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Distribution-oblivious Online Algorithms for Age-of-Information Penalty Minimization

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***Abstract*—The ever-increasing needs of supporting real-time applications have spurred new studies on minimizing Age-of-Information (AoI), a novel metric characterizing the data freshness of the system. This work studies the single-queue information update system and strengthens the seminal results of Sun *et al.* on the following fronts: (i) When designing the optimal offline schemes with full knowledge of the delay distributions, a new *fixed-point-based* method is proposed with *quadratic convergence rate*, an order-of-magnitude improvement over the state-of-the-art; (ii) When the distributional knowledge is unavailable (which is the norm in practice), two new low-complexity online algorithms are proposed, which provably attain the optimal average AoI penalty; and (iii) the online schemes also admit a modular architecture, which allows the designer to *upgrade* certain components to handle additional practical challenges. Two such upgrades are proposed for the situations: (iii.1) The AoI penalty function is also unknown and must be estimated on the fly, and (iii.2) the unknown delay distribution is Markovian instead of i.i.d. The performance of our schemes is either provably optimal or within 3% of the omniscient optimal offline solutions in all simulation scenarios.**

***Index Terms*—Age-of-information, online algorithm, fixed-point equation, stochastic approximation algorithm.**

I. Introduction

Thanks to the accelerating growth of networked systems in the past decades, the capability of providing real-time status updates has been the cornerstone of many important practical systems. Examples include remote health monitoring, GPS location tracking and closed-loop drone control. Recent development of the Internet of Things (IoT) also promises real-time communication between numerous devices [2].

Since stale data is often of less value, it is crucial to optimize the *data freshness* of the system. An elementary approach is to transmit as many updates as possible. This, however, may clog the network and consume excessive energy. Recently, Age-of-Information (AoI) was introduced to characterize the level of information freshness [3], which has since been the foundation of many studies on data freshness control.

Early AoI minimization works studied the model where update packets arrive at the destination according to specific stochastic processes. [4], [5] studied the *generate-at-will* model and showed that to minimize the average AoI, the source node often has to *wait before sending the next packet* even when the channel/queue is currently idle. [6], [7] unified the freshness control [4] and remote estimation settings of [8] under a new *remote control* setting and derived the optimal joint source-&-destination policy. [9] found the optimal scheduling policy of a joint network cost and AoI minimization problem when multiple independent queues may share a common network cost constraint.

This work revisits and significantly strengthens the existing results [4], [5] with the following contributions: (i) When designing the optimal offline schemes with full knowledge of the delay distributions, all existing results [4]–[9] used a bisection search to find the optimal policy, which exhibits linear convergence rate. In contrast, *we propose a fixed-point-based method of computing the optimal policy under any arbitrarily given AoI penalty function, which exhibits quadratic convergence rate, an order-of-magnitude improvement over the state of the art.*

(ii) In most prior works [4]–[6], [8]–[12], the knowledge of delay distribution is required before one can numerically find the optimal waiting policy. In practice it may be difficult to know the delay distribution *a priori* since each sample (i.e., packet transmission) takes a full round-trip-time to complete and one may need many samples to accurately estimate the probability density function, which can be exceedingly time consuming. Furthermore, the delay distribution is constantly subject to network topology changes and traffic fluctuations [13], which further complicate the task of learning the distribution. To address these issues, *this work derives two new low-complexity online algorithms for arbitrarily given AoI penalty functions, and they provably converge to the optimum without knowing the delay distribution*, a result that could have substantial impact on practical protocol designs.

(iii) The new online schemes admit a modular architecture, which allows the designer to *upgrade* certain components to tackle additional practical challenges. Two such upgrades are proposed for the following two useful situations: Situation #1: Existing works [4], [5] assumed that the AoI penalty function $\gamma(\cdot)$ is known *a priori*. However, in practice, transmission decision is often made at the source but the penalty is often incurred at the destination. Therefore, it could be difficult for the source to know the AoI penalty function $\gamma(\cdot)$ a priori. A real-life analogy is that a vendor s (stands for the source) understands that less fresh produce will make his/her customer d (stands for the destination) unhappy but s may not know how unhappy d would be until d eventually receives the (not-so-fresh) produce. Furthermore, even d may not know how he/she will react to the stale produce until he/she actually receives the delivery. As a result, the AoI penalty function is not known

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in advance and our goal is to design a near-optimal online scheme with zero knowledge of either the delay distribution or the $\gamma(\cdot)$ penalty function. In this work, we design such a scheme by leveraging the *monotonic regression* method to estimate any arbitrary, non-linear $\gamma(\cdot)$, which further broadens the applicability.

Situation #2: The online schemes in (ii) provably converge to optimality for any unknown i.i.d. delay. In practice, the delay process may exhibit some memory/Markov behavior. We have devised a more versatile scheme based on K-nearest-neighbors (KNN). In our extensive simulation, even under the most challenging setting of arbitrary unknown Markov delay distribution and zero knowledge of $\gamma(\cdot)$, the performance of the resulting scheme is always within 2% of the omniscient offline optimal solution.

The rest of the paper is organized as follows. In Sec. II, we make detailed comparison to existing works. In Sec. III, we present our system model and problem formulation. In Sec. IV, we derive the analytical results for the optimal offline policy. Sec. V describes two online algorithms that are provably convergent to the optimum under any unknown i.i.d. delay distributions. Some practical issues are addressed in Secs. VI (under i.i.d. delay setting) and VII (under Markov delay setting). Numerical results are reported in Sec. VIII, and we conclude our work in Sec. IX. Most of the proofs will be provided in the appendices.

II. Related Works

One approach of handling unknown delay distribution is to apply reinforcement learning (RL) [14], [15]. However, none of these RL-based AoI minimization schemes has a provable optimality guarantee and can be strictly suboptimal in many cases. For example, while exhibiting some promising performance, the RL scheme in [15] is not able to converge to the optimal scheme in any of the experiments in [15]. In contrast, this work proposes two adaptive schemes that converge to optimality both analytically and in numerical experiments.

Additionally, some previous works proposed online algorithms with bounded regret or provable performance [16], [17]. However, they all studied the simplest *linear* age penalty function or assumed the transmission delay is *deterministic*. For example, a provably optimal online algorithm was derived in [17] with the focus exclusively on linear AoI penalty function. This works allows for *non-linear* AoI penalty function and *random* transmission delay simultaneously. In terms of regret minimization, since our schemes converge to optimality, they can be viewed as *no-regret* policies, a strict improvement over bounded-regret solutions [16].

III. Model and Formulation

A. System Model with Two-way Delay

Consider the system in Fig. 1, which comprises a source, a destination, a forward source-to-destination (s2d) channel and a backward destination-to-source (d2s) channel. We assume the following ACK-based model: At any time instant $t \in \mathbb{R}_+$, the source can generate a (status) update packet and transmit it to the destination, the *generate-at-will* model [4], [5], [18]. When the destination receives the update packet, an ACK is transmitted back to the source immediately. Once the source receives the ACK, then it can either transmit the next update packet immediately or wait for an arbitrary (but finite) amount of time. After the next transmission, it again waits for ACK. The process repeats itself indefinitely.

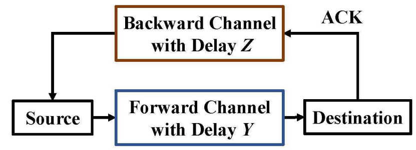


Fig. 1: Our system model with two-way delay.

Both the s2d and d2s channels incur some random delay.[1] We assume all packets are time stamped and describe the detailed system evolution as follows.

Time sequences: The system consists of three discrete-indexed real-valued non-negative random processes X_i, Y_i, and Z_i, for all $i \geq 0$. X_i is the waiting time of the i-th update packet at the source; Y_i (resp. Z_i) is the random delay for the i-th use of the s2d (resp. d2s) channel.

The instant when the i-th waiting time is over is denoted by S_i. That is, at time S_i, the i-th packet is generated and immediately transmitted. It is delivered to the destination at time D_i. The source will receive its ACK at time A_i. The values of (S_i, D_i, A_i) refer to the absolute time instants while the values of (X_i, Y_i, Z_i) represent the lengths of the intervals. They are related by the following equations: Initialize $A_0 = X_0 = Y_0 = Z_0 = 0$. For all $i \geq 1$, we have $S_i = A_{i-1} + X_i$, $D_i = S_i + Y_i$, and $A_i = D_i + Z_i$. We call the time interval $[A_{i-1}, A_i)$ as the *i-th round*, which consists of the i-th waiting time X_i at the source, the i-th forward delay Y_i and backward delay Z_i. See Fig. 2. The AoI $\Delta(t)$ is defined by

$$\Delta(t) \triangleq t - \max\{S_i : D_i \leq t\}. \tag{1}$$

The *AoI penalty function* $\gamma(\cdot) : [0, \infty) \to [0, \infty)$ quantifies the cost of stale data. Three popular choices are: (i) linear $\gamma_{\text{lin}}(\Delta) = \Delta$ [4]; (ii) exponential $\gamma_{\text{exp}}(\Delta) = e^{a\Delta} - 1$ for some constant $a > 0$ [5]; and (iii) quadratic $\gamma_{\text{qdr}}(\Delta) = \Delta^2$ [4]. Our results hold for any choice of $\gamma(\cdot)$ satisfying the technical assumption described in the next paragraph.

Technical assumptions: We assume (i) there exist finite $y_{\max}$, $z_{\max}$, and $yz_{\min} > 0$ such that $\mathbb{P}(Y_i \leq y_{\max}) = \mathbb{P}(Z_i \leq z_{\max}) = \mathbb{P}(Y_i + Z_i \geq yz_{\min}) = 1$; (ii) (Y_i, Z_i) can be of arbitrary joint distribution $\mathbb{P}_{YZ}$ but the vector random process $\{(Y_i, Z_i) : i \geq 1\}$ is stationary, Markov and ergodic; (iii) The AoI penalty function $\gamma(\cdot) : [0, \infty) \to [0, \infty)$ is a continuous and *strictly increasing* function satisfying $\gamma(0) = 0$.

[1] If we assume the d2s delay is zero with probability one, then the setting is identical that of [4]. The consideration of random d2s delay is to provide additional flexibility if needed.

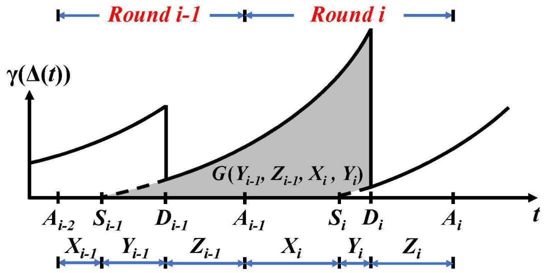


Fig. 2: Evolution of the AoI penalty function $\gamma(\Delta(t))$.

B. The Objective

Our goal is to minimize the long-term average AoI penalty:

$$\beta^* \triangleq \inf_{\{X_i\}} \limsup_{T\to\infty} \frac{1}{T}\int_0^T \mathbb{E}\{\gamma(\Delta(t))\}\, dt. \quad (2)$$

To simplify (2), we define two deterministic functions:

$$G(y', z', x, y) \triangleq \int_0^{y'+z'+x+y} \gamma(t)dt - \int_0^{y} \gamma(t)dt \quad (3)$$

$$G_1(y', z', x) \triangleq \mathbb{E}\{G(y', z', x, Y_i)|Y_{i-1} = y', Z_{i-1} = z'\} \quad (4)$$

where $G_1(y', z', x)$ is the conditional expectation of $G(y', z', x, Y_i)$ over Y_i given $Y_{i-1} = y'$ and $Z_{i-1} = z'$. The intuition behind (3) is that the shaded area in Fig. 2 is characterized by $G(Y_{i-1}, Z_{i-1}, X_i, Y_i)$. By noticing that the overall area underneath $\gamma(\Delta(t))$ can be decomposed as a summation of smaller sub-areas with shapes similar to the shaded area $G(Y_{i-1}, Z_{i-1}, X_i, Y_i)$ in Fig. 2, the optimization problem in (2) can be rewritten as

$$\beta^* = \inf_{\{X_i\}} \limsup_{n\to\infty} \frac{\sum_{i=1}^{n} \mathbb{E}\{G(Y_{i-1}, Z_{i-1}, X_i, Y_i)\}}{\sum_{i=1}^{n} \mathbb{E}\{Y_{i-1} + Z_{i-1} + X_i\}}. \quad (5)$$

Since (5) is a Markov decision problem with stationary, Markov and ergodic $\{(Y_i, Z_i)\}$, it suffices to find the optimal policy for the *single-round* optimization problem instead (see [4], [5] for the detailed derivation). The optimization problem (5) can thus be simplified as

$$\beta^* = \inf_{X_i} \frac{\mathbb{E}\{G_1(Y_{i-1}, Z_{i-1}, X_i)\}}{\mathbb{E}\{Y_{i-1} + Z_{i-1} + X_i\}} \quad (6)$$

where the numerator of (6) follows from (4).

We conclude this subsection by defining a constant β_{ZW} that would be useful for subsequent discussion.

$$\beta_{\text{ZW}} \triangleq \frac{\mathbb{E}\{G_1(Y_{i-1}, Z_{i-1}, 0)\}}{\mathbb{E}\{Y_{i-1} + Z_{i-1} + 0\}}. \quad (7)$$

That is, β_{ZW} is the objective function value in (6) when evaluated using a Zero-Wait policy.

Lemma 1: We must have $\beta_{\text{ZW}} < \infty$.

Proof: By the assumption about $y_{\max}$, $z_{\max}$, $yz_{\min}$, and the monotonicity of $\gamma(\cdot)$, we have $\mathbb{E}\{Y_{i-1}\}+\mathbb{E}\{Z_{i-1}\} \geq yz_{\min} > 0$ and $\mathbb{E}\{G_1(Y_{i-1}, Z_{i-1}, 0)\} \leq \mathbb{E}\{G_1(y_{\max}, z_{\max}, 0)\} < \infty$. As a result, $\beta_{\text{ZW}} < \infty$. ■

IV. Analytical Results

A. A Hitting-time-based Policy

At time A_{i-1}, the source has the knowledge of the past delays Y_{i-1} and Z_{i-1} since all packets are time stamped. As a result, we can write any waiting time rule $X_i = \phi(Y_{i-1}, Z_{i-1})$ as a function of (Y_{i-1}, Z_{i-1}).

Definition 1: We say a scheme A is *of finite expected duration* (FED) if $\mathbb{E}\{\phi(Y_{i-1}, Z_{i-1})\} < \infty$.

Practically speaking, it is crucial that the waiting time of each packet transmission has finite expectation. We thus limit the domain of the optimization problem of (6) to FED schemes only and ignore schemes that are not of FED.

Once we specify a waiting time function $\phi(Y_{i-1}, Z_{i-1})$, the resulting[2] averaged AoI penalty, not necessarily the minimum one, becomes

$$\text{Avg. AoI Penalty: } \frac{\mathbb{E}\{G_1(Y_{i-1}, Z_{i-1}, \phi(Y_{i-1}, Z_{i-1}))\}}{\mathbb{E}\{Y_{i-1} + Z_{i-1} + \phi(Y_{i-1}, Z_{i-1})\}}. \quad (8)$$

For any FED scheme A, we denote its average AoI penalty by β_A, which is evaluated by (8).

Define

$$\beta_{\text{UB}} \triangleq \lim_{t\to\infty} \gamma(t) \quad (9)$$

Note that the constant β_{UB} can be either a finite constant $0 < \beta_{\text{UB}} < \infty$ or infinity $\beta_{\text{UB}} = \infty$ if $\gamma(\cdot)$ grows unbounded. We then have the following lemma.

Lemma 2: For any arbitrary FED scheme A with average AoI penalty β_A, if $\beta_A < \infty$, then $\beta_A \in [0, \beta_{\text{UB}})$.

Proof: See Appendix A. ■

The intuition of Lemma 2 is that β_{UB} is a strict upper bound for the performance of any reasonable scheme A (excluding those poorly designed schemes having infinite average AoI penalty $\beta_A = \infty$).

Note that since zero-wait is a FED policy, by combining Lemmas 1 and 2, we have

$$0 \leq \beta^* \leq \beta_{\text{ZW}} < \beta_{\text{UB}} \leq \infty. \quad (10)$$

We now describe a special scheme Γ_β. For any given $\beta \in [0, \beta_{\text{UB}})$, scheme Γ_β has the following special decision rule:

$$X_i = \phi_{\Gamma,\beta}(Y_{i-1}, Z_{i-1}) \triangleq \inf\left\{t > 0 : \frac{d}{dt}G_1(Y_{i-1}, Z_{i-1}, t) > \beta\right\}. \quad (11)$$

By (4), $G_1(Y_{i-1}, Z_{i-1}, t)$ is the expected AoI penalty (the shaded area in Fig. 2) if the i-th waiting time is $X_i = t$. Therefore, the decision rule $\phi_{\Gamma,\beta}$ in (11) essentially chooses the hitting time for which the *growth rate*[3] of the expected AoI penalty $G_1(Y_{i-1}, Z_{i-1}, t)$ first hits the threshold β.

An important remark is that *the input parameter β of the above scheme Γ_β must be within $[0, \beta_{UB})$*. The reason is due to the following lemmas.

[2] The scheduling rule ϕ can be deterministic or randomized. In case of the latter, the expectation in (8) takes the average over the randomness in ϕ.

[3] As will be shown in Lemma 3, for any given (y', z'), $G_1(y', z', t)$ is differentiable with respect to t.

Lemma 3: For any given $y', z', t < \infty$, we have

$$\frac{d}{dt}G_1(y', z', t) = \mathbb{E}\left\{\gamma(y' + z' + t + Y_i)|Y_{i-1} = y', Z_{i-1} = z'\right\}. \quad (12)$$

Proof: See Appendix B. ■

Lemma 4: For any arbitrary $\beta \in [0, \beta_{\text{UB}})$, there exists a $t_{\text{UB}} < \infty$ such that $\frac{d}{dt}G_1(y', z', t_{\text{UB}}) > \beta$ for all y', z'. Conversely, for any $\beta \geq \beta_{\text{UB}}$, $\frac{d}{dt}G_1(y', z', t) \leq \beta$ for all finite $t, y', z' < \infty$ values. See Appendix C for the proof.

Lemma 4 implies that for any $\beta \in [0, \beta_{\text{UB}})$, the X_i value computed by (11) satisfies $X_i \leq t_{\text{UB}} < \infty$ almost surely, and the resulting Γ_β is thus a FED scheme. On the other hand, for any $\beta \geq \beta_{\text{UB}}$, the second half of Lemma 4 implies that the X_i value computed by (11) is always infinite. The resulting scheme thus has $\mathbb{P}(X_i = \infty) = 1$ and is catastrophic to the system. As a result, every time we describe/use the Γ_β scheme, it is critical to ensure the input parameter satisfying $\beta \in [0, \beta_{\text{UB}})$.

For this scheme Γ_β, we use $f_\Gamma(\beta)$ to denote its average AoI penalty, which can be computed by substituting the ϕ in (8) with the $\phi_{\Gamma,\beta}$ in (11). The input argument "(β)" highlights the fact that the average AoI penalty of the decision rule $\phi_{\Gamma,\beta}$ is a function of the hitting time threshold β.

Proposition 1: For any FED scheme A with scheduling rule ϕ_A and the corresponding average AoI penalty $\beta_A < \infty$, the following inequality must hold: $f_\Gamma(\beta_A) \leq \beta_A$.

The physical interpretation of this proposition is as follows. For scheme A that satisfies $\beta_A < \infty$, its average AoI penalty must also satisfy $\beta_A \in [0, \beta_{\text{UB}})$ by Lemma 2. Since the new scheme Γ_β in (11) can take any arbitrary $\beta \in [0, \beta_{\text{UB}})$ as input, we can use β_A as the hitting time threshold in (11). Then $f_\Gamma(\beta_A)$, the AoI penalty of the new scheme Γ_{β_A}, will be no worse than the average AoI penalty β_A of the original scheme A.

Proof: We provide high-level sketches. The details are relegated to Appendix D.

For schemes A and Γ_{β_A}, recall that $\phi_A(Y_{i-1}, Z_{i-1})$ and $\phi_{\Gamma,\beta_A}(Y_{i-1}, Z_{i-1})$ are the waiting times for schemes A and Γ_{β_A}, respectively. For simplicity, we use ϕ_A and ϕ_{Γ,β_A} as shorthand by dropping the input arguments (Y_{i-1}, Z_{i-1}).

Suppose we are in the event of $\phi_{\Gamma,\beta_A} \leq \phi_A$, i.e., the scheme Γ_{β_A} sends the i-th update earlier than the scheme A. *During the interval $(\phi_{\Gamma,\beta_A}, \phi_A]$, the growth rate of $G_1(Y_{i-1}, Z_{i-1}, t)$ is strictly higher than β_A*. The reason is as follows. By the definition of ϕ_{Γ,β_A} in (11), the growth rate of $G_1(Y_{i-1}, Z_{i-1}, t)$ at time $t = \phi_{\Gamma,\beta_A}$ is greater than or equal to β_A. Since the growth rate of $G_1(Y_{i-1}, Z_{i-1}, t)$ is strictly increasing (due to strictly increasing $\gamma(\cdot)$ and by Lemma 3), the growth rate of $G_1(Y_{i-1}, Z_{i-1}, t)$ is strictly larger than β_A during $(\phi_{\Gamma,\beta_A}, \phi_A]$. Compared to the original scheme A, the new scheme Γ_{β_A} avoids "higher-than-β_A" AoI penalty accumulation rates during the interval $(\phi_{\Gamma,\beta_A}, \phi_A]$, which in turn helps make its average AoI penalty $f_\Gamma(\beta_A)$ smaller than the benchmark β_A.

Similarly, in the event of $0 \leq \phi_A < \phi_{\Gamma,\beta_A}$, during the interval $(\phi_A, \phi_{\Gamma,\beta_A}]$, the new scheme Γ_{β_A} will experience "no-higher-than-β_A" AoI penalty accumulation rates since the growth rate of $G_1(Y_{i-1}, Z_{i-1}, t)$ has not hit β_A yet during $t \in (\phi_A, \phi_{\Gamma,\beta_A}]$, which again helps make $f_\Gamma(\beta_A)$ lower than β_A.

Since in either case the average AoI penalty of Γ_{β_A} has improved over the benchmark β_A, we have proven Proposition 1. ■

Recall that β^* is the minimum of (6). Since $\beta^* \in [0, \beta_{\text{UB}})$ by (10), Proposition 1 implies $\beta^* \geq f_\Gamma(\beta^*)$. Since Γ_{β^*} is yet another scheme, (6) implies $\beta^* \leq f_\Gamma(\beta^*)$. Jointly we have

Corollary 1: The minimum average AoI penalty value β^* is a root of the fixed-point equation

$$\beta = f_\Gamma(\beta) \quad (13)$$

over the domain $[0, \beta_{\text{UB}})$.

One can complement this corollary by the following Lemma.

Lemma 5: For any given penalty function $\gamma(\cdot)$, the equation $\beta = f_\Gamma(\beta)$ has a unique root in the domain $\beta \in [0, \beta_{\text{UB}})$.

The proof of Lemma 5 is relegated to Appendix E. Jointly Corollary 1 and Lemma 5 show that the task of finding β^* can be found by solving the fixed-point equation (13) over the domain $\beta \in [0, \beta_{\text{UB}})$. Secondly, if we know the value of β^*, then we can obtain the optimal policy by plugging β^* into the hitting time rule $\phi_{\Gamma,\beta}(\cdot,\cdot)$ in (11). Namely, the fixed-point equation not only finds the β^* but also finds a β^*-attaining optimal policy.

Remark 1: Corollary 1 is of similar form to [4, Theorem 3] and [5, Theorem 1]. However, the way we derive Corollary 1 is new. In [4], [5], the authors first defined the corresponding Lagrangian, then reformulated and solved it as a convex optimization problem, and finally showed that it admits no duality gap. In contrast, we first prove an intuitive result in Proposition 1 and the optimality conditions then follow suit naturally.

Remark 2: The function $f_\Gamma(\beta)$ can be computed easily by (3), (4), (8), (11), together with the complete knowledge of distribution $\mathbb{P}_{Y_{i-1},Z_{i-1}}$.

Remark 3: Since β for the scheme Γ_β must satisfy $\beta \in [0, \beta_{\text{UB}})$, the corresponding average penalty value $f_\Gamma(\beta)$ is defined only over the domain $[0, \beta_{\text{UB}})$. It is possible to extend the domain of $f_\Gamma(\beta)$ beyond $[0, \beta_{\text{UB}})$ by defining $f_\Gamma(\beta_{\text{UB}}) \triangleq \lim_{\beta\to\beta_{\text{UB}}} f_\Gamma(\beta)$. Under this extended domain $[0, \beta_{\text{UB}}]$, it is possible to have another fixed point $\beta_{\text{UB}} = f_\Gamma(\beta_{\text{UB}})$ as observed in [19]. At the same time, as explained in Lemma 4, any $\beta \geq \beta_{\text{UB}}$ will lead to schemes with $\mathbb{P}(X_i = \infty) = 1$ and such extended domain $[0, \beta_{\text{UB}}]$ is thus considered practically irrelevant.

B. Fast Fixed-point Iteration for Computing β^*

We now present a new way of computing β^* using (13).

Proposition 2: Set $\beta_0 = 0$ and iteratively compute $\beta_i = f_\Gamma(\beta_{i-1})$ for all $i = 1, 2, 3, \cdots$. The resulting sequence $\{\beta_i : i \geq 1\}$ is non-increasing and converges to the optimal β^*. Furthermore, if we also assume $f_\Gamma(\beta)$ is doubly continuously differentiable in an open neighborhood of β^*,[4] then the convergence speed of this iterative computation is quadratic.

Proof: We first note that our function $f_\Gamma(\beta)$ is defined only over the domain $[0, \beta_{\text{UB}})$. As a result, we first need to prove that $\beta_i \in [0, \beta_{\text{UB}})$ for all $i \geq 1$. Since $\beta_0 = 0 \in [0, \beta_{\text{UB}})$, the corresponding scheme $X_i = \phi_{\Gamma,\beta}(Y_{i-1}, Z_{i-1})$ in (11) always leads to $X_i = 0$, the zero-wait policy. As a result, $\beta_1 = f_\Gamma(0) = \beta_{\text{ZW}}$. By (10), we have $\beta_1 \in [0, \beta_{\text{UB}})$.

We now prove that $\beta_{i+1} \leq \beta_i \leq \beta_1 < \beta_{\text{UB}}$ for all $i \geq 1$. For any $i \geq 1$, since $\beta_i = f_\Gamma(\beta_{i-1})$, the value of β_i is the average AoI penalty for the scheme $\Gamma_{\beta_{i-1}}$. If we temporarily call the scheme $\Gamma_{\beta_{i-1}}$ as Scheme A, then we can apply Proposition 1 and obtain

$$f_\Gamma(\beta_i) \leq \beta_i. \tag{14}$$

The sequence $\{\beta_i : i \geq 1\}$ is thus non-increasing. Since $\beta_i \geq \beta^*$ for all $i \geq 1$, the sequence converges and we also have $\lim_{i\to\infty} \beta_i \in [0, \beta_{\text{UB}})$.

Since $\lim_{i\to\infty} \beta_i$ must be a root of $\beta = f_\Gamma(\beta)$, Lemma 5 implies $\lim_{i\to\infty} \beta_i = \beta^*$.

We now establish the quadratic convergence by proving the following inequality for all $i \geq i_0$, where i_0 is the first time β_i enters the neighborhood of β^* for which $f_\Gamma(\beta)$ is doubly differentiable.

Applying Taylor's expansion to $f_\Gamma(\beta)$ near β^*, we have:

$$\begin{aligned}
&\beta_{i+1} - \beta^* = f_\Gamma(\beta_i) - \beta^* \\
&= \left(f_\Gamma(\beta^*) + (\beta_i - \beta^*) f'_\Gamma(\beta^*) + \frac{f''_\Gamma(z_i)}{2} (\beta_i - \beta^*)^2 \right) - \beta^*
\end{aligned}$$

for some $z_i \in [\beta^*, \beta_i]$. Note that since $f_\Gamma(\beta)$ is doubly continuous in an open neighborhood containing β^* and since β^* minimizes $f_\Gamma(\beta)$, we must have $f'_\Gamma(\beta^*) = 0$. Then, by (i) $f_\Gamma(\beta^*) = \beta^*$ and (ii) $f'_\Gamma(\beta^*) = 0$ we have

$$\beta_{i+1} - \beta^* = \frac{f''_\Gamma(z_i)}{2} (\beta_i - \beta^*)^2. \tag{15}$$

Since $z_i \in [\beta^*, \beta_i] \subseteq [0, \beta_1]$, (15) implies

$$|\beta_{i+1} - \beta^*| \leq \left(\max_{z \in [\beta^*, \beta_1]} \frac{|f''_\Gamma(z)|}{2} \right) \cdot |\beta_i - \beta^*|^2, \ \forall i \geq i_0 \tag{16}$$

We can further relax the condition $i \geq i_0$ by noting that

$$\begin{aligned}
&|\beta_{i+1} - \beta^*| \\
&\leq \max\left(\max_{z \in [\beta^*, \beta_1]} \frac{|f''_\Gamma(z)|}{2}, \frac{|\beta_1 - \beta^*|}{(\beta_{i_0-1} - \beta^*)^2} \right) \cdot |\beta_i - \beta^*|^2, \\
&\quad \forall i \geq 1
\end{aligned} \tag{17}$$

which uses the fact that for all $i \in [1, i_0)$, we have

$$|\beta_{i+1} - \beta^*| \leq |\beta_1 - \beta^*|, \text{ and } \frac{|\beta_i - \beta^*|^2}{|\beta_{i_0-1} - \beta^*|^2} \geq 1 \tag{18}$$

due to the monotonicity of $\{\beta_i\}$. The inequality in (17) implies quadratic convergence rate of $\{\beta_i\}$. ■

[4]For instance, if (i) γ is doubly continuously differentiable and (ii) Y_i and Z_i are discrete random variables with $N_Y < \infty$ and $N_Z < \infty$ points having strictly positive probabilities, then $f_\Gamma(\beta)$ is doubly continuously differentiable for the entire domain $(0, \beta_{\text{UB}})$ expect for up to $N_Y \cdot N_Z$ points. Then as long as the optimal β^* does not fall into any of the $N_Y \cdot N_Z$ points, then this assumption holds. Another scenario for which such assumption holds is if both Y and Z are well-behaved continuous random variables, e.g., both being exponential or both being log-normal, etc. In this scenario, $f_\Gamma(\beta)$ is doubly continuously differentiable for the entire domain $(0, \beta_{\text{UB}})$ except for a single point $\beta_{\text{singular}} = \sup\{\beta \geq 0 : f_\Gamma(\beta) = \beta_{\text{ZW}}\}$.

V. Two Distribution-oblivious Online Algorithms for the I.I.D. Delay Setting

In the sequel, we propose two online algorithms that do not need the detailed probability distribution $\mathbb{P}_{Y_{i-1}, Z_{i-1}}$. This section considers the simpler setting in which the delay (vector) process $\{(Y_i, Z_i) : i \geq 1\}$ is *i.i.d.,* and we derive two online algorithms that are provably convergent to the optimal solution. Some practical issues will then be discussed in Sec. VI. The general case where $\{(Y_i, Z_i) : i \geq 1\}$ can be any ergodic stationary *Markov* process is considered in Sec. VII, where we explain how the designed algorithms can be seamlessly modified to accommodate the Markovian delay even though we no longer have provable convergence.

Before proceeding, we introduce a few new notations necessary when describing the algorithm. For any $\beta \in [0, \beta_{\text{UB}})$ and any $0 \leq y', z' < \infty$, we define

$$g_1(y', z', \beta) \triangleq G_1(y', z', \phi_{\Gamma,\beta}(y', z')) \tag{19}$$

$$g_2(y', z', \beta) \triangleq y' + z' + \phi_{\Gamma,\beta}(y', z') \tag{20}$$

$$\overline{g}_1(\beta) \triangleq \mathbb{E}_{Y_{i-1}, Z_{i-1}} \{g_1(Y_{i-1}, Z_{i-1}, \beta)\} \tag{21}$$

$$\overline{g}_2(\beta) \triangleq \mathbb{E}_{Y_{i-1}, Z_{i-1}} \{g_2(Y_{i-1}, Z_{i-1}, \beta)\} \tag{22}$$

Recalling that $f_\Gamma(\beta)$ is the average AoI penalty when $X_i = \phi_{\Gamma,\beta}(Y_{i-1}, Z_{i-1})$, by (8) we have

$$f_\Gamma(\beta) = \frac{\overline{g}_1(\beta)}{\overline{g}_2(\beta)}, \quad \forall \beta \in [0, \beta_{\text{UB}}). \tag{23}$$

A. Algorithm 1: Fixed-point-iteration-based Solution

The detailed step is described in Algorithm 1. At the beginning of the i-th round (Line 4), the algorithm updates the value β_i, see Lines 6 and 8, and then use (11) to compute the waiting time $X_i = \phi_{\Gamma,\beta_i}(Y_{i-1}, Z_{i-1})$ (Line 10), and update two register values (Lines 12 and 13). Then wait for X_i time before sending out the i-th packet. After sending the packet, source waits for the ACK of the i-th packet (Line 14) and the iteration continues.

We now elaborate how to compute β_i in Algorithm 1 in Lines 6 and 8, which would then be used to find X_i in Line 10. We initialize $\beta_1 = \beta_2 = 0$ in Line 6. For $i \geq 3$, we use the $g_1(y', z', \beta)$ and $g_2(y', z', \beta)$ functions defined in (19) and (20), respectively, and compute

$$\beta_i = \frac{\sum_{j=1}^{i-1} g_1(Y_{j-1}, Z_{j-1}, \beta_j)}{\sum_{j=1}^{i-1} g_2(Y_{j-1}, Z_{j-1}, \beta_j)} = \frac{\sum_{j=1}^{i-1} g_1(Y_{j-1}, Z_{j-1}, \beta_j)}{S_{i-1}}. \tag{24}$$

The denominator of (24) is derived by noting that

$$\begin{aligned}
&g_2(Y_{j-1}, Z_{j-1}, \beta_j) = Y_{j-1} + Z_{j-1} + \phi_{\Gamma,\beta_j}(Y_{j-1}, Z_{j-1}) \\
&= Y_{j-1} + Z_{j-1} + X_j = S_j - S_{j-1}, \ \forall j \in [1, i-1]
\end{aligned} \tag{25}$$

and hence the denominator of (24) can be simplified by the telescoping argument $\sum_{j=1}^{i-1} g_2(Y_{j-1}, Z_{j-1}, \beta_j) = S_{i-1}$. That is why we use the register $\mathbf{snd.time}$ to record the latest "send time" in Line 13 and then use this value as the denominator in Line 8.

In fact, one can prove that the $\mathbf{snd.time}$ of round 1 is always $S_1 = 0$ and the $\mathbf{snd.time}$ of round 2 is $S_2 > 0$. Note that in the beginning of round i, we use the $\mathbf{snd.time}$ value of the *previous round* as the denominator of β_i in Line 8, also see (24). Therefore, only at round $i = 3$ can we start to have a strictly positive denominator in Line 8 (and in (24)). That is why we hardwire $\beta_1 = \beta_2 = 0$ in Line 6.

Algorithm 1 Fixed-point-iteration-based online algorithm

Universal input for every round: A set of statistics of Y, denoted as SY_γ
Per-round input: (Y_{i-1}, Z_{i-1})
Per-round output: Waiting time X_i in the i-th round

1: Initialize $Y_0 = Z_0 = A_0 = 0$ (see Sec. III-A)
2: Maintain two scalar registers for $\mathbf{snd.time}$ and $\mathbf{sum.AoI.pnlty}$
3: Initialize $\mathbf{snd.time} = \mathbf{sum.AoI.pnlty} = 0$
4: **for** time instant A_{i-1}, i.e., the beginning of round $i = 1, 2, 3, \cdots$ **do**
5:     **if** $i \leq 2$ **then**
6:         $\beta_i = 0$
7:     **else**
8:         $\beta_i = \frac{\mathbf{sum.AoI.pnlty}}{\mathbf{snd.time}}$, which implements (24)
9:     **end if**
10:     Use (11) and SY_γ to compute $X_i = \phi_{\Gamma,\beta_i}(Y_{i-1}, Z_{i-1})$
11:     Use (19) and SY_γ to compute $g_1(Y_{i-1}, Z_{i-1}, \beta_i)$
12:     Update $\mathbf{sum.AoI.pnlty} = \mathbf{sum.AoI.pnlty} + g_1(Y_{i-1}, Z_{i-1}, \beta_i)$
13:     Update $\mathbf{snd.time} = \mathbf{snd.time} + Y_{i-1} + Z_{i-1} + X_i$
14:     Wait for X_i time, send the i-th packet, and wait for the ACK to start the next round
15: **end for**

The numerator of (24) can also be simplified. That is, there is no need to repeat the summation $\sum_{j=1}^{i-1} g_1(Y_{j-1}, Z_{j-1}, \beta_j)$ for each i. Instead, we only need to "update" the sum by adding the increment from the previous upper limit $i-2$ to the new upper limit $i-1$, as shown in Line 12 of Algorithm 1. By combining these two simplifications, the actual update of β_i is carried out in Line 8 of Algorithm 1.

B. Intuition of Algorithm 1

Note that each $g_1(Y_{j-1}, Z_{j-1}, \beta_j)$ term in the summation can be viewed as the *empirical AoI penalty* experienced during time interval (S_{j-1}, S_j). As a result, (24) computes the ratio of the past total AoI penalty over the past duration $[0, S_{i-1}]$, which is essentially the *empirical average AoI penalty.* We then use it as the new threshold β_i to decide the $X_i = \phi_{\Gamma,\beta_i}(Y_{i-1}, Z_{i-1})$ for the i-th round. This closely follows the spirit of the fixed-point iteration

$$\beta_i = f_\Gamma(\beta_{i-1}) = \frac{\overline{g}_1(\beta_{i-1})}{\overline{g}_2(\beta_{i-1})} \tag{26}$$

in Proposition 2. The differences between (24) and (26) are (i) (24) not only depends on β_{i-1} but also on $\{\beta_j : j \leq i-1\}$ and (ii) (24) uses the empirical $g_1(Y_{j-1}, Z_{j-1}, \beta_j)$ and $g_2(Y_{j-1}, Z_{j-1}, \beta_j)$ rather than the expectations $\overline{g}_1(\beta_{i-1})$ and $\overline{g}_2(\beta_{i-1})$. Therefore, $\{\beta_i\}$ in (24) is a *random process* but $\{\beta_i\}$ in Proposition 2 (also in (26)) is a deterministic sequence.

C. Knowledge Required to Run Algorithm 1

In order to run Algorithm 1, we need to compute $\phi_{\Gamma,\beta}(y', z', \beta)$ and $g_1(y', z', \beta)$ using (11) and (19), respectively. Recall that in this section (Sec. V) we only consider the *i.i.d.* vector random process $\{(Y_i, Z_i) : i \geq 1\}$. By combining (3), (4), (11), (12) and (19), we have a simplified version

$$\phi_{\Gamma,\beta}(y', z') = \inf\Big\{t > 0 : \mathbb{E}\{\gamma(y' + z' + t + Y)\} > \beta\Big\} \tag{27}$$

$$g_1(y', z', \beta) = \mathbb{E}_Y\left\{\int_Y^{y'+z'+\phi_{\Gamma,\beta}(y',z')+Y} \gamma(t)dt\right\}. \tag{28}$$

Both still require some knowledge of the statistics of Y. That is why in Algorithm 1, we use a notation SY_γ to denote the needed Statistics of Y (SY) and we clearly indicate that SY_γ is needed in Lines 10 and 11. As we will see, the set of needed statistics depends on the AoI penalty function $\gamma(\cdot)$ and this is why we have γ in the subscript of SY_γ.

We discuss the cases of the most popular penalty functions $\gamma_{\text{lin}}(\cdot)$, $\gamma_{\text{qdr}}(\cdot)$, and $\gamma_{\text{exp}}(\cdot)$. Note that the proposed algorithm is not limited to the above choices and can be tailored for other choices of $\gamma(\cdot)$ satisfying the technical assumption (iii) described in Sec. III-A. One just has to analyze the needed SY_γ separately for other classes of penalty functions.

Case 1: Linear penalty $\gamma_{lin}(\Delta) = \Delta$. We use $\phi^{\text{SY}}_{\text{lin},\beta}(y', z')$ to denote the waiting time function $\phi_{\Gamma,\beta}(y', z')$ specialized for $\gamma_{\text{lin}}(\Delta)$. Similarly, $g^{\text{SY}}_{\text{lin},1}(y', z', \beta)$ denotes the $g_1(y', z', \beta)$ specialized for $\gamma_{\text{lin}}(\Delta)$. The superscript "SY" indicates that this function requires (knowing) some Statistics of $\mathbb{P}_Y$. Applying simple calculus to (3), (4), (11), and (19) shows that

$$\phi^{\text{SY}}_{\text{lin},\beta}(y', z') = \max\left(\beta - \mathbb{E}\{Y\} - y' - z', 0\right) \tag{29}$$

$$g^{\text{SY}}_{\text{lin},1}(y', z', \beta) = \frac{\left(y' + z' + \phi^{\text{SY}}_{\text{lin},\beta}(y', z')\right)^2}{2} + \left(y' + z' + \phi^{\text{SY}}_{\text{lin},\beta}(y', z')\right)\mathbb{E}\{Y\}. \tag{30}$$

From (29) and (30), it is clear that to calculate $\phi^{\text{SY}}_{\text{lin},\beta}(y', z')$ and $g^{\text{SY}}_{\text{lin},1}(y', z', \beta)$, the only statistical knowledge we need is a scalar $\mathsf{SY}_\gamma = \mathbb{E}\{Y\}$.

Case 2: Exponential penalty $\gamma_{exp}(\Delta) = e^{a\Delta} - 1$ for a constant a that is known globally. Similar to the previous case, we use $\phi^{\text{SY}}_{\text{exp},\beta}(y', z')$ and $g^{\text{SY}}_{\text{exp},1}(y', z', \beta)$ to describe the

$\phi_{\Gamma,\beta}(y',z')$ and $g_1(y',z',\beta)$ specialized for $\gamma_{\text{exp}}(\Delta)$. Again, applying simple calculus to (3), (4), (11), and (19) shows that

$$\begin{aligned}&\phi^{\text{SY}}_{\text{exp},\beta}(y',z')\\&=\max\left(\frac{\ln(\beta+1)-\ln\left(\mathbb{E}\{e^{aY}\}\right)}{a}-y'-z',0\right)\end{aligned}\quad(31)$$

$$\begin{aligned}&g^{\text{SY}}_{\text{exp},1}(y',z',\beta)\\&=\frac{\left(e^{a\left(y'+z'+\phi^{\text{SY}}_{\text{exp},\beta}(y',z')\right)}-1\right)\mathbb{E}\{e^{aY}\}}{a}\\&\quad-\left(y'+z'+\phi^{\text{SY}}_{\text{exp},\beta}(y',z')\right).\end{aligned}\quad(32)$$

From (31) and (32), it is clear that to calculate $\phi^{\text{SY}}_{\text{exp},\beta}(y',z')$ and $g^{\text{SY}}_{\text{exp},1}(y',z',\beta)$, the only statistical knowledge we need is a scalar $\mathsf{SY}_\gamma=\mathbb{E}\{e^{aY}\}$ where a is the exponent of the $\gamma_{\text{exp}}(\cdot)$ that is known globally in advance.

Case 3: Quadratic $\gamma_{qdr}(\Delta)=\Delta^2$. We use $\phi^{\text{SY}}_{\text{qdr},\beta}(y',z')$ and $g^{\text{SY}}_{\text{qdr},1}(y',z',\beta)$ to describe the $\phi_{\Gamma,\beta}(y',z')$ and $g_1(y',z',\beta)$ specialized for $\gamma_{\text{qdr}}(\Delta)$. We then have

$$\begin{aligned}w_{\text{qdr}}(\beta)\triangleq{}&\mathbb{1}_{\{\beta+(\mathbb{E}\{Y\})^2\geq\mathbb{E}\{Y^2\}\}}\cdot\\&\left(\sqrt{\beta+(\mathbb{E}\{Y\})^2-\mathbb{E}\{Y^2\}}-\mathbb{E}\{Y\}\right)\end{aligned}\quad(33)$$

$$\phi^{\text{SY}}_{\text{qdr},\beta}(y',z')=\max\left(w_{\text{qdr}}(\beta)-y'-z',0\right)\quad(34)$$

$$\begin{aligned}g^{\text{SY}}_{\text{qdr},1}(y',z',\beta)={}&\frac{\left(y'+z'+\phi^{\text{SY}}_{\text{qdr},\beta}(y',z')\right)^3}{3}\\&+\left(y'+z'+\phi^{\text{SY}}_{\text{qdr},\beta}(y',z')\right)^2\mathbb{E}\{Y\}\\&+\left(y'+z'+\phi^{\text{SY}}_{\text{qdr},\beta}(y',z')\right)\mathbb{E}\{Y^2\}.\end{aligned}\quad(35)$$

From (33) to (35), it is clear that to calculate $\phi^{\text{SY}}_{\text{qdr},\beta}(y',z')$ and $g^{\text{SY}}_{\text{qdr},1}(y',z',\beta)$, the only statistical knowledge we need is a pair $\mathsf{SY}_\gamma=\left(\mathbb{E}\{Y\},\mathbb{E}\{Y^2\}\right)$.

Depending on which $\gamma(\cdot)$ is considered, the corresponding $\phi_{\Gamma,\beta}(\cdot,\cdot)$ and $g_1(\cdot,\cdot,\cdot)$ functions in Lines 10 and 11 are different. For ease of exposition, we introduce the following notation.

(i) $\Xi^{\text{SY}}_{\text{lin}}$ denotes Algorithm 1 when specialized for $\gamma_{\text{lin}}(\Delta)=\Delta$. In this case $\mathsf{SY}_\gamma=\mathbb{E}\{Y\}$.
(ii) $\Xi^{\text{SY}}_{\text{qdr}}$ denotes Algorithm 1 when specialized for $\gamma_{\text{qdr}}(\Delta)=\Delta^2$. In this case $\mathsf{SY}_\gamma=\left(\mathbb{E}\{Y\},\mathbb{E}\{Y^2\}\right)$.
(iii) $\Xi^{\text{SY}}_{\text{exp}}$ denotes Algorithm 1 when specialized for $\gamma_{\text{exp}}(\Delta)=e^{a\Delta}-1$. In this case $\mathsf{SY}_\gamma=\mathbb{E}\{e^{aY}\}$.

One remarkable feature of Algorithm 1 is that instead of requiring the knowledge of the entire delay distribution (e.g., pdf or cdf or pmf), it requires only a scalar statistic (Cases 1 and 2) or a pair of statistics (Case 3).

D. Feasibility and Convergence of Algorithm 1

One implicit but key assumption in Algorithm 1 is that when describing the $\phi_{\Gamma,\beta}$ scheme, we require the input parameter β to be in the range $[0,\beta_{\text{UB}})$. Therefore, the feasibility of Algorithm 1 hinges on that all β_i computed in Line 8 are in the range $[0,\beta_{\text{UB}})$. We affirm this feasibility condition as follows.

Lemma 6: For any $\gamma(\cdot)$ function, the random process β_i computed by the iterative formula (24) satisfies

$$\sup\{\beta_i: i\in[1,\infty)\}\leq\beta_{\max}<\beta_{\text{UB}}\quad(36)$$

almost surely for some constant $\beta_{\max}$.

Proof: See Appendix G. ■

For example, for the linear, quadratic, and exponential $\gamma(\cdot)$, we have $\beta_{\text{UB}}=\infty$. Since β_i is the empirical average AoI penalty (which remains finite all the time), the condition $\beta_i<\beta_{\text{UB}}$ is trivially true. However, for the signal-agnostic sampling of the Ornstein-Uhlenbeck (OU) process [19], the equivalent AoI penalty is $\gamma(\Delta)=\frac{\sigma^2}{2\theta}(1-e^{-2\theta\Delta})$, and the corresponding $\beta_{\text{UB}}=\frac{\sigma^2}{2\theta}$. As a result, the inequality $\beta_i<\beta_{\text{UB}}$ becomes a non-trivial condition that needs to carefully examined. Lemma 6 guarantees $\beta_i<\beta_{\text{UB}}$ always holds regardless whether $\beta_{\text{UB}}\triangleq\lim_{\Delta\to\infty}\gamma(\Delta)$ is infinite or finite.

We now present the optimality results.

Proposition 3 (Convergence in probability): There exist $\alpha\in(0,0.5)$ and $c_1,c_2,c_3,c_4>0$ such that $\forall i\geq1$,

$$\mathbb{P}\left(\beta_{i+1}<\beta^*-c_1\cdot i^{-(0.5-\alpha)}\right)\leq c_2\cdot\exp\left(-c_3\cdot i^{2\alpha}\right)\quad(37)$$

$$\mathbb{E}\{\beta_i-\beta^*\}\leq c_4\cdot i^{-(0.5-\alpha)}\quad(38)$$

Proof: See Appendices H and I. ■

Corollary 2 (Convergence in L^2): The random process $\{\beta_i\}$ computed in (24) converges to β^* in L^2.

Proof: See Appendix J. ■

E. Algorithm 2: A Root-finding-based Online Algorithm

A close look at Algorithm 1 shows that it consists of two components. Firstly, create a (random) sequence β_i that eventually converges to the optimal β^*. Secondly, use each β_i to compute the waiting time $X_i=\phi_{\Gamma,\beta_i}(Y_{i-1},Z_{i-1})$ for the i-th round. This decoupled structure immediately prompts the following question: Can we design a different online algorithm of computing β_i that also converges to β^*? If so, then the scheme will eventually have the optimal β^* and the waiting time $X_i=\phi_{\Gamma,\beta^*}(Y_{i-1},Z_{i-1})$ will also become optimal.

This observation prompts the second online algorithm that uses the Robbins-Monro algorithm to compute/update β_i. All the subsequent discussion for this new algorithm assumes $\beta_{\text{UB}}=\lim_{t\to\infty}\gamma(t)=\infty$. The reason why we impose this non-trivial assumption will be provided in Sec. V-G.

By Lemma 5, β^* is the unique root of (13). Since $\overline{g}_1(\beta)$, and $\overline{g}_2(\beta)$ in (21) and (22) are both finite for any $\beta\in[0,\beta_{\text{UB}})=[0,\infty)$, β^* is also the unique root of the equation

$$\beta\cdot\overline{g}_2(\beta)-\overline{g}_1(\beta)=0.\quad(39)$$

Since $\overline{g}_1(\beta)$ and $\overline{g}_2(\beta)$ take the *expectations* of the functions $g_1(\cdot,\cdot,\cdot)$ and $g_2(\cdot,\cdot,\cdot)$, respectively, the task of finding β^* can be solved by the Robbins-Monro algorithms [20], [21] that find the root of (39), which results in our new Algorithm 2.

Algorithms 1 and 2 are very similar. Specifically, both use the first two rounds $i\leq2$ for initialization. The computed β_i is then used to compute the waiting time X_i (see Lines 8 and

10 of Algorithm 1 and Lines 8 and 10 of Algorithm 2) in the same way. The main difference between Algorithms 1 and 2 is how β_i is computed. For any step-size parameter $\eta > 0$, for all $i \geq 3$, Line 8 of Algorithm 2 essentially computes

$$\beta_i = \beta_{i-1} - \frac{\eta}{i} \cdot \left(\beta_{i-1} \cdot g_2(Y_{i-2}, Z_{i-2}, \beta_{i-1}) - g_1(Y_{i-2}, Z_{i-2}, \beta_{i-1})\right) \quad (40)$$

where $g_1(\cdot,\cdot,\cdot)$ and $g_2(\cdot,\cdot,\cdot)$ are defined in (19) and (20), respectively.

Algorithm 2 Root-finding-based (Robbins-Monro) online algorithm

Universal input for every round: η and SY_γ (a set of statistics of Y)
Per-round input: (Y_{i-1}, Z_{i-1})
Per-round output: Waiting time X_i in the i-th round

1: Initialize $Y_0 = Z_0 = A_0 = 0$ (see Sec. III-A)
2: Maintain two scalar registers $\mathbf{curr.g1}$ and $\mathbf{curr.g2}$
3: Initialize $\mathbf{curr.g1} = \mathbf{curr.g2} = 0$
4: **for** time instant A_{i-1}, i.e., the beginning of round $i = 1, 2, 3, \cdots$ **do**
5:     **if** $i \leq 2$ **then**
6:         $\beta_i = 0$
7:     **else**
8:         $\beta_i = \beta_{i-1} - \frac{\eta}{i} \cdot (\beta_{i-1} \cdot \mathbf{curr.g2} - \mathbf{curr.g1})$ based on (40)
9:     **end if**
10:     Use (11) and SY_γ to compute $X_i = \phi_{\Gamma,\beta_i}(Y_{i-1}, Z_{i-1})$
11:     Use (19), β_i, SY_γ to compute $\mathbf{curr.g1} = g_1(Y_{i-1}, Z_{i-1}, \beta_i)$
12:     $\mathbf{curr.g2} = Y_{i-1} + Z_{i-1} + X_i$ (see (20))
13:     Wait for X_i time, then send the i-th packet, and wait for the ACK to start the next round
14: **end for**

The update rule in (40) follows from standard Robbins-Monro algorithm proposed in [20] since conditioning on the previous β_{i-1} value, we have

$$\mathbb{E}\{\beta_{i-1} \cdot g_2(Y_{i-2}, Z_{i-2}, \beta_{i-1}) - g_1(Y_{i-2}, Z_{i-2}, \beta_{i-1}) | \beta_{i-1}\}$$
$$= \beta_{i-1} \cdot \mathbb{E}\{g_2(Y_{i-2}, Z_{i-2}, \beta_{i-1}) | \beta_{i-1}\} - \mathbb{E}\{g_1(Y_{i-2}, Z_{i-2}, \beta_{i-1}) | \beta_{i-1}\} \quad (41)$$
$$= \beta_{i-1} \cdot \overline{g}_2(\beta_{i-1}) - \overline{g}_1(\beta_{i-1}) \quad (42)$$

where (42) follows from the facts that (i) $\{(Y_{i-2}, Z_{i-2})\}$ is independent of $\{(Y_j, Z_j)\}_{i=0}^{j-3}$; (ii) β_{i-1} in (40) was computed by the history of $\{(Y_j, Z_j)\}_{i=0}^{j-3}$ and is thus independent of Y_{i-2} and Z_{i-2}; and (iii) the definitions in (21) and (22).

Blum [21] proved that the standard Robbins-Monro algorithm (i.e., $\{\beta_i\}$ computed by (40)) converges to the unique root (i.e., β^*) almost surely, provided that the following three conditions are met.

(i) $\{\beta_i\}$ computed by (40) is uniformly bounded.
(ii) $\beta \cdot \overline{g}_2(\beta) - \overline{g}_1(\beta)$ is non-decreasing.
(iii) $0 < \frac{d}{d\beta}\left(\beta \cdot \overline{g}_2(\beta) - \overline{g}_1(\beta)\right)\Big|_{\beta=\beta^*} < \infty$.

By proving that all three conditions hold in our AoI penalty minimization setting, we have

Proposition 4 (Almost sure convergence): For any $\eta > 0$, the sequence $\{\beta_i\}$ computed in (40) converges to β^* almost surely.

Proof: See Appendix M. ■

F. Knowledge Required to Run Algorithm 2

Algorithm 2 requires the computation of $g_1(y', z', \beta)$, $g_2(y', z', \beta)$ and $\phi_{\Gamma,\beta}(y', z')$ for any $(Y_{i-1} = y', Z_{i-1} = z')$. Since $g_2(y', z', \beta) = y' + z' + \phi_{\Gamma,\beta}(y', z')$ (see (20)), it is clear that Algorithm 2 only needs to compute $g_1(y', z', \beta)$ and $\phi_{\Gamma,\beta}(y', z')$, which is also needed by Algorithm 1. From the discussion in V-C, we conclude that Algorithm 2 requires the same statistics of Y (i.e., SY_γ) as Algorithm 1.

For ease of future reference, we introduce another three notations for the corresponding online algorithms.

(i) $\Lambda^{\mathsf{SY}}_{\text{lin}}$ denotes Algorithm 2 when specialized for the linear AoI penalty function $\gamma_{\text{lin}}(\Delta) = \Delta$;
(ii) $\Lambda^{\mathsf{SY}}_{\text{qdr}}$ denotes Algorithm 2 when specialized for the quadratic AoI penalty function $\gamma_{\text{qdr}}(\Delta) = \Delta^2$;
(iii) $\Lambda^{\mathsf{SY}}_{\text{exp}}$ denotes Algorithm 2 when specialized for the exponential AoI penalty function $\gamma_{\text{exp}}(\Delta) = e^{a\Delta} - 1$

where we use the Λ schemes for the Robbins-Monro-based solution in this subsection (Ξ is reserved for the fixed-point-based scheme in Sec. V-A). The superscript SY indicates that these three schemes require the Statistics of Y.

G. Two Critical Differences between Algorithms 1 and 2

Difference #1: Algorithm 2 can be applied only if the AoI penalty function $\gamma(\cdot)$ satisfies $\beta_{\text{UB}} = \lim_{t\to\infty} \gamma(t) = \infty$ while Algorithm 1 does not require this restrictive assumption. The reason is as follows. Suppose $\beta_{\text{UB}} < \infty$. We note that the update rule (40) starts from the previous β_{i-1} value and then adds a random disturbance term. In the initial rounds (when i is still small), the step size $\frac{\eta}{i}$ is still large. Therefore, the new β_i after random perturbation may be outside the target range $[0, \beta_{\text{UB}})$. The impact of possibly having $\beta_i \geq \beta_{\text{UB}}$ is catastrophic since it immediately results in an infinite waiting time $X_i = \phi_{\Gamma,\beta_i}(\cdot,\cdot)$ (see the discussion on β_{UB} in Sec. IV-A) that halts the entire system. This is the reason why Algorithm 2, at least in its current form, is feasible only under the assumption $\beta_{\text{UB}} = \infty$, which guarantees $\beta_i < \beta_{\text{UB}} = \infty$ will always be within the right range.

One way to avoid having $\beta_i \geq \beta_{\text{UB}}$ is to choose a small η to begin with, see (40). However, choosing a small η will adversely affect the convergence speed even though eventually it still converges to optimum. This leads to the second main difference.

Difference #2: Finding the right step size $\frac{\eta}{i}$ that balances the convergence speed and stability is important for the Robbins-Monroe algorithms. For comparison, there is no step-size parameter in the fixed-point-based solution in Algorithm 1. In a broad sense, the fixed-point update rule in Algorithm 1 which uses the old empirical average as the new β_i is able to "self-regulate" the perturbation of each update step without an explicitly specified step size needed in a Robbins-Monro algorithm.

Remark 4: For the scenario of $\beta_{\text{UB}} < \infty$, it is possible to add a projection operation to the Robbins-Monro algorithm when updating β_i in (40). One natural choice is to project β_i back to $[0, \beta_{\text{UB}}]$ so that all $\beta_i \leq \beta_{\text{UB}}$. However, such immediate modification still does not work since we need $\beta_i < \beta_{\text{UB}}$ with strict inequality (note that $\beta_i = \beta_{\text{UB}}$ still leads to $X_i = \infty$). As will be seen in the numerical results, Algorithm 1 offers superior/equal performance to Algorithm 2 while being more robust on all scenarios without the need of fine tuning step sizes. We thus leave the generalization of Algorithm 2 for the scenario of $\beta_{\text{UB}} < \infty$ as future work.

VI. Addressing Practical Issues in the I.I.D. Delay Setting

A. Using Running Average to Replace SY_γ

Note that both Algorithm 1 (Sec. V-A) and Algorithm 2 (Sec. V-E) require the knowledge of SY_γ, some statistics of Y, to run. Since estimating a statistic is much easier than learning the entire distribution $\mathbb{P}_Y$, we can further substitute the values of SY_γ in Algorithms 1 and 2 by the running empirical averages to obtain truly distribution-oblivious online algorithms.

For example, consider the linear $\gamma_{\text{lin}}(\Delta) = \Delta$, in which case $\mathsf{SY}_\gamma = \mathbb{E}\{Y\}$. We first set the window size $N_{\text{RA}} > 0$. Then, for $i = 1$ (at the beginning of the first round), we use the value 0 as the estimate of $\mathbb{E}\{Y\}$. For $i \geq 2$, we compute the running empirical average that has a fixed window size N_{RA}:

$$\frac{1}{\min(i-1, N_{\text{RA}})} \sum_{j=\min(1, i-N_{\text{RA}})}^{i-1} Y_j. \tag{43}$$

The complexity of computing the running empirical average is $O(1)$ per slot, since for every round one can simply add the latest term Y_{i-1} and subtract the oldest term $Y_{i-N_{\text{RA}}}$.

Once we replace the statistics SY_γ by its running average, we can carry out the computation of Lines 10 and 11 of Algorithm 1 (resp. Algorithm 2) without knowing the true value of SY_γ. Similar RA-based substitution can be applied to other non-linear $\gamma(\cdot)$ as well, including $\gamma_{\text{qdr}}(\cdot)$ and $\gamma_{\text{exp}}(\cdot)$. We denote the resulting online algorithms by $\Xi^{\text{RA}}_{\text{lin}}$, $\Xi^{\text{RA}}_{\text{qdr}}$ and $\Xi^{\text{RA}}_{\text{exp}}$ (resp. $\Lambda^{\text{RA}}_{\text{lin}}$, $\Lambda^{\text{RA}}_{\text{qdr}}$ and $\Lambda^{\text{RA}}_{\text{exp}}$) for the fixed-point-based (resp. Robbins-Monro-based) algorithm.

B. The Case of Unknown AoI Penalty Function $\gamma(\cdot)$

In all the previous sections, we implicitly assume that the source has perfect knowledge of the AoI penalty function $\gamma(\cdot)$, which may not always hold in real world. In particular, the AoI penalty depends on the specific application that is currently running *at the destination*. Although the source is able to compute the value of the AoI using the time stamps, it may not have access to the application layer ($\gamma(\Delta)$ value) *at the destination*. Even the destination may not have full knowledge of its own AoI penalty function. See the discussion in Sec. I. This section addresses this important practical need of estimating $\gamma(\cdot)$ on the fly.

For any i, when the destination receives the update packet at time $S_i + Y_i$, if we denote $(Y_{i-1} = y', Z_{i-1} = z', X_i = x, Y_i = y)$ and define $p_i \triangleq y' + z' + x + y$, then the peak AoI penalty at that time is $\gamma(p_i)$ (see Fig. 2). Suppose the destination does not have full knowledge of $\gamma(\cdot)$. Instead, destination can observe how good/poor the system state is at that time instant and use it to estimate the scalar AoI penalty value $\gamma(p_i)$. For example, say at time $S_i + Y_i$, the system is on the brink of major disruption due to the inability to receive update packets for a long time (large p_i value), then the AoI penalty $\gamma(p_i)$ value is likely to be large. On the contrary, if at the time of receiving the latest packet, the system is still functioning normally, then the $\gamma(p_i)$ value is likely to be small. Since destination estimates $\gamma(p_i)$ by observing the system state, we assume that destination knows the value of $q_i = \gamma(p_i) + n_i$, where n_i's are i.i.d. zero-mean Gaussian random variables that represent the estimation/observation error. The values of the pair (p_i, q_i) is then fed back to the source via ACK.

At the i-th round, the source maintains the set of the past $N_{\hat{\gamma}}$ observations $\mathcal{S}_{\hat{\gamma}} = \{(p_j, q_j)\}_{j=i-N_{\hat{\gamma}}}^{i-1}$. Any (p_i, q_i) outside of this set is considered *too old* and is excluded from consideration. Our goal is to estimate the true $\gamma(\cdot)$ using the noisy observations $\mathcal{S}_{\hat{\gamma}}$.

Step 1: Since the penalty function $\gamma(\cdot)$ is non-negative and satisfies $\gamma(0) = 0$, we first add the point $(0, 0)$ to the set $\mathcal{S}_{\hat{\gamma}}$.

Step 2: We sort and relabel the elements of $\mathcal{S}_{\hat{\gamma}}$ in an ascending order of p_i and thus we have $0 = p_1 \leq p_2 \leq \cdots \leq p_{N_{\hat{\gamma}}+1}$. Note that since $\gamma(\cdot)$ is an increasing function, the sequence $\{\gamma(p_i)\}$ after sorting is also increasing. However, with noisy observation $q_i = \gamma(p_i) + n_i$ the corresponding $\{q_i\}_{i=1}^{N_{\hat{\gamma}}+1}$ may not be an increasing sequence.

Step 3: Since the AoI analysis and scheduler designs rely heavily on the assumption that $\gamma(\cdot)$ is monotonically increasing, our plan is to first solve the following quadratic programming problem

$$\begin{aligned} &\min_{\{\hat{q}_i\}_{i=1}^{N_{\hat{\gamma}}+1}} \sum_{i=1}^{N_{\hat{\gamma}}+1} (\hat{q}_i - q_i)^2 \\ &\text{subject to } \hat{q}_i \leq \hat{q}_{i+1},\ 1 \leq i \leq N_{\hat{\gamma}} \end{aligned} \tag{44}$$

that gives us a new sequence of pairs $(p_i, \hat{q}_i)$ that is non-decreasing in both coordinates. We then set the estimated AoI penalty function $\hat{\gamma}(\cdot)$ to be a piece-wise linear function with $N_{\hat{\gamma}}$ pieces. For all $i \in [1, N_{\hat{\gamma}} - 1]$, the i-th line segment is connecting the two pairs $(p_i, \hat{q}_i)$ and $(p_{i+1}, \hat{q}_{i+1})$. The last segment ($i = N_{\hat{\gamma}}$) is starting from $(p_{N_{\hat{\gamma}}}, \hat{q}_{N_{\hat{\gamma}}})$, going through $(p_{N_{\hat{\gamma}}+1}, \hat{q}_{N_{\hat{\gamma}}+1})$, and extending all the way to infinity. Mathematically, we can write $\hat{\gamma}(\cdot)$ as follows.

$$\hat{\gamma}(t) = \begin{cases} \hat{q}_i + \frac{t-p_i}{p_{i+1}-p_i}(\hat{q}_{i+1} - \hat{q}_i) & \text{if } t \in [p_i, p_{i+1}], \\ & 1 \leq i \leq N_{\hat{\gamma}} \\ \hat{q}_{N_{\hat{\gamma}}+1} + \frac{t-p_{N_{\hat{\gamma}}+1}}{p_{N_{\hat{\gamma}}+1}-p_{N_{\hat{\gamma}}}}(\hat{q}_{N_{\hat{\gamma}}+1} - \hat{q}_{N_{\hat{\gamma}}}) & \text{if } t \geq p_{N_{\hat{\gamma}}+1}. \end{cases} \tag{45}$$

In the literature, performing Step 3 to obtain the estimate $\hat{\gamma}(\cdot)$ is termed *monotonic regression* or *isotonic regression.* [5]

Step 4: After obtaining $\hat{\gamma}(\cdot)$, we can carry out the online Algorithms 1 and 2 easily by substituting $\gamma(\cdot)$ by $\hat{\gamma}(\cdot)$. Specifically, since the $\hat{\gamma}(\cdot)$ is piece-wise linear, two major substitutions are needed that are different from our previous discussion of γ_{lin}, γ_{exp}, and γ_{qdr} in Sec. V-C. First, we replace the waiting time $X_i = \phi_{\Gamma,\beta_i}(y', z')$ in Algorithms 1 and 2 with

$$X_i = \phi^{\text{RA}}_{\hat{\gamma},\beta_i}(y', z') \triangleq \inf\left\{t > 0 : \frac{\sum_{j=i-N_{\text{sum}}}^{i-1} \hat{\gamma}\left(y' + z' + t + Y_j\right)}{N_{\text{sum}}} > \beta_i\right\} \quad (46)$$

where $N_{\text{sum}} \leq N_{\hat{\gamma}}$ samples from the past Y_i are plugged into the estimated $\hat{\gamma}(\cdot)$ to approximate the expectation $\mathbb{E}(\gamma(y' + z' + t + Y))$ in (11) and (12).

Step 5: The second substitution is for computing $g_1(y', z', \beta_i)$. We first note that by (4) and (19), the value $g_1(y', z', \beta_i)$ in Algorithms 1 and 2 is *the expectation of an integral with random upper and lower limits.* As a result, we again use the combination of running average (RA) and estimated $\hat{\gamma}(\cdot)$ to estimate the $g_1(y', z', \beta_i)$, and we have

$$g^{\text{RA}}_{\hat{\gamma},1}(y', z', \beta_i) \triangleq \frac{\sum_{j=i-N_{\text{sum}}}^{i-1} \left(\mathsf{Cumulative.trapezoidal.integral}_j\right)}{N_{\text{sum}}} \quad (47)$$

where the summation and division are to compute the running empirical average, and we use the following cumulative trapezoidal integral

$$\mathsf{Cumulative.trapezoidal.integral}_j \triangleq \sum_{n=1}^{N_{\text{trapezoid}}} \frac{\hat{\gamma}(x_j^{n-1}) + \hat{\gamma}(x_j^n)}{2}\delta_j \quad (48)$$

in which $\{x_j^n\}_{n=0}^{N_{\text{trapezoid}}}$ is a uniformly spaced partition points of $[Y_j, Y_j + y' + z' + \phi^{\text{RA}}_{\hat{\gamma},\beta_i}(y', z')]$ (based on the j-th sample Y_j and the previously computed $\phi^{\text{RA}}_{\hat{\gamma},\beta_i}(y', z')$ value in (46)) and δ_j is the corresponding spacing between x_j^{n-1} and x_j^n. Mathematically,

$$\delta_j \triangleq \frac{y' + z' + \phi^{\text{RA}}_{\hat{\gamma},\beta_i}(y', z')}{N_{\text{trapezoid}}} \quad (49)$$

$$x_j^n \triangleq Y_j + n\delta_j, \ \forall n \in [0, N_{\text{trapezoid}}]. \quad (50)$$

After the modification described in Steps 1 to 5, both Algorithms 1 and 2 can be carried out for arbitrary unknown $\gamma(\cdot)$, not limited to the previous $\gamma_{\text{lin}}(\cdot)$, $\gamma_{\text{qdr}}(\cdot)$ or $\gamma_{\text{exp}}(\cdot)$.

We now analyze the complexity incurred of estimating $\hat{\gamma}(\cdot)$. In Step 1, we add a point $(0, 0)$ to the set $\mathcal{S}_{\hat{\gamma}}$ and hence the complexity of $O(1)$. For Step 2, we only need to *update* the sorted list so that per-time-slot complexity is only $\log N_{\hat{\gamma}}$. To carry out the monotonic regression in Step 3, one may use the active set algorithms in [22] that have a complexity of $O(N_{\hat{\gamma}})$.

[5] Our derivation assumes $\gamma(\cdot)$ to be strictly increasing and Step 3 only guarantees a non-decreasing $\hat{\gamma}(\cdot)$. Nonetheless, if we have enough observations (i.e., large $N_{\hat{\gamma}}$), $\hat{\gamma}(\cdot)$ is very well-behaved (very close to being strictly increasing). As will be shown in the simulation results (Sec. VIII), the online algorithms using $\hat{\gamma}(\cdot)$ still achieve satisfactory performance.

For Step 4, given (y', z', β_i), evaluating the summation in (46) takes $O(N_{\text{sum}})$ time, and to compute the infimum t value in (46) we use the bisection method over the interval $[0, t_{\text{large}}]$ for a sufficiently large $t_{\text{large}} > 0$. Depending on the desired precision $\varrho > 0$, the combined complexity of (46) is $O(\log(t_{\text{large}}/\varrho)N_{\text{sum}})$. For Step 5, since the trapezoidal integral is done for each Y_j and we have N_{sum} such Y_j, together the complexity is $O(N_{\text{sum}} \cdot N_{\text{trapezoid}})$. In sum, the per-round complexity of this new modification is $O\left(N_{\hat{\gamma}} + \log N_{\hat{\gamma}} + N_{\text{sum}} \cdot (\log(t_{\text{large}}/\varrho) + N_{\text{trapezoid}})\right)$.

We denote $\Xi^{\text{RA}}_{\hat{\gamma}}$ (resp. $\Lambda^{\text{RA}}_{\hat{\gamma}}$) to be the online algorithm that uses the above 5 steps to modify the original Algorithm 1 (resp. Algorithm 2). The resulting algorithm is fully distribution- and AoI-penalty-oblivious.

VII. Addressing Practical Issues in the Markov Delay Setting

A. Using KNN to Estimate Conditional Probabilities

In this section, we turn our attention to the case where $\{(Y_i, Z_i)\}$ is a stationary and ergodic Markov process (not necessarily i.i.d.).

By examining the proposed $\Xi^{\text{RA}}_{\hat{\gamma}}$ (resp. $\Lambda^{\text{RA}}_{\hat{\gamma}}$) in Sec. VI, we notice that the proposed schemes have a desirable "modular" structure. Namely, the estimation of the unknown $\gamma(\cdot)$ is separated from the actual evaluation of the functions $\phi_{\Gamma,\beta}$ and g_1 in (46) and (47), respectively. With a Markov delay setting, we can thus reuse the first 3 steps to estimate $\gamma(\cdot)$ and only modify Steps 4 and 5 accordingly.

In the i.i.d.-based computation (46) and (47), we use all the past N_{sum} observations $\{Y_j\}_{j=i-N_{\text{sum}}}^{i-1}$ to compute the running empirical average. For the Markovian delay setting, we propose using the k-nearest neighbors (KNN) algorithm that computes $\mathbb{P}_{Y_i|Y_{i-1},Z_{i-1}}$ by considering a set of k neighboring points that are the nearest to the given $(Y_{i-1}, Z_{i-1}) = (y', z')$ [23].

Specifically, at the i-th round, among all the past N_{sum} observed delay values $\{(Y_{j-1}, Z_{j-1}, Y_j)\}_{j=i-N_{\text{sum}}}^{i-1}$, we select N_{KNN} points such that the first two coordinates (Y_{j-1}, Z_{i-1}) are of the shortest Euclidean distance to the latest observation $(Y_{i-1}, Z_{i-1}) = (y', z')$. Then the *Modified Step 4* computes X_i in Algorithms 1 and 2 using

$$X_i = \phi^{\text{KNN}}_{\hat{\gamma},\beta_i}(y', z') \triangleq \inf\left\{t > 0 : \frac{\sum_{j=1}^{N_{\text{KNN}}} \hat{\gamma}\left(y' + z' + t + Y_j^{\text{select}}\right)}{N_{\text{KNN}}} > \beta_i\right\}. \quad (51)$$

Eq. (51) is almost identical to (46) except that Y_j^{select} are the third coordinates of the $N_{\text{KNN}} \leq N_{\text{sum}}$ samples nearest to (y', z') instead of all N_{sum} past samples.

Modified Step 5: Replace $g_1(y', z', \beta_i)$ in Algorithms 1 and 2 with

$$g^{\text{KNN}}_{\hat{\gamma},1}(y', z', \beta_i) \triangleq \frac{\sum_{j=1}^{N_{\text{KNN}}} \left(\mathsf{Cumulative.trapezoidal.integral}_j\right)}{N_{\text{KNN}}} \quad (52)$$

where the cumulative trapezoidal integral is computed in the same way as in (48), but replaces the Y_j used in (48) with the

Y_j^{select} from the N_{KNN} samples nearest to (y', z'). Since the modified Steps 4 and 5 are very similar to the Steps 4 and 5 in Sec. VI-B, the KNN version has similar complexity as the RA version.

We use $\Xi_{\hat{\gamma}}^{\text{KNN}}$ (resp. $\Lambda_{\hat{\gamma}}^{\text{KNN}}$) to denote the online algorithm that uses the modified Algorithm 1 (resp. modified Algorithm 2). Table I summarizes the notations used for the fixed-point-based online algorithms. All the algorithms can be divided to two major categories, depending on whether it is developed exclusively for the i.i.d. delay models or for the more general Markov models; and whether it assumes the full knowledge of $\gamma(\cdot)$ or involves the penalty estimation $\hat{\gamma}(\cdot)$ component. Finally, we also distinguish the cases based on whether some statistics of $\mathbb{P}_Y$ are known *a priori* versus the truly distribution-oblivious setting for which the algorithm has absolutely zero knowledge of the distribution or statistics.

As can be seen, four major sets of algorithms are developed. We start from the most basic version that requires both the statistics and $\gamma(\cdot)$. Next we introduce the running average version that still requires $\gamma(\cdot)$. Then we introduce the component of estimated $\hat{\gamma}(\cdot)$. Finally, we relax the i.i.d. assumption and extend the results to the Markov delay models. Four other combinations are not considered in this work and they correspond to the "– – – " parts in Table I. If desired, our approaches can be easily applied to those settings as well.

VIII. Simulation Results

A. I.I.D. Delay

In this section, we consider *i.i.d.* $\{(Y_i, Z_i) : i \geq 1\}$ with Y_i and Z_i being independent log-normal random variables with $(\mu_Y, \sigma_Y^2) = (0.5, 0.25)$ and $(\mu_Z, \sigma_Z^2) = (0.5, 0.5)$. We consider the quadratic AoI penalty function $\gamma_{\text{qdr}}(\Delta) = \Delta^2$.

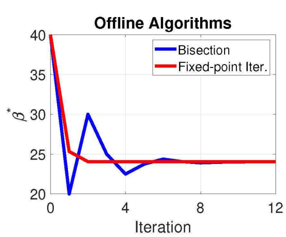


Fig. 3: Offline computation for β^* under the i.i.d. delay.

The trajectories of the *offline* fixed-point computation $\beta_{i+1} = f_\Gamma(\beta_i)$, described in Sec. IV-B, versus the bisection method are plotted in Fig. 3. The advantage of our scheme is twofold. Firstly it converges faster than the bisection method. Secondly, as proved in Proposition 2, the sequence $\{\beta_i\}$ is non-increasing and thus does not fluctuate as in the case of the bisection search.

We also run the *fixed-point-iteration-based* online algorithm $\Xi_{\text{qdr}}^{\text{SY}}$. Fig. 4a plots the evolution of β_i versus i and benchmarks β_i against β^* (the red dashed line). The three curves in Fig. 4a are generated by different random seeds. For each curve, β_i is within 8% of β^* after just 10^3 iterations. Since it is an online algorithm, *it means that using our distribution-oblivious scheme, after sending just 1000 update packets, the average AoI penalty of the underlying system (over the last 1000 packets) is already within* 8% *of the best offline solution that requires complete knowledge of the delay distributions.* The gap is less than 4% after 10^4 iterations.

Note that Fig. 4a traces the evolution of the β_i computed by $\Xi_{\text{qdr}}^{\text{SY}}$. The value of β_i is then fed into (11) to compute the waiting time X_i. Fig. 4a shows that β_i converges to the optimal choice β^* but does not evaluate how close the empirical AoI penalty, resulting from these choices β_i, is to the optimal/minimal AoI penalty. To directly examine the penalty performance, we compute the *observed avg. AoI penalty* $\frac{\int_0^{D_i} \gamma(t)dt}{D_i}$ for every $i \geq 1$. The red horizontal dashed line is the AoI penalty achieved by the best possible offline algorithm. Similar to Fig. 4a, the observed avg. AoI penalty is within 7% of β^* after just 10^3 iterations, and the difference is less than 3% after 10^4 iterations.

Next we run the *root-finding-based* online algorithm $\Lambda_{\text{qdr}}^{\text{SY}}$ where we choose the step-size to be $\frac{0.5}{i}$ for the i-th update.[6] The results are presented in Figs. 4c and 4d. Compared with the fixed-point-iteration-based scheme $\Xi_{\text{qdr}}^{\text{SY}}$, β_i using $\Lambda_{\text{qdr}}^{\text{SY}}$ is generally higher for the first 1000 iterations (see Fig. 4a versus Fig. 4c) and eventually converges to the optimal β^*. We also directly examine the resulting empirical AoI in Fig. 4d. Even though the β_i chosen by the Robbins-Monro algorithm is generally larger, their impact on the average AoI performance is not significant. That is, after the 100 iterations, the empirical AoI of both $\Xi_{\text{qdr}}^{\text{SY}}$ and $\Lambda_{\text{qdr}}^{\text{SY}}$ are very close to each other. This relative insensitivity to the β_i choice could be explained as follows. Recall that both Ξ^{SY} and Λ^{SY} choose their waiting time X_i based on the same water-filling rule (11). Therefore, for a wide range of β_i we will choose to *zero-wait* $\phi_{\text{qdr},\beta_i}^{\text{SY}}(y', z') = 0$ when y' and z' in (11) are large. As a result, different β_i values have impacts only in the scenarios of small $(Y_{i-1}, Z_{i-1}) = (y', z')$ and thus the actual AoI penalty performance is not very sensitive to the β_i value. Since the fixed-point-iteration-based and root-finding-based online algorithms achieve similar performance, for the rest of the paper we will only present the results of the *fixed-point-iteration-based* online algorithms.

We then consider the case of using the running average-based online algorithm $\Xi_{\text{qdr}}^{\text{RA}}$. We set $N_{\text{RA}} = 10^3$ and the results are plotted in Figs. 4e and 4f. Comparing $\Xi_{\text{qdr}}^{\text{SY}}$ and $\Xi_{\text{qdr}}^{\text{RA}}$, we observe that for the first 100 iterations, with an insufficient number of samples, $\Xi_{\text{qdr}}^{\text{RA}}$ does not have an accurate estimate of SY_γ, and hence the resulting β_i is slightly higher than $\Xi_{\text{qdr}}^{\text{SY}}$ (see Figs. 4a and 4e). However, the slightly higher parameter β_i choices do not impact much on the actual empirical AoI penalty. The curves in Figs. 4b and 4f are almost identical. After the first 100 iterations, both $\Xi_{\text{qdr}}^{\text{SY}}$ and $\Xi_{\text{qdr}}^{\text{RA}}$ have similar β_i and similar empirical AoI penalty, and eventually converge to the optimal value. This confirms the benefits of using the

[6] The step-size has to be carefully determined to balance the convergence speed and numerical stability. For example, if we set the step-size to be $\frac{0.01}{i}$, then β_i is still unable to converge to β^* even after 10^6 iterations. On the other hand, if $\frac{3}{i}$ is picked, then fatal instability is observed in our simulation, i.e., the observed avg. AoI could grow as high as 10^{41}, which leads to numeric overflow. The fixed-point-based online algorithm, however, does not have such an issue. Also see the discussion in Sec. V-G.

TABLE I: Summary of the fixed-point-based online algorithms. Each of the fixed-point-based algorithm Ξ has a stochastic approximation Robbins-Monro-based counterpart, denoted by the symbol Λ (instead of Ξ).

	Derived under i.i.d. delay		Derived under Markov delay	
	Requiring statistics SY_γ	Estimating statistics SY_γ	Requiring statistics SY_γ	Estimating statistics SY_γ
γ	• $\phi^{\mathsf{SY}}_{\text{lin},\beta_i}$, $\phi^{\mathsf{SY}}_{\text{qdr},\beta_i}$, $\phi^{\mathsf{SY}}_{\text{exp},\beta_i}$ derived according to (29), (33), (31) • Policies: $\Xi^{\mathsf{SY}}_{\text{lin}}$, $\Xi^{\mathsf{SY}}_{\text{qdr}}$, $\Xi^{\mathsf{SY}}_{\text{exp}}$	• $\phi^{\mathsf{RA}}_{\text{lin},\beta_i}$, $\phi^{\mathsf{RA}}_{\text{qdr},\beta_i}$, $\phi^{\mathsf{RA}}_{\text{exp},\beta_i}$ use running emp. avg. to replace SY_γ, according to Sec. VI-A • Policies: $\Xi^{\mathsf{RA}}_{\text{lin}}$, $\Xi^{\mathsf{RA}}_{\text{qdr}}$, $\Xi^{\mathsf{RA}}_{\text{exp}}$	– – – – – – – –	– – – – – – – – – – –
$\hat{\gamma}$	– – – – – – – – – – – – –	• $\phi^{\mathsf{RA}}_{\hat{\gamma},\beta_i}$ estimates $\gamma(\cdot)$ and uses running emp. avg. according to (46) • Policy: $\Xi^{\mathsf{RA}}_{\hat{\gamma}}$	– – – – – – – –	• $\phi^{\mathsf{KNN}}_{\hat{\gamma},\beta_i}$ estimates $\gamma(\cdot)$ and uses KNN according to (51) • Policy: $\Xi^{\mathsf{KNN}}_{\hat{\gamma}}$

running empirical average as a substitute of the delay statistics SY_γ.

A more interesting scenario is when $\gamma(\cdot)$ is unknown. In our simulation, the destination observes $q_i = \gamma(p_i) + n_i$ where $p_i = Y_{i-1} + Z_{i-1} + X_i + Y_i$. We assume $\gamma(\Delta) = \Delta^2$ but this fact is unknown to the source/destination pair, and we also assume the observation error n_i's are i.i.d. Gaussian random variables with mean zero and variance 0.05. Other parameters are set as follows: $N_{\hat{\gamma}} = 10^3$, $N_{\text{sum}} = 200$ and $N_{\text{trapezoid}} = 10^3$. Fig. 5a plots the resulting $\hat{\gamma}(\cdot)$ and the true underlying $\gamma(\cdot)$ (those scattered red points are the elements in the set $\mathcal{S}_{\hat{\gamma}}$). As can be seen from Figs. 5a and 5b (magnified version), $\hat{\gamma}(\cdot)$ is non-decreasing and sufficiently close to $\gamma(\cdot)$. At the end of iteration ($i = 10^6$), β_i from every curve in Fig. 5c is within 4% of β^*. Meanwhile, because the actual AoI penalty performance is less sensitive to the β_i value, the observed avg. AoI penalty is within 1% of the best offline algorithm that knows both the distribution $\mathbb{P}_Y$ and the $\gamma(\cdot)$. This demonstrates the superior performance of $\Xi^{\mathsf{RA}}_{\hat{\gamma}}$, which is not only distribution-oblivious, but it is also able to estimate $\gamma(\cdot)$ in an online manner.

Finally, even though $\Xi^{\text{KNN}}_{\hat{\gamma}}$ is originally derived assuming Markov delay (see Sec. VII-A and Table I), we use it here to examine its performance under the i.i.d. delay scenario. The associated parameters are set as follows: $N_{\hat{\gamma}} = 10^3$, $N_{\text{KNN}} = 100$ and $N_{\text{trapezoid}} = 10^3$. At $i = 10^6$ iteration, for each random seed β_i is 15% away from β^*, see Fig. 6a. The reason is that with N_{KNN} set to be a small number 100, the estimation of the expectations is not as accurate as the running average scheme (which has $N_{\text{sum}} = 200$ observations), which leads to larger error of β_i, see see Figs. 4e and 6a. Nonetheless, the observed avg. AoI penalty is only 3% away from the offline optimum (see Fig. 6b), which shows the effectiveness and robustness of $\Xi^{\text{KNN}}_{\hat{\gamma}}$ even under the i.i.d. delay setting.

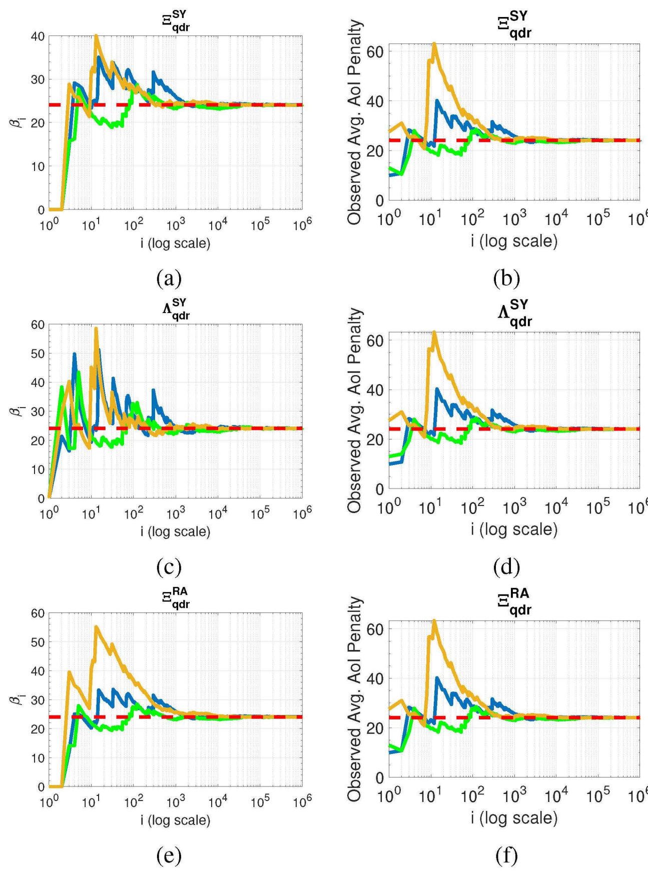


Fig. 4: Left: Evolution of β_i using the online algorithm. Right: Evolution of the observed AoI penalty using the online algorithm. Different curves represent different random seeds. The horizontal red dashed line represents β^*.

B. Markov Delay

In this section, we simulate *Markov* $\{(Y_i, Z_i) : i \geq 1\}$. Specifically, a stationary discrete Markov chain is considered: We set $\mathbb{P}(Z_i = 1) = 1$, $\mathbb{P}(Y_i = 1) = \mathbb{P}(Y_i = 2) = \mathbb{P}(Y_i = 3) = \frac{1}{3}$ and the transition matrix for Y_i is

$$\begin{bmatrix} 0.95 & 0.025 & 0.025 \\ 0.025 & 0.95 & 0.025 \\ 0.025 & 0.025 & 0.95 \end{bmatrix}. \quad (53)$$

The exponential AoI penalty function $\gamma_{\text{exp}}(\Delta) = e^{2\Delta} - 1$ is considered. The offline optimal hitting time threshold β^* and the corresponding optimal average AoI penalty is found by the fixed-point computation $\beta_{i+1} = f_\Gamma(\beta_i)$ in Sec. IV-B.

Next, we examine the performance degradation when the delay process is *Markov*, but the source *wrongly believes that the process is i.i.d.* That is, we run the best i.i.d.-delay-assuming online algorithm $\Xi^{\mathsf{SY}}_{\text{exp}}$ while directly feeding the true value of $\mathsf{SY}_\gamma = \mathbb{E}\{e^{aY}\}$ to the algorithm, and plot the trace of the observed avg. AoI penalty. Here we do not plot the trace

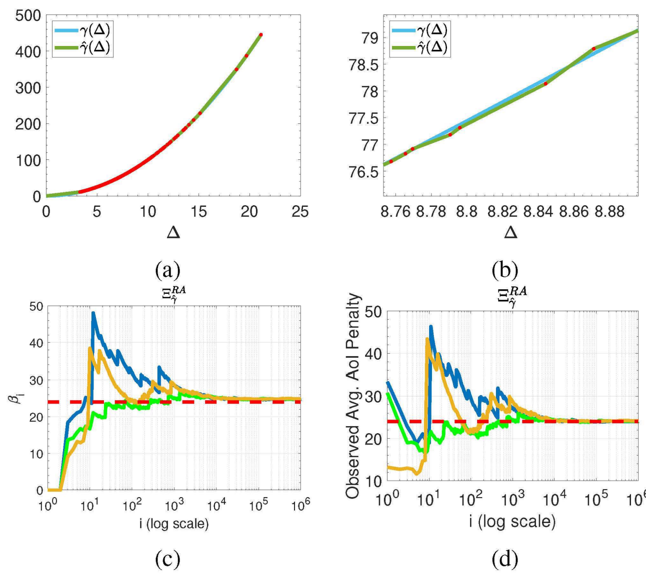


Fig. 5: Simulation results using the online algorithm $\Xi^{\text{RA}}_{\hat{\gamma}}$ under the i.i.d. delay.

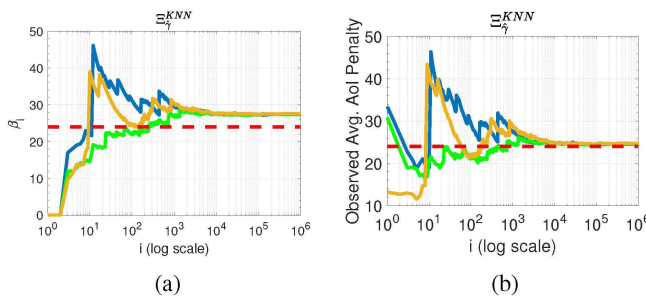


Fig. 6: Simulation results using the online algorithm $\Xi^{\text{KNN}}_{\hat{\gamma}}$ under the i.i.d. delay.

of β_i for $\Xi^{\text{SY}}_{\text{exp}}$ scheme anymore. The reason is that even if there is a genie that gives $\Xi^{\text{SY}}_{\text{exp}}$ the ideal β^* value, the scheme will still compute a suboptimal X_i since when computing X_i, the i.i.d.-delay-assuming scheme (incorrectly) takes the expectation of the marginal distribution $\mathbb{P}_Y$ in (11) and (12) while an optimal Markov-delay-assuming scheme, given the ideal β^* value, would take the expectation over the conditional distribution $\mathbb{P}_{Y_i|Y_{i-1},Z_{i-1}}$ instead. As a result, how close β_i is to β^* has little indication of how good the performance of the i.i.d.-based $\Xi^{\text{SY}}_{\text{exp}}$ scheme is when applied to a Markov setting. The only meaningful metric is to directly measure the observed avg. AoI penalty of different schemes. As shown in Fig. 7a, at the end of iteration the avg. AoI penalty of the i.i.d.-based $\Xi^{\text{SY}}_{\text{exp}}$ is 11% away from optimal offline Markov scheme.

We then run the online algorithm $\Xi^{\text{KNN}}_{\hat{\gamma}}$ with the associated parameters $N_{\hat{\gamma}} = 10^3$, $N_{\text{KNN}} = 100$ and $N_{\text{trapezoid}} = 10^3$. Without knowing that we are dealing with Markovian delay and without the knowledge of the penalty function $\gamma(\cdot)$, our scheme $\Xi^{\text{KNN}}_{\hat{\gamma}}$ performs extremely well. Fig. 8a plots the resulting $\hat{\gamma}(\cdot)$ and the true underlying $\gamma(\cdot)$ (those scattered red points are the elements in the set $\mathcal{S}_{\hat{\gamma}}$). This time, the estimator $\hat{\gamma}(\cdot)$ automatically adapts to a different underlying $\gamma(\cdot)$. The scheme $\Xi^{\text{KNN}}_{\hat{\gamma}}$ leads to 11% higher β_i compared with β^* (see Fig. 8c) while the observed avg. AoI penalty, arguably the more important metric, is within 2% of best possible offline solution (see Fig. 8d). This again shows the strength of the online algorithm $\Xi^{\text{KNN}}_{\hat{\gamma}}$.

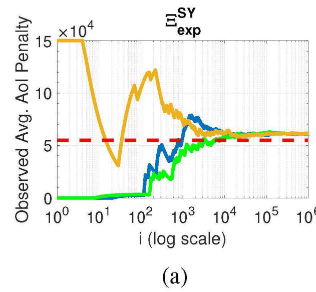


Fig. 7: Simulation results using the online algorithm $\Xi^{\text{SY}}_{\text{exp}}$ under the Markov delay.

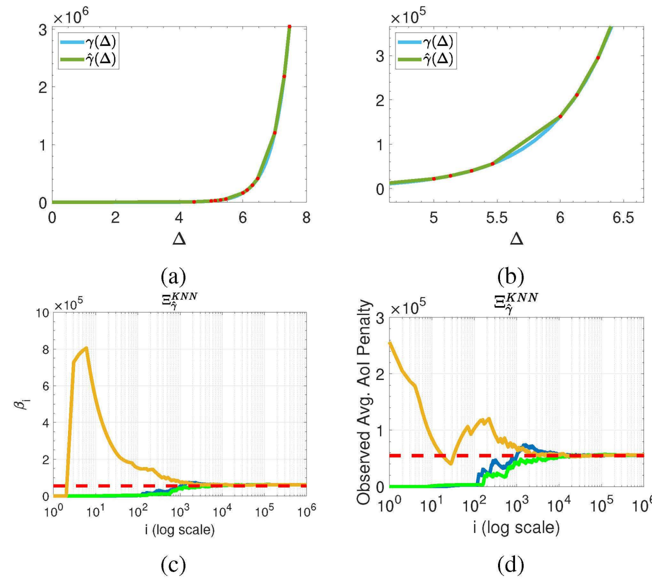


Fig. 8: Simulation results using the online algorithm $\Xi^{\text{KNN}}_{\hat{\gamma}}$ under the Markov delay.

C. Bounded $\gamma(\cdot)$ for the OU Process

One critical difference between the fixed-point-based Algorithm 1 and the Robbins-Monro-based Algorithm 2 is that the former is capable of handling bounded $\gamma(\cdot)$ while the latter is not. In this section, we consider the bounded $\gamma_{\text{OU}}(t) = \frac{\sigma^2}{2\theta}(1 - e^{-2\theta t})$ corresponding to signal-agnostic sampling for the OU process [19] and the corresponding $\beta_{\text{UB}} = \lim_{\Delta\to\infty} \gamma(\Delta) = \frac{\sigma^2}{2\theta} < \infty$.

Following the same manner in Sec.V-C, we use $\phi^{\text{SY}}_{\text{OU},\beta}(y', z')$ to denote the waiting time function $\phi_{\Gamma,\beta}(y', z')$ specialized for the OU-process penalty. Similarly, $g^{\text{SY}}_{\text{OU},1}(y', z', \beta)$ denotes the empirical AoI penalty function $g_1(y', z', \beta)$ specialized for the OU-process $\gamma_{\text{OU}}(\cdot)$. Applying simple calculus to (3), (4), (11), and (19) shows that for $\beta < \beta_{\text{UB}} = \frac{\sigma^2}{2\theta}$, we have

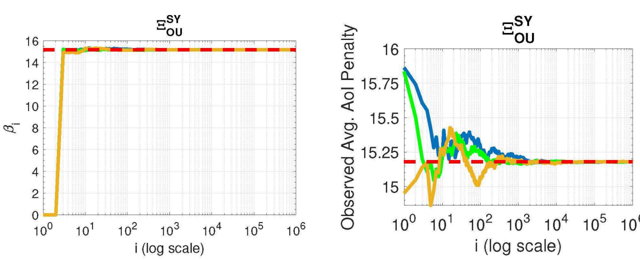
$$\phi^{\text{SY}}_{\text{OU},\beta}(y', z') = \max\left(\frac{1}{2\theta}\ln\left(\frac{\mathbb{E}\{e^{-2\theta Y}\}}{1-\frac{2\theta}{\sigma^2}\beta}\right) - y' - z', 0\right) \quad (54)$$

$$g^{\text{SY}}_{\text{OU},1}(y', z', \beta) = \frac{\sigma^2}{2\theta}\left(y' + z' + \phi^{\text{SY}}_{\text{OU},\beta}(y', z')\right) - \left(\frac{\sigma}{2\theta}\right)^2 \cdot \left(1 - e^{-2\theta\left(y'+z'+\phi^{\text{SY}}_{\text{OU},\beta}(y',z')\right)}\right) \cdot \mathbb{E}\{e^{-2\theta Y}\}. \quad (55)$$

From (54) and (55), it is clear that to calculate $\phi^{\text{SY}}_{\text{OU},\beta}(y', z')$ and $g^{\text{SY}}_{\text{OU},1}(y', z', \beta)$, the only statistical knowledge we need is a scalar $\mathsf{SY}_\gamma = \mathbb{E}\{e^{-2\theta Y}\}$, which can be well estimated in practice. $\Xi^{\text{SY}}_{\text{OU}}$ denotes Algorithm 1 when specialized for the OU-process AoI Penalty function $\gamma_{\text{OU}}(\Delta) = \frac{\sigma^2}{2\theta}(1 - e^{-2\theta\Delta})$.

We consider the same log-normal delay as in Sec. VIII-A and set $\sigma = 4$ and $\theta = 0.5$. The simulation results running $\Xi^{\text{SY}}_{\text{OU}}$ are presented in Figs. 9a and 9b.

As shown in Fig. 9a, we always have $\beta_i < \beta_{\text{UB}} = 16$ (as proved by Lemma 6), which demonstrates the applicability of Algorithm 1 under the bounded $\gamma(\cdot)$. Moreover, compared with the unbounded $\gamma_{\text{qdr}}(\cdot)$ (Figs. 4a and 4b), the convergence rate of Algorithm 1 for the bounded γ_{OU} seems even faster. Specifically, β_i is already within 1% of β^* after just 10 iterations, and the difference is less than 0.4% after 10^2 iterations. The observed avg. AoI penalty (Fig. 9b) is within 1.3% of offline optimum after 10^2 iterations.



(a) (b)

Fig. 9: Simulation results using the bounded $\gamma_{\text{OU}}(\cdot)$ corresponding to the OU process.

IX. Conclusion

We have studied the AoI minimization problem based on a new fixed-point-based framework, and derived the corresponding optimal waiting policy. We have also developed the first provably optimal distribution-oblivious online algorithms on AoI minimization for arbitrary AoI penalty functions, which may be bounded or unbounded. Additionally, we have addressed several practical issues in the i.i.d. delay and Markov delay settings, including proposing an effective solution to estimating the AoI penalty function $\gamma(\cdot)$ using monotonic regression. Simulation results verify the effectiveness of the proposed schemes.

Appendix A
Proof of Lemma 2

Consider the following two cases.

Case 1: $\beta_{\text{UB}} = \infty$. This case is obviously true since Lemma 2 considers an FED scheme A with finite $\beta_A < \infty$.

Case 2: $0 < \beta_{\text{UB}} < \infty$. For any finite $y', z', x, y < \infty$, by (3), we have

$$G(y', z', x, y) = \int_y^{y+x+y'+z'} \gamma(t)dt < \int_y^{y+x+y'+z'} \beta_{\text{UB}} \cdot dt = \beta_{\text{UB}} \cdot (x + y' + z') \quad (56)$$

where the inequality follows since we assume $\gamma(t)$ is strictly increasing and β_{UB} is the limit. As a result, from (4) and (56) we have

$$G_1(y', z', x) < \beta_{\text{UB}} \cdot (x + y' + z') \quad (57)$$

for any finite x, y', z'. From (57), since $\mathbb{E}\{X_i\} < \infty$ and (Y_{i-1}, Z_{i-1}) are of bounded support, we must have

$$\mathbb{E}\left\{G_1(Y_{i-1}, Z_{i-1}, \phi_A(Y_{i-1}, Z_{i-1}))\right\} < \beta_{\text{UB}} \cdot \mathbb{E}\left\{Y_{i-1} + Z_{i-1} + \phi_A(Y_{i-1}, Z_{i-1})\right\} \quad (58)$$

with strict inequality. Finally, since $\mathbb{E}\{Y_{i-1} + Z_{i-1}\} > 0$, we can move the expected duration of (58) to the left-hand side and have

$$\beta_A \triangleq \frac{\mathbb{E}\left\{G_1(Y_{i-1}, Z_{i-1}, \phi_A(Y_{i-1}, Z_{i-1}))\right\}}{\mathbb{E}\left\{Y_{i-1} + Z_{i-1} + \phi_A(Y_{i-1}, Z_{i-1})\right\}} < \beta_{\text{UB}}. \quad (59)$$

The proof of Lemma 2 is thus complete.

Appendix B
Proof of Lemma 3

For any given $T > 0$, we will prove that Lemma 3 holds if we replace the range of $t \in (0, \infty)$ inside Lemma 3 by $t \in (0, T)$. Once this is proven, we simply let $T \to \infty$ and we obtain our desired result.

Given any $(Y_{i-1} = y', Z_{i-1} = z')$ and any $t \in (0, T)$, from (4) we have

$$\frac{d}{dt}G_1(y', z', t) = \frac{d}{dt}\mathbb{E}_Y\left\{G(y', z', t, Y)|Y_{i-1} = y', Z_{i-1} = z'\right\} \quad (60)$$

which involves differentiation of a conditional expectation. We then observe that $G(y', z', t, Y_i)$ satisfies the following three conditions.

(i) $\mathbb{E}_{Y_i}\left\{G(y', z', t, Y_i))|Y_{i-1} = y', Z_{i-1} = z'\right\} < \infty$ for all $t \in (0, T)$, namely, $G(y', z', t, Y_i)$ is a Lebesgue-integrable function of Y_i for each $t \in (0, T)$. This is true because of the assumption that $Y \triangleq Y_i$, $Y' \triangleq Y_{i-1}$, and $Z' \triangleq Z_{i-1}$ all have bounded support, $t < T$, and the function $G(\cdot, \cdot, \cdot, \cdot)$ is strictly increasing for all four input variables (due to $\gamma(\cdot)$ being strictly increasing).

(ii) Given any $Y = y$ and any $t \in (0, T)$, since γ is continuous, we immediately have

$$\frac{d}{dt}G(y', z', t, y) = \gamma(y' + z' + t + y) \quad (61)$$

by (3) and the first fundamental theorem of calculus.

(iii) Since Y and Z are of bounded support, $\mathbb{P}(Y \leq y_{\max}, Z \leq z_{\max}) = 1$. Given any $Y = y$ and any $t \in (0, T)$, we then have

$$\left| \frac{d}{dt} G(y', z', t, y) \right| = \gamma \left(y' + z' + t + y \right) \tag{62}$$

$$\leq \gamma \left(y_{\max} + z_{\max} + T + y_{\max} \right) \tag{63}$$

where (62) follows from (61) and (63) holds since γ is strictly increasing. By (63), for any $Y = y$ and any $t \in (0, T)$, there exists a constant (and hence a Lebesgue-integrable function of Y) that upper bounds $\left| \frac{d}{dt} G(y', z', t, y) \right|$.

Since $G(y', z', t, Y)$ satisfies the above three conditions, by Leibniz's integral rule [24], we can interchange the differentiation and the expectation. Finally, we have

$$\begin{aligned} &\frac{d}{dt} \mathbb{E}_Y \left\{ G(y', z', t, Y) | Y_{i-1} = y', Z_{i-1} = z' \right\} \\ &= \mathbb{E}_Y \left\{ \frac{d}{dt} G(y', z', t, Y) | Y_{i-1} = y', Z_{i-1} = z' \right\} \\ &= \mathbb{E}_Y \left\{ \gamma(y' + z' + t + Y) | Y_{i-1} = y', Z_{i-1} = z' \right\} \end{aligned} \tag{64}$$

where (64) follows from (61). Lemma 3 follows from (60) and (64).

APPENDIX C
PROOF OF LEMMA 4

We first prove the first half of Lemma 4. By Lemma 3, it is sufficient to show that for any $\beta \in [0, \beta_{\text{UB}})$, there exists a $t_{\text{UB}} < \infty$ such that $\mathbb{E}\{\gamma(t_{\text{UB}} + y' + z' + Y_i) | Y_{i-1} = y', Z_{i-1} = z'\} > \beta$ for all y', z'. Since $\gamma(\cdot)$ is continuous and strictly increasing and $\beta < \beta_{\text{UB}}$, we can choose $t_{\text{UB}} \triangleq \gamma^{-1}(\beta)$ and we thus have

$$\begin{aligned} &\mathbb{E}\{\gamma(t_{\text{UB}} + y' + z' + Y_i) | Y_{i-1} = y', Z_{i-1} = z'\} \\ &\geq \gamma(t_{\text{UB}} + y z_{\min}) > \gamma(t_{\text{UB}}) = \beta. \end{aligned} \tag{65}$$

The first half of the proof is complete.

On the other hand, since $\sup_{t \to \infty} \gamma(t) = \beta_{\text{UB}}$, for any $\beta \geq \beta_{\text{UB}}$ and any finite $t, y', z' < \infty$, we have

$$\mathbb{E}\{\gamma(t + y' + z' + Y_i) | Y_{i-1} = y', Z_{i-1} = z'\} \leq \beta_{\text{UB}} \leq \beta. \tag{66}$$

By Lemma 3, the second half of the proof of Lemma 4 is complete.

APPENDIX D
PROOF OF PROPOSITION 1

The proof of Proposition 1 will need the following lemma.

Lemma 7: For any positive finite constants $p_1, T_1, r_1, p_2, T_2, r_2, \tau, r_\tau > 0$, we have the following two "$\Longrightarrow$" statements:

$$\frac{p_1 T_1 r_1 + p_2 (T_2 r_2 + \tau r_\tau)}{p_1 T_1 + p_2 (T_2 + \tau)} \leq r_\tau \tag{67}$$

$$\Longrightarrow \frac{p_1 T_1 r_1 + p_2 T_2 r_2}{p_1 T_1 + p_2 T_2} \leq \frac{p_1 T_1 r_1 + p_2 (T_2 r_2 + \tau r_\tau)}{p_1 T_1 + p_2 (T_2 + \tau)} \tag{68}$$

and

$$\frac{p_1 T_1 r_1 + p_2 T_2 r_2}{p_1 T_1 + p_2 T_2} \geq r_\tau \tag{69}$$

$$\Longrightarrow \frac{p_1 T_1 r_1 + p_2 (T_2 r_2 + \tau r_\tau)}{p_1 T_1 + p_2 (T_2 + \tau)} \leq \frac{p_1 T_1 r_1 + p_2 T_2 r_2}{p_1 T_1 + p_2 T_2}. \tag{70}$$

Proof: Consider any arbitrary positive and finite constants $A, B, a, b > 0$. It is straightforward to verify the following equivalent statements.

$$\frac{a}{b} \geq \frac{A + a}{B + b} \Longleftrightarrow \frac{a}{b} \geq \frac{A}{B} \Longleftrightarrow \frac{A + a}{B + b} \geq \frac{A}{B} \tag{71}$$

By choosing $A = p_2 \tau r_\tau$, $B = p_2 \tau$, $a = p_1 T_1 r_1 + p_2 T_2 r_2$, $b = p_1 T_1 + p_2 T_2$, and using the "$\Longleftarrow$" direction of the first $\Longleftrightarrow$ relationship in (71), we have proven the relationship in (69) and (70).

By choosing $A = p_1 T_1 r_1 + p_2 T_2 r_2$, $B = p_1 T_1 + p_2 T_2$, $a = p_2 \tau r_\tau$, $b = p_2 \tau$, and using the "$\Longrightarrow$" direction of both $\Longleftrightarrow$ relationships in (71), we have proven the relationship in (67) and (68). ■

For schemes A and Γ_{β_A}, recall that $\phi_A(Y_{i-1}, Z_{i-1})$ and $\phi_{\Gamma,\beta_A}(Y_{i-1}, Z_{i-1})$ are the waiting times for schemes A and Γ_{β_A}, respectively. For simplicity, we use ϕ_A and ϕ_{Γ,β_A} as shorthand by dropping the input arguments (Y_{i-1}, Z_{i-1}).

Suppose we are in the event of $\phi_{\Gamma,\beta_A} \leq \phi_A$, i.e., the scheme Γ_{β_A} sends the i-th update earlier than the scheme A. *During the interval $(\phi_{\Gamma,\beta_A}, \phi_A]$, the growth rate of $G_1(Y_{i-1}, Z_{i-1}, t)$ is strictly higher than β_A.* The reason is as follows. By the definition of ϕ_{Γ,β_A} in (11), the growth rate of $G_1(Y_{i-1}, Z_{i-1}, t)$ at time $t = \phi_{\Gamma,\beta_A}$ is either greater than or equal to β_A if ϕ_{Γ,β_A} is zero, or is equal to β_A if ϕ_{Γ,β_A} is strictly greater than zero. Since the growth rate of $G_1(Y_{i-1}, Z_{i-1}, t)$ is strictly increasing (due to strictly increasing $\gamma(\cdot)$ and by Lemma 3), in either case the growth rate of $G_1(Y_{i-1}, Z_{i-1}, t)$ is strictly larger than β_A during $(\phi_{\Gamma,\beta_A}, \phi_A]$. Compared to the original scheme A, the new scheme Γ_{β_A} avoids "higher-than-β_A" average during the interval $(\phi_{\Gamma,\beta_A}, \phi_A]$, which in turn helps make its average AoI penalty $f_\Gamma(\beta_A)$ smaller than the benchmark β_A.

Mathematically speaking, the average AoI penalty is the ratio of two expectations. If we use a simplified probabilistic model for discussion, then the left-hand side of (67) in Lemma 7 is indeed a ratio of two expectations. In the event with probability p_2, there is a duration of length τ with average growth rate within that duration of τ being r_τ. The left-hand side of (67) is how we calculate the overall average AoI penalty. The statement in (67) then says that if the penalty growth rate r_τ in the small duration τ is larger than the current average, then we always have (68). That is, by avoiding this duration of τ, the new average (the left-hand side of (68)) is better than the original average AoI penalty (the right-hand side of (68)).

Similarly, in the event of $0 \leq \phi_A < \phi_{\Gamma,\beta_A}$, during the interval $(\phi_A, \phi_{\Gamma,\beta_A}]$, the new scheme Γ_{β_A} will experience "no-higher-than-β_A" growth rate since the growth rate of $G_1(Y_{i-1}, Z_{i-1}, t)$ has not hit β_A yet for $t \in (\phi_A, \phi_{\Gamma,\beta_A}]$, which again helps make $f_\Gamma(\beta_A)$ lower than β_A.

Mathematically speaking, the left-hand side of (69) represents the current average AoI penalty, and the inequality (69) says that if the growth rate r_τ of a duration τ is smaller than the current average, then by adding a duration of length τ that has the penalty growth rate r_τ, the new average (the left-hand side of (70)) is again lower than the original average AoI penalty (the right hand side of (70)).

Since in either case the average AoI penalty of Γ_{β_A} has improved over the benchmark β_A, we have proven Proposition 1.

Appendix E
Proof of Lemma 5

From Corollary 1, we know β^* is one root of $\beta = f_\Gamma(\beta)$ within the domain $\beta \in [0, \beta_{\text{UB}})$. Suppose that there exists another root $\beta_0 \in [0, \beta_{\text{UB}})$ and $\beta_0 \neq \beta^*$.

Case 1: If $\beta_0 < \beta^*$, then we have the following contradiction

$$\beta_0 < \beta^* \leq f_\Gamma(\beta_0) = \beta_0 \tag{72}$$

where the "$\leq$" follows from (6).

Lemma 8: For any arbitrarily given penalty function $\gamma(\cdot)$ and any $\beta \in [0, \beta_{\text{UB}})$, we always have

$$(\beta - \beta^*)\overline{g}_2(\beta^*) \leq \beta \cdot \overline{g}_2(\beta) - \overline{g}_1(\beta) \tag{73}$$

regardless whether $\beta < \beta^*$ or $\beta \geq \beta^*$.

Proof: See Appendix F. ■

Case 2: Next we consider the case of $\beta^* < \beta_0$, which implies

$$0 < (\beta_0 - \beta^*)\overline{g}_2(\beta^*) \tag{74}$$

since $\overline{g}_2(\beta^*) \geq \mathbb{E}\{Y + Z\} > 0$. At the same time, if we substitute $\beta = \beta_0$ in (73) in Lemma 8, we have

$$(\beta_0 - \beta^*)\overline{g}_2(\beta^*) \leq \beta_0 \cdot \overline{g}_2(\beta_0) - \overline{g}_1(\beta_0). \tag{75}$$

Finally, β_0 being a fixed point implies

$$\beta_0 = \frac{\overline{g}_1(\beta_0)}{\overline{g}_2(\beta_0)} \Longleftrightarrow \beta_0 \cdot \overline{g}_2(\beta_0) - \overline{g}_1(\beta_0) = 0 \tag{76}$$

since by Lemma 4, we have $\overline{g}_2(\beta_0) < \infty$ as long as $\beta_0 \in [0, \beta_{\text{UB}})$. Concatenating the above three inequalities (74) to (76) implies the contradiction $0 < 0$. As a result, no such β_0 exists. The proof of uniqueness is complete.

Appendix F
Proof of Lemma 8

We prove Lemma 8 by first showing that for any $\beta_L \leq \beta_U \in [0, \beta_{\text{UB}})$,

$$\begin{aligned}\beta_L\left(\overline{g}_2(\beta_U) - \overline{g}_2(\beta_L)\right) &\leq \overline{g}_1(\beta_U) - \overline{g}_1(\beta_L)\\ &\leq \beta_U\left(\overline{g}_2(\beta_U) - \overline{g}_2(\beta_L)\right).\end{aligned} \tag{77}$$

By noticing that $\overline{g}_1(\beta)$ and $\overline{g}_2(\beta)$ are both non-decreasing function with respect to β, from (77) we immediately have

$$\overline{g}_1(\beta_1) - \overline{g}_1(\beta_2) \leq \beta_1\left(\overline{g}_2(\beta_1) - \overline{g}_2(\beta_2)\right). \tag{78}$$

for all $\beta_1, \beta_2 \in [0, \beta_{\text{UB}})$ regardless of whether $\beta_1 > \beta_2$ or $\beta_1 \leq \beta_2$. By choosing β_1 to be an arbitrary β value and setting $\beta_2 = \beta^*$, the optimal β value, we then have

$$\overline{g}_1(\beta) - \overline{g}_1(\beta^*) \leq \beta\left(\overline{g}_2(\beta) - \overline{g}_2(\beta^*)\right). \tag{79}$$

By Corollary 1 and (23), we have $\overline{g}_1(\beta^*) = \beta^* \cdot \overline{g}_2(\beta^*)$. Eq. (73) in Lemma 8 then follows directly from (79).

In the sequel, we prove (77). From (11), given any $(Y_{i-1} = y', Z_{i-1} = z')$, $\phi_{\Gamma,\beta}(y', z')$ is non-decreasing in β and hence we must have $\phi_{\Gamma,\beta_L}(y', z') \leq \phi_{\Gamma,\beta_U}(y', z')$. From (3), (4), (19) and (21), we then have

$$\begin{aligned}&\overline{g}_1(\beta_U) - \overline{g}_1(\beta_L)\\ &= \mathbb{E}\Bigg\{\mathbb{1}_{\{0<\phi_{\Gamma,\beta_L}(Y_{i-1},Z_{i-1})\leq\phi_{\Gamma,\beta_U}(Y_{i-1},Z_{i-1})\}}\cdot\\ &\quad\mathbb{E}\left\{\int_{\phi_{\Gamma,\beta_L}(Y_{i-1},Z_{i-1})+Y_{i-1}+Z_{i-1}+Y}^{\phi_{\Gamma,\beta_U}(Y_{i-1},Z_{i-1})+Y_{i-1}+Z_{i-1}+Y}\gamma(t)dt\middle|Y_{i-1},Z_{i-1}\right\}\Bigg\}\\ &\quad+\mathbb{E}\Bigg\{\mathbb{1}_{\{0=\phi_{\Gamma,\beta_L}(Y_{i-1},Z_{i-1})<\phi_{\Gamma,\beta_U}(Y_{i-1},Z_{i-1})\}}\cdot\\ &\quad\mathbb{E}\left\{\int_{Y_{i-1}+Z_{i-1}+Y}^{\phi_{\Gamma,\beta_U}(Y_{i-1},Z_{i-1})+Y_{i-1}+Z_{i-1}+Y}\gamma(t)dt\middle|Y_{i-1},Z_{i-1}\right\}\Bigg\}\\ &\quad+\mathbb{E}\Bigg\{\mathbb{1}_{\{0=\phi_{\Gamma,\beta_L}(Y_{i-1},Z_{i-1})=\phi_{\Gamma,\beta_U}(Y_{i-1},Z_{i-1})\}}\cdot 0\Bigg\}\end{aligned} \tag{80}$$

where (80) considers three *partitioning events* that discuss the order relationship among the three values: 0 versus $\phi_{\Gamma,\beta_L}(Y_{i-1}, Z_{i-1})$ versus $\phi_{\Gamma,\beta_U}(Y_{i-1}, Z_{i-1})$.

By similarly decomposing the expectation according to its three partitioning events, we also have

$$\begin{aligned}&\overline{g}_2(\beta_U) - \overline{g}_2(\beta_L)\\ &= \mathbb{E}\Bigg\{\mathbb{1}_{\{0<\phi_{\Gamma,\beta_L}(Y_{i-1},Z_{i-1})\leq\phi_{\Gamma,\beta_U}(Y_{i-1},Z_{i-1})\}}\cdot\\ &\qquad\Big(\phi_{\Gamma,\beta_U}(Y_{i-1},Z_{i-1}) - \phi_{\Gamma,\beta_L}(Y_{i-1},Z_{i-1})\Big)\Bigg\}\\ &\quad+\mathbb{E}\Bigg\{\mathbb{1}_{\{0=\phi_{\Gamma,\beta_L}(Y_{i-1},Z_{i-1})<\phi_{\Gamma,\beta_U}(Y_{i-1},Z_{i-1})\}}\cdot\\ &\qquad\phi_{\Gamma,\beta_U}(Y_{i-1},Z_{i-1})\Bigg\}\\ &\quad+\mathbb{E}\Bigg\{\mathbb{1}_{\{0=\phi_{\Gamma,\beta_L}(Y_{i-1},Z_{i-1})=\phi_{\Gamma,\beta_U}(Y_{i-1},Z_{i-1})\}}\cdot 0\Bigg\}.\end{aligned} \tag{81}$$

Under the first event in (80) $\{0 < \phi_{\Gamma,\beta_L}(Y_{i-1}, Z_{i-1}) \leq \phi_{\Gamma,\beta_U}(Y_{i-1}, Z_{i-1})\}$, we have

$$\begin{aligned}&\mathbb{E}\left\{\int_{\phi_{\Gamma,\beta_L}(Y_{i-1},Z_{i-1})+Y_{i-1}+Z_{i-1}+Y}^{\phi_{\Gamma,\beta_U}(Y_{i-1},Z_{i-1})+Y_{i-1}+Z_{i-1}+Y}\gamma(t)dt\middle|Y_{i-1},Z_{i-1}\right\}\\ &\leq \left(\phi_{\Gamma,\beta_U}(Y_{i-1},Z_{i-1}) - \phi_{\Gamma,\beta_L}(Y_{i-1},Z_{i-1})\right)\cdot\\ &\mathbb{E}\left\{\gamma(\phi_{\Gamma,\beta_U}(Y_{i-1},Z_{i-1}) + Y_{i-1} + Z_{i-1} + Y)\middle|Y_{i-1},Z_{i-1}\right\}\end{aligned} \tag{82}$$

$$= \left(\phi_{\Gamma,\beta_U}(Y_{i-1},Z_{i-1}) - \phi_{\Gamma,\beta_L}(Y_{i-1},Z_{i-1})\right)\cdot\beta_U \tag{83}$$

where (82) follows from the fact that γ is strictly increasing. Since $\phi_{\Gamma,\beta_U}(Y_{i-1}, Z_{i-1}) > 0$, from the definition of $\phi_{\Gamma,\beta}(Y_{i-1}, Z_{i-1})$ in (11) and the result in (12), $\mathbb{E}\left\{\gamma(\phi_{\Gamma,\beta_U}(Y_{i-1}, Z_{i-1}) + Y_{i-1} + Z_{i-1} + Y)\middle|Y_{i-1}, Z_{i-1}\right\} = \beta_U$ and thus (83) holds.

The same arguments also imply

$$\mathbb{E}\left\{\int_{\phi_{\Gamma,\beta_L}(Y_{i-1},Z_{i-1})+Y_{i-1}+Z_{i-1}+Y}^{\phi_{\Gamma,\beta_U}(Y_{i-1},Z_{i-1})+Y_{i-1}+Z_{i-1}+Y}\gamma(t)dt\Big|Y_{i-1},Z_{i-1}\right\}$$
$$\geq (\phi_{\Gamma,\beta_U}(Y_{i-1},Z_{i-1})-\phi_{\Gamma,\beta_L}(Y_{i-1},Z_{i-1}))\cdot$$
$$\mathbb{E}\left\{\gamma(\phi_{\Gamma,\beta_L}(Y_{i-1},Z_{i-1})+Y_{i-1}+Z_{i-1}+Y)\Big|Y_{i-1},Z_{i-1}\right\} \tag{84}$$
$$= (\phi_{\Gamma,\beta_U}(Y_{i-1},Z_{i-1})-\phi_{\Gamma,\beta_L}(Y_{i-1},Z_{i-1}))\cdot\beta_L \tag{85}$$

That is, instead of upper bounding the expectation, we now lower bound it.

Now consider the second event in (80) $\{0=\phi_{\Gamma,\beta_L}(Y_{i-1},Z_{i-1})<\phi_{\Gamma,\beta_U}(Y_{i-1},Z_{i-1})\}$. We have

$$\mathbb{E}\left\{\int_{Y_{i-1}+Z_{i-1}+Y}^{\phi_{\Gamma,\beta_U}(Y_{i-1},Z_{i-1})+Y_{i-1}+Z_{i-1}+Y}\gamma(t)dt\Big|Y_{i-1},Z_{i-1}\right\}$$
$$\leq \phi_{\Gamma,\beta_U}(Y_{i-1},Z_{i-1})\cdot$$
$$\mathbb{E}\left\{\gamma(\phi_{\Gamma,\beta_U}(Y_{i-1},Z_{i-1})+Y_{i-1}+Z_{i-1}+Y)\Big|Y_{i-1},Z_{i-1}\right\} \tag{86}$$
$$= \phi_{\Gamma,\beta_U}(Y_{i-1},Z_{i-1})\cdot\beta_U \tag{87}$$

where (86) holds since γ is strictly increasing, and (87) holds since $\phi_{\Gamma,\beta_U}(Y_{i-1},Z_{i-1})>0$ and thus $\mathbb{E}\{\gamma(\phi_{\Gamma,\beta_U}(Y_{i-1},Z_{i-1})+Y_{i-1}+Z_{i-1}+Y)\big|Y_{i-1},Z_{i-1}\}=\beta_U$. Similarly, we also have

$$\mathbb{E}\left\{\int_{Y_{i-1}+Z_{i-1}+Y}^{\phi_{\Gamma,\beta_U}(Y_{i-1},Z_{i-1})+Y_{i-1}+Z_{i-1}+Y}\gamma(t)dt\Big|Y_{i-1},Z_{i-1}\right\}$$
$$\geq \phi_{\Gamma,\beta_U}(Y_{i-1},Z_{i-1})\cdot$$
$$\mathbb{E}\left\{\gamma(Y_{i-1}+Z_{i-1}+Y)\Big|Y_{i-1},Z_{i-1}\right\} \tag{88}$$
$$\geq \phi_{\Gamma,\beta_U}(Y_{i-1},Z_{i-1})\cdot\beta_L \tag{89}$$

where the last inequality uses the fact that since $\phi_{\Gamma,\beta_L}(Y_{i-1},Z_{i-1})=0$ we must have $\mathbb{E}\{\gamma(Y_{i-1}+Z_{i-1}+Y)\big|Y_{i-1},Z_{i-1}\}\geq\beta_L$.

From (80), (83) and (87), we have

$$\overline{g}_1(\beta_U)-\overline{g}_1(\beta_L)$$
$$\leq \mathbb{E}\Big\{\mathbb{1}_{\{0<\phi_{\Gamma,\beta_L}(Y_{i-1},Z_{i-1})\leq\phi_{\Gamma,\beta_U}(Y_{i-1},Z_{i-1})\}}\cdot$$
$$\beta_U\cdot(\phi_{\Gamma,\beta_U}(Y_{i-1},Z_{i-1})-\phi_{\Gamma,\beta_L}(Y_{i-1},Z_{i-1}))\Big\}$$
$$+\mathbb{E}\Big\{\mathbb{1}_{\{0=\phi_{\Gamma,\beta_L}(Y_{i-1},Z_{i-1})<\phi_{\Gamma,\beta_U}(Y_{i-1},Z_{i-1})\}}\cdot$$
$$\beta_U\cdot\phi_{\Gamma,\beta_U}(Y_{i-1},Z_{i-1})\Big\}$$
$$=\beta_U\left(\overline{g}_2(\beta_U)-\overline{g}_2(\beta_L)\right) \tag{90}$$

where (90) follows from (81).

From (80), (85) and (89), we have

$$\overline{g}_1(\beta_U)-\overline{g}_1(\beta_L)$$
$$\geq \mathbb{E}\Big\{\mathbb{1}_{\{0<\phi_{\Gamma,\beta_L}(Y_{i-1},Z_{i-1})\leq\phi_{\Gamma,\beta_U}(Y_{i-1},Z_{i-1})\}}\cdot$$
$$\beta_L\cdot(\phi_{\Gamma,\beta_U}(Y_{i-1},Z_{i-1})-\phi_{\Gamma,\beta_L}(Y_{i-1},Z_{i-1}))\Big\}$$
$$+\mathbb{E}\Big\{\mathbb{1}_{\{0=\phi_{\Gamma,\beta_L}(Y_{i-1},Z_{i-1})<\phi_{\Gamma,\beta_U}(Y_{i-1},Z_{i-1})\}}\cdot$$
$$\beta_L\cdot\phi_{\Gamma,\beta_U}(Y_{i-1},Z_{i-1})\Big\}$$
$$=\beta_L\left(\overline{g}_2(\beta_U)-\overline{g}_2(\beta_L)\right) \tag{91}$$

where (91) follows from (81). Jointly we have proven (77).

APPENDIX G
PROOF OF LEMMA 6

We first prove

Lemma 9: For any given $(Y_{i-1}=y',Z_{i-1}=z')$ and for any $\beta\in[0,\beta_{\text{UB}})$,

$$\frac{g_1(y',z',\beta)}{g_2(y',z',\beta)} \tag{92}$$

is non-decreasing with respect to β.

Proof: Define a positive function

$$\tilde{g}_1(w)=\mathbb{E}\left\{\int_Y^{w+Y}\gamma(t)dt\Big|Y_{i-1}=y',Z_{i-1}=z'\right\} \tag{93}$$

which satisfies $\tilde{g}_1(0)=0$ and $\tilde{g}_1(g_2(y',z',\beta))=g_1(y',z',\beta)$. We then have

$$\frac{g_1(y',z',\beta)}{g_2(y',z',\beta)}=\frac{\tilde{g}_1(g_2(y',z',\beta))}{g_2(y',z',\beta)}. \tag{94}$$

From (94), since $g_2(y',z',\beta)$ is a non-decreasing function of β (because $\phi_{\Gamma,\beta}(y',z')$ is a non-decreasing function of β, see (11)), if we can show that

$$\frac{\tilde{g}_1(w)}{w} \tag{95}$$

is a non-decreasing function of w, then the term in (94) is a non-decreasing function of β and the proof would be complete. We now prove that $\frac{\tilde{g}_1(w)}{w}$ is a non-decreasing function of w.

Using Leibniz's integral rule as in Lemma 3, for any $w>0$, the derivative of $\tilde{g}_1(w)$ can be computed by

$$\frac{d}{dw}\tilde{g}_1(w)=\mathbb{E}\{\gamma(w+Y)\big|Y_{i-1}=y',Z_{i-1}=z'\} \tag{96}$$

which is strictly increasing since γ is strictly increasing. From (96), since the derivative of $\tilde{g}_1(w)$ is increasing, $\tilde{g}_1(w)$ is a convex function of w. By the property of a convex function, for any $0<w_1<w_2$, we must have

$$\frac{\tilde{g}_1(w_1)-\tilde{g}_1(0)}{w_1-0}\leq\frac{\tilde{g}_1(w_2)-\tilde{g}_1(0)}{w_2-0}. \tag{97}$$

Since $\tilde{g}_1(0)=0$, from (97) we know $\frac{\tilde{g}_1(w)}{w}$ is a non-decreasing function of w. ■

Lemma 10: Recall that $y_{\max}$ and $z_{\max}$ are the upper bounds of the random variables Y and Z. Define

$$\beta_{\max} \triangleq \gamma(2y_{\max} + z_{\max} + 1) < \beta_{\text{UB}}. \tag{98}$$

For any arbitrary $(Y_{i-1} = y', Z_{i-1} = z')$ and any arbitrary $\beta \leq \beta_{\max}$, we always have

$$\frac{g_1(y', z', \beta)}{g_2(y', z', \beta)} \leq \frac{g_1(y', z', \beta_{\max})}{g_2(y', z', \beta_{\max})} \leq \beta_{\max}. \tag{99}$$

Proof: The first inequality in (99) holds by Lemma 9. We now prove the second inequality.

Proceeding from the proof of Lemma 9, for any given $(Y_{i-1} = y', Z_{i-1} = z')$, we have

$$\beta_{\max} = \mathbb{E}\{\gamma(w+Y)\big|Y_{i-1} = y', Z_{i-1} = z'\}\Big|_{w=g_2(y',z',\beta_{\max})} \tag{100}$$

which follows from (i) the definition $g_2(y', z', \beta_{\max}) = \phi_{\Gamma,\beta_{\max}}(y', z') + y' + z'$ in (19), (ii) because the $\beta_{\max}$ in (98) is *sufficiently large*, by the definition of $\phi_{\Gamma,\beta}(y', z')$ in (11), we always have $\phi_{\Gamma,\beta_{\max}}(y', z') > 0$ for any (y', z'). (i) and (ii) jointly imply that the expectation is indeed $\beta_{\max}$.

We then have

$$\begin{aligned}
&\frac{g_1(y', z', \beta_{\max})}{g_2(y', z', \beta_{\max})} = \frac{\tilde{g}_1(g_2(y', z', \beta_{\max}))}{g_2(y', z', \beta_{\max})} \\
&= \frac{\tilde{g}_1(g_2(y', z', \beta_{\max})) - \tilde{g}_1(0)}{g_2(y', z', \beta_{\max}) - 0} \\
&\leq \frac{d}{dw}\tilde{g}_1(w)\Big|_{w=g_2(y',z',\beta_{\max})} \\
&= \mathbb{E}\{\gamma(w+Y)\big|Y_{i-1} = y', Z_{i-1} = z'\}\Big|_{w=g_2(y',z',\beta_{\max})} \\
&= \beta_{\max}.
\end{aligned} \tag{101}$$

where the first equality follows from the definition of $\tilde{g}_1$ in (93); the second equality follows from the fact that $\tilde{g}_1(0) = 0$; the first inequality follows from the property of the convex function $\tilde{g}_1(w)$ and the fact that $\tilde{g}_1(0) = 0 < g_2(y', z', \beta_{\max})$; the third equality follows from (96); the fourth equality follows from the discussion after (100). The second equality in (99) is thus proved. ∎

Since $0 < \beta_{\max}$, Lemma 6 holds clearly for $i = 1$ and 2. For any $i \geq 3$, by (24) we have

$$\beta_i \leq \max_{j\in[1,i-1]} \frac{g_1(Y_{j-1}, Z_{j-1}, \beta_j)}{g_2(Y_{j-1}, Z_{j-1}, \beta_j)} \leq \beta_{\max} \tag{102}$$

where the last inequality follows from iteratively applying Lemma 10 for all $i \geq 3$. The proof is complete.

APPENDIX H
PROOF OF (37) IN PROPOSITION 3

The proof of Proposition 3 consists of two halves, the proof of (37) (Appendix H) and the proof of (38) (Appendix I). We first prove (37).

We define two random processes M_i and N_i as follows. Set $M_0 = N_0 = 0$ and for any $i \geq 1$,

$$M_i = \sum_{j=1}^{i-1} \left(g_1(Y_{j-1}, Z_{j-1}, \beta_j) - \overline{g}_1(\beta_j)\right) \tag{103}$$

$$N_i = \sum_{j=1}^{i-1} \left(g_2(Y_{j-1}, Z_{j-1}, \beta_j) - \overline{g}_2(\beta_j)\right). \tag{104}$$

Define $\mathcal{F}_i \triangleq \{(Y_j, Z_j) : j \leq i-2\}$ as the set of all the previous forward and backward channel delays up to the $(i-2)$-th packet.

Lemma 11: $\{M_i\}$ and $\{N_i\}$ are martingales with respect to $\mathcal{F}_i$.

Proof: First, since $\{Y_i\}$, $\{Z_i\}$ and $\{\beta_i\}$ are all bounded, we have $\mathbb{E}\{|M_i|\} < \infty$ and $\mathbb{E}\{|N_i|\} < \infty$.

We then have

$$\begin{aligned}
&\mathbb{E}\{M_i - M_{i-1}|\mathcal{F}_{i-1}\} \\
&= \mathbb{E}\{g_1(Y_{i-2}, Z_{i-2}, \beta_{i-1}) - \overline{g}_1(\beta_{i-1})|\mathcal{F}_{i-1}\} &(105)\\
&= \mathbb{E}\{g_1(Y_{i-2}, Z_{i-2}, \beta_{i-1})|\mathcal{F}_{i-1}\} - \overline{g}_1(\beta_{i-1}) = 0 &(106)
\end{aligned}$$

where the first equality in (106) follows from the fact that β_{i-1} is completely determined by $\mathcal{F}_{i-1}$ (see (24) and the definition of $\mathcal{F}_i$ in the above); and the second equality in (106) follows from $\{(Y_{i-2}, Z_{i-2})\}$ being i.i.d. and independent of $\mathcal{F}_{i-1}$. Similar reasoning gives $\mathbb{E}\{N_i - N_{i-1}|\mathcal{F}_{i-1}\} = 0$. ∎

Lemma 12: For $\alpha > 0$ and for all $i \geq 1$, there exist two positive constants $k_1, k_2 > 0$ such that (107) and (108) hold.

$$\begin{aligned}
&\mathbb{P}\left(\sum_{j=1}^{i} g_1(Y_{j-1}, Z_{j-1}, \beta_j) < -i^{(0.5+\alpha)} + \sum_{j=1}^{i} \overline{g}_1(\beta_j)\right) \\
&\leq \exp\left(\frac{-i^{2\alpha}}{2(k_1)^2}\right) &(107)\\
&\mathbb{P}\left(\sum_{j=1}^{i} g_2(Y_{j-1}, Z_{j-1}, \beta_j) > i^{(0.5+\alpha)} + \sum_{j=1}^{i} \overline{g}_2(\beta_j)\right) \\
&\leq \exp\left(\frac{-i^{2\alpha}}{2(k_2)^2}\right). &(108)
\end{aligned}$$

Though admitting a complicated form, the intuition of Lemma 12 is simple. Because M_i and N_i are Martingales, the growth rates of both M_i and N_i should be within $\pm i^{0.5+\alpha}$ with close-to-one probability. Nonetheless because we only need one side of it, we bound the probability of M_i being too small in (107) and bound the probability of N_i being too large in (108).

Proof: Recall that $\mathbb{P}(Y \leq y_{\max}, Z \leq y_{\max}) = 1$. We then have

$$\begin{aligned}
&|M_i - M_{i-1}| = |g_1(Y_{i-2}, Z_{i-2}, \beta_{i-1}) - \overline{g}_1(\beta_{i-1})| &(109)\\
&\leq g_1(y_{\max}, z_{\max}, \beta_{\max}) + \overline{g}_1(\beta_{\max}) &(110)
\end{aligned}$$

where (109) follows from (103); and (110) follows from $\{Y_i\}$, $\{Z_i\}$ and $\{\beta_i\}$ all being bounded and $g_1(\cdot,\cdot,\beta)$ and $\overline{g}_1(\beta)$ are non-decreasing in β.

Similarly, we have

$$|N_i - N_{i-1}| \leq g_2(y_{\max}, z_{\max}, \beta_{\max}) + \overline{g}_2(\beta_{\max}). \tag{111}$$

Since $\{M_i\}$ and $\{N_i\}$ are martingales satisfying (110) and (111), by Azuma's inequality [25], there exist positive constants k_1 and k_2 such that (107) and (108) hold. ■

Lemma 13: For $\alpha > 0$ and $i \geq 1$, there exists a positive constant $k_3 > 0$ such that

$$\mathbb{P}\left(\frac{\sum_{j=1}^{i} g_1(Y_{j-1}, Z_{j-1}, \beta_j)}{\sum_{j=1}^{i} g_2(Y_{j-1}, Z_{j-1}, \beta_j)} < \frac{-i^{(0.5+\alpha)} + \sum_{j=1}^{i} \overline{g}_1(\beta_j)}{i^{(0.5+\alpha)} + \sum_{j=1}^{i} \overline{g}_2(\beta_j)}\right) \leq 2\exp\left(\frac{-i^{2\alpha}}{2(k_3)^2}\right). \quad (112)$$

Proof: We first define 3 events.

$$A_1 \triangleq \left\{\sum_{j=1}^{i} g_1(Y_{j-1}, Z_{j-1}, \beta_j) < -i^{(0.5+\alpha)} + \sum_{j=1}^{i} \overline{g}_1(\beta_j)\right\} \quad (113)$$

$$A_2 \triangleq \left\{\sum_{j=1}^{i} g_2(Y_{j-1}, Z_{j-1}, \beta_j) > i^{(0.5+\alpha)} + \sum_{j=1}^{i} \overline{g}_2(\beta_j)\right\} \quad (114)$$

$$A \triangleq \left\{\frac{\sum_{j=1}^{i} g_1(Y_{j-1}, Z_{j-1}, \beta_j)}{\sum_{j=1}^{i} g_2(Y_{j-1}, Z_{j-1}, \beta_j)} < \frac{-i^{(0.5+\alpha)} + \sum_{j=1}^{i} \overline{g}_1(\beta_j)}{i^{(0.5+\alpha)} + \sum_{j=1}^{i} \overline{g}_2(\beta_j)}\right\} \quad (115)$$

Eq. (112) is equivalent to $\mathbb{P}(A) \leq 2\exp\left(\frac{-i^{2\alpha}}{2\cdot(k_3)^2}\right)$.

We then observe that $(A_1)^c \cap (A_2)^c \subseteq A^c$, which implies $\mathbb{P}(A) \leq P(A_1 \cup A_2)$. By the union bound and by choosing $k_3 = \max(k_1, k_2)$, we have completed the proof. ■

Lemma 14: For any positive constants $a, b, c, d > 0$, if $c - d \geq 0$, then we have

$$\frac{c-d}{a+b} \geq (c-d)\left(\frac{1}{a} - \frac{b}{a^2}\right) \geq \frac{c}{a} - \frac{d}{a} - \frac{bc}{a^2}. \quad (116)$$

Proof: The second inequality can be easily proved by observing

$$(c-d)\left(\frac{1}{a} - \frac{b}{a^2}\right) = \frac{c}{a} - \frac{d}{a} - \frac{bc}{a^2} + \frac{bd}{a^2} \geq \frac{c}{a} - \frac{d}{a} - \frac{bc}{a^2}. \quad (117)$$

We now prove the first inequality. We have

$$\begin{aligned} & a^2 \geq a^2 - b^2 = (a-b)(a+b) \\ \Longrightarrow \quad & \frac{1}{a+b} \geq \frac{a-b}{a^2} = \frac{1}{a} - \frac{b}{a^2} \\ \Longrightarrow \quad & \frac{c-d}{a+b} \geq (c-d)\left(\frac{1}{a} - \frac{b}{a^2}\right). \end{aligned}$$

■

Lemma 15: Given any $\alpha \in (0, 0.5)$, define

$$I_1 \triangleq \left\lceil \overline{g}_1(0)^{\left(\frac{-1}{0.5-\alpha}\right)} \right\rceil. \quad (118)$$

Then, for all $i \geq I_1$,

$$\overline{g}_1(0) \cdot i \geq i^{(0.5+\alpha)}. \quad (119)$$

Proof: Eq. (119) holds if

$$i^{(0.5-\alpha)} \geq \frac{1}{\overline{g}_1(0)} \quad (120)$$

$$\Longleftrightarrow \quad \ln(i) \geq \frac{1}{(0.5-\alpha)} \cdot \ln\left(\frac{1}{\overline{g}_1(0)}\right) \quad (121)$$

$$\Longleftrightarrow \quad i \geq \overline{g}_1(0)^{\left(\frac{-1}{0.5-\alpha}\right)}. \quad (122)$$

■

Lemma 16: Given any $\alpha \in (0, 0.5)$, for $i \geq I_1$,

$$\mathbb{P}\left(\frac{\sum_{j=1}^{i} g_1(Y_{j-1}, Z_{j-1}, \beta_j)}{\sum_{j=1}^{i} g_2(Y_{j-1}, Z_{j-1}, \beta_j)} < \frac{\sum_{j=1}^{i} \overline{g}_1(\beta_j)}{\sum_{j=1}^{i} \overline{g}_2(\beta_j)} - \frac{i^{-(0.5-\alpha)}}{\overline{g}_2(0)} - \frac{\overline{g}_1(\beta_{\max}) \cdot i^{-(0.5-\alpha)}}{(\overline{g}_2(0))^2}\right) \leq 2\exp\left(\frac{-i^{2\alpha}}{2(k_3)^2}\right). \quad (123)$$

Proof: The proof is a directly combination of Lemmas 13 to 15. Given any $\alpha \in (0, 0.5)$ and $i \geq I_1$, for notational simplicity, we set

$$a \triangleq \sum_{j=1}^{i} \overline{g}_2(\beta_j) \geq i \cdot \overline{g}_2(0) \quad (124)$$

$$b = d \triangleq i^{(0.5+\alpha)} \quad (125)$$

$$i \cdot \overline{g}_1(0) \leq c \triangleq \sum_{j=1}^{i} \overline{g}_1(\beta_j) \leq i \cdot \overline{g}_1(\beta_{\max}) \quad (126)$$

where the inequalities in (124) and (126) follow from the fact that $\overline{g}_1(\beta)$ and $\overline{g}_2(\beta)$ are both non-decreasing in β.

The reason why we define a to d is that the event in (112) involves an inequality, for which the right-hand side is exactly $\frac{c-d}{a+b}$. Note that a, b, c, d are positive constants. Further, from (125) and (126), we have $c - d \geq i \cdot \overline{g}_1(0) - i^{(0.5+\alpha)} \geq 0$ since we consider the case where $i \geq I_1$. Since a, b, c and d are all positive, we can use Lemma 14 in the proof.

From (124) and (126), we have

$$-\frac{d}{a} = -\frac{d}{\sum_{j=1}^{i} \overline{g}_2(\beta_j)} \geq -\frac{i^{(0.5+\alpha)}}{i \cdot \overline{g}_2(0)} = -\frac{i^{-(0.5-\alpha)}}{\overline{g}_2(0)} \quad (127)$$

and

$$-\frac{bc}{a^2} \geq -\frac{b \cdot i \cdot \overline{g}_1(\beta_{\max})}{a^2} \geq -\frac{i^{(0.5+\alpha)} \cdot i \cdot \overline{g}_1(\beta_{\max})}{(i \cdot \overline{g}_2(0))^2} = \frac{\overline{g}_1(\beta_{\max}) \cdot i^{-(0.5-\alpha)}}{(\overline{g}_2(0))^2}. \quad (128)$$

Putting them together, we have

$$2\exp\left(\frac{-i^{2\alpha}}{2(k_3)^2}\right)$$
$$\geq \mathbb{P}\left(\frac{\sum_{j=1}^{i} g_1(Y_{j-1},Z_{j-1},\beta_j)}{\sum_{j=1}^{i} g_2(Y_{j-1},Z_{j-1},\beta_j)} < \frac{c-d}{a+b}\right) \tag{129}$$
$$\geq \mathbb{P}\left(\frac{\sum_{j=1}^{i} g_1(Y_{j-1},Z_{j-1},\beta_j)}{\sum_{j=1}^{i} g_2(Y_{j-1},Z_{j-1},\beta_j)} < \frac{c}{a}-\frac{d}{a}-\frac{bc}{a^2}\right) \tag{130}$$
$$\geq \mathbb{P}\Bigg(\frac{\sum_{j=1}^{i} g_1(Y_{j-1},Z_{j-1},\beta_j)}{\sum_{j=1}^{i} g_2(Y_{j-1},Z_{j-1},\beta_j)} < \frac{\sum_{j=1}^{i}\overline{g}_1(\beta_j)}{\sum_{j=1}^{i}\overline{g}_2(\beta_j)} - \frac{i^{-(0.5-\alpha)}}{\overline{g}_2(0)} - \frac{\overline{g}_1(\beta_{\max})\cdot i^{-(0.5-\alpha)}}{(\overline{g}_2(0))^2}\Bigg) \tag{131}$$

where (129) follows from Lemma 13; (130) follows from Lemma 14; (131) follows from the definitions in (124), (125) and (126) and the inequalities in (127) and (128). ■

Since

$$\beta^* = \min_{\beta}\frac{\overline{g}_1(\beta)}{\overline{g}_2(\beta)} \leq \frac{\overline{g}_2(\beta_j)}{\overline{g}_2(\beta_j)} \tag{132}$$

for any β_j (see the discussions in Proposition 1 and Corollary 1), we must have

$$\beta^* \leq \frac{\sum_{j=1}^{i}\overline{g}_1(\beta_j)}{\sum_{j=1}^{i}\overline{g}_2(\beta_j)}. \tag{133}$$

Continuing from (129) and (133), we have

Lemma 17: Given any $\alpha \in (0, 0.5)$, for $i \geq I_1$, there exists a constant k_4 such that

$$\mathbb{P}\left(\frac{\sum_{j=1}^{i} g_1(Y_{j-1},Z_{j-1},\beta_j)}{\sum_{j=1}^{i} g_2(Y_{j-1},Z_{j-1},\beta_j)} < \beta^* - k_4\cdot i^{-(0.5-\alpha)}\right)$$
$$\leq 2\exp\left(\frac{-i^{2\alpha}}{2(k_3)^2}\right) \tag{134}$$

Proof: Ineq. (134) follows directly from (131) and (133) by setting

$$k_4 \triangleq \frac{1}{\overline{g}_2(0)} + \frac{\overline{g}_1(\beta_{\max})}{(\overline{g}_2(0))^2}. \tag{135}$$

■

By the β_i update rule in (24), Lemma 17 can be rewritten as

$$\mathbb{P}\left(\beta_{i+1} < \beta^* - k_4\cdot i^{-(0.5-\alpha)}\right)$$
$$\leq 2\exp\left(\frac{-i^{2\alpha}}{2(k_3)^2}\right), \quad \forall i > I_1. \tag{136}$$

Given any $\alpha \in (0, 0.5)$, we set the positive constants in (37) in the following way:

$$c_1 \triangleq k_4 = \frac{1}{\overline{g}_2(0)} + \frac{\overline{g}_1(\beta_{\max})}{(\overline{g}_2(0))^2} \tag{137}$$
$$c_2 \triangleq \max\left(2, \exp\left(c_3\cdot(I_1)^{2\alpha}\right)\right) \tag{138}$$
$$c_3 \triangleq \frac{1}{2(k_3)^2}. \tag{139}$$

The above specific choices of c_1 to c_3 plus the inequality (136), we have proven (37).

Appendix I
Proof of (38) in Proposition 3

Lemma 18: For any positive constant $a > 0$ and any non-negative constants, $b, c, d \geq 0$, we have

$$\frac{c+d}{a+b} \leq (c+d)(\frac{1}{a}-\frac{b}{a^2}+\frac{b^2}{a^3})$$
$$\leq \frac{c}{a}+\frac{d}{a}-\frac{bc}{a^2}+\frac{b^2c}{a^3}+\frac{b^2d}{a^3}. \tag{140}$$

Proof: The second inequality in (140) follows from expanding the previous term and adding a non-negative term $\frac{bd}{a^2}$. We hence only need to prove the first inequality in (140).

$$a^3 \leq a^3 + b^3 = (a+b)(a^2-ab+b^2) \tag{141}$$
$$\Longrightarrow \quad \frac{1}{a+b} \leq \frac{a^2-ab+b^2}{a^3} = \frac{1}{a}-\frac{b}{a^2}+\frac{b^2}{a^3} \tag{142}$$
$$\Longrightarrow \quad \frac{c+d}{a+b} \leq (c+d)(\frac{1}{a}-\frac{b}{a^2}+\frac{b^2}{a^3}). \tag{143}$$

■

We define

$$g_{2,\min} \triangleq y z_{\min}. \tag{144}$$

Since $g_2(\cdot,\cdot,\cdot)$ is non-decreasing with respect to all three input variables and $\mathbb{P}(Y+Z > yz_{\min}) = 1$, we have $\mathbb{P}(g_2(Y,Z,\beta) > g_{2,\min}) = 1$.

Similarly, we define

$$g_{2,\max} \triangleq g_2(y_{\max}, z_{\max}, \beta_{\max}) \tag{145}$$
$$g_{1,\max} \triangleq g_1(y_{\max}, z_{\max}, \beta_{\max}) \tag{146}$$

such that $\mathbb{P}(g_2(Y,Z,\beta) \leq g_{2,\max}) = 1$ and $\mathbb{P}(g_1(Y,Z,\beta) \leq g_{1,\max}) = 1$, where $\beta_{\max}$ is first defined in (98).

Lemma 19: There exists a positive constant $k_5 > 0$ such that for all $i \geq 3$, we have

$$\beta_{i+1} = \frac{\sum_{j=1}^{i} g_1(Y_{j-1},Z_{j-1},\beta_j)}{\sum_{j=1}^{i} g_2(Y_{j-1},Z_{j-1},\beta_j)} \tag{147}$$
$$\leq \beta_i - \frac{\beta_i\cdot g_2(Y_{i-1},Z_{i-1},\beta_i) - g_1(Y_{i-1},Z_{i-1},\beta_i)}{\sum_{j=1}^{i-1} g_2(Y_{j-1},Z_{j-1},\beta_j)} + \frac{k_5}{(i-1)^2}. \tag{148}$$

Proof: Eq. (147) follows from (24). We now prove (148). For notational simplicity, we set

$$a \triangleq \sum_{j=1}^{i-1} g_2(Y_{j-1},Z_{j-1},\beta_j) \tag{149}$$
$$b \triangleq g_2(Y_{i-1},Z_{i-1},\beta_i) \tag{150}$$
$$c \triangleq \sum_{j=1}^{i-1} g_1(Y_{j-1},Z_{j-1},\beta_j) \tag{151}$$
$$d \triangleq g_1(Y_{i-1},Z_{i-1},\beta_i). \tag{152}$$

From (24) and since $i \geq 3$, we have

$$\frac{c}{a} = \beta_i. \tag{153}$$

Next, we have

$$\frac{d}{a} = \frac{g_1(Y_{i-1}, Z_{i-1}, \beta_i)}{\sum_{j=1}^{i-1} g_2(Y_{j-1}, Z_{j-1}, \beta_j)}. \tag{154}$$

Then, from (153) we have

$$-\frac{bc}{a^2} = -(\frac{c}{a})(\frac{b}{a}) = (-\beta_i)(\frac{b}{a}) = \frac{-\beta_i \cdot g_2(Y_{i-1}, Z_{i-1}, \beta_i)}{\sum_{j=1}^{i-1} g_2(Y_{j-1}, Z_{j-1}, \beta_j)}. \tag{155}$$

Further,

$$\frac{b^2 c}{a^3} = (\frac{b}{a})^2(\frac{c}{a}) = \left(\frac{g_2(Y_{i-1}, Z_{i-1}, \beta_i)}{\sum_{j=1}^{i-1} g_2(Y_{j-1}, Z_{j-1}, \beta_j)}\right)^2 \cdot \beta_i \tag{156}$$

$$\leq \left(\frac{g_{2,\text{max}}}{(i-1) g_{2,\text{min}}}\right)^2 \cdot (\beta_{\text{max}}) = \frac{l_1}{(i-1)^2} \tag{157}$$

where

$$l_1 \triangleq \beta_{\text{max}} \cdot \left(\frac{g_{2,\text{max}}}{g_{2,\text{min}}}\right)^2 \tag{158}$$

is a positive constant. Eq. (156) follows from the definitions in (149), (150) and (151); the inequality in (157) follows from (144) and (145) and $\beta_i \leq \beta_{\text{max}}$ in Lemma 6; the equality in (157) follows from (158). Similarly,

$$\frac{b^2 d}{a^3} = \frac{\left(g_2(Y_{i-1}, Z_{i-1}, \beta_i)\right)^2 g_1(Y_{i-1}, Z_{i-1}, \beta_i)}{\left(\sum_{j=1}^{i-1} g_2(Y_{j-1}, Z_{j-1}, \beta_j)\right)^3}$$
$$\leq \frac{\left(g_{2,\text{max}}\right)^2 g_{1,\text{max}}}{\left((i-1) \cdot g_{2,\text{min}}\right)^3} = \frac{l_2}{(i-1)^3} \tag{159}$$

where

$$l_2 \triangleq \frac{\left(g_{2,\text{max}}\right)^2 g_{1,\text{max}}}{\left(g_{2,\text{min}}\right)^3} \tag{160}$$

is a positive constant.

Finally, we have

$$(147) = \frac{c+d}{a+b} \leq \frac{c}{a} + \frac{d}{a} - \frac{bc}{a^2} + \frac{b^2 c}{a^3} + \frac{b^2 d}{a^3} \tag{161}$$

$$\leq \beta_i + \frac{g_1(Y_{i-1}, Z_{i-1}, \beta_i)}{\sum_{j=1}^{i-1} g_2(Y_{j-1}, Z_{j-1}, \beta_j)} - \frac{\beta_i \cdot g_2(Y_{i-1}, Z_{i-1}, \beta_i)}{\sum_{j=1}^{i-1} g_2(Y_{j-1}, Z_{j-1}, \beta_j)} + \frac{l_1}{(i-1)^2} + \frac{l_2}{(i-1)^3} \tag{162}$$

$$\leq \beta_i - \frac{\beta_i \cdot g_2(Y_{i-1}, Z_{i-1}, \beta_i) - g_1(Y_{i-1}, Z_{i-1}, \beta_i)}{\sum_{j=1}^{i-1} g_2(Y_{j-1}, Z_{j-1}, \beta_j)} + \frac{k_5}{(i-1)^2} \tag{163}$$

where $k_5 \triangleq l_1 + l_2$. Eq. (161) follows from (140); (162) follows from (153), (154), (155), (157) and (159); (163) follows from $k_5 \triangleq l_1 + l_2$. Lemma 19 is proved. ■

Recall that $\mathcal{F}_i \triangleq \{(Y_j, Z_j) : j \leq i-2\}$ is the set of all the previous forward and backward channel delays up to the $(i-2)$-th packet. Since β_i is completely determined by $\mathcal{F}_i$ (see (24)), if we take the conditional expectation $\mathbb{E}_{Y_{i-1}, Z_{i-1}}\{\cdot|\mathcal{F}_i\}$ and subtract β^* from both sides of (148), we get

$$\mathbb{E}\{\beta_{i+1} - \beta^* | \mathcal{F}_i\} \leq \beta_i - \frac{\beta_i \cdot \overline{g}_2(\beta_i) - \overline{g}_1(\beta_i)}{\sum_{j=1}^{i-1} g_2(Y_{j-1}, Z_{j-1}, \beta_j)} - \beta^* + \frac{k_5}{(i-1)^2}. \tag{164}$$

From (73), we also have

$$-\left(\beta_i \cdot \overline{g}_2(\beta_i) - \overline{g}_1(\beta_i)\right) \leq -\left(\beta_i - \beta^*\right) \overline{g}_2(\beta^*) \tag{165}$$

Eqs. (164) and (165) jointly imply

Lemma 20: For all $i \geq 3$,

$$\mathbb{E}\{\beta_{i+1} - \beta^* | \mathcal{F}_i\} \leq (\beta_i - \beta^*) \left(1 - \frac{\overline{g}_2(\beta^*)}{\sum_{j=1}^{i-1} g_2(Y_{j-1}, Z_{j-1}, \beta_j)}\right) + \frac{k_5}{(i-1)^2}. \tag{166}$$

The intuition is that in average $(\beta_i - \beta^*)$ will have a tendency to shrink by a factor that is strictly less than 1, if we ignore the $k_5/(i-1)^2$ term. Namely, the difference to β^* would shrink gradually in a way similar to having negative drift in Lyapunov analysis. However, the subtlety of this equation is that the factor is a random variable that depends on the historical values (Y_1, Z_1) to (Y_{i-2}, Z_{i-2}) (recall that we set $Y_0 = Z_0 = \beta_0 = 0$). Therefore, the shrinking factor and the target term $(\beta_i - \beta^*)$ is highly correlated. Therefore, it is not possible to take the expectation of the right-hand side of (166) and hope to bootstrap the results to show $\mathbb{E}\{\beta_i - \beta^*\}$ is always decreasing.

In addition to the correlation between the shrinking factor and the target term $(\beta_i - \beta^*)$, the second complication is that there is no guarantee that $\beta_i - \beta^*$ is positive. If we are shrinking the $\beta_i - \beta^*$ term when $\beta_i - \beta^* < 0$, it could actually make the overall expectation $\mathbb{E}\{\beta_i - \beta^*\}$ bigger since the right-hand side of (166) (multiplying $\beta_i - \beta^*$, a negative term, by a factor that is less than one) grows larger than the original $\beta_i - \beta^*$. This is against the goal of proving that $\lim_{i \to \infty} \mathbb{E}\{\beta_i - \beta^*\} \leq 0$. To overcome these two subtleties, further derivation is provided in the sequel.

Lemma 21: Define

$$I_2 \triangleq \max\left(3, \left\lceil \frac{\overline{g}_2(\beta^*)}{g_{2,\text{min}}} + 1 \right\rceil\right). \tag{167}$$

For all $i \geq I_2$, we have

$$1 - \frac{\overline{g}_2(\beta^*)}{\sum_{j=1}^{i-1} g_2(Y_{j-1}, Z_{j-1}, \beta_j)} \geq 0. \tag{168}$$

Proof: First, we notice that (168) holds if

$$\sum_{j=1}^{i-1} g_2(Y_{j-1}, Z_{j-1}, \beta_j) \geq \overline{g}_2(\beta^*). \tag{169}$$

For $i \geq 3$, if

$$i \geq \frac{\overline{g}_2(\beta^*)}{g_{2,\text{min}}} + 1 \tag{170}$$

then

$$\sum_{j=1}^{i-1} g_2(Y_{j-1}, Z_{j-1}, \beta_j) \geq (i-1) \cdot g_{2,\min} \geq \overline{g}_2(\beta^*) \quad (171)$$

and hence (168) holds. The second inequality in (171) follows from the condition in (170). Therefore, Lemma 21 follows from the definition of I_2 in (167). ■

If we further bound the right-hand-side of (166), we have

Lemma 22: Given any $\alpha \in (0, 0.5)$, there exist positive constants k_6 and k_7 such that for all $i \geq I_2$,

$$\begin{aligned}
&\mathbb{E}\{\beta_{i+1} - \beta^* | \mathcal{F}_i\} \\
&\leq (\beta_i - \beta^*) \left(1 - \frac{\overline{g}_2(0)}{(i-1) \cdot g_{2,\max}}\right) + \frac{k_5}{(i-1)^2} \\
&\quad + \mathbb{1}_{\{\beta^* - c_1 \cdot (i-1)^{-(0.5-\alpha)} \leq \beta_i < \beta^*\}} \cdot \frac{k_6}{(i-1)^{(1.5-\alpha)}} \\
&\quad + \mathbb{1}_{\{0 \leq \beta_i < \beta^* - c_1 \cdot (i-1)^{-(0.5-\alpha)}\}} \cdot \frac{k_7}{(i-1)} \quad (172)
\end{aligned}$$

where the expression of c_1 can be found in (137) (c_1 is the same constant used in (37)).

Proof: See Appendix K. ■

That is, the shrinking factor of $(\beta_i - \beta^*)$, which was a random variable in (166), now becomes a deterministic constant in (172). Taking the expectation from both sides of (172), we have

$$\begin{aligned}
&\mathbb{E}\{\beta_{i+1} - \beta^* | \mathcal{F}_i\} \\
&\leq \mathbb{E}\{\beta_i - \beta^*\} \left(1 - \frac{\overline{g}_2(0)}{(i-1) \cdot g_{2,\max}}\right) + \frac{k_5}{(i-1)^2} \\
&\quad + \frac{k_6}{(i-1)^{(1.5-\alpha)}} + \frac{k_7}{(i-1)} \cdot c_2 \cdot \exp\left(-c_3 \cdot (i-1)^{2\alpha}\right). \quad (173)
\end{aligned}$$

where (173) follows from (172), the fact that $\mathbb{P}(\cdot) \leq 1$ and the result in (37).

The next step is to notice that the among the last three terms of (173), the second term $\frac{k_6}{(i-1)^{1.5-\alpha}}$ decreases to 0 the most slowly. Therefore, we have

Lemma 23: Given any $\alpha \in (0, 0.5)$, there exists a constant k_8 such that for all $i \geq I_2$,

$$\begin{aligned}
&\mathbb{E}\{\beta_{i+1} - \beta^*\} \\
&\leq \mathbb{E}\{\beta_i - \beta^*\} \left(1 - \frac{\overline{g}_2(0)}{(i-1) \cdot g_{2,\max}}\right) \\
&\quad + \frac{k_8}{(i-1)^{(1.5-\alpha)}}. \quad (174)
\end{aligned}$$

Proof: See Appendix L. ■

Define $E_i \triangleq \mathbb{E}\{\beta_i - \beta^*\}$. Then, from (174), we have for $i \geq I_2$,

$$E_{i+1} \leq (1 - \frac{e}{i-1})E_i + \frac{k_8}{(i-1)^{(1.5-\alpha)}} \quad (175)$$

$$\leq (1 - \frac{e}{i})E_i + \frac{k_8}{(i-1)^{(1.5-\alpha)}} \quad (176)$$

where

$$0 < e \triangleq \frac{\overline{g}_2(0)}{g_{2,\max}} < 1. \quad (177)$$

We are now ready to prove (38). Recall the definition of $e \in (0, 1)$ in (177). Since $e \in (0, 1)$, there exists $\alpha \in (0, 0.5)$ such that $\alpha \in (0.5 - e, 0.5)$.

Consider $\alpha \in (0.5 - e, 0.5)$. We then define $b \triangleq 0.5 - \alpha$. Note that $b \in (0, 0.5)$ since $\alpha \in (0, 0.5)$.

We set the term c_4 in (38) to be

$$c_4 \triangleq \max\left(\beta_{\max} \cdot (I_2)^{(0.5-\alpha)}, \sup_{i \geq I_2} \left(\frac{k_8}{e-b}\right) \cdot \left(\frac{i}{i-1}\right)^{b+1}\right) \quad (178)$$

where the second term in max operation is finite since $\lim_{i\to\infty}(\frac{i}{i-1})^{b+1} = 1$. Hence, c_4 must be finite. In the following, we prove that (38) holds by considering two cases.

Case 1: When $i < I_2$, we have

$$\begin{aligned}
E_i = \mathbb{E}\{\beta_i - \beta^*\} \leq \beta_{\max} &= \frac{\beta_{\max} \cdot (I_2)^{(0.5-\alpha)}}{(I_2)^{(0.5-\alpha)}} \\
&\leq \frac{c_4}{(I_2)^{(0.5-\alpha)}} \quad (179)
\end{aligned}$$

where the first inequality follows from $\beta_i \leq \beta_{\max}$ almost surely, and the last inequality follows from the definition of c_4 in (178), $i < I_2$ and $\alpha \in (0, 0.5)$.

Case 2: When $i \geq I_2$, we will prove (38) holds using mathematical induction.

First, from the last inequality of (179), when $i = I_2$, (38) holds. Now suppose (38) holds for $I_2 \leq i \leq i_0$. Then for $i = i_0 + 1$, we have

$$\begin{aligned}
E_{i_0+1} &\leq (1 - \frac{e}{i_0})E_i + \frac{k_8}{(i_0-1)^{(1.5-\alpha)}} \quad (180) \\
&\leq (1 - \frac{e}{i_0})\left(\frac{c_4}{(i_0)^{(0.5-\alpha)}}\right) + \frac{k_8}{(i_0-1)^{(1.5-\alpha)}} \quad (181) \\
&\leq \frac{c_4}{(i_0+1)^{(0.5-\alpha)}} \quad (182)
\end{aligned}$$

where (180) follows from (176); (181) follows from the induction hypothesis that (38) holds for $I_2 \leq i \leq i_0$; (182) holds for the following reasons.

Rearranging the inequality in (182), it is sufficient to show that the positive constant c_4 defined in (178) satisfies the following inequality for all $i \geq I_2$.

$$\frac{k_8}{c_4} \cdot \frac{1}{(i-1)^{(b+1)}} \leq \frac{1}{(i+1)^b} - \frac{1}{i^b} + \frac{e}{i^{(b+1)}} \quad (183)$$

Since $0 < b \triangleq 0.5 - \alpha$, the function x^{-b} is convex for the range of $x \in (0, \infty)$. As a result, for any given i value, by comparing x^{-b} to the tangent line at $x = i$, we have

$$x^{-b} \geq i^{-b} - b \cdot i^{(-b-1)}(x - i) \quad (184)$$

Plugging $x = i + 1$ into (184), we have

$$(i+1)^{-b} \geq i^{-b} - b \cdot i^{(-b-1)} \quad (185)$$

From (185), the right-hand-side of (183) thus satisfies

$$\begin{aligned}
&(i+1)^{-b} - i^{-b} + e \cdot i^{(-b-1)} \\
&\geq (-b) \cdot i^{(-b-1)} + e \cdot i^{(-b-1)} = (e-b) \cdot i^{(-b-1)} \quad (186)
\end{aligned}$$

Comparing (183) and (186), it is clear that if there exists a finite c_4 satisfying

$$c_4 \geq \sup_{i\geq I_2} \left(\frac{k_8}{e-b}\right)\cdot\left(\frac{i}{i-1}\right)^{b+1} \tag{187}$$

then such a c_4 will satisfy (183). Since we define c_4 as in (178), (183) (and thus (182)) holds. The proof of (38) is complete.

Appendix J
Proof of Corollary 2

From (37), we have

$$\lim_{i\to\infty} \mathbb{E}\{(\beta^* - \beta_i)^+\} = 0. \tag{188}$$

From (38) we have

$$\lim_{i\to\infty} \mathbb{E}\{\beta_i - \beta^*\} \leq 0. \tag{189}$$

By the following inequality $0 \leq |a-b| \leq (a-b)+2\cdot(b-a)^+$, we thus have

$$\lim_{i\to\infty} \mathbb{E}\{|\beta_i - \beta^*|\} = 0. \tag{190}$$

Next, for $i \geq 1$, we have

$$\mathbb{E}\{(\beta_i - \beta^*)^2\} \leq \mathbb{E}\{(\beta_{\max} - 0)\cdot|\beta_i - \beta^*|\} \tag{191}$$
$$= \beta_{\max}\cdot\mathbb{E}\{|\beta_i - \beta^*|\} \tag{192}$$

where (191) follows from Lemma 6. Jointly (190) and (192) imply Corollary 2.

Appendix K
Proof of Lemma 22

For notational simplicity, we first define two functions

$$h_1(\beta) \triangleq (\beta-\beta^*)\cdot\left(1 - \frac{\overline{g}_2(\beta^*)}{\sum_{j=1}^{i-1} g_2(Y_{j-1}, Z_{j-1}, \beta_j)}\right) \tag{193}$$

$$h_2(\beta) \triangleq (\beta-\beta^*)\cdot\left(1 - \frac{\overline{g}_2(0)}{(i-1)\cdot g_{2,\max}}\right) \tag{194}$$

where $h_1(\beta)$ represents the right-hand-side of (166) (if ignoring $k_5/(i-1)^2$). Our goal is to first upper bound $h_1(\beta)$ using $h_2(\beta)$, and then add $k_5/(i-1)^2$ back. Recall that in Lemma 21, we have proved that when $i \geq I_2$, $1 - \frac{\overline{g}_2(\beta^*)}{\sum_{j=1}^{i-1} g_2(Y_{j-1},Z_{j-1},\beta_j)} \geq 0$. Therefore, the terms after $(\beta-\beta^*)$ in (193) and (194) are both non-negative. Also note that by the monotonicity of $\overline{g}_2$ and g_2 functions, we have

$$\frac{\overline{g}_2(\beta^*)}{\sum_{j=1}^{i-1} g_2(Y_{j-1}, Z_{j-1}, \beta_j)} - \frac{\overline{g}_2(0)}{(i-1)\cdot g_{2,\max}} \geq 0. \tag{195}$$

Since

$$h_2(\beta) - h_1(\beta) = (\beta-\beta^*)\cdot \left(\frac{\overline{g}_2(\beta^*)}{\sum_{j=1}^{i-1} g_2(Y_{j-1}, Z_{j-1}, \beta_j)} - \frac{\overline{g}_2(0)}{(i-1)\cdot g_{2,\max}}\right) \tag{196}$$

we have

$$h_1(\beta) \leq h_2(\beta), \quad \text{if } \beta-\beta^* \geq 0. \tag{197}$$

Note that if $\beta < \beta^*$, we will have $h_2(\beta) \leq h_1(\beta)$. As a result, to remove the conditioning inequality in (197), we add a couple of indicator functions and write

$$\begin{aligned} &h_1(\beta) \leq h_2(\beta) \\ &+ \mathbb{1}_{\{\beta\in[\beta^*-c_1\cdot(i-1)^{-(0.5-\alpha)},\beta^*)\}}\cdot(h_1(\beta)-h_2(\beta)) \\ &+ \mathbb{1}_{\{\beta\in[0,\beta^*-c_1\cdot(i-1)^{-(0.5-\alpha)})\}}\cdot(h_1(\beta)-h_2(\beta)) \end{aligned} \tag{198}$$

by considering two partitioning events when $\beta < \beta^*$. In the following we further upper bound the second and the third term in (198).

Case 1: For $\beta \in [\beta^* - c_1\cdot(i-1)^{-(0.5-\alpha)}, \beta^*)$, we have

$$\begin{aligned} &h_1(\beta) - h_2(\beta) \\ &\leq h_1(\beta) - h_2(\beta)\Big|_{\beta=\beta^*-c_1\cdot(i-1)^{-(0.5-\alpha)}} && (199) \\ &= c_1\cdot(i-1)^{-(0.5-\alpha)}\cdot \\ &\quad\left(\frac{\overline{g}_2(\beta^*)}{\sum_{j=1}^{i-1} g_2(Y_{j-1}, Z_{j-1}, \beta_j)} - \frac{\overline{g}_2(0)}{(i-1)\cdot g_{2,\max}}\right) && (200) \\ &\leq c_1\cdot(i-1)^{-(0.5-\alpha)}\cdot\left(\frac{\overline{g}_2(\beta_{\max})}{(i-1)\cdot g_{2,\min}} - \frac{\overline{g}_2(0)}{(i-1)\cdot g_{2,\max}}\right) && (201) \\ &= \frac{k_6}{(i-1)^{(1.5-\alpha)}} && (202) \end{aligned}$$

where (199) follows from that the largest distance between two linear functions ($h_1(\beta)$ and $h_2(\beta)$) happens at the furthest point away from the intersecting point $\beta = \beta^*$; (202) follows from (201) by setting

$$k_6 \triangleq c_1\cdot\left(\frac{\overline{g}_2(\beta_{\max})}{g_{2,\min}} - \frac{\overline{g}_2(0)}{g_{2,\max}}\right). \tag{203}$$

Case 2: For $\beta \in [0, \beta^* - c_1\cdot(i-1)^{-(0.5-\alpha)})$, we have

$$\begin{aligned} &h_1(\beta) - h_2(\beta) \\ &\leq h_1(\beta) - h_2(\beta)\Big|_{\beta=0} && (204) \\ &= \beta^*\cdot\left(\frac{\overline{g}_2(\beta^*)}{\sum_{j=1}^{i-1} g_2(Y_{j-1}, Z_{j-1}, \beta_j)} - \frac{\overline{g}_2(0)}{(i-1)\cdot g_{2,\max}}\right) && (205) \\ &\leq \beta^*\cdot\left(\frac{\overline{g}_2(\beta_{\max})}{(i-1)\cdot g_{2,\min}} - \frac{\overline{g}_2(0)}{(i-1)\cdot g_{2,\max}}\right) && (206) \\ &= \frac{k_7}{(i-1)} && (207) \end{aligned}$$

where (204) follows from the same reasoning as in (199); (206) follows from the monotonicity of $\overline{g}_2$ and g_2; (207) follows from (206) by setting

$$k_7 \triangleq \frac{\overline{g}_2(\beta_{\max})}{g_{2,\min}} - \frac{\overline{g}_2(0)}{g_{2,\max}}. \tag{208}$$

By combining (198), (202) and (207) with the above discussion, we have

$$\begin{aligned} h_1(\beta) \leq h_2(\beta) &+ \mathbb{1}_{\{\beta\in[\beta^*-c_1\cdot(i-1)^{-(0.5-\alpha)},\beta^*)\}}\cdot\frac{k_6}{(i-1)^{(1.5-\alpha)}} \\ &+ \mathbb{1}_{\{\beta\in[0,\beta^*-c_1\cdot(i-1)^{-(0.5-\alpha)})\}}\cdot\frac{k_7}{(i-1)} \end{aligned} \tag{209}$$

Eq. (172) follows from upper bounding the right-hand-side of (166) using the result in (209).

Appendix L
Proof of Lemma 23

Given any $\alpha \in (0, 0.5)$, for all $i \geq 2$, we have

$$\frac{k_5}{(i-1)^2} \leq \frac{k_5}{(i-1)^{(1.5-\alpha)}} \tag{210}$$

since $0 < 1.5 - \alpha < 2$.

Next, there exists a positive integer $I \geq 2$ such that for all $i \geq I$, we have

$$\exp\left(-c_3 \cdot (i-1)^{2\alpha}\right) \leq \frac{1}{(i-1)^{(0.5-\alpha)}} \tag{211}$$

since the former is exponentially decreasing and the latter is is polynomially decreasing. Hence, we define

$$\theta \triangleq \max\left(1, \max_{2\leq i\leq I} (i-1)^{(0.5-\alpha)} \cdot \exp\left(-c_3 \cdot (i-1)^{2\alpha}\right)\right). \tag{212}$$

One can easily verify that for all $i \geq 2$, we have

$$\exp\left(-c_3 \cdot (i-1)^{2\alpha}\right) \leq \frac{\theta}{(i-1)^{(0.5-\alpha)}}. \tag{213}$$

From (210) and (213), Eq. (174) follows from (173) by setting

$$k_8 \triangleq k_5 + k_6 + k_7 \cdot c_2 \cdot \theta. \tag{214}$$

Appendix M
Proof of Proposition 4

Blum [21] proved that the standard Robbins-Monro algorithm (i.e., $\{\beta_i\}$ computed by (40)) converges to the unique root (i.e., β^*) almost surely, provided that the following three conditions are met.

(i) $\{\beta_i\}$ computed by (40) is uniformly bounded.
(ii) $\beta \cdot \overline{g}_2(\beta) - \overline{g}_1(\beta)$ is non-decreasing.
(iii) $0 < \frac{d}{d\beta}\left(\beta \cdot \overline{g}_2(\beta) - \overline{g}_1(\beta)\right)\Big|_{\beta=\beta^*} < \infty$.

We will prove that all three conditions hold in our AoI penalty minimization setting.

Conditions (ii) and (iii) are satisfied from the following lemma.

Lemma 24: Under the assumption of $\beta_{\text{UB}} = \infty$, $\beta \cdot \overline{g}_2(\beta) - \overline{g}_1(\beta)$ is a continuous and strictly increasing function of β. Furthermore, its value is strictly negative when $\beta = 0$ and it approaches ∞ when $\beta \to \infty$.

Proof: See Appendix N. ■

The following lemma proves that Condition (i) also holds.

Lemma 25: There exist four positive values $\overline{\beta}, \delta_+, \delta_{-,1}, \delta_{-,2} > 0$ and one negative value $\underline{\beta} < 0$ such that (i) if $\beta_i \geq \overline{\beta} > 0$, then $\beta_{i+1} \leq \beta_i$; (ii) if $0 \leq \beta_i \leq \overline{\beta}$, then

$$\beta_{i+1} \leq \beta_i + \delta_+; \tag{215}$$

(iii) if $\underline{\beta} \leq \beta_i < 0$, then

$$\beta_i \leq \beta_{i+1} \leq \beta_i + \delta_{-,1}; \tag{216}$$

and (iv) if $\beta_i \leq \underline{\beta} < 0$, then $\beta_i \leq \beta_{i+1} \leq \delta_{-,2}$.

Note that this lemma immediately implies uniform boundedness. In particular, we will have

$$\beta_i \leq \beta_{\text{RM,UB}} \triangleq \max\left(\overline{\beta} + \delta_+, \delta_{-,1}, \delta_{-,2}\right). \tag{217}$$

The proof is done by induction. Since $\beta_0 = 0$, we have (217) for $i = 0$; by (i) and (ii), we have $\beta_{i+1} \leq \beta_{\text{RM,UB}}$ if $\beta_i \geq 0$; and by (iii) and (iv) we have $\beta_{i+1} \leq \beta_{\text{RM,UB}}$ if $\beta_i \leq 0$. The proof of (217) is complete.

We will also have $\forall i \geq 0$,

$$\beta_i \geq \beta_{\text{RM,LB}} \triangleq \min\left(\underline{\beta}, -\eta \cdot \beta_{\text{RM,UB}} \cdot g_2(y_{\max}, z_{\max}, \beta_{\text{RM,UB}})\right) \tag{218}$$

where η is the parameter used in the step size $\frac{\eta}{i}$. The proof is carried out again by induction. Since $\beta_0 = 0$, we have (218) for $i = 0$. If $\beta_i \geq 0$, then because β_i is upper bounded by $\beta_{\text{RM,UB}} > 0$, from (40) it is clear that

$$\beta_{i+1} \geq \beta_i - \eta \cdot \beta_{\text{RM,UB}} \cdot g_2(y_{\max}, z_{\max}, \beta_{\text{RM,UB}}) \geq \beta_{\text{RM,LB}}. \tag{219}$$

If $\beta_i \leq 0$, by (iii) and (iv), $\beta_{i+1} \geq \beta_i \geq \beta_{\text{RM,LB}}$. The proof of (218) is complete.

From (217) and (218), Condition (i) (uniform boundedness of β_i) holds and the proof of Proposition 4 is complete.

The rest of the proof is thus to show that Lemma 25 is true.

Statement (i) of Lemma 25 can be proved as follows. Define

$$\overline{\beta} \triangleq \mathbb{E}\left\{\gamma(y_{\max} + z_{\max} + Y + 1)\right\} > 0. \tag{220}$$

If $\beta_i \geq \overline{\beta}$, from the definition of $\phi_{\Gamma,\beta_i}(y', z')$ in (11) and the result in (12), we must have $\phi_{\Gamma,\beta_i}(y', z') > 0$ for any $(Y_{i-1} = y', Z_{i-1} = z')$. The following Lemma 26 directly leads to $\beta_{i+1} \leq \beta_i$ and Statement (i) of Lemma 25 is thus proved.

Lemma 26: Given any $(Y_{i-1} = y', Z_{i-1} = z')$, if $\phi_{\Gamma,\beta_i}(y', z') > 0$, then we have $\beta_{i+1} \leq \beta_i$.

Proof: If $\phi_{\Gamma,\beta_i}(y', z') > 0$, then at time $A_{i-1} + \phi_{\Gamma,\beta_i}(y', z')$, the AoI penalty growth rate is exactly β_i. Since the AoI penalty growth rate is strictly increasing, during the time $[A_{i-1}, A_{i-1} + \phi_{\Gamma,\beta_i}(y', z'))$ the AoI penalty growth rate is strictly lower than β_i. As a result, $\beta_i \cdot g_2(y', z', \phi_{\Gamma,\beta_i}(y', z')) - g_1(y', z', \phi_{\Gamma,\beta_i}(y', z'))$ is strictly positive. By (40), $\beta_{i+1} \leq \beta_i$. ■

Statement (ii) of Lemma 25 can be proved as follows. First, we present the following

Corollary 3: Given any $(Y_{i-1} = y', Z_{i-1} = z')$, $\beta_{i+1} > \beta_i$ only if $\phi_{\Gamma,\beta_i}(y', z') = 0$.

Corollary 3 directly follows from Lemma 26.

Define

$$\delta_+ \triangleq \eta \cdot g_{1,\beta=0,\max} \tag{221}$$

where

$$g_{1,\beta=0,\max} \triangleq \int_0^{2y_{\max}+z_{\max}} \gamma(t)dt. \tag{222}$$

Note that for any $(Y_{i-1} = y', Z_{i-1} = z')$, we have

$$g_1(y', z', 0) \leq g_{1,\beta=0,\max} \tag{223}$$

using the definition of $g_1(\cdot,\cdot,\cdot)$ in (19) and $\mathbb{P}(Y \leq y_{\max}, Z \leq y_{\max}) = 1$.

Recall that we focus on the scenario of $0 \leq \beta_i \leq \overline{\beta}$, we then have

$$\beta_{i+1} - \beta_i = -\frac{\eta}{i+1} \cdot (\beta_i \cdot g_2(y', z', \beta_i) - g_1(y', z', \beta_i))$$
$$\leq \frac{\eta}{i+1} \cdot g_1(y', z', 0) \tag{224}$$
$$\leq \eta \cdot g_{1,\beta=0,\max} \tag{225}$$
$$\leq \delta_+ \tag{226}$$

where (224) holds because of the following reasons. If $\beta_{i+1} \leq \beta_i$, then (224) holds naturally since $g_1(y', z', 0) \geq 0$. If $\beta_{i+1} \geq \beta_i$, then we first notice that since $\beta_i \geq 0$ in our scenario, the left-hand side of (224) is no larger than $\frac{\eta}{i+1} \cdot g_1(y', z', \beta_i)$. However, by Corollary 3, $\beta_{i+1} \geq \beta_i$ only occurs when $\phi_{\Gamma,\beta_i}(y', z') = 0 = \phi_{\Gamma,0}(y', z')$. Therefore, $g_1(y', z', \beta_i) = g_1(y', z', 0)$ if $\beta_{i+1} \geq \beta_i$. We thus have (224).

Ineq. (225) follows by removing the denominator $(i+1)$ in (224) and by (223); (226) follows from the definition of δ_+ in (221). Ineq. (226) immediately implies the Statement (ii) of Lemma 25.

The first inequality $\beta_i \leq \beta_{i+1}$ in Statements (iii) and (iv) can be proved as follows. Since $\beta_i < 0$, the term $\beta_{i+1} - \beta_i = -\frac{\eta}{i+1} \cdot (\beta_i \cdot g_2(y', z', \beta_i) - g_1(y', z', \beta_i))$ is strictly positive and we therefore have $\beta_i \leq \beta_{i+1}$ (see the update formula in (40)).

The second inequality in Statement (iii) of Lemma 25 can be proved as follows. Next, we define

$$\underline{\beta} \triangleq -(1 + \eta \cdot (y_{\max} + z_{\max})) \cdot \eta \cdot g_{1,\beta=0,\max} \tag{227}$$

where $g_{1,\beta=0,\max}$ is defined in (222) and

$$\delta_{-,1} \triangleq \eta \cdot \left(|\underline{\beta}| \cdot (y_{\max} + z_{\max}) + g_{1,\beta=0,\max}\right). \tag{228}$$

If $\underline{\beta} \leq \beta_i < 0$, then

$$\beta_{i+1} - \beta_i = -\frac{\eta}{i+1} \cdot (\beta_i \cdot g_2(y', z', \beta_i) - g_1(y', z', \beta_i))$$
$$\leq \eta \cdot \left(|\underline{\beta}| \cdot g_2(y', z', \beta_i) + g_1(y', z', \beta_i)\right) \tag{229}$$
$$\leq \eta \cdot \left(|\underline{\beta}| \cdot (y_{\max} + z_{\max}) + g_{1,\beta=0,\max}\right) \tag{230}$$
$$= \delta_{-,1} \tag{231}$$

where (229) holds since $\eta > 0$ and $\underline{\beta} \leq \beta_i < 0$; Since $\beta_i < 0$, for any $(Y_{i-1} = y', Z_{i-1} = z')$ we have $\phi_{\Gamma,\beta_i}(y', z') = 0$ and hence (i) $g_2(y', z', \beta_i) = y' + z' \leq y_{\max} + z_{\max}$, and (ii) $g_1(y', z', \beta_i) \leq g_{1,\beta=0,\max}$; (230) therefore follows from (229); (231) follows from the definition of $\delta_{-,1}$ in (228). The second inequality in Statement (iii) of Lemma 25 is proved.

The second inequality in Statement (iv) of Lemma 25 can be proved as follows. We consider two cases depending on the index i of β_i,

Case 1: $1 \leq i \leq \lceil \eta \cdot (y_{\max} + z_{\max}) + 1 \rceil - 1$. We derive a (loose) upper bound of $|\beta_{i+1}|$ in this case. From (40), we have

$$|\beta_{i+1}| \leq |\beta_i| \cdot (1 + \eta \cdot g_2(y_{\max}, z_{\max}, |\beta_i|)) + \eta \cdot g_1(y_{\max}, z_{\max}, |\beta_i|) \tag{232}$$

by simple algebra. From (232), since the right-hand-side is an increasing function of $|\beta_i|$, the upper bound of $|\beta_i|$ is also increasing, and hence we only need to consider the bound for $|\beta_{i+1}|$ when $i = \lceil \eta \cdot (y_{\max} + z_{\max}) + 1 \rceil - 1$. Since $\beta_1 = 0$, the upper bound can be iteratively computed as follows. Define a function of β

$$v(\beta) \triangleq \beta \cdot (1 + \eta \cdot g_2(y_{\max}, z_{\max}, \beta)) + \eta \cdot g_1(y_{\max}, z_{\max}, \beta). \tag{233}$$

Then run the following Algorithm 3.

Algorithm 3 Derive the upper bound of $|\beta_{\lceil \eta \cdot (y_{\max} + z_{\max}) + 1 \rceil}|$

Universal input for every round: η, $y_{\max}$, $z_{\max}$ and SY_γ (a set of statistics of Y)
Output: The upper bound of $|\beta_{\lceil \eta \cdot (y_{\max} + z_{\max}) + 1 \rceil}|$

1: Initialize $\mu_1 = 0$
2: Maintain a scalar register μ_i
3: **for** round $i = 1, 2, ..., \lceil \eta \cdot (y_{\max} + z_{\max}) + 1 \rceil - 1$ **do**
4: Use (19), (20), η, $y_{\max}$, $z_{\max}$, μ_i, SY_γ to compute $g_1(y_{\max}, z_{\max}, \mu_i)$ and $g_2(y_{\max}, z_{\max}, \mu_i)$
5: Use (233) to compute $\mu_{i+1} = v(\mu_i)$
6: **end for**
7: **Return** $\mu_{\lceil \eta \cdot (y_{\max} + z_{\max}) + 1 \rceil}$ as the upper bound of $|\beta_{\lceil \eta \cdot (y_{\max} + z_{\max}) + 1 \rceil}|$

We then define

$$\delta_{-,2} \triangleq \mu_{\lceil \eta \cdot (y_{\max} + z_{\max}) + 1 \rceil} \tag{234}$$

and therefore $\beta_{i+1} \leq |\beta_{i+1}| \leq \delta_{-,2}$ for all $1 \leq i \leq \lceil \eta \cdot (y_{\max} + z_{\max}) + 1 \rceil - 1$ in Case 1.

Case 2: $i \geq \lceil \eta \cdot (y_{\max} + z_{\max}) + 1 \rceil$. In this case, we will show that if $\beta_i \leq \underline{\beta} < 0$, then $\beta_{i+1} < 0$. This, together with the discussion in Case 1 will complete the proof of the second inequality in Statement (iv) of Lemma 25. Specifically, we have

$$\beta_{i+1} = \beta_i - \frac{\eta}{i+1} (\beta_i \cdot g_2(y', z', 0) - g_1(y', z', 0)) \tag{235}$$
$$\leq \beta_i \cdot \left(1 - \frac{\eta}{i+1} g_2(y', z', 0)\right) + \eta \cdot g_1(y', z', 0) \tag{236}$$

where (235) is by (40) and the fact that when $\beta_i < 0$, $\phi_{\Gamma,\beta_i}(y', z') = 0$ and we can thus replace the β_i inside (235) by 0. Ineq. (236) is by changing the coefficient in front of $g_1(y', z', 0)$ from $\eta/(i+1)$ to η.

Since $g_2(y', z', 0) = y' + z' \leq y_{\max} + z_{\max}$ and since we consider only those i satisfying $i + 1 \geq \eta \cdot (y_{\max} + z_{\max}) + 1$, we have

$$1 - \frac{\eta}{i+1} g_2(y', z', 0) \geq 1 - \frac{\eta \cdot (y_{\max} + z_{\max})}{1 + \eta \cdot (y_{\max} + z_{\max})} > 0. \tag{237}$$

Using (237) and continuing from (236), since $\beta_i < 0$ we have

$$\beta_{i+1} \leq \beta_i \cdot \left(1 - \frac{\eta \cdot (y_{\max} + z_{\max})}{1 + \eta \cdot (y_{\max} + z_{\max})}\right) + \eta \cdot g_{1,\beta=0,\max}. \tag{238}$$

Finally because $\beta_i \leq \underline{\beta} = -(1 + \eta \cdot (y_{\max} + z_{\max})) \cdot \eta \cdot g_{1,\beta=0,\max}$, plugging this inequality into (238) we have $\beta_{i+1} \leq 0$.

Appendix N
Proof of Lemma 24

For any $\beta \geq 0$, we define

$$q(\beta) \triangleq \beta \cdot \overline{g}_2(\beta) - \overline{g}_1(\beta). \tag{239}$$

By the definition of g_1 and g_2 functions, we immediately have

$$\begin{aligned} q(\beta) = \beta \cdot \mathbb{E}\{Y_{i-1} + Z_{i-1} + \phi_{\Gamma,\beta}(Y_{i-1}, Z_{i-1})\} \\ -\mathbb{E}\{G_1(Y_{i-1}, Z_{i-1}, \phi_{\Gamma,\beta}(Y_{i-1}, Z_{i-1})\}. \end{aligned} \tag{240}$$

We first argue that $q(\beta)$ can be viewed as the objective function of the following maximization problem:

$$\begin{aligned} q(\beta) = \sup_{X_i \geq 0} \beta \cdot \mathbb{E}\{Y_{i-1} + Z_{i-1} + X_i\} \\ - \mathbb{E}\{G_1(Y_{i-1}, Z_{i-1}, X_i)\} \\ X_i \text{ is computed by a FED scheme.} \end{aligned} \tag{241}$$

The reason is very similar to that of Proposition 1. Namely, because the scheme Γ_β sends the packet when the marginal increase rate of G_1 is larger than β, the waiting time decision $\phi_{\Gamma,\beta}$ also maximizes the (241) since any further waiting will decrease the objective value. Since we focus on the FED scheme, $\mathbb{E}\{Y_{i-1} + Z_{i-1} + X_i\} < \infty$, see Definition 1. As result, $q(\beta) < \infty$ for any finite $\beta \geq 0$.

Once (241) is established, we note that $q(\beta)$ is a supremum of a set of linear functions of β. Furthermore, because $g_2(Y_{i-1}, Z_{i-1}, \beta) \geq Y_{i-1} + Z_{i-1} \geq yz_{\min}$, the set of linear functions are all of strictly positive slopes. Furthermore, if we hardwire $X_i = 0$ (one instance of the optimization domain), the constant term satisfies $0 \geq -\mathbb{E}\{G_1(Y_{i-1}, Z_{i-1}, 0)\} > -\infty$. Jointly we thus have that $q(\beta)$ must be (i) continuous; (ii) strictly increasing; (iii) convex; and (iv) $\lim_{\beta\to\infty} q(\beta) = \infty$. Finally we notice that when $\beta = 0$, we have $q(0) = -G_1(Y_{i-1}, Z_{i-1}, 0)$. Note that $-G_1(Y_{i-1}, Z_{i-1},)$ is strictly negative since the duration of each round $Y_{i-1} + Z_{i-1} \geq yz_{\min} > 0$ and the AoI penalty function is strictly increasing while satisfying $\gamma(0) = 0$. This thus implies $q(0) < 0$. The proof is complete.

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