde{\boldsymbol{\omega}}^A{}_B \;=\; [g]^A{}_C \,\boldsymbol{\omega}^C{}_D \,[\bar{g}]^D{}_B - \mathbf{d}\,[g]^A{}_C \,[\bar{g}]^C{}_B \tag{67}$$

Suppressing the indices in equation [67]:

$$\tilde{\boldsymbol{\omega}} \;=\; g\boldsymbol{\omega}\bar{g} - \mathbf{d}g\bar{g} \tag{68}$$

Note that eq.(63) is solved by $\boldsymbol{\omega}^A{}_B = 0$. Applying the group transformation to this particular solution, we have

$$\tilde{\boldsymbol{\omega}}^A{}_B \;=\; -\left(\mathbf{d}\,[g]^A{}_C\right)[\bar{g}]^C{}_B \tag{69}$$

This is the *pure gauge* form of the Maurer-Cartan connection.

Check: Substituting $\tilde{\boldsymbol{\omega}}^A{}_B$ into the Maurer-Cartan equation,

$$
\begin{aligned}
\mathbf{d}\tilde{\boldsymbol{\omega}}^A{}_B - \tilde{\boldsymbol{\omega}}^C{}_B \wedge \tilde{\boldsymbol{\omega}}^A{}_C \;=\;& \mathbf{d}\left(-[\bar{g}]^C{}_B\,\mathbf{d}\,[g]^A{}_C\right) \\
& - \left(-[\bar{g}]^D{}_B\,\mathbf{d}\,[g]^C{}_D\right) \wedge \left(-[\bar{g}]^E{}_C\,\mathbf{d}\,[g]^A{}_E\right) \\
=\;& -\mathbf{d}\,[\bar{g}]^C{}_B \wedge \mathbf{d}\,[g]^A{}_C \\
& - \left([\bar{g}]^D{}_B\,\mathbf{d}\,[g]^C{}_D\right) \wedge [\bar{g}]^E{}_C\,\mathbf{d}\,[g]^A{}_E \\
=\;& -\mathbf{d}\,[\bar{g}]^C{}_B \wedge \mathbf{d}\,[g]^A{}_C \\
& +\mathbf{d}\,[\bar{g}]^D{}_B\,[\bar{g}]^C{}_D \wedge [\bar{g}]^E{}_C\,\mathbf{d}\,[g]^A{}_E \\
=\;& -\mathbf{d}\,[\bar{g}]^C{}_B \wedge \mathbf{d}\,[g]^A{}_C + \mathbf{d}\,[\bar{g}]^D{}_B \wedge \mathbf{d}\,[g]^A{}_D \\
=\;& 0
\end{aligned}
$$

Now suppose we allow the more general form of eq.(67) but *change* the connection

so that it no longer satisfies the Maurer-Cartan equation, but instead we have

$$\mathbf{d}\boldsymbol{\omega}^A{}_B - \boldsymbol{\omega}^C{}_B \wedge \boldsymbol{\omega}^A{}_C \;=\; \mathbf{R}^A{}_B \tag{70}$$

where $\mathbf{R}^A{}_B$ is some 2-form. Then transforming the connection in the left side of

eq.(70) according to eq.(67), we find how $\mathbf{R}^A{}_B$ transforms,

$$
\begin{aligned}
\tilde{\mathbf{R}}^A{}_B &= \mathbf{d}\tilde{\boldsymbol{\omega}}^A{}_B - \tilde{\boldsymbol{\omega}}^C{}_B \wedge \tilde{\boldsymbol{\omega}}^A{}_C \\[4pt]
&= \mathbf{d}\left([g]^A{}_C\,\boldsymbol{\omega}^C{}_D\,[\bar{g}]^D{}_B - [\bar{g}]^C{}_B\,\mathbf{d}\,[g]^A{}_C\right) \\
&\quad -\left([g]^C{}_E\,\boldsymbol{\omega}^E{}_D\,[\bar{g}]^D{}_B - [\bar{g}]^D{}_B\,\mathbf{d}\,[g]^C{}_D\right) \\
&\quad \wedge\left([g]^A{}_F\,\boldsymbol{\omega}^F{}_G\,[\bar{g}]^G{}_C - [\bar{g}]^F{}_C\,\mathbf{d}\,[g]^A{}_F\right) \\[6pt]
&= \mathbf{d}\,[g]^A{}_C\,\boldsymbol{\omega}^C{}_D\,[\bar{g}]^D{}_B + [g]^A{}_C\,\mathbf{d}\boldsymbol{\omega}^C{}_D\,[\bar{g}]^D{}_B - [g]^A{}_C\,\boldsymbol{\omega}^C{}_D\mathbf{d}\,[\bar{g}]^D{}_B \\
&\quad -\mathbf{d}\,[\bar{g}]^C{}_B\,\mathbf{d}\,[g]^A{}_C - [g]^C{}_E\,\boldsymbol{\omega}^E{}_D\,[\bar{g}]^D{}_B \wedge [g]^A{}_F\,\boldsymbol{\omega}^F{}_G\,[\bar{g}]^G{}_C \\
&\quad + [g]^C{}_E\,\boldsymbol{\omega}^E{}_D\,[\bar{g}]^D{}_B \wedge [\bar{g}]^F{}_C\,\mathbf{d}\,[g]^A{}_F \\
&\quad + [\bar{g}]^D{}_B\,\mathbf{d}\,[g]^C{}_D \wedge [g]^A{}_F\,\boldsymbol{\omega}^F{}_G\,[\bar{g}]^G{}_C \\
&\quad - [\bar{g}]^D{}_B\,\mathbf{d}\,[g]^C{}_D \wedge [\bar{g}]^F{}_C\,\mathbf{d}\,[g]^A{}_F \\[6pt]
&= \mathbf{d}\,[g]^A{}_C\,\boldsymbol{\omega}^C{}_D\,[\bar{g}]^D{}_B + [g]^A{}_C\,\mathbf{d}\boldsymbol{\omega}^C{}_D\,[\bar{g}]^D{}_B \\
&\quad - [g]^A{}_C\,\boldsymbol{\omega}^C{}_D\mathbf{d}\,[\bar{g}]^D{}_B - \mathbf{d}\,[\bar{g}]^C{}_B\,\mathbf{d}\,[g]^A{}_C \\
&\quad - [g]^A{}_F\,\boldsymbol{\omega}^E{}_D \wedge \boldsymbol{\omega}^F{}_E\,[\bar{g}]^D{}_B + \boldsymbol{\omega}^E{}_D\,[\bar{g}]^D{}_B \wedge \mathbf{d}\,[g]^A{}_E \\
&\quad + [\bar{g}]^D{}_B\left(-[g]^C{}_D\,\mathbf{d}\,[\bar{g}]^G{}_C\right) \wedge [g]^A{}_F\,\boldsymbol{\omega}^F{}_G \\
&\quad - \left(-\mathbf{d}\,[\bar{g}]^D{}_B\,[g]^C{}_D\right) \wedge [\bar{g}]^F{}_C\,\mathbf{d}\,[g]^A{}_F \\[6pt]
&= [g]^A{}_C\,\mathbf{d}\boldsymbol{\omega}^C{}_D\,[\bar{g}]^D{}_B - [g]^A{}_F\,\boldsymbol{\omega}^E{}_D \wedge \boldsymbol{\omega}^F{}_E\,[\bar{g}]^D{}_B \\
&\quad +\mathbf{d}\,[g]^A{}_C\,\boldsymbol{\omega}^C{}_D\,[\bar{g}]^D{}_B - [g]^A{}_C\,\boldsymbol{\omega}^C{}_D\mathbf{d}\,[\bar{g}]^D{}_B - \mathbf{d}\,[\bar{g}]^C{}_B\,\mathbf{d}\,[g]^A{}_C \\
&\quad +\boldsymbol{\omega}^E{}_D\,[\bar{g}]^D{}_B \wedge \mathbf{d}\,[g]^A{}_E + [g]^A{}_F\,\boldsymbol{\omega}^F{}_G \wedge \mathbf{d}\,[\bar{g}]^G{}_B + \mathbf{d}\,[\bar{g}]^D{}_B\,\mathbf{d}\,[g]^A{}_D \\[6pt]
&= [g]^A{}_C\left(\mathbf{d}\boldsymbol{\omega}^C{}_D - \boldsymbol{\omega}^E{}_D \wedge \boldsymbol{\omega}^C{}_E\right)[\bar{g}]^D{}_B \\
&\quad +\mathbf{d}\,[g]^A{}_C\,\boldsymbol{\omega}^C{}_D\,[\bar{g}]^D{}_B - [g]^A{}_C\,\boldsymbol{\omega}^C{}_D\mathbf{d}\,[\bar{g}]^D{}_B - \mathbf{d}\,[\bar{g}]^C{}_B\,\mathbf{d}\,[g]^A{}_C \\
&\quad -\mathbf{d}\,[g]^A{}_C \wedge \boldsymbol{\omega}^C{}_D\,[\bar{g}]^D{}_B + [g]^A{}_C\,\boldsymbol{\omega}^C{}_D \wedge \mathbf{d}\,[\bar{g}]^D{}_B + \mathbf{d}\,[\bar{g}]^C{}_B\,\mathbf{d}\,[g]^A{}_C \\[6pt]
&= [g]^A{}_C\left(\mathbf{d}\boldsymbol{\omega}^C{}_D - \boldsymbol{\omega}^E{}_D \wedge \boldsymbol{\omega}^C{}_E\right)[\bar{g}]^D{}_B
\end{aligned}
$$

and finally

$$\tilde{\mathbf{R}}^A{}_B = [g]^A{}_C \, \mathbf{R}^C{}_D \, [\bar{g}]^D{}_B \tag{71}$$

Hence $\mathbf{R}^A{}_B$ transforms as a tensor under $\mathcal{G}$.

Geodesic Equation

The mathematical underpinnings of the notion of **curvature** were first nailed down in a 3-index symbol found in an influential paper by Elwin Bruno Christoffel, who in 1917 defined parallel transport of a vector in Riemannian geometry [9].

On a curved manifold, the directions of parallel transport of a vector do not commute. A smooth curve γ is an *autoparallel* and non-accelerating if its tangent vector field v^σ satisfies

$$\mathrm{v}^\alpha D_\alpha \mathrm{v}^\sigma \;=\; 0$$

where D_α is the *covariant derivative.*

We already remarked that gravity manifests itself by giving space-time a curvature.This fact can be be derived from the commutator of a pair of covariant derivatives:

$$[D_\mu, D_\nu]\,\mathbf{e}^a \;=\; R^a{}_{b\mu\nu}\mathbf{e}^b \tag{72}$$

$$[D_\mu, D_\nu]\,u^\alpha \;=\; D_\mu D_\nu u^\alpha - D_\nu D_\mu u^\alpha$$

Computing two derivatives,

$$
\begin{aligned}
D_\mu D_\nu u^\alpha &= D_\mu \left(\partial_\nu u^\alpha + u^\beta \Gamma^\alpha_{\beta\nu} \right) \\
&= \partial_\mu \left(\partial_\nu u^\alpha + u^\beta \Gamma^\alpha_{\beta\nu} \right) + \left(\partial_\nu u^\rho + u^\beta \Gamma^\rho_{\beta\nu} \right) \Gamma^\alpha_{\rho\mu} - \left(\partial_\rho u^\alpha + u^\beta \Gamma^\alpha_{\rho\nu} \right) \Gamma^\rho_{\nu\mu}
\end{aligned}
$$

so the commutator gives

$$
\begin{aligned}
\left[D_\mu, D_\nu \right] u^\alpha &= \partial_\mu \left(\partial_\nu u^\alpha + u^\beta \Gamma^\alpha_{\beta\nu} \right) + \left(\partial_\nu u^\rho + u^\beta \Gamma^\rho_{\beta\nu} \right) \Gamma^\alpha_{\rho\mu} - \left(\partial_\rho u^\alpha + u^\beta \Gamma^\alpha_{\rho\nu} \right) \Gamma^\rho_{\nu\mu} \\
&\quad - (\mu \longleftrightarrow \nu) \\
&= \partial_\mu \left(\partial_\nu u^\alpha + u^\beta \Gamma^\alpha_{\beta\nu} \right) - \partial_\nu \left(\partial_\mu u^\alpha + u^\beta \Gamma^\alpha_{\beta\mu} \right) + \left(\partial_\nu u^\rho + u^\beta \Gamma^\rho_{\beta\nu} \right) \Gamma^\alpha_{\rho\mu} \\
&\quad - \left(\partial_\mu u^\rho + u^\beta \Gamma^\rho_{\beta\mu} \right) \Gamma^\alpha_{\rho\nu} - \left(\partial_\rho u^\alpha + u^\beta \Gamma^\alpha_{\rho\nu} \right) \Gamma^\rho_{\nu\mu} \\
&\quad + \left(\partial_\rho u^\alpha + u^\beta \Gamma^\alpha_{\rho\nu} \right) \Gamma^\rho_{\mu\nu} \\
&= \cancel{\partial_\mu \partial_\nu u^\alpha} + \partial_\mu \left(u^\beta \Gamma^\alpha_{\beta\nu} \right) - \cancel{\partial_\nu \partial_\mu u^\alpha} - \partial_\nu \left(u^\beta \Gamma^\alpha_{\beta\mu} \right) \\
&\quad + \left(\partial_\nu u^\rho + u^\beta \Gamma^\rho_{\beta\nu} \right) \Gamma^\alpha_{\rho\mu} - \left(\partial_\mu u^\rho + u^\beta \Gamma^\rho_{\beta\mu} \right) \Gamma^\alpha_{\rho\nu} \\
&\quad - \left(\partial_\rho u^\alpha + u^\beta \Gamma^\alpha_{\rho\nu} \right) \Gamma^\rho_{\nu\mu} + \left(\partial_\rho u^\alpha + u^\beta \Gamma^\alpha_{\rho\nu} \right) \Gamma^\rho_{\mu\nu} \\
&= u^\beta \partial_\mu \Gamma^\alpha_{\beta\nu} - u^\beta \partial_\nu \Gamma^\alpha_{\beta\mu} + u^\beta \Gamma^\rho_{\beta\nu} \Gamma^\alpha_{\rho\mu} - u^\beta \Gamma^\rho_{\beta\mu} \Gamma^\alpha_{\rho\nu} \\
&\quad + u^\beta \Gamma^\alpha_{\rho\nu} \left(\cancelto{0}{\Gamma^\rho_{\mu\nu} - \Gamma^\rho_{\nu\mu}} \right) + \cancel{\Gamma^\alpha_{\beta\nu} \left(\partial_\mu u^\beta \right)} - \cancel{\Gamma^\alpha_{\beta\nu} \left(\partial_\mu u^\beta \right)} \\
&\quad + \cancel{\Gamma^\alpha_{\beta\mu} \left(\partial_\nu u^\beta \right)} - \cancel{\Gamma^\alpha_{\beta\mu} \left(\partial_\nu u^\beta \right)} + \Gamma^\rho_{\mu\nu} \left(\cancelto{0}{\partial_\rho u^\alpha - \partial_\rho u^\alpha} \right) \\
&= \left(\partial_\mu \Gamma^\alpha_{\beta\nu} - \partial_\nu \Gamma^\alpha_{\beta\mu} + \Gamma^\rho_{\beta\nu} \Gamma^\alpha_{\rho\mu} - \Gamma^\rho_{\beta\mu} \Gamma^\alpha_{\rho\nu} \right) u^\beta \\
&= R^\alpha{}_{\beta\nu\mu} u^\beta
\end{aligned}
$$

Therefore the Riemann curvature tensor is given by

$$
R^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda} \Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda} \Gamma^\lambda_{\mu\sigma} \tag{73}
$$

The above equation [73] is a special case of a more general equation [70].

A **geodesic** is by definition the extremal point of an action. So, a vanishing variation of the arc length $d\tau$ given by $-c^2 d\tau^2 = g'_{\alpha\beta} dx'^\alpha dx'^\beta$ (set $c = 1$) yields the geodesic equation. Then, the "action" to be extremized is the arc length,

$$
\begin{aligned}
\tau \;&=\; \int d\tau \\
&=\; \int \sqrt{-g_{\mu\nu} dx^\mu dx^\nu} \\
&=\; \int \sqrt{-g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}} \, d\tau
\end{aligned}
$$

This is reparameterization invariant. Here we parameterize the curve by proper time, $x^\alpha(\tau)$, but it works for any $x^\alpha(\lambda)$.

$$\begin{aligned}
0 &= \delta S \\
&= -\frac{1}{2} \int_{\tau_1}^{\tau_2} \frac{1}{\mathrm{d}\tau}\, \delta\left(g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu\right) \\
&= -\frac{1}{2} \int_{\tau_1}^{\tau_2} \frac{1}{\mathrm{d}\tau}\, \left(\partial_\alpha g_{\mu\nu}\, \delta x^\alpha\, \mathrm{d}x^\mu \mathrm{d}x^\nu + 2 g_{\mu\nu}\, \mathrm{d}x^\mu\, \mathrm{d}\left(\delta x^\nu\right)\right) \\
&= -\int_{\tau_1}^{\tau_2} \left(\frac{1}{2}\partial_\alpha g_{\mu\nu} \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\delta x^\alpha + g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}\left(\delta x^\nu\right)}{\mathrm{d}\tau}\right)\mathrm{d}\tau \\
&= \int_{\tau_1}^{\tau_2} \left[-\frac{1}{2}\partial_\alpha g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\delta x^\alpha + \frac{\mathrm{d}}{\mathrm{d}\tau}\left(g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\right)\delta x^\nu\right]\mathrm{d}\tau - \left[\left(g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\right)\delta x^\nu\right]_{\tau_1}^{\tau_2} \, 0 \\
&= \int_{\tau_1}^{\tau_2} \left[-\frac{1}{2}\partial_\alpha g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} + \frac{\mathrm{d}}{\mathrm{d}\tau}\left(g_{\mu\alpha}\frac{\mathrm{d}x^\mu}{\mathrm{d}s}\right)\right]\delta x^\alpha\, \mathrm{d}\tau \\
&= \int_{\tau_1}^{\tau_2} \left[-\frac{1}{2}\partial_\alpha g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} + \frac{\mathrm{d}g_{\mu\alpha}}{\mathrm{d}\tau}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} + g_{\mu\alpha}\frac{\mathrm{d}^2x^\mu}{\mathrm{d}\tau^2}\right]\delta x^\alpha\, \mathrm{d}\tau \\
&= \int_{\tau_1}^{\tau_2} \left[-\frac{1}{2}\partial_\alpha g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} + \partial_\nu g_{\mu\alpha}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} + g_{\mu\alpha}\frac{\mathrm{d}^2x^\mu}{\mathrm{d}\tau^2}\right]\delta x^\alpha\, \mathrm{d}\tau \\
&= \int_{\tau_1}^{\tau_2} \left[-\frac{1}{2}\partial_\alpha g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} + \frac{1}{2}\partial_\nu g_{\mu\alpha}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\right. \\
&\qquad \left. + +\frac{1}{2}\partial_\mu g_{\nu\alpha}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} + g_{\mu\alpha}\frac{\mathrm{d}^2x^\mu}{\mathrm{d}\tau^2}\right]\delta x^\alpha\, \mathrm{d}\tau \\
&= \int_{\tau_1}^{\tau_2} \left[\frac{1}{2}\left(\partial_\mu g_{\nu\alpha} + \partial_\nu g_{\mu\alpha} - \partial_\alpha g_{\mu\nu}\right)\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} + g_{\mu\alpha}\frac{\mathrm{d}^2x^\mu}{\mathrm{d}\tau^2}\right]\delta x^\alpha\, \mathrm{d}\tau
\end{aligned}$$

which is true for all δx^α . Contracting with $g^{\sigma\alpha}$:

$$
\begin{aligned}
0 &= \frac{1}{2}g^{\sigma\alpha}\left(\partial_\mu g_{\nu\alpha} + \partial_\nu g_{\mu\alpha} - \partial_\alpha g_{\mu\nu}\right)\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} + g^{\sigma\alpha}g_{\mu\alpha}\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2} \\
&= \frac{1}{2}g^{\sigma\alpha}\left(\partial_\mu g_{\nu\alpha} + \partial_\nu g_{\mu\alpha} - \partial_\alpha g_{\mu\nu}\right)\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} + \delta^\sigma{}_\mu\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2} \\
&= \Gamma^\sigma{}_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} + \frac{\mathrm{d}^2 x^\sigma}{\mathrm{d}\tau^2}
\end{aligned}
$$

How does the geodesic equation transform with change of coordinates, say, from $\{x^\mu\}$ to $\{x'^\mu\}$?

$$
\begin{aligned}
0 &= \frac{\mathrm{d}^2 x^\sigma}{\mathrm{d}\tau^2} + \Gamma^\sigma{}_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} \\
&= \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial x^\sigma}{\partial x'^\rho}\frac{\mathrm{d}x'^\rho}{\mathrm{d}\tau}\right) + \Gamma^\sigma{}_{\alpha\beta}\frac{\partial x^\alpha}{\partial x'^\rho}\frac{\partial x^\beta}{\partial x'^\delta}\left(\frac{\mathrm{d}x'^\rho}{\mathrm{d}\tau}\frac{\mathrm{d}x'^\delta}{\mathrm{d}\tau}\right) \\
&= \left(\frac{\partial x^\sigma}{\partial x'^\rho}\frac{\mathrm{d}^2 x'^\rho}{\mathrm{d}\tau^2} + \frac{\partial^2 x^\sigma}{\partial x'^\rho \partial x'^\delta}\frac{\mathrm{d}x'^\rho}{\mathrm{d}\tau}\frac{\mathrm{d}x'^\delta}{\mathrm{d}\tau}\right) + \Gamma^\sigma{}_{\alpha\beta}\frac{\partial x^\alpha}{\partial x'^\rho}\frac{\partial x^\beta}{\partial x'^\sigma}\left(\frac{\mathrm{d}x'^\rho}{\mathrm{d}\tau}\frac{\mathrm{d}x'^\sigma}{\mathrm{d}\tau}\right)
\end{aligned}
$$

Multiply by $\frac{\partial x'^\gamma}{\partial x^\kappa}$:

$$
0 = \frac{\partial x'^\gamma}{\partial x^\kappa}\left(\frac{\partial x^\sigma}{\partial x'^\rho}\frac{\mathrm{d}^2 x'^\rho}{\mathrm{d}\tau^2} + \frac{\partial^2 x^\sigma}{\partial x'^\rho \partial x'^\delta}\frac{\mathrm{d}x'^\rho}{\mathrm{d}\tau}\frac{\mathrm{d}x'^\delta}{\mathrm{d}\tau}\right) + \Gamma^\sigma{}_{\alpha\beta}\frac{\partial x'^\gamma}{\partial x^\kappa}\frac{\partial x^\alpha}{\partial x'^\rho}\frac{\partial x^\beta}{\partial x'^\sigma}\left(\frac{\mathrm{d}x'^\rho}{\mathrm{d}\tau}\frac{\mathrm{d}x'^\sigma}{\mathrm{d}\tau}\right)
$$

Contract σ, κ indices :

$$
0 = \frac{\mathrm{d}^2 x'^\gamma}{\mathrm{d}\tau^2} + \left(\frac{\partial x'^\gamma}{\partial x^\delta}\frac{\partial x^\delta}{\partial x'^\rho \partial x'^\sigma} + \Gamma^\delta{}_{\alpha\beta}\frac{\partial x'^\gamma}{\partial x^\delta}\frac{\partial x^\alpha}{\partial x'^\rho}\frac{\partial x^\beta}{\partial x'^\sigma}\right)\frac{\mathrm{d}x'^\rho}{\mathrm{d}\tau}\frac{\mathrm{d}x'^\sigma}{\mathrm{d}\tau}
$$

If we demand that $\Gamma^\sigma{}_{\mu\nu}$ transforms *inhomogenously i.e.*

$$
\Gamma'^\gamma{}_{\rho\sigma} = \Gamma^\delta{}_{\alpha\beta}\frac{\partial x'^\gamma}{\partial x^\delta}\frac{\partial x^\alpha}{\partial x'^\rho}\frac{\partial x^\beta}{\partial x'^\sigma} + \frac{\partial x'^\gamma}{\partial x^\delta}\frac{\partial x^\delta}{\partial x'^\rho \partial x'^\sigma}
$$

The geodesic equation takes the form of an autoparallel,

$$
\begin{aligned}
0 &= \frac{\mathrm{d}^2 x'^\gamma}{\mathrm{d}\tau^2} + \Gamma'^\gamma_{\rho\sigma} \frac{\mathrm{d}x'^\rho}{\mathrm{d}\tau} \frac{\mathrm{d}x'^\sigma}{\mathrm{d}\tau} \\
&= v^\alpha D_\alpha v^\gamma
\end{aligned}
$$

For every tangent vector $v^\sigma = \frac{dx^\sigma}{d\tau}$ to a curve $\gamma(t)$, the acceleration vector field a^σ is a measure of the changes of the direction of the curve itself,

$$
a^\sigma = v^\alpha D_\alpha v^\sigma = \frac{\mathrm{d}v^\sigma}{\mathrm{d}\tau} + \Gamma^\sigma_{\mu\nu} \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} \frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}
$$

The curves that satisfy $a^\sigma = v^\alpha D_\alpha v^\sigma = 0$ are called *geodesics*. The covariant derivative of a vector field is or

$$
D_\mu v^\sigma = \partial_\mu v^\sigma + \Gamma^\beta_{\rho\mu} v^\rho
$$

Consider a geodesics $x^\mu(\tau)$ that is infinitesimally separated from another geodesic $y^\mu(\tau)$:

$$
y^\mu(\tau) = x^\mu(\tau) + \epsilon^\mu(\tau)
$$

Explicitly, the geodesics in question are:

$$
\begin{aligned}
0 &= \frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2} + \Gamma^\mu_{\alpha\beta}(x^\mu) \frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau} \frac{\mathrm{d}x^\beta}{\mathrm{d}\tau} \\
0 &= \frac{\mathrm{d}^2 y^\mu}{\mathrm{d}\tau^2} + \Gamma^\mu_{\alpha\beta}(x^\mu + \epsilon^\mu) \frac{\mathrm{d}y^\alpha}{\mathrm{d}\tau} \frac{\mathrm{d}y^\beta}{\mathrm{d}\tau}
\end{aligned}
$$

Carryout a Taylor expansion in the second equation above of the connection to first

order in ϵ:

$$
\begin{aligned}
0 &= \frac{\mathrm{d}^2 y^\mu}{\mathrm{d}\tau^2} + \Gamma^\mu{}_{\alpha\beta}\left(x^\mu + \epsilon^\mu\right) \frac{\mathrm{d}y^\alpha}{\mathrm{d}\tau} \frac{\mathrm{d}y^\beta}{\mathrm{d}\tau} \\
&= \left(\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2} + \frac{\mathrm{d}^2 \epsilon^\mu}{\mathrm{d}\tau^2}\right) + \Gamma^\mu{}_{\alpha\beta}\left(x^\mu + \epsilon^\mu\right)\left(\frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau} + \frac{\mathrm{d}\epsilon^\alpha}{\mathrm{d}\tau}\right)\left(\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau} + \frac{\mathrm{d}\epsilon^\beta}{\mathrm{d}\tau}\right) \\
&= \left(\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2} + \frac{\mathrm{d}^2 \epsilon^\mu}{\mathrm{d}\tau^2}\right) + \left[\Gamma^\mu{}_{\alpha\beta}\left(x\right) + \epsilon^\nu \partial_\nu \Gamma^\mu{}_{\alpha\beta}\left(x\right) + \cdots\right] \\
&\quad \times \left(\frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau} \frac{\mathrm{d}x^\beta}{\mathrm{d}\tau} + \frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau} \frac{\mathrm{d}\epsilon^\beta}{\mathrm{d}\tau} + \frac{\mathrm{d}\epsilon^\alpha}{\mathrm{d}\tau} \frac{\mathrm{d}x^\beta}{\mathrm{d}\tau} + \frac{\mathrm{d}\epsilon^\alpha}{\mathrm{d}\tau} \frac{\mathrm{d}\epsilon^\beta}{\mathrm{d}\tau}\right) \\
&= \left(\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2} + \frac{\mathrm{d}^2 \epsilon^\mu}{\mathrm{d}\tau^2}\right) \\
&\quad + \left[\Gamma^\mu{}_{\alpha\beta}\left(x\right) + \epsilon^\nu \partial_\nu \Gamma^\mu{}_{\alpha\beta}\left(x\right) + \cdots\right]\left(\frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau} \frac{\mathrm{d}x^\beta}{\mathrm{d}\tau} + 2\frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau} \frac{\mathrm{d}\epsilon^\beta}{\mathrm{d}\tau} + \cdots\right)
\end{aligned}
$$

Subtract the first geodesic equation from the last line above:

$$
\frac{\mathrm{d}^2 \epsilon^\mu}{\mathrm{d}\tau^2} + \epsilon^\nu \left(\partial_\nu \Gamma^\mu{}_{\alpha\beta}\right) \frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau} \frac{\mathrm{d}x^\beta}{\mathrm{d}\tau} + 2\Gamma^\mu{}_{\alpha\beta} \frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau} \frac{\mathrm{d}\epsilon^\beta}{\mathrm{d}\tau} = 0
$$

But the covariant derivative of the the vector ϵ^μ along $x^\mu\left(\tau\right)$ is given by

$$
\frac{D\epsilon^\mu}{\mathrm{d}\tau} = \frac{\mathrm{d}\epsilon^\mu}{\mathrm{d}\tau} + \Gamma^\mu{}_{\alpha\beta} \frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau} \epsilon^\beta
$$

Taking a second covariant derivative:

$$
\begin{aligned}
\frac{D^2 \epsilon^\mu}{d\tau^2} &= \frac{d}{d\tau}\left(\frac{D\epsilon^\mu}{d\tau}\right) + \Gamma^\mu{}_{\lambda\sigma}\frac{dx^\lambda}{d\tau}\frac{D\epsilon^\sigma}{d\tau} \\
&= \left[\frac{d^2\epsilon^\mu}{d\tau^2} + \Gamma^\mu{}_{\alpha\beta}\left(\epsilon^\beta\frac{d^2 x^\alpha}{d\tau^2} + \frac{dx^\alpha}{d\tau}\frac{d\epsilon^\beta}{d\tau}\right) + \frac{dx^\nu}{d\tau}\left(\partial_\nu \Gamma^\mu{}_{\alpha\beta}\right)\frac{dx^\alpha}{d\tau}\epsilon^\beta\right] \\
&\quad + \Gamma^\mu{}_{\lambda\sigma}\frac{dx^\lambda}{d\tau}\left(\frac{d\epsilon^\sigma}{d\tau} + \Gamma^\sigma{}_{\alpha\beta}\frac{dx^\alpha}{d\tau}\epsilon^\beta\right) \\
&= \left(\frac{d^2\epsilon^\mu}{d\tau^2} + \Gamma^\mu{}_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{d\epsilon^\beta}{d\tau} + \Gamma^\mu{}_{\lambda\sigma}\frac{dx^\lambda}{d\tau}\frac{d\epsilon^\sigma}{d\tau}\right) + \epsilon^\beta \Gamma^\mu{}_{\alpha\beta}\frac{d^2 x^\alpha}{d\tau^2} \\
&\quad + \left(\partial_\nu \Gamma^\mu{}_{\alpha\beta}\right)\frac{dx^\nu}{d\tau}\frac{dx^\alpha}{d\tau}\epsilon^\beta + \Gamma^\mu{}_{\lambda\sigma}\Gamma^\sigma{}_{\alpha\beta}\frac{dx^\lambda}{d\tau}\frac{dx^\alpha}{d\tau}\epsilon^\beta \\
&= \left(\frac{d^2\epsilon^\mu}{d\tau^2} + \epsilon^\nu\left(\partial_\nu \Gamma^\mu{}_{\alpha\beta}\right)\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} + 2\Gamma^\mu{}_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{d\epsilon^\beta}{d\tau}\right) - \epsilon^\nu\left(\partial_\nu \Gamma^\mu{}_{\alpha\beta}\right)\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} \\
&\quad + \epsilon^\beta \Gamma^\mu{}_{\alpha\beta}\frac{d^2 x^\alpha}{d\tau^2} + \left(\partial_\nu \Gamma^\mu{}_{\alpha\beta}\right)\frac{dx^\nu}{d\tau}\frac{dx^\alpha}{d\tau}\epsilon^\beta + \Gamma^\mu{}_{\lambda\sigma}\Gamma^\sigma{}_{\alpha\beta}\frac{dx^\lambda}{d\tau}\frac{dx^\alpha}{d\tau}\epsilon^\beta \\
&= \epsilon^\beta \Gamma^\mu{}_{\alpha\beta}\frac{d^2 x^\alpha}{d\tau^2} + \left[\epsilon^\nu \partial_\beta \Gamma^\mu{}_{\alpha\nu} - \epsilon^\nu \partial_\nu \Gamma^\mu{}_{\alpha\beta} + \Gamma^\mu{}_{\beta\sigma}\Gamma^\sigma{}_{\alpha\rho}\epsilon^\rho\right]\frac{dx^\beta}{d\tau}\frac{dx^\alpha}{d\tau} \\
&= -\epsilon^\nu \Gamma^\mu{}_{\rho\nu}\Gamma^\rho{}_{\alpha\beta}\frac{dx^\beta}{d\tau}\frac{dx^\alpha}{d\tau} + \left[\epsilon^\nu \partial_\beta \Gamma^\mu{}_{\alpha\nu} - \epsilon^\nu \partial_\nu \Gamma^\mu{}_{\alpha\beta} + \Gamma^\mu{}_{\beta\sigma}\Gamma^\sigma{}_{\alpha\rho}\epsilon^\rho\right]\frac{dx^\beta}{d\tau}\frac{dx^\alpha}{d\tau} \\
&= \left[\partial_\beta \Gamma^\mu{}_{\alpha\nu} + \Gamma^\mu{}_{\beta\sigma}\Gamma^\sigma{}_{\alpha\nu} - \partial_\nu \Gamma^\mu{}_{\alpha\beta} - \Gamma^\mu{}_{\sigma\nu}\Gamma^\sigma{}_{\alpha\beta}\right]\frac{dx^\beta}{d\tau}\frac{dx^\alpha}{d\tau}\epsilon^\nu \\
&= -\left[\partial_\nu \Gamma^\mu{}_{\alpha\beta} - \partial_\beta \Gamma^\mu{}_{\alpha\nu} - \Gamma^\mu{}_{\beta\sigma}\Gamma^\sigma{}_{\alpha\nu} + \Gamma^\mu{}_{\sigma\nu}\Gamma^\sigma{}_{\alpha\beta}\right]\frac{dx^\beta}{d\tau}\frac{dx^\alpha}{d\tau}\epsilon^\nu
\end{aligned}
$$

Write the above result more compactly:

$$
\frac{D^2 \epsilon^\mu}{d\tau^2} = -\epsilon^\nu R^\mu{}_{\nu\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}
$$

where we define a fourth-rank tensor , the Riemann curvature tensor as follows

$$
R^\mu{}_{\nu\alpha\beta} = \partial_\nu \Gamma^\mu{}_{\alpha\beta} - \partial_\beta \Gamma^\mu{}_{\alpha\nu} - \Gamma^\mu{}_{\beta\sigma}\Gamma^\sigma{}_{\alpha\nu} + \Gamma^\mu{}_{\sigma\nu}\Gamma^\sigma{}_{\alpha\beta}
$$

ϵ^ν is the perpendicular distance from one worldline to the other.

Compatible Connection

A **connection** is a structure which specifies how tensors are transported along a curve on a manifold. A metric compatible connection preserves lengths and angles under parallel transport. In other words, the covariant derivative of the compatible metric must vanish *i.e.*

$$
\begin{aligned}
0 &= D_\lambda g_{\alpha\beta} \\
&= \partial_\lambda g_{\alpha\beta} - g_{\rho\beta}\Gamma^\rho_{\ \alpha\lambda} - g_{\rho\alpha}\Gamma^\rho_{\ \beta\lambda} \\
&= \partial_\lambda g_{\alpha\beta} - \Gamma_{\beta\alpha\lambda} - \Gamma_{\alpha\beta\lambda}
\end{aligned}
$$

The Christoffel connection $\Gamma^\rho_{\ \beta\lambda}$ ensures covariance *w.r.t* local $GL(n)$ transformations. For an infinitesimal displacement $\mathbf{d}x^\mu$

$$
\boldsymbol{\Gamma}^\rho_{\ \alpha} = \Gamma^\rho_{\ \alpha\mu}\mathbf{d}x^\mu
$$

where $\Gamma^\rho_{\ \alpha\mu} = \Gamma^\rho_{\ \mu\alpha}$. To see why the Christoffel symbols are symmetric in their lower indices, notice that an arbitrary vector $\mathbf{v}$ can be expressed as

$$
\mathbf{v} = \mathrm{v}^\nu \hat{\mathbf{e}}_\nu \tag{74}
$$

where $\hat{\mathbf{e}}_\nu = \overrightarrow{\frac{\partial}{\partial x^\nu}}$ is a coordinate basis. The derivative of the vector *w.r.t* an arbitrary coordinate x^μ is

$$
\begin{aligned}
\partial_\mu \mathbf{v} &= \partial_\mu \left(\mathrm{v}^\nu \hat{\mathbf{e}}_\nu \right) \\
&= \left(\partial_\mu \mathrm{v}^\nu \right) \hat{\mathbf{e}}_\nu + \mathrm{v}^\nu \left(\partial_\mu \hat{\mathbf{e}}_\nu \right) \\
&= \left(\partial_\mu \mathrm{v}^\nu \right) \hat{\mathbf{e}}_\nu + \left(\mathrm{v}^\nu \Gamma^\rho_{\ \mu\nu} \right) \hat{\mathbf{e}}_\rho
\end{aligned}
$$

where the quantity $\Gamma^{\rho}{}_{\mu\nu}$ is defined by

$$\partial_{\mu}\hat{\mathbf{e}}_{\nu} \;=\; \Gamma^{\rho}{}_{\mu\nu}\hat{\mathbf{e}}_{\rho}$$

Since for any function f, $(\partial_{\mu}\hat{\mathbf{e}}_{\nu})\,(f) = \frac{\partial^2 f}{\partial x^{\mu}\partial x^{\nu}} = \partial_{\mu}\hat{\mathbf{e}}_{\nu}\,(f)$, it follows that

$$\Gamma^{\rho}{}_{\mu\nu} = \Gamma^{\rho}{}_{\nu\mu} \tag{75}$$

Notice that this symmetry is a coordinate invariant condition because the inhomogeneous part of the transformation law for $\Gamma^{\alpha}{}_{\mu\nu}$ is symmetric. The remaining matrix, $\mathbf{\Gamma}^{\rho}{}_{\alpha}$, is an element of the Lie algebra of $GL\,(n)$. Therefore, the connection gives the infinitesimal general linear transformation of a vector transported through the displacement $\mathbf{d}x^{\mu}$.

To determine the Christoffel connection, we permute the indices as follows

$$\partial_{\lambda}g_{\alpha\beta} - \Gamma_{\beta\alpha\lambda} - \Gamma_{\alpha\beta\lambda} \;=\; 0$$
$$\partial_{\alpha}g_{\lambda\beta} - \Gamma_{\lambda\beta\alpha} - \Gamma_{\beta\lambda\alpha} \;=\; 0$$
$$\partial_{\beta}g_{\alpha\lambda} - \Gamma_{\alpha\lambda\beta} - \Gamma_{\lambda\alpha\beta} \;=\; 0$$

Adding the first two equations and subtracting the third gives

$$\begin{aligned}
0 &= (\partial_{\lambda}g_{\alpha\beta} - \Gamma_{\beta\alpha\lambda} - \Gamma_{\alpha\beta\lambda}) + (\partial_{\alpha}g_{\lambda\beta} - \Gamma_{\lambda\beta\alpha} - \Gamma_{\beta\lambda\alpha}) - (\partial_{\beta}g_{\alpha\lambda} - \Gamma_{\alpha\lambda\beta} - \Gamma_{\lambda\alpha\beta}) \\
&= (\partial_{\lambda}g_{\alpha\beta} + \partial_{\alpha}g_{\lambda\beta} - \partial_{\beta}g_{\alpha\lambda}) - (\Gamma_{\beta\alpha\lambda} + \Gamma_{\beta\lambda\alpha})
\end{aligned}$$

Hence

$$\Gamma_{\beta\alpha\lambda} \;=\; \frac{1}{2}\left(\partial_{\lambda}g_{\alpha\beta} + \partial_{\alpha}g_{\lambda\beta} - \partial_{\beta}g_{\alpha\lambda}\right)$$

Raising the first index

$$\Gamma^{\beta}_{\alpha\lambda} = \frac{1}{2}g^{\beta\nu}\left(\partial_\lambda g_{\alpha\nu} + \partial_\alpha g_{\lambda\nu} - \partial_\nu g_{\alpha\lambda}\right) \tag{76}$$

Take a trace on α, β

$$\begin{aligned}
\Gamma^{\alpha}_{\lambda\alpha} &= \frac{1}{2}g^{\alpha\nu}\left(\partial_\lambda g_{\alpha\nu} + \partial_\alpha g_{\lambda\nu} - \partial_\nu g_{\alpha\lambda}\right) \\
&= \frac{1}{2}\left(g^{\alpha\nu}\partial_\lambda g_{\alpha\nu} + g^{\alpha\nu}\partial_\alpha g_{\lambda\nu} - g^{\alpha\nu}\partial_\nu g_{\alpha\lambda}\right) \\
&= \frac{1}{2}\left(g^{\alpha\nu}\partial_\lambda g_{\alpha\nu}\right) \\
&= \frac{1}{2}\frac{1}{g}\partial_\lambda\left(g\right) \\
&= \partial_\lambda\left(\ln\sqrt{g}\right)
\end{aligned}$$

Therefore

$$\partial_\nu \Gamma^{\alpha}_{\lambda\alpha} = \partial_\nu\partial_\lambda\left(\ln\sqrt{g}\right)$$

V. YANG-MILLS GAUGE THEORY

Let us take the structure group, $\mathcal{G}$ to be $SU(n)$ and let G_a denote its generators

$$\mathbf{A} = \mathbf{A}^a G_a \tag{77}$$

where the gauge fields $\mathbf{A}$ are Lie algebra valued connection 1-forms.

$\mathbf{A}$ has the meaning of a **connection** on a fiber bundle because it acts on the components of a field $\boldsymbol{\psi}$ with respect to some reference frame. Acting on ψ^α, $\alpha = 1, \cdots, n$, the gauge fields produce infinitesimal $SU(n)$ transformations that are given by linear combinations of the generators $\mathbf{A}^a G_a$. The generators in a matrix representation transform as

$$\psi^\beta \;\rightarrow\; \mathbf{A}^b \, [G_b]^\alpha{}_\beta \, \psi^\beta$$

To work out the transformation properties of $\mathbf{A}$, we demand covariance

$$\tilde{\mathbf{D}}\tilde{\psi} \;=\; \widetilde{\mathbf{D}\psi}$$

Let g be an element of $SU(n)$ acting on ψ

$$\tilde{\psi} \;=\; g\,\psi$$

Covariance requires

$$\left(\mathbf{d} + \tilde{\mathbf{A}}\right)(g\,\psi) \;=\; g\,(\mathbf{d} + \mathbf{A})\,\psi$$
$$(\mathbf{d}g)\,\psi + g\mathbf{d}\psi + \tilde{\mathbf{A}}\,(g\,\psi) \;=\; g\mathbf{d}\psi + g\mathbf{A}\psi$$
$$\Longrightarrow \tilde{\mathbf{A}}\,(g\,\psi) \;=\; g\mathbf{A}\psi - (\mathbf{d}g)\,\psi$$

Since ψ is arbitrary, we conclude that the local $SU\left(n\right)$ gauge transformation for the gauge potential is

$$\tilde{\mathbf{A}} \;=\; g\mathbf{A}g^{-1} - \left(\mathbf{d}g\right)g^{-1}$$

The *pure gauge* form is therefore

$$\tilde{\mathbf{A}} = -\left(\mathbf{d}g\right)g^{-1}$$

The **curvature** 2-form $\mathbf{F}$ that corresponds to the connection $\mathbf{A}$, called the **field strength** must be independent of the local section. The Yang-Mills field strength takes the form

$$\mathbf{F} \;=\; \mathbf{d}\mathbf{A} + \mathbf{A}\wedge\mathbf{A}$$

Invoking equation [77]:

$$
\begin{aligned}
\mathbf{F} \;&=\; \mathbf{d}\mathbf{A} + \mathbf{A}\wedge\mathbf{A} \\
&=\; \mathbf{d}\mathbf{A} + \mathbf{A}^a G_a \wedge \mathbf{A}^b G_b \\
&=\; \mathbf{d}\mathbf{A} + \mathbf{A}^a \wedge \mathbf{A}^b G_a G_b \\
&=\; \mathbf{d}\mathbf{A} + \frac{1}{2}\left(\mathbf{A}^a \wedge \mathbf{A}^b - \mathbf{A}^b \wedge \mathbf{A}^a\right)G_a G_b \\
&=\; \mathbf{d}\mathbf{A} + \frac{1}{2}\left(\mathbf{A}^a \wedge \mathbf{A}^b G_a G_b - \mathbf{A}^b \wedge \mathbf{A}^a G_a G_b\right) \\
&=\; \mathbf{d}\mathbf{A} + \frac{1}{2}\left(\mathbf{A}^a \wedge \mathbf{A}^b G_a G_b - \mathbf{A}^a \wedge \mathbf{A}^b G_b G_a\right) \\
&=\; \mathbf{d}\mathbf{A} + \frac{1}{2}\mathbf{A}^a \wedge \mathbf{A}^b \left(G_a G_b - G_b G_a\right) \\
&=\; \mathbf{d}\mathbf{A} + \frac{1}{2}\mathbf{A}^a \wedge \mathbf{A}^b \left[G_a, G_b\right] \\
\Rightarrow \mathbf{F}^c G_c \;&=\; \mathbf{d}\mathbf{A}^c G_c + \frac{1}{2}\mathbf{A}^a \wedge \mathbf{A}^b c^c_{\;\;ab} G_c
\end{aligned}
$$

which is true for all G_a. Dropping the generators,

$$\mathbf{F}^c \quad = \quad \mathbf{dA}^c + \frac{1}{2}c^c{}_{ab}\mathbf{A}^a \wedge \mathbf{A}^b \tag{78}$$

The integrability condition for equation [78] is

$$
\begin{aligned}
\mathbf{dF}^c \quad &= \quad \mathbf{d}^2\mathbf{A}^c + \frac{1}{2}c^c{}_{ab}\mathbf{dA}^a \wedge \mathbf{A}^b - \frac{1}{2}c^c{}_{ab}\mathbf{A}^a \wedge \mathbf{dA}^b \\
&= \quad \frac{1}{2}c^c{}_{ab}\left(\mathbf{F}^a - \frac{1}{2}c^a{}_{de}\mathbf{A}^d \wedge \mathbf{A}^e\right) \wedge \mathbf{A}^b - \frac{1}{2}c^c{}_{ab}\mathbf{A}^a \wedge \left(\mathbf{F}^b - \frac{1}{2}c^b{}_{de}\mathbf{A}^d \wedge \mathbf{A}^e\right) \\
&= \quad \frac{1}{2}c^c{}_{ab}\mathbf{F}^a \wedge \mathbf{A}^b - \left(\frac{1}{4}c^c{}_{ab}c^a{}_{de}\mathbf{A}^d \wedge \mathbf{A}^e \wedge \mathbf{A}^b\right) - \frac{1}{2}c^c{}_{ab}\mathbf{A}^a \wedge \mathbf{F}^b \\
&\quad + \left(\frac{1}{4}c^c{}_{ab}c^b{}_{de}\mathbf{A}^a \wedge \mathbf{A}^d \wedge \mathbf{A}^e\right)
\end{aligned}
$$

Using the Jacobi identity $c^c{}_{a[b}c^a{}_{de]} = 0$ to simply the terms in brackets on the last line above gives

$$
\begin{aligned}
\frac{1}{4}c^c{}_{ab}c^a{}_{de}\mathbf{A}^d \wedge \mathbf{A}^e \wedge \mathbf{A}^b - \frac{1}{4}c^c{}_{ab}c^b{}_{de}\mathbf{A}^a \wedge \mathbf{A}^d \wedge \mathbf{A}^e \quad &= \quad \frac{1}{2}c^c{}_{ab}c^a{}_{de}\mathbf{A}^b \wedge \mathbf{A}^d \wedge \mathbf{A}^e \\
&= \quad \frac{1}{2}c^c{}_{a[b}c^a{}_{de]}\mathbf{A}^b \wedge \mathbf{A}^d \wedge \mathbf{A}^e \\
&= \quad 0
\end{aligned}
$$

This leaves us with

$$\mathbf{dF}^c \quad = \quad \frac{1}{2}c^c{}_{ab}\mathbf{F}^a \wedge \mathbf{A}^b - \frac{1}{2}c^c{}_{ab}\mathbf{A}^a \wedge \mathbf{F}^b \tag{79}$$

We know that $\mathbf{A}^a$ is a connection–a Lie-algebra-valued 1-form. In the basis of the Lie algebra

$$\mathbf{A}^a\left[G_a\right]^A{}_B$$

so if we use the adjoint representation for the generators,

$$[G_a]^c{}_b = c^c{}_{ab}$$

the connection in the adjoint representation is the combination

$$\boldsymbol{\alpha}^c{}_b \equiv \mathbf{A}^a c^c{}_{ab}$$

To differentiate a *vector*, we may write

$$
\begin{aligned}
\mathbf{D}v^a &= \mathbf{d}v^a + v^b \boldsymbol{\alpha}^a{}_b \\
&= \mathbf{d}v^a + v^b c^a{}_{cb} \mathbf{A}^c
\end{aligned}
$$

Returning to the integrability condition, equation [79]

$$
\begin{aligned}
0 &= \mathbf{dF}^c - \frac{1}{2}c^c{}_{ab}\mathbf{F}^a \wedge \mathbf{A}^b + \frac{1}{2}c^c{}_{ab}\mathbf{A}^a \wedge \mathbf{F}^b \\
&= \mathbf{dF}^c + \mathbf{F}^a \wedge c^c{}_{ba}\mathbf{A}^b \\
&= \mathbf{dF}^c + \mathbf{F}^a \wedge \boldsymbol{\alpha}^c{}_a
\end{aligned}
$$

The *Bianchi identity* for the Yang-Mills field is therefore

$$\mathbf{DF}^c = \mathbf{dF}^c + \mathbf{F}^a \wedge \boldsymbol{\alpha}^c{}_a = 0 \tag{80}$$

Equation [80] is in fact a generalization of Maxwell's electromagnetic theory. For electromagnetism, the structure group is the Abelian Lie group $U(1)$ with vanishing structure constants, reducing to

$$\mathbf{dF}^c = 0$$

For the U(1) case, equation [78] is

$$\mathbf{F} = \mathbf{dA} \tag{81}$$

More explicitly

$$
\begin{aligned}
\mathbf{F} &= \mathbf{dA} \\
&= \frac{1}{2} F_{\mu\nu} \mathbf{d}x^\mu \wedge \mathbf{d}x^\nu \\
&= E_i \, \mathbf{d}x^0 \wedge \mathbf{d}x^i + \frac{1}{2} \varepsilon_{ijk} \, B^k \, \mathbf{d}x^i \wedge \mathbf{d}x^j
\end{aligned}
$$

This field strength is covariant under a *coordinate* transformation

$$
\begin{aligned}
F'_{\mu\nu} &= \frac{\partial A'_\nu}{\partial x'^\mu} - \frac{\partial A'_\mu}{\partial x'^\nu} \\
&= \frac{\partial}{\partial x'^\mu} \left(\frac{\partial x^\gamma}{\partial x'^\nu} A_\gamma \right) - \frac{\partial}{\partial x'^\nu} \left(\frac{\partial x^\delta}{\partial x'^\mu} A_\delta \right) \\
&= \frac{\partial x^\gamma}{\partial x'^\mu} \frac{\partial x^\delta}{\partial x'^\nu} \frac{\partial A_\gamma}{\partial x^\delta} - \frac{\partial x^\delta}{\partial x'^\mu} \frac{\partial x^\gamma}{\partial x'^\nu} \frac{\partial A_\delta}{\partial x^\gamma} \\
&= \frac{\partial x^\delta}{\partial x'^\mu} \frac{\partial x^\gamma}{\partial x'^\nu} \left(\frac{\partial A_\gamma}{\partial x^\delta} - \frac{\partial A_\delta}{\partial x^\gamma} \right) \\
&= \frac{\partial x^\delta}{\partial x'^\mu} \frac{\partial x^\gamma}{\partial x'^\nu} F_{\delta\gamma}
\end{aligned}
$$

Furthermore, field strength F is invariant under a $U(1)$ *gauge* transformation of the electromagnetic four potential *i.e*

$$A_\alpha \longrightarrow A_\alpha + \partial_\alpha \phi \tag{82}$$

with ϕ being a scalar field

$$
\begin{aligned}
F_{\mu\nu} \quad &\longrightarrow \quad \partial_\mu \left(A_\nu + \partial_\nu\phi\right) - \partial_\nu \left(A_\mu + \partial_\mu\phi\right) \\
&= \quad \partial_\mu A_\nu + \partial_\mu\partial_\nu\phi - \partial_\nu A_\mu - \partial_\nu\partial_\mu\phi \\
&= \quad \partial_\mu A_\nu - \partial_\nu A_\mu + \overset{0}{\left(\partial_\mu\partial_\nu\phi - \partial_\nu\partial_\mu\phi\right)} \\
&= \quad \partial_\mu A_\nu - \partial_\nu A_\mu \\
&= \quad F_{\mu\nu}
\end{aligned}
$$

It follow from equation [81] that

$$
\begin{aligned}
\mathbf{dF} \quad &= \quad \mathbf{d}^2\mathbf{A} \\
&\equiv \quad 0
\end{aligned}
$$

Therefore

$$
\frac{1}{3!} \left(\partial_\mu F_{\nu\alpha} + \partial_\alpha F_{\mu\nu} + \partial_\nu F_{\alpha\mu}\right) \mathbf{d}x^\mu \wedge \mathbf{d}x^\nu \wedge \mathbf{d}x^\alpha \quad = \quad 0
$$

The above relationship is the $U(1)$ **Bianchi identity**

$$
\partial_\mu F_{\nu\alpha} + \partial_\alpha F_{\mu\nu} + \partial_\nu F_{\alpha\mu} \quad = \quad 0 \tag{83}
$$

Maxwell's equations follow from a variation of the action. The Lagrangian of a free electromagnetic field is

$$
\mathcal{L}_M \quad = \quad -\frac{1}{4} g^{\mu\nu} g^{\alpha\beta} F_{\mu\alpha} F_{\nu\beta} \tag{84}
$$

U(1)-Bundle Formulation of Electromagnetism

The *gauge postulate*, is arguably the spell that binds field theory. A famous and elegant way of re-casting Maxwell's electromagnetic theory is as a principal $U(1)$-bundle over a *base* manifold $\mathcal{M}^4$.

The $U(1)$-bundle arises as the *gauge group* of charged matter fields. The vector potential A_μ in equation [82], a 1-form, is <u>immeasurable</u> since physical experiments cannot distinguish between A_α and $A_\alpha + \partial_\alpha \psi$, where ψ is a $U(1)$-valued function on $\mathcal{M}$. In other words, only *equivalence classes* of A_α are relevant and gauge transformations amount to changing the bundle cross sections

$$\tilde{\mathbf{A}} \;=\; \psi \mathbf{A} \psi^{-1} - i \mathbf{d}\psi \, \psi^{-1} \tag{85}$$

For $U(1)$ we have

$$\psi \;=\; \mathrm{e}^{-i\phi} \tag{86}$$

Substituting equation [86] into equation [85] gives

$$\tilde{\mathbf{A}} \;=\; \mathbf{A} + \mathbf{d}\phi$$

In particular, the *gauge group* of Maxwell's electromagnetic theory is $U(1)$ which is a Lie group consisting of all complex numbers z such that

$$|z| \;=\; 1$$

$\mathfrak{u}(1)$ is the gauge Lie algebra with $\mathbf{A} \in \mathfrak{u}(1)$

Recall that a 1-form $\mathbf{A}$ on a principal fiber bundle is ***horizontal*** if its integral

along any curve on the bundle is independent of **lifting**. We can therefore write the gauge potentials in terms of Lie algebra valued 1-forms

$$\mathbf{A} \;=\; \left[A^a{}_\mu(x)\right] G_a\, \mathbf{d}x^\mu$$

where G_a are generators subject to the normalization condition

$$\mathrm{Tr}\,(G_a G_b) \;=\; \frac{1}{2}\delta_{ab}$$

If the gauge group $\mathcal{G} = U(N)$, then the generators G_a are traceless $N \times N$ Hermitian matrices.

By virtue of the Poincaré lemma, the gauge potentials $\tilde{\mathbf{A}}$ and $\mathbf{A}$ for $U(1)$ differ by a closed 1-form. This means that *field strength* $\mathbf{F}$ is independent of section of the bundle $\mathbf{A}$. Furthermore, $F_{\mu\nu}$ is the **local curvature** in direct analogy to the *Riemann curvature* tensor $R^\mu{}_{\nu\alpha\beta}$. In differential geometry, $R^\mu{}_{\nu\alpha\beta}$ is a measure of the failure of the parallel transport of a vector v^μ around an infinitesimal loop C to return to its original position p_0

$$\triangle\mathrm{v}^\mu|_{p_0} \;=\; -\frac{1}{2}\oint_C \mathrm{v}^\nu R^\mu{}_{\nu\alpha\beta}\,\mathbf{d}x^\alpha \wedge \mathbf{d}x^\beta$$

In that sense the strength of the gauge fields, $F_{\mu\nu}$, represent a curvature on a principal fiber bundle $\mathbb{E} = \mathcal{M}^4 \times \mathcal{G}$ where $\mathcal{G} = U(1)$ and $\mathcal{M}^4$ is the space-time *base manifold*.

The gauge potential A_μ is a *connection* and therefore describes parallel transport on the principal fiber bundle $\mathbb{E}$. This identification underscores the celebrated result of a diffraction experiment that was suggested by David Bohm and his student Yakir Aharonov in 1959. An electron beam is split into two beams that are transmitted through two metallic tubes at different potentials. The beams are collected on a

screen and the interference pattern is observed [10].

In the Aharonov-Bohm experiment, the electron phase difference is computed from the loop integral

$$\alpha \;=\; q \oint_C A_\mu\,(x)\,\mathbf{d}x^\mu$$

where C denotes a closed contour. Since a change of α by an integer multiple of 2π does not change the diffraction pattern on the screen, the physically meaningful quantity is the **phase factor** Φ

$$\Phi \;=\; \exp\left(iq \oint_{\partial\mathcal{M}} A_\mu\,(x)\,\mathbf{d}x^\mu\right) \tag{87}$$

By Stokes theorem

$$\Phi \;=\; \exp\left(\frac{i}{2}q \int_{\mathcal{M}} F_{\mu\nu}\mathbf{d}x^\mu \wedge \mathbf{d}x^\nu\right) \approx \mathbf{1} - F_{\mu\nu}\mathbf{d}x^\mu \wedge \mathbf{d}x^\nu$$

where $F_{\mu\nu}$ denotes the components of the *field strength* which is defined when the contour is contracted to a point. $\mathbf{d}x^\mu \wedge \mathbf{d}x^\nu$ is the infinitesimal surface whose boundary is the close curve. The analogue of the Ricci identity in GR is given by

$$[D_\mu, D_\nu] \;=\; F_{\mu\nu}$$

where D_μ is the gauge-covariant derivative and is given by

$$D_\mu\psi \;=\; \partial_\mu\psi - iA_\mu\psi$$

As we will see in a little more detail in the next sections, the **Cartan equations**

describe the physics of electromagnetic fields over a curved base manifold $\mathcal{M}$

$$\begin{aligned}
\mathbf{d}\boldsymbol{\omega}^a{}_b &= \boldsymbol{\omega}^c{}_b \wedge \boldsymbol{\omega}^a{}_c + \mathbf{R}^a{}_b \\
\mathbf{d}\mathbf{e}^a &= \mathbf{e}^b \wedge \boldsymbol{\omega}^a{}_b + \mathbf{T}^a \\
\mathbf{d}\mathbf{A}_\alpha &= \mathbf{F}_\alpha
\end{aligned}$$

For a Yang-Mills gauge theory, the **Wilson loop** Φ_W depends on both the representation of the gauge fields as well as the value along the contour

$$\Phi_W = \mathscr{P} \exp \left(iq \oint_{\partial\mathcal{M}} \left[A^a{}_\mu (x) \right] G_a \, \mathbf{d}x^\mu \right)$$

where $\mathscr{P}$ denotes path-ordering.

Yang-Mills Action

The dynamics of a physical system in the Lagrangian formalism are computed from an n-form in n dimensions

$$\mathcal{L} (\boldsymbol{\psi}, \mathbf{d}\boldsymbol{\psi})$$

which is normally a function of the fields $\boldsymbol{\psi}$ and their first derivatives. To make $\mathbf{A}$ dynamical, another Lagrangian n-form $\mathcal{L}_{\text{int}}$ is introduced. Schematically, the Yang-Mills action functional takes the form

$$S = \int \mathcal{L}_{\text{free}} + \mathcal{L}_{\text{int}}$$

Analogous to Maxwell's electromagnetism, the Yang-Mills action is constructed out of the $\mathfrak{su}(n)$-valued 2-form $\mathbf{F}$ which is covariant $w.r.t$ local $SU(n)$ gauge transformations

$$
\begin{aligned}
\tilde{\mathbf{F}} \;=\;& \mathbf{d}\tilde{\mathbf{A}} + \tilde{\mathbf{A}} \wedge \tilde{\mathbf{A}} \\[2mm]
=\;& \mathbf{d}\left(g\mathbf{A}g^{-1} - (\mathbf{d}g)\,g^{-1}\right) + \left(g\mathbf{A}g^{-1} - (\mathbf{d}g)\,g^{-1}\right) \wedge \left(g\mathbf{A}g^{-1} - (\mathbf{d}g)\,g^{-1}\right) \\[2mm]
=\;& (\mathbf{d}g)\,\mathbf{A}g^{-1} + g\,(\mathbf{d}\mathbf{A})\,g^{-1} - g\mathbf{A}\left(\mathbf{d}g^{-1}\right) + (\mathbf{d}g) \wedge \left(\mathbf{d}g^{-1}\right) \\
& + \left(g\mathbf{A}g^{-1}\right) \wedge \left(g\mathbf{A}g^{-1}\right) - \left(g\mathbf{A}g^{-1}\right) \wedge (\mathbf{d}g)\,g^{-1} - (\mathbf{d}g)\,g^{-1} \wedge \left(g\mathbf{A}g^{-1}\right) \\
& + (\mathbf{d}g)\,g^{-1} \wedge (\mathbf{d}g)\,g^{-1} \\[2mm]
=\;& (\mathbf{d}g)\,\mathbf{A}g^{-1} + g\,(\mathbf{d}\mathbf{A})\,g^{-1} - g\mathbf{A}\left(\mathbf{d}g^{-1}\right) + (\mathbf{d}g) \wedge \left(\mathbf{d}g^{-1}\right) \\
& + \left(g\mathbf{A}g^{-1}\right) \wedge \left(g\mathbf{A}g^{-1}\right) + g\mathbf{A} \wedge (\mathbf{d}g)\,gg^{-1} - (\mathbf{d}g) \wedge \mathbf{A}g^{-1} \\
& + (\mathbf{d}g)\,g^{-1} \wedge (\mathbf{d}g)\,g^{-1} \\[2mm]
=\;& g\,(\mathbf{d}\mathbf{A})\,g^{-1} + g\,(\mathbf{A} \wedge \mathbf{A})\,g^{-1} + \cancel{(\mathbf{d}g) \wedge \mathbf{A}g^{-1}} - \cancel{(\mathbf{d}g) \wedge \mathbf{A}g^{-1}} \\
& - \cancel{g\mathbf{A} \wedge \left(\mathbf{d}g^{-1}\right)} + \cancel{g\mathbf{A} \wedge \left(\mathbf{d}g^{-1}\right)} + \cancel{(\mathbf{d}g) \wedge \left(\mathbf{d}g^{-1}\right)} - \cancel{(\mathbf{d}g) \wedge \left(\mathbf{d}g^{-1}\right)} \\[2mm]
=\;& g\,(\mathbf{d}\mathbf{A} + \mathbf{A} \wedge \mathbf{A})\,g^{-1}
\end{aligned}
$$

Therefore

$$
\tilde{\mathbf{F}} \;=\; g\,(\mathbf{F})\,g^{-1}
$$

transforms as a tensor under $SU(n)$. In 4-dimensions, the free electromagnetic action in equation [84] is the integral of a 4-form

$$
S \;=\; -\frac{1}{2}\int \mathbf{F} \wedge {}^{*}\mathbf{F} \;=\; -\frac{1}{2}\int \left(\mathbf{E}^2 - \mathbf{B}^2\right)\boldsymbol{\Phi} \tag{88}
$$

where $\boldsymbol{\Phi}$ is the volume form. Without the Hodge star operator in equation [88] which depends on the spacetime metric, the only $SU(n)$-invariant action permitted is a

topological invariant, called the **Pontryagin character**.

$$S \;=\; -\frac{1}{2}\int \mathbf{F}\wedge\mathbf{F} \;=\; \int (\mathbf{E}\cdot\mathbf{B})\,\boldsymbol{\Phi}$$

Varying the above action does not yield equations of motion because it can be expressed as a total divergence

$$
\begin{aligned}
\delta S &= -\int \delta\mathbf{F}\wedge\mathbf{F}\\
&= -\int \mathbf{d}\,(\delta\mathbf{A})\wedge\mathbf{F}\\
&= -\int [\mathbf{d}\,(\delta\mathbf{A}\wedge\mathbf{F}) - (\delta\mathbf{A})\wedge\mathbf{dF}]\\
&= -\int \mathbf{d}\,(\delta\mathbf{A}\wedge\mathbf{F})\\
&= 0
\end{aligned}
$$

The Action functional for the Yang-Mills field is

$$S_{YM} = \int_V \left(\frac{1}{2}\mathbf{F}^A\wedge {}^*\mathbf{F}^B - \kappa\mathbf{A}^A\wedge {}^*\mathbf{J}^B\right) K_{AB} \tag{89}$$

where K_{AB} is the $\mathfrak{su}\,(n)$ Killing form and $\mathbf{J}$ is a covariantly conserved current.

A variation of the action *w.r.t* the gauge potentials yields the *inhomogeneous* field

equations,

$$
\begin{aligned}
0 \; = \; & \delta S_{YM} \\[4pt]
= \; & \int_V \left[\delta \mathbf{F}^A \wedge {}^*\mathbf{F}^B - \kappa \delta \mathbf{A}^A \wedge {}^*\mathbf{J}^B \right] K_{AB} \\[4pt]
= \; & \int_V \left[\left\{ \mathbf{d}\left(\delta \mathbf{A}_A \right) + \frac{1}{2} c_A{}^{CD} \left(\delta \mathbf{A}_C \right) \mathbf{A}_D + \frac{1}{2} c_A{}^{CD} \mathbf{A}_C \delta \mathbf{A}_D \right\} \wedge {}^*\mathbf{F}^A - \kappa \delta \mathbf{A}^A \wedge {}^*\mathbf{J}^B \right] \\[4pt]
= \; & \int_V \left[\left\{ \mathbf{d}\left(\delta \mathbf{A}_A \right) + c_A{}^{CD} \mathbf{A}_C \left(\delta \mathbf{A}_D \right) \right\} \wedge {}^*\mathbf{F}^A - \kappa \delta \mathbf{A}^A \wedge {}^*\mathbf{J}^B \right] \\[4pt]
= \; & \int_V \left(\mathbf{D}\left(\delta \mathbf{A}_A \right) \wedge {}^*\mathbf{F}^A - \kappa \delta \mathbf{A}^A \wedge {}^*\mathbf{J}^B \right) \\[4pt]
= \; & \int_V \left[\mathbf{D}\left(\delta \mathbf{A}_A \wedge {}^*\mathbf{F}^A \right) \right] + \int_V \delta \mathbf{A}_A \wedge \left[\mathbf{D}{}^*\mathbf{F}^A - \kappa\,{}^*\mathbf{J}^A \right] \\[4pt]
= \; & \int_V \mathbf{d}\left(\delta \mathbf{A}_A \wedge {}^*\mathbf{F}^A \right) + \int_V \delta \mathbf{A}_A \wedge \left[\mathbf{D}{}^*\mathbf{F}^A - \kappa\,{}^*\mathbf{J}^A \right] \\[4pt]
= \; & \int_V \delta \mathbf{A}_A \wedge \left[\mathbf{D}{}^*\mathbf{F}^A - \kappa\,{}^*\mathbf{J}^A \right]
\end{aligned}
$$

The last line holds for all $\delta \mathbf{A}_A$. This leaves us with

$$
\mathbf{D}{}^*\mathbf{F}^A \;\; = \;\; \kappa\,{}^*\mathbf{J}^A
$$

Taking its dual gives us our field equation,

$$
{}^*\mathbf{D}{}^*\mathbf{F}^A \;\; = \;\; \kappa\,\mathbf{J}^A
$$

where ${}^*\mathbf{D}{}^*\mathbf{F}^A$ is the covariant divergence.

Quotient construction of Yang-Mills gauge theory

The **Cartan quotient method** is a simple way to construct geometries with continuous local symmetries. Starting with a Lie group $\mathcal{G}$, a principal fiber bundle is constructed by taking the quotient of $\mathcal{G}$ by a normal Lie subgroup, $\mathcal{H}$. This subgroup becomes the local symmetry of the quotient manifold.

If n is the dimension of $\mathcal{G}$ and m the dimension of $\mathcal{H}$, then the quotient manifold is $(n-m)$-dimensional. Since the Maurer-Cartan equations along with their integrability condition constitute the Lie algebra of a Lie group, a local gauge theory arises from the Maurer-Cartan structure equations for the product of $\mathcal{G}$ with the Poincaré group, $\mathcal{P} \times \mathcal{G}$:

$$\mathbf{d}\tilde{\boldsymbol{\omega}}^a{}_b \ = \ \tilde{\boldsymbol{\omega}}^c{}_b \wedge \tilde{\boldsymbol{\omega}}^a{}_c$$

$$\mathbf{d}\tilde{\mathbf{e}}^a \ = \ \tilde{\mathbf{e}}^b \wedge \tilde{\boldsymbol{\omega}}^a{}_b$$

$$\mathbf{d}\tilde{\mathbf{A}}_\alpha \ = \ -\frac{1}{2}c_\alpha{}^{\beta\delta}\tilde{\mathbf{A}}_\beta \wedge \tilde{\mathbf{A}}_\delta$$

In general, the geometry is curved. To achieve this, we generalize the connection

$$\tilde{\boldsymbol{\omega}}^a{}_b \ \rightarrow \ \boldsymbol{\omega}^a{}_b$$

$$\tilde{\mathbf{e}}^a \ \rightarrow \ \mathbf{e}^a$$

$$\tilde{\mathbf{A}}_\alpha \ \rightarrow \ \mathbf{A}_\alpha = A_{\alpha b}\mathbf{e}^b$$

With a generalized connection, we arrive at the Cartan structure equations

$$\mathbf{d\omega}^a{}_b = \boldsymbol{\omega}^c{}_b \wedge \boldsymbol{\omega}^a{}_c + \mathbf{R}^a{}_b$$

$$\mathbf{de}^a = \mathbf{e}^b \wedge \boldsymbol{\omega}^a{}_b + \mathbf{T}^a$$

$$\mathbf{dA}_\alpha = -\frac{1}{2}c_\alpha{}^{\beta\delta}\mathbf{A}_\beta \wedge \mathbf{A}_\delta + \mathbf{F}_\alpha$$

where for Riemannian geometry we take $\mathbf{T}^a = 0$.

For the curvature $\mathbf{R}^a{}_b$ and field strength $\mathbf{F}_\alpha$ to represent curvatures of the geometry, we require them to be independent of lifting. To guarantee local Lorentz and $\mathcal{G} = SU(2) \times U(1)$ symmetries for instance, we take the quotient $(\mathcal{P} \times \mathcal{G}) / (\mathcal{L} \times \mathcal{G})$, so that the horizontal directions are spanned by the solder form alone. Therefore, the curvature and field strength are

$$\mathbf{R}^a{}_b = \frac{1}{2}R^a{}_{bcd}\mathbf{e}^c \wedge \mathbf{e}^d \tag{90}$$

$$\mathbf{F}_\alpha = \frac{1}{2}F_{\alpha cd}\mathbf{e}^c \wedge \mathbf{e}^d \tag{91}$$

The Bianchi identity for the gauge potential $\mathbf{A}_\beta$ is

$$
\begin{aligned}
0 &\equiv \mathbf{d}^2\mathbf{A}_\beta \\
&= -\frac{1}{2}c_\beta{}^{\varsigma\delta}\mathbf{dA}_\varsigma \wedge \mathbf{A}_\delta + \frac{1}{2}c_\beta{}^{\varsigma\delta}\mathbf{A}_\varsigma \wedge \mathbf{dA}_\delta + \mathbf{dF}_\beta \\
&= \mathbf{dF}_\beta + c_\beta{}^{\varsigma\delta}\mathbf{A}_\varsigma \wedge \left(-\frac{1}{2}c_\delta{}^{\epsilon\varphi}\mathbf{A}_\epsilon \wedge \mathbf{A}_\varphi + \mathbf{F}_\delta\right) \\
&= \mathbf{dF}_\beta - \frac{1}{2}c_\delta{}^{[\epsilon\varphi}c_\beta{}^{\varsigma]\delta}\mathbf{A}_\varsigma \wedge \mathbf{A}_\epsilon \wedge \mathbf{A}_\varphi + c_\beta{}^{\varsigma\delta}\mathbf{A}_\varsigma \wedge \mathbf{F}_\delta \\
&= \mathbf{dF}_\beta + c_\beta{}^{\varsigma\delta}\mathbf{A}_\varsigma \wedge \mathbf{F}_\delta \\
&= \mathbf{DF}_\beta
\end{aligned}
$$

where $c_\delta{}^{[\epsilon\varphi}c_\beta{}^{\varsigma]\delta} = 0$ is the Jacobi identity. Therefore,

$$
\begin{aligned}
\mathbf{DF}_\alpha &= \mathbf{dF}_\alpha + c_\alpha{}^{\mu\beta}\mathbf{A}_\mu \wedge \mathbf{F}_\beta \\
&= \mathbf{dF}_\alpha + \boldsymbol{\omega}^\beta{}_\alpha \wedge \mathbf{F}_\beta
\end{aligned}
$$

which is equation [80].

$$
\begin{aligned}
\mathbf{D}\left(\frac{1}{2}F_{\beta ab}\mathbf{e}^a \wedge \mathbf{e}^b\right) &= \mathbf{d}\left(\frac{1}{2}F_{\beta ab}\mathbf{e}^a \wedge \mathbf{e}^b\right) + \boldsymbol{\omega}^\delta{}_\beta \wedge \left(\frac{1}{2}F_{\delta ab}\mathbf{e}^a \wedge \mathbf{e}^b\right) \\
&= \frac{1}{2}\left(\mathbf{d}F_{\beta ab}\mathbf{e}^a \wedge \mathbf{e}^b + F_{\beta ab}\mathbf{de}^a \wedge \mathbf{e}^b - F_{\beta ab}\mathbf{e}^a \wedge \mathbf{de}^b\right) \\
&\quad + \frac{1}{2}\left(F_{\delta ab}\boldsymbol{\omega}^\delta{}_\beta \wedge \mathbf{e}^a \wedge \mathbf{e}^b\right) \\
&= \frac{1}{2}\left(\mathbf{d}F_{\beta ab}\mathbf{e}^a \wedge \mathbf{e}^b - F_{\beta eb}\boldsymbol{\omega}^e{}_c \wedge \mathbf{e}^c \wedge \mathbf{e}^b\right) \\
&\quad + \frac{1}{2}\left(-F_{\beta ab}\boldsymbol{\omega}^b{}_c \wedge \mathbf{e}^a \wedge \mathbf{e}^c + F_{\delta ab}\boldsymbol{\omega}^\delta{}_\beta \wedge \mathbf{e}^a \wedge \mathbf{e}^b\right) \\
&= \frac{1}{2}\left(\mathbf{d}F_{\beta cd} - F_{\beta ed}\boldsymbol{\omega}^e{}_c - F_{\beta ce}\boldsymbol{\omega}^e{}_d + F_{\delta cd}\boldsymbol{\omega}^\delta{}_\beta\right) \wedge \mathbf{e}^c \wedge \mathbf{e}^d \\
&= \frac{1}{2}\left(\mathscr{D}F_{\beta cd}\right) \wedge \mathbf{e}^c \wedge \mathbf{e}^d
\end{aligned}
$$

where

$$
\mathscr{D}F_{\beta cd} = \mathbf{d}F_{\beta cd} - F_{\beta ed}\boldsymbol{\omega}^e{}_c - F_{\beta ce}\boldsymbol{\omega}^e{}_d + F_{\delta cd}\boldsymbol{\omega}^\delta{}_\beta \tag{92}
$$

VI. THE EINSTEIN-HILBERT ACTION

The Einstein-Hilbert action which is invariant under arbitrary coordinate transformations describes how gravity couples to matter fields

$$S \;=\; \frac{1}{2\kappa} \int R \sqrt{-g} \mathrm{d}^4 x + \int \mathcal{L}_M \sqrt{-g} \mathrm{d}^4 x \tag{93}$$

By the principle of least action, we demand the vanishing of the variation of the action *w.r.t* the inverse metric

$$
\begin{aligned}
0 \;&=\; \delta S \\
&=\; \frac{1}{2\kappa} \int \delta_{g^{\mu\nu}} \left(\sqrt{-g}\, R \right) \mathrm{d}^4 x + \int \delta_{g^{\mu\nu}} \left(\sqrt{-g} \mathcal{L}_M \right) \mathrm{d}^4 x \\
&=\; \int \left[\frac{1}{2\kappa} \left(\frac{\delta R}{\delta g^{\mu\nu}} + \frac{R}{\sqrt{-g}} \frac{\delta \sqrt{-g}}{\delta g^{\mu\nu}} \right) + \frac{1}{2} \frac{1}{\sqrt{-g}} \frac{\delta \left(\sqrt{-g} \mathcal{L}_M \right)}{\delta g^{\mu\nu}} \right] \delta g_{\mu\nu} \sqrt{-g} \mathrm{d}^4 x
\end{aligned}
$$

Note:

$$\delta_{g^{\mu\nu}} \left(\sqrt{-g}\, R \right) = \left(\frac{\delta R}{\delta g^{\mu\nu}} + \frac{R}{\sqrt{-g}} \frac{\delta \sqrt{-g}}{\delta g^{\mu\nu}} \right) \delta g_{\mu\nu}$$

Variation of the Ricci Scalar

On the left hand side, the Ricci scalar is defined as

$$R \;=\; g^{\alpha\mu} R_{\beta\mu} \tag{94}$$

where $R_{\beta\mu}$ is the Ricci tensor which is related to the Riemann tensor

$$R_{\beta\mu} \;\equiv\; R^{\sigma}{}_{\beta\sigma\mu} \;=\; \partial_\sigma \Gamma^{\sigma}{}_{\beta\mu} - \partial_\beta \Gamma^{\sigma}{}_{\sigma\mu} + \Gamma^{\sigma}{}_{\sigma\nu} \Gamma^{\nu}{}_{\beta\mu} - \Gamma^{\sigma}{}_{\beta\nu} \Gamma^{\nu}{}_{\sigma\mu} \tag{95}$$

The variation of the Riemann curvature tensor follows from above

$$
\begin{aligned}
\delta R^{\rho}{}_{\sigma\mu\nu} &= \partial_{\mu}\left(\delta\Gamma^{\rho}{}_{\nu\sigma}\right) - \partial_{\nu}\left(\delta\Gamma^{\rho}{}_{\mu\sigma}\right) + \left(\delta\Gamma^{\rho}{}_{\mu\lambda}\right)\Gamma^{\lambda}{}_{\nu\sigma} + \Gamma^{\rho}{}_{\mu\lambda}\left(\delta\Gamma^{\lambda}{}_{\nu\sigma}\right) \\
&\quad - \left(\delta\Gamma^{\rho}{}_{\nu\lambda}\right)\Gamma^{\lambda}{}_{\mu\sigma} - \Gamma^{\rho}{}_{\nu\lambda}\left(\delta\Gamma^{\lambda}{}_{\mu\sigma}\right) \\
&= \left[\partial_{\mu}\left(\delta\Gamma^{\rho}{}_{\nu\sigma}\right) + \Gamma^{\rho}{}_{\mu\lambda}\left(\delta\Gamma^{\lambda}{}_{\nu\sigma}\right) - \Gamma^{\lambda}{}_{\mu\sigma}\left(\delta\Gamma^{\rho}{}_{\nu\lambda}\right) - \Gamma^{\lambda}{}_{\mu\nu}\left(\delta\Gamma^{\rho}{}_{\lambda\sigma}\right)\right] \\
&\quad - \left[\partial_{\nu}\left(\delta\Gamma^{\rho}{}_{\mu\sigma}\right) + \Gamma^{\rho}{}_{\nu\lambda}\left(\delta\Gamma^{\lambda}{}_{\mu\sigma}\right) - \Gamma^{\lambda}{}_{\nu\sigma}\left(\delta\Gamma^{\rho}{}_{\mu\lambda}\right) - \Gamma^{\lambda}{}_{\nu\mu}\left(\delta\Gamma^{\rho}{}_{\lambda\sigma}\right)\right] \\
&= D_{\mu}\left(\delta\Gamma^{\rho}{}_{\sigma\nu}\right) - D_{\nu}\left(\delta\Gamma^{\rho}{}_{\sigma\mu}\right)
\end{aligned}
$$

But

$$
\begin{aligned}
\delta\Gamma^{\lambda}{}_{\mu\nu} &= \delta\left(g^{\lambda\delta}\Gamma_{\mu\nu\delta}\right) \\
&= \left(\delta g^{\lambda\delta}\right)\Gamma_{\mu\nu\delta} + g^{\lambda\delta}\left(\delta\Gamma_{\mu\nu\delta}\right) \\
&= -g^{\lambda\rho}g^{\sigma\delta}\left(\delta g_{\rho\sigma}\right)\Gamma_{\mu\nu\delta} + g^{\lambda\rho}\left(\delta\Gamma_{\mu\nu\rho}\right) \\
&= -g^{\lambda\rho}\left(\delta g_{\rho\sigma}\right)\Gamma^{\sigma}{}_{\mu\nu} + \frac{1}{2}g^{\lambda\rho}\delta\left(\partial_{\nu}g_{\mu\rho} + \partial_{\mu}g_{\nu\rho} - \partial_{\rho}g_{\mu\nu}\right) \\
&= -g^{\lambda\rho}\left(\delta g_{\rho\sigma}\right)\Gamma^{\sigma}{}_{\mu\nu} + \frac{1}{2}g^{\lambda\rho}\left[\partial_{\nu}\left(\delta g_{\mu\rho}\right) + \partial_{\mu}\left(\delta g_{\nu\rho}\right) - \partial_{\rho}\left(\delta g_{\mu\nu}\right)\right] \\
&= \frac{1}{2}g^{\lambda\rho}\left[\partial_{\nu}\left(\delta g_{\mu\rho}\right) + \partial_{\mu}\left(\delta g_{\nu\rho}\right) - \partial_{\rho}\left(\delta g_{\mu\nu}\right) - 2\Gamma^{\sigma}{}_{\mu\nu}\left(\delta g_{\rho\sigma}\right)\right] \\
&= \frac{1}{2}g^{\lambda\rho}\left\{\partial_{\nu}\left(\delta g_{\mu\rho}\right) - \Gamma^{\alpha}{}_{\mu\nu}\left(\delta g_{\alpha\rho}\right) - \Gamma^{\alpha}{}_{\nu\rho}\left(\delta g_{\alpha\mu}\right)\right\} \\
&\quad + \frac{1}{2}g^{\lambda\rho}\left\{\partial_{\mu}\left(\delta g_{\nu\rho}\right) - \Gamma^{\alpha}{}_{\nu\mu}\left(\delta g_{\alpha\rho}\right) - \Gamma^{\alpha}{}_{\rho\mu}\left(\delta g_{\alpha\nu}\right)\right\} \\
&\quad - \frac{1}{2}g^{\lambda\rho}\left\{\partial_{\rho}\left(\delta g_{\mu\nu}\right) - \Gamma^{\alpha}{}_{\mu\rho}\left(\delta g_{\alpha\nu}\right) - \Gamma^{\alpha}{}_{\nu\rho}\left(\delta g_{\alpha\mu}\right)\right\} \\
&= \frac{1}{2}g^{\lambda\rho}\left[D_{\nu}\left(\delta g_{\mu\rho}\right) + D_{\mu}\left(\delta g_{\nu\rho}\right) - D_{\rho}\left(\delta g_{\mu\nu}\right)\right]
\end{aligned}
$$

And

$$
\begin{aligned}
g^{\sigma\nu}\delta R_{\sigma\nu} &= g^{\sigma\nu}D_\rho\left(\delta\Gamma^\rho_{\sigma\nu}\right) - g^{\sigma\nu}D_\nu\left(\delta\Gamma^\rho_{\sigma\rho}\right) \\
&= D_\rho\left(g^{\sigma\nu}\delta\Gamma^\rho_{\sigma\nu}\right) - D_\rho\left(g^{\sigma\rho}\delta\Gamma^\mu_{\sigma\mu}\right) \\
&= D_\rho\left(g^{\sigma\nu}\delta\Gamma^\rho_{\sigma\nu} - g^{\sigma\rho}\delta\Gamma^\mu_{\sigma\mu}\right)
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\delta R &= R_{\mu\nu}\delta g^{\mu\nu} + g^{\mu\nu}\delta R_{\mu\nu} \\
&= R_{\mu\nu}\delta g^{\mu\nu} + D_\beta\left(g^{\mu\nu}\delta\Gamma^\beta_{\nu\mu} - g^{\mu\beta}\delta\Gamma^\nu_{\nu\mu}\right)
\end{aligned}
$$

For an arbitrary vector , $\mathcal{A}^\rho$

$$
\mathcal{A}^\rho = g^{\mu\nu}\delta\Gamma^\rho_{\nu\mu} - g^{\mu\rho}\delta\Gamma^\nu_{\nu\mu}
$$

Multiplication by $\sqrt{-g}$ yields a total derivative. So

$$
\sqrt{-g}\,D_\rho\mathcal{A}^\rho = D_\rho\left(\sqrt{-g}\,\mathcal{A}^\rho\right) = \partial_\rho\left(\sqrt{-g}\,\mathcal{A}^\rho\right)
$$

By invoking Stokes' theorem, we are left with a boundary term which does not vanish in general. However, we may require our variation of the metric to vanish in a region of interest

$$
\delta R = R_{\mu\nu}\delta g^{\mu\nu}
$$

Variation of the metric determinant

To determine the variation of the second term on the LHS , we note that the determinant of the metric tensor is given by

$$
\begin{aligned}
g &= \det\left(g_{\rho\sigma}\right) \\
&= -\varepsilon^{\alpha\beta\gamma\delta} g_{\alpha 0} g_{\beta 1} g_{\gamma 2} g_{\delta 3} \\
&= \frac{1}{4!}\varepsilon^{\alpha\beta\gamma\delta}\varepsilon^{\alpha\beta\gamma\delta} g_{\alpha 0} g_{\beta 1} g_{\gamma 2} g_{\delta 3}
\end{aligned}
$$

But $\varepsilon^{\alpha\beta\gamma\delta}$ is a covariant antisymmetric tensor with weight -1

$$
\begin{aligned}
\varepsilon_{\kappa\lambda\mu\nu} &= \varepsilon^{\alpha\beta\gamma\delta} g_{\alpha\kappa} g_{\beta\lambda} g_{\gamma\mu} g_{\delta\nu} \\
&= \varepsilon^{\kappa\lambda\mu\nu} \frac{\partial x'^{\alpha}}{\partial x^{\kappa}} \frac{\partial x'^{\beta}}{\partial x^{\lambda}} \frac{\partial x'^{\gamma}}{\partial x^{\mu}} \frac{\partial x'^{\delta}}{\partial x^{\nu}} \\
&= \varepsilon^{\kappa\lambda\mu\nu} \left|\frac{\partial x'}{\partial x}\right| \\
&= -g\varepsilon^{\kappa\lambda\mu\nu}
\end{aligned}
$$

It can be shown that the variation of a metric determinant is given by

$$
\delta g = g\, g^{\mu\nu}\, \delta g_{\mu\nu}
$$

So

$$
\begin{aligned}
\delta\sqrt{-g} &= -\frac{1}{2\sqrt{-g}}\delta g \\
&= -\frac{1}{2\sqrt{-g}}\left(g\, g^{\mu\nu}\, \delta g_{\mu\nu}\right) \\
&= \frac{1}{2}\sqrt{-g}\left(g^{\mu\nu}\, \delta g_{\mu\nu}\right)
\end{aligned}
$$

But

$$g_{\mu\nu}\,\delta g^{\mu\nu} \;=\; -g^{\mu\nu}\,\delta g_{\mu\nu}$$

Therefore

$$\delta\sqrt{-g} \;=\; -\frac{1}{2}\sqrt{-g}\,(g_{\mu\nu}\,\delta g^{\mu\nu})$$

In conclusion,

$$\frac{1}{\sqrt{-g}}\delta_{g_{\mu\nu}}\sqrt{-g} \;=\; -\frac{1}{2}g_{\mu\nu}\delta g^{\mu\nu} \tag{96}$$

Collecting Variation Results

Collecting results:

$$\delta R \;=\; R_{\mu\nu}\delta g^{\mu\nu} + D_\beta\left(g^{\mu\nu}\delta\Gamma^\beta_{\nu\mu} - g^{\mu\beta}\delta\Gamma^\nu_{\nu\mu}\right) \tag{97}$$

$$\delta_{g_{\mu\nu}}\sqrt{-g} \;=\; -\frac{1}{2}\sqrt{-g}\,g_{\mu\nu} \tag{98}$$

Returning to the variation,

$$
\begin{aligned}
0 \;=\; & \delta S \\
=\; & \int\left[\frac{1}{2\kappa}\left\{\frac{\delta R}{\delta g^{\mu\nu}} + \frac{R}{\sqrt{-g}}\frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}}\right\} + \frac{1}{\sqrt{-g}}\frac{\delta\left(\sqrt{-g}\mathcal{L}_M\right)}{\delta g^{\mu\nu}}\right]\delta g^{\mu\nu}\sqrt{-g}\,\mathrm{d}^4x \\
=\; & \int\left[\frac{1}{2\kappa}\left\{R_{\mu\nu} + \frac{R}{\sqrt{-g}}\left(-\frac{1}{2}\sqrt{-g}\,g_{\mu\nu}\right)\right\} + \frac{1}{\sqrt{-g}}\frac{\delta\left(\sqrt{-g}\mathcal{L}_M\right)}{\delta g^{\mu\nu}}\right]\delta g^{\mu\nu}\sqrt{-g}\,\mathrm{d}^4x \\
& + \int\left[\frac{1}{2\kappa}D_\beta\left(g^{\mu\nu}\delta\Gamma^\beta_{\nu\mu} - g^{\mu\beta}\delta\Gamma^\nu_{\nu\mu}\right)\right]\sqrt{-g}\,\mathrm{d}^4x
\end{aligned}
$$

Integrate the final term by parts,

$$
\begin{aligned}
Final\ term \ &= \ \frac{1}{2\kappa} \int D_\beta \left(g^{\mu\nu}\delta\Gamma^\beta_{\ \nu\mu} - g^{\mu\beta}\delta\Gamma^\nu_{\ \nu\mu} \right) \sqrt{-g}\,\mathrm{d}^4x \\
&= \ \frac{1}{2\kappa} \int_{\delta V} \left[\left(g^{\mu\nu}\delta\Gamma^\beta_{\ \nu\mu} - g^{\mu\beta}\delta\Gamma^\nu_{\ \nu\mu} \right) \sqrt{-g} \right]\,\mathrm{d}^4x \\
&\quad - \frac{1}{2\kappa} \int_V \left(g^{\mu\nu}\delta\Gamma^\beta_{\ \nu\mu} - g^{\mu\beta}\delta\Gamma^\nu_{\ \nu\mu} \right) \cancel{D_\beta\sqrt{-g}}\mathrm{d}^4x \\
&= \ 0
\end{aligned}
$$

where the surface term vanishes because $\delta g^{\mu\nu}$ vanishes on the boundary and $\delta\Gamma^\beta_{\ \mu\nu} = 0$ whenever $\delta g^{\mu\nu} = 0$.

Since the above relationship holds for any variation $\delta g^{\mu\nu}$, we have

$$
\frac{1}{\kappa}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R \right) \ = \ \frac{-2}{\sqrt{-g}}\frac{\delta\left(\sqrt{-g}\mathcal{L}_M\right)}{\delta g^{\mu\nu}} \tag{99}
$$

The stress-energy tensor $T_{\mu\nu}$ is defined as the metric variation of the matter action

$$
\begin{aligned}
T_{\mu\nu} \ &:= \ \frac{-2}{\sqrt{-g}}\frac{\delta\left(\sqrt{-g}\mathcal{L}_M\right)}{\delta g^{\mu\nu}} \\
&= \ \frac{-2}{\sqrt{-g}}\frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}}\mathcal{L}_M - 2\frac{\delta\mathcal{L}_M}{\delta g^{\mu\nu}}
\end{aligned}
$$

Substituting this result yields the Einstein field equations

$$
R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R \ = \ \kappa T_{\mu\nu} \tag{100}
$$

where

$$
\kappa \ = \ \frac{8\pi G}{c^4}
$$

In this scheme, Newton's force of gravity is the Riemann curvature of the 4-dimensional spacetime manifold.

Energy Momentum Tensor

We assume that the fields vanish at the boundary of a given volume , or at infinity, the field equations are given by

$$
\begin{aligned}
0 &= \int \delta_{A_\alpha} \mathcal{L}_M \\
&= -\int \frac{1}{2} F^{\mu\nu} \delta_{A_\alpha} F_{\mu\nu} \\
&= -\int \frac{1}{2} F^{\mu\nu} \delta_{A_\alpha} \left(\partial_\mu A_\nu - \partial_\nu A_\mu \right) \\
&= -\int F^{\mu\nu} \partial_\mu \delta A_\nu
\end{aligned}
$$

where of course $\delta_A A_\nu = \delta A_\nu$. Integrating by parts

$$
\begin{aligned}
0 &= -\int \partial_\mu \left(F^{\mu\nu} \delta A_\nu \right) + \int \left(\partial_\mu F^{\mu\nu} \right) \delta A_\nu \\
0 &= -F^{\mu\nu} \delta A_\nu |_{boundary} + \int \left(\partial_\mu F^{\mu\nu} \right) \delta A_\nu
\end{aligned}
$$

The first term vanishes because we *choose* the variation to be zero on the boundary. From the last term we can conclude that $\partial_\mu F^{\mu\nu} = 0$ because throughout the volume

of integration δA_ν is arbitrary. The stress-energy tensor is given by

$$
\begin{aligned}
T_{\mu\nu} &= -2\delta_{g_{\mu\nu}}\mathcal{L}_M + g_{\mu\nu}\mathcal{L}_M \\
&= -2\delta_{g_{\mu\nu}}\left(\frac{1}{4}g^{\mu\nu}g^{\alpha\beta}F_{\mu\alpha}F_{\nu\beta}\right) + g_{\mu\nu}\left(\frac{1}{4}g^{\gamma\delta}g^{\alpha\beta}F_{\gamma\alpha}F_{\delta\beta}\right) \\
&= -\frac{1}{2}F_{\mu\alpha}F_{\nu\beta}\delta_{g_{\mu\nu}}\left(g^{\mu\nu}g^{\alpha\beta}\right) + \frac{1}{4}g_{\mu\nu}F^{\delta\beta}F_{\delta\beta} \\
&= -\frac{1}{2}F_{\mu\alpha}F_{\nu\beta}\delta_{g_{\mu\nu}}\left(g^{\mu\nu}g^{\alpha\beta}\right) + \frac{1}{4}g_{\mu\nu}F^{\alpha\beta}F_{\alpha\beta} \\
&= -F_{\mu\alpha}F_{\nu}{}^{\alpha} + \frac{1}{4}g_{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}
\end{aligned}
$$

The stress-energy tensor is trace-free:

$$
\begin{aligned}
T^{\mu}{}_{\mu} &= -F^{\mu\alpha}F_{\mu\alpha} + \frac{1}{4}g^{\mu\nu}g_{\mu\nu}F^{\alpha\beta}F_{\alpha\beta} \\
&= -F^{\mu\alpha}F_{\mu\alpha} + \frac{1}{4}(4)F^{\alpha\beta}F_{\alpha\beta} \\
&= 0
\end{aligned}
$$

Conservation implies that the covariant derivative of energy-momentum tensor vanishes

$$
D_\mu\left(\sqrt{-g}\,T^{\mu}{}_{\nu}\right) = \partial_\mu\left(\sqrt{-g}\,T^{\mu}{}_{\nu}\right) - \Gamma^{\rho}{}_{\nu\mu}\left(\sqrt{-g}\,T^{\mu}{}_{\rho}\right) = 0
$$

Consequently, in a gravitational field, there exists no conservation law corresponding to global internal symmetries

$$
\begin{aligned}
0 &= D_\nu T^{\mu\nu} \\
&= 2\, D_\nu F^{\mu\alpha}\, g_{\alpha\beta}\, F^{\nu\beta} + 2\, F^{\mu\alpha}\, g_{\alpha\beta}\, D_\nu F^{\nu\beta} - g^{\mu\nu}\, D_\nu F_{\sigma\rho}\, F^{\sigma\rho} \\
&= 2\, (D_\sigma F^{\mu\alpha}\, g_{\alpha\rho}\, F^{\sigma\rho} + D_\rho F^{\mu\alpha}\, g_{\alpha\sigma}\, F^{\rho\sigma}) \\
&\quad + 2\, F^{\mu\alpha}\, g_{\alpha\beta}\, D_\nu F^{\nu\beta} - g^{\mu\nu}\, D_\nu F_{\sigma\rho}\, F^{\sigma\rho} \\
&= 2\, g^{\mu\nu}\, (D_\sigma F_{\nu\rho} + D_\rho F_{\sigma\nu})\, F^{\rho\sigma} + 2\, F^{\mu\alpha}\, g_{\alpha\beta}\, D_\nu F^{\nu\beta} - g^{\mu\nu}\, D_\nu F_{\sigma\rho}\, F^{\sigma\rho} \\
&= 2\, g^{\mu\nu}\, (D_\sigma F_{\nu\rho} + D_\rho F_{\sigma\nu} + D_\nu F_{\rho\sigma})\, F^{\rho\sigma} + 2\, F^{\mu\alpha}\, g_{\alpha\beta}\, D_\nu F^{\nu\beta} \\
&= 2\, F^\mu{}_\beta\, D_\nu F^{\nu\beta}
\end{aligned}
$$

where β runs from 0 to 4. Consequently

$$
F^\mu{}_\beta\, D_\nu F^{\nu\beta} = 0 \tag{101}
$$

Assuming a non-vanishing determinant for coefficients in equation [101], the equation of motion for the Maxwell field is

$$
D_\nu F^{\nu\beta} = 0 \tag{102}
$$

Equation [102] is a direct consequences of Einstein equation $G_{\mu\nu} = 0$.

VII. MATTER IN GRAVITY THEORIES

In general relativity the coupling of matter sources to gravity is accomplished by making the matter action invariant under general coordinate transformations, then adding it to the Einstein-Hilbert action. In formulations of general relativity based on the conformal group, there may be additional conditions. Here we examine $SU(N)$ gauge theories as sources for gravity in a large class of spaces of doubled dimension.

Doubled dimension of spacetime arises in various contexts, frequently related to the idea of a relativistic phase space. Born reciprocity (M. Born, [11, 12]) was one early suggestion aimed at unifying relativity and quantum theory. The reciprocity involves the scaled symplectic exchange $x^\alpha \to ap^\alpha, p^\beta \to -bx^\beta$, thereby preserving Hamilton's equations. Further developments include the study of Kähler manifolds, with mutually compatible metric, symplectic, and complex structures such that any two of the structures yield the third.

In 1982, using a gauge theory approach to gravity, Ivanov and Niederle [13, 14] showed that general relativity can arise in a space of doubled dimension called bi-conformal space. Generalizing the 8-dimensional quotient of the conformal group of spacetime by its homogeneous Weyl subgroup led the authors to a class of curved geometries. From an action quadratic in the curvatures they found that suitable constraints reduced the field equations to the Einstein equation in 4-dimensions.

The group quotient approach used in [13, 14] generalizes to arbitrary dimension n and signature (p, q). The quotient of conformal group of an n dimensional space by its Weyl subgroup yields a $2n$ dimensional biconformal manifold. In [15] it was shown that biconformal spaces of any dimension $2n$ admit an action *linear* in the curvature with field equations reducing to the vacuum Einstein equation in n dimensions. Unlike the other double field theories cited below, this reduction in both field count and the number of independent variables occurs *by virtue of the field equations*. Starting from

the most general action linear in the curvatures–which takes essentially the same form in any dimension–it has now been shown that the field equations generically lead to scale invariant general relativity on the co-tangent bundle [15, 6]. As in Riemannian geometry, these biconformal spaces are taken to be torsion free.

Because biconformal spaces have natural symplectic structure [16] they give an arena appropriate to quantum problems [17], and when they are required to fully reproduce the properties of phase space, the $(3, 1)$ signature of spacetime emerges necessarliy from an originally Euclidean space [18, 16, 19]. It is further shown in [6, 16, 19, 20] that these spaces are generically Kähler, and that they share the properties of double field theories.

Some years after the first biconformal spaces, another form of doubled dimension called double field theories arose as a means of making the $O(d, d)$ symmetry of T-duality manifest. By introducing scalars to produce an additional d dimensions, Duff [21]] doubled the X(σ,τ) string variables to make this $O(d, d)$ symmetry manifest. Siegel brought the idea to full fruition by deriving results from superstring theory [22, 23, 24]. Allowing fields to depend on all $2d$ coordinates, Siegel introduced generalized Lie brackets, gauge transformations, covariant derivatives, and a section condition on the full doubled space, thereby introducing torsions and curvatures in addition to the manifest T-duality. By restricting half the coordinates–called imposing a *section condition*–one recovers the d-dimensional theory.

In any of these doubled dimension gravity theories it is desirable to understand matter couplings, and preferable to introduce the matter fields *ab initio* in the doubled space. Carrying out such an investigation in biconformal spaces simultaneously gives results applicable to other doubled spaces. Because biconformal gravity is a gauge theory, it allows direct extension of the symmetry to include gravitational sources from $SU(N)$ gauge theories. However, such Yang-Mills type sources must be written

in the space of doubled dimension. This means the introduction of far more potential source fields, $\frac{2n(2n-1)}{2} \times (N^2 - 1)$ instead of only $\frac{n(n-1)}{2} \times (N^2 - 1)$, each depending on $2n$ independent variables instead of only n. There are two principal questions we address here. First is the question of the correct form of the Yang-Mills action in the doubled dimension. We find that the usual $\int tr \left(\boldsymbol{\mathcal{F}} \, ^* \boldsymbol{\mathcal{F}} \right)$ form must be augmented with a "twist" to reproduce familiar results, similar to the twist found in studies of double field theories [25, 26, 27, 28]. The second question is whether the increase in fields and independent variables spoils the gravitational reduction, or at the other extreme, shares the reduction to n dimensions with gravity. We find that both gravity and sources reduce to n-dimensions. The reduction again occurs by virtue of the field equations, with no need for section conditions.

A further result arises from this investigation. With the exceptions of a study of biconformal supersymmetry [29], which necessarily includes matter fields, and of a brief study [30] with scalar field sources, previous solutions and further properties of biconformal gravity [14, 15, 6, 16, 19, 31, 32] have been based on pure gravity solutions. The field equations for pure biconformal gravity arise by variation of the gauge fields, and it has not been necessary to introduce a metric. However, the actions for Yang-Mills theories involve the Hodge dual and therefore a metric. But biconformal spaces possess both conformal and Kähler structures, it is not clear which of these should take precedence in defining the orthonormality of the basis gauge fields. The ambiguity in identifying the metric is confounded with the specification of the biconformal matter action. The outcome of the current investigation is that only the Killing form reproduces the expected coupling to general relativity.

Concretely, biconformal spaces are spanned by two sets of frame fields, $(\mathbf{e}^a, \mathbf{f}_b)$, called the solder form and the co-solder form. In solutions, the solder form, $\mathbf{e}^a$, reduces via the field equations to the usual solder form on spacetime. In order to write any

Yang-Mills action we are compelled to place an orthonormality condition on these forms,

$$
\begin{aligned}
\langle \mathbf{e}^a, \mathbf{e}^b \rangle &= M^{ab} \\
\langle \mathbf{e}^a, \mathbf{f}_b \rangle &= M^a{}_b \\
\langle \mathbf{f}_a, \mathbf{f}_b \rangle &= M_{ab}
\end{aligned}
$$

where some invariant matrix,

$$
M^{AB} = \begin{pmatrix} M^{ab} & M^a{}_b \\ M_a{}^b & M_{ab} \end{pmatrix}
$$

must be specified. A central hurdle in the course of our study was that there are two natural candidates for M^{AB}: the Killing form of the conformal group restricted to the base manifold, and the Kähler metric of the Kähler structure. We carry out the search for a suitable Yang-Mills action for each candidate symmetric form. Ultimately, we find that only the Killing form can give the expected coupling to gravitation. It is therefore the Killing form that provides the orthonormality of the solder and co-solder forms throughout the remainder of the paper.

Before proceeding with our investigation of $SU(N)$ sources in biconformal, double-field-theory, or Kahler gravity, we look briefly at sources in general relativity with Yang-Mills sources as a gauge theory. This displays our general approach to gauging, as well as the reduced result we hope to achieve. In general relativity, we must extend the symmetry of source actions to general coordinate from global Lorentz. By analogy, we expect that including matter in conformally based theories may require some additional conditions.

Yang-Mills matter in general relativity

Defining a projection on the quotient of the $\frac{n(n+1)}{2}$-dimensional Poincarè group $\mathcal{P}$ by its $\frac{n(n-1)}{2}$-dimensional Lorentz subgroup $\mathcal{L}$ gives a principal fiber bundle with Lorentz fibers over an n-dimensional Minkowski spacetime. Generalizing the base space by changing the Maurer-Cartan connection forms of the Poincarè group and perhaps changing the manifold, the fiber structure is maintained by demanding horizontality of the curvature and torsion. The result is a Riemann-Cartan geometry characterized by curvature and torsion with local Lorentz symmetry.

Concretely, the generalization of the connection $\left(\tilde{\mathbf{e}}^b, \tilde{\boldsymbol{\omega}}^a{}_b \right) \Rightarrow \left(\mathbf{e}^b, \boldsymbol{\omega}^a{}_b \right)$ takes the Maurer-Cartan equations of the Poincarè group,

$$
\begin{aligned}
\mathbf{d}\tilde{\boldsymbol{\omega}}^a{}_b &= \tilde{\boldsymbol{\omega}}^c{}_b \wedge \tilde{\boldsymbol{\omega}}^a{}_c \\
\mathbf{d}\tilde{\mathbf{e}}^a &= \tilde{\mathbf{e}}^b \wedge \tilde{\boldsymbol{\omega}}^a{}_b
\end{aligned}
$$

to the Cartan equations,

$$
\mathbf{d}\boldsymbol{\omega}^a{}_b = \boldsymbol{\omega}^c{}_b \wedge \boldsymbol{\omega}^a{}_c + \mathbf{R}^a{}_b \tag{103}
$$

$$
\mathbf{d}\mathbf{e}^a = \mathbf{e}^b \wedge \boldsymbol{\omega}^a{}_b + \mathbf{T}^a \tag{104}
$$

where horizontality of the curvature $\mathbf{R}^{ab}$ and the torsion $\mathbf{T}^a$ is captured by omitting any occurrence of the spin connection when writing them expressly as 2-forms

$$
\begin{aligned}
\mathbf{R}^a{}_b &= \frac{1}{2} R^a{}_{bcd} \mathbf{e}^c \wedge \mathbf{e}^d \\
\mathbf{T}^a &= \frac{1}{2} T^a{}_{bc} \mathbf{e}^b \wedge \mathbf{e}^c
\end{aligned}
$$

Horizontality insures the survival of the principal fiber bundle by guaranteeing that

integrals of the curvatures over an area, or equivalenty of the connection forms over closed curves, are independent of lifting.

Completing the description of the Riemann-Cartan geometry is the demand for integrability of the Cartan equations, which follows by exterior differentiation of Eqs.(103) and (104):

$$\mathbf{DR}^a{}_b = 0$$
$$\mathbf{DT}^a = \mathbf{e}^b \wedge \mathbf{R}^a{}_b$$

When torsion vanishes, this construction describes a general n-dimensional Riemannian spacetime with local Lorentz symmetry.

To include an additional $SU(N)$ Yang-Mills symmetry in the fiber bundle we extend the $\mathcal{P}/\mathcal{L}$ quotient to the quotient of the product $\mathcal{P} \times SU(N)$ by the product $\mathcal{L} \times SU(N)$:

$$\mathcal{P} \times SU(N)/\mathcal{L} \times SU(N)$$

and carry out the same procedure. This still results in an n-dimensional spacetime but now the fibers of the principal bundle are isomorphic to $\mathcal{L} \times SU(N)$. The Cartan Eqs.(103) and (104) are augmented by a third equation,

$$\mathbf{dA}^i = -\frac{1}{2}c^i{}_{jk}\mathbf{A}^j \wedge \mathbf{A}^k + \mathbf{F}^i$$

where indices beginning with i have range $i, j, k, \ldots = 1, 2, \ldots, N^2 - 1$, and $\mathbf{F}^i$ is horizontal

$$\mathbf{F}^i = \frac{1}{2}F^i{}_{ab}\mathbf{e}^a \wedge \mathbf{e}^b$$

The integrability condition is $\boldsymbol{\mathcal{D}}\mathbf{F}^i = 0$, where $\boldsymbol{\mathcal{D}}$ is the $SU(N)$ covariant derivative.

Here, indices from the first part of the alphabet have range $a, b, \ldots = 1, \ldots, n$.

To build a physical theory, we write an action functional using any of the tensor fields arising from the construction, $\mathbf{R}^a{}_b, \mathbf{T}^a, \mathbf{F}^i, \mathbf{e}^a, \eta_{ab}, e_{ab\cdots c}$, together with any representations of the original group.

In any dimension of spacetime, the action coupling the $SU(N)$ Yang-Mills field to general relativity is written as

$$
\begin{aligned}
S &= S_{GR} + S_{\text{YM}} \\
&= \int \mathbf{R}^{ab} \wedge \mathbf{e}^c \wedge \ldots \wedge \mathbf{e}^d e_{abc\ldots d} - \frac{\kappa}{2} \int \mathbf{F}^i \wedge {}^* \mathbf{F}_i
\end{aligned}
$$

where ${}^*\mathbf{F}_i$ is the Hodge dual of the 2-form $\mathbf{F}_i$. We vary the action with respect to the solder form, $\mathbf{e}^a$, the spin connection, $\boldsymbol{\omega}^a{}_b$, and the Yang-Mills connection $\mathbf{A}^i$. Making the usual assumptions for the gravity theory to reduce to general relativity, this results in

$$
\begin{aligned}
R_{ab} - \frac{1}{2}\eta_{ab}R &= \kappa\left(\eta^{cd}F^i{}_{ac}F_{i\,bd} - \frac{1}{4}\eta_{ab}F^{i\,cd}F_{i\,cd}\right) \\
\tilde{D}^c F^i{}_{ac} &= 0
\end{aligned}
\tag{105}
$$

where $\tilde{D}^c$ is covariant with respect to both local Lorentz and local $SU(N)$ transformations. These methods generalize immediately to additional internal symmetries, such as the $SU(3) \times SU(2) \times U(1)$ of the standard model.

Our main result is to show that Eqs.(105) in n-dimensions follow from the field equations of biconformal gravity coupled to a twisted Yang-Mills matter action, formulated in $2n$-dimensions.

Sources in doubled dimension

We now turn to biconformal gravity [14, 32, 15, 6], double field theory [21, 22, 23, 24, 42, 44], or gravity on a Kähler manifold [16, 6]. Each of these cases starts as a fully $2n$-dimensional theory but ultimately describes gravity on an n-dimensional submanifold. It is desirable to have a fully $2n$-dimensional form of the matter action which nonetheless also reduces to the expected n-dimensional source as a consequence of the field equations. It is this condition we address. We discuss the issue in biconformal space, since biconformal gravity is already a gauge theory and it generically includes the structures of both double field theory and Kähler manifolds [16, 6].

For matter fields we restrict our attention to Yang-Mills type sources. We find that although the usual form of $2n$-dimensional Yang-Mills action gives nonstandard coupling to gravity, including a "twist" matrix in the action corrects the problem.

Biconformal gravity arises as follows. The quotient of the conformal group $\mathcal{C}_{p,q} = SO(p+1, q+1)$ of an $SO(p,q)$-symmetric space ($p + q = n$, metric η_{ab}) by its homogeneous Weyl subgroup, $\mathcal{W}_{p,q} \equiv SO(p,q) \times SO(1,1)$, leads to a $2n$-dimensional homogeneous space with local $\mathcal{W}_{p,q}$ symmetry. This homogeneous space, discussed in [14] and [32] and studied extensively in [6, 16, 19], is found to have compatible symplectic, metric and complex structures, making it Kähler [16]. In addition, the restriction of the Killing form to the base manifold is nondegenerate and scale invariant, and the volume form of the base manifold is scale invariant. The homogeneous space and its curved generalizations are called biconformal spaces.

Ivanov and Niederle [14], wrote a gravity theory on an 8-dimensional biconformal space, using the curvature-quadratic action of Weyl gravity. By a suitable restriction of the coordinate transformations of the extra 4-dimensions, they showed that 4-dimensional general relativity describes the remaining subspace. Subsequently, Wehner and Wheeler [15] introduced a class of $\mathcal{W}$-invariant actions *linear* in the cur-

vatures, defining biconformal gravity. Curvature-linear actions are possible because the $2n$-dimensional volume element is scale invariant. Unlike the 4-dimensional theories above with actions quadratic in the curvature, the linear action functionals take the same form in any dimension. The doubled dimension is understood in terms of the symplectic structure, leading to a phase space interpretation for generic solutions. Lagrangian submanifolds represent the physical spacetime and have the original dimension. The class of torsion-free biconformal spaces has been shown to reduce to general relativity on the cotangent bundle of spacetime [6]. These reductions of the model work for any signature (p, q).

The most general action linear in the biconformal curvatures is given by

$$S = \lambda \int e_{ac\cdots d}{}^{be\cdots f} \left(\alpha \boldsymbol{\Omega}^a{}_b + \beta \delta^a_b \boldsymbol{\Omega} + \gamma \mathbf{e}^a \wedge \mathbf{f}_b \right) \wedge \mathbf{e}^c \wedge \cdots \wedge \mathbf{e}^d \wedge \mathbf{f}_e \wedge \cdots \wedge \mathbf{f}_f \quad (106)$$

where $\boldsymbol{\Omega}^a{}_b$ is the curvature of the spin connection and $\boldsymbol{\Omega}$ is the dilatational curvature. Here $\lambda = \frac{(-1)^n}{(n-1)!(n-1)!}$ is a convenient constant, chosen to eliminate a combinatoric factor and to make our sign conventions agree with [6]. The cotangent bundle is spanned by the pair, $(\mathbf{e}^a, \mathbf{f}_b)$, called the solder form and the co-solder form, respectively. The variation is taken with respect to all $\frac{(n+1)(n+2)}{2}$ gauge fields.

The reduction of a fully $2n$-dimensional gravity theory to dependence only on the fields of n-dimensional gravity is a remarkable feature of biconformal gravity. While it has been shown to be a double field theory [21, 22, 23, 24, 43, 44, 6], double field theories require the assumption that fields depend on only half the coordinates. This artificial constraint is called a section condition. In sharp contrast, biconformal solutions do *not* require a section condition, *reducing as a consequence of the field equations* of torsion-free biconformal spaces. Thus, using the torsion-free field equations, the components of the $\frac{(n+1)(n+2)}{2}$ curvatures–initially dependent on $2n$ independent coordinates–reduce to the usual Riemannian curvature tensor in n

dimensions. Correspondingly, the n-dim solder form determines all fields, up to coordinate and gauge transformations. Generic, torsion-free, vacuum solutions describe n-dimensional scale-covariant general relativity on the co-tangent bundle.

As noted in the introduction, with the exceptions biconformal supergravity [29] and a scalar field example [41], studies of biconformal spaces [32, 15, 6, 18, 16, 19, 17, 31, 20] have considered pure gravity biconformal spaces, leading to vacuum general relativity.

Here we consider $SU(N)$ Yang-Mills fields as gravitational sources. The central issue is to show that even with a completely general $SU(N)$ gauge theory over a $2n$-dimensional biconformal space, a full $2n \rightarrow n$ reduction occurs, both for gravity and the Yang-Mills field.

As with the Riemann-Cartan construction of general relativity above, the development of biconformal spaces from group symmetry makes it straightforward to include the additional symmetry of sources. By extending the quotient to

$$\mathcal{M}^{2n} = \mathcal{C}_{p,q} \times SU(N) / \mathcal{W}_{p,q} \times SU(N)$$

the local symmetry is enlarged by $SU(N)$ and we may add a Yang-Mills or similar action to Eq.(106). This construction gives the form of the Yang-Mills field in terms of the potentials, but not the form of the action.

There are then two basic parts to our investigation.

First, we must determine a suitable $2n$-dimensional action functional for the sources. To accomplish this requires two interdependent specifications:

- Find a form of the Yang-Mills action which gives the usual n-dimensional Yang-Mills source to the Einstein tensor.

- Fix the orthornormality condition for the solder and co-solder form, $M^{AB} =$

$\left\langle \tilde{\mathbf{e}}^A, \tilde{\mathbf{e}}^B \right\rangle$ where $\tilde{\mathbf{e}}^A = (\mathbf{e}^a, \mathbf{f}_b)$. This determines for form of the Hodge dual and ultimately the metric variation of the matter action.

In accomplishing these steps we show that the standard Yang-Mills action

$$S^0_{YM} = \int tr \left(\mathbf{F} \wedge^* \mathbf{F} \right)$$

cannot give the right couplings, and find a satisfactory modification

$$S_{YM} = \int tr \left(\bar{\mathbf{F}} \wedge^* \mathbf{F} \right)$$

where the twisted field, $\bar{\mathbf{F}}$ is defined below. The twisted action, together with the restricted Killing form as orthonormal metric, give the desired reduction.

The second challenge is then to use the field equations to show:

- The number of field components in $2n$ dimensions reduces to the expected number on n dimensional spacetime.

- The functional dependence of the fields reduces from $2n$ to n independent variables.

- The gravitational source is the usual Yang-Mills stress-energy tensor.

- The $SU(N)$ field equation is the usual n-dim Yang-Mills field equation.

The remainder of our presentation proceeds as follows. In the next Section, we introduce our notation and other conventions. In Section 3, we show that the usual form of Yang-Mills action, S^0_{YM}, cannot produce the usual coupling to gravity. The twisted form $\bar{\mathbf{F}}$ is developed in Section 4, and the variation of S_{YM} carried out. The Yang-Mills potentials are identified and varied in Section 5. The next Section contains the reduction of the gravitational field equations, as far as possible with the presence of

sources. This reduction closely follows reference [6]. We find that the field equations imply certain restrictions that must be applied to the matter sources.

The reduction of the number of fields and the number of independent variables is shown in Section 7 to follow from the full gravitational equations. We find that the reduction of fields that is necessary in the purely gravitational sector also forces reduction of the source fields. The Section concludes with the emergence of both the usual Yang-Mills gravitational source in n-dimensions, and the usual n-dimensional Yang-Mills equation. The final Section includes a brief review of the main results.

VIII. NOTATION AND CONVENTIONS

Conventions with biconformal tensors

Differential forms

The co-tangent space of biconformal manifolds are spanned by two sets of opposite conformal weight orthonormal frame fields, $\tilde{\mathbf{e}}^A = (\mathbf{e}^a, \mathbf{f}_b)$, with lowercase Latin indices $a, b, \ldots = 1, 2, \ldots, n$ indicating the use of these frames and upper case Latin $A, B, \ldots = 1, 2, \ldots, 2n$ to denote the pair. Coordinate indices are lower case Greek, $\mu, \nu, \ldots = 1, 2, \ldots, n$ so that we have, for example,

$$\mathbf{e}^a = e_\mu{}^a \mathbf{d}x^\mu + e^{\mu a} \mathbf{d}y_\mu$$

A general 2-form may be written in the orthonormal basis as

$$\mathcal{F} \;=\; \frac{1}{2} F_{ab} \mathbf{e}^a \wedge \mathbf{e}^b + F^a{}_b \mathbf{f}_a \wedge \mathbf{e}^b + \frac{1}{2} F^{ab} \mathbf{f}_a \wedge \mathbf{f}_b$$

It is important to realize that $F_{ab}, F^a{}_b$ and F^{ab} are distinct fields. Therefore, we cannot raise and lower indices in the usual way unless we choose different names for the separate independent components. As compensation for this, the raised or lowered position of an index reflects its conformal weight. Thus, F_{ab} has weight -2 while F^{ab} has weight $+2$. When practical these distinct fields will be given different names,

$$\mathcal{F} \;=\; \frac{1}{2} F_{ab} \mathbf{e}^a \wedge \mathbf{e}^b + G^a{}_b \mathbf{f}_a \wedge \mathbf{e}^b + H^{ab} \mathbf{f}_a \wedge \mathbf{f}_b$$

but this can lead to an unnecessary profusion of field names.

When we need to explicitly refer to internal $SU(N)$ indices, the field and its components will be given an additional index from the lower case Latin set $\{i, j, k\}$. Other lower case Latin indices refer to the null orthonormal frame field, $\left(\mathbf{e}^b, \mathbf{f}_a\right)$. Thus $F^i{}_{ab}$ represents the components of $\frac{1}{2}F^i{}_{ab}G_i\mathbf{e}^a \wedge \mathbf{e}^b$ where G_i is a generator of $SU(N)$ as i runs from 1 to $N^2 - 1$. In most cases the internal index is suppressed.

Differential forms are written in boldface and always multiplied with the wedge product. For brevity in some longer expressions we omit the explicit wedge between forms. Thus, for example,

$$\mathbf{f}_a \wedge \mathbf{f}_b \wedge \mathbf{f}_c \wedge \mathbf{e}^d \wedge \mathbf{e}^e \iff \mathbf{f}_{abc}\mathbf{e}^{de}$$

The bold font shows that these are differential forms, and therefore are to be wedged together.

As a compromise between keeping track of conformal weights, while being able to assess the symmetry or antisymmetry of components, we introduce a weight $+1$ basis $\mathbf{e}^A \equiv \left(\mathbf{e}^a, \eta^{ab}\mathbf{f}_b\right)$, where η_{ab} is the n-dimensional metric of the original $SO(p, q)$-symmetric space, *not* the metric of the biconformal space. Thus, we may write

$$
\begin{aligned}
\boldsymbol{\mathcal{F}} &= \frac{1}{2}F_{AB}\mathbf{e}^A \wedge \mathbf{e}^B \\
&= \frac{1}{2}F_{ab}\mathbf{e}^a \wedge \mathbf{e}^b + \mathcal{G}_{ab}\eta^{ac}\mathbf{f}_c \wedge \mathbf{e}^b + \frac{1}{2}\mathcal{H}_{ab}\eta^{ac}\eta^{bd}\mathbf{f}_c \wedge \mathbf{f}_d
\end{aligned}
$$

where we have defined

$$
\begin{aligned}
\mathcal{G}_{ab} &\equiv \eta_{ac}G^c{}_b \\
\mathcal{H}_{ab} &\equiv \eta_{ac}\eta_{bd}H^{cd}
\end{aligned}
$$

The use of a different font is important because we are not using the biconformal

metric, K_{AB}, to change index positions. It is the original index positions and font, $G^a{}_b$, H^{ab}, that are the proper field components.

The matrix components F_{AB} are then written as

$$
\begin{aligned}
F_{AB} &= \begin{pmatrix} F_{ab} & F_a{}^b \\ F^a{}_b & F^{ab} \end{pmatrix} \\
&= \begin{pmatrix} F_{ab} & -\mathcal{G}_{ba} \\ \mathcal{G}_{ab} & \mathcal{H}_{ab} \end{pmatrix}
\end{aligned}
\tag{107}
$$

where the form of the upper right corner follows because

$$
\begin{aligned}
-\mathcal{G}_{ba}\mathbf{e}^a \wedge \eta^{bc}\mathbf{f}_c &= \mathcal{G}_{ba}\eta^{bc}\mathbf{f}_c \wedge \mathbf{e}^a \\
&= G^c{}_a\mathbf{f}_c \wedge \mathbf{e}^a
\end{aligned}
$$

With this convention, we can meaningfully define the transpose of matrices. Specifically, while the transpose of

$$
\begin{pmatrix} F_{ab} & F_a{}^b \\ F^a{}_b & F^{ab} \end{pmatrix}
$$

is ill-defined because of the mixed indices on the off-diagonal terms, the transpose of F_{AB} in the weight $+1$ basis is

$$
\begin{pmatrix} F_{ab} & -\mathcal{G}_{ba} \\ \mathcal{G}_{ab} & \mathcal{H}_{ab} \end{pmatrix}^t = \begin{pmatrix} F_{ba} & \mathcal{G}_{ba} \\ -\mathcal{G}_{ab} & \mathcal{H}_{ba} \end{pmatrix}
\tag{108}
$$

For $\mathcal{H}_{ba} = -\mathcal{H}_{ab}$ and $F_{ba} = -F_{ab}$ this is manifestly antisymmetric, as befits a 2-form. Notice that the effect of two transposes is the identity, so this operation provides an involutive automorphism even though η_{ab} is not the biconformal metric.

From $[\mathcal{G}_{ab}]^t = \mathcal{G}_{ba}$ we have

$$[\eta_{ac}G^c{}_b]^t = [\eta_{bc}G^c{}_a]$$

and therefore, $[G^d{}_b]^t = \eta^{ad}\eta_{bc}G^c{}_a$.

Metric

Because all quadrants of the metric are used in the variation, it is desirable to retain both the name and index positions throughout. Since

$$K_{AB} = \begin{pmatrix} K_{ab} & K_a{}^b \\ K^a{}_b & K^{ab} \end{pmatrix}$$

contains all index positions, the inverse metric and its components are written with an overbar,

$$\bar{K}^{AB} = \begin{pmatrix} \bar{K}^{ab} & \bar{K}^a{}_b \\ \bar{K}_a{}^b & \bar{K}_{ab} \end{pmatrix}$$

Thus K_{ab} is the first quadrant of the metric, while $\bar{K}_{ab}$ is the final quadrant of the inverse metric. Here any changes of index position must be indicated with an explicit factor of η_{ab} or η^{ab}

Volume form

It is convenient to define a volume form as $\mathbf{\Phi} \equiv {}^*1$ but in defining the Hodge dual operation a number of ambiguities need to be clarified. Because up and down indices have distinct conformal weight, we may partially order the indices on the Levi-Civita tensor. We establish the following conventions:

1. All factors of $\mathbf{f}_a$ are written first, followed by all of the $\mathbf{e}^b$.

2. The Levi-Civita *tensor* is written as $e^{ab\cdots c}{}_{de\cdots f}$, with all n up indices first. The partially ordered antisymmetric *symbol* is written as $\varepsilon^{ab\cdots c}{}_{de\cdots f}$.

3. When taking the dual of a p-form, we sum the components on the first p indices of the Levi-Civita tensor, then introduce factors of -1 to move indices to the default positions. For example, the dual of $\boldsymbol{H} = H^a{}_b \mathbf{f}_a \wedge \mathbf{e}^b$ is

$$
\begin{aligned}
{}^*\boldsymbol{H} &= \frac{1}{(n-1)!\,(n-1)!} H^a{}_b e_a{}^{bc\cdots d}{}_{e\cdots f} \mathbf{f}_c \wedge \cdots \wedge \mathbf{f}_d \wedge \mathbf{e}^e \wedge \cdots \wedge \mathbf{e}^f \\
&= \frac{(-1)^n}{(n-1)!\,(n-1)!} H^a{}_b e^{bc\cdots d}{}_{ae\cdots f} \mathbf{f}_c \wedge \cdots \wedge \mathbf{f}_d \wedge \mathbf{e}^e \wedge \cdots \wedge \mathbf{e}^f \\
&= \frac{(-1)^n}{(n-1)!\,(n-1)!} H^a{}_b e^{bc\cdots d}{}_{ae\cdots f} \mathbf{f}_{c\cdots d} \mathbf{e}^{e\cdots f}
\end{aligned}
$$

Notice that in the final step, the number of wedged forms in $\mathbf{f}_{c\cdots d}$ may be inferred from the Levi-Civita tensor. Since the Levi-Civita tensor always has n up and n down indices, the number of basis forms of each type is unambiguous. For example,

$$
e^{abc\cdots d}{}_{e\cdots f} \mathbf{f}_{c\cdots d} \wedge \mathbf{e}^{e\cdots f} = e^{abc\cdots d}{}_{e\cdots f} \underbrace{\mathbf{f}_c \wedge \ldots \wedge \mathbf{f}_d}_{n-2} \wedge \underbrace{\mathbf{e}^e \wedge \ldots \wedge \mathbf{e}^f}_{n}
$$

is a $2n-2$ form that includes the wedge product of $n-2$ factors of $\mathbf{f}_a$ and n factors of $\mathbf{e}^a$.

4. An m-form is a polynomial with each term having different numbers of $\mathbf{e}$'s and $\mathbf{f}$'s, we write the terms in order of increasing number of $\mathbf{f}$'s.

5. The partial ordering of indices on the Levi-Civita tensor reduces the normalization of the volume element from $\frac{1}{(2n)!}$ to $\frac{1}{n!n!}$.

It has been noted elsewhere [45] that there are alternative duals in biconformal space. For instance, we may use the symplectic form instead of the metric to connect indices.

The difference resides in the relative signs between the $\mathbf{e}^a$, $\mathbf{f}_a$, and mixed terms. Here we use only the Hodge dual, taking care to keep the correct signs.

With these conventions in mind, we define

$$
\begin{aligned}
\mathbf{\Phi} \;\equiv\;& {}^{*}1 \\
=\;& \frac{1}{n!n!} e^{c\cdots d}{}_{e\cdots f} \mathbf{f}_{c\cdots d} \wedge \mathbf{e}^{e\cdots f} \\
=\;& \frac{1}{n!n!} \sqrt{K} \varepsilon^{c\cdots d}{}_{e\cdots f} \mathbf{f}_{c\cdots d} \wedge \mathbf{e}^{e\cdots f}
\end{aligned}
\tag{109}
$$

and consequently,

$$
\begin{aligned}
\mathbf{f}_{c\cdots d} \wedge \mathbf{e}^{e\cdots f} \;=\;& \frac{1}{\sqrt{K}} \varepsilon^{e\cdots f}{}_{c\cdots d} \mathbf{\Phi} \\
=\;& \bar{e}_{c\cdots d}{}^{e\cdots f} \mathbf{\Phi}
\end{aligned}
\tag{110}
$$

where the overbar denotes the contravariant form of the Levi-Civita tensor. The contravariant form satisfies

$$
e^{a\cdots b}{}_{c\cdots d} \bar{e}_{a\cdots b}{}^{c\cdots d} \;=\; n!n!
$$

We also need the reduction formulas,

$$
\begin{aligned}
e^{e\cdots f}{}_{mnc\cdots d} \bar{e}^{ghc\cdots d}{}_{e\cdots f} \;=\;& n!\,(n-2)!\left(\delta_m^g \delta_n^h - \delta_m^h \delta_n^g\right) \\
e^{mc\cdots d}{}_{ne\cdots f} \bar{e}^{he\cdots f}{}_{gc\cdots d} \;=\;& (n-1)!\,(n-1)!\,\delta_n^h \delta_g^m
\end{aligned}
\tag{111}
$$

Further notation

The antisymmetric projection operator mapping for $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ tensors,

$$
\begin{aligned}
\Delta_{db}^{ac} &\equiv \frac{1}{2}\left(\delta_d^a \delta_b^c - \eta^{ac}\eta_{db}\right) \\
&= \frac{1}{2}\eta^{ce}\eta_{bf}\left(\delta_d^a \delta_e^f - \delta_e^a \delta_d^f\right)
\end{aligned}
$$

arises frequently.

Conventions for invariant matrices

Possible actions can be constructed using curvatures naturally arising in a theory, together with any invariant tensors consistent with the gauging. The biconformal gauging of the conformal group has a surprising number of invariant objects. These invariant structures arise from internal symmetries of the conformal group are induced into generic biconformal spaces [16].

The conformally invariant Killing form, restricted to the base manifold:

$$
K_{AB} = \begin{pmatrix} 0 & \delta_a{}^b \\ \delta^a{}_b & 0 \end{pmatrix} \tag{112}
$$

The symplectic form, underlying dilatations,

$$
\Omega_{AB} = \begin{pmatrix} 0 & \delta_a{}^b \\ -\delta^a{}_b & 0 \end{pmatrix} \tag{113}
$$

Interestingly, this form manifests Born reciprocity [11, 12]. The complex structure, arising from the symmetry between translations of the origin and translations of the

point at infinity (i.e., special conformal transformations),

$$J^A{}_B \quad := \quad \begin{pmatrix} 0 & -\eta^{ab} \\ \eta_{ab} & 0 \end{pmatrix} \tag{114}$$

The symmetric Kähler form, arising from the compatibility, $g\left(u,v\right) = \Omega\left(u,Jv\right)$, of the symplectic and complex structures,

$$\begin{aligned} g_{AB} &= \Omega_{AC} J^C{}_B \\ &= \begin{pmatrix} 0 & \delta_a{}^c \\ -\delta^a{}_c & 0 \end{pmatrix} \begin{pmatrix} 0 & -\eta^{cb} \\ \eta_{cb} & 0 \end{pmatrix} \\ &= \begin{pmatrix} \eta_{ab} & 0 \\ 0 & \eta^{ab} \end{pmatrix} \end{aligned} \tag{115}$$

These three Kähler structures satisfy

$$g\left(u,v\right) = \Omega\left(u,Jv\right) \tag{116}$$

Notice that the symmetric Kähler form is not invariant under the conformal structure.

Kähler or Killing?

As noted in the introduction, vacuum biconformal gravity depends only on the variation of the gauge fields and does not require introduction of a metric while Yang-Mills actions make use of the Hodge dual and a metric is required. Given the presence of two natural symmetric forms in biconformal spaces, i.e., the scale invariant Killing form, Eq.(112), and the Kähler metric, Eq.(115), we must make a choice of how to specify the orthonormality relation of the solder and co-solder forms. With $\tilde{\mathbf{e}}^A = (\mathbf{e}^a, \mathbf{f}_b)$, and

using an overbar to denote the inverse, we may specify orthonormality by either

$$\left\langle \tilde{\mathbf{e}}^A, \tilde{\mathbf{e}}^B \right\rangle \equiv \bar{K}^{AB} \tag{117}$$

or

$$\left\langle \tilde{\mathbf{e}}^A, \tilde{\mathbf{e}}^B \right\rangle \equiv \bar{g}^{AB} \tag{118}$$

but clearly not both. Once we choose either Eq.(117) or Eq.(118), the corresponding symmetric form becomes the metric, with its variation following from the variation of the solder and co-solder forms.

This choice between the symmetric Killing and Kähler forms is not arbitrary. We therefore studied both cases, ultimately showing that the correct matter couplings arise only if we use the *Killing form*, K_{AB}, to determine orthonormality of the basis forms. By inverting the components of the basis forms, we then have the metric in arbitrary coordinates (indicated by *coordinate* indices M, N, distinct from *orthonormal* indices A, B)

$$K_{MN} = \tilde{e}_M{}^A \tilde{e}_N{}^B K_{AB} \tag{119}$$

This is our final choice of metric, and is the only choice which yields the expected Yang-Mills source for general relativity.

The gravitational field equations follow by variation of the connection forms, including the solder and co-solder forms $\tilde{e}_M{}^A$. The variation of the metric then follows from Eq.(119). Until the variation is complete, we need the general form of the inverse metric,

$$\bar{K}^{AB} = \begin{pmatrix} \bar{K}^{ab} & \bar{K}^a{}_b \\ \bar{K}_a{}^b & \bar{K}_{ab} \end{pmatrix}$$

Once the variations are expressed in terms of $\delta \bar{K}^{AB}$ or $\delta \bar{g}^{AB}$ any remaining components may be returned to the orthonormal form, Eq.(112) or Eq.(115).

Let the solder and co-solder variations be given by

$$\delta \mathbf{e}^a = A^a{}_c \mathbf{e}^c + B^{ac} \mathbf{f}_c \tag{120}$$

$$\delta \mathbf{f}_c = C_{cd} \mathbf{e}^d + D_c{}^d \mathbf{f}_d \tag{121}$$

Then for variation of the components of the Killing metric we expand Eq.(117)

$$\begin{aligned}
\delta \bar{K}^{ab} &= \delta \left\langle \mathbf{e}^a, \mathbf{e}^b \right\rangle \\
&= \left\langle A^a{}_c \mathbf{e}^c + B^{ac} \mathbf{f}_c, \mathbf{e}^b \right\rangle + \left\langle \mathbf{e}^a, A^b{}_c \mathbf{e}^c + B^{bc} \mathbf{f}_c \right\rangle \\
&= A^a{}_c \bar{K}^{cb} + B^{ac} \bar{K}_c{}^b + A^b{}_c \bar{K}^{ac} + B^{bc} \bar{K}^a{}_c
\end{aligned}$$

Since the coefficients $A^a{}_c, B^{ac}, C_{cd}, D_c{}^d$ now represent the variation, we may return the remaining components of $\bar{K}^{AB}$ to the null orthonormal form of Eq.(112),

$$\delta \bar{K}^{ab} = B^{ab} + B^{ba}$$

Computing the remaining components in the same way, we arrive at the full set,

$$\begin{aligned}
\delta \bar{K}^{ab} &= B^{ab} + B^{ba} \\
\delta \bar{K}^a{}_b &= A^a{}_b + D_b{}^a \\
\delta \bar{K}_a{}^b &= D_a{}^b + A^b{}_a \\
\delta \bar{K}_{ab} &= C_{ab} + C_{ba}
\end{aligned} \tag{122}$$

We also considered the analogous calculation if we were to choose the Kähler inner

product, Eq.(118). This variation gives

$$
\begin{aligned}
\delta \bar{g}^{ab} &= A^a{}_c \eta^{cb} + A^b{}_c \eta^{ac} \\
\delta \bar{g}^a{}_b &= B^{ac}\eta_{bc} + C_{bc}\eta^{ac} \\
\delta \bar{g}_a{}^b &= C_{ac}\eta^{cb} + B^{bc}\eta_{ac} \\
\delta \bar{g}_{ab} &= D_a{}^c \eta_{cb} + D_b{}^c \eta_{ac}
\end{aligned}
\tag{123}
$$

As we indicate when we perform the gravitational variation, Eq.(161) in Section 6 below, it is the B^{ab} part of the variation which ultimately reduces to the Einstein equation. From Eqs.(122) and (123) above we see that the different possible choices for the metric lead to completely different sources for gravity. Using the Killing form to define orthonormality of the basis, it is the coefficient of the $\delta \bar{K}^{ab}$ variation that provides the gravitational source, while the opposite choice of the Kähler form couples the coefficients of the cross-terms $\delta \bar{g}^a{}_b$ and $\delta \bar{g}_a{}^b$ to gravity. Only one of these can give the usual Yang-Mills energy tensor. To determine which, we checked each of the two inner products, for each proposed action functional below until it became clear that *we must use the Killing form.*

Rather than presenting these distinct variations here, we continue with the Killing metric and its variation, as given by Eqs.(112), (117), and (122). Details of the failure of the Kähler choice are given in Appendix A.

IX. THE YANG-MILLS ACTION

In spacetime, the action for a Yang-Mills field may be written as

$$S_{\text{YM}} \;=\; -\frac{\kappa}{2} \int tr \boldsymbol{\mathcal{F}} \wedge^{*} \boldsymbol{\mathcal{F}} \tag{124}$$

and it is natural to consider the same form in biconformal space. However, as we show in this Section, this usual form leads to nonstandard coupling to gravity. We show in the next Section that a twisted action is required to give the usual coupling.

Expanded into independent components in the $\tilde{\mathbf{e}}^A$ basis,

$$\boldsymbol{\mathcal{F}}^{i} \;=\; \frac{1}{2}\mathcal{F}^{i}{}_{ab}\mathbf{e}^{a} \wedge \mathbf{e}^{b} + \mathcal{F}^{ia}{}_{b}\mathbf{f}_{a} \wedge \mathbf{e}^{b} + \frac{1}{2}\mathcal{F}^{iab}\mathbf{f}_{a} \wedge \mathbf{f}_{b}$$

where i is an index of the internal Lie algebra. This index can be suppressed without loss of generality in the action and gravitational field equations. Also, for the Yang-Mills field it proves more transparent to give the three coefficients distinct names. Finally, since we must insure antisymmetry of the twisted field, it is most transparent to use the uniform weight basis, $\mathbf{e}^A = \left(\mathbf{e}^a, \eta^{bc}\mathbf{f}_c\right) = \left(\mathbf{e}^a, \mathbf{f}^b\right)$. We therefore write

$$\boldsymbol{\mathcal{F}} \;=\; \frac{1}{2}\mathcal{F}_{AB}\mathbf{e}^{A} \wedge \mathbf{e}^{b} \;=\; \frac{1}{2}F_{ab}\mathbf{e}^{a} \wedge \mathbf{e}^{b} + \mathcal{G}_{ab}\mathbf{f}^{a} \wedge \mathbf{e}^{b} + \frac{1}{2}\mathcal{H}_{ab}\mathbf{f}^{a} \wedge \mathbf{f}^{b} \tag{125}$$

In discussions of the Yang-Mills field equations (as opposed to the gravitational equations), the internal index becomes important and will be shown where necessary.

Note that F_{ab} and $\mathcal{H}_{ab}$ are antisymmetric. The cross-term may be written as

$$\begin{aligned}
G^{a}{}_{b}\mathbf{f}_{a} \wedge \mathbf{e}^{b} &\;=\; \frac{1}{2}\left(G^{a}{}_{b}\right)\mathbf{f}_{a} \wedge \mathbf{e}^{b} + \frac{1}{2}\left(-G^{b}{}_{a}\right)\mathbf{e}^{a} \wedge \mathbf{f}_{b} \\
&\;=\; \mathcal{G}_{ab}\mathbf{f}^{a} \wedge \mathbf{e}^{b}
\end{aligned}$$

where $\mathcal{G}_{ab}$ may be asymmetric. As a matrix, $\mathcal{F}_{AB}$ takes the form given in Eq.(107).

The Hodge dual and standard Yang-Mills action

With these factors in mind, we find the Hodge dual of the Yang-Mills field. In general terms the Hodge dual of a 2-form is given by

$$
\begin{aligned}
{}^*\boldsymbol{\mathcal{F}} &= {}^*\left(\frac{1}{2}\mathcal{F}_{AB}\tilde{\mathbf{e}}^A \wedge \tilde{\mathbf{e}}^B\right) \\
&= \frac{1}{(2n-2)!}\frac{1}{2}\mathcal{F}_{AB}\bar{K}^{AC}\bar{K}^{BD}\varepsilon_{CDE\ldots F}\tilde{\mathbf{e}}^E \wedge \ldots \wedge \tilde{\mathbf{e}}^F
\end{aligned}
$$

However, we need to separate the distinct quadrants of each inverse metric. Expanding each upper case index $A, B, \ldots$, as a raised, weight $+1$ index and a lowered weight -1 index in turn, then collecting like terms leads to

$$
\begin{aligned}
{}^*\boldsymbol{\mathcal{F}} &= \frac{1}{n!\,(n-2)!}\left(\frac{1}{2}F_{ab}\bar{K}^{am}\bar{K}^{bn} + G^a{}_b\bar{K}_a{}^m\bar{K}^{bn} + \frac{1}{2}H^{ab}\bar{K}_a{}^m\bar{K}_b{}^n\right) \\
&\quad \times \varepsilon^{c\cdots d}{}_{mne\cdots f}\mathbf{f}_{c\cdots d}\mathbf{e}^{e\cdots f} \\
&\quad + \frac{(-1)^{n-1}}{(n-1)!\,(n-1)!}\left(\frac{1}{2}F_{ab}\bar{K}^a{}_m\bar{K}^{bn} + G^a{}_b\bar{K}_{am}\bar{K}^{bn} + \frac{1}{2}H^{ab}\bar{K}_{am}\bar{K}_b{}^n\right) \\
&\quad \times \varepsilon^{mc\cdots d}{}_{ne\cdots f}\mathbf{f}_{c\cdots d}\mathbf{e}^{e\cdots f} \\
&\quad + \frac{(-1)^n}{(n-1)!\,(n-1)!}\left(\frac{1}{2}F_{ab}\bar{K}^{am}\bar{K}^b{}_n + G^a{}_b\bar{K}_a{}^m\bar{K}^b{}_n + \frac{1}{2}H^{ab}\bar{K}_a{}^m\bar{K}_{bn}\right) \\
&\quad \times \varepsilon^{nc\cdots d}{}_{me\cdots f}\mathbf{f}_{c\cdots d}\mathbf{e}^{e\cdots f} \\
&\quad + \frac{1}{n!\,(n-2)!}\left(\frac{1}{2}F_{ab}\bar{K}^a{}_m\bar{K}^b{}_n + G^a{}_b\bar{K}_{am}\bar{K}^b{}_n + \frac{1}{2}H^{ab}\bar{K}_{am}\bar{K}_{bn}\right) \\
&\quad \times \varepsilon^{mnc\cdots d}{}_{e\cdots f}\mathbf{f}_{c\cdots d}\mathbf{e}^{e\cdots f} \qquad (126)
\end{aligned}
$$

where one readily sees the advantage of omitting the wedge between forms, $\mathbf{e}^e \wedge \ldots \wedge \mathbf{e}^f \longleftrightarrow \mathbf{e}^{e\cdots f}$.

We form the usual Yang-Mills Lagrangian density as the wedge product, $\boldsymbol{\mathcal{F}} \wedge {}^*\boldsymbol{\mathcal{F}}$,

eliminating the basis forms in favor of the volume form $\mathbf{\Phi}$. After a bit of algebra, we find

$$
\begin{aligned}
\boldsymbol{\mathcal{F}} \wedge {}^{*}\boldsymbol{\mathcal{F}} \;=\; & \left(\frac{1}{2} F_{mn} \bar{K}^{am} \bar{K}^{bn} + G^{m}{}_{n} \bar{K}^{a}{}_{m} \bar{K}^{bn} + \frac{1}{2} H^{mn} \bar{K}^{a}{}_{m} \bar{K}^{b}{}_{n} \right) F_{ab} \mathbf{\Phi} \\
& + \left(F_{mn} \bar{K}_{a}{}^{m} \bar{K}^{bn} + G^{m}{}_{n} \left(\bar{K}_{am} \bar{K}^{bn} - \bar{K}_{a}{}^{n} \bar{K}^{b}{}_{m} \right) + H^{mn} \bar{K}_{am} \bar{K}^{b}{}_{n} \right) G^{a}{}_{b} \mathbf{\Phi} \\
& + \left(\frac{1}{2} F_{mn} \bar{K}_{a}{}^{m} \bar{K}_{b}{}^{n} + G^{m}{}_{n} \bar{K}_{am} \bar{K}_{b}{}^{n} + \frac{1}{2} H^{mn} \bar{K}_{am} \bar{K}_{bn} \right) H^{ab} \mathbf{\Phi} \qquad (127)
\end{aligned}
$$

and the matter action is given by Eq.(124). The full action is the combination of Eq.(106) and Eq.(124),

$$
S = S_{G} + S_{YM}
$$

Substituting the null orthonormal form of the Killing metric, Eq.(112) into Eq.(127) the lagrange density reduces to

$$
\boldsymbol{\mathcal{F}} \, {}^{*}\boldsymbol{\mathcal{F}} = \left(H^{ab} F_{ab} - G^{b}{}_{a} G^{a}{}_{b} \right) \mathbf{\Phi} \qquad (128)
$$

We find that this Hodge dual form of the action is identical to the form of the action given in [29], despite the claim in [29] that the action is independent of the metric. The presence of the metric is concealed in [29] because the Killing metric in this basis is comprised of Kronecker deltas.

Although varying the Yang-Mills potentials in Eq.(127), or equivalently in Eq.(128), yields the usual Yang-Mills field equation, we show below that varying the metric gives a nonstandard coupling to gravity. In the next Section we define a twisted action that gives the usual coupling to gravity, while still leading to the correct Yang-Mills field equations.

The failure of the $\mathcal{F} \wedge^* \mathcal{F}$ action

Variation of the usual Lagrangian Eq.(127) gives

$$
\begin{aligned}
\delta_g \int \boldsymbol{\mathcal{F}} \wedge {}^* \boldsymbol{\mathcal{F}} \;=\; & \int \left(F_{mn} F_{ab} \delta \bar{K}^{am} \eta^{bn} \right) \boldsymbol{\Phi} \\
& + \left(G^m{}_n F_{ab} \eta^{bn} \delta \bar{K}^a{}_m + F_{mn} G^a{}_b \eta^{bn} \delta \bar{K}_a{}^m \right) \boldsymbol{\Phi} \\
& + \left(G^m{}_n G^a{}_b \eta^{bn} \delta \bar{K}_{am} + G^m{}_n G^a{}_b \eta_{am} \delta \bar{K}^{bn} \right) \boldsymbol{\Phi} \\
& + \left(H^{mn} G^a{}_b \eta_{am} \delta \bar{K}^b{}_n + G^m{}_n H^{ab} \eta_{am} \delta \bar{K}_b{}^n \right) \boldsymbol{\Phi} \\
& + \left(H^{mn} H^{ab} \eta_{bn} \delta \bar{K}_{am} \right) \boldsymbol{\Phi}
\end{aligned}
$$

where after variation we returned the remaining inverse metric components to the orthonormal form. The metric variations are now given by Eq.(122). Substituting, the variation yields the gravitational field equations,

$$
\begin{aligned}
\alpha \Omega^n{}_b{}^b{}_m - \alpha \Omega^a{}_b{}^b{}_a \delta^n_m & \\
+ \beta \Omega^n{}_m - \beta \Omega^a{}_a \delta^n_m + \Lambda \delta^n_m &= -\kappa \left(H^{bn} F_{bm} - G^b{}_m G^n{}_b \right) \\
& \quad -\kappa \left(\frac{1}{2} \delta^n_m \left(F_{bc} H^{bc} - G^b{}_c G^c{}_b \right) \right) \qquad (129) \\
\alpha \Omega^a{}_n{}^m{}_a - \alpha \Omega^a{}_b{}^b{}_a \delta^m_n & \\
+ \beta \Omega^m{}_n - \beta \Omega^a{}_a \delta^m_n + \Lambda \delta^m_n &= -\kappa \left(H^{bm} F_{bn} - G^b{}_n G^m{}_b \right) \\
& \quad -\kappa \left(\frac{1}{2} \delta^m_n \left(F_{bc} H^{bc} - G^b{}_c G^c{}_b \right) \right) \qquad (130) \\
\alpha \Omega^a{}_{nam} + \beta \Omega_{nm} &= \kappa \left(F_{am} G^a{}_n + F_{an} G^a{}_m \right) \qquad (131) \\
\alpha \Omega^n{}_b{}^{bm} + \beta \Omega^{nm} &= -\kappa \left(H^{nb} G^m{}_b + H^{mb} G^n{}_b \right) \qquad (132)
\end{aligned}
$$

The problem with this coupling to gravity becomes evident when the gravitational equations require vanishing momentum curvatures on the left side of Eq.(132). With

this, Eq.(132) implies

$$H^{nb}G^m{}_b + H^{mb}G^n{}_b \;\;=\;\; 0$$

The failure of this result is not immediate, but the reduction of Eqs.(129) and (130) gives a second constraint on H^{ab} and $G^a{}_b$. The two constraints lead, at the most general, to vanishing H^{ab} and symmetric $G^a{}_b$, leaving the source for the Einstein equation, Eq.(131) linear in F_{ab} and therefore incompatible with the usual energy source for general relativity.

The problem cannot be altered by a different choice of the inner product (see Appendix A), but must lie in the use of the usual action. In the context of other double field theories, a twist allows dimensional reduction to preserve gauging of supersymmetries [25, 27, 28]. Here we find that including a twist insures consistency under dimensional reduction not only for supersymmetry, but also gives the correct coupling of Yang-Mills sources to gravity.

We now turn to the definition of the twisted Yang-Mills action, and its metric variation. Then, in Section 5, we vary the Yang-Mills potentials. The remainder of our presentation details the reduction of the full set of field equations to reproduce scale covariant general relativity with Yang-Mills sources in the usual n-dimensional form.

X. METRIC VARIATION

Here we consider the metric variation of the twisted Yang-Mills action. Thus, instead of the usual spacetime action, Eq.(124), we consider a biconformal Yang-Mills theory with an action functional of the form

$$S_{\text{TYM}} = -\frac{\kappa}{2} \int tr \bar{\mathcal{F}} \wedge^* \mathcal{F} \tag{133}$$

where $*$ is the usual Hodge dual, $\mathcal{F}$ is a curvature 2-form, $\bar{\mathcal{F}}$ is a twisted conjugate curvature, and the trace is over the $SU(N)$ generators. The twist matrix is formed using both the Killing metric and the Kähler form, $K^A{}_B \equiv \bar{K}^{AC} g_{CB}$. While the twist matrix is similar to that used to preserve supersymmetry in other double field theories, [25, 27, 28], we define the twisted Yang-Mills field by

$$\bar{\mathcal{F}}_{AB} = \frac{1}{2} \left(K_A{}^C \mathcal{F}_{CB} + \mathcal{F}_{AC} K^C{}_B \right)$$

We find that this form is necessary to preserve the antisymmetry of the field while giving the required interchange of source fields. Variation of the inverse Killing form $\bar{K}^{AB}$ then gives the source for the gravitational field equations.

Details of the twist

The twist is accomplished using $K^A{}_B \equiv \bar{K}^{AC} g_{CB}$ where $K^A{}_B = K_B{}^A$, since both g_{AB} and K_{AB} are symmetric. In the null-orthonormal form and the $\mathbf{e}^A$ basis, this matrix is simply

$$K^A{}_B = K_B{}^A = \begin{pmatrix} 0 & \delta_b^a \\ \delta_b^a & 0 \end{pmatrix}$$

and the required form of the field is

$$\begin{aligned}
\bar{\mathcal{F}}_{AB} &= \frac{1}{2}\left(K_A{}^C\mathcal{F}_{CB} + \mathcal{F}_{AC}K^C{}_B\right) \\
&= \begin{pmatrix} \mathcal{G}_{[ab]} & \frac{1}{2}\left(F_{ab} + \mathcal{H}_{ab}\right) \\ \frac{1}{2}\left(F_{ab} + \mathcal{H}_{ab}\right) & \mathcal{G}_{[ab]} \end{pmatrix}
\end{aligned} \tag{134}$$

where $\mathcal{F}_{AB}$ is given by Eq.(107). Note that this transformation maintains the anti-symmetry while interchanging the diagonal and anti-diagonal elements.

However, the reduced form of the twisted field in the orthonormal frame given in Eq.(134) is insufficient. Until we complete the metric variation we must use the generic form of the metric in computing the twist matrix,

$$\begin{aligned}
K^A{}_B &= \begin{pmatrix} \bar{K}^{ac} & \bar{K}^a{}_e\eta^{ec} \\ \eta^{ae}\bar{K}_e{}^c & \eta^{ae}\eta^{cf}\bar{K}_{ef} \end{pmatrix} \begin{pmatrix} \eta_{cb} & 0 \\ 0 & \eta_{bc} \end{pmatrix} \\
&= \begin{pmatrix} \bar{K}^{ac}\eta_{cb} & \bar{K}^a{}_b \\ \bar{K}^a{}_b & \eta^{ae}\bar{K}_{eb} \end{pmatrix} \\
K_A{}^B &= \begin{pmatrix} \eta_{ad}\bar{K}^{db} & \bar{K}_a{}^b \\ \bar{K}_a{}^b & \bar{K}_{ae}\eta^{eb} \end{pmatrix}
\end{aligned}$$

with symmetry given by $\bar{K}^a{}_b = \eta^{ae}\bar{K}_e{}^c\eta_{cb}$ (see Appendix B for details of the symmetry). Then $\bar{\mathcal{F}}_{AB}$ becomes

$$\bar{\mathcal{F}}_{AB} = \frac{1}{2}\left(K_A{}^C \mathcal{F}_{CB} + \mathcal{F}_{AC} K^C{}_B\right)$$

$$= \frac{1}{2}\begin{pmatrix} \eta_{ad}\bar{K}^{dc}F_{cb} + \bar{K}_a{}^c\mathcal{G}_{cb} & -\eta_{ad}\bar{K}^{dc}\mathcal{G}_{bc} + \bar{K}_a{}^c\mathcal{H}_{cb} \\ \eta_{ad}\bar{K}^d{}_e\eta^{ec}F_{cb} + \bar{K}_{ae}\eta^{ec}\mathcal{G}_{cb} & -\eta_{ad}\bar{K}^d{}_e\eta^{ec}\mathcal{G}_{bc} + \bar{K}_{ae}\eta^{ec}\mathcal{H}_{cb} \end{pmatrix}$$

$$+ \frac{1}{2}\begin{pmatrix} F_{ac}\bar{K}^{ce}\eta_{eb} - \mathcal{G}_{ca}\eta^{cd}\bar{K}_d{}^e\eta_{eb} & F_{ac}\bar{K}^c{}_b - \mathcal{G}_{ca}\eta^{ce}\bar{K}_{eb} \\ \mathcal{G}_{ac}\bar{K}^{ce}\eta_{eb} + \mathcal{H}_{ac}\eta^{cd}\bar{K}_d{}^e\eta_{eb} & \mathcal{G}_{ac}\bar{K}^c{}_b + \mathcal{H}_{ac}\eta^{ce}\bar{K}_{eb} \end{pmatrix} \qquad (135)$$

and we check that 135 agrees with Eq.(134) when we restore the null orthonormal frame for the metric. The twisted field is simpler when written as a 2-form,

$$\bar{\mathcal{F}} = \frac{1}{2}\bar{\mathcal{F}}_{AB}\mathbf{e}^A \wedge \mathbf{e}^B$$

$$= \frac{1}{2}\left(F_{ac}\bar{K}^{ce}\eta_{eb} + \bar{K}_a{}^c\mathcal{G}_{cb}\right)\mathbf{e}^a \wedge \mathbf{e}^b$$

$$+ \frac{1}{2}\left(F_{ac}\bar{K}^c{}_b\eta^{bf} + \bar{K}_a{}^c\mathcal{H}_{cb}\eta^{bf} - \mathcal{G}_{ca}\eta^{ce}\bar{K}_{eb}\eta^{bf} - \mathcal{G}_{bc}\bar{K}^{ce}\eta_{ea}\eta^{bf}\right)\mathbf{e}^a \wedge \mathbf{f}_f$$

$$+ \frac{1}{2}\left(\mathcal{H}_{ac}\eta^{ce}\bar{K}_{eb} + \mathcal{G}_{ac}\bar{K}^c{}_b\right)\eta^{af}\mathbf{f}_f \wedge \eta^{bg}\mathbf{f}_g \qquad (136)$$

We may interchange $\bar{K}_a{}^b$ and $\bar{K}^b{}_a$ when convenient since these have identical variations and both restrict to δ^a_b in the null orthonormal basis.

The action

The dual field is given by Eq.(126). To avoid duplication of indices when we wedge the dual field together with the twisted field, we rename the indices in the twisted

field

$$\bar{\mathcal{F}} = \frac{1}{2}\left(F_{rq}\bar{K}^{qt}\eta_{ts} + \bar{K}_r{}^q\mathcal{G}_{qs}\right)\mathbf{e}^r \wedge \mathbf{e}^s$$

$$+ \frac{1}{2}\left(F_{rc}\bar{K}^c{}_s + \bar{K}_r{}^c\mathcal{H}_{cs}\right)\eta^{sw}\mathbf{e}^r \wedge \mathbf{f}_w$$

$$+ \frac{1}{2}\left(-\mathcal{G}_{sq}\bar{K}^{qt}\eta_{tr} - \mathcal{G}_{qr}\eta^{qt}\bar{K}_{ts}\right)\eta^{sw}\mathbf{e}^r \wedge \mathbf{f}_w$$

$$+ \frac{1}{2}\left(\mathcal{G}_{rq}\bar{K}^q{}_s + \mathcal{H}_{rq}\eta^{qt}\bar{K}_{ts}\right)\eta^{rw}\eta^{sx}\mathbf{f}_w \wedge \mathbf{f}_x$$

Each term of the wedge product is proportional to the volume form, using $\mathbf{f}_{c\cdots d}\wedge\mathbf{e}^{e\cdots f} = \bar{e}_{c\cdots d}{}^{e\cdots f}\mathbf{\Phi}$. Then, replacing the double Levi-Civita tensors with Kronecker deltas according to Eqs.(111), we fully distribute the lengthy expression. We summarize the essential features here, but details including the full wedge product $\bar{\mathcal{F}}\wedge{}^*\mathcal{F}$ and its reduction to $\bar{\mathcal{F}}\wedge{}^*\mathcal{F}\big|_{contributing}$ below are given in Appendix C.

Since we are interested only in the variation, and will return the metric to the null orthonormal form of Eq.(112) after variation, certain terms clearly do not contribute to the field equations. For example, in the product

$$\left(-\mathcal{G}_{sq}\bar{K}^{qt}\eta_{tr} - \mathcal{G}_{qr}\eta^{qt}\bar{K}_{ts}\right)\eta^{sw}\left(\frac{1}{2}F_{ab}\bar{K}^a{}_m\bar{K}^{bn} + \frac{1}{2}\mathcal{H}_{gh}\eta^{ga}\eta^{hb}\bar{K}_{am}\bar{K}_b{}^n\right)$$

none of the four terms after distribution will contribute to the field equation because the variation of any one factor of metric components always leaves an unvaried $\bar{K}_{ts}$ or $\bar{K}^{bn}$,

$$\delta\left(\bar{K}^a{}_m\bar{K}_{ts}\bar{K}^{bn}\right) = \delta\bar{K}^a{}_m\bar{K}_{ts}\bar{K}^{bn}\!\!\!\nearrow^0 + \bar{K}^a{}_m\delta\bar{K}_{ts}\bar{K}^{bn}\!\!\!\nearrow^0 + \bar{K}^a{}_m\bar{K}_{ts}\delta\bar{K}^{bn}\!\!\!\nearrow^0$$

Dropping such terms, we are only required to vary terms linear in $\bar{K}_{ab}$ or $\bar{K}^{ab}$, or cubic in the off diagonal components $\bar{K}^a{}_b$. Collecting these and using the symmetries

of the fields, finally yields,

$$
\begin{aligned}
\bar{\mathcal{F}} \wedge {}^*\mathcal{F}\big|_0 &= \left(\frac{1}{2} F_{bc} F_{da} \eta^{ac} + F_{bc} \mathcal{H}^{ac} \eta_{ad} + \frac{1}{2} \left(\mathcal{G}_{ad} - 2\mathcal{G}_{da} \right) \eta_{bc} \mathcal{G}^{ac} \right) \bar{K}^{bd} \Phi \\
&\quad + \left(\mathcal{G}_{ba} H^{dc} + F_{ab} \mathcal{G}^{cd} \right) \bar{K}^b{}_e \bar{K}^e{}_d \bar{K}^a{}_c \Phi \\
&\quad + \left(\left(\frac{1}{2} \mathcal{H}_{ac} + F_{ac} \right) \eta^{cd} \mathcal{H}^{ab} + \frac{1}{2} \left(\mathcal{G}^{ba} - 2\mathcal{G}^{ab} \right) \eta^{cd} \mathcal{G}_{ca} \right) \bar{K}_{bd} \Phi \quad (137)
\end{aligned}
$$

We may now vary the metric.

To carry out the variation of the Yang-Mills potentials we may write the action in the null orthonormal frame. This form is still contained in the expression above, given by the purely off-diagonal terms, cubic in $\bar{K}^a{}_b$. This simpler form of the action follows immediately as

$$
S_{TYM} = \kappa \int tr \left(\bar{\mathcal{F}} \wedge {}^*\mathcal{F} \right) = \kappa \int tr \left(\mathcal{G}^{ab} \left(\mathcal{H}_{ab} + F_{ab} \right) \right) \Phi \qquad (138)
$$

The variation of the potentials is carried out in Section .

Metric variation

Using the variation of the Killing metric given in Eq.(122) and the variation of the volume form given by

$$
\begin{aligned}
\delta \Phi &= -\frac{1}{2} K_{AB} \delta \bar{K}^{AB} \Phi \\
&= -\frac{1}{2} \left[K_{ab} \left(B^{ab} + B^{ba} \right) + K^a{}_b \left(D_a{}^b + A^b{}_a \right) \right] \Phi \\
&\quad - \frac{1}{2} \left[K_b{}^a \left(A^b{}_a + D_a{}^b \right) + K^{ab} \left(C_{ab} + C_{ba} \right) \right] \Phi \\
&= -\delta^a{}_b \left(A^b{}_a + D_a{}^b \right) \Phi
\end{aligned}
$$

the variation of Eq.(137) yields

$$
\begin{aligned}
\delta \left(\bar{\mathcal{F}} \wedge {}^*\mathcal{F} \right) &= \left(\frac{1}{2} F_{bc} F_{da} \eta^{ac} + F_{bc} \mathcal{H}^{ac} \eta_{ad} + \frac{1}{2} \left(\mathcal{G}_{ad} - 2\mathcal{G}_{da} \right) \eta_{bc} \mathcal{G}^{ac} \right) 2 B^{(bd)} \boldsymbol{\Phi} \\
&\quad + \left(\left(2\mathcal{G}^{ad} - \mathcal{G}^{da} \right) F_{ab} - \left(2\mathcal{G}^{da} - \mathcal{G}^{ad} \right) \mathcal{H}_{ab} - \left(F_{ac} + \mathcal{H}_{ac} \right) \mathcal{G}^{ac} \delta^d{}_b \right) \\
&\quad \times \left(A^b{}_d + D_d{}^b \right) \boldsymbol{\Phi} \\
&\quad + \left(\frac{1}{2} \mathcal{H}_{ac} \mathcal{H}^{ab} \eta^{cd} + F_{ac} \mathcal{H}^{ab} \eta^{cd} + \frac{1}{2} \left(\mathcal{G}^{ba} - 2\mathcal{G}^{ab} \right) \eta^{cd} \mathcal{G}_{ca} \right) 2 C_{(bd)} \boldsymbol{\Phi} \quad (139)
\end{aligned}
$$

This variation couples to the $(\mathbf{e}^a, \mathbf{f}_a)$ variation of the gravity action, Eq.(106). The resulting field equations and their reduction to the gravity theory are given below. Before considering reduction of the field equations, we turn to variation of the Yang-Mills potentials to find the Yang-Mills equations.

XI. YANG-MILLS FIELD EQUATIONS

We also need to vary the potential to find the Yang-Mills field equations. We start with the action of Eq.(138) in the null orthonormal basis,

$$S_{TYM} \;\; = \;\; \kappa \int tr \bar{\boldsymbol{\mathcal{F}}}^i \wedge {}^* \boldsymbol{\mathcal{F}}^i = \kappa \int \mathcal{G}^{iab} \left(\mathcal{H}_{i\,ab} + F_{i\,ab} \right) \boldsymbol{\Phi} \tag{140}$$

In this Section, we make the internal symmetry explicit, varying S_{YM} with respect to the $SU(N)$ potentials. It is most convenient to work in the $\bar{\mathbf{e}}^A = (\mathbf{e}^a, \mathbf{f}_b)$ basis. Internal indices are labeled with letters i, j, k, while frame indices are chosen from the beginning of the alphabet, $a, b, c, \ldots$.

The field components and potentials

The $SU(N)$ field is given in the $\bar{\mathbf{e}}^A$ basis by Eq.(141),

$$\boldsymbol{\mathcal{F}}^i \;\; = \;\; \frac{1}{2} F^i{}_{ab} \mathbf{e}^a \wedge \mathbf{e}^b + \mathcal{G}^i{}_{cb} \eta^{ac} \mathbf{f}_a \wedge \mathbf{e}^b + \frac{1}{2} \mathcal{H}^i{}_{ab} \eta^{ac} \eta^{bd} \mathbf{f}_c \wedge \mathbf{f}_c \tag{141}$$

where as a matrix,

$$\mathcal{F}^i{}_{AB} \;\; = \;\; \begin{pmatrix} F^i{}_{ab} & G^i{}_a{}^b \\ G^{i\,a}{}_b & \mathcal{H}^i{}_{ab} \end{pmatrix}$$

with $G^i{}_a{}^b = -G^{i\,a}{}_b$.

The field is given in terms of its $U(1)$ or $SU(N)$ potential by the Cartan equation,

$$\boldsymbol{\mathcal{F}}^i \;\; = \;\; \mathbf{d}\boldsymbol{\mathcal{A}}^i - \frac{1}{2} c^i{}_{jk} \boldsymbol{\mathcal{A}}^j \wedge \boldsymbol{\mathcal{A}}^k$$

where the potentials are biconformal 1-forms,

$$\boldsymbol{\mathcal{A}}^i = A^i{}_a \mathbf{e}^a + B^{i\,a} \mathbf{f}_a$$

In terms of $A^i{}_a, B^{i\,a}$, the field becomes

$$
\begin{aligned}
\boldsymbol{\mathcal{F}}^i &= \mathbf{D}A^i{}_a \wedge \mathbf{e}^a - \frac{1}{2}\boldsymbol{\mathcal{A}}^i{}_k \wedge A^k{}_b \mathbf{e}^b \\
&\quad + \mathbf{D}B^{i\,a} \wedge \mathbf{f}_a + B^{i\,a}\mathbf{S}_a - \frac{1}{2}\boldsymbol{\mathcal{A}}^i{}_k \wedge B^{k\,b}\mathbf{f}_b
\end{aligned}
\tag{142}
$$

where $\boldsymbol{\mathcal{A}}^i{}_k$ is the $SU(N)$ connection in the adjoint representation,

$$\boldsymbol{\mathcal{A}}^i{}_k \equiv c^i{}_{jk}\boldsymbol{\mathcal{A}}^j$$

and the Weyl-covariant derivatives of the potentials are given by

$$
\begin{aligned}
\mathbf{D}A^i{}_a &= \mathbf{d}A^i{}_a - \boldsymbol{\omega}^a{}_c A^i{}_a + A^i{}_a \boldsymbol{\omega} \\
\mathbf{D}B^{i\,a} &= \mathbf{d}B^{i\,a} - B^{i\,a}\boldsymbol{\omega}^c{}_a - B^{i\,a}\boldsymbol{\omega}
\end{aligned}
$$

Note that the covariant derivative of η^{ab} does not necessarily vanish, $\mathbf{D}\eta^{ab} = \mathbf{d}\eta^{ab} - 2\boldsymbol{\omega}\eta^{ab}$ where $\mathbf{d}\eta^{ab}$ takes into account the conformal equivalence class, $\eta_{ab} \in \{e^{2\varphi}\eta^0_{ab}|\,\varphi = \varphi(x,y)\}$.

Now we separate Eq.(142) into its independent $\mathbf{e}^a \wedge \mathbf{e}^b$, $\mathbf{f}_a \wedge \mathbf{e}^b$ and $\mathbf{f}_a \wedge \mathbf{f}_b$ parts. We begin by expanding the exterior derivatives in the orthonormal basis

$$
\begin{aligned}
\mathbf{d}A^i{}_a &= A^i{}_{a,b}\mathbf{e}^b + A^i{}_a{}^{,b}\mathbf{f}_b \\
\mathbf{d}B^{i\,a} &= B^{i\,a}{}_{,b}\mathbf{e}^b + B^{i\,a,b}\mathbf{f}_b
\end{aligned}
$$

and similarly the covariant exterior derivatives

$$\mathbf{D}A^i{}_a = A^i{}_{a;b}\mathbf{e}^b + A^i{}_a{}^{;b}\mathbf{f}_b$$

$$\mathbf{D}B^{i\,a} = B^{i\,a}{}_{;b}\mathbf{e}^b + B^{i\,a;b}\mathbf{f}_b$$

We also expand the $SU(N)$ connection, $\boldsymbol{\mathcal{A}}^j G_j$, in the adjoint representation:

$$\boldsymbol{\mathcal{A}}^i{}_k = c^i{}_{jk}A^j{}_a\mathbf{e}^a + c^i{}_{jk}B^{j\,a}\mathbf{f}_a$$

$$= \boldsymbol{\alpha}^i{}_k + \boldsymbol{\beta}^i{}_k$$

Finally, writing the general form of the co-torsion

$$\mathbf{S}_a = \frac{1}{2}S_{abc}\mathbf{e}^b \wedge \mathbf{e}^c + S_a{}^b{}_c\mathbf{f}_b \wedge \mathbf{e}^c + \frac{1}{2}S_a{}^{bc}\mathbf{f}_b \wedge \mathbf{f}_c$$

we arrive the component fields in terms of the potentials:

$$F^i{}_{ab} = A^i{}_{b;a} - A^i{}_{a;b} - c^i{}_{jk}A^j{}_a A^k{}_b + B^{i\,c}S_{cab} \tag{143}$$

$$G^{i\,a}{}_b = A^i{}_b{}^{;a} - B^{i\,a}{}_{;b} - c^i{}_{jk}B^{j\,a}A^k{}_b + B^{i\,c}S_c{}^a{}_b \tag{144}$$

$$H^{i\,ab} = B^{i\,b;a} - B^{i\,a;b} - c^i{}_{jk}B^{j\,a}B^{k\,b} + B^{i\,c}S_c{}^{ab} \tag{145}$$

These expressions are covariant with respect to both $SU(N)$ and Weyl group transformations.

The field equations for the potentials

Using Eqs.(143)-(145) in the action, Eq.(140), the variation of $A^i{}_a$ gives

$$
\begin{aligned}
\delta_A S_{TYM} &= \kappa \int \left(\delta_A \mathcal{G}^{iab} \left(\mathcal{H}_{i\,ab} + F_{i\,ab} \right) \right) \boldsymbol{\Phi} \\
&\quad + \kappa \int \mathcal{G}^{iab} \left(\widehat{\delta_A \mathcal{H}_{i\,ab}} + \delta_A F_{i\,ab} \right) \boldsymbol{\Phi} \\
&= \kappa \int \left(\delta A^k{}_b \eta^{bc} \left(-\left(\mathcal{H}_{k\,ac} + F_{k\,ac} \right)^{;a} - c^i{}_{jk} B^{j\,a} \left(\mathcal{H}_{i\,ac} + F_{i\,ac} \right) \right) \right) \boldsymbol{\Phi} \\
&\quad + \kappa \int \delta A^k{}_b \left(-\left(\mathcal{G}_k{}^{ab} - \mathcal{G}_k{}^{ba} \right)_{;a} - c^i{}_{kj} A^j{}_a \left(\mathcal{G}_i{}^{ba} - \mathcal{G}_i{}^{ab} \right) \right) \boldsymbol{\Phi}
\end{aligned}
$$

Therefore the field equation becomes

$$
\begin{aligned}
0 &= \eta^{bc} \left(\mathcal{H}_{k\,ac} + F_{k\,ac} \right)^{;a} + \eta^{bc} \left(\mathcal{H}_{i\,ac} + F_{i\,ac} \right) \beta^i{}_k{}^a \\
&\quad + \left(\mathcal{G}_k{}^{ab} - \mathcal{G}_k{}^{ba} \right)_{;a} + \left(\mathcal{G}_i{}^{ab} - \mathcal{G}_i{}^{ba} \right) \alpha^i{}_{ka}
\end{aligned}
$$

For the $B^{i\,a}$ variation, recalling that i, j, are internal indices and a, b, c, e orthonormal indices, we find

$$
\begin{aligned}
0 &= \eta^{bc} \left(\left(\mathcal{H}_{j\,ac} + F_{j\,ac} \right)_{;b} + \alpha^i{}_{jb} \left(\mathcal{H}_{i\,ac} + F_{i\,ac} \right) + S_a{}^e{}_b \left(\mathcal{H}_{j\,ec} + F_{j\,ec} \right) \right) \\
&\quad + \left(\mathcal{G}_{j\,ab} - \mathcal{G}_{j\,ba} \right)^{;b} + \left(\mathcal{G}_{i\,ab} - \mathcal{G}_{i\,ba} \right) \beta^i{}_j{}^b + \frac{1}{2} \left(\mathcal{G}_{j\,bc} - \mathcal{G}_{j\,cb} \right) S_a{}^{bc} \\
&\quad + \frac{1}{2} \left(\mathcal{G}_j{}^{bc} - \mathcal{G}_j{}^{cb} \right) S_{abc}
\end{aligned}
$$

Notice that there is no dynamical equation for the symmetric part of $\mathcal{G}^k{}_{cd}$. Moreover, the action depends only on the antisymmetric part of $\mathcal{G}^k{}_{cd}$. Therefore, from here on we assume that like $F^i{}_{ab}$ and $H^{i\,ab}$, $\mathcal{G}^k{}_{cd}$ is antisymmetric, $\mathcal{G}^k{}_{ab} \equiv \mathcal{G}^k{}_{[ab]}$. Also notice that there is no separate Yang-Mills field equation for F_{ab} and H_{ab}. Both field equations contain only their sum, $\mathcal{H}^k{}_{ab} + F^k{}_{ab}$, although F_{ab} and $\mathcal{H}_{ab}$ enter the

gravitational equations separately. We therefore define a new field,

$$K^k_{\ ab} \ \equiv \ \frac{1}{2}\left(\mathcal{H}^k_{\ ab} + F^k_{\ ab}\right)$$

In terms of these, the field equations become

$$0 \ = \ \eta^{bc}K_{k\ ac}^{\ \ ;a} + \eta^{bc}K_{i\ ac}\beta^i_{\ k}^{\ \ a} + \mathcal{G}_k^{\ ab}_{\ \ ;a} + \mathcal{G}_i^{\ ab}\alpha^i_{\ ka} \tag{146}$$

$$0 \ = \ \eta^{bc}\left(K_{j\ ac;b} + K_{i\ ac}\alpha^i_{\ jb} + K_{j\ ec}S_a^{\ e}_{\ \ b}\right)$$
$$+\mathcal{G}_{j\ ab}^{\ \ ;b} + \mathcal{G}_{i\ ab}\beta^i_{\ j}^{\ \ b} + \frac{1}{2}\mathcal{G}_{j\ bc}S_a^{\ bc} + \frac{1}{2}\mathcal{G}_j^{\ bc}S_{abc} \tag{147}$$

These have the expected form of a divergence of the field strength.

The field equations of an $SU(N)$ gauge theory on a $2n$-dimensional space given by Eqs.(146) and (147) together with the gravitational sources of Eqs.(**??**) complete the first stage of our investigation. These expressions give a satisfactory formulation of biconformal Yang-Mills theory.

The second stage of this study is to understand how the $SU(N)$ gravitational sources together with their field equations affect the gravitational and Yang-Mills solutions. Specifically, we want to know if the reduction to the co-tangent bundle of n-dimensional spacetime still occurs, and if so, whether the usual Yang-Mills field equations and gravitational sources result.

This is indeed what happens. Once we have reduced the gravitational field equations to more simply describe the underlying geometry, we will return to Eqs.(146) and (147). Our final result is to show that the reduction of the underlying geometry to the co-tangent bundle of general relativity simultaneously forces the reduction of the source equations to the usual Yang-Mills sources for general relativity.

In the next Section we carry out the gravitational reduction. Since the pure gravitational field case is presented in detail elsewhere [**?**], we are able to simply

state some of the conclusions, focussing on those features which are different in the presence of matter. Subsequently, in Sec.(**??**), we give particular attention to the resulting twofold reduction of the $SU(N)$ fields: (1) from $(N^2 - 1) \times \frac{2n(2n-1)}{2}$ field components to $(N^2 - 1) \times \frac{n(n-1)}{2}$ components, and (2) the restriction of the number of effective independent variables, $\boldsymbol{F}(x, y) \rightarrow \boldsymbol{F}(x)$.

XII. REDUCING THE GRAVITY EQUATIONS

Although the reduction of the gravitational field equations has been presented in detail elsewhere [6] and we only highlight the features which differ in the presence of matter, the presentation is still somewhat lengthy. To aid in following the discussion, we begin with a short description of the steps before providing details. The basic steps are the following:

- Present the structure equations and Bianchi identities and their immediate consequences for the form of the torsion-free biconformal curvatures. By manipulating the field equations, we arrive at a reduced form for the components of the curvature tensors. (These calculations involve the sources in essential ways that require more detailed presentation here.)

- The Cartan structure equations show that the solder form $\mathbf{e}^a$ is in involution. The Frobenius theorem therefore allows us to set $\mathbf{e}^a = e_\mu{}^a \mathbf{d}x^\mu$ and study the restricted solution on the $x^\mu = constant$ submanifolds spanned by n additional coordinates y_μ. Because of the reduced form of the curvature components, this restricted solution completely determines the $\mathbf{e}^a = 0$ pieces of the connection forms. We then extend the connection forms back to the full biconformal space. At this point, the connection forms have far fewer than their original number of components.

- Substitute the reduced connection forms into the structure equations and impose the field equations. This is done one structure equation at a time, each time reducing the degrees of freedom of the full system. Ultimately, all connection and curvature components are determined by the solder form, $\mathbf{e}^a$, and the components of the solder form itself only depend on half the original coordinates, $e_\mu{}^a (x, y) \Rightarrow e_\mu{}^a (x)$.

Field equations for the twisted action

The full action is

$$
\begin{aligned}
S &= S_G + S_{YM} \\
&= \lambda \int e_{ac\cdots d}{}^{be\cdots f} \left(\alpha \boldsymbol{\Omega}^a{}_b + \beta \delta^a{}_b \boldsymbol{\Omega} + \gamma \mathbf{e}^a \wedge \mathbf{f}_b \right) \wedge \mathbf{e}^c \wedge \cdots \wedge \mathbf{e}^d \wedge \mathbf{f}_e \wedge \cdots \wedge \mathbf{f}_f \\
&\quad - \frac{\kappa}{2} \int tr \overline{\mathcal{F}} \wedge{}^* \mathcal{F}
\end{aligned}
\tag{148}
$$

We choose the combinatoric factor λ so the final coupling is κ.

Curvatures, Bianchi identities, and gravity variation

The gravitational field equations are given by varying S_G in Eq.(148) with respect to all connection 1-forms, then combining with the Yang-Mills sources found in Eq.(139). The curvature components are given in terms of the connection by the Cartan structure equations,

$$
\mathbf{d}\boldsymbol{\omega}^a{}_b = \boldsymbol{\omega}^c{}_b \boldsymbol{\omega}^a{}_c + 2\Delta^{ac}_{db}\mathbf{f}_c \mathbf{e}^d + \boldsymbol{\Omega}^a{}_b \tag{149}
$$

$$
\mathbf{d}\mathbf{e}^a = \mathbf{e}^c \boldsymbol{\omega}^a{}_c + \boldsymbol{\omega}\mathbf{e}^a + \mathbf{T}^a \tag{150}
$$

$$
\mathbf{d}\mathbf{f}_a = \boldsymbol{\omega}^c{}_a \mathbf{f}_c + \mathbf{f}_a \boldsymbol{\omega} + \mathbf{S}_a \tag{151}
$$

$$
\mathbf{d}\boldsymbol{\omega} = \mathbf{e}^c \mathbf{f}_c + \boldsymbol{\Omega} \tag{152}
$$

We also have the integrability conditions–generalized Bianchi identities–which follow from the Poincarè lemma, $\mathbf{d}^2 \equiv 0$. Exterior differentiation of the Cartan equations, Eqs.(149)-(152) yields conditions which must be satisfied in order for solutions to

exist. These take the form

$$\mathbf{D}\mathbf{\Omega}^a{}_b \;=\; 2\Delta^{ac}_{db}\mathbf{f}_c\mathbf{T}^d - 2\Delta^{ac}_{db}\mathbf{S}_c\mathbf{e}^d \tag{153}$$

$$\mathbf{D}\mathbf{T}^a \;=\; \mathbf{e}^c\mathbf{\Omega}^a{}_c - \mathbf{\Omega}\mathbf{e}^a \tag{154}$$

$$\mathbf{D}\mathbf{S}_a \;=\; -\mathbf{\Omega}^c{}_a\mathbf{f}_c + \mathbf{f}_a\mathbf{\Omega} \tag{155}$$

$$\mathbf{D}\mathbf{\Omega} \;=\; -\mathbf{T}^c\mathbf{f}_c + \mathbf{e}^c\mathbf{S}_c \tag{156}$$

Conversely, the converse to the Poincarè lemma tells us that in a star-shaped region the integrability conditions Eqs.(153)-(156) imply the original Cartan equations, up to boundary terms.

For example, consider a Newtonian force written as minus the gradient of a potential, $\mathbf{F} = -\mathbf{d}V$. The Poincarè lemma shows that this equation has no solution unless

$$0 \;\equiv\; \mathbf{d}^2V = -\mathbf{d}\mathbf{F}$$

so the force must be curl-free. Conversely, we may solve the this "Bianchi identity" first. In a star-shaped region any curl-free force must be a gradient, $\mathbf{F} = \mathbf{d}U$. This reproduces the potential up to a "boundary" term satisfying

$$\mathbf{d}\,(U + V) \;=\; 0$$
$$U + V \;=\; constant$$

This in turn provides the usual additive constant to the potential. These principles hold for any set of equations where we may apply the Poincarè lemma and its converse, allowing us use both the original equations and their integrability conditions throughout the development of a solution. In the end, one set of equations cannot be

satisfied without simultaneously satisfying the other up to boundary terms.

The variation of the gravitational part of the action, Eq.(148), is discussed in detail in [?], and involves only variation of the connection forms, so we simply state the result. The variation of the spin connnection $\boldsymbol{\omega}^a{}_b$ and Weyl vector $\boldsymbol{\omega}$ give

$$T^{ae}{}_e - T^{ea}{}_e - S_e{}^{ae} = 0 \tag{157}$$

$$T^a{}_{ca} + S_c{}^a{}_a - S_a{}^a{}_c = 0 \tag{158}$$

$$\alpha \Delta^{ar}_{sb} \left(T^{mb}{}_a - \delta^m_a T^{eb}{}_e - \delta^m_a S_c{}^{bc} \right) = 0 \tag{159}$$

$$\alpha \Delta^{ar}_{sb} \left(\delta^b_c T^d{}_{ad} + S_c{}^b{}_a - \delta^b_c S_d{}^d{}_a \right) = 0 \tag{160}$$

and these acquire no sources since the Yang-Mills action is independent of the these connection forms, as noted in [15]. The variation of the solder and co-solder forms lead to

$$\left[\alpha \left(\Omega^n{}_b{}^b{}_m - \Omega^a{}_b{}^b{}_a \delta^n_m \right) + \beta \left(\Omega^n{}_m - \Omega^a{}_a \delta^n_m \right) + \Lambda \delta^n_m \right] A^m{}_n$$

$$\left[\alpha \left(\Omega^a{}_n{}^m{}_a - \Omega^a{}_b{}^b{}_a \delta^m_n \right) + \beta \left(\Omega^m{}_n - \Omega^a{}_a \delta^m_n \right) + \Lambda \delta^m_n \right] D_n{}^m$$

$$- \left[\alpha \Omega^a{}_{nam} + \beta \Omega_{nm} \right] B^{mn}$$

$$\left[\alpha \Omega^n{}_b{}^{bm} + \beta \Omega^{nm} \right] C_{mn} \tag{161}$$

with the arbitrary variations $A^m{}_n, B^{mn}, C_{mn}$ and $D_n{}^m$ defined in Eqs.(120) and (121). In [?] the expressions above are equated to zero, but they now acquire sources.

Combined equations

Equating corresponding parts of Eqs.(**??**) and Eqs.(161) and symmetrizing appropriately,

$$\left[\alpha \left(\Omega^a{}_c{}^c{}_b - \Omega^c{}_d{}^d{}_c \delta^a_b\right) + \beta \left(\Omega^a{}_b - \Omega^c{}_c \delta^a_b\right) + \Lambda \delta^a_b\right] = \kappa W^a{}_b \qquad (162)$$

$$\left[\alpha \left(\Omega^c{}_b{}^a{}_c - \Omega^c{}_d{}^d{}_c \delta^a_b\right) + \beta \left(\Omega^a{}_b - \Omega^c{}_c \delta^a_b\right) + \Lambda \delta^a_b\right] = \kappa W^a{}_b \qquad (163)$$

$$-\left[\alpha \Omega^c{}_{bca} + \beta \Omega_{ba}\right] = \kappa T_{ab} \qquad (164)$$

$$\left[\alpha \Omega^b{}_c{}^{ca} + \beta \Omega^{ba}\right] = \kappa S^{ab} \qquad (165)$$

where, recalling the antisymmetry of $\mathcal{G}^{ab}$,

$$T_{ab} \equiv tr\left(F_{ac}F_{bd}\eta^{cd} + H_{ac}F_{bd}\eta^{cd} + H_{bc}F_{ad}\eta^{cd} + 3\eta^{cd}\mathcal{G}_{ac}\mathcal{G}_{bd}\right) \qquad (166)$$

$$S^{ab} \equiv \eta^{bd}\eta^{ac}tr\left(\eta^{ef}H_{ec}H_{fd} + \eta^{ef}H_{ec}F_{fd} + \eta^{ef}H_{ed}F_{fc} + 3\eta^{fe}\mathcal{G}_{ec}\mathcal{G}_{fd}\right) \qquad (167)$$

$$W^a{}_b \equiv tr\left(3\mathcal{G}^{ca}F_{cb} + 3\mathcal{G}^{ca}\mathcal{H}_{cb} - \left(F_{cd} + \mathcal{H}_{cd}\right)\mathcal{G}^{cd}\delta^a{}_b\right) \qquad (168)$$

In addition, we have the field equations for the $U\left(1\right)$ field,

$$0 = \eta^{bc}K_{k\,ac}{}^{;a} + \eta^{bc}K_{i\,ac}\beta^i{}_k{}^a + \mathcal{G}_k{}^{ab}{}_{;a} + \mathcal{G}_i{}^{ab}\alpha^i{}_{ka} \qquad (169)$$

$$0 = \eta^{bc}\left(K_{j\,ac;b} + K_{i\,ac}\alpha^i{}_{jb} + K_{j\,ec}S_a{}^e{}_b\right)$$

$$+\mathcal{G}_{j\,ab}{}^{;b} + \mathcal{G}_{i\,ab}\beta^i{}_j{}^b + \frac{1}{2}\mathcal{G}_{j\,bc}S_a{}^{bc} + \frac{1}{2}\mathcal{G}_j{}^{bc}S_{abc} \qquad (170)$$

Our gravitational solution now follows many of the steps presented in detail in [**?**].

Solving the field equations for the twisted action

For the remainder of the gravitational solution, the particular forms of T_{ab}, S^{ab} and $W^a{}_b$ make little difference; indeed, the solution of this Section holds for the metric variation of any sources at all. While the form of these source tensors varies with the fields and the matter action, the positions in which they occur in the field equations and their symmetries follow knowing only the variation of $\bar{K}^{AB}$. For the present, this is all we need.

We first turn to the consequences of vanishing torsion, $\mathbf{T}^a = 0$.

Vanishing torsion

Similarly to general relativity, with vanishing torsion the torsion Bianchi identity, Eq.(154), simplifies to an algebraic relation,

$$0 \;=\; \mathbf{e}^c \mathbf{\Omega}^a{}_c - \mathbf{\Omega} \mathbf{e}^a$$

which must hold independently of any sources. The algebraic condition $\mathbf{e}^c \mathbf{\Omega}^a{}_c = \mathbf{\Omega} \mathbf{e}^c$ expands to three independent component equations,

$$\Omega^a{}_{[bcd]} \;=\; \delta^a_{[b} \Omega_{cd]} \tag{171}$$

$$\Omega^a{}_b{}^c{}_d - \Omega^a{}_d{}^c{}_b \;=\; \delta^a_b \Omega^c{}_d - \delta^a_d \Omega^c{}_b \tag{172}$$

$$\Omega^a{}_b{}^{cd} \;=\; \delta^a_b \Omega^{cd} \tag{173}$$

Since $\eta_{ea}\Omega^a_{\ b}^{\ \ cd} = -\eta_{ba}\Omega^a_{\ e}^{\ \ cd}$, the ab trace of Eq.(173) leaves $\Omega^{cd} = 0$. Therefore each term vanishes separately,

$$\Omega^a_{\ b}^{\ \ cd} = 0 \tag{174}$$

$$\Omega^{cd} = 0 \tag{175}$$

The ad contraction of Eq.(172) gives

$$\Omega^a_{\ b}^{\ \ c}_{\ a} = -(n-1)\,\Omega^c_{\ \ b} \tag{176}$$

Combining Eqs.(174) and (175) with Eq.(165) we immediately find that the gravitational fields force a constraint on the source fields. This is our *first source constraint*:

$$S^{ab} = 0 \tag{177}$$

We next look at the field equations for the curvature.

Curvature equations

We now combine the vanishing torsion simplifications with the curvature and dilatation field equations, Eqs.(162) and (163). The reduction of these equations begins by noting that the difference between Eq.(162) and Eq.(163) immediately gives equality of the traces,

$$\Omega^c_{\ b}^{\ a}_{\ c} = \Omega^a_{\ c}^{\ c}_{\ b} \tag{178}$$

Next, formally lowering an index in Eq.(172)

$$\eta_{ea}\Omega^a{}_b{}^c{}_d - \eta_{ea}\Omega^a{}_d{}^c{}_b = \eta_{eb}\Omega^c{}_d - \eta_{ed}\Omega^c{}_b$$

we cycle *ebd*, then add the first two and subtract the third. Using the the antisymmetry of the curvature on the first two indices, $\eta_{ea}\Omega^a{}_d{}^c{}_b = -\eta_{da}\Omega^a{}_e{}^c{}_b$ we find

$$\Omega^a{}_e{}^c{}_d = -2\Delta^{ab}_{de}\Omega^c{}_b \tag{179}$$

Substituting Eq.(179) into the trace symmetry, Eq.(178) to the two contractions of Eq.(179) constrains the cross-dilatation,

$$-\delta^a_d\Omega^c{}_c + \eta^{ae}\eta_{cd}\Omega^c{}_e = -(n-1)\,\Omega^a{}_d$$

Contracting with η_{ba}, we see that the antisymmetric part vanishes,

$$(n-2)\left(\eta_{bc}\Omega^c{}_d - \eta_{cd}\Omega^c{}_b\right) = 0$$

in dimensions greater than 2, while an explicit check confirms the vanishing antisymmetry in 2-dimensions as well. Therefore, the symmetric part, $\eta_{bc}\Omega^c{}_d + \eta_{dc}\Omega^c{}_b = \frac{2}{n}\eta_{bd}\Omega^c{}_c$, becomes a solution for the full cross-dilatation in terms of its trace,

$$\Omega^c{}_d = \frac{1}{n}\delta^a_b\Omega^c{}_c \tag{180}$$

This, in turn, combines with Eq.(179) to give the cross-curvature in terms of the trace of the dilatation,

$$\Omega^a{}_b{}^c{}_d = -\frac{2}{n}\Delta^{ac}_{db}\Omega^e{}_e \tag{181}$$

We have one remaining cross-curvature field equation, Eq.(162), which couples the cross-dilatation trace, $\Omega^a{}_a$, to the Yang-Mills source fields. Using Eqs.(180) and (181) to replace the cross-curvature and the cross-dilatation in Eq.(162), and simplifying,

$$\frac{1}{n}\left(n-1\right)\left[\left(\left(n-1\right)\alpha-\beta\right)\Omega^c{}_c+n\Lambda\right]\delta^a_b \;\; = \;\; W^a{}_b$$

so that $W^a{}_b = f\delta^a_b$ for some function f. The constant Λ is given by $\Lambda = \left(n-1\right)\alpha - \beta + n^2\gamma$. Contracting then substituting back, the gravitational field equations force a *second source constraint*:

$$W^a{}_b \;\; = \;\; \frac{1}{n}W^c{}_c\delta^a_b \tag{182}$$

where

$$W^c{}_c \;\; = \;\; \left(n-1\right)\left[\left(\left(n-1\right)\alpha-\beta\right)\Omega^c{}_c + n\Lambda\right]$$

The traced source tensor on the right, $W^c{}_c$, therefore drives the entire cross-curvature and cross-dilatation. It is striking that the only source dependence for these components is the Yang-Mills Lagrangian density,

$$W^a{}_a \;\; = \;\; 3\mathcal{G}^{i\,ac}\left(\mathcal{H}_{i\,ac} + F_{i\,ac}\right) = 3\mathcal{L}$$

Spacetime terms

Finally, we combine the remaining field equation, Eq.(164),

$$\alpha\Omega^c{}_{bca} + \beta\Omega_{ba} = -\kappa T_{ab}$$

and the corresponding part of the vanishing torsion Bianchi, Eq.(171), which expanded becomes

$$\Omega^a{}_{bcd} + \Omega^a{}_{cdb} + \Omega^a{}_{dbc} = \delta^a_b \Omega_{cd} + \delta^a_c \Omega_{db} + \delta^a_d \Omega_{bc}$$

The ac trace reduces this to

$$\Omega^c{}_{bcd} - \Omega^c{}_{dcb} = -(n-2)\,\Omega_{bd}$$

Combining this with the antisymmetric part of the field equation, $\alpha\left(\Omega^a{}_{nam} - \Omega^a{}_{man}\right) = -2\beta\Omega_{nm}$ shows that

$$((n-2)\,\alpha - 2\beta)\,\Omega_{ab} = 0$$

so that generically (i.e., unless $(n-2)\,\alpha = 2\beta$), the spacetime dilatation vanishes. Note that this is true for any symmetric source tensor, so spacetime dilatation is never driven by ordinary matter. As a result,

$$\Omega^c{}_{acb} = \Omega^c{}_{(a|c|b)} = -\frac{\kappa}{\alpha}T_{ab} \tag{183}$$

$$\Omega_{nm} = 0 \tag{184}$$

Dilatation

Having reduced the dilatational curvature to a single function,

$$\boldsymbol{\Omega} = \chi\mathbf{e}^a \wedge \mathbf{f}_a \tag{185}$$

where $\chi \equiv -\frac{1}{n}\Omega^a{}_a$, we can now use the dilatational integrability condition, Eq.(156), to press further. Substituting Eq.(185) into the Bianchi identity, Eq.(156),

$$
\begin{aligned}
0 &= \mathbf{d\Omega} - \mathbf{e}^b \wedge \mathbf{S}_b \\
&= \mathbf{d\chi} \wedge \mathbf{e}^a \wedge \mathbf{f}_a - (1+\chi)\,\mathbf{e}^a \wedge \mathbf{S}_a
\end{aligned}
$$

where we have used $\mathbf{d}\left(\mathbf{e}^a \wedge \mathbf{f}_a\right) = \mathbf{D}\left(\mathbf{e}^a \wedge \mathbf{f}_a\right) = \boldsymbol{T}^a \wedge \mathbf{f}_a - \mathbf{e}^a \wedge \mathbf{S}_a$. Setting $\mathbf{d\chi} = \chi_c \mathbf{e}^c + \chi^c \mathbf{f}_c$, expanding the co-torsion into components, and combining like forms yields three independent equations,

$$(1+\chi)\,S_{[acd]} = 0 \tag{186}$$

$$(1+\chi)\left(S_c{}^a{}_d - S_d{}^a{}_c\right) = \chi_d \delta_c^a - \chi_c \delta_d^a \tag{187}$$

$$(1+\chi)\,S_a{}^{cd} = \chi^d \delta_a^c - \chi^c \delta_a^d \tag{188}$$

We may now use the co-torsion field equations to gain insight into χ.

With vanishing torsion, the co-torsion field equations Eqs.(157)-(160) reduce to

$$S_e{}^{ae} = 0 \tag{189}$$

$$S_c{}^a{}_a - S_a{}^a{}_c = 0 \tag{190}$$

$$\alpha \Delta_{sb}^{ar}\left(S_c{}^b{}_a - \delta_c^b S_d{}^d{}_a\right) = 0 \tag{191}$$

Using the field equation Eq.(190) to replace the co-torsion terms in the trace of the Bianchi identity, Eq.(187), gives

$$(1+\chi)\left(S_c{}^a{}_a - S_a{}^a{}_c\right) = -(n-1)\,\chi_c$$

$$\chi_c = 0$$

Then, combining Eq.(189) with the *ad* trace of Eq.(188),

$$(1 + \chi) \, S_a{}^{ca} = -(n-1) \, \chi^c$$

$$\chi^c = 0$$

and therefore,

$$\mathbf{d}\chi = 0 \tag{192}$$

The dilatation therefore takes the form

$$\mathbf{\Omega} = \chi \mathbf{e}^a \wedge \mathbf{f}_a$$

with χ *constant.* The remainder of the development of the solution continues as in the homogeneous case but with a different constant value,

$$\chi = \frac{1}{(n-1)\,\alpha - \beta} \left(\frac{1}{n-1}\Lambda - \frac{\kappa}{n\,(n-1)} W^c{}_c \right)$$

for the magnitude of the dilatation cross-term.

Importantly, the constancy of χ implies the constancy of $W^c{}_c$, and via the second source constraint, Eq.(182), the (mostly zero) constancy of all of $W^a{}_b$.

The Frobenius theorem and the final reduction

With vanishing torsion, Eq.(150) shows that the solder form becomes involute, and we may write

$$\mathbf{e}^a = e_\alpha{}^a \mathbf{d}x^\alpha$$

where x^α comprise n of the $2n$ coordinates. Holding x^α constant, $\mathbf{e}^a = 0$, and the residual field equations describe a submanifold. Here the discussion exactly parallels that of [6]:

1. Solve for the connection on the $\mathbf{e}^a = 0$ submanifold. Here, because the curvature and dilatation vanish (see Eqs.(174) and (175)), we may gauge the restricted components of the spin connection and Weyl vector to zero. A careful coordinate choice puts the submanifold basis in the form $\mathbf{h}_a = e_a{}^\mu \mathbf{dy}_\mu$.

2. Now let x^α vary, extending the solution back to the full biconformal space. This allows all connection forms to acquire an additional $\mathbf{d}x^\alpha$ or $\mathbf{e}^a$ term,

$$
\begin{aligned}
\boldsymbol{\omega}^a{}_b &= \boldsymbol{\omega}^a{}_{bc}\mathbf{e}^c \\
\mathbf{e}^a &= e_\alpha{}^a\mathbf{d}x^\alpha \\
\mathbf{f}_a &= e_a{}^\mu\mathbf{d}y_\mu + c_{ab}\mathbf{e}^b \\
\boldsymbol{\omega} &= W_a\mathbf{e}^a
\end{aligned}
\tag{193}
$$

3. Note that Eq.(150) is now purely quadratic in $\mathbf{e}^a$, and therefore requires the coefficients to depend only on the x-coordinates, $e_\alpha{}^a = e_\alpha{}^a(x)$. Solving for the connection separates it into a compatible piece and a Weyl vector piece,

$$
\boldsymbol{\omega}^a{}_b = \boldsymbol{\alpha}^a{}_b - 2\Delta^{ac}_{db}W_c\mathbf{e}^d
$$

where $\mathbf{d}\mathbf{e}^a = \mathbf{e}^b\boldsymbol{\alpha}^a{}_b$.

4. Substitute these reduced forms of $\mathbf{e}^a, \mathbf{f}_b$ into the dilatation, Eq.(152) and solve for the Weyl vector. This yields

$$
\boldsymbol{\omega} = -\left(1 + \chi\right)y_a\mathbf{e}^a
$$

where $y_a = e_a{}^\mu y_\mu$.

These steps give the final expressions for the connection forms, except for the form of $c_{ab} = c_{ba}$ in the expansion of the co-solder form $\mathbf{f}_a$.

We note that the submanifolds found by setting either $\mathbf{e}^a = 0$ by holding x^μ constant, or $\mathbf{h}_a = 0$ by holding y_μ constant are Lagrangian submanifolds.

The curvature

The final steps in the gravitational reduction are to substitute the partial solution for the connection forms Eq.(193) into Eqs.(149) and (151) to impose the final field equations.

To express the remaining undetermined component of the curvature, $\Omega^a{}_{bcd}$, we define the Schouten tensor

$$\boldsymbol{\mathcal{R}}_a = \frac{1}{n-2}\left(R_{ab} - \frac{1}{2(n-1)}\eta_{ab}R\right)\mathbf{e}^b$$

where $R_{ab} = (n-2)\mathcal{R}_{ab} + \eta_{ab}\mathcal{R}$ is the Ricci tensor. The generalization of the Schouten tensor to an integrable Weyl geometry is then (see [33])

$$\boldsymbol{\mathscr{R}}_a = \boldsymbol{\mathcal{R}}_a + \mathbf{D}_{(\alpha,x)}W_a + W_a\boldsymbol{\omega} - \frac{1}{2}\eta_{ab}W^2\mathbf{e}^b$$

In terms of the Schouten tensor, the decomposition of the Riemann curvature 2-form into the Weyl curvature 2-form and trace parts is

$$\mathbf{R}^a{}_b = \mathbf{C}^a{}_b - 2\Delta^{ac}_{db}\boldsymbol{\mathcal{R}}_c \wedge \mathbf{e}^d$$

Because of the manifest involution of $\mathbf{h}_a = e_a{}^\mu(x)\,\mathbf{d}y_\mu$, the subspace spanned by the solder form, $\mathbf{e}^a$, is a submanifold. Because $\Omega_{ab} = 0$ the submanifold geometry is

always an *integrable* Weyl geometry, so the Weyl vector may be removed from the spacetime submanifold by a gauge transformation. The spacetime submanifold is simply a Riemannian geometry with local scale invariance.

Now, introducing the reduced form of the connection into Eq.(149) and imposing the corresponding field equation, Eq,(164) shows that

$$\frac{1}{1+\chi}\mathscr{R}_{ab} + c_{ab} = -\frac{\kappa}{\alpha}T_{ab} \tag{194}$$

with the full spacetime component of the biconformal curvature given by the Weyl curvature, $\Omega^a{}_{bcd} = C^a{}_{bcd}$.

The co-torsion

A similar introduction of the reduced connection into Eq.(151) for the co-torsion shows that the momentum and cross-terms vanish, while (following the somewhat intricate calculation of [6] the remaining component is given by

$$\mathbf{S}_a = \mathbf{d}_{(x)}\mathbf{c}_a + \mathbf{c}_b \wedge \boldsymbol{\omega}^b{}_a + \boldsymbol{\omega} \wedge \mathbf{c}_a \tag{195}$$

where $\mathbf{c}_a$, in turn, is determined by Eq.(194).

Expanding $\mathbf{c}_a$ fully to separate the Weyl vector parts,

$$
\begin{aligned}
\mathbf{c}_a &= -\frac{1}{1+\chi}\left(\mathcal{R}_a + \mathbf{D}_{(\alpha,x)}W_a + W_a\boldsymbol{\omega} - \frac{1}{2}\eta_{ab}W^2\mathbf{e}^b\right) - \frac{\kappa}{\alpha}\mathbf{T}_a \\
&= \mathbf{b}_a - \frac{1}{1+\chi}\left(\mathbf{D}_{(\alpha,x)}W_a + W_a\boldsymbol{\omega} - \frac{1}{2}\eta_{ab}W^2\mathbf{e}^b\right)
\end{aligned}
$$

where $\mathbf{T}_a = T_{ab}\mathbf{e}^b$ is the remaining source field and $\mathbf{b}_a \equiv -\frac{1}{1+\chi}\mathcal{R}_a - \frac{\kappa}{\alpha}\mathbf{T}_a$. Then

substituting into Eq.(195), after multiple cancellations the co-torsion becomes

$$
\begin{aligned}
\mathbf{S}_a &= \frac{1}{1+\chi} W_b \mathbf{R}^b{}_a - \mathbf{D}_{(\alpha,x)} \left(\frac{1}{1+\chi} \boldsymbol{\mathcal{R}}_a + \frac{\kappa}{\alpha} \mathbf{T}_a \right) \\
&\quad + 2\Delta_{ca}^{db} W_d \left(\frac{1}{1+\chi} \boldsymbol{\mathcal{R}}_b + \frac{\kappa}{\alpha} \mathbf{T}_b \right) \wedge \mathbf{e}^c
\end{aligned}
\tag{196}
$$

with the cross term and momentum term of the co-torsion vanishing.

This result is quite similar to an integrability condition. It is shown in [33] that the condition for the existence of a conformal gauge in which the Einstein equation, $G_{ab} = \kappa T_{ab}$, holds is

$$
0 = \phi_{,b} \mathbf{R}^b{}_a - \mathbf{D}_{(\alpha,x)} \left(\boldsymbol{\mathcal{R}}_a - \kappa \boldsymbol{\mathcal{T}}_a \right) + 2\Delta_{ca}^{db} \phi_{,d} \left(\boldsymbol{\mathcal{R}}_b - \kappa \boldsymbol{\mathcal{T}}_b \right) \wedge \mathbf{e}^c
\tag{197}
$$

where

$$
\boldsymbol{\mathcal{T}}_a = \frac{1}{n-2} \left(T_{ab} - \frac{1}{n-1} T \eta_{ab} \right)
$$

When $\boldsymbol{\mathcal{T}}_a = 0$, Eq.(197) reduces to the well-known condition, $\mathbf{D}_{(\alpha,x)} \boldsymbol{\mathcal{R}}_a - \varphi_{,b} \mathbf{C}^b{}_a = 0$, for the existence of a Ricci flat conformal gauge.

There are two differences between Eq.(196) and Eq.(197). First, the co-torsion on the left hand side of Eq.(196) obstructs the integrability condition, Eq.(197), and we cannot set $\mathbf{S}_a = 0$ because the Triviality Theorem shows that when both torsion and co-torsion vanish, biconformal space must be trivial. The second difference is that the Weyl vector on the right is not integrable on the *full* biconformal space.

These issues have a common solution. The part of structure equation for the co-solder form involving $\mathbf{h}_a$ is

$$
\mathbf{dh}_a = \boldsymbol{\omega}^c{}_a \wedge \mathbf{h}_c + \mathbf{h}_a \wedge \boldsymbol{\omega}
\tag{198}
$$

Therefore, as briefly noted above, $\mathbf{h}_a = e_a{}^\mu \mathbf{d} y_\mu$ is in involution. Holding $y_\mu = y_\mu^0$ constant shows that $\mathbf{e}^a$ spans a submanifold. On that submanifold, the Weyl vector becomes exact,

$$\boldsymbol{\omega} = W_a \mathbf{e}^a = \mathbf{d}\left(y_\mu^0 x^\mu\right)$$

This means that on the $y_\mu = y_\mu^0$ spacetime submanifold, the right side takes the form of the integrability condition.

At the same time, we may use the form of $\mathbf{S}_a$ as the covariant derivative of $\mathbf{c}_a$, Eq.(195) with a suitable choice of c_{ab}. In [6] it is shown that c_{ab} is symmetric and divergence free. While the interpretation given in [6] of a phenomenological energy tensor is consistent, it is at odds with the more fundamental interpretation of sources given here. Instead, we identify c_{ab} as proportional to the Minkowski metric–the only invariant, symmetric tensor available. It is also divergence free with respect to the compatible connection, since $D_{(\alpha,x)}\eta_{ab} = 0$. However, as noted in the previous Section the fully biconformal-covariant derivative of η_{ab} does not necessarily vanish. Since by Eq.(195) the co-torsion is given by the full biconformal derivative of c_{ab}, the identification $c_{ab} = \Lambda_0 \eta_{ab}$, implies

$$\mathbf{S}_a = 2\left(1 + \chi\right) \Delta_{ea}^{bc} y_b \Lambda_0 \eta_{cd} \mathbf{e}^d \wedge \mathbf{e}^e$$

thereby avoiding the Triviality Theorem. This residual form of the co-torsion may now be combined into the right hand side of Eq.(196).

Combining these observations, on the $\mathbf{h}_a = 0$ spacetime submanifold with $c_{ab} = \Lambda_0 \eta_{ab}$, setting $\phi_{,a} = y_a^0$, and using $\mathbf{D}_{(\alpha,x)}\left(\Lambda_0 \eta_{ab}\right) = 0$, it follows that

$$\begin{aligned}
0 &= \frac{1}{1+\chi}\phi_{,b}\mathbf{R}^b{}_a - \mathbf{D}_{(\alpha,x)}\left(\frac{1}{1+\chi}\boldsymbol{\mathcal{R}}_a + \frac{\kappa}{\alpha}\mathbf{T}_a + \Lambda_0 \eta_{ab}\mathbf{e}^b\right) \\
&\quad + 2\Delta_{ca}^{db}\phi_{,d}\left(\frac{1}{1+\chi}\boldsymbol{\mathcal{R}}_b + \frac{\kappa}{\alpha}\mathbf{T}_b + \Lambda_0 \eta_{be}\mathbf{e}^e\right) \wedge \mathbf{e}^c
\end{aligned} \tag{199}$$

This is now the condition for the existence of a conformal transformation such that

$$\frac{1}{1+\chi}\boldsymbol{\mathcal{R}}_b + \Lambda_0 \eta_{be}\mathbf{e}^e \;=\; -\frac{\kappa}{\alpha}\mathbf{T}_b$$

Expressed in terms of the Einstein tensor,

$$G_{ab} + \Lambda_C \eta_{ab} \;=\; -(n-2)\frac{\kappa\,(1+\chi)}{\alpha}\,(T_{ab} - \eta_{ab}T) \qquad (200)$$

where the net effect of c_{ab} is a cosmological constant, $\Lambda_C = -(n-1)(n-2)(1+\chi)\Lambda_0$.

If we make the conformal transformation that produces Eq.(200), the co-torsion equation (199) reduces to $\phi_{,b}\mathbf{R}^b{}_a = 0$. In generic spacetimes this requires $W_\mu = y^0_\mu = 0$.

XIII. THE SOURCE FOR GRAVITY

The reduction of sources forced by coupling to gravity

The necessary source constraints from the gravitational couplings, Eqs.(177) and (182), may be written as

$$0 \ = \ \mathcal{H}^i{}_{ac}\mathcal{H}_{i\,bd}\eta^{cd} + \left(\mathcal{H}^i{}_{ac}F_{i\,bd} + \mathcal{H}^i{}_{bc}F_{i\,ad}\right)\eta^{cd} + 3\mathcal{G}^i{}_{ac}\mathcal{G}_{i\,bd}\eta^{cd} \qquad (201)$$

$$0 \ = \ \mathcal{G}^{i\,ca}\left(F^i{}_{cb} + \mathcal{H}^i{}_{cb}\right) - \frac{1}{n}\delta^a{}_b\mathcal{G}^{i\,cd}\left(F^i{}_{cd} + \mathcal{H}^i{}_{cd}\right) \qquad (202)$$

where the full contraction of the second, $\mathcal{G}^{i\,cd}\left(F^i{}_{cd} + \mathcal{H}^i{}_{cd}\right)$, is constant.

These conditions must continue to hold for small physical variations of the independent potentials, $A^i{}_a$ and B^{ia}. We may imagine two nearby solutions differing only in one or both of the potentials and look at their difference. The change in $F^i{}_{ab}$ as we change $A^i{}_a$ is given by

$$\begin{aligned} \delta F^i{}_{ab} \ &= \ \delta A^i{}_{b;a} - \delta A^i{}_{a;b} - c^i{}_{jk}\delta A^j{}_a A^k{}_b - c^i{}_{jk}A^j{}_a\delta A^k{}_b \\ &= \ \left(\delta A^i{}_{b;a} - \alpha^i{}_{ka}\delta A^k{}_b\right) - \left(\delta A^i{}_{a;b} - \alpha^i{}_{jb}\delta A^j{}_a\right) \\ &= \ \mathcal{D}_a\left(\delta A^i{}_b\right) - \mathcal{D}_b\left(\delta A^i{}_a\right) \end{aligned}$$

where $\mathcal{D}_a$ is covariant with respect to local Lorentz, dilatational and $SU\left(N\right)$ transformations. Similarly we find for $G^{i\,a}{}_b$ and $H^{i\,ab}$,

$$\begin{aligned} \delta_A G^{i\,a}{}_b \ &= \ \mathcal{D}^a\left(\delta A^i{}_b\right) \\ \delta_A H^{i\,ab} \ &= \ 0 \end{aligned}$$

Of course, under changes of gauge, these fields are invariant.

The conditions (201) and (202) must continue to hold throughout such small changes. Substituting these variations into the first constraint,

$$0 = \delta F^i{}_{ed} \left(\mathcal{H}_{i\,ac} \delta^e_b + \mathcal{H}_{i\,bc} \delta^e_a \right) \eta^{cd} + 3 \left(\eta_{ae} \delta G^{i\,e}{}_c \mathcal{G}_{i\,bd} + \mathcal{G}_{i\,ac} \eta_{be} \delta G^{i\,e}{}_d \right) \eta^{cd} \qquad (203)$$

The first term of Eq.(201) has dropped out because $H^i{}^{ab}$ is independent of $A^i{}_a$.

Now we expand Eq.(203) in terms of the variation $\delta A^i{}_a$ and its derivatives,

$$\begin{aligned}
0 \;=\; & \delta A^i{}_{d,e} \left(\mathcal{H}_{i\,ac} \delta^e_b \eta^{cd} + \mathcal{H}_{i\,bc} \delta^e_a \eta^{cd} - \mathcal{H}_{i\,ac} \delta^d_b \eta^{ce} - \mathcal{H}_{i\,bc} \delta^d_a \eta^{ce} \right) \\
& + \delta A^i{}_c {}^{,e} 3 \left(\eta_{ae} \mathcal{G}_{i\,bd} + \mathcal{G}_{i\,ad} \eta_{be} \right) \eta^{cd} \\
& + \delta A^k{}_f \left(-\omega^f{}_{de} + W_e \delta^f_d + \alpha^i{}_{kd} \delta^f_e \right) \\
& \times \left(\mathcal{H}_{k\,ac} \delta^e_b \eta^{cd} + \mathcal{H}_{k\,bc} \delta^e_a \eta^{cd} - \mathcal{H}_{k\,ac} \delta^d_b \eta^{ce} - \mathcal{H}_{k\,bc} \delta^d_a \eta^{ce} \right) \\
& + \delta A^k{}_f \left(3 c^i{}_{kj} B^{j\,e} \right) \left(\eta_{ae} \mathcal{G}_{i\,bd} \eta^{fd} + \mathcal{G}_{i\,ad} \eta_{be} \eta^{fd} \right)
\end{aligned}$$

where we collect terms proportional to $\delta A^k{}_f, \delta A^k{}_{f,e}$ and $\delta A^k{}_f {}^{,e}$ separately, noting that the gravitational solution reduces the y-covariant derivative to a partial, $\delta A^i{}_b {}^{;a} = \delta A^i{}_b {}^{,a}$.

While the field equations determine the second derivatives of the potentials, the potential itself and its first derivative are arbitrary initial conditions on any Cauchy surface. Therefore, the three variations $\delta A^k{}_f, \delta A^k{}_{f,e}$ and $\delta A^k{}_f {}^{,e}$ are independent,

and the coefficient of each must vanish separately:

$$0 = \mathcal{H}_{i\,ac}\delta_b^e\eta^{cd} + \mathcal{H}_{i\,bc}\delta_a^e\eta^{cd} - \mathcal{H}_{i\,ac}\delta_b^d\eta^{ce} - \mathcal{H}_{i\,bc}\delta_a^d\eta^{ce} \tag{204}$$

$$0 = 3\left(\eta_{ae}\mathcal{G}_{i\,bd} + \mathcal{G}_{i\,ad}\eta_{be}\right)\eta^{cd} \tag{205}$$

$$\begin{aligned}
0 = &\left(-\omega^f_{\ de} + W_e\delta_d^f + \alpha^i_{\ kd}\delta_e^f\right) \\
&\times \left(\mathcal{H}_{k\,ac}\delta_b^e\eta^{cd} + \mathcal{H}_{k\,bc}\delta_a^e\eta^{cd} - \mathcal{H}_{k\,ac}\delta_b^d\eta^{ce} - \mathcal{H}_{k\,bc}\delta_a^d\eta^{ce}\right) \\
&+ \left(3c^i_{\ kj}B^{j\,e}\right)\left(\eta_{ae}\mathcal{G}_{i\,bd}\eta^{fd} + \mathcal{G}_{i\,ad}\eta_{be}\eta^{fd}\right)
\end{aligned} \tag{206}$$

For the x-derivative part of the constraint, Eq.(204), we contract eb and lower the d index to show that $\mathcal{H}_{i\,ac}$ must vanish,

$$0 = n\mathcal{H}_{i\,ac}$$

Similarly, contracting ac in Eq.(205) expressing the independence of the y-derivative, shows that $\mathcal{G}_{i\,be}$ must also vanish.

$$0 = 3\mathcal{G}_{i\,be}$$

With these two conditions, the final equation Eq.(206) is identically satisfied.

These conditions satisfy both gravitational conditions on the sources, Eqs.(177) and (182).

The source for gravity

We have shown that

$$\frac{1}{1+\chi}\mathcal{R}_b + \Lambda_0\eta_{be}\mathbf{e}^e = -\frac{\kappa}{\alpha}\mathbf{T}_b$$

Expressed in terms of the Einstein tensor this is

$$G_{ab} + \Lambda_C \eta_{ab} \;\; = \;\; -\left(n-2\right)\left(1+\chi\right)\frac{\kappa}{\alpha}\left(T_{ab} - \eta_{ab}T\right)$$

where the vanishing of $\mathcal{H}_{i\,ab}$ and $\mathcal{G}_{i\,ab}$ leave us with

$$T_{ab} \;\; = \;\; F^i{}_{ca}F_{i\,db}\eta^{cd}$$

The trace of this is well-known to be gauge dependent, and the conformal symmetry requires the energy tensor to be trace free. Therefore, we are justified in adjusting the $SU\left(N\right)$ gauge to give

$$G_{ab} + \Lambda_C \eta_{ab} \;\; = \;\; -\lambda\left(F^i{}_{ca}F_{i\,db}\eta^{cd} - \frac{1}{4}\eta_{ab}\left(\eta^{ce}\eta^{df}F^i{}_{cd}F_{i\,ef}\right)\right)$$

where

$$\lambda = \left(n-2\right)\left(1+\chi\right)\frac{\kappa}{\alpha}$$

The Yang-Mills equation

With $\mathcal{H}^k{}_{ab} = G^{i\,a}{}_b = 0$, Eqs.(146) and (147) reduce to

$$0 \;\; = \;\; \eta^{bc}F_{k\,ac}{}^{;a} + \eta^{bc}F_{i\,ac}\beta^i{}_k{}^a$$

$$0 \;\; = \;\; \eta^{bc}\left(F_{j\,ac;b} + F_{i\,ac}\alpha^i{}_{jb}\right)$$

where

$$\boldsymbol{\mathcal{A}}^i{}_k \;\; = \;\; c^i{}_{jk}A^j{}_a\mathbf{e}^a + c^i{}_{jk}B^{j\,a}\mathbf{f}_a$$

$$= \;\; \boldsymbol{\alpha}^i{}_k + \boldsymbol{\beta}^i{}_k$$

We may use these results and the form of the co-torsion,

$$\mathbf{S}_a = 2\left(1+\chi\right)\Delta^{bc}_{ea}y_b\Lambda\eta_{cd}\mathbf{e}^d\wedge\mathbf{e}^e$$

to solve for the potentials,

$$
\begin{aligned}
F^i{}_{ab} &= A^i{}_{b;a} - A^i{}_{a;b} - c^i{}_{jk}A^j{}_a A^k{}_b + \left(1+\chi\right)\Lambda B^{i\,c}\left(\Delta^{gf}_{bc}y_g\eta_{fa} - \Delta^{gf}_{ac}y_g\eta_{fb}\right)\\
0 &= A^i{}_b{}^{,a} - B^i{}_{\,;b}^{\,a} - c^i{}_{jk}B^{j\,a}A^k{}_b\\
0 &= B^{i\;b,a} - B^{i\;a,b} - c^i{}_{jk}B^{j\,a}B^{k\,b}
\end{aligned}
\tag{207}
$$

The third equation is the vanishing of the Yang-Mills field strength on the y-submanifold,

$$\mathbf{d}_{(y)}\mathbf{B}^i = -\frac{1}{2}c^i{}_{jk}\mathbf{B}^j\wedge\mathbf{B}^k$$

so that $\mathbf{B}^k$ is a pure-gauge connection for any fixed x^α. Therefore, for each x_0^α we may choose an $SU\left(N\right)$ gauge $\Lambda\left(x_0^\alpha,y_\beta\right)$ such that $\mathbf{B}^k = 0$. But this makes the value of $\mathbf{B}^k$ independent of x^α as well, so $\mathbf{B}^k = 0$ everywhere. As a result, the fields in terms of the potentials reduce to

$$
\begin{aligned}
F^i{}_{ab} &= A^i{}_{b;a} - A^i{}_{a;b} - c^i{}_{jk}A^j{}_a A^k{}_b\\
A^i{}_b{}^{,a} &= 0
\end{aligned}
$$

Now, when we write the field equations in terms of the potentials and set $B^{k\,a} = 0$, we have

$$
\begin{aligned}
0 &= F_{k\,ab}{}^{,a}\\
0 &= \eta^{bc}\left(F_{j\,ac;b} + F_{i\,ac}c^i{}_{jk}A^j{}_a\right)
\end{aligned}
$$

The first shows that $F_{k\ ab}$ is independent of y_α and the second shows it to be covariantly divergence free.

XIV. CONCLUSIONS

Gravitational field theories in doubled dimensions include biconformal gravity [14, 32, 15, 6], double field theory [22, 23, 24, 21, 42, 44], and gravity on a Kähler manifold [16, 6]. Each of these cases starts as a fully $2n$-dimensional theory but ultimately is intended to describe gravity on an n-dimensional submanifold. We have found a satisfactory $2n$-dimensional form of Yang-Mills matter sources and shown that they also reduce to the expected n-dimensional sources *as a consequence of the field equations*. Our gravitational reduction and the consequent reduction of the Yang-Mills fields and field equations *does* **not** *require a section condition*.

While we discussed the issue in biconformal space, our results hold in the related forms of double field theory and Kähler manifolds [16, 6].

For matter fields we restrict our attention to gauged $SU(N)$ sources (Yang-Mills type). While we find that the usual form of $2n$-dimensional Yang-Mills action gives incorrect coupling to gravity, including a "twist" matrix in the action corrects the problem.

For the gravitational fields we use the most general action linear in the biconformal curvatures. The variation is taken with respect to all $\frac{(n+1)(n+2)}{2}$ conformal gauge fields. In the absence of sources, the use of the gravitational field equations to reduce fully $2n$-dimensional gravity theory to dependence only on the fields of n-dimensional gravity is well established. The field equations of torsion-free biconformal space restrict the $\frac{1}{2}(n+1)(n+2)$ curvature components, each initially dependent on $2n$ independent coordinates, to the usual locally scale covariant Riemannian curvature tensor in n dimensions. Ultimately, the n-dim solder form determines all fields, up to coordinate and gauge transformations. Generic, torsion-free, vacuum solutions describe n-dimensional scale-covariant general relativity on the co-tangent bundle.

Here we have shown that the same reduction occurs when gauged $SU(N)$ field

strengths are included as matter sources. The result goes well beyond any previous work. With two exceptions [41, 29] studies of biconformal spaces [32, 15, 29, 17, 18, 16, 19, 6, 31, 20] have considered the pure gravity biconformal spaces, leading to vacuum general relativity. With $SU(N)$ Yang-Mills fields as gravitational sources, the central issue was to show that a completely general $SU(N)$ gauge theory over a $2n$-dimensional biconformal space does not disrupt the gravitational reduction to general relativity, but rather itself reduces to a suitable n-dim gravitational source and Yang-Mills field equation.

As with the Riemann-Cartan construction of general relativity above, the development of biconformal spaces from group symmetry made it straightforward to include the additional symmetry of sources. By extending the quotient to

$$\mathcal{M}^{2n} = [SO(p+1, q+1) \times SU(N)] / [SO(p, q) \times SO(1, 1) \times SU(N)]$$

the local symmetry is enlarged by $SU(N)$. We considered the effects of adding an $SU(N)$ action to the gravitational action Eq.(106). As central results we successfully showed:

1. The number of field components in $2n$ dimensions reduces by a factor of $\frac{n-2}{2(2n-1)}$ to the expected number $\frac{n(n-1)}{2}(N^2 - 1)$ on n-dimensional spacetime.

2. The functional dependence of the fields reduces from $2n$ to n independent variables.

3. The usual form of Yang-Mills stress-energy tensor provides the source for the scale-covariant Einstein equation on n-dimensional spacetime.

4. The usual Yang-Mills field equation holds on the spacetime submanifold.

To accomplish these goals we required two interdependent intermediate steps:

1. We considered alternate forms of $2n$-dimensional Yang-Mills action, showing that the usual action, $S^0_{YM} = \int tr \left(\mathbf{F} \wedge^* \mathbf{F}\right)$ gives nonstandard coupling to gravity. Instead, including a "twist" matrix in the action $S_{YM} = \int tr \left(\bar{\mathbf{F}} \wedge^* \mathbf{F}\right)$ with twisted form

$$\bar{\mathcal{F}}_{AB} = \frac{1}{2} \left(K_A{}^C \mathcal{F}_{CB} + \mathcal{F}_{AC} K^C{}_B \right)$$

leads to both the usual n-dimensional Yang-Mills source to the Einstein tensor and the usual Yang-Mills equation for the $SU(N)$ fields. A similar twist has been found in other double field theory studies [25, 27, 28] in order to enforce supersymmetry. Here, the twist is required for the bosonic fields alone. Interestingly, the twist matrix $K^A{}_B = \bar{K}^{AC} g_{CB}$ makes use of both the Kähler and Killing forms, g_{AB} and K_{AB}, respectively.

2. We considered two naturally occurring inner products for the orthonormal frame fields: the restriction to the base manifold of the Killing form, and the Kähler metric. We showed the Kähler form cannot lead to the usual field equations while the variation of the Killing form in the twisted action gives usual Yang-Mills equations and usual coupling to gravity. Previous results in biconformal gravity did not require the inner product.

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APPENDICES

Appendix A: Failure of the Kähler inner product

In this Appendix, we show that using the Kähler metric to define orthonormality of the solder and co-solder forms cannot lead to the usual Yang-Mills gravitational coupling to gravity, for either the usual Yang-Mills action or the twisted Yang-Mills action.

We first find the Hodge dual of the Yang-Mills field here using the inverse Kähler metric $\bar{g}^{AB}$. In general terms the Hodge dual of a 2-form is given by Eq.(126), where we now substitute the Kähler metric,

$$
\begin{aligned}
{}^{*}\boldsymbol{\mathcal{F}} \;=\;& \frac{1}{n!\,(n-2)!}\left(\frac{1}{2}F_{ab}\bar{g}^{am}\bar{g}^{bn} + G^{a}{}_{b}\bar{g}_{a}{}^{m}\bar{g}^{bn} + \frac{1}{2}H^{ab}\bar{g}_{a}{}^{m}\bar{g}_{b}{}^{n}\right) \\
& \times \varepsilon^{c\cdots d}{}_{mne\cdots f}\mathbf{f}_{c\cdots d}\mathbf{e}^{e\cdots f} \\
& + \frac{(-1)^{n-1}}{(n-1)!\,(n-1)!}\left(\frac{1}{2}F_{ab}\bar{g}^{a}{}_{m}\bar{g}^{bn} + G^{a}{}_{b}\bar{g}_{am}\bar{g}^{bn} + \frac{1}{2}H^{ab}\bar{g}_{am}\bar{g}_{b}{}^{n}\right) \\
& \times \varepsilon^{mc\cdots d}{}_{ne\cdots f}\mathbf{f}_{c\cdots d}\mathbf{e}^{e\cdots f} \\
& + \frac{(-1)^{n}}{(n-1)!\,(n-1)!}\left(\frac{1}{2}F_{ab}\bar{g}^{am}\bar{g}^{b}{}_{n} + G^{a}{}_{b}\bar{g}_{a}{}^{m}\bar{g}^{b}{}_{n} + \frac{1}{2}H^{ab}\bar{g}_{a}{}^{m}\bar{g}_{bn}\right) \\
& \times \varepsilon^{nc\cdots d}{}_{me\cdots f}\mathbf{f}_{c\cdots d}\mathbf{e}^{e\cdots f} \\
& + \frac{1}{n!\,(n-2)!}\left(\frac{1}{2}F_{ab}\bar{g}^{a}{}_{m}\bar{g}^{b}{}_{n} + G^{a}{}_{b}\bar{g}_{am}\bar{g}^{b}{}_{n} + \frac{1}{2}H^{ab}\bar{g}_{am}\bar{g}_{bn}\right) \\
& \times \varepsilon^{mnc\cdots d}{}_{e\cdots f}\mathbf{f}_{c\cdots d}\mathbf{e}^{e\cdots f}
\end{aligned}
$$

Forming the usual Yang-Mills Lagrangian density as the wedge product, $\boldsymbol{\mathcal{F}} \wedge {}^{*}\boldsymbol{\mathcal{F}}$, and eliminating the basis forms in favor of the volume form $\boldsymbol{\Phi}$ yields

$$\boldsymbol{\mathcal{F}} \wedge {}^*\boldsymbol{\mathcal{F}} \;=\; \left(\frac{1}{2} F_{mn} \bar{g}^{am} \bar{g}^{bn} + G^m{}_n \bar{g}^a{}_m \bar{g}^{bn} + \frac{1}{2} H^{mn} \bar{g}^a{}_m \bar{g}^b{}_n \right) F_{ab} \boldsymbol{\Phi}$$
$$+ \left(F_{mn} \bar{g}_a{}^m \bar{g}^{bn} + G^m{}_n \left(\bar{g}_{am} \bar{g}^{bn} - \bar{g}_a{}^n \bar{g}^b{}_m \right) + H^{mn} \bar{g}_{am} \bar{g}^b{}_n \right) G^a{}_b \boldsymbol{\Phi}$$
$$+ \left(\frac{1}{2} F_{mn} \bar{g}_a{}^m \bar{g}_b{}^n + G^m{}_n \bar{g}_{am} \bar{g}_b{}^n + \frac{1}{2} H^{mn} \bar{g}_{am} \bar{g}_{bn} \right) H^{ab} \boldsymbol{\Phi} \qquad (208)$$

For the diagonal form of the Kähler metric, Eq.(115), this Lagrange density reduces to

$$\boldsymbol{\mathcal{F}} \wedge {}^*\boldsymbol{\mathcal{F}} \;=\; \left(\frac{1}{2} \eta^{am} \eta^{bn} F_{ab} F_{mn} + \eta_{am} \eta^{bn} G^m{}_n G^a{}_b + \frac{1}{2} \eta_{am} \eta_{bn} H^{mn} H^{ab} \right) \boldsymbol{\Phi} \quad (209)$$

Varying the potentials in Eq.(209) yields the usual Yang-Mills field equation. However, metric variation of Eq.(208) gives a nonstandard coupling to gravity. Varying and reducing the integral of Eq.(208), the resulting gravitational field equations are:

$$\alpha \left(\Omega^n{}_b{}^b{}_m - \Omega^a{}_b{}^b{}_a \delta^n_m \right)$$
$$+ \beta \left(\Omega^n{}_m - \Omega^a{}_a \delta^n_m \right) + \Lambda \delta^n_m \;=\; - \left(2 \eta^{be} F_{me} F_{ab} + 2 G^d{}_a G^e{}_m \eta_{de} \right) \eta^{an} \qquad (210)$$
$$\alpha \left(\Omega^a{}_n{}^m{}_a - \Omega^a{}_b{}^b{}_a \delta^m_n \right)$$
$$+ \beta \left(\Omega^m{}_n - \Omega^a{}_a \delta^m_n \right) + \Lambda \delta^m_n \;=\; - \left(2 G^d{}_c G^n{}_b \eta_{dm} \eta^{bc} + 2 \eta_{bc} \eta_{dm} H^{dc} H^{nb} \right) \;(211)$$
$$\alpha \Omega^a{}_{nam} + \beta \Omega_{nm} \;=\; \left(2 \eta^{ca} G^b{}_a F_{mc} + 2 \eta_{ac} G^c{}_m H^{ab} \right) \eta_{bn} \qquad (212)$$
$$\alpha \Omega^n{}_b{}^{bm} + \beta \Omega^{nm} \;=\; - \left(2 \eta^{ca} G^m{}_a F_{bc} + 2 \eta_{ac} G^c{}_b H^{am} \right) \eta^{nb} \qquad (213)$$

The remaining four field equations involving the torsion and co-torsion are unchanged.

Notice that the sources on the right sides of Eqs.(212) and (213) differ only in the overall sign. This means that both the spacetime curvature, $\Omega^a{}_{nam}$, and the momentum space curvature, $\Omega^n{}_b{}^{bm}$, are driven with equal strength. Thus, if spacetime

curvature is nonzero, the momentum space must also be correspondingly curved. In torsion-free solutions the left hand side of Eq.(213) vanishes independently of the sources, implying a constraint on the source fields. Thus, for the $\int \boldsymbol{\mathcal{F}} \wedge^* \boldsymbol{\mathcal{F}}$ source and the Kähler case,

$$\eta^{ca} G^m{}_a F_{bc} + \eta_{ac} G^c{}_b H^{am} = 0$$

and this immediately shows that the source for the Einstein equation, Eq.(212), vanishes.

The necessity for equal spacetime and momentum curvatures suggests the possibility implementing Born reciprocity. This idea will be explored elsewhere. However, momentum curvature also requires some part of the torsion to be nonvanishing, and this in turn requires a different gravitational solution than that known to reproduce general relativity. Thus, we cannot maintain vanishing torsion without forcing the spacetime source to vanish.

In addition to the inescapability of torsion and momentum space curvature, the independence of the sources to Eqs.(210) and (211) also raises issues with the Kähler variation, because the method of solution employed in [6] makes use of the near identity of these two equations. At the very least, an entirely different form of reduction of the equations would be required, with no guarantee that the Einstein equation would emerge.

A similar calculation shows that the same difficulties arise from the twisted form of the action using the Kähler metric.

These issues do not arise with the Killing variation, Eqs.(122)–the source for the spacetime curvature and momentum space curvature are independent while remaining two variations are identical. The use of the Killing form as metric also makes good geometric sense, since it arises directly as a symmetric form in the Lie algebra and

thus as metric of the co-tangent space. As such, it respects the conformal invariance of the full model. The Kähler structure, by contrast, reflects symmetries and dynamical properties within the conformal group and depend for their existence on the solution on the biconformal space. It is not conformally invariant.

Appendix B: Symmetry of the twist matrix

Writing the twist matrix while keeping factors of η^{ab} explicit, we have

$$
K^A{}_B = \begin{pmatrix} \bar{K}^{ac} & \bar{K}^a{}_e \eta^{ec} \\ \eta^{ae} \bar{K}_e{}^c & \eta^{ae} \eta^{cf} \bar{K}_{ef} \end{pmatrix} \begin{pmatrix} \eta_{cb} & 0 \\ 0 & \eta_{cm} \eta_{bn} \eta^{mn} \end{pmatrix}
$$

$$
= \begin{pmatrix} \bar{K}^{ac} \eta_{cb} & \bar{K}^a{}_b \\ \eta^{ae} \bar{K}_e{}^c \eta_{cb} & \eta^{ae} \bar{K}_{eb} \end{pmatrix}
$$

$$
K_A{}^B = \begin{pmatrix} \eta_{ad} \bar{K}^{db} & \bar{K}_a{}^b \\ \eta_{ad} \bar{K}^d{}_e \eta^{eb} & \bar{K}_{ae} \eta^{eb} \end{pmatrix}
$$

Symmetry of $K^A{}_B$ follows from the symmetry of the Killing and Kähler forms, $K_{AB} = K_{BA}, g_{AB} = g_{BA}$:

$$
K^A{}_B \equiv \bar{K}^{AC} g_{CB} = g_{BC} \bar{K}^{CA} = K_B{}^A
$$

Comparing the expressions,

$$
K^A{}_B = K_B{}^A
$$

$$
\begin{pmatrix} \bar{K}^{ac} \eta_{cb} & \bar{K}^a{}_b \\ \eta^{ae} \bar{K}_e{}^c \eta_{cb} & \eta^{ae} \bar{K}_{eb} \end{pmatrix} = \begin{pmatrix} \eta_{bc} \bar{K}^{ca} & \eta_{bc} \bar{K}^c{}_e \eta^{ea} \\ \bar{K}_b{}^a & \bar{K}_{bc} \eta^{ca} \end{pmatrix}
$$

Therefore, from the upper right quadrant we must have $\bar{K}^a{}_b = \eta_{bc}\bar{K}^c{}_e\eta^{ea}$ and similarly $\eta^{ae}\bar{K}_e{}^c\eta_{cb} = \bar{K}_b{}^a$ from the lower left. However we also have symmetry $\bar{K}^{AB} = \bar{K}^{BA}$ of $\bar{K}^{AB}$ itself:

$$\bar{K}^{AB} = \begin{pmatrix} \bar{K}^{ab} & \bar{K}^a{}_e\eta^{eb} \\ \eta^{ae}\bar{K}_e{}^b & \eta^{ae}\eta^{bf}\bar{K}_{ef} \end{pmatrix}$$

$$\left[\bar{K}^t\right]^{BA} = \begin{pmatrix} \bar{K}^{ba} & \eta^{be}\bar{K}_e{}^a \\ \bar{K}^b{}_e\eta^{ea} & \eta^{ae}\eta^{bf}\bar{K}_{ef} \end{pmatrix}$$

This shows that $\bar{K}^a{}_e\eta^{eb} = \eta^{be}\bar{K}_e{}^a$ from which it follows that

$$\bar{K}^a{}_b = \bar{K}_b{}^a$$

Combine this with $\eta^{ae}\bar{K}_e{}^c\eta_{cb} = \bar{K}_b{}^a$ and we see that all forms are equivalent,

$$\eta^{ae}\bar{K}_e{}^c\eta_{cb} = \bar{K}_b{}^a = \bar{K}^a{}_b = \eta_{bc}\bar{K}^c{}_e\eta^{ea}$$

With this, we may write

$$K^A{}_B = \begin{pmatrix} \bar{K}^{ac}\eta_{cb} & \bar{K}^a{}_b \\ \bar{K}^a{}_b & \eta^{ae}\bar{K}_{eb} \end{pmatrix}$$

$$K_B{}^A = \begin{pmatrix} \eta_{bd}\bar{K}^{da} & \bar{K}_b{}^a \\ \bar{K}_b{}^a & \bar{K}_{be}\eta^{ea} \end{pmatrix} = K^A{}_B$$

Appendix C: Details of the metric variation of the action

The variation of the twisted action is lengthy and includes some subtleties, so we include details here.

We have the dual field,

$$
\begin{aligned}
{}^{*}\boldsymbol{\mathcal{F}} \;=\;& \frac{1}{n!\,(n-2)!}\left(\frac{1}{2}F_{ab}\bar{K}^{am}\bar{K}^{bn}+\mathcal{G}_{gb}\eta^{ga}\bar{K}_{a}{}^{m}\bar{K}^{bn}+\frac{1}{2}\mathcal{H}_{gh}\eta^{ga}\bar{K}_{a}{}^{m}\eta^{hb}\bar{K}_{b}{}^{n}\right)\\
&\times\varepsilon^{c\cdots d}{}_{mne\cdots f}\mathbf{f}_{c\cdots d}\mathbf{e}^{e\cdots f}\\
&+\frac{(-1)^{n-1}}{(n-1)!\,(n-1)!}\left(\frac{1}{2}F_{ab}\bar{K}^{a}{}_{m}\bar{K}^{bn}+\mathcal{G}_{gb}\eta^{ga}\bar{K}_{am}\bar{K}^{bn}+\frac{1}{2}\mathcal{H}_{gh}\eta^{ga}\eta^{hb}\bar{K}_{am}\bar{K}_{b}{}^{n}\right)\\
&\times\varepsilon^{mc\cdots d}{}_{ne\cdots f}\mathbf{f}_{c\cdots d}\mathbf{e}^{e\cdots f}\\
&+\frac{(-1)^{n}}{(n-1)!\,(n-1)!}\left(\frac{1}{2}F_{ab}\bar{K}^{am}\bar{K}^{b}{}_{n}+\mathcal{G}_{gb}\eta^{ga}\bar{K}_{a}{}^{m}\bar{K}^{b}{}_{n}+\frac{1}{2}\mathcal{H}_{gh}\eta^{ga}\eta^{hb}\bar{K}_{a}{}^{m}\bar{K}_{bn}\right)\\
&\times\varepsilon^{nc\cdots d}{}_{me\cdots f}\mathbf{f}_{c\cdots d}\mathbf{e}^{e\cdots f}\\
&+\frac{1}{n!\,(n-2)!}\left(\frac{1}{2}F_{ab}\bar{K}^{a}{}_{m}\bar{K}^{b}{}_{n}+\mathcal{G}_{gb}\eta^{ga}\bar{K}_{am}\bar{K}^{b}{}_{n}+\frac{1}{2}\mathcal{H}_{gh}\eta^{ga}\eta^{hb}\bar{K}_{am}\bar{K}_{bn}\right)\\
&\times\varepsilon^{mnc\cdots d}{}_{e\cdots f}\mathbf{f}_{c\cdots d}\mathbf{e}^{e\cdots f}
\end{aligned}
$$

and, changing indices to avoid duplication, the barred field,

$$
\begin{aligned}
\bar{\boldsymbol{\mathcal{F}}} \;=\;& \frac{1}{2}\left(F_{rq}\bar{K}^{qt}\eta_{ts}+\bar{K}_{r}{}^{q}\mathcal{G}_{qs}\right)\mathbf{e}^{r}\wedge\mathbf{e}^{s}\\
&+\frac{1}{2}\left(F_{rc}\bar{K}^{c}{}_{s}+\bar{K}_{r}{}^{c}\mathcal{H}_{cs}\right)\eta^{sw}\mathbf{e}^{r}\wedge\mathbf{f}_{w}\\
&+\frac{1}{2}\left(-\mathcal{G}_{sq}\bar{K}^{qt}\eta_{tr}-\mathcal{G}_{qr}\eta^{qt}\bar{K}_{ts}\right)\eta^{sw}\mathbf{e}^{r}\wedge\mathbf{f}_{w}\\
&+\frac{1}{2}\left(\mathcal{G}_{rq}\bar{K}^{q}{}_{s}+\mathcal{H}_{rq}\eta^{qt}\bar{K}_{ts}\right)\eta^{rw}\eta^{sx}\mathbf{f}_{w}\wedge\mathbf{f}_{x}
\end{aligned}
$$

We must wedge these together, then vary the metric.

Wedging,

$$
\begin{aligned}
\bar{\mathcal{F}} \wedge {}^*\mathcal{F} \;=\; & \frac{1}{2}\frac{1}{n!\,(n-2)!}\left(F_{rq}\bar{K}^{qt}\eta_{ts} + \bar{K}_r{}^q\mathcal{G}_{qs}\right) \\
& \times \left(\mathcal{G}_{gb}\eta^{ga}\bar{K}_a{}^m\bar{K}^{bn} + \frac{1}{2}\mathcal{H}_{gh}\eta^{ga}\bar{K}_a{}^m\eta^{hb}\bar{K}_b{}^n\right)\varepsilon^{c\cdots d}{}_{mne\cdots f}\mathbf{f}_{c\cdots d}\mathbf{e}^{rse\cdots f} \\
& -\frac{1}{2}\frac{1}{(n-1)!\,(n-1)!}\left(F_{rc}\bar{K}^c{}_s + \bar{K}_r{}^c\mathcal{H}_{cs}\right)\eta^{sw} \\
& \times \left(\frac{1}{2}F_{ab}\bar{K}^a{}_m\bar{K}^{bn} + \frac{1}{2}\mathcal{H}_{gh}\eta^{ga}\eta^{hb}\bar{K}_{am}\bar{K}_b{}^n\right)\varepsilon^{mc\cdots d}{}_{ne\cdots f}\mathbf{f}_{wc\cdots d}\mathbf{e}^{re\cdots f} \\
& +\frac{1}{2}\frac{1}{(n-1)!\,(n-1)!}\left(F_{rc}\bar{K}^c{}_s + \bar{K}_r{}^c\mathcal{H}_{cs}\right)\eta^{sw} \\
& \times \left(\frac{1}{2}F_{ab}\bar{K}^{am}\bar{K}^b{}_n + \mathcal{G}_{gb}\eta^{ga}\bar{K}_a{}^m\bar{K}^b{}_n + \frac{1}{2}\mathcal{H}_{gh}\eta^{ga}\eta^{hb}\bar{K}_a{}^m\bar{K}_{bn}\right) \\
& \times\varepsilon^{nc\cdots d}{}_{me\cdots f}\mathbf{f}_{wc\cdots d}\mathbf{e}^{re\cdots f} \\
& -\frac{1}{2}\frac{1}{(n-1)!\,(n-1)!}\left(-\mathcal{G}_{sq}\bar{K}^{qt}\eta_{tr} - \mathcal{G}_{qr}\eta^{qt}\bar{K}_{ts}\right)\eta^{sw} \\
& \times \left(\frac{1}{2}F_{ab}\bar{K}^a{}_m\bar{K}^{bn} + \frac{1}{2}\mathcal{H}_{gh}\eta^{ga}\eta^{hb}\bar{K}_{am}\bar{K}_b{}^n\right)\varepsilon^{mc\cdots d}{}_{ne\cdots f}\mathbf{f}_{wc\cdots d}\mathbf{e}^{re\cdots f} \\
& +\frac{1}{2}\frac{1}{(n-1)!\,(n-1)!}\left(-\mathcal{G}_{sq}\bar{K}^{qt}\eta_{tr} - \mathcal{G}_{qr}\eta^{qt}\bar{K}_{ts}\right)\eta^{sw} \\
& \times \left(\frac{1}{2}F_{ab}\bar{K}^{am}\bar{K}^b{}_n + \mathcal{G}_{gb}\eta^{ga}\bar{K}_a{}^m\bar{K}^b{}_n + \frac{1}{2}\mathcal{H}_{gh}\eta^{ga}\eta^{hb}\bar{K}_a{}^m\bar{K}_{bn}\right) \\
& \times\varepsilon^{nc\cdots d}{}_{me\cdots f}\mathbf{f}_{wc\cdots d}\mathbf{e}^{re\cdots f} \\
& +\frac{1}{2}\frac{1}{n!\,(n-2)!}\left(\mathcal{G}_{rq}\bar{K}^q{}_s + \mathcal{H}_{rq}\eta^{qt}\bar{K}_{ts}\right)\eta^{rw}\eta^{sx} \\
& \times \left(\frac{1}{2}F_{ab}\bar{K}^a{}_m\bar{K}^b{}_n + \mathcal{G}_{gb}\eta^{ga}\bar{K}_{am}\bar{K}^b{}_n\right)\varepsilon^{mnc\cdots d}{}_{e\cdots f}\mathbf{f}_{wxc\cdots d}\mathbf{e}^{e\cdots f}
\end{aligned}
$$

Now use the relation between the basis forms and the volume element, and the Kronecker reduction of pairs of Levi-Civita tensors,

$$
\begin{aligned}
\mathbf{f}_{c\cdots d} \wedge \mathbf{e}^{e\cdots f} \;=\;& \bar{e}_{c\cdots d}{}^{e\cdots f}\mathbf{\Phi} \\
\varepsilon^{mnc\cdots d}{}_{e\cdots f}\bar{e}_{pqc\cdots d}{}^{e\cdots f} \;=\;& n!\,(n-2)!\left(\delta^m_p\delta^n_q - \delta^n_p\delta^m_q\right) \\
\varepsilon^{mc\cdots d}{}_{ne\cdots f}\bar{e}_{pc\cdots d}{}^{qe\cdots f} \;=\;& (n-1)!\,(n-1)!\,\delta^m_p\delta^q_n
\end{aligned}
$$

to reduce the Lagrange density to

$$
\begin{aligned}
\bar{\boldsymbol{\mathcal{F}}} \wedge {}^{*}\boldsymbol{\mathcal{F}} \;=\;\;
& \frac{1}{2}\left(F_{rq}\bar{K}^{qt}\eta_{ts} + \bar{K}_{r}{}^{q}\mathcal{G}_{qs}\right)\left(\delta^{r}_{m}\delta^{s}_{n} - \delta^{s}_{m}\delta^{r}_{n}\right) \\
& \times \left(\mathcal{G}_{gb}\eta^{ga}\bar{K}_{a}{}^{m}\bar{K}^{bn} + \frac{1}{2}\mathcal{H}_{gh}\eta^{ga}\bar{K}_{a}{}^{m}\eta^{hb}\bar{K}_{b}{}^{n}\right)\boldsymbol{\Phi} \\
& -\frac{1}{2}\left(F_{rc}\bar{K}^{c}{}_{s} + \bar{K}_{r}{}^{c}\mathcal{H}_{cs}\right)\eta^{sw} \\
& \times \left(\frac{1}{2}F_{ab}\bar{K}^{a}{}_{m}\bar{K}^{bn} + \frac{1}{2}\mathcal{H}_{gh}\eta^{ga}\eta^{hb}\bar{K}_{am}\bar{K}_{b}{}^{n}\right)\delta^{m}_{w}\delta^{r}_{n}\boldsymbol{\Phi} \\
& +\frac{1}{2}\left(F_{rc}\bar{K}^{c}{}_{s} + \bar{K}_{r}{}^{c}\mathcal{H}_{cs}\right)\eta^{sw}\delta^{n}_{w}\delta^{r}_{m} \\
& \times \left(\frac{1}{2}F_{ab}\bar{K}^{am}\bar{K}^{b}{}_{n} + \mathcal{G}_{gb}\eta^{ga}\bar{K}_{a}{}^{m}\bar{K}^{b}{}_{n} + \frac{1}{2}\mathcal{H}_{gh}\eta^{ga}\eta^{hb}\bar{K}_{a}{}^{m}\bar{K}_{bn}\right)\boldsymbol{\Phi} \\
& -\frac{1}{2}\left(-\mathcal{G}_{sq}\bar{K}^{qt}\eta_{tr} - \mathcal{G}_{qr}\eta^{qt}\bar{K}_{ts}\right) \\
& \times \eta^{sw}\left(\frac{1}{2}F_{ab}\bar{K}^{a}{}_{m}\bar{K}^{bn} + \frac{1}{2}\mathcal{H}_{gh}\eta^{ga}\eta^{hb}\bar{K}_{am}\bar{K}_{b}{}^{n}\right)\delta^{m}_{w}\delta^{r}_{n}\boldsymbol{\Phi} \\
& +\frac{1}{2}\left(-\mathcal{G}_{sq}\bar{K}^{qt}\eta_{tr} - \mathcal{G}_{qr}\eta^{qt}\bar{K}_{ts}\right)\eta^{sw}\delta^{n}_{w}\delta^{r}_{m} \\
& \times \left(\frac{1}{2}F_{ab}\bar{K}^{am}\bar{K}^{b}{}_{n} + \mathcal{G}_{gb}\eta^{ga}\bar{K}_{a}{}^{m}\bar{K}^{b}{}_{n} + \frac{1}{2}\mathcal{H}_{gh}\eta^{ga}\eta^{hb}\bar{K}_{a}{}^{m}\bar{K}_{bn}\right)\boldsymbol{\Phi} \\
& +\frac{1}{2}\left(\mathcal{G}_{rq}\bar{K}^{q}{}_{s} + \mathcal{H}_{rq}\eta^{qt}\bar{K}_{ts}\right)\left(\delta^{m}_{w}\delta^{n}_{x} - \delta^{n}_{w}\delta^{m}_{x}\right)\eta^{rw}\eta^{sx} \\
& \times \left(\frac{1}{2}F_{ab}\bar{K}^{a}{}_{m}\bar{K}^{b}{}_{n} + \mathcal{G}_{gb}\eta^{ga}\bar{K}_{am}\bar{K}^{b}{}_{n}\right)\boldsymbol{\Phi}
\end{aligned}
$$

Then, absorbing the η_{ab} and δ_b^a factors,

$$
\begin{aligned}
\bar{\mathcal{F}} \wedge {}^*\mathcal{F} \;=\; & \frac{1}{2}\left(F_{mq}\bar{K}^{qt}\eta_{tn} - F_{nq}\bar{K}^{qt}\eta_{tm} + \bar{K}_m{}^q\mathcal{G}_{qn} - \bar{K}_n{}^q\mathcal{G}_{qm}\right) \\
& \times \left(\mathcal{G}_{gb}\eta^{ga}\bar{K}_a{}^m\bar{K}^{bn} + \frac{1}{2}\mathcal{H}_{gh}\eta^{ga}\bar{K}_a{}^m\eta^{hb}\bar{K}_b{}^n\right)\mathbf{\Phi} \\
& -\frac{1}{2}\left(F_{nc}\bar{K}^c{}_s + \bar{K}_n{}^c\mathcal{H}_{cs}\right) \\
& \times \eta^{sm}\left(\frac{1}{2}F_{ab}\bar{K}^a{}_m\bar{K}^{bn} + \frac{1}{2}\mathcal{H}_{gh}\eta^{ga}\eta^{hb}\bar{K}_{am}\bar{K}_b{}^n\right)\mathbf{\Phi} \\
& +\frac{1}{2}\left(F_{mc}\bar{K}^c{}_s + \bar{K}_m{}^c\mathcal{H}_{cs}\right) \\
& \times \eta^{sn}\left(\frac{1}{2}F_{ab}\bar{K}^{am}\bar{K}^b{}_n + \mathcal{G}_{gb}\eta^{ga}\bar{K}_a{}^m\bar{K}^b{}_n + \frac{1}{2}\mathcal{H}_{gh}\eta^{ga}\eta^{hb}\bar{K}_a{}^m\bar{K}_{bn}\right)\mathbf{\Phi} \\
& -\frac{1}{2}\left(-\mathcal{G}_{sq}\bar{K}^{qt}\eta_{tn} - \mathcal{G}_{qn}\eta^{qt}\bar{K}_{ts}\right) \\
& \times \eta^{sm}\left(\frac{1}{2}F_{ab}\bar{K}^a{}_m\bar{K}^{bn} + \frac{1}{2}\mathcal{H}_{gh}\eta^{ga}\eta^{hb}\bar{K}_{am}\bar{K}_b{}^n\right)\mathbf{\Phi} \\
& +\frac{1}{2}\left(-\mathcal{G}_{sq}\bar{K}^{qt}\eta_{tm} - \mathcal{G}_{qm}\eta^{qt}\bar{K}_{ts}\right) \\
& \times \eta^{sn}\left(\frac{1}{2}F_{ab}\bar{K}^{am}\bar{K}^b{}_n + \mathcal{G}_{gb}\eta^{ga}\bar{K}_a{}^m\bar{K}^b{}_n + \frac{1}{2}\mathcal{H}_{gh}\eta^{ga}\eta^{hb}\bar{K}_a{}^m\bar{K}_{bn}\right)\mathbf{\Phi} \\
& +\frac{1}{2}\left(\mathcal{G}_{rq}\bar{K}^q{}_s\eta^{rm}\eta^{sn} - \mathcal{G}_{rq}\bar{K}^q{}_s\eta^{rn}\eta^{sm}\right) \\
& +\frac{1}{2}\left(\mathcal{H}_{rq}\eta^{qt}\bar{K}_{ts}\eta^{rm}\eta^{sn} - \mathcal{H}_{rq}\eta^{qt}\bar{K}_{ts}\eta^{rn}\eta^{sm}\right) \\
& \times \left(\frac{1}{2}F_{ab}\bar{K}^a{}_m\bar{K}^b{}_n + \mathcal{G}_{gb}\eta^{ga}\bar{K}_{am}\bar{K}^b{}_n\right)\mathbf{\Phi}
\end{aligned}
$$

After varying the metric, the null-orthonormal form of the metric is restored, so we may anticipate this and drop terms in the product such as $\mathcal{G}_{gb}\eta^{ga}\bar{K}_a{}^m\bar{K}^{bn}\mathcal{G}_{sq}\bar{K}^{qt}\eta_{tr}$ which will ultimately vanish. Terms with two or more factors of $\bar{K}^{ab}$ and/or $\bar{K}_{ab}$ will vanish, so we have only terms with one of these and two off diagonal factors such as $\bar{K}_a{}^b$, or terms with three off diagonal factors. In all cases where there is one factor of $\bar{K}^{ab}$ or $\bar{K}_{ab}$, it is this factor that must be varied so the off diagonal factors may be replaced. For example, once the null orthonormal basis is restored, the only surviving

term of the variation of

$$F_{mq}\eta_{tn}\frac{1}{2}\mathcal{H}_{gh}\eta^{ga}\eta^{hb}\bar{K}^{qt}\bar{K}_a{}^m\bar{K}_b{}^n$$

will be

$$F_{mq}\eta_{tn}\frac{1}{2}\mathcal{H}_{gh}\eta^{ga}\eta^{hb}\delta\bar{K}^{qt}\bar{K}_a{}^m\bar{K}_b{}^n$$

which we may immediately write as

$$F_{mq}\eta_{tn}\frac{1}{2}\mathcal{H}_{gh}\eta^{ga}\eta^{hb}\delta\bar{K}^{qt}\delta_a{}^m\delta_b{}^n = \frac{1}{2}\left(F_{aq}\eta_{tb}\mathcal{H}_{gh}\eta^{ga}\eta^{hb}\right)\delta\bar{K}^{qt}$$

Terms with three off-diagonal components of the metric must be retained until after variation.

Distributing fully, then making these reductions where possible, we collect terms

$$
\begin{aligned}
\bar{\mathcal{F}}\wedge{}^*\mathcal{F} \;=\; & \frac{1}{2}\left[\frac{1}{2}F_{ec}\eta^{cb}F_{db} - \frac{1}{2}F_{ec}\eta^{ca}F_{ad} + \frac{1}{2}F_{ad}\eta_{eb}H^{ab}\right. \\
& \left. + \frac{1}{2}\mathcal{H}_{ec}\eta^{cb}F_{db} - \frac{1}{2}F_{bd}\eta_{ea}H^{ab} - \frac{1}{2}\mathcal{H}_{ec}\eta^{ca}F_{ad}\bar{K}^{de}\right]\mathbf{\Phi} \\
& + \frac{1}{2}\left(-\mathcal{G}_{ea}\mathcal{G}_{cd}\eta^{ca} + \mathcal{G}_{ae}\mathcal{G}_{cd}\eta^{ca} - \mathcal{G}_{cd}\eta_{ea}\eta^{cb}\mathcal{G}_{gb}\eta^{ga}\right)\bar{K}^{de}\mathbf{\Phi} \\
& + \frac{1}{2}\left[-\frac{1}{2}F_{bc}\eta^{ce}H^{db} - \mathcal{H}_{bc}\eta^{ce}\frac{1}{2}H^{db} + \frac{1}{2}F_{ac}\eta^{ce}H^{ad}\right. \\
& \left. + \frac{1}{2}\mathcal{H}_{ac}\eta^{ce}H^{ad} + \frac{1}{2}H^{ad}\eta^{eb}F_{ab} - \frac{1}{2}H^{bd}\eta^{ea}F_{ab}\bar{K}_{de}\right]\mathbf{\Phi} \\
& + \frac{1}{2}\left(\mathcal{G}_{rs}\eta^{re}\eta^{sb}\mathcal{G}_{gb}\eta^{gd} - \mathcal{G}_{rs}\eta^{rb}\eta^{se}\mathcal{G}_{gb}\eta^{gd} - \mathcal{G}_{qa}\eta^{qd}\eta^{eb}\mathcal{G}_{gb}\eta^{ga}\right)\bar{K}_{de}\mathbf{\Phi} \\
& + \frac{1}{2}\left(\frac{1}{2}\mathcal{G}_{qn}H^{ab}\delta_m^c - \frac{1}{2}\mathcal{G}_{qm}H^{ab}\delta_n^c + F_{mq}\eta^{cb}\mathcal{G}_{gn}\eta^{ga}\right)\bar{K}^q{}_c\bar{K}_a{}^m\bar{K}_b{}^n\mathbf{\Phi} \\
& + \frac{1}{2}\left(\mathcal{H}_{qs}\eta^{sb}\mathcal{G}_{gn}\eta^{ga}\delta_m^c + \frac{1}{2}\mathcal{G}_{rq}\eta^{ra}\eta^{cb}F_{mn} - \frac{1}{2}\mathcal{G}_{rq}\eta^{rb}\eta^{ca}F_{mn}\right) \\
& \times \bar{K}^m{}_a\bar{K}^n{}_b\bar{K}^q{}_c\mathbf{\Phi}
\end{aligned}
$$

and consolidate using symmetries,

$$
\begin{aligned}
\bar{\mathcal{F}} \wedge {}^{*}\mathcal{F} \;=\; & \frac{1}{2}\left(F_{dc}\eta^{cb}F_{eb} + 2F_{ad}H^{ab}\eta_{be}\right)\bar{K}^{de}\boldsymbol{\Phi} \\
& +\frac{1}{2}\left(\mathcal{G}_{ae}\mathcal{G}_{cd}\eta^{ca} - 2\mathcal{G}_{ea}\mathcal{G}_{cd}\eta^{ca}\right)\bar{K}^{de}\boldsymbol{\Phi} \\
& +\frac{1}{2}\left(\mathcal{H}_{ab}H^{ad}\eta^{be} + 2F_{ac}H^{ad}\eta^{ce}\right)\bar{K}_{de}\boldsymbol{\Phi} \\
& +\frac{1}{2}\left(\mathcal{G}_{ca}\mathcal{G}_{db}\eta^{ab} - 2\mathcal{G}_{da}\mathcal{G}_{bc}\eta^{ab}\right)\eta^{de}\eta^{cf}\bar{K}_{ef}\boldsymbol{\Phi} \\
& +\frac{1}{2}\left(\frac{1}{2}\mathcal{G}_{dn}H^{ab}\delta^{c}_{m} - \frac{1}{2}\mathcal{G}_{dm}H^{ab}\delta^{c}_{n} + F_{md}\eta^{cb}\mathcal{G}_{gn}\eta^{ga}\right)\bar{K}_{a}{}^{m}\bar{K}_{b}{}^{n}\bar{K}^{d}{}_{c}\boldsymbol{\Phi} \\
& +\frac{1}{2}\left(\mathcal{H}_{ds}\eta^{sb}\mathcal{G}_{gn}\eta^{ga}\delta^{c}_{m} + \frac{1}{2}\mathcal{G}_{rd}\eta^{ra}\eta^{cb}F_{mn} - \frac{1}{2}\mathcal{G}_{rd}\eta^{rb}\eta^{ca}F_{mn}\right) \\
& \times \bar{K}^{m}{}_{a}\bar{K}^{n}{}_{b}\bar{K}^{d}{}_{c}\boldsymbol{\Phi}
\end{aligned}
$$

Checking the limit in the orthonormal basis, we find the correct form,

$$
\bar{\mathcal{F}} \wedge {}^{*}\mathcal{F} = \left(\mathcal{G}_{ab}H^{ab} + F_{ab}\eta^{ac}\eta^{bd}\mathcal{G}_{cd}\right)\boldsymbol{\Phi}
$$

Proceeding, we vary the metric

$$
\begin{aligned}
\delta \left(\bar{\boldsymbol{\mathcal{F}}} \wedge {}^* \boldsymbol{\mathcal{F}} \right) \;=\; & \frac{1}{2} \left(F_{dc} \eta^{cb} F_{eb} + 2 F_{ad} H^{ab} \eta_{be} + \left(\mathcal{G}_{ae} - 2 \mathcal{G}_{ea} \right) \mathcal{G}_{cd} \eta^{ca} \right) \delta \bar{K}^{de} \boldsymbol{\Phi} \\
& + \frac{1}{2} \bar{K}_b{}^n \bar{K}^d{}_c \left(\frac{1}{2} \mathcal{G}_{db} H^{ab} \delta^c_m - \frac{1}{2} \mathcal{G}_{dm} H^{ab} \delta^c_b + F_{md} \eta^{cb} \mathcal{G}_{gb} \eta^{ga} \right) \delta \bar{K}_a{}^m \boldsymbol{\Phi} \\
& + \frac{1}{2} \left(\frac{1}{2} \mathcal{G}_{dn} H^{ab} \delta^c_m - \frac{1}{2} \mathcal{G}_{dm} H^{ab} \delta^c_n + F_{md} \eta^{cb} \mathcal{G}_{gn} \eta^{ga} \right) \bar{K}_a{}^m \delta \bar{K}_b{}^n \bar{K}^d{}_c \boldsymbol{\Phi} \\
& + \frac{1}{2} \left(\frac{1}{2} \mathcal{G}_{dn} H^{ab} \delta^c_m - \frac{1}{2} \mathcal{G}_{dm} H^{ab} \delta^c_n + F_{md} \eta^{cb} \mathcal{G}_{gn} \eta^{ga} \right) \bar{K}_a{}^m \bar{K}_b{}^n \delta \bar{K}^d{}_c \boldsymbol{\Phi} \\
& + \frac{1}{2} \left(\mathcal{H}_{ds} \eta^{sb} \mathcal{G}_{gn} \eta^{ga} \delta^c_m + \frac{1}{2} \mathcal{G}_{rd} \eta^{ra} \eta^{cb} F_{mn} - \frac{1}{2} \mathcal{G}_{rd} \eta^{rb} \eta^{ca} F_{mn} \right) \\
& \times \delta \bar{K}^m{}_a \bar{K}^n{}_b \bar{K}^d{}_c \boldsymbol{\Phi} \\
& + \frac{1}{2} \left(\mathcal{H}_{ds} \eta^{sb} \mathcal{G}_{gn} \eta^{ga} \delta^c_m + \frac{1}{2} \mathcal{G}_{rd} \eta^{ra} \eta^{cb} F_{mn} - \frac{1}{2} \mathcal{G}_{rd} \eta^{rb} \eta^{ca} F_{mn} \right) \\
& \times \bar{K}^m{}_a \delta \bar{K}^n{}_b \bar{K}^d{}_c \boldsymbol{\Phi} \\
& + \frac{1}{2} \left(\mathcal{H}_{ds} \eta^{sb} \mathcal{G}_{gn} \eta^{ga} \delta^c_m + \frac{1}{2} \mathcal{G}_{rd} \eta^{ra} \eta^{cb} F_{mn} - \frac{1}{2} \mathcal{G}_{rd} \eta^{rb} \eta^{ca} F_{mn} \right) \\
& \times \bar{K}^m{}_a \bar{K}^n{}_b \delta \bar{K}^d{}_c \boldsymbol{\Phi} \\
& + \frac{1}{2} \left(\mathcal{H}_{ab} H^{ad} \eta^{be} + 2 F_{ac} H^{ad} \eta^{ce} \right) \bar{K}_{de} \boldsymbol{\Phi} \\
& + \frac{1}{2} \mathcal{G}_{ca} \eta^{ab} \left(\mathcal{G}_{db} - 2 \mathcal{G}_{bd} \right) \eta^{de} \eta^{cf} \bar{K}_{ef} \boldsymbol{\Phi} \\
& + \left(\mathcal{G}_{ab} H^{ab} + F_{ab} \eta^{ac} \eta^{bd} \mathcal{G}_{cd} \right) \delta \boldsymbol{\Phi}
\end{aligned}
$$

At this point we may replace remaining unvaried metric components, leaving

$$
\begin{aligned}
\delta\left(\bar{\boldsymbol{\mathcal{F}}} \wedge {}^{*}\boldsymbol{\mathcal{F}}\right) \;=\;& \frac{1}{2}\left(F_{dc}\eta^{cb}F_{eb} + 2F_{ad}H^{ab}\eta_{be} + \left(\mathcal{G}_{ae} - 2\mathcal{G}_{ea}\right)\mathcal{G}_{cd}\eta^{ca}\right)\delta\bar{K}^{de}\boldsymbol{\Phi} \\
&+\frac{1}{2}\left(\frac{1}{2}\mathcal{G}_{cb}H^{ab}\delta^{c}_{m} - \frac{1}{2}\mathcal{G}_{cm}H^{ab}\delta^{c}_{b} + F_{mc}\eta^{cb}\mathcal{G}_{gb}\eta^{ga}\right)\delta\bar{K}_{a}{}^{m}\boldsymbol{\Phi} \\
&+\frac{1}{2}\left(\frac{1}{2}\mathcal{G}_{an}H^{ab} - \frac{1}{2}\mathcal{G}_{na}H^{ab} + F_{ac}\eta^{cn}\mathcal{G}_{gb}\eta^{ga}\right)\delta\bar{K}_{b}{}^{n}\boldsymbol{\Phi} \\
&+\frac{1}{2}\left(\frac{1}{2}\mathcal{G}_{db}H^{cb} - \frac{1}{2}\mathcal{G}_{da}H^{ac} + F_{ad}\eta^{cb}\mathcal{G}_{gb}\eta^{ga}\right)\bar{K}_{b}{}^{n}\delta\bar{K}^{d}{}_{c}\boldsymbol{\Phi} \\
&+\frac{1}{2}\left(\mathcal{H}_{cs}\eta^{sb}\mathcal{G}_{gb}\eta^{ga}\delta^{c}_{m} + \frac{1}{2}\mathcal{G}_{rc}\eta^{ra}\eta^{cb}F_{mb} - \frac{1}{2}\mathcal{G}_{rc}\eta^{rb}\eta^{ca}F_{mb}\right)\delta\bar{K}^{m}{}_{a}\boldsymbol{\Phi} \\
&+\frac{1}{2}\left(\mathcal{H}_{cs}\eta^{sb}\mathcal{G}_{gn}\eta^{gc} + \frac{1}{2}\mathcal{G}_{rc}\eta^{ra}\eta^{cb}F_{an} - \frac{1}{2}\mathcal{G}_{rc}\eta^{rb}\eta^{ca}F_{an}\right)\delta\bar{K}^{n}{}_{b}\boldsymbol{\Phi} \\
&+\frac{1}{2}\left(\mathcal{H}_{ds}\eta^{sb}\mathcal{G}_{gb}\eta^{gc} + \frac{1}{2}\mathcal{G}_{rd}\eta^{ra}\eta^{cb}F_{ab} - \frac{1}{2}\mathcal{G}_{rd}\eta^{rb}\eta^{ca}F_{ab}\right)\delta\bar{K}^{d}{}_{c}\boldsymbol{\Phi} \\
&+\frac{1}{2}\left(\left(\mathcal{H}_{ab}H^{ae}\eta^{bf} + 2F_{ac}H^{ae}\eta^{cf}\right) + \mathcal{G}_{ca}\eta^{ab}\left(\mathcal{G}_{db} - 2\mathcal{G}_{bd}\right)\eta^{de}\eta^{cf}\right)\bar{K}_{ef}\boldsymbol{\Phi} \\
&+\left(\mathcal{G}_{ab}H^{ab} + F_{ab}\eta^{ac}\eta^{bd}\mathcal{G}_{cd}\right)\delta\boldsymbol{\Phi}
\end{aligned}
$$

Finally, we collect all terms by type of variation,

$$
\begin{aligned}
\delta \left(\bar{\mathcal{F}} \wedge {}^* \mathcal{F} \right) \; = \; & \frac{1}{2} \left(F_{dc} \eta^{cb} F_{eb} + 2 F_{ad} H^{ab} \eta_{be} + \left(\mathcal{G}_{ae} - 2 \mathcal{G}_{ea} \right) \mathcal{G}_{cd} \eta^{ca} \right) \delta \bar{K}^{de} \boldsymbol{\Phi} \\
& + \frac{1}{2} \left(\frac{1}{2} \mathcal{G}_{cb} H^{nb} \delta^c_m - \frac{1}{2} \mathcal{G}_{cm} H^{nb} \delta^c_b + F_{mc} \eta^{cb} \mathcal{G}_{gb} \eta^{gn} \right) \delta \bar{K}_n{}^m \boldsymbol{\Phi} \\
& + \frac{1}{2} \left(\frac{1}{2} \mathcal{G}_{am} H^{an} - \frac{1}{2} \mathcal{G}_{ma} H^{an} + F_{ac} \eta^{cn} \mathcal{G}_{gm} \eta^{ga} \right) \delta \bar{K}_n{}^m \boldsymbol{\Phi} \\
& + \frac{1}{2} \left(\frac{1}{2} \mathcal{G}_{mb} H^{nb} - \frac{1}{2} \mathcal{G}_{ma} H^{an} + F_{am} \eta^{nb} \mathcal{G}_{gb} \eta^{ga} \right) \delta \bar{K}^m{}_n \boldsymbol{\Phi} \\
& + \left(\mathcal{H}_{cs} \eta^{sb} \mathcal{G}_{gb} \eta^{gn} \delta^c_m + \frac{1}{2} \mathcal{G}_{rc} \eta^{rn} \eta^{cb} F_{mb} - \frac{1}{2} \mathcal{G}_{rc} \eta^{rb} \eta^{cn} F_{mb} \right) \\
& \times \frac{1}{2} \delta \bar{K}^m{}_n \boldsymbol{\Phi} \\
& + \left(\mathcal{H}_{cs} \eta^{sn} \mathcal{G}_{gm} \eta^{gc} + \frac{1}{2} \mathcal{G}_{rc} \eta^{ra} \eta^{cn} F_{am} - \frac{1}{2} \mathcal{G}_{rc} \eta^{rn} \eta^{ca} F_{am} \right) \\
& \times \frac{1}{2} \delta \bar{K}^m{}_n \boldsymbol{\Phi} \\
& + \left(\mathcal{H}_{ms} \eta^{sb} \mathcal{G}_{gb} \eta^{gn} + \frac{1}{2} \mathcal{G}_{rm} \eta^{ra} \eta^{nb} F_{ab} - \frac{1}{2} \mathcal{G}_{rm} \eta^{rb} \eta^{na} F_{ab} \right) \\
& \times \frac{1}{2} \delta \bar{K}^m{}_n \boldsymbol{\Phi} \\
& + \left(\left(\mathcal{H}_{ab} H^{ae} \eta^{bf} + 2 F_{ac} H^{ae} \eta^{cf} \right) + \mathcal{G}_{ca} \eta^{ab} \left(\mathcal{G}_{db} - 2 \mathcal{G}_{bd} \right) \eta^{de} \eta^{cf} \right) \\
& \times \frac{1}{2} \bar{K}_{ef} \boldsymbol{\Phi} \\
& + \left(\mathcal{G}_{ab} H^{ab} + F_{ab} \eta^{ac} \eta^{bd} \mathcal{G}_{cd} \right) \delta \boldsymbol{\Phi}
\end{aligned}
$$

This allows further simplifications, then including the variation of the volume form

we have the final result,

$$
\begin{aligned}
\delta \left(\bar{\mathcal{F}} \wedge {}^{*}\mathcal{F} \right) \;=\; & \frac{1}{2} \left(F_{dc}\eta^{cb}F_{eb} + 2F_{ad}H^{ab}\eta_{be} \right) \delta\bar{K}^{de}\boldsymbol{\Phi} \\
& + \frac{1}{2} \left(\mathcal{G}_{ae} - 2\mathcal{G}_{ea} \right) \mathcal{G}_{cd}\eta^{ca}\delta\bar{K}^{de}\boldsymbol{\Phi} \\
& + H^{na} \left(2\mathcal{G}_{ma} - \mathcal{G}_{am} \right) \delta\bar{K}_{n}{}^{m}\boldsymbol{\Phi} \\
& + F_{ma} \left(\mathcal{G}^{na} - 2\mathcal{G}^{an} \right) \delta\bar{K}^{m}{}_{n}\boldsymbol{\Phi} \\
& + \frac{1}{2} \left(\mathcal{H}_{ab}H^{ae}\eta^{bf} + 2F_{ac}H^{ae}\eta^{cf} \right) \bar{K}_{ef}\boldsymbol{\Phi} \\
& + \frac{1}{2}\mathcal{G}_{ca}\eta^{ab} \left(\mathcal{G}_{db} - 2\mathcal{G}_{bd} \right) \eta^{de}\eta^{cf}\bar{K}_{ef}\boldsymbol{\Phi} \\
& + \left(\mathcal{G}_{ab}H^{ab} + F_{ab}\eta^{ac}\eta^{bd}\mathcal{G}_{cd} \right) \delta\boldsymbol{\Phi}
\end{aligned}
$$