THE DIFFERENTIALGEOMETRY SOFTWARE PROJECT

Examples of the Birkhoff Theorem and its Generalizations

Synopsis

In the following I demonstrate three versions of Birkhoff's theorem, which is a characterization of spherically symmetric solutions of the Einstein equations. The three versions considered here correspond to taking the "Einstein equations" to be: (1) the vacuum Einstein equations; (2) the Einstein equations with a cosmological constant (3) the Einstein-Maxwell equations I will restrict my attention to 4-dimensional spacetimes. For recent references with additional results, details, and references to the literature, see [1, 2].

For any of the 3 versions, the gist of the theorem is as follows. A 4-dimensional spacetime is said to be spherically symmetric if it admits an SO(3) isometry group with orbits which are spacelike 2-spheres. Because the codimension of the orbits of the isometry group is two, one expects that spherically symmetric solutions of the field equations would be characterized by PDEs in two independent variables, leading to solutions depending upon arbitrary functions of one variable; these functions corresponding to a choice of spherically symmetric initial data. But Birkhoff's theorem asserts that (modulo coordinate freedom) the spherically symmetric solutions form a finite-dimensional family of solutions - the Schwarzschild metric and its generalizations. The reason for this is that spherically symmetric solutions to the Einstein equations always admit at least one additional continuous symmetry whose group orbits are orthogonal to the spherical orbits. Consequently, the symmetry-reduced field equations are actually ODEs instead of PDEs. The existence of an additional Killing vector for spherically symmetric solutions of the Einstein equations is another way of stating the Birkhoff theorem.

In this worksheet I will:

- Characterize spherically symmetric metrics
- Compute the general spherically symmetric solution for the vacuum Einstein equations.

- Compute the general spherically symmetric solution for the vacuum Einstein equations including a cosmological constant.
- Compute the general spherically symmetric solution for the Einstein-Maxwell equations.

Spherically symmetric metrics

I will define a spherically symmetric metric to be one which admits an SO(3) isometry group with orbits which are spacelike 2-spheres. Locally, such spaces can be topologically identified with (an open subset of) $R^2 \times S^2$, with coordinates $(t, r) \in R^2$ which are invariants under the group action, and spherical coordinates $(\theta, \phi) \in S^2$ which transform in the usua way with respect to SO(3). In such coordinates the spherically symmetric metric is a "warped product",

$$g_{ab} = t_{ab} + \rho\, s_{ab},$$

where t_{ab} is any Lorentz signature metric on R^2, s_{ab} is the standard constant curvature ("round") metric on S^2, and ρ is a function on R^2.

Load the DifferentialGeometry package and the Tensor sub-package.

```
> with(DifferentialGeometry): with(Tensor): with(LieAlgebras): with(GroupActions):
> interface(typesetting = extended):
> Preferences("TensorDisplay", 1):
```

Initialize the coordinate chart.

```
> DGsetup([t, r, theta, phi], M);
```

$$\textit{Manifold: M} \quad (1.1$$

Define the metric.

```
M > g := evalDG(2*f(t,r)*dt &s dr + rho(t, r)*(dtheta &t dtheta + sin(theta)^2*dphi &t
    dphi));
```

$$g := f(t, r)\, dt \otimes dr + f(t, r)\, dr \otimes dt + \rho(t, r)\, d\theta \otimes d\theta + \rho(t, r) \sin(\theta)^2\, d\phi \otimes d\phi \quad (1.2$$

This metric depends upon the specification of three functions of two variables, $f(t, r)$ and $\rho(t, r)$. The most general metric includes *tt* and *rr* components, as we shall show below. For convenience in the next computation we have chosen a null coordinate system on R^2 in which those two components vanish.

We can verify that this metric is spherically symmetric by computing its infinitesimal isometry generators - its Killing vector fields. A direct assault on the Killing equations for this metric is possible.

```
M > KV0 := KillingVectors(g);
```

$$KV0 := \left[\frac{\sqrt{-1+\cos(2\,\theta)}}{\sin(\theta)}\,\partial_\phi,\ -\frac{\sqrt{-1+\cos(2\,\theta)}\,\cos(\phi)}{\sin(\theta)}\,\partial_\theta + \frac{\sqrt{-1+\cos(2\,\theta)}\,\sin(\phi)\,\cos(\theta)}{\sin(\theta)^2}\,\partial_\phi,\ \frac{\sqrt{-1+\cos(2\,\theta)}\,\sin(\phi)}{\sin(\theta)}\,\partial_\theta + \frac{\sqrt{-1+\cos(2\,\theta)}\,\cos(\phi)\,\cos(\theta)}{\sin(\theta)^2}\,\partial_\phi \right] \quad (1.3$$

```
M > KV := simplify(evalDG(sqrt(2)/I*~eval(KV0, cos(2*theta) = cos(theta)^2 - sin(theta)^2)
    )) assuming theta > 0, theta < Pi;
```

$$KV := \left[2\,\partial_\phi,\ -2\cos(\phi)\,\partial_\theta + \frac{2\sin(\phi)\cos(\theta)}{\sin(\theta)}\,\partial_\phi,\ 2\sin(\phi)\,\partial_\theta + \frac{2\cos(\phi)\cos(\theta)}{\sin(\theta)}\,\partial_\phi \right] \quad (1.4$$

Here we compute the commutators of the Killing vector fields, which determines the Lie algebra of the isometry group.

```
M > LieAlgebraData(KV);
```

$$[e1, e2] = 2\,e3,\ [e1, e3] = -2\,e2,\ [e2, e3] = 2\,e1 \quad (1.5$$

Here we recognize that the isometry algebra is the Lie algebra of SO(3). (This can be made more obvious with a change of basis $e3 \rightarrow \frac{1}{2} e3$.) The orbits are the 2-spheres $t = const.$, $r = const.$, which are spacelike provided $\rho(t, r) > 0$.

Conversely, we can show that, up to a choice of coordinates, the metric g is defined by its invariance with respect to the vector fields generating the SO(3) action on the 2-sphere.

Sym2 is a basis for the set of rank-2 symmetric tensors. Sym2inv is a basis for the set of rank-2 symmetric tensors which are invariant under the symmetries generated by the Killing vector fields. The general spherically symmetric rank-2 symmetric tensor is then a linear combination from Sym2inv with coefficients which are arbitrary functions of the scalar invariants *t* and *r*.

```
M > Sym2 := GenerateSymmetricTensors([dt, dr, dtheta, dphi], 2):
M > Sym2inv := InvariantGeometricObjectFields(KV, Sym2, output = "list");
```

$$Sym2inv := \left[d\theta \otimes d\theta - \left(-\frac{1}{2} + \frac{\cos(2\,\theta)}{2}\right) d\phi \otimes d\phi, dr \otimes dr, \left(\frac{1}{2}\right) dt \otimes dr + \left(\frac{1}{2}\right) dr \otimes dt, dt \otimes dt\right] \quad (1.6)$$

```
M > simplify(eval(Sym2inv, [cos(2*theta) = cos(theta)^2 - sin(theta)^2]));
```

$$\left[d\theta \otimes d\theta + \sin(\theta)^2\, d\phi \otimes d\phi, dr \otimes dr, \left(\frac{1}{2}\right) dt \otimes dr + \left(\frac{1}{2}\right) dr \otimes dt, dt \otimes dt\right] \quad (1.7)$$

The subsequent analysis depends upon the nature of the first derivatives of $\rho(t, r)$. We distinguish three possibilities: in some open set, either (1) $d\rho$ is non-null; (2) $d\rho$ is non-zero and null; (3) $d\rho = 0$.

In case (1) we can choose the r coordinate such that $\rho(t, r) = r^2$ and we pick the *t* coordinate so that *(r, t)* are orthogonal coordinates on R^2. We denote this metric as *g1*.

```
M > g1 := evalDG( f(t, r)*dt &t dt + h(t,r)*dr &t dr + r^2*(dtheta &t dtheta + sin(theta)
    ^2*dphi &t dphi));
```

$$g1 := f(t, r)\, dt \otimes dt + h(t, r)\, dr \otimes dr + r^2\, d\theta \otimes d\theta + r^2 \sin(\theta)^2\, d\phi \otimes d\phi \quad (1.8)$$

In case (2) we can still choose $\rho(t, r) = r^2$ but, if we do, the hypersurfaces $r = const.$ are null, so that the metric takes the form:

```
M > g2  := evalDG(f(t, r) * dr &t dr + 2*h(t, r) * dt &s dr + r^2*(dtheta &t dtheta + sin
    (theta)^2*dphi &t dphi));
```

$$g2 := h(t, r)\, dt \otimes dr + h(t, r)\, dr \otimes dt + f(t, r)\, dr \otimes dr + r^2\, d\theta \otimes d\theta + r^2 \sin(\theta)^2\, d\phi \otimes d\phi \quad (1.9)$$

In case (3) the spacetime is geometrically the product $R^2 \times S^2$, there is a single sphere of radius *r0*. For later convenience we use a null coordinate system in the R^2.

```
M > g3 := evalDG( 2*h(t,r)*dt &s dr + r0^2*(dtheta &t dtheta + sin(theta)^2*dphi &t dphi))
```

;

$$g3 := h(t,r)\, dt \otimes dr + h(t,r)\, dr \otimes dt + r0^2\, d\theta \otimes d\theta + r0^2 \sin(\theta)^2\, d\phi \otimes d\phi \tag{1.10}$$

Birkhoff for vacuum

The classical Birkhoff theorem – evidently first proved by Jebsen [2] – asserts that a spherically symmetric solution to the vacuum Einstein equations is isometric to (an open submanifold of) the Schwarzschild spacetime. We prove this by considering the vacuum Einstein equations, which we write as the vanishing of the Ricci tensor, $R_{ij}(g) = 0$, in each of the three cases shown above.

Case 1.

Here is the Ricci tensor in case (1).

```
M > RT1:=RicciTensor(g1);
```

$$RT1 := \frac{1}{4 f(t,r)\, h(t,r)^2\, r}\Bigg(-2\, h(t,r) f(t,r) \left(\frac{\partial^2}{\partial r^2} f(t,r)\right) r - 2\, h(t,r) \left(\frac{\partial^2}{\partial t^2} h(t,r)\right) f(t,r)\, r + \left(\frac{\partial}{\partial r} f(t,r)\right)^2 h(t,r)\, r + f(t,r) \left(\left(\frac{\partial}{\partial r} h(t,r)\right) r - 4\, h(t,r)\right) \left(\frac{\partial}{\partial r} f(t,r)\right) + r \left(\frac{\partial}{\partial t} h(t,r)\right) \left(f(t,r) \left(\frac{\partial}{\partial t} h(t,r)\right) + h(t,r) \left(\frac{\partial}{\partial t} f(t,r)\right)\right)\Bigg) dt \otimes dt + \frac{\frac{\partial}{\partial t} h(t,r)}{r\, h(t,r)}\, dt \otimes dr + \frac{\frac{\partial}{\partial t} h(t,r)}{r\, h(t,r)}\, dr \otimes dt + \frac{1}{4 f(t,r)^2\, h(t,r)\, r}\Bigg(\left(\frac{\partial}{\partial t} h(t,r)\right)^2 f(t,r)\, r + \left(\frac{\partial}{\partial t} h(t,r)\right) h(t,r) \left(\frac{\partial}{\partial t} f(t,r)\right) r + \left(\frac{\partial}{\partial r} f(t,r)\right)^2 h(t,r)\, r + \left(\frac{\partial}{\partial r} f(t,r)\right) \left(\frac{\partial}{\partial r} h(t,r)\right) f(t,r)\, r - 2\, h(t,r) \left(\frac{\partial^2}{\partial t^2} h(t,r)\right) f(t,r)\, r - 2\, h(t,r) f(t,r) \left(\frac{\partial^2}{\partial r^2} f(t,r)\right) r + 4 \left(\frac{\partial}{\partial r} h(t,r)\right) f(t,r)^2\Bigg) dr \otimes dr \tag{2.1}$$

$$+\frac{-\left(\frac{\partial}{\partial r} f(t,r)\right) r\,h(t,r) + f(t,r)\left(\left(\frac{\partial}{\partial r} h(t,r)\right) r + 2\,h(t,r)^2 - 2\,h(t,r)\right)}{2\,f(t,r)\,h(t,r)^2}\,d\theta \otimes d\theta$$

$$+\frac{\sin(\theta)^2\left(-\left(\frac{\partial}{\partial r} f(t,r)\right) r\,h(t,r) + f(t,r)\left(\left(\frac{\partial}{\partial r} h(t,r)\right) r + 2\,h(t,r)^2 - 2\,h(t,r)\right)\right)}{2\,h(t,r)^2 f(t,r)}\,d\phi \otimes d\phi$$

Here is the general solution to the vacuum Einstein equations in this case:

```
M > DGsolve(RT1, g1);
```

$$\left\{\frac{(r + \_C1)\,\_F1(t)}{r}\,dt \otimes dt + \frac{r}{r + \_C1}\,dr \otimes dr + r^2\,d\theta \otimes d\theta + r^2 \sin(\theta)^2\,d\phi \otimes d\phi\right\} \tag{2.2}$$

With an appropriate choice of the *t* coordinate (corresponding to setting _F1(t) = -1) and with an appropriate choice of the constant _C1 (i.e., _C1 = -2m), this is (an open subspace of) the Schwarzschild solution.

Case 2.

Here is the Ricci tensor in case (2):

```
M > RT2 := RicciTensor(g2);
```

$$RT2 := \frac{h(t,r)\left(\frac{\partial^2}{\partial t^2} f(t,r)\right) - 2\,h(t,r)\left(\frac{\partial^2}{\partial r \partial t} h(t,r)\right) - \left(\frac{\partial}{\partial t} h(t,r)\right)\left(-2\,\frac{\partial}{\partial r} h(t,r) + \frac{\partial}{\partial t} f(t,r)\right)}{2\,h(t,r)^2}\,dt \otimes dr \tag{2.3}$$

$$+\frac{h(t,r)\left(\frac{\partial^2}{\partial t^2} f(t,r)\right) - 2\,h(t,r)\left(\frac{\partial^2}{\partial r \partial t} h(t,r)\right) - \left(\frac{\partial}{\partial t} h(t,r)\right)\left(-2\,\frac{\partial}{\partial r} h(t,r) + \frac{\partial}{\partial t} f(t,r)\right)}{2\,h(t,r)^2}\,dr \otimes dt$$

$$+\frac{1}{2\,h(t,r)^3\,r}\left(\left(\frac{\partial^2}{\partial t^2}\,f(t,r)\right)h(t,r)\,f(t,r)\,r-2\,h(t,r)\,f(t,r)\left(\frac{\partial^2}{\partial r\partial t}\,h(t,r)\right)r-\left(-2\,\frac{\partial}{\partial r}\,h(t,r)+\frac{\partial}{\partial t}\,f(t,r)\right)\left(r\,f(t,r)\left(\frac{\partial}{\partial t}\,h(t,r)\right)+2\,h(t,r)^2\right)\right)dr\otimes dr+d\theta\otimes d\theta+\sin(\theta)^2\,d\phi\otimes d\phi$$

From $R_{\theta\theta} = 1 = R_{\phi\phi}/\sin(\theta)^2$ we see that there are no vacuum solutions in this case.

Case 3.

Here is the Ricci tensor in case (3):

```
M > RT3 := RicciTensor(g3);
```

$$RT3 := \frac{\left(\frac{\partial}{\partial t}\,h(t,r)\right)\left(\frac{\partial}{\partial r}\,h(t,r)\right)-h(t,r)\left(\frac{\partial^2}{\partial r\partial t}\,h(t,r)\right)}{h(t,r)^2}\,dt\otimes dr+\frac{\left(\frac{\partial}{\partial t}\,h(t,r)\right)\left(\frac{\partial}{\partial r}\,h(t,r)\right)-h(t,r)\left(\frac{\partial^2}{\partial r\partial t}\,h(t,r)\right)}{h(t,r)^2}\,dr\otimes dt+d\theta\otimes d\theta+\sin(\theta)^2\,d\phi\otimes d\phi \quad (2.4$$

Once again, from $R_{\theta\theta} = 1 = R_{\phi\phi}/\sin(\theta)^2$ we see that there are no vacuum solutions in this case.
Thus only case (1) admits vacuum solutions, which are isometric to (a subspace of) the Schwarzschild solution. This is the classical version of the Birkhoff theorem.

Birkhoff with cosmological constant

This generalization of the Birkhoff theorem characterizes spherically symmetric solutions of the Einstein equations with a cosmological constant. These equations can be written as $R_{ij} = \lambda\, g_{ij}$. From the contracted Bianchi identity, λ must be a constant.

Case 1.

Case (1) leads to the Kottler-Weyl metric, which can be viewed as the generalization of the Schwarzschild solution to include a cosmological constant. The Einstein equations in this case are *E1 = 0*, where

```
M > E1 := evalDG(RT1 - lambda*g1);
```

$$E1 := \frac{1}{4\,h(t,r)^2 f(t,r)\,r}\Bigg(-2\,h(t,r)\,f(t,r)\left(\frac{\partial^2}{\partial r^2} f(t,r)\right) r - 2\,h(t,r)\left(\frac{\partial^2}{\partial t^2} h(t,r)\right) f(t,r)\,r + \left(\frac{\partial}{\partial r} f(t,r)\right)^2 h(t,r)\,r + f(t,r)\left(\left(\frac{\partial}{\partial r} h(t,r)\right) r - 4\,h(t,r)\right)\left(\frac{\partial}{\partial r} f(t,r)\right) - 4\,r\left(\lambda\,f(t,r)^2\,h(t,r)^2 - \frac{\left(\frac{\partial}{\partial t} h(t,r)\right)^2 f(t,r)}{4} - \frac{\left(\frac{\partial}{\partial t} h(t,r)\right)\left(\frac{\partial}{\partial t} f(t,r)\right) h(t,r)}{4}\right)\Bigg)\,dt \otimes dt + \frac{\frac{\partial}{\partial t} h(t,r)}{r\,h(t,r)}\,dt \otimes dr + \frac{\frac{\partial}{\partial t} h(t,r)}{r\,h(t,r)}\,dr \otimes dt + \frac{1}{4\,f(t,r)^2\,h(t,r)\,r}\Bigg(-4\,\lambda\,f(t,r)^2\,h(t,r)^2\,r + \left(\frac{\partial}{\partial t} h(t,r)\right)^2 f(t,r)\,r + \left(\frac{\partial}{\partial t} h(t,r)\right) h(t,r)\left(\frac{\partial}{\partial t} f(t,r)\right) r + \left(\frac{\partial}{\partial r} f(t,r)\right)\left(\frac{\partial}{\partial r} h(t,r)\right) f(t,r)\,r - 2\,h(t,r)\left(\frac{\partial^2}{\partial t^2} h(t,r)\right) f(t,r)\,r - 2\,h(t,r)\,f(t,r)\left(\frac{\partial^2}{\partial r^2} f(t,r)\right) r + \left(\frac{\partial}{\partial r} f(t,r)\right)^2 h(t,r)\,r + 4\left(\frac{\partial}{\partial r} h(t,r)\right) f(t,r)^2\Bigg)\,dr \otimes dr + \frac{-\left(\frac{\partial}{\partial r} f(t,r)\right) r\,h(t,r) - 2\left(-\frac{\left(\frac{\partial}{\partial r} h(t,r)\right) r}{2} + \left(\lambda\,r^2 - 1\right) h(t,r)^2 + h(t,r)\right) f(t,r)}{2\,h(t,r)^2 f(t,r)}\,d\theta \otimes d\theta \tag{3.1}$$

$$-\frac{\sin(\theta)^2\left(\frac{\left(\frac{\partial}{\partial r} f(t,r)\right) r\, h(t,r)}{2}+\left(-\frac{\left(\frac{\partial}{\partial r} h(t,r)\right) r}{2}+(\lambda r^2-1)\, h(t,r)^2+h(t,r)\right) f(t,r)\right)}{f(t,r)\, h(t,r)^2}\, d\phi\otimes d\phi$$

The general solution to these equations is:

```
M > DGsolve(E1, g1);
```

$$\left\{\frac{\_F1(t)\,(\lambda r^3-3\,\_C1-3\,r)}{r}\, dt\otimes dt-\frac{3\,r}{\lambda r^3-3\,\_C1-3\,r}\, dr\otimes dr+r^2\, d\theta\otimes d\theta+r^2\sin(\theta)^2\, d\phi\otimes d\phi\right\} \quad (3.2$$

With an appropriate choice of the *t* coordinate and constant of integration _C1 this is the Kottler-Weyl metric.

Case2.

Case 2 does not admit a solution to the field equations - see the $\theta\theta$ and $\phi\phi$ components of the Einstein equations, which se the constant $\lambda = \frac{1}{r^2}$.

```
M > E2 := evalDG(RT2 - lambda*g2);
```

$$E2 := \frac{h(t,r)\left(\frac{\partial^2}{\partial t^2} f(t,r)\right)-2\,h(t,r)\left(\frac{\partial^2}{\partial r\partial t} h(t,r)\right)+\left(2\frac{\partial}{\partial r} h(t,r)-\frac{\partial}{\partial t} f(t,r)\right)\left(\frac{\partial}{\partial t} h(t,r)\right)-2\,\lambda\, h(t,r)^3}{2\,h(t,r)^2}\, dt \quad (3.3$$

$$\otimes dr$$

$$+\frac{h(t,r)\left(\frac{\partial^2}{\partial t^2} f(t,r)\right)-2\,h(t,r)\left(\frac{\partial^2}{\partial r\partial t} h(t,r)\right)+\left(2\frac{\partial}{\partial r} h(t,r)-\frac{\partial}{\partial t} f(t,r)\right)\left(\frac{\partial}{\partial t} h(t,r)\right)-2\,\lambda\, h(t,r)^3}{2\,h(t,r)^2}\, dr$$

$$\otimes dt-\left(\lambda f(t,r)-\frac{1}{2\,h(t,r)^3\, r}\left(\left(\frac{\partial^2}{\partial t^2} f(t,r)\right) h(t,r)\, f(t,r)\, r-2\,h(t,r)\, f(t,r)\left(\frac{\partial^2}{\partial r\partial t} h(t,r)\right) r\right.\right.$$

$$-2\left(\frac{r f(t,r)\left(\frac{\partial}{\partial t} h(t,r)\right)}{2}+h(t,r)^2\right)\left(-2\frac{\partial}{\partial r} h(t,r)+\frac{\partial}{\partial t} f(t,r)\right)\Bigg)\Bigg] dr\otimes dr-\left[\lambda r^2-1\right] d\theta\otimes d\theta+\left(-\lambda r^2+1\right)\sin(\theta)^2 d\phi\otimes d\phi$$

```
M > evalDG(E2, [D_theta, D_theta]);
```

$$\frac{h(t,r)\left(\frac{\partial^2}{\partial t^2} f(t,r)\right)-2h(t,r)\left(\frac{\partial^2}{\partial r\partial t} h(t,r)\right)+\left(2\frac{\partial}{\partial r} h(t,r)-\frac{\partial}{\partial t} f(t,r)\right)\left(\frac{\partial}{\partial t} h(t,r)\right)-2\lambda h(t,r)^3}{2h(t,r)^2} dt\otimes dr \quad (3.4$$

$$+\frac{h(t,r)\left(\frac{\partial^2}{\partial t^2} f(t,r)\right)-2h(t,r)\left(\frac{\partial^2}{\partial r\partial t} h(t,r)\right)+\left(2\frac{\partial}{\partial r} h(t,r)-\frac{\partial}{\partial t} f(t,r)\right)\left(\frac{\partial}{\partial t} h(t,r)\right)-2\lambda h(t,r)^3}{2h(t,r)^2} dr$$

$$\otimes dt-\Bigg[\lambda f(t,r)-\frac{1}{2h(t,r)^3 r}\Bigg(\left(\frac{\partial^2}{\partial t^2} f(t,r)\right)h(t,r)f(t,r)r-2h(t,r)f(t,r)\left(\frac{\partial^2}{\partial r\partial t} h(t,r)\right)r$$

$$-2\left(\frac{r f(t,r)\left(\frac{\partial}{\partial t} h(t,r)\right)}{2}+h(t,r)^2\right)\left(-2\frac{\partial}{\partial r} h(t,r)+\frac{\partial}{\partial t} f(t,r)\right)\Bigg)\Bigg] dr\otimes dr-\left[\lambda r^2-1\right] d\theta\otimes d\theta+\left(-\lambda r^2+1\right)\sin(\theta)^2 d\phi\otimes d\phi$$

Case 3.

Case 3 admits a solution which is the product of a round, Riemannian 2-sphere with a Lorentzian 2-manifold of constant positive curvature.

```
> E3 := evalDG(RT3 - lambda*g3);
```

(3.5

$$E3 := \frac{-\lambda\, h(t,r)^3 + \left(\frac{\partial}{\partial t} h(t,r)\right)\left(\frac{\partial}{\partial r} h(t,r)\right) - h(t,r)\left(\frac{\partial^2}{\partial r \partial t} h(t,r)\right)}{h(t,r)^2}\, dt \otimes dr + \frac{-\lambda\, h(t,r)^3 + \left(\frac{\partial}{\partial t} h(t,r)\right)\left(\frac{\partial}{\partial r} h(t,r)\right) - h(t,r)\left(\frac{\partial^2}{\partial r \partial t} h(t,r)\right)}{h(t,r)^2}\, dr \otimes dt - \left|\lambda\, r0^2 - 1\right|\, d\theta \otimes d\theta + \left(-\lambda\, r0^2 + 1\right)\sin(\theta)^2\, d\phi \otimes d\phi \quad (3.5$$

```
M > g3lambdasol :=DGsolve(E3, g3, {r0, f(t,r), h(t,r)});
```

$$g3lambdasol := \left\{\frac{2\,\_C2\,\_C3}{\cosh(\_C2\, t + \_C3\, r + \_C1)^2\,\lambda}\, dt \otimes dr + \frac{2\,\_C2\,\_C3}{\cosh(\_C2\, t + \_C3\, r + \_C1)^2\,\lambda}\, dr \otimes dt + \frac{1}{\lambda}\, d\theta \otimes d\theta + \frac{\sin(\theta)^2}{\lambda}\, d\phi \otimes d\phi\right\} \quad (3.6$$

This solution is isometric to the Nariai metric. Aside from the cosmological constant there are no free parameters in this solution - the integration constants _C1, _C2, _C3 can be absorbed into a a redefinition of the coordinates (t, r).

Birkhoff for electrovacuum

The Einstein-Maxwell equations are given by (in appropriate units)

$$G_{ab} = F_{ac}F_b{}^c - \frac{1}{4}F_{mn}F^{mn}g_{ab}, \qquad \nabla^b F_{ab} = 0 = \nabla_{[a}F_{bc]},$$

where G_{ab} is the Einstein tensor of the metric g_{ab}, and $F_{ab} = F_{[ab]}$ is the electromagnetic field.

If a spacetime metric features in a solution to these equations it is called an "electrovacuum", generalizing the Ricci-flat "vacuum" metrics. A simple and elegant way to find all spherically symmetric electrovacua is to solve the Rainich conditions for non-null electrovacua. Here non-null means that at least one of the scalar invariants

$F_{ab}F^{ab}$ and $\varepsilon^{abcd}F_{ab}F_{cd}$
is non-vanishing. There are no spherically symmetric null electrovacua, which we demonstrate at the end of this section.

Case 1.

Compute the Rainich conditions. RC1 and RC2 are the algebraic conditions; RC1 is too long to display here. RC3 is the differential condition, which is automatically satisfied.

```
M > RC1, RC2, RC3 := RainichConditions(g1, output = "equations"):
M > RC2;
```

$$\frac{1}{2 r^2 f(t,r)^2 h(t,r)^2}\Bigg(-2 r^2 f(t,r)\, h(t,r) \left(\frac{\partial^2}{\partial r^2} f(t,r)\right) - 2 r^2 \left(\frac{\partial^2}{\partial t^2} h(t,r)\right) f(t,r)\, h(t,r) + r^2 h(t,r) \left(\frac{\partial}{\partial r} f(t,r)\right)^2 + r f(t,r)\left(\left(\frac{\partial}{\partial r} h(t,r)\right) r - 4 h(t,r)\right)\left(\frac{\partial}{\partial r} f(t,r)\right) + 4\left(\frac{\partial}{\partial r} h(t,r)\right) f(t,r)^2 r + r^2 f(t,r)\left(\frac{\partial}{\partial t} h(t,r)\right)^2 + r^2 h(t,r)\left(\frac{\partial}{\partial t} f(t,r)\right)\left(\frac{\partial}{\partial t} h(t,r)\right) + 4 f(t,r)^2 h(t,r)\,(h(t,r)-1)\Bigg) \quad (4.1$$

```
M > RC3;
```

$$0\, dt \wedge dr \quad (4.2$$

Solve the Rainich conditions.

```
M > g1sol:=DGsolve([RC1, RC2&t dt, RC3], g1);
```

$$g1sol := \left\{ \_C1\, \_F2(r)\, dr \otimes dr + r^2\, d\theta \otimes d\theta + r^2 \sin(\theta)^2\, d\phi \otimes d\phi,\ \frac{(r + \_C1)\, \_F1(t)}{r}\, dt \otimes dt + \frac{r}{r + \_C1}\, dr \otimes dr + r^2\, d\theta \otimes d\theta + r^2 \sin(\theta)^2\, d\phi \otimes d\phi,\ \frac{\_F1(t)\,(\_C2\, r + r^2 - \_C1)}{r^2}\, dt \otimes dt + \frac{r^2}{\_C2\, r + r^2 - \_C1}\, dr \otimes dr + r^2\, d\theta \otimes d\theta + r^2 \sin(\theta)^2\, d\phi \otimes d\phi \right\} \quad (4.3$$

The first "solution" is not a metric. The second solution is trivial - it's Ricci-flat, i.e., a vacuum solution.

```
M > RicciTensor(g1sol[2]);
```

$$0\, dt \otimes dt \quad (4.4$$

The third solution is (locally, and up to a choice of the *t* coordinate) the Reissner-Nordstrom solution. We use the Rainich theory to reconstruct the electromagnetic field from the metric satisfying the Rainich conditions..

```
M > g1RN :=  eval(g1sol[3], [_C1 = -q^2, _C2 = -2*m, _F1(t) = -1]);
```

$$g1RN := -\frac{-2\, m\, r + q^2 + r^2}{r^2}\, dt \otimes dt + \frac{r^2}{-2\, m\, r + q^2 + r^2}\, dr \otimes dr + r^2\, d\theta \otimes d\theta + r^2 \sin(\theta)^2\, d\phi \otimes d\phi \quad (4.5$$

```
M > Ric := RicciTensor(g1RN);
```

$$Ric := \frac{-2\, m\, q^2\, r + q^4 + q^2\, r^2}{r^6}\, dt \otimes dt + \frac{q^2}{2\, m\, r^3 - q^2\, r^2 - r^4}\, dr \otimes dr + \frac{q^2}{r^2}\, d\theta \otimes d\theta + \frac{\sin(\theta)^2\, q^2}{r^2}\, d\phi \otimes d\phi \quad (4.6$$

```
M > DRic := CovariantDerivative(Ric, Christoffel(g1RN));
```

$$DRic := \frac{8\, m\, q^2\, r - 4\, q^4 - 4\, q^2\, r^2}{r^7}\, dt \otimes dt \otimes dr - \frac{4\, q^2}{\left(2\, m\, r - q^2 - r^2\right) r^3}\, dr \otimes dr \otimes dr - \frac{2\, q^2}{r^3}\, dr \otimes d\theta \otimes d\theta$$
$$- \frac{2 \sin(\theta)^2\, q^2}{r^3}\, dr \otimes d\phi \otimes d\phi - \frac{2\, q^2}{r^3}\, d\theta \otimes dr \otimes d\theta - \frac{4\, q^2}{r^3}\, d\theta \otimes d\theta \otimes dr - \frac{2 \sin(\theta)^2\, q^2}{r^3}\, d\phi \otimes dr \otimes d\phi$$
$$- \frac{4 \sin(\theta)^2\, q^2}{r^3}\, d\phi \otimes d\phi \otimes dr \quad (4.7$$

```
M > F := RainichElectromagneticField(g1RN, Ric, DRic) assuming q > 0, m > 0, r > 0, theta
    > 0, theta < Pi;
```

$$F := \frac{\cos(\_C1)\, \sqrt{2}\, q}{r^2}\, dt \wedge dr + \sin(\_C1) \sin(\theta)\, \sqrt{2}\, q\, d\theta \wedge d\phi \quad (4.8$$

Up to the usual duality rotation symmetry, this is the Coulomb field of the point charge featuring the Reissner-Nordstrom spacetime.

We conclude by verifying the Einstein-Maxwell equations for this solution.

Einstein equations:

```
M > G := EinsteinTensor(g1RN):
M > T := EnergyMomentumTensor("Electromagnetic", g1RN, F):
M > evalDG(G - T);
```

$$0 \tag{4.9}$$

Maxwell equations:

```
M > MatterFieldEquations("Electromagnetic", g1RN, F);
```

$$0\, \partial_t,\, 0\, dt \wedge dr \wedge d\theta \tag{4.10}$$

Case 2.

This case admits no solution to the Rainich conditions. We show this by taking a few components of the Rainich conditions and checking that they are inconsistent.

```
M > RC1, RC2, RC3 := RainichConditions(g2, output = "equations"):
M > RC2;
```

$$\frac{h(t,r)\left(\frac{\partial^2}{\partial t^2} f(t,r)\right) - 2\, h(t,r)\left(\frac{\partial^2}{\partial r \partial t} h(t,r)\right) - \left(\frac{\partial}{\partial t} h(t,r)\right)\left(-2\, \frac{\partial}{\partial r} h(t,r) + \frac{\partial}{\partial t} f(t,r)\right)}{h(t,r)^3} + \frac{2}{r^2} \tag{4.11}$$

```
M > RC3;
```

$$0\, dt \wedge dr \tag{4.12}$$

```
M > eq1 := Hook([D_t, D_r], RC1);
```

$$eq1 := \frac{1}{8\, h(t,r)^5\, r^4}\left(\left(r^2\, h(t,r)\left(\frac{\partial^2}{\partial t^2} f(t,r)\right) - 2\, r^2\, h(t,r)\left(\frac{\partial^2}{\partial r \partial t} h(t,r)\right) - r^2\left(-2\, \frac{\partial}{\partial r} h(t,r) + \frac{\partial}{\partial t} f(t,r)\right)\left(\frac{\partial}{\partial t} h(t,r)\right) + 2\, h(t,r)^3\right)\left(r^2\, h(t,r)\left(\frac{\partial^2}{\partial t^2} f(t,r)\right) - 2\, r^2\, h(t,r)\left(\frac{\partial^2}{\partial r \partial t} h(t,r)\right) - r^2\left(-2\, \frac{\partial}{\partial r} h(t,r) + \frac{\partial}{\partial t} f(t,r)\right)\left(\frac{\partial}{\partial t}\right.\right.\right. \tag{4.13}$$

$$h(t,r)\Big) - 2\,h(t,r)^3\Big)\Big)$$

```
M > eq2 := Hook([D_r, D_r], RC1 );
```

$$eq2 := \frac{1}{4\,h(t,r)^6\,r}\Bigg(\Bigg(-\left(\frac{\partial^2}{\partial t^2}\,f(t,r)\right)h(t,r)\,f(t,r)\,r + 2\,h(t,r)\,f(t,r)\left(\frac{\partial^2}{\partial r\partial t}\,h(t,r)\right)r + \left(-2\,\frac{\partial}{\partial r}\,h(t,r) + \frac{\partial}{\partial t}\,f(t,r)\right)\left(r\,f(t,r)\left(\frac{\partial}{\partial t}\,h(t,r)\right) + 4\,h(t,r)^2\right)\Bigg)\Bigg(-h(t,r)\left(\frac{\partial^2}{\partial t^2}\,f(t,r)\right) + 2\,h(t,r)\left(\frac{\partial^2}{\partial r\partial t}\,h(t,r)\right) + \left(\frac{\partial}{\partial t}\,h(t,r)\right)\left(-2\,\frac{\partial}{\partial r}\,h(t,r) + \frac{\partial}{\partial t}\,f(t,r)\right)\Bigg)\Bigg) - \frac{1}{8\,h(t,r)^6\,r^4}\Bigg(f(t,r)\Bigg(4\left(\frac{\partial^2}{\partial r\partial t}\,h(t,r)\right)^2 h(t,r)^2\,r^4 + 4\,r^4\Bigg(-h(t,r)\left(\frac{\partial^2}{\partial t^2}\,f(t,r)\right) + \left(\frac{\partial}{\partial t}\,h(t,r)\right)\left(-2\,\frac{\partial}{\partial r}\,h(t,r) + \frac{\partial}{\partial t}\,f(t,r)\right)\Bigg)h(t,r)\left(\frac{\partial^2}{\partial r\partial t}\,h(t,r)\right) + \left(\frac{\partial^2}{\partial t^2}\,f(t,r)\right)^2 h(t,r)^2\,r^4 - 2\left(\frac{\partial}{\partial t}\,h(t,r)\right)r^4\,h(t,r)\left(-2\,\frac{\partial}{\partial r}\,h(t,r) + \frac{\partial}{\partial t}\,f(t,r)\right)\left(\frac{\partial^2}{\partial t^2}\,f(t,r)\right) + r^4\left(-2\,\frac{\partial}{\partial r}\,h(t,r) + \frac{\partial}{\partial t}\,f(t,r)\right)^2\left(\frac{\partial}{\partial t}\,h(t,r)\right)^2 + 4\,h(t,r)^6\Bigg)\Bigg) \quad (4.14$$

```
M > eq3 := Hook([D_theta, D_theta], RC1);
```

$$eq3 := -\frac{1}{8\,r^2\,h(t,r)^6}\Bigg(\Bigg(r^2\,h(t,r)\left(\frac{\partial^2}{\partial t^2}\,f(t,r)\right) - 2\,r^2\,h(t,r)\left(\frac{\partial^2}{\partial r\partial t}\,h(t,r)\right) - r^2\left(-2\,\frac{\partial}{\partial r}\,h(t,r) + \frac{\partial}{\partial t}\,f(t,r)\right)\left(\frac{\partial}{\partial t}\,h(t,r)\right) + 2\,h(t,r)^3\Bigg)\Bigg(r^2\,h(t,r)\left(\frac{\partial^2}{\partial t^2}\,f(t,r)\right) - 2\,r^2\,h(t,r)\left(\frac{\partial^2}{\partial r\partial t}\,h(t,r)\right) - r^2\left(-2\,\frac{\partial}{\partial r}\,h(t,r) + \frac{\partial}{\partial t}\,f(t,r)\right)\left(\frac{\partial}{\partial t}\,h(t,r)\right) - 2\,h(t,r)^3\Bigg)\Bigg) \quad (4.15$$

```
M > pdsolve([eq1, eq2]);
Warning: System is inconsistent
```

We elaborate on this inconsistency by resolving two of the Rainich conditions for second derivatives of h followed by substitution into another Rainich condition, which yields an equation which has no acceptable solution.

Here we resolve eq1 and eq3 for a second derivative of h(t,r).

```
M > sol_13 := pdsolve([eq1, eq3]);
```

$$sol\_13 := \left[\left[\frac{\partial^2}{\partial r \partial t} h(t,r) = \frac{r^2 h(t,r)\left(\frac{\partial^2}{\partial t^2} f(t,r)\right) + 2\left(\frac{\partial}{\partial t} h(t,r)\right)\left(\frac{\partial}{\partial r} h(t,r)\right) r^2 - r^2\left(\frac{\partial}{\partial t} f(t,r)\right)\left(\frac{\partial}{\partial t} h(t,r)\right) + 2\, h(t,r)^3}{2\, r^2\, h(t,r)}\right]\right], \left[\left[\frac{\partial^2}{\partial r \partial t} h(t,r) = \frac{r^2 h(t,r)\left(\frac{\partial^2}{\partial t^2} f(t,r)\right) + 2\left(\frac{\partial}{\partial t} h(t,r)\right)\left(\frac{\partial}{\partial r} h(t,r)\right) r^2 - r^2\left(\frac{\partial}{\partial t} f(t,r)\right)\left(\frac{\partial}{\partial t} h(t,r)\right) - 2\, h(t,r)^3}{2\, r^2\, h(t,r)}\right]\right] \quad (4.16$$

```
> simplify(sol_13[1][1][1] - sol_13[2][1][1])
```

$$0 = \frac{2\, h(t,r)^2}{r^2} \quad (4.17$$

The solution *h(t, r) = 0* is not acceptable since the metric becomes degenerate and the original PDEs become undefined.

Case 3.

This is the case where the metric on $R^2 \times S^2$ is the metric product of two 2-dimensional constant curvature metrics.

Compute the Rainich conditions (output suppressed) and solve them:

```
M > RC3_1, RC3_2, RC3_3 := RainichConditions(g3, output = "equations"):
M > g3sol:=DGsolve([RC3_1, RC3_2&t dt, RC3_3], g3);
```

$$g3sol := \left\{ -\frac{2\, r0^2\, \_C2\, \_C3}{\cosh(\_C2\, t + \_C3\, r + \_C1)^2}\, dt \otimes dr - \frac{2\, r0^2\, \_C2\, \_C3}{\cosh(\_C2\, t + \_C3\, r + \_C1)^2}\, dr \otimes dt + r0^2\, d\theta \otimes d\theta + r0^2 \sin(\theta)^2\, d\phi \otimes d\phi \right\} \quad (4.18)$$

The integration constants _C1, _C2, _C3 can all be eliminated by an appropriate choice of the *t* and *r* coordinates.

```
M > g3sol1 := eval(g3sol[1], {_C1=0, _C2 = 1, _C3 = 1});
```

$$g3sol1 := -\frac{2\, r0^2}{\cosh(r+t)^2}\, dt \otimes dr - \frac{2\, r0^2}{\cosh(r+t)^2}\, dr \otimes dt + r0^2\, d\theta \otimes d\theta + r0^2 \sin(\theta)^2\, d\phi \otimes d\phi \quad (4.19)$$

Reconstruct the electromagnetic field from the metric satisfying the Rainich conditions.

```
M > F0:=RainichElectromagneticField(g3sol1);
```

$$F0 := \frac{2\cos(\_C1)\sqrt{\operatorname{csgn}\left(\frac{1}{r0^2}\right)+1}\; r0^2}{\sqrt{\frac{r0^2}{\cosh(r+t)^4}}\; \cosh(r+t)^4}\, dt \wedge dr + \frac{\sin(\_C1)\sqrt{\frac{\cosh(r+t)^4}{r0^8 \sin(\theta)^2}}\; r0^6 \sin(\theta)^2 \sqrt{\operatorname{csgn}\left(\frac{1}{r0^2}\right)+1}}{\sqrt{\frac{r0^2}{\cosh(r+t)^4}}\; \cosh(r+t)^4}\, d\theta \wedge d\phi \quad (4.20)$$

```
M > F3 := simplify(eval(F0, _C1 = 0)) assuming r0 > 0, t::real, r::real;
```

$$F3 := \frac{2\sqrt{2}\; r0}{\cosh(r+t)^2}\, dt \wedge dr + 0\, d\theta \wedge d\phi \quad (4.21)$$

Verify the Einstein-Maxwell equations.

```
M > G3 := EinsteinTensor(g3sol1);
```

$$G3 := \frac{\cosh(r+t)^2}{2\, r0^4}\, \partial_t \otimes \partial_r + \frac{\cosh(r+t)^2}{2\, r0^4}\, \partial_r \otimes \partial_t + \frac{1}{r0^4}\, \partial_\theta \otimes \partial_\theta + \frac{1}{r0^4 \sin(\theta)^2}\, \partial_\phi \otimes \partial_\phi \quad (4.22)$$

```
M > T3 := EnergyMomentumTensor("Electromagnetic", g3sol1, F3);
```

$$T3 := \frac{\cosh(r+t)^2}{2\,r0^4}\,\partial_t \otimes \partial_r + \frac{\cosh(r+t)^2}{2\,r0^4}\,\partial_r \otimes \partial_t + \frac{1}{r0^4}\,\partial_\theta \otimes \partial_\theta + \frac{1}{r0^4 \sin(\theta)^2}\,\partial_\phi \otimes \partial_\phi \quad (4.23$$

```
M > evalDG(G3 - T3);
```

$$0 \quad (4.24$$

```
M > MatterFieldEquations("Electromagnetic", g3sol1, F3);
```

$$0\,\partial_t,\ 0\,dt \wedge dr \wedge d\theta \quad (4.25$$

This solution, known as the Bertotti-Robinson solution, is again a product of constant curvature spaces (this time a negative curvature for R^2). Notice that the Bertotti-Robinson solution looks remarkably similar to the Nariai solution in these coordinates. But these two solutions are not isometric. Indeed, the Nariai solution does not solve the Rainich conditions. An easy way to see this is to compare their Ricci scalars. The key difference physically is that the Nariai solution is an Einstein space while the Bertotti-Robinson solution is an electrovacuum.

```
S > g3sol;
```

$$\left\{ -\frac{2\,r0^2\,\_C2\,\_C3}{\cosh(\_C2\,t+\_C3\,r+\_C1)^2}\,dt \otimes dr - \frac{2\,r0^2\,\_C2\,\_C3}{\cosh(\_C2\,t+\_C3\,r+\_C1)^2}\,dr \otimes dt + r0^2\,d\theta \otimes d\theta + r0^2 \sin(\theta)^2\,d\phi \otimes d \right. \quad (4.26$$

$$\left. \left\{ \phi \right. \qquad , \right\}$$

```
M > g3lambdasol;
```

$$\left\{ \frac{2\,\_C2\,\_C3}{\cosh(\_C2\,t+\_C3\,r+\_C1)^2\,\lambda}\,dt \otimes dr + \frac{2\,\_C2\,\_C3}{\cosh(\_C2\,t+\_C3\,r+\_C1)^2\,\lambda}\,dr \otimes dt + \frac{1}{\lambda}\,d\theta \otimes d\theta + \frac{\sin(\theta)^2}{\lambda}\,d\phi \otimes d\phi \right\} \quad (4.27$$

```
M > RainichConditions(g3sol[1]);
```

$$true \quad (4.28$$

```
M > RainichConditions(g3lambdasol[1]);
```

$$false \quad (4.29$$

```
M > RicciScalar(g3lambdasol[1]);
```

$$4\,\lambda \quad (4.30$$

```
M > RicciScalar(g3sol[1]);
```

$$0 \tag{4.31}$$

The key difference geometrically is that the Nariai solution is the product of two positive curvature spaces while the Bertotti-Robinson solution is the product of a positive curvature and a negative curvature space.

Null Electrovacua

Finally, let us consider the possibility of spherically symmetric null electrovacua. A necessary condition for a null electrovacuum is that the Ricci tensor should be non-vanishing and null [3]:

$$R_{ab}R^{bc} = 0 .$$

Such spacetimes correspond to "pure radiation" solutions of the Einstein equations (if the energy density is non-negative). This condition will be used here to rule out the possibility of spherically symmetric null electrovacua.

Case 1.

For this case it is convenient to make a change of t coordinate $t \rightarrow t + w(u, r)$ with $w(u,r)$ chosen so that ∂_r is null. The metric takes the following form.

```
M > DGsetup([u, r, theta, phi], M1null);
```

Manifold: M1null (4.1.1)

```
M > g1null := evalDG(f(u,r)*du &t du - 2*h(u,r)*du &s dr + r^2*(dtheta &t dtheta + sin
    (theta)^2*dphi &t dphi));
```

$$g1null := f(u, r)\, du \otimes du - h(u, r)\, du \otimes dr - h(u, r)\, dr \otimes du + r^2\, d\theta \otimes d\theta + r^2 \sin(\theta)^2\, d\phi \otimes d\phi \tag{4.1.2}$$

The functions f and $h \neq 0$ are arbitrary.

Compute the Ricci tensor and its square. Set the latter to zero and solve.

```
M > RT1null := RicciTensor(g1null):
M > RT1squared := TensorInnerProduct(g1null, RT1null, RT1null, tensorindices=[1]):
M > DGsolve(RT1squared, g1null);
```

$$\left\{ -\left(\_F1(u)^2 - \frac{\_F2(u)}{r} \right) du \otimes du - \_F1(u)\, du \otimes dr - \_F1(u)\, dr \otimes du + r^2\, d\theta \otimes d\theta + r^2 \sin(\theta)^2\, d\phi \otimes d\phi \right\} \quad (4.1.3)$$

With an appropriate choice of coordinate u this is the Vaidya metric, which is known not to be an electrovacuum [3].

Case 2.

Here we find there are no null Ricci tensors. The metric, the Ricci tensor, and the square of the Ricci tensor are as follows.

```
M > g2;
```

$$h(t,r)\, dt \otimes dr + h(t,r)\, dr \otimes dt + f(t,r)\, dr \otimes dr + r^2\, d\theta \otimes d\theta + r^2 \sin(\theta)^2\, d\phi \otimes d\phi \quad (4.1.4)$$

```
M > RT2 := RicciTensor(g2);
```

$$RT2 := \frac{h(t,r)\left(\frac{\partial^2}{\partial t^2} f(t,r)\right) - 2\, h(t,r)\left(\frac{\partial^2}{\partial r \partial t} h(t,r)\right) - \left(\frac{\partial}{\partial t} h(t,r)\right)\left(-2\frac{\partial}{\partial r} h(t,r) + \frac{\partial}{\partial t} f(t,r)\right)}{2\, h(t,r)^2}\, dt \otimes dr \quad (4.1.5)$$
$$+ \frac{h(t,r)\left(\frac{\partial^2}{\partial t^2} f(t,r)\right) - 2\, h(t,r)\left(\frac{\partial^2}{\partial r \partial t} h(t,r)\right) - \left(\frac{\partial}{\partial t} h(t,r)\right)\left(-2\frac{\partial}{\partial r} h(t,r) + \frac{\partial}{\partial t} f(t,r)\right)}{2\, h(t,r)^2}\, dr \otimes dt$$
$$+ \frac{1}{2\, h(t,r)^3\, r}\left(\left(\frac{\partial^2}{\partial t^2} f(t,r)\right) h(t,r)\, f(t,r)\, r - 2\, h(t,r)\, f(t,r)\left(\frac{\partial^2}{\partial r \partial t} h(t,r)\right) r - \left(-2\frac{\partial}{\partial r} h(t,r) + \frac{\partial}{\partial t} f(t,r)\right)\left(r\, f(t,r)\left(\frac{\partial}{\partial t} h(t,r)\right) + 2\, h(t,r)^2\right)\right) dr \otimes dr + d\theta \otimes d\theta + \sin(\theta)^2\, d\phi \otimes d\phi$$

```
M > RT2squared := TensorInnerProduct(g2, RT2, RT2, tensorindices=[1]);
```

$$RT2squared := \frac{\left(h(t,r)\left(\frac{\partial^2}{\partial t^2} f(t,r)\right) - 2\,h(t,r)\left(\frac{\partial^2}{\partial r \partial t} h(t,r)\right) - \left(\frac{\partial}{\partial t} h(t,r)\right)\left(-2\frac{\partial}{\partial r} h(t,r) + \frac{\partial}{\partial t} f(t,r)\right)\right)^2}{4\,h(t,r)^5}\, dt \otimes dr + \frac{\left(h(t,r)\left(\frac{\partial^2}{\partial t^2} f(t,r)\right) - 2\,h(t,r)\left(\frac{\partial^2}{\partial r \partial t} h(t,r)\right) - \left(\frac{\partial}{\partial t} h(t,r)\right)\left(-2\frac{\partial}{\partial r} h(t,r) + \frac{\partial}{\partial t} f(t,r)\right)\right)^2}{4\,h(t,r)^5}\, dr \otimes dt + \frac{1}{4\,h(t,r)^6\, r}\left(\left(h(t,r)\left(\frac{\partial^2}{\partial t^2} f(t,r)\right) - 2\,h(t,r)\left(\frac{\partial^2}{\partial r \partial t} h(t,r)\right) - \left(\frac{\partial}{\partial t} h(t,r)\right)\left(-2\frac{\partial}{\partial r} h(t,r) + \frac{\partial}{\partial t} f(t,r)\right)\right)\left(\left(\frac{\partial^2}{\partial t^2} f(t,r)\right) h(t,r) f(t,r)\, r - 2\,h(t,r) f(t,r)\left(\frac{\partial^2}{\partial r \partial t} h(t,r)\right) r - \left(-2\frac{\partial}{\partial r} h(t,r) + \frac{\partial}{\partial t} f(t,r)\right)\left(r f(t,r)\left(\frac{\partial}{\partial t} h(t,r)\right) + 4\,h(t,r)^2\right)\right)\right) dr \otimes dr + \frac{1}{r^2}\, d\theta \otimes d\theta + \frac{\sin(\theta)^2}{r^2}\, d\phi \otimes d\phi \quad \textbf{(4.1.6)}$$

The spherical components cannot vanish.

```
M > evalDG(RT2squared, [D_theta, D_theta]);
```

$$\frac{\left(h(t,r)\left(\frac{\partial^2}{\partial t^2} f(t,r)\right) - 2\,h(t,r)\left(\frac{\partial^2}{\partial r \partial t} h(t,r)\right) - \left(\frac{\partial}{\partial t} h(t,r)\right)\left(-2\frac{\partial}{\partial r} h(t,r) + \frac{\partial}{\partial t} f(t,r)\right)\right)^2}{4\,h(t,r)^5}\, dt \otimes dr + \frac{\left(h(t,r)\left(\frac{\partial^2}{\partial t^2} f(t,r)\right) - 2\,h(t,r)\left(\frac{\partial^2}{\partial r \partial t} h(t,r)\right) - \left(\frac{\partial}{\partial t} h(t,r)\right)\left(-2\frac{\partial}{\partial r} h(t,r) + \frac{\partial}{\partial t} f(t,r)\right)\right)^2}{4\,h(t,r)^5}\, dr \otimes dt + \frac{1}{4\,h(t,r)^6\, r}\left(\left(h(t,r)\left(\frac{\partial^2}{\partial t^2} f(t,r)\right) - 2\,h(t,r)\left(\frac{\partial^2}{\partial r \partial t} h(t,r)\right) - \left(\frac{\partial}{\partial t} h(t,r)\right)\left(-2\frac{\partial}{\partial r} h(t,r) + \frac{\partial}{\partial t} f(t,r)\right)\right)\left(\left(\frac{\partial^2}{\partial t^2} f(t,r)\right) h(t,r) f(t,r)\, r - 2\,h(t,r) f(t,r)\left(\frac{\partial^2}{\partial r \partial t} h(t,r)\right) r - \left(-2\frac{\partial}{\partial r} h(t,r) + \frac{\partial}{\partial t} f(t,r)\right)\left(r f(t, \quad \textbf{(4.1.7)}$$

$$r)\left(\frac{\partial}{\partial t}h(t,r)\right)+4h(t,r)^2\Big)\Big)\Big)\,dr\otimes dr+\frac{1}{r^2}\,d\theta\otimes d\theta+\frac{\sin(\theta)^2}{r^2}\,d\phi\otimes d\phi$$

Case 3.

Again we find there are no null Ricci tensors. The metric, the Ricci tensor, and the square of the Ricci tensor are as follows.

```
M > g3;
```

$$h(t,r)\,dt\otimes dr+h(t,r)\,dr\otimes dt+r0^2\,d\theta\otimes d\theta+r0^2\sin(\theta)^2\,d\phi\otimes d\phi \qquad (4.1.8)$$

```
M > RT3 := RicciTensor(g3);
```

$$RT3 := \frac{\left(\frac{\partial}{\partial t}h(t,r)\right)\left(\frac{\partial}{\partial r}h(t,r)\right)-h(t,r)\left(\frac{\partial^2}{\partial r\partial t}h(t,r)\right)}{h(t,r)^2}\,dt\otimes dr + \frac{\left(\frac{\partial}{\partial t}h(t,r)\right)\left(\frac{\partial}{\partial r}h(t,r)\right)-h(t,r)\left(\frac{\partial^2}{\partial r\partial t}h(t,r)\right)}{h(t,r)^2}\,dr\otimes dt+d\theta\otimes d\theta+\sin(\theta)^2\,d\phi\otimes d\phi \qquad (4.1.9)$$

```
M > TensorInnerProduct(g3, RT3, RT3, tensorindices=[1]);
```

$$\frac{\left(\left(\frac{\partial}{\partial t}h(t,r)\right)\left(\frac{\partial}{\partial r}h(t,r)\right)-h(t,r)\left(\frac{\partial^2}{\partial r\partial t}h(t,r)\right)\right)^2}{h(t,r)^5}\,dt\otimes dr + \frac{\left(\left(\frac{\partial}{\partial t}h(t,r)\right)\left(\frac{\partial}{\partial r}h(t,r)\right)-h(t,r)\left(\frac{\partial^2}{\partial r\partial t}h(t,r)\right)\right)^2}{h(t,r)^5}\,dr\otimes dt+\frac{1}{r0^2}\,d\theta\otimes d\theta+\frac{\sin(\theta)^2}{r0^2}\,d\phi\otimes d\phi \qquad (4.1.10)$$

Commands Illustrated

- KillingVectors, LieAlgebraData, GenerateSymmetricTensors, InvariantGeometricObjectFields, RicciTensor, DGsolve, RainichConditions, RainichElectromagneticField

References

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Release notes

• Worksheet was executed using Maple 2018 and DifferentialGeometry build "USU001-1460 9:25:48.66 Fri 01/03/2020"

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