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Pre-Service Teachers' Understanding of Geometric Reflections in Terms of Motion and Mapping View

Murat Akarsu
Purdue University

PRE-SERVICE TEACHERS' UNDERSTANDING OF GEOMETRIC REFLECTIONS IN TERMS OF MOTION AND MAPPING VIEW

by

Murat Akarsu

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STATEMENT OF COMMITTEE APPROVAL

Dr. Signe Kastberg, Chair
Department of Curriculum and Instruction
Dr. Rachael Kenney
Department of Curriculum and Instruction
Dr. Jill Newton
Department of Curriculum and Instruction
Dr. Jacqueline Sack
Department of Urban Education

Approved by:
Dr. Janet M. Alsup
Head of the Graduate Program

This thesis is dedicated to my mother Ayse, and my father Mehmet.

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ABSTRACT

Author: Akarsu, Murat, J. PhD
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Title: Pre-service Teachers' Understanding of Geometric Reflections: Motion and Mapping View
Major Professor: Dr. Signe Kastberg

In this manuscript, I describe a study of pre-service secondary mathematics teachers' (PTs) understanding of geometric reflection in terms of a motion and a mapping views. PTs often have a motion view of geometric reflection based on their understanding of reflection line, domain, and plane. A motion view is a preliminary perspective developed prior to the construction of a mapping view. PTs need a mapping view of geometric reflection, and to be conscious of sub-concepts of a mapping view involved reflection line, domain, and plane. However, there is no clear evidence documenting how a learner's motion view evolves to produce a mapping view. A clinical interview methodology was used to describe how mental structures occur in the movement between PTs' motion view and the mapping view. Also, factors critical to the transition from a motion view to a mapping view were explored. Four case studies were constructed from transcript audio records, videos, and written works. Ongoing and retrospective analyses using Dubinsky's action, process, object and schema (APOS) framework were used to examine PTs' mental structures. The results indicated that the motion view transforms into the mapping view through the development of mental structures associated with three important sub-concepts of geometric reflection. These three sub-concepts are reflection line, domain, and plane. The results further indicated that there are series of factors that impact the development from the motion view to the mapping view. These factors are perpendicularity and equidistance properties, the role of reflection line, type of figures (circle, semicircle, interior and exterior points of the figures), the operational definition of the plane, and relations between figure and plane.

CHAPTER 1: INTRODUCTION

In the last two decades, there has been growing interest in teaching and learning geometric transformations (Flanagan, 2001; Glass, 2001; Yanik, 2006). According to the National Council of Teachers of Mathematics' (NCTM, 2000), "Instructional programs from pre-kindergarten through grade 12 should enable all students to apply transformations and use symmetry to analyze mathematical situations" (NCTM, 2000, p. 41). Studying geometric transformations provides advantages for students, enabling them to form the basis of a number of mathematical concepts (e.g., functions, symmetry, congruence), and develop their mathematical and cognitive skills (Clements, Battista, Sarama & Swaminathan, 1997; Hollebrands, 2003; Portnoy, Grundmeier & Graham, 2006; Yanik & Flores, 2009). These potential outcomes demonstrate the important role of geometric transformations in geometry and measurement learning. Understanding the meaning and role of reflection in geometry, it is crucial for pre-service teachers (PTs) preparing to teach the concept of geometric reflection.

Several researchers have documented issues concerning PTs' understanding of geometric reflection (Flanagan, 2001; Glass, 2001; Harper, 2002; Mhloo &Schafer, 2013; Portnoy, Grundmeier & Graham, 2006; Yanik, 2006). Previous studies focused on PTs' challenges in describing geometric reflection, identifying the reflection line, and using strategies for performing geometric reflection (Harper, 2002; Mhloo &Schafer, 2013; Portnoy, Grundmeier & Graham, 2006). Recent studies have hypothesized two understanding of geometric reflection: motion and mapping views (Flanagan, 2001; Glass, 2001; Yanik, 2006). In motion view, PTs used the reflection line without using the properties of perpendicularity and equidistance, consider the domain as a single points of the figure, and see the points or figures are “separated from the plane” (Yanik & Flores, p. 55). On the other hand, in mapping view, PTs know the role of reflection line using the properties of perpendicularity and equidistance, consider “the domain as all points in the plane” (Yanik & Flores, p. 42), and see the points or figures are subset of the plane. According to Flanagan (2001) and Yanik’s (2006) findings, there are three critical sub-concepts of motion view and mapping view: reflection line, domain, and plane. I began calling motion and mapping views because this term represents the idea that the students picture geometric reflection in their minds.

Flanagan (2001), Glass (2001) and Yanik (2006) found that both high school students and PTs tend to have a motion view rather than a mapping view of geometric reflection. This finding implies that learners have difficulty understanding the role of the reflection line that defines the geometric reflection (e.g., understanding the relationships between corresponding pre-image and image points and the reflection line; and use of equidistance and perpendicular properties of geometric reflection). For instance, in Hollebrands’ (2004) study, when high school students were asked to reflect a quadrilateral [ABCD] over the oblique reflection line [BA] (see Figure 1), most students did not use the equidistance and perpendicular properties of geometric reflection to position the points of the figure correctly. From this example, the students have difficulty with equidistance and perpendicularity properties of geometric reflection indicating that they have not developed a mental structure for the role of reflection line.

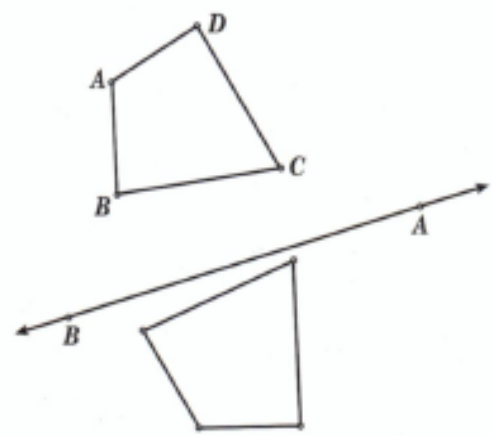


Figure 1. An example of students' work on a reflection task (Hollebrands, 2004, p. 209).

A motion view of geometric reflection also implies that one applies a geometric reflection to a single point or figure as a domain rather than all points in the plane. For instance, in Yanik's (2006) study, when asked to reflect a trapezoid over the reflection line, a PT chose only the sides and vertices of the trapezoid for performing the geometric reflection. In response to Yanik's suggestion to consider selecting unmarked points on the trapezoid, the PT stated that for reflection, only points on the perimeter of the figure are important. The PT justified this thinking by claiming, "inside and outside points will not change anything for the reflection" (p. 99). From this empirical evidence, I infer that the PT considered reflection as involving a single figure rather all points in the plane.

A motion view of geometric reflection also implies that one applies geometric reflections without understanding what constitutes the plane, making it difficult to operate within it. I hypothesize that PTs might conceptualize definitions of some constructs, such as points, line, domain, and plane metaphorically (i.e., know definitions verbally without being able to operate with them to perform geometric reflection) rather than mathematically. For example, when PTs talk about figures in geometric reflection, they often speak of the figure as "separate from the plane" (Yanik & Flores, p. 55), not as a subset of the plane. Specifically, in Yanik's (2014) study of sixth grade students' "concept images" (Tall & Vinner, 1981, p. 152) of translations in a non-technological environment, most of the students had difficulty defining "concept definitions" of translations in a non-technological environment. To investigate the reason behind this finding, Yanik interviewed two teachers and found that, to help students make sense of translation before introducing a formal definition, both teachers used motion-oriented examples from the textbook or self-generated examples such as "walking from one place to another," "the movement of a bicycle or a car," and "movement of a ball" (p. 45). Such examples may led students to think of the figure to be translated as separate from, rather than as a subset of, the plane. A mapping view is a key factor in understanding geometric reflections because it does not require visualizing the movement of a figure to consider the effect of the transformation on the set of points (Flanagan, 2001). I hypothesize that PTs' understanding of the role of the reflection line, of the domain as all points in the plane, and of the plane and its relationship with geometric figures is a critical sub-concept for developing a mapping view.

Statement of the Problem

Only a few studies have addressed PTs' understandings of geometric transformations (translation, rotation, reflection and dilation). Instead much of the research has focused on PTs' strategies for understanding geometric transformations. Hollebrands (2003; 2004; 2007) and Yanik (2011, 2013, 2014) investigated PTs' learning and understanding of geometric transformations in a dynamic geometry environment. Their findings revealed that PTs often have a motion view based on their understanding of parameters (e.g., vector, reflection line), domain, and plane. In Yanik's (2006) study, PTs defined reflection "as a movement of all points rather than mapping of the plane onto itself" (Yanik, 2006, p.138), showing that PTs considered points as "separate from the plane" (Yanik & Flores, p. 55) rather than parts of the plane. Taken together, these empirical studies show that both students and PTs usually have a motion view of geometric reflections. Hence, PTs need a mapping view of geometric reflections, and to be conscious of sub-concepts of mapping view involved reflection line, domain, and plane (Hollebrands, 2003, 2007; Mhloo, 2013; Yanik, 2006). Further, to help PTs to develop this understanding, explorations of motion and mapping views of geometric transformations in general, and reflections in particular, are needed. Accordingly, the present study is designed to investigate PTs' motion and mapping views and contribute to current research by offering insights into the development of understanding of geometric reflection.

One tool to investigate the development of geometric reflection is Action-Process-Object-Schema (APOS) theory (Dubinsky, 1991). The theory focuses on tracing mental structures to provide models for exploring how individuals understand mathematical concepts. To identify mental structures involved in the understanding of geometric reflections, it is necessary to identify sub-concepts in the development of understanding that can serve as milestones. Using APOS theory, I aim to unpack PTs' current understandings of geometric reflections using sub-concepts involved reflection line, domain, and plane as either a motion or a mapping view further identify the factors that facilitate the development of mental structures from a motion to a mapping view. I hypothesize that a student's transition from a motion view to a mapping view involves the development of a collection of geometric concepts not yet explored in research on geometric reflections.

Research Question

To explore PTs' mental structures of geometric reflection, I use the (APOS) theoretical perspective as the conceptual framework. Because I am interested in the preparation of future K-12 mathematics teachers, I focus on secondary mathematics PTs in this study. The research questions that guide this study are as follows:

1. How does a motion view of geometric reflections develop into a mapping view for secondary mathematics PTs?

2. What factors facilitate PTs' development of a motion view of geometric reflections into a mapping view?

Significance of the Study

In the mathematics education literature, there is little research, which focuses on how learners understand geometric reflections. APOS theory has been used effectively in studies of students' geometric understanding. Flanagan (2001) used APOS theory to examine secondary students' understandings of translations, reflections, rotations and dilations; Yanik (2006) used APOS theory to analyze elementary PTs' understandings of rigid transformations. In this study I will use APOS theory to analyze PTs' understanding of geometric reflections to explain their mental structures development. Further, I will explain how the APOS components (action, process, object and schema) of a motion view may be used to produce a mapping view. I will then apply APOS theory to construct a genetic decomposition (GD) of PTs' geometric reflection schemas. To date, no one has explicated a mapping view of geometric transformations in general or of geometric reflection specifically. Also, there is no clear evidence documenting how a learner's motion view evolves to produce a mapping view. By examining PTs' evolving mental structures, this study will outline the progression of PTs' conceptualizations and factors facilitate this change. Addressing the research questions of this study will produce insights that are useful for incorporating ways to support PTs' understanding of geometric reflection and thus prepare PTs to teach this concept effectively. More importantly, the findings and conclusions of this study will answer questions raised by the research community in regard to how a motion view evolves and generates a mapping view.

CHAPTER 2. THEORETICAL FRAMEWORK AND LITERATURE REVIEW

To explore PTs' understanding of geometric reflection, it is important to investigate how PTs understand and develop the sub-concepts of geometric reflections, including the reflection line, domain, and plane. In previous studies, PTs have demonstrated a limited understanding of these sub-concepts of geometric reflection (Flanagan, 2001; Yanik, 2006). This study aims to reveal and explain how PTs' mental structures develop from a motion view to a mapping view, and to identify what factors facilitate this development. The purpose of this chapter is to review the literature related to the understanding of geometric reflection and describe my theoretical framework. First, I review the existing research related to the concept of geometric reflection. This is followed by an explanation of the APOS theoretical framework. Finally, I explain students' and PTs' difficulties and the reasons for these difficulties related to the understanding of geometric reflection.

The Concept of Geometric Reflection

Among the four main geometric transformations (translations, reflections, rotations and dilations), reflections provide an important underpinning for learning other geometric transformations (Flanagan, 2001). The term *geometric reflection* has several meanings in mathematics. According to Boyd, Cummings, Malloy, Carter, and Flores (2004), geometric reflection refers to a transformation representing a flip of figures around a point, line, or plane. Another meaning derived from the definition of function, involves a mapping view. This view is informed by formal mathematics, in which a geometric reflection on a plane is a one-to-one correspondence from the set of points in the plane onto itself" (Hollebrands, 2003, p. 56). Understanding the domain and range of a geometric reflection involves understanding that all points in the plane are mapped to other points in the plane. From this viewpoint, the transformation is not limited to a figure or a point. Instead, as Yanik and Flores (2009) state, one maps "all points in the plane to other points in the plane rather than moving images/points from their original locations to different locations" (p. 42). One-to-one means that if there are two different elements in the domain (two points K and L such that $K \neq L$) then the output for K under the geometric reflection will be different from the output for L under the same reflection ($R(K) \neq R(L)$ where R is a reflection) (Hollebrands, 2003). A geometric reflection being onto means that every element in the range (every point K in the plane) has a corresponding element in the domain (a point L in the plane) such that $R(L) = K$.

Students' notions of geometric reflection have been understood in terms of the broad notions "motion" and "mapping" (Edwards, 2003; Hollebrands, 2003; Mhlolo & Schafer, 2013; Yanik & Flores, 2009). In 2003, Edwards used the context of science and the work of DiSessa (1996) to illustrate how students reason about the physical motions of objects. She then linked this reasoning to rigid ways of operating with motions in a researcher-designed micro world, claiming that learners reasoned about geometric reflection as "physical motions of geometric figures on top of the plane" (Edwards, 2003, p. 8). This is different than thinking about transformations of the plane. This idea was taken up by Hollebrands (2003) in her discussion of the nature of students' understanding of geometric transformations (translations, reflections, rotations, and dilations). She described that students with a motion view have difficulty understanding the role of the reflection line that defines the geometric reflection (understanding the relationships between corresponding pre-image and image points and the reflection line; and the use of equidistance and perpendicular properties of geometric reflection). Students with motion view performed a geometric reflection to a single point or figure rather than all points in the plane. Students with a motion view considered the plane as a background on top of which the geometric figures can be manipulated. This view is erroneous because the plane is a set of infinite points, and geometric figures are not "separate from the plane" (Yanik & Flores, p. 55) but a part of points on it (Edwards, 2003; Hollebrands, 2003; Yanik, 2006).

Hollebrands (2003) described an alternative view of transformations as a "mapping." (p. 58). Students with a mapping view have the role of reflection line when performing geometric reflection. In addition, they understand that the geometric reflection affects all points in the plane, rather than a single point or figure. Students with a mapping view do not conceptualize movement on a plane when reflecting a figure, instead they see the plane as including an infinite number of points where the geometric figure is a subset of the plane. Moreover, having the role of reflection line and use of properties of equidistance and perpendicularity, considering the domain as all points in the plane, and having the relationship between points or figures and the plane are crucial sub-concepts of the mapping view applied to a geometric reflection.

Action-Process-Object-Schema (APOS) Theory

A model with carefully defined structures offers a suitable conceptual framework to understand and reason about PTs' mental structures. I intend to use APOS theory (Dubinsky, 1991), which focuses on mental structures, to investigate PTs' understanding of geometric reflection in my study. I reason that APOS's five mechanisms (interiorization, coordination, encapsulation, reversal, and de-encapsulation) for building four mental structures (action, process, object, and schema) can help me to describe the mental structures that students make to develop their understanding of geometric reflection. APOS theory was developed by Dubinsky and his colleagues (Arnon et al., 2014; Asiala et al., 1996; Dubinsky, 1991) by extending Piaget's reflective abstraction theory and applying it to mathematical thinking. Dubinsky proposed that the use of reflective abstraction provides a significant tool to explain the development of mental structures for advanced mathematical topics. Asiale et al., (1996) explained the construction of mathematical knowledge as follows:

> An individual's mathematical knowledge is her or his tendency to respond to perceived mathematical problem situations by reflecting on problems and their solutions in a social context and by constructing or reconstructing mathematical actions, processes and objects and organizing these in schemas to use in dealing with the situations. (p. 7)

From this we can see that someone's mathematical understanding is determine by their mental organized mental structures, and their application of these mental structures toward a given problem. The fundamental purpose for developing APOS theory was to develop a model to explore and explain learners' mental structures in their understanding of any mathematical concept based on their constructing of mental structures as actions, processes, objects, or schemas. To explain PTs' mental structures, in particular, how PTs' develop geometric reflection as a mathematics concept (e.g., a motion view, and a mapping view), I need to describe the development of PTs' mental structures. APOS theory is a powerful tool to describe how a person constructs a mathematical concept. Such a description is called a genetic decomposition (GD), which is a hypothetical model of the mental structures that PTs needed to develop a mathematical concept (Arnon et al., 2014; Asiala et al., 1996; Dubinksy, 1991). More specifically, a GD is a model that illustrates how mental structures of actions, processes, and objects form a schema, and how these structures are organized to create a schema for a mathematical concept.

According to Dubinsky (1991), there are five mental mechanisms to describe the construction of mental structures of actions, processes, and objects, and the organization of these into schemas: interiorization, coordination, reversal, encapsulation, and de-encapsulation (see Figure 2). Asiala et al. (1996) described the construction of mental structures involved as follows:

> We consider that understanding a mathematical concept begins with manipulating previously constructed mental or physical objects to form actions; actions are then interiorized to form processes which are then encapsulated to form objects. Objects can be de-encapsulated back to the processes from which they were formed. Finally, actions, processes and objects can be organized in schemas. (p. 9).

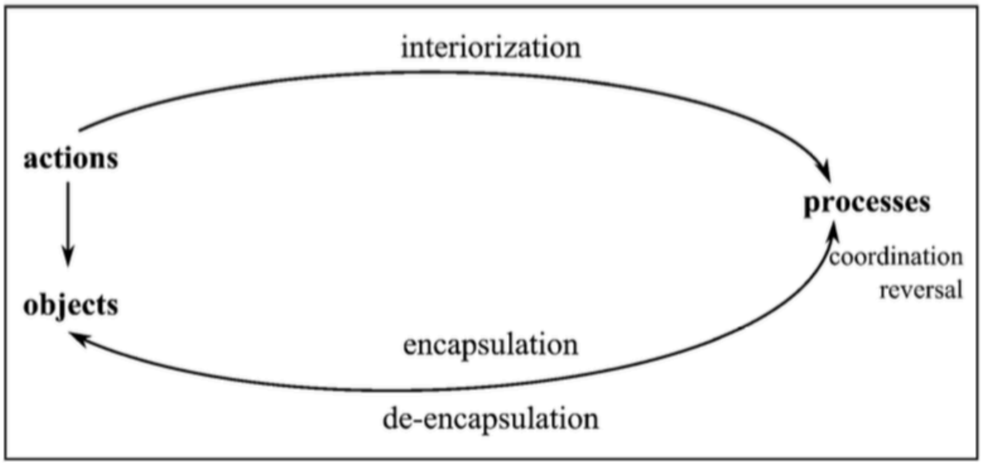


Figure 2. Mental Structures and their constructions (Asiala et al., 1996, p.9)

Formation of mental structures of schemas is a dynamic and progressive procedure. Figure 2 shows that interiorization, coordination, encapsulation, reversal and de-encapsulation are significant mental mechanisms for creating mental structures through reconstruction or revision of previously constructed mental structures. Arnon et al. (2014) stated that the more connections a student establishes among mental structures, the more deeply s/he can understand a mathematical concept because these connections form a schema or schemas to help the student to make sense of mathematical situations in relation to the concept. In the following section, I will explain the construction of mental structures with mechanisms in detail.

The Action Structure

Action is the starting place for understanding a new mathematical concept. Arnon et al. (2014) defined action as “external in the sense that each step of transformations needs to be performed explicitly and guided by external instructions; additionally, each step prompts the next, that is, the steps of the action cannot yet be imagined and none can be skipped” (p. 19). In other words, action is an indispensable component for development of mental structures, because it is the interiorization of actions that result in a mental structure called process. If a student does not have an opportunity to act in a particular problem situation, the mental mechanism (interiorization) is unlikely to be triggered. Without interiorization, the mental structure of process will not be developed.

According to Flanagan (2001) and Yanik (2006), an action structure of a transformation involves three important sub-concepts. First, I hypothesize that students whose mental structures are limited to action have difficulty with all three sub-concepts (reflection line, domain, plane). Students with an action structure of reflection line consider pre-image and image points of figures, but do not yet consider the properties of equidistance and perpendicularity of geometric reflections. For example, when a problem asked students to perform a reflection of a rectangle over the line of reflection, they had problems determining where to position the figure or points without considering that the line of reflection would be equidistant to and the perpendicular bisector of the segments.

Second, students with an action structure of domain think of the domain as a single figure rather than as all points in the plane. In particular, students see only the labeled points of the figure for reflections. For instance, when a problem asked students to perform a reflection of a rectangle, students considered only the labeled points, vertices and sides of the rectangle rather than the whole rectangle with interior and exterior points (Yanik, 2006). From this view, I hypothesize that students see the plane as an empty space and did not consider anything else except the given points, figures, and interior points.

Third, according to Flanagan (2001) and Yanik (2006), to apply a reflection to a figure, students with an action structure of plane may simply perform a reflection as “a movement rather than a mapping of the plane onto itself" (Yanik & Flores, p. 46). For example, when students were asked to perform a reflection for a quadrilateral ABCD over the line of reflection EF (see Figure 3), their responses indicated that they were applying the reflection to points A, B, C, and D as a movement, and then connecting the points with segments to create a quadrilateral. From this view, I hypothesize students see the figure or points as "separate from the plane" (Yanik & Flores, p. 55) rather than part of the plane or a subset of the plane.

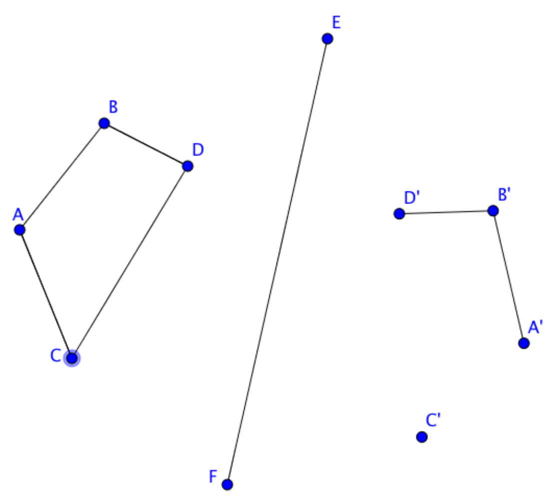


Figure 3. An example of reflection for quadrilateral

The Process Structure

Dubinsky et al., (2005a) defined a process as the result of interiorized action. "A process is a mental structure that performs the same operation as the action being interiorized, but wholly in the mind of the individual, thus enabling her or him to imagine performing the transformation without having to execute each step explicitly." (p. 339). When an action is iterated, or associated with other actions, then the action can be interiorized into a process. Interiorization is a mechanism that makes this mental change possible. According to Dubinsky (1991), the interiorization can be defined as a reconstruction and organization of actions to make sense of perceived phenomena during and as a result of action. At the process structure, students do not need to perform every step explicitly and can skip steps. For example, students can build new mental structure or combine two or more mental structures together to conceptualize the new concept. One these new processes have been created, an individual can coordinate them with previously existing processes. These connections can occur in a variety of ways in terms of alignment, including by reversing or generalizing previously existing processes.

According to Flanagan (2001), a transformation as a process structure involves three important sub-concepts. First, students with the process structure of reflection line know that the reflection line defines a geometric reflection and its relationships with pre-image and image points by using the properties of equidistance and perpendicularity. When the student performs a reflection, s/he can find a line of reflection that is the same distance between pre-image and image points of the figure, connects corresponding pre-image and image points, and is perpendicular to the reflection line. Second, students with a process structure of domain have started to think about the domain as all points in the plane rather than only the labeled points, vertices or sides on a single figure. Third, students with a process structure of plane think of the points or figures as a part of or a subset of the plane rather than points or figures that can be moved.

The Object Structure

Processes act in a similar fashion to the notion of objects, as actions do to processes. When someone reflects on a process and is able to construct transformations using their understanding, then the person has evolved to thinking of the process as an object. This is described by saying that the process has been encapsulated to an object. Arnon et al., (2014) described encapsulation as a mental mechanism to change a process structure to an object structure. According to Arnon et al., (2014), encapsulation is an important mechanism for constructing new concepts, and actions or processes maybe encapsulated to build more difficult concepts. When a process is encapsulated into an object structure, it can be de-encapsulated, which means that an individual may go back to the process structure that gave rise to the object. De-encapsulation is a helpful mechanism for going to a process structure from an object structure to modify the existing processes that were the basis of the object structure.

According to Flanagan (2001), an object structure of geometric transformations involves conceiving geometric transformations as a function that reflects all points in a plane. Flanagan stated that geometric transformations as a function might help students to explore the relationships between pre-image and image figures and composite reflections to investigate their effects. Students with the object structure can also explain their reasoning about how composite geometric reflection produce other geometric transformations such as translation and rotation, and explained that what properties are preserved under composite geometric reflection. For instance, if a student was asked to reflect a triangle by using two parallel lines of reflection, the student would need to identify that after two reflections, first image and final image would be geometric translations. This means that the composition of two geometric reflections produce other geometric transformations such as geometric translations.

The Schema Structure

According to Piaget and Garcia (1989), the schema is the highest level of mathematical structure. When actions, processes and objects are connected for a single concept in the learner's mind, they constitute schemas (Arnon et al., 2014). In other words, a schema is a collection of mental structures such as actions, processes, and objects, and connections among these mental structures. When connections among mental structures are increased, learners' mathematical structures become more systematized and sophisticated.

Schemas are coordinated with each other to provide different ways of reasoning in order to make sense of mathematical situations. It should be noted that each individual's schema might have a different structure because individuals can form different kinds of relations among the structures of a schema (Arnon et al., 2014). According to Flanagan (2001), a schema of geometric reflection allows a student to reason about theorems and geometric proofs that involve statements such as: if a point K is reflected to point K' over the reflection line, the reflection line is the perpendicular bisector of the segment KK', or for a point K on the reflection line, the segment KM is reflected to KM', so the segments are congruent and the triangle KMM' is isosceles (see Figure 4).

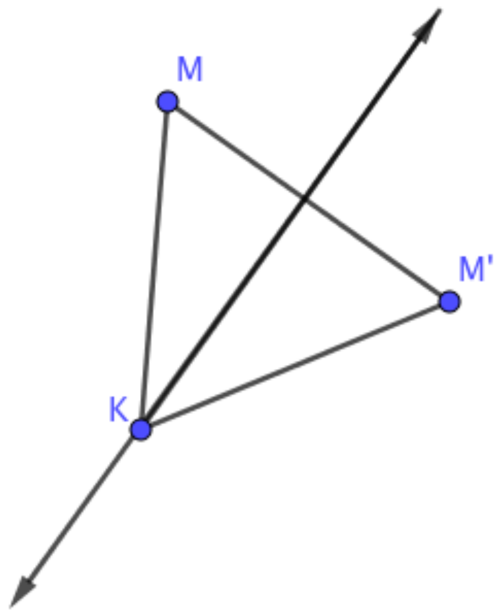


Figure 4. KMM' is an isosceles triangle

Flanagan's (2001) and Yanik's (2006) studies using APOS theory made a significant contribution to describing how students understand geometric transformations. However, no research has explained how a motion view is transformed to a mapping view in geometric reflections. The APOS framework was used to build a GD of PTs' understanding of geometric reflections. This research addresses the change of and explores how PTs' mental structures of a motion view are related to their mental structures of a mapping view, and what factors affect their transition from one level to the next.

Students' Understanding of the Concept of Geometric Reflection

Existing research on geometric reflections provides useful findings identifying challenges that students encounter in understanding geometric reflection (Boulter & Kirby, 1994; Dixon, 1997; Edwards & Zazkis, 1993; Guven, 2012; Hollebrands, 2003, 2004; Mhlolo & Schafer, 2013). These studies have demonstrated that most students have a motion view rather than a mapping view of geometric reflection. Hollebrands (2003) conducted a seven-week instructional unit using The Geometer's Sketchpad (Jackiw, 2001) to explore 16 tenth grade students' understandings of translation, rotation, reflection, and dilation. She found that understanding domain is critical to understanding geometric reflection as a mapping. In Hollebrands' study (2003), all 16 students demonstrated a motion view. They considered domain as a single figure, rather than considering all points in the plane as the domain. This is evidence of an action structure of the domain for geometric reflections. This conception may impact the development of a mapping view. For instance, when a student was asked to identify all points on a trapezoid ABCD that would be fixed under a reflection, the student said that only points C and D would be fixed (Hollebrands, 2003) (see Figure 5). Hollebrands asked the student to identify all other fixed points besides, C and D. The student said "none" (Hollebrands, 2003, p. 62). This indicated that the student thought of the two labeled points on the reflection line as fixed rather than all points in the plane as fixed when performing geometric reflections. I inferred from this explanation to mean that the students had an action structure of domain for geometric reflections. I hypothesize that considering all points in the plane, as the domain is a critical sub concept, which is necessary to conceptualize geometric reflection as a mapping view.

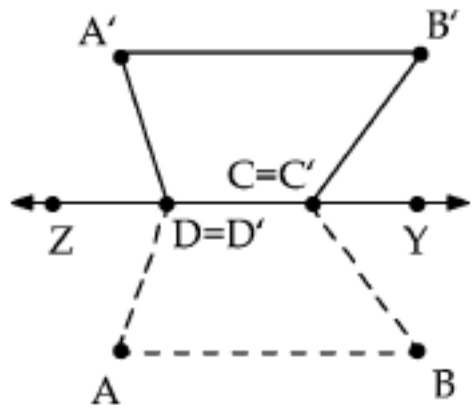


Figure 5. A question from the text that asked students to identify all points that were fixed under the geometric reflection (Hollebrands, 2003, p. 62).

Hollebrands (2004) analyzed six tenth-grade students' work on tasks with translations, reflections, and rotations. The analysis of the students' work indicated that most students reflected the polygon without considering the relations between pre-image and image points and the reflection line (role of reflection line) (see Figure 6). This means that they did not use the properties of equidistance and perpendicularity of geometric reflection to position the points of the polygon correctly over an oblique reflection line. I inferred from this work is that the students had difficulty to understand the role of reflection line. This is evidence that students had an action structure of reflection line for geometric reflection. Additionally, Hollebrands stated that performing geometric reflections with an oblique reflection line is more difficult than a reflection with a horizontal or vertical reflection line.

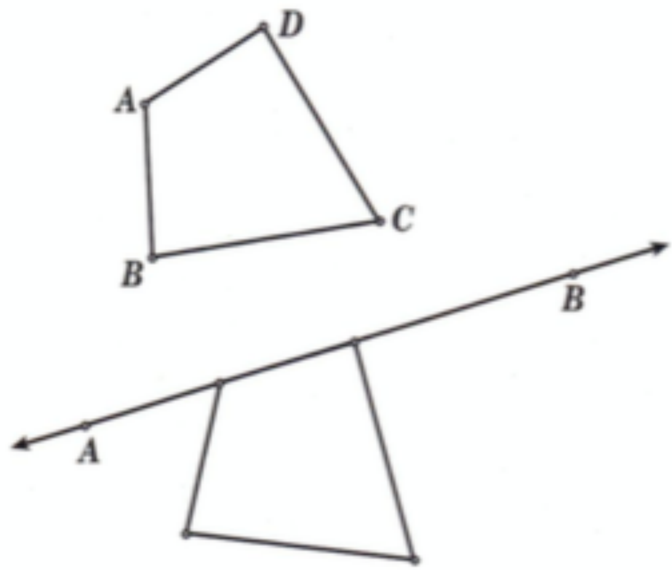


Figure 6. Sample of students' work on the reflection task (Hollebrands, 2003, p. 62).

Mhlolo and Schafer (2013) analyzed 235 grade 11 students' preconceptions of reflective symmetry through the lens of the taxonomy of structure of the observed learning outcome of a reflection symmetry task. Their findings are similar to that of Hollebrands' (2003) findings, in that their analysis of the students' work showed that 85% of their answers indicated a motion view of geometric reflections. This is due to the fact that they considered the geometrical figures as "separated from the plane" (Yanik & Flores, p. 55), rather than a part of the plane. I inferred that students have difficulty in understanding the relationship between the points of the figure and the plane. This is evidence that students had an action structure of the plane. Second, 43% of the students performed their geometric reflections over the y-axis and/or x-axis correctly. However, none of the students succeeded to perform a geometric reflection over an oblique reflection line (y=x or y= -x). This means that students had difficulty with understanding the role of the reflection line. This result is consistent with Hollebrands' (2004) study. I infer that having properties of equidistance and perpendicularity could help students to understand the relationship between pre-image and image points and the reflection line when they try to perform a geometric reflection over an oblique reflection line. I hypothesize that if students understand relationships between pre-image and image points and a line of reflection, they will notice invariant properties such as having the same length, same angle, same area, etc. under the reflection.

Boulter and Kirby (1994) identified holistic and analytic strategies learners' use on geometric reflection tasks. Holistic strategy is described as focusing on a whole figure in a geometric reflection. Analytic strategy, on the other hand, focuses on partitions of a figure (e.g., vertices, sides, points) in a geometric reflection. They found that the use of analytic strategy was most successful for performing geometric reflection based on students' scores on a transformational geometry test. I interpreted that partitioning the figure as vertices, sides, or points may help students to think about the figure as a collection of points rather than as a whole shape. When students consider a figure as a collection of points, they might consider the plane as a set of infinite points, and see domain as all points in the plane rather than a single figure. The use of an analytic strategy may lead students to have a process structure of domain in geometric reflections. On the other hand, when students use the holistic strategy, they might position the figure over the reflection line incorrectly. When students reflect the whole figure, they might not use the properties of equidistance and perpendicularity to position the figure over the reflection line because they might think that drawing the whole figure over the reflection line is a correct way to perform a geometric reflection without using the equidistance and perpendicularity properties of geometric reflection. This means that they have difficulty with the role of the reflection line. From my perspective, the use of a holistic strategy suggests an action structure of the domain.

Pre-service Mathematics Teachers' Understanding of the Concept of Geometric Reflection

Existing research has shed light on PTs perceptions and knowledge of geometric transformations such as translation, reflection, and rotation (Edwards & Zazkis, 1993; Harper, 2002; Portnoy, Grundmeier, & Graham, 2006; Yanik, 2006; Yanik & Flores, 2009). Yanik (2006) documented four PTs' development of understanding of geometric transformations including translations, reflections, rotations, and dilations through a teaching experiment in a dynamic software environment (e.g., Geometer's Sketchpad). The results indicated that all four participants had a motion view of geometric reflection. One challenge for the participants was that they have a limited understanding of the reflection line, namely that it is what defines a reflection. Specifically, PTs have difficulty determining how to use the reflection line, along with the properties of equidistance and perpendicularity because these properties are important for identifying how to position the points or figure correctly. PTs, who have difficulty with the role of the reflection line, had an action structure of the reflection line. Another challenge for the PTs was that PTs considered the domain as the points of a single figure, rather than all points in the plane when they perform a geometric reflection. The use of a dynamic geometry environment may support PTs' understanding of the domain as a single figure since they need to select a figure from the plane to apply a reflection. Selecting a figure from the domain by using technology may allow PTs to think of reflection as applied to that particular figure, which will reinforce the motion view. The result is consistent with

Flanagan's (2001) findings. In addition, Portnoy et al., (2006) investigated 19 PTs understanding of geometric transformations, including translation, reflection, and rotation. The PTs defined reflection as flipping a figure on the plane or "taking something and putting it another place in the plane" (p. 201). These explanations indicated that they considered the geometric reflections as a movement of the points or figures. This means that PTs consider the figures or points are "separated from the plane" (Yanik & Flores, p. 55) rather than part of the plane. This is evidence for an action structure of plane.

Harper (2002) investigated four PTs' knowledge of geometric transformations, including translation, reflection and rotation, using a dynamic geometry environment (Geometer's Sketchpad). She found that all four PTs have difficulty to find the reflection line between a pre-image and image figure. They do not consider the role of the reflection line by using the properties of equidistance and perpendicularity when they perform geometric reflection. These results indicated that PTs had an action structure of reflection line. Likewise, Edwards and Zaskis (1993) analyzed 14 PTs' performance in two-dimensional geometric transformation tasks, including rotation and reflection in a dynamic geometry environment (i.e., microworlds computer). The PTs were asked to rotate and reflect different objects such as a book, a pen, an L-shape and a flag. The results showed that PTs had difficulties with the role of the reflection line (namely the relationship between pre-image and image points and the reflection line). For instance, several PTs did not consider the properties of equidistance and perpendicularity to position the L-shape correctly (see Figure 7). This result showed that PTs had an action structure of reflection line. I infer that understanding the role of the reflection line and considering the domain as all points in the plane are crucial factors from moving from a motion view to a mapping view.

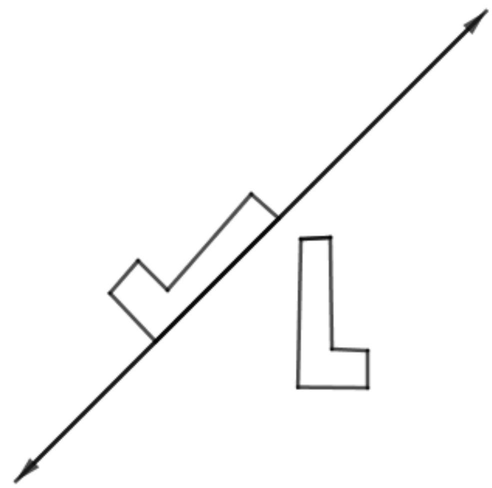


Figure 7. Sample of students' work on the reflection task (Edwards & Zazkis, 1993, p. 141).

Use of dynamic geometric software (DGS), such as Geometers' Sketchpad (Jackiw, 2001) and GeoGebra (Hohenwarter, 2001) may have some limitations for promoting PTs' understanding of geometric reflections as a mapping view. For instance, when they click on the "reflect about line" to perform a reflection, it is possible that they think of the line of reflection as a tool rather than a geometrical object. In other words, when students use features of dynamic geometric software such as the dragging modality, they may focus on the physical representations of the figures (e.g., movement of figures). Also, use of DGS may support students' understanding of the domain as a single figure since they need to select a figure or point from the plane to apply the reflection. Selecting a figure or a point from the domain by using technology may allow students to think of reflection as applied to that particular figure or point rather than as all points in the plane (Hollebrands, 2003). This analysis clearly demonstrates that use of DGS may not be beneficial to move students from a motion view to a mapping view. Therefore, I will not use any DGS in my study.

The results of existing studies indicate that both students and PTs have a motion view of geometric reflections. One reason for this problem might be lack of content in textbooks (Zorin, 2011). In the following, I discuss my examination of the treatment of geometric reflections in the middle grade (6, 7, 8) textbooks of the National Science Foundation (NSF) dounded Curriculum of Mathematics Project (CMP) Textbook series (including teacher's guides and student's editions).

Textbook Analysis

Among the important factors affecting student learning are textbooks (Begle, 1973; Schmidt et al., 2001; Valverde, Bianchi, & Wolfe, 2002), which are significant in determining the content of a topic and how it is presented to students. According to the National Research Council (NRC, 2004), a textbook analysis should start with content analysis to identify connections between standards and the effectiveness of the textbooks. Content analysis focuses on levels of cognitive demand, the nature of lesson presentations, and the types of tasks and activities for students provided in each lesson (Thaqi & Gimenez, 2016; Zorin, 2011). Many educators have reported that they are dissatisfied with content emphases in textbooks (Ball, 1993; Jones, 2004, Ma, 1999). Because teachers use textbooks as instructional guidelines for teaching mathematical concepts, and students rely on textbooks for helpful exercises and examples to develop their understanding of mathematical concepts, weak coverage of mathematical concepts in textbooks is highly problematic. Inadequate content presents difficulties for teachers, who must find ways to supplement the content.

The NCTM (2000) has suggested that it is important to provide tasks that help students develop problem solving skills, critical thinking skills, and the ability to carry out reasoning and proofs because “tasks convey messages about what mathematics is and what doing mathematics entails” (NCTM, 1991, p. 24). In order to reach these goals, it is important to provide high-level complex tasks that provide multiple ways for students to think conceptually, such as those that involve communication and explaining mathematical ideas, conjecturing, generalizing, and justifying strategies while interpreting and framing mathematical problems and drawing conclusions (Silver & Stein, 1996). To learn mathematical topics conceptually, students need to solve high-level complex tasks. Analyzing the content of the textbooks is therefore important because it is a major determinant of the classroom curriculum.

Studies show that students and PTs have a motion view rather than a mapping view of geometric reflections based on their understandings of the roles of the reflection line, domain, and plane (Flanagan, 2001; Yanik, 2006). One reason for this problem might be lack of content in textbooks (Zorin, 2011). In this section, I discuss my examination of the treatment of geometric reflections in the middle-grade (6, 7, 8) textbooks of the National Science Foundation (NSF) funded Curriculum of Mathematics Project (CMP) Textbook series (including teacher’s guides and student’s editions). These books were selected based on their popularity in the USA. The focus of this examination was on treatment of geometric reflections in terms of motion and mapping views.

The geometric reflection unit begins with a section called “Investigations,” which is divided into several subsections including “Investigation overview,” “Goals and Standards,” and “Problem-solving Activities,” starting with definitions, a list of properties, and practice problems for students to explore the concepts. In the "Investigations" section, symmetry and transformations (e.g., reflections, rotations, translations, and dilations) are discussed.

Analysis of the definitions of geometric reflection in CMP textbooks indicated that they support a motion view rather than mapping view. For instance, the CMP3 teacher's guide describes geometric reflections as occurring "if a reflection in a line maps the figure exactly onto itself" (CMP3, p. 13). From this description a student would be likely to infer that a geometric reflection involves only the given figure rather than all points in the plane. In this way, CMP textbooks may promote students' understanding of domain as a single figure rather than as all points in the plane in performing a geometric reflection. Another likelihood is that the description "a reflection line maps the figure" may result in students' understanding the reflection line as reflecting a figure as a whole rather than the points that constitute the figure. I hypothesize from the literature that the perception that a figure is being reflected as a whole may support a motion view rather than a mapping view, because to have a mapping view of performing a geometric reflection, students need to consider all points in the plane rather than only given figures (Boulter & Kirby, 1994; Flanagan, 2001; Yanik, 2006). On the other hand, the teacher's guide for the CMP3 textbook provides a mapping perspective of domain as illustrated in the following suggested explanation and question:

> If you draw a figure and a line of reflection on a piece paper, every point on the paper has an image on the other side of the line of reflections. You can think of the piece of paper as a plane that goes on forever. When there is a line of reflection every point in the plane has an image point. You can picture a copy of the plane flipping in the line of reflection, while carrying the figure with it. Are any points unmoved by a line reflection? (Yes; only the points on the line of reflection) (Teacher's Guide CMP3, p.5)

This perspective clearly explains that to perform a geometric reflection, students need to consider all points in the plane rather than only given figures. However, an analysis of all tasks in the CMP3 teacher's guide revealed that, in many cases, students are asked to reflect a figure and its image without considering the effect of the geometric reflection on other points in the plane.

An example of a task in a CMP textbook (see Figure 8) shows support for students' construction of rules for geometric reflection on a coordinate plane. This task promotes procedural understanding based on noticing changes in points on the perimeter of a given figure, and points on the image of the figure. The task calls for constructing a rule for finding the image of a general point (x, y) under a geometric reflection over the x-axis or the y-axis. For instance, a reflection in the y-axis is equivalent to mapping (x, y) to (-x, y), and a reflection in the x-axis is equivalent to mapping (x, y) to (x, -y).

A Copy and complete the table showing the coordinates of points *A*–*E* and their images under a reflection in the *y*-axis.

Point	*A*	*B*	*C*	*D*	*E*
Original Coordinates	(0, 0)	(2, 4)	■	■	■
Coordinates After a Reflection	■	■	■	■	■

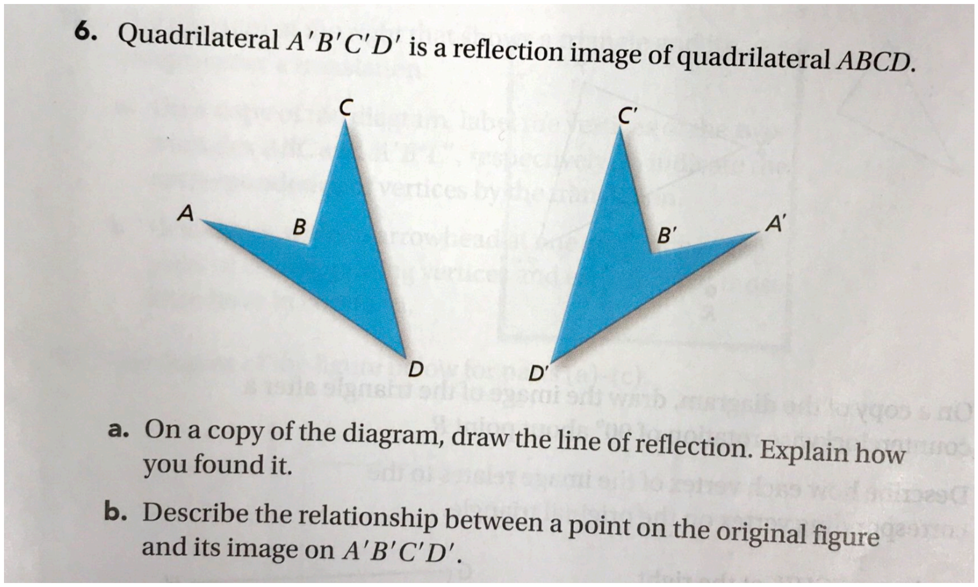
1. Write a rule relating coordinates of key points and their images after a reflection in the *y*-axis: $(x, y) \rightarrow (■, ■)$.
2. Would your rule give the correct coordinates if the flag started in the second, third, or fourth quadrant? Justify your answer with sketches and samples of coordinates that match.
3. **a.** Do any points remain unchanged under this reflection? Explain.

   b. Do the flag and its image make a symmetric design?

Figure 8. Example task in CMP3 textbooks (p. 51).

In this example, the authors were trying to help students to first focus on points that can be seen and then extend the rule that they construct to all points by asking, “Do any points remain unchanged under this reflection?” This question relates to the whole plane in that it specifically asks whether “any points remain unchanged,” which is referring to every point in the plane. Nevertheless, this example suggests that even a text identified as supporting conceptual development may support a motion view by focusing on perimeter points rather than all points in the plane. Thus, students’ attention is directed to the seen, rather than to the unseen points. In addition, analysis of the teacher’s guide for CMP textbooks indicated that there is not even one example of an explanation for performing a geometric reflection that emphasizes reflecting all points in the plane rather than a single figure. I contend that asking “Do any points remain unchanged under the reflection” requires students to consider all points in the plane when performing a geometric reflection. Before being asked this kind of question, students need practice with specific tasks (e.g. circle tasks, inside and outside colored tasks etc.) to develop a mental construction of domain and plane.

Some questions call for drawing the reflection line between the pre-image and image figures (see Figure 9). This approach is important for understanding the role of the reflection line, as learners can use the equidistance property to position where to place the reflection line between the pre-image and image figures. This is done by selecting vertices or a few main points. This approach is important for students to consider a figure as a collection of points on a plane, rather than as a whole discrete figure. Performing a geometric reflection in this way might also be helpful for students to make the connection between geometric reflections and mathematical functions, by understanding that geometric reflections are functions. However, the CMP textbooks do not go further to illustrate relationships between geometric reflections and mathematical functions in terms of a mapping view to encourage learners to consider a figure as a collection of points on a plane rather than as a whole discrete figure.


6. Quadrilateral $A'B'C'D'$ is a reflection image of quadrilateral $ABCD$.

a. On a copy of the diagram, draw the line of reflection. Explain how you found it.

b. Describe the relationship between a point on the original figure and its image on $A'B'C'D'$.

Figure 9. Example task in CMP3 textbook (p. 85).

In this example, the authors were trying to help students to think about figures as collections of points in that asking them to “explain how you found it.” This was intended to lead students to explain that they selected vertices or a few main points to determine where to place the reflection line, rather than that they reflected the whole figure. Additionally, asking about the relationship between pre-image and image points would help students to think about some of the properties of geometric reflections such as equidistance and perpendicularity.

In summary, these textbooks were meant to support conceptual development, but it is possible that the focus of the textbook design on inductive conceptual development may inadvertently support a motion view, not a mapping view. Providing formal mathematical definitions explicitly in textbooks rather than approaching them implicitly is important for helping students understand mathematical concepts (Tossavainen, Suomalainen, & Mäkäläinen, 2017). Analysis of the CMP textbooks reveals that geometric reflections are not formally defined but instead treated indirectly (e.g., introducing geometric reflections with instructions to find coordinates over the x-axis or y axis). Edwards (1990) found that middle school students encountered difficulties in both executing and identifying transformations because of not knowing their definitions. In these textbooks, geometric reflection exercises are predominantly related to movement of the figures or points, which supports a motion view rather than a mapping view.

Conclusion of Literature Review

The results of existing studies indicate that both students and PTs have a motion view of geometric reflections. There is no research that explains how students' and PTs' schema evolve from a motion view to a mapping view of geometric reflections. Based on my own reinterpretation of the existing literature through the lens of the APOS theory, I suggest that the students and PTs have a motion view of geometric reflections. These studies have helped me to analyze students' and PTs' mental constructions in terms of the motion view of geometric transformations. Yet potential exists for the development of a mapping view. To explore such development, I seek to answer the following two research questions.

1. How does a motion view of geometric reflections develop into a mapping view for secondary mathematics PTs?
2. What factors facilitate PTs' development of a motion view of geometric reflections into a mapping view?

CHAPTER 3. METHODS

The purpose of my study is to investigate pre-service secondary mathematics teachers' (PTs') understanding of geometric reflection. To do this, I aim to develop a genetic decomposition (GD) (Arnon et al., 2014) to identify how PTs' mental structures occur in the transition between a motion view and a mapping view and what factors effectively facilitate this transition. Based on the literature, I found that there are three important sub-concepts that I need to consider in creating my preliminary genetic decomposition (PGD): reflection line, domain, and plane. In this chapter, I will define genetic decomposition (GD), and describe my PGD; describe the sub-concepts that involved in the PGD, and why I hypothesize that process structures are necessary to create a mental structure for a mapping view of geometric reflections. Next, I will explain my interview methodology, data collection procedures. Finally, I will explain how I analyzed the interview data using APOS theory.

Genetic Decomposition

A GD is a hypothetical model that explains how a mental structure occurs and how actions, processes, objects and other schemas are organized to describe an individual's mental structures (Arnon et al., 2014; Asiala et al., 1996). A preliminary genetic decomposition (PGD) is an initial model of mental structures hypothesized by the researcher. This model informs the design of the interview and tasks used to investigate hypotheses about PTs' mental structures. My PGD was created based on the mathematics education literature focused specifically on geometric reflections (Flanagan, 2001; Yanik, 2006) and inferences that I drew from the literature. My PGD was adapted as evidence of

PTs' mental structures was obtained through interviews. After each interview, I developed a new GD.

Preliminary Genetic Decomposition

The mapping view is a hypothesized mental structure for geometric reflections as described in mathematics education literature (Flanagan, 2001; Yanik 2006). While researchers have hypothesized the existence of the mapping view for geometric reflections, no participant has been identified as having the mapping view. Instead reports have identified and described participants' motion view. I begin by explaining three critical sub-concepts involved in geometric reflections: reflection line, domain, and plane. These sub-concepts are necessary for developing a mapping view. Mental structures of these sub-concepts (reflection line, domain, and plane) in some combinations of action, and process create the conditions for constructing mental structures of mapping view for geometric reflection.

I hypothesized that to develop a mental structure for geometric reflection, PTs begin with a motion view. This hypothesis is based on the fact that researchers have identified students and teachers with a motion view and characterized the need for this initial view. Similar to results reported by Flanagan (2001) and Yanik (2006), I anticipated that PTs must have a process structure for the role of a reflection line before constructing a mental structure for geometric reflection to have mapping view. PTs with a process structure knew that the reflection line is an essential sub-concept for performing geometric reflection. I make this conjecture because the reflection line is what defines the reflection. PTs with the process structure also have an understanding of the relationship between pre-image and image points and reflection line. For example, in her study of tenth grade students, Hollebrands (2004) asked her students to reflect a quadrilateral [ABCD] over the reflection line [BA] (see Figure 10). As in figure 10, most students applied the reflection without considering that pre-image points and image points should be equidistant and perpendicular from the reflection line. This task could be used to investigate the notions of equidistance and perpendicularity. Students who have mental structures of the properties of equidistance and perpendicularity, which indicated that they have process structure of reflection line. These students who are struggling with equidistance are demonstrating that they have not developed a mental structure at a process structure level yet. Based on these findings, I hypothesize that PTs may not have had the opportunity to develop a mental structure for a reflection line, including whether or not the reflection line is necessary for a geometric reflection. From this example, I can specifically say that the students appear to show no evidence of a process structure.

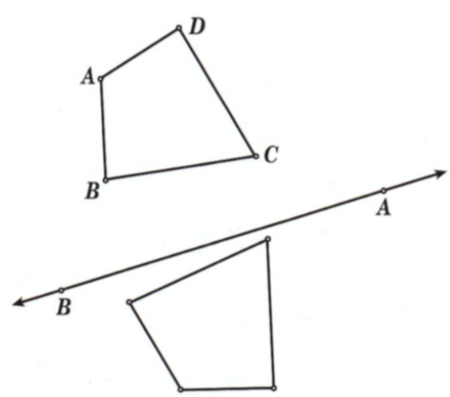


Figure 10. An example of students' work on a reflection task (Hollebrands, 2004, p.209).

To develop a mental structure for geometric reflection, PTs further need to explain and reason about relationships between pre-image, image points/figures and a line of reflection, such as observing that "a reflection line is a perpendicular bisector of the segment joining corresponding pre-image and image points," and "corresponding pre-image and image points are equidistant from the line of reflection under the reflection" (Yanik, 2006, p. 91). For example, Yanik (2006) created several points in GeoGebra, and then asked a student to investigate what would happen if they moved the reflection line HG to the right (see Figure 11). The student stated that if the reflection line was moved to the right, the distances between pre-image and image points would be the same (e.g., |MD|=|DM'|). Yanik also asked that what would occur if image points were placed on the reflection line (resulting in a zero vector). The student stated, "pre-image and image points would coincide" (p. 98). Based on Yanik's (2006) findings, I hypothesized that PTs would know that the reflection line has a role in mapping points in the plane to itself under geometric reflection. I anticipate that PTs who have this mental structure (mapping points on a figure across the reflection line) for geometric reflections have a process structure for reflection line. From this perspective, I hypothesized that having the role of the reflection line is a crucial sub-concept for the geometric reflection, and essential for moving from a motion view to a mapping view.

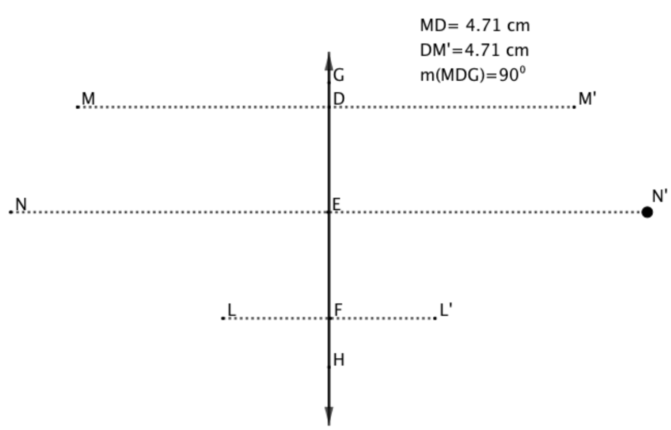


Figure 11. An example of a reflection task (Yanik, 2006, p.96).

I hypothesize that PTs must have a process structure of domain to have a mapping view. To build comprehensive knowledge of domain, according to Flanagan (2001) and Yanik (2006), PTs with a process structure consider the domain as being composed of all points in the plane rather than as a single figure. For instance, when Yanik asked a student to reflect a trapezoid over a reflection line, the student selected the sides and vertices of the trapezoid rather than all of the points in the plane (e.g., points that were inside or outside of the trapezoid). From this empirical work, I derive the idea that students consider the domain as a single figure (point, line segment, polygon), which is consistent with a motion view. This is in contrast with considering that PTs must consider the domain as all points in the plane rather than only a subset of points. Hence, the domain is a critical sub-concept for geometric reflections, and essential for moving from a motion view to a mapping view.

I hypothesize that PTs must have a process structure of plane to have a mapping view. PTs with a process structure have an understanding of the relationship between the points or figures and the plane (i.e., points or figures are a subset of the plane). Research shows that PTs with a motion view think of reflection as a "movement of points or figures on a plane, rather than a mapping of the plane onto itself" (Yanik & Flores, 2009, p. 46). In particular, PTs with a motion view see the plane as an empty space or separate from geometric points or figures (Flanagan, 2001; Yanik, 2006). For example, in Yanik's (2006) report, a student was asked to reflect a triangle over the reflection line. The student selected three edges to reflect over the reflection line as a first step. Then, the student connected these image points to construct a triangle. The student stated, "the only changes were about the location of the pre-image points under the reflection" (p. 97).

This example illustrates that the student was thinking of reflection as moving a subset of the image points and relocating them on the plane (or locating the image points). From this finding, I inferred that students consider reflection as a movement of points/figures consistent with a motion view, rather than as "a mapping of the plane onto itself" (Yanik & Flores, p. 46) consistent with a mapping view. Based on these findings, I anticipated that PTs would need to understand that the plane is a set of infinite points, and geometric objects are not "separate from the plane" (Yanik & Flores, p. 55) but a subset of points in it. Hence, the plane is a critical sub-concept for the geometric reflections, and essential for moving from a motion view to mapping view.

Following Flanagan's (2001) and Yanik's (2006) descriptions, I hypothesized that if PTs decide to use a procedure or memorized facts in the given tasks, they have an action structure because they are using a formula or memorized facts to solve the tasks without reasoning. On the other hand, if PTs apply a geometric reflection knowing the role of the reflection line, I posited that they will reach a process structure because they will be able to use the equidistance and perpendicularity properties for performing a geometric reflection and to describe relations among pre-image and image points/figures and the reflection line.

Based on Flanagan's (2001) and Yanik's (2006) studies, I posited that a mental structure for geometric reflections occurs from the coordination of actions and processes (see Figure 12). I hypothesized that the arrows bridging the motion view to the mapping view illustrate the changes that need to occur with respect to PTs mental structures. The motion view incorporates three different sub-concepts including reflection line, domain, and plane. For instance, under the motion view, a PT is considering the domain to be a single figure; whereas under the mapping view the entire plane is considered as the domain. This latter feature is a different mental structure for domain, and we can see that an existing mental structure must change or evolve in order to move towards the mapping view. This is also true for mental structures related to the reflection line, and the plane (which are both also illustrated below). When these mental structures are organized in a structured form, a mapping view for geometric reflections may develop.

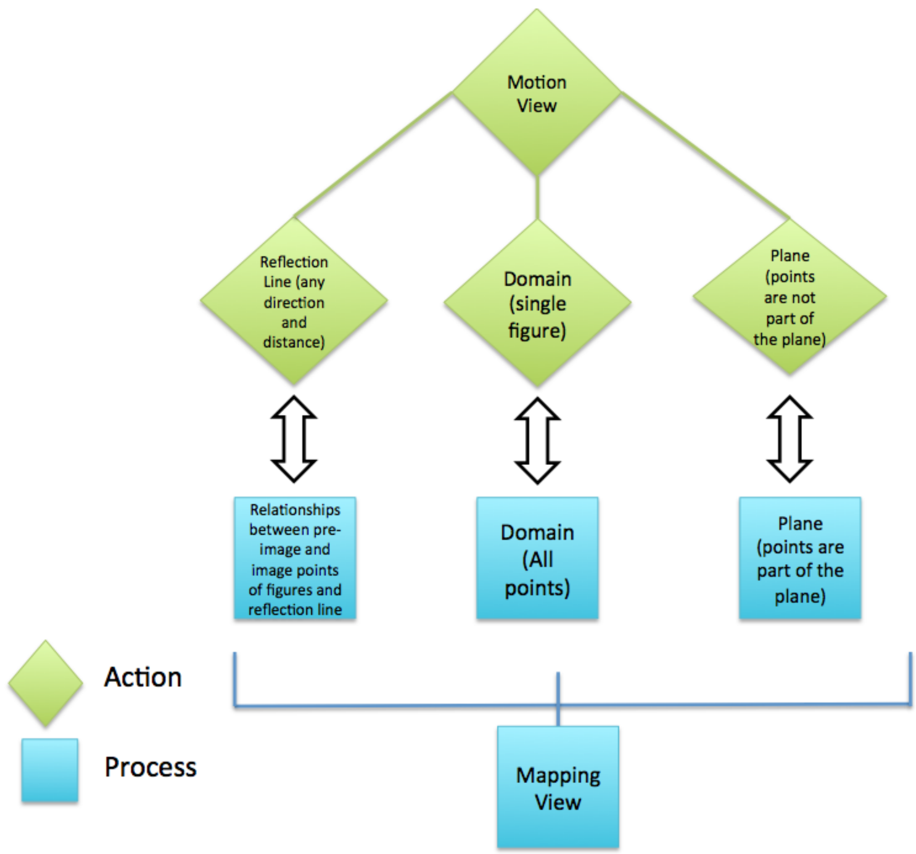


Figure 12. Preliminary genetic decomposition.

For this study, I hypothesized that PTs' mental structures of geometric reflection could be inferred by their answers to complex, non-routine geometric reflection tasks, involving reflection line, domain, and plane. I posited that PTs' challenges in solving the tasks would provide evidence of structural components. I also hypothesized that, if PTs reason to solve complex, non-routine geometric reflection tasks and connect actions and process structures, the resulting the mental structure for geometric reflection will have a mapping view.

Clinical Interview Methodology

To collect my data I used a clinical interview methodology that was developed on the basis of Piaget's (1975) work (Clements, 2000; Goldin, 2000). Based on this methodology, I could describe how PTs' understanding of the concept of geometric reflection develops, and how mental structures occur in the movement between PTs' a motion view and a mapping view. I could also determine what factors are critical to the transition from one structure to the next and to construct new structures from existing mental structures. To do this, I observed PTs' external representations to conjecture about their mental structures. However, focusing only on PTs' external representations was not sufficient to identify mental structures; therefore, I conducted interviews to explore what actions they performed and why. Their explanations in conjunction with these external representations helped me to determine their mental structures. Thus, the interview methodology was used to investigate PTs' current mental structures. I could then design a genetic decomposition of each PT's mental structures to understand how mental structure develop in the movement between the motion view and the mapping view. The interviews included open-ended questions to elicit PTs' understanding and reasoning processes. To evaluate and determine PTs' mental structures, I used APOS theory.

Participants and Settings

Selection of Participants

Six PTs enrolled in a course in the Curriculum and Instruction Department at a large public Midwestern university in the United States were selected as potential participants for the study. To choose the participants, I obtained the permission of the instructor to attend a class meeting, at which I briefly described my research study, including my main goal, the setting, the procedures, the content of the study, and the risks and benefits. I explained that I needed four to six participants to complete my data collection within a month, which would consist of one initial interview, and three subsequent interviews. I informed the participants that they would to be paid $20 for each interview, with no fiscal compensation for the initial interview. When they completed the three interviews, they would receive $60 for their participation. A participant who chose to discontinue participation during the interview process would be compensated $20.

I further stressed that participating would not impact their grades, and their identities would be protected according to IRB guidelines. Then, I distributed recruitment letters (See Appendix A), which included a description of the primary purpose of the study and its setting, procedures, content, and risks and benefits of participating. The recruitment letter included my contact information so that PTs who were willing to participate in this study could reach me. I asked them to complete the recruitment letter and to return it to me if they were interested in participating, or if they had any questions about the study to contact me. Based on the responses that I received, I contacted six interested PTs to schedule initial interview sessions. Four PTs were finally selected, based on their willingness and ability to explain their thought processes. I excluded two

PTs because they did not meet the threshold for verbal expression of ideas, so I would not be able to obtain sufficient external evidence to hypothesize about their mental structures.

Settings for Interviews

The initial and exploratory interviews were conducted in one-on-one sessions, which provide more reliable data on individuals than group interviews. In-group interviewing, it is possible that one participant might dominate others, and then I might not be able to observe each participant's schema development. By focusing on each PT separately, I could investigate each participant's reasoning and thought processes, which provided more details and comprehensive understanding to determine each PT's mental structure according to APOS theory.

The data were collected outside of class meetings in a seminar/conference room setting on the campus of a Midwestern university. The room was reserved each week for each interview session. All interviews were recorded using two different video cameras, with one camera focused on the researcher and participant's interactions, and one camera focused on the participant's working area. The video recording captured the PTs' utterances, gestures, manipulations, and speech characteristics. The data collection timeline is shown in table 1.

Table 1 *Data Collection Timetable for Main Study*

Dates (2017)	**Interview Schedule for Study**
September 14-15	Initial Interview was conducted
September 15-16	Ongoing Analysis
September 18-20	Interview 1 was conducted
September 18-20	Ongoing Analysis
September 21-29	Interview 2 was conducted
September 22-30	Ongoing Analysis
September 30-October 4	Interview 3 was conducted
October 4-30	All interviews were transcribed
November 1-30	Secondary Analysis

Methods of Data Collection

Initial Clinical Interviews

The main purpose of the initial interview (See Appendix B) was to gather background knowledge about participants and to analyze PTs' willingness and ability to explain their mathematical thinking and ideas. On the basis of this information, I choose PTs for interviews. Data from these interviews were not used to investigate PTs' understanding of geometric reflection. After the volunteers returned the recruitment letter to me, I contacted those who were interested to schedule the initial interviews. The initial interview questions were identical for each participant and lasted approximately thirty minutes (see Table 2).

Table 2 *Initial Interview Questions*

Initial Interview Questions (Appendix B)	Question Type Provides data about	Purpose is to find out
Q1, Q2, Q3, Q4, Q5	participants' identity and educational goals	participants' interest in learning and teaching
Q6	participants' mathematics background	how the participants' backgrounds fit what the researcher looking for
Q7, Q8, Q9	how participants describe understanding a mathematical concept	participants' meaning of the understanding and their performance
Q10, Q11, Q12, Q13, Q14, Q15, Q16, Q17, Q18, Q19	participants' definition of geometric reflections and their experiences with geometric reflections	the nature of participants' definitions for geometric reflections. Knowing conceptual definitions is important for a mathematical concept (Tossavainen, Suomalainen, & Makalainen, 2017).

Explanatory Clinical Interviews

I met with each participant once a week for four weeks, conducting one initial interview and three additional interviews. Each of the three one-on-one interview sessions took approximately 45 minutes to one hour. All audio recordings, videotapes, observation notes, written works, and initial hypotheses were kept in a Dropbox folder to share with my advisor. After meeting with all participants in a week, I spent three or four days editing and reordering the tasks and testing my hypotheses for the next round of interview sessions. Also, after each interview, I took notes based on PTs' interactions with the tasks. In particular, I noted such statements as "this question is easier than the previous one" in relation to whether the reflection line was vertical, horizontal, or oblique relative to the bottom of the page. From these findings, I changed the order of the tasks for the next participant to make questions progress from easier to more difficult. In this way, I continuously refined my interview process over the course of the data collection.

The main purpose of the interviews was to collect evidence of PTs' mental structures in order to investigate how PTs moved from a motion view to a mapping view and developed the sub-concepts of geometric reflection including reflection line, domain, and plane. In order to gather evidence of PTs' mental structures, I used the APOS structures of mental actions and processes of geometric reflections to create the model for my GD for each PT's geometric reflection. Table 3 shows the summary of all tasks that I designed for the interviews. In the following, I describe each interview.

Explanatory Interview 1

Previous studies of PTs' understanding of geometric reflections have revealed that PTs have difficulties with foundational concepts of geometric reflections such as the reflection line, domain, and plane (Edwards & Zazkis, 1993; Flanagan, 2001; Yanik, 2006). The first interview was designed to explore PTs' mental structures of a motion view of geometric reflections in terms of reflection line and domain. For this purpose, I referred to existing literature and PGD to design my tasks (see Table 3).

Table 3 *Interview Questions 1*

Interview Questions 1 (Appendix C)	Reference	Purpose
Q1-Q2	Researcher developed	Learn the definition of geometric reflection
Q3-Q5	Molina (1992)	Determine where to place the reflection line
Q6-Q16	Researcher developed	Determine the role of the reflection line, and explore the concept of the domain
Q17-Q31	Researcher developed	Determine the role of the reflection line, and explore the concept of the domain
Q32-Q47	Researcher developed	Determine the role of the reflection line and explore the concept of the domain
Q47-Q50	Researcher developed	Learn the definition of geometric reflection and have experiences with the tasks

For the interview, the PTs were given tasks adapted from Molina, (1992) while other tasks were created by the researcher (See Appendix C). Questions in the interview were designed to determine PTs' concept of geometric reflections in terms of actions, and processes. I hypothesized that to reach a mapping view, all PTs must be at least at the process structure. In particular, PTs were asked to identify whether two figures constitute a reflection, and to justify their stance. If the participant could not identify whether two figures constituted a reflection, then I explored how to explain their reasoning by asking probing questions.

I also gave the PT a series of open and closed shapes to perform a reflection with a line, triangle, trapezoid, rectangle, etc. Next, I ask the PT to perform a reflection, and explain how the image was positioned in the plane. This task was given to investigate the participants' use of the reflection line, which was positioned in a variety of ways (e.g., vertical, horizontal, and oblique) relative to the bottom of the page. Based on the findings of Schultz and Austin (1983), and Hollebrands (2004), the position of the reflection line is considered a significant factor in performing a reflection for the participants.

Participants were also asked to describe a reflection (e.g., What information do you need to perform a reflection?), (How would you define a geometric reflection?), (Where have you seen something like this before?) (See Appendix B for more examples). During the interviews, PTs were asked follow up questions to obtain as much information as possible about their understanding of geometric reflection. Explanations of the role of the reflection line in positioning the figure, and explanations of relationships between the pre-image and image points or figures based on properties of geometric reflection constituted evidence for the process structure. If PTs performed a reflection using the role of the reflection line without regard for distance from it or direction, this conception of the role of the reflection line constituted evidence for the action structure.

Explanatory Interview 2

For the second interview, the PTs were given a set of tasks adapted from Glass (2001) and Molina (1992), while other tasks were created by the researcher (see Table 4) (See Appendix D). The main purpose of the tasks was to collect evidence of PTs' mental structures of actions and processes, building from the reflection line, domain, and plane. Some of the tasks were designed for PTs to recall their knowledge of the reflection line to explore their conception of the geometric reflection, which is important to develop a mental structure for domain and plane.

Table 4 *Interview questions 2*

Interview Questions 2 (Appendix D)	Reference	Purpose
Q1-Q5	Researcher developed	Determine the role of reflection line, and explore the concept of the domain
Q6-Q10	Glass (2001)	Determine the role of reflection line, and explore the concept of the domain
Q10-Q26	Researcher developed	Determine the concept of the domain
Q27-Q28	Researcher developed	Determine the concept of the plane
Q29-Q32	Researcher developed	Determine the concept of the domain
Q33-35	Researcher developed	Determine the concept of the plane
Q36-Q39	Researcher developed	Determine the concept of the domain
Q40-Q43	Researcher developed	Determine the concept of the plane
Q44-Q49	Researcher developed	Determine the concept of the domain
Q50-Q54	Molina (1992)	Determine the role of reflection line
Q55-Q56	Researcher developed	Learn the definition of geometric reflection

To review what participants learned from the first interview, I first asked PTs to perform a reflection with open and closed figures to explore the role of the reflection line. I then asked additional follow-up questions to explore the concept of domain: “Is there any point outside the figure?” “If yes, where do the points outside the figure go when you have a reflection?” “Is there any point inside the figure?” “If yes, where do the points inside the figure go when you have a reflection?” (See Appendix D, Task 14). If a PT explained the position of the points in performing a geometric reflection, such as a response in which she considered inside and outside points (i.e., had begun to consider all points in the domain rather than view it as a single figure), I hypothesized that the PT was at the process structure for the concept of the domain. I also asked PTs to explain how they determined where to place points. If they explained that the reflection line is used to map a pre-image point to its image point for performing a geometric reflection, I hypothesized that the PTs were at the process structure for the reflection line.

I then asked the participants to perform a reflection on a plane containing a trapezoid, which was filled with a blue color over the line of reflection (See Appendix D, Task 17). With this task, I aimed to help PTs to consider the inside points rather than only the vertices and sides of the trapezoid. I asked follow-up questions to probe the concept of the domain such as “Is this a reflection? Why or Why not?” “Which points of the figure are reflected?” If PTs explained that only the vertices, sides and interior points were reflected, I inferred that the participant was still at the action structure because the participant did not consider the effect of the reflection on points outside the figure.

In addition, I asked PTs to perform a reflection on a plane containing a trapezoid filled with a yellow color over an oblique reflection line relative to the bottom of the page (See Appendix D, Task 36). This task sought to investigate whether PTs’ understood the role of the reflection line as points mapping onto themselves as a result of the reflection. I asked follow-up questions to elicit PTs’ mental structures to gain further evidence of their understanding of the role of the reflection line such as “What would happen if you move the reflection line horizontally to the left? What is constant? What is changing?” “What would happen if you move the reflection line vertically to the left?” If PTs explained that the reflection line is useful to map a pre-image point onto itself under the reflection, I inferred that they had reached a process structure, which is necessary for developing the mapping view.

The PTs were also given several tasks to identify a reflection line. For instance, they were given two figures (pre-image and image), and then were asked to find the reflection line based on the given figures. The reason behind this task was to explore the notion that a reflection line defines the reflection and points that could be mapped onto themselves for performing a geometric reflection. I asked, “How did you decide where to place the reflection line?” as a follow up question to see their reasoning (e.g., a reflection line is a perpendicular bisector of the segment joining corresponding pre-image and image points etc.).

Explanatory Interview 3

For the third interview, the PTs were given a set of tasks adapted from Molina (1992), and Yanik (2006), and other tasks that I created (see Table 5) (See Appendix E). The main purpose of these tasks was to gather evidence of PTs’ current mental structures of actions, processes or schemas of a mapping view, and how PTs coordinated and interrelated their schemas for the concepts of reflection line, domain, and plane to develop a schema for a mapping view.

Table 5 *Interview questions 3*

Interview Questions 3 (Appendix E)	Reference	Purpose
Q1-Q12	Researcher developed	Determine the role of reflection line, and explore the concept of the domain
Q13-Q24	Researcher developed	Determine the concept of the plane
Q25-Q29	Researcher developed	Determine the concept of the plane
Q30-Q33	Yanik (2006)	Determine the concept of the domain and plane
Q34-Q37	Researcher developed	Determine the concept of the domain and plane
Q38-Q47	Molina (1992)	Explore the object structure

To review what the participants had learned from the first and second interviews, I asked them to perform a reflection with a closed figure to explore the role of the reflection line. Then, I asked the PTs to reflect a circle with a yellow triangle inside over the reflection line (See Appendix E, Task 6). The reason behind this was to help PTs understand that the plane involves a set of points, and when they reflect a figure, they need to consider all points in the plane to map them under the reflection. I then asked follow-up questions to see their thought processes about domain and plane. For example, I asked, “Is there any point on the image? If yes, where do the points on the image go when you have reflection?” “Is there any point on the white part of the plane? If yes, where did those points go when you have a reflection?” If PTs applied the reflection and stated that all points inside the figure, on the figure, and outside the figure are reflected to the other part of the plane, I hypothesized that they had reached the process structure, which is necessary for a mapping view.

In the following tasks, PTs were given a rectangle on the left side of the reflection line and, a triangle and a circle on the right side of the reflection line. The question was "find the image over the line of reflection. Is this a reflection? Why or Why not? The reason behind this task was to demonstrate that when they reflected the rectangle over the right side of the line of reflection, they also needed to consider reflecting other points and figures over the left side of reflection line. I inferred that a participant who did so was at the process structure. This task was helpful to prompt them to think that performing a geometric reflection involves reflecting every point in the plane, which means the participant needs to consider reflecting all points in the plane (mapping view) rather than reflecting a single figure or points (motion view).

In the following task, to determine whether the PTs reached the process structure for domain and plane, I asked them to perform a reflection inside and outside the yellow area of the figure (See Appendix E, Task 25). When the participant reflected the figure, I asked follow-up questions to investigate her mental structures in terms of actions and processes. For instance, I asked, "Is this a reflection? Why or Why not?" "Is there any point on the figure? If yes, where do the points on the figure go when you have reflection?" "Is there any other point on the figure? If yes, where do the points on the figure go when you have reflection?" The reason behind this task was to see whether PTs have mental structures about domain and plane in terms of process structures. If the PTs reflected the figure over the reflection line and then explained that when a reflection is performed, one considers points on the figure, inside the figure, and outside the figure, in effect they were considering all points in the plane to perform the reflection. Based on this perspective, I inferred that PTs reached the process structure of domain. During these three interviews, if PTs understood the relationship between points/figures and the plane, which means that they perceived points/figures as part of the plane rather than "separate from the plane" (Yanik & Flores, p. 55), I concluded that the PTs had also reached the process structure for the plane.

The PTs were asked to complete a series of tasks during the three interviews to invoke and explore the development of their schemas for geometric reflections from a motion view to a mapping view. If PTs unpacked their mental structures of reflection line, domain, and plane in terms of actions and processes to reason about the tasks, I inferred that they reached a schema structure of geometric reflections, which was a mapping view.

Method of Data Analysis

After the initial interview with each participant, I transcribed the audio records and viewed the videotapes to investigate the PT's background knowledge about geometric reflections (e.g., definitions of geometric reflections, and examples given for geometric reflections). I repeated this process after each interview to analyze whether each participant had an action or process structure of reflection line, domain, and plane for geometric reflections. I also looked for factors that were helpful for transitioning from one structure to the next (relationships between pre-image and image points/figure and the reflection line, equidistance and perpendicularity properties of geometric reflections, operational definition of plane, type of figures, relationships between points or figures and the plane). After transcribing all the audio records, and viewing the videotapes and written works, I started to formulate hypotheses based on participants' responses and interactions with the tasks and used these hypotheses to investigate what mental structures participants developed during the interviews. I also kept observation notes throughout the three interviews to gather evidence of PTs' mental structures on geometric reflections in terms of APOS theory.

APOS theory was applied to the transcripts of the audio of each interview session, and then I constructed genetic decomposition of PTs' geometric reflection schemas. Next, I described common patterns among PTs, and created a table to identify evidence points (Arnon et al., 2014; Dubinsky, 1991). Ongoing and retrospective analyses were used to analyze data using the APOS theory.

Ongoing Analysis

Based on preliminary analysis of interview 1, I revised interview 2. This process, called ongoing analysis, was repeated between the second and third interviews. The main purpose of the ongoing analysis was to build a model of each PT's mental structures for specific concepts such as reflection line, domain and plane. Another purpose of the ongoing analysis was to analyze participants' ways of thinking. These findings helped me to analyze PTs' current understanding of geometric reflections and provided a basis for organizing the next interview questions.

After completing an ongoing analysis of each participant, I tested the preliminary genetic decomposition by using APOS theory. For example, my preliminary genetic decomposition indicated that all participants would have an action structure of the reflection line. After the first interview, I found that all participants had an action and a process structure of the reflection line. I also found that knowing equidistance and perpendicularity properties and the relationship between pre-image and image points of figures and the reflection line are important factors to help participants to move from an action structure to a process structure. After testing the preliminary genetic decomposition, I created additional hypotheses to be tested in the subsequent interview. For instance, I hypothesized that all of the participants had an action structure of the domain after the first interview. After the first interview, I found that some of the participants had a process structure of the domain because in that they considered all points in the plane to perform a geometric reflection. Therefore, I needed to create new hypotheses for the remaining interviews to test tasks again or add several tasks to ask participants to move to the next structure to determine who already had a process structure for domain. I repeated ongoing analysis and testing of hypotheses after each interview.

Retrospective Analysis

Retrospective analysis began after completing the three interviews. The main goal of this analysis was to collect in-depth information about the development of PTs' mental structures of geometric reflections. During this analysis, I sought to find answers to such questions as "How does a motion view of geometric reflections develop into a mapping view for secondary mathematics PTs?" and "What factors facilitate PTs' development of a motion view of geometric reflections into a mapping view?" To answer these research questions, I analyzed data and developed a preliminary genetic decomposition to find evidence to support it. Table 6 below shows a sample of my tentative analysis of each PT's case. The first column indicates the code that I assigned, either action or process. The second column provides interview excerpts from the original transcript to illustrate each code. The third column includes my reasons for inferences and related comments, and the last column includes any extra notes.

Table 6 *Sample of Data Analysis*

Codes	Transcript Excerpt	Reasons and Comments	Notes
Action Process	**R:** Find the image of the triangle after performing reflection over the line, and explain how you determined where to place the figure? **A PT:** I selected three vertices and reflected over the reflection line. Then, I connected them together to draw triangle. **R:** How did you reflect these points? **A PT:** To find the image of a point, A, under the reflection, it is necessary to draw a line through B perpendicular to the line of reflection. B' would be the same distance from the line of reflection as B. **R:** Is there any point inside the figure? **A PT:** No. There are three points labeled that I selected to perform a reflection.	The PT's answer shows that the PT seemed to understand the role of reflection line (e.g., mapping points over the reflection line) using perpendicularity and equidistance properties. Based on this explanation, the PT is at the process structure for reflection line. However, PT only selected vertices and sides, rather than all points in the domain to apply the reflection. At this point, the PT is at the action structure for domain.	The participant did not go through each step (e.g., mapping all points on the perimeter of the figure, inside and outside the figure) to perform a reflection. PT directly reflected vertices over the reflection line and explained the role of reflection. The participant had interiorized his action into a process structure.

After organizing my notes on the video records, I examined the written work to find evidence of each PT's mental structures in terms of actions, and processes. The main goal of this process was to interpret each PT's mental structures based on my preliminary genetic decomposition. In my analysis, I specifically explained why I interpreted a given response as an action or as a process.

I assigned codes based on PTs' oral and written responses to the tasks, and made an outline for each concept. For instance, after interview 1, I found evidence of each PT's mental structures (with respect to the motion view) for sub-concepts of geometric reflections (i.e., reflection line, domain and plane). Then, I coded each specific answer as an action or process. For example, if a PT performed a reflection without considering perpendicular or equidistance properties of geometric reflections, I interpreted this as evidence of an action structure. If a PT performed a reflection using equidistance between pre-image and image points/figure, and explained the relationships between pre-image and image points and the line of reflection (e.g. the reflection line is used to map pre-image points to its image points under reflection), this evidence was coded as a process structure.

After examining the data for each interview, I looked at mental structures of actions and processes to address differences between the participants' understanding of the role of reflection line and the concept of the domain and plane. If necessary, when addressing these differences, in particular those that seemed unique, I replayed the interview videos and reviewed the transcriptions to gather additional evidence. Afterwards I constructed a revised genetic decomposition of geometric reflections based on new evidence in comparison with the preliminary genetic decomposition.

Trustworthiness of the Data

To ensure the validity of the collected data, I used triangulation. The use of transcriptions, video recordings, and my observation notes allowed the collected data to be validated. This evidence was used to construct models of the PT's actions and processes for geometric reflections. These models allowed me to look for commonalities, as well as difference amongst all PTs. In Merriam (2009), we also see that multiple interviews with the same participant serves as triangulation. Additionally, Prior to submitting my dissertation, each PT had two weeks to review my work related to them; however no participants requested any changes. These strategies ensured the accuracy of my data from exploring PSMTs' mental actions, and processes to construct geometric reflection structure.

CHAPTER 4. RESULTS

The purpose of this chapter is to describe four pre-service secondary mathematics teachers' (PTs) understanding of geometric reflections using action-process-object-schema (APOS) theory. The results are written in the form of case studies of Linda, Emily, Michael, and John (pseudonyms). These cases explain how four PTs' motion views of geometric reflections evolved into mapping views and the factors that facilitated these developments. Each case has three parts, including a brief introduction of the participants' experiences with geometric reflections and their mathematical background.

The first part shows evidence and my interpretations of the mental structures that the four PTs used related to their understanding of a reflection line in terms of the action and process structure. In this part, the role of a reflection line and its importance will be discussed for geometric reflection. Likewise, the second part provides evidence and my interpretations of the mental structures about the PTs' understanding of domain in terms of the action and process structure. In this part, the role of domain and its importance will be discussed for geometric reflection (e.g., exploring the domain of reflection as all points in the plane rather than as a single figure). The third part presents evidence and my interpretations of the mental structures used by the PTs related to their understanding of the plane in terms of the action and process structure. In this part, the plane of the reflection and its importance will be discussed for geometric reflection (e.g., geometric figures are not "separate from the plane" (Yanik & Flores, p. 55)—rather they are part of the plane). The four PTs' understanding of geometric reflection will be discussed in the sections that follow. Whenever I used the term "R" in this study refers to the "Researcher".

Linda's Understanding of Geometric Reflections

Introduction

Linda (a pseudonym) was a 21-year-old, majoring in mathematics education. She took a geometry course in the fall of 2015. This course focused on how to prove (perform proofs) using theorems, propositions, definitions of lines and points, and different shapes. Prior to this geometry course, she took several theoretical mathematics courses (e.g., introduction to statistics and probability, Calculus I, Calculus II, and abstract algebra). In general, she described herself as an average mathematics student; she earned A's and B's in all of her mathematics and mathematics education courses.

According to Linda, her experiences with mathematics in high school were practice type instruction. In college, when she took high-level mathematics classes, she started to love mathematics. Then, she decided to become a mathematics teacher. When discussing her understanding of a mathematics concept, Linda stated, "If I actually understand it [referring to a mathematics concept], I will be able to do it continuously and consistently. Then I will understand what I am actually doing, not just go through steps but to understand why I go from each step to each step" (Line 66-69; 09/14/2017, Initial Interview). She explained that solving more examples and watching online videos was helpful in understanding a mathematical concept.

Linda recalled being exposed to the concept of geometric reflection in her high school geometry course. She did not recall any concepts related to geometric reflections in her college geometry course. She described geometric reflection as "[reflecting] across the y-axis [so] that it [referring to a figure] would have to look like in a mirror" (Line 167-168; 09/14/2017, Initial Interview) (see Figure 13). She also described a reflection line as “the line in which you are reflecting your shape” (Line 199-200; 09/14/2017, Initial Interview), and explained the role of the reflection line as “knowing where you are going to reflect” (Line 212-213; 09/14/2017, Initial Interview). When she was asked about the properties of geometric reflections, she stated that geometric reflections preserve the “shape,” the lengths of the sides, and the measurements of the angles, and they do not preserve the orientation. From Linda’s explanation, I inferred that to perform a geometric reflection, a reflection line, a figure and a plane are necessary. Linda also defined a plane as “a surface [on] which points and lines can be drawn,” and it is also “unlimited” (Line 235; 09/14/2017, Initial Interview).

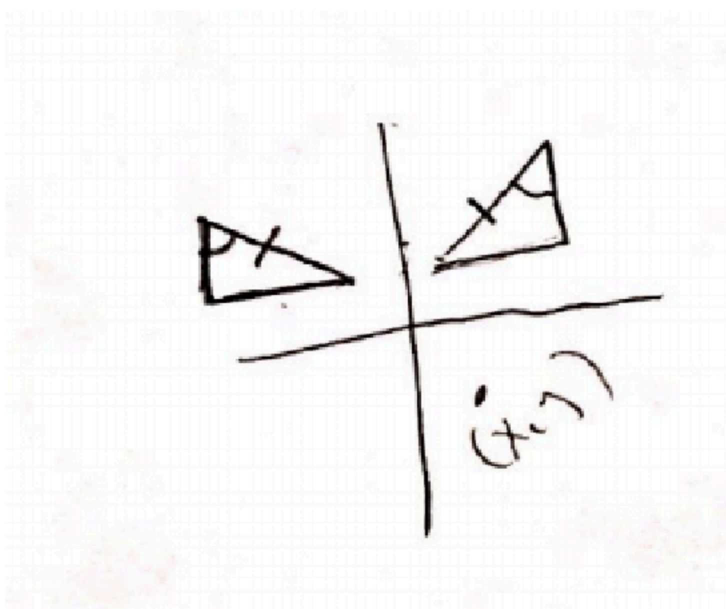


Figure 13. Linda drawing of a reflection.

Linda’s Understanding of Reflection Line

I administered several tasks to examine Linda’s mental structures about reflection line. Evidence from Linda’s performance on two of the tasks was consistent with her having action and process structures of the reflection line. For the first task, she was given a task with two figures without a reflection line and was asked: “Is this reflection? Why or why not?” Linda stated that it did not represent a reflection. She drew a reflection line between the given two figures and stated, “this distance [referring to distance from point A to the reflection line] from the line [referring to reflection line that she drew] to the point [referring to the point A] is not the same as this distance from this point [referring to point A`] to the line of reflection” (Line 24-25; 09/19/2017, Interview 1) (see Figure 14).

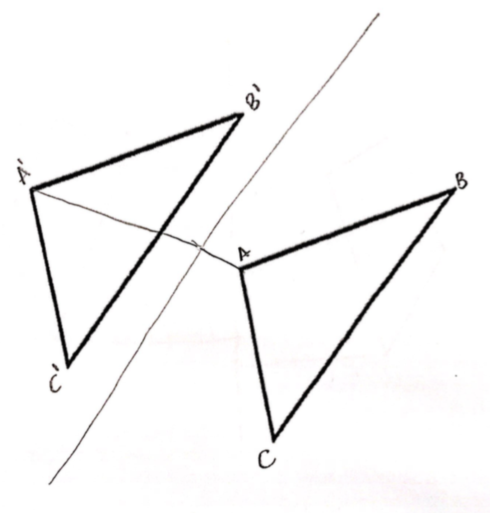


Figure 14. Linda’s drawing of reflection line between given figures.

When she drew the reflection line suggested that her understanding of a geometric reflection included a reflection line. Her drawing the reflection line is evidence that she had an action structure for the reflection line. She knows that a reflection line is a useful tool for being able to determine whether an image (A`B`C`) is the reflection of the pre-image (ABC) (see Figure 14).

Later in the same interview, she was given a triangle with an oblique reflection line and asked to perform a geometric reflection (see Figure 15). Linda selected the closest point (referring to point A) to the line of reflection. She used an index card to measure the distance from A to the reflection line, and from the line of reflection to A`. She stated that the distance between A to the line of reflection and the line of reflection to A` would be the "same" and "perpendicular." Then, she reflected the remaining vertices (point B, and point C) and connected them to make a triangle. To understand her mental structures related to performing geometric reflection, she was asked to explain how she determined where to place the figure. Linda explained:

> I knew that these two vertices [referring to point B and C] of the triangles were the same distance away from the line [referring to the reflection line], so I made sure that they [referring to point B and C] were even together at the same line [referring to the IBCI line], or on the same line, I guess. Then I decided that this [referring to point A] is the same distance from the line of reflection (Line 92-95; 09/19/2017, Interview 1).

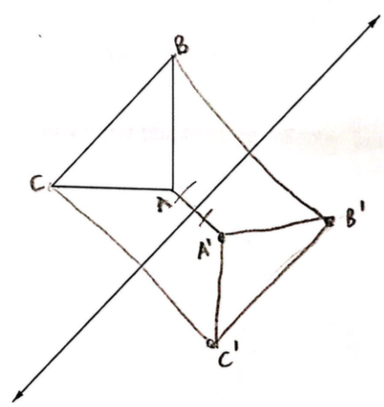


Figure 15. Linda's drawing of a reflection on an oblique reflection line.

Linda's explanations were evidence that her understanding of the reflection line included a method to map points on a figure across the reflection line. Her action of mapping the points was evidence that she had a process structure of the reflection line because Linda reflected the figure as a collection of parts (e.g., points, vertices, sides) rather than as a whole figure. I determined that her drawing means that Linda knew the relationships between pre-image and image points of the figures and the reflection line. She progressed beyond just identifying the line of reflection as an essential component of reflection towards the idea that she needed particular points, mapped in a particular way through her use of properties of equidistance and perpendicularity. This constitutes a process structure of a reflection line.

During the first interview, Linda performed geometric reflections with vertical, horizontal and oblique reflection lines. She consistently used equidistance and perpendicularity properties when she performed or explained a geometric reflection. She knew how to perform a reflection by measuring the distance between a point of the figure and reflection line, and she also knew that each point of the figure would be perpendicular to the reflection line. At the end of the first interview, she described geometric reflection as “using the line of reflection and finding the distance between the points you are trying to reflect and the line of reflection. Then you would reverse that to the other side of the line of reflection, and make sure the distances are the same” (Line 326-329; 09/19/2017, Interview 1). Based on this definition, Linda started to describe geometric reflections by referencing the role of the reflection line and properties of equidistant, and perpendicularity. In comparing to the definition of geometric reflection she gave in the initial interview, she revealed mental structures by using the reflection line and properties of perpendicularity and equidistance of geometric reflection in the first interview tasks. Hence, Linda had an action and a process structure of reflection line. The analysis of the first interview indicated that the reflection line is a significant sub-concept for the concept of geometric reflection and essential to move from a motion view to a mapping view. In the following diagram, I describe how she had process structure of the reflection line by unpacking her mental structures through the lens of APOS theory (see Figure 16).

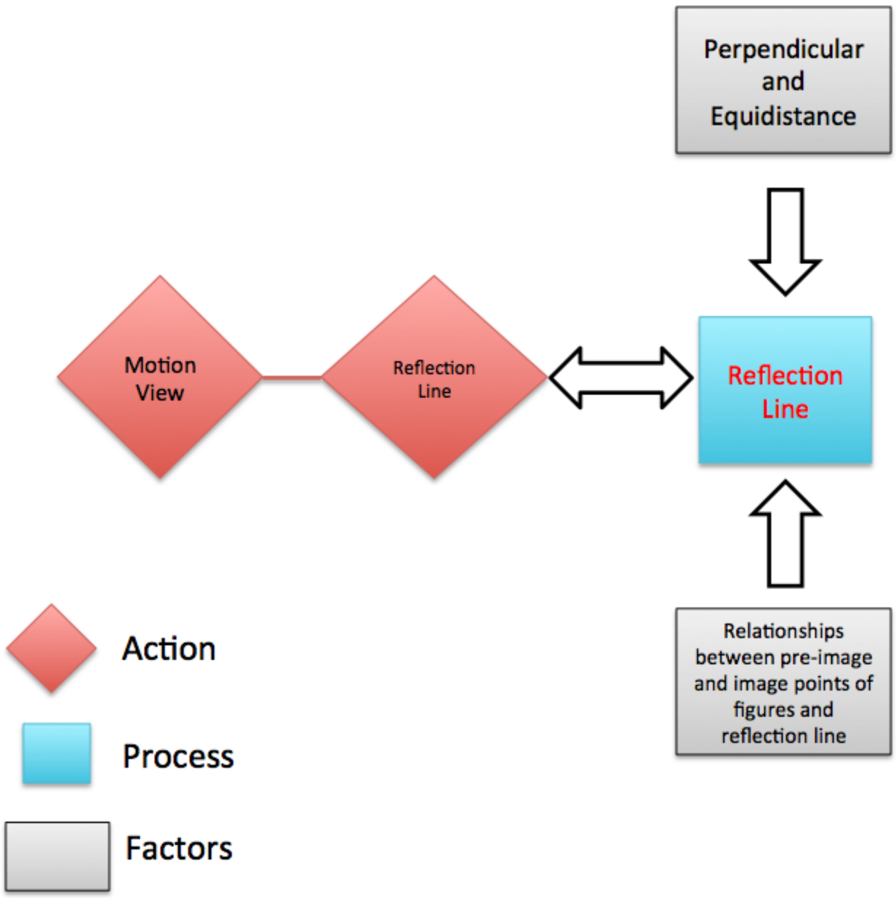


Figure 16. Linda's mental structures of reflection line.

As previously mentioned in my preliminary genetic decomposition (PGD), I hypothesized that PTs begin with a motion view of the reflection line as an action structure to develop a mental structure for reflection line as a process structure. PTs need to know two critical factors involved with the role of the reflection line and the relationship between the pre-image and image points and the reflection line. These two factors are sufficient to have a process structure of reflection line in geometric reflection.

The process structure of reflection line involved action structure of reflection line as well. I investigated how Linda coordinated her mental structures with respect to the role of the reflection line and the relationship between the pre-image and image points and the reflection line to perform geometric reflections. During the first interview, Linda had mental structures for the properties of equidistance and perpendicularity and the relationship between pre-image and image points and the reflection line. Therefore, it is clear that Linda had an action and a process structure of reflection line for performing geometric reflections.

Linda's Understanding of Domain

Several tasks were used to examine Linda's mental structures in terms of the concept of a domain. Evidence from the tasks demonstrated that Linda had an action and process structure of the domain of geometric reflection. My analysis of Linda's understanding of the domain is based on four tasks from the first, second and third interviews. In the first interview, she was given a square with an oblique reflection line and was asked to "find the figure after performing a reflection across the line" (see Figure 17). Linda reflected four vertices of a square by measuring the distance between each of the vertices to the reflection line using an index card (e.g., A, B, C, D) by she then connected the vertices of the reflection to make the square. She further used an index card to position other points (G, H, I) to reflect.

When I asked her to explain what points she reflected, she elaborated her ideas in the following conversation:

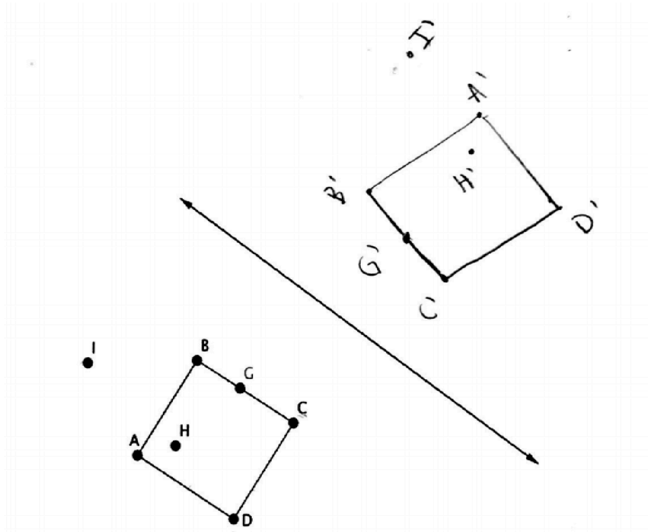


Figure 17. Linda's drawing of a reflection on an oblique line.

> L: I reflected A, B, C, and D, and then G came along on line BC because it was on the original line BC, and then I reflected H, and I as well.
>
> R: Okay, are there any other points being reflected beside those, like A, B, C, D, G, H, and I?
>
> L: All the other points on the lines A, B, C, [and] D.
>
> R: Okay. Are there any other points that are reflected beside [these] lines [referring to points on the segments AB, BC, CD, AD]?
>
> L: I do not believe so (Line 290-296; 09/19/2017, Interview 1).

Linda's explanation and drawing suggests that she considered labeled points on the perimeter of the figure, all points on the perimeter, and one point (point H) in the interior figure and one point (point I) on the exterior figure to perform the geometric reflection. To reflect labeled points on the task, she used an index card to measure the distance between pre-image and image points and the reflection line. I interpreted her explanations to mean that for Linda geometric reflections reflect given points on the domain rather than all points in the plane. This is evidence of an action structure of the domain since the focus is on labeled points rather than all points in the plane.

Further evidence of the domain as an action structure was gathered during the second interview. I asked Linda to reflect a circle with an oblique reflection line. Linda stated, “it was difficult to reflect the circle since there were no vertices to measure distance.” (Line 65; 09/21/2017, Interview 2) (see Figure 18). She measured the distance the point on the circle (point B) to the reflection line and reflected point B over the reflection line by using an index card (see Figure 19a). Then, she measured the diameter of the circle (see Figure 19b) reflecting the point A over the reflection line. Linda applied the same process for point C and D. She then drew the circle. She was using equidistance and perpendicularity properties to perform reflections. Additionally, she noted, “it was difficult to reflect the circle with four points,” and explained that “selecting more points would help to reflect the circle more accurately.” (Line 82-83; 09/21/2017, Interview 2)

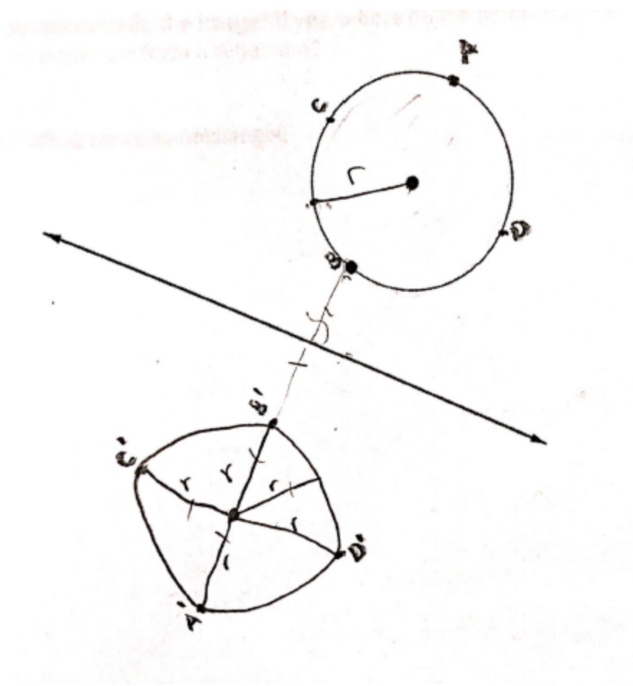


Figure 18. Linda’s drawing of a reflection on a circle task.

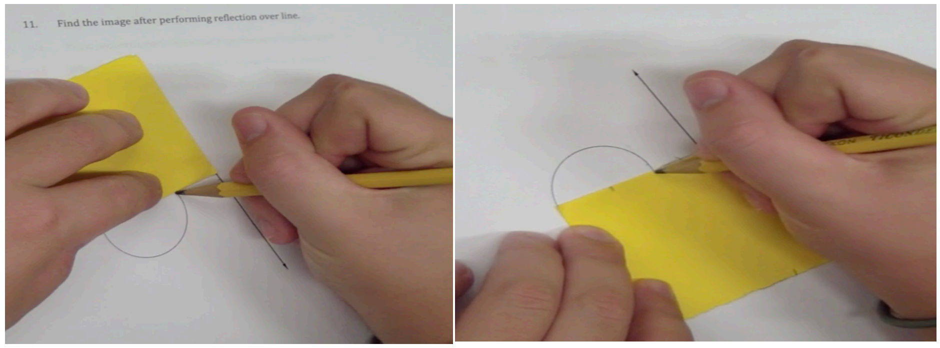


Figure 19 a,b. Linda’s drawing of a reflection on a circle task.

When Linda completed the reflection, I asked her to explain her approach.

> R: How many points did you reflect?
>
> L: I reflected all the points on the circle.
>
> R: Okay. Are there any other points being reflected beside perimeter of the circle points?
>
> L: Nope. I do not believe so. Just on the circle. Oh, well, sorry, I guess the center of the circle I reflected. Reflect this [referring to center point of the circle of pre-image] as well if that is a point. I guess I reflected that [referring to center point of the circle of pre-image] across the line of reflection.
>
> R: Okay. Did you only reflect the center point?
>
> L: No. So you would reflect all of the points inside the circle. (Line 79-87; 09/21/2017, Interview 2).

Linda’s explanations suggest that she started to think of reflecting points interior to the circle. Linda explained her reasoning in the following quote:

> I thought to find the center of the circle, and reflecting that point [referring to the center of the circle], and then just finding the radius and then drawing the circle. So then I started to think about ... if you actually are going to reflect the center across then it would make sense because you would obviously reflect the center and if I were going to draw the radius, then I would reflect all the points on them and the perimeter of the circle as well (Line 113-118; 09/21/2017, Interview 2).

Linda's conclusion in the section now include points interior to the figure, at least when the figure is a circle for performing geometric reflections. Her explanation revealed what seemed to be a new insight about geometric reflections. Reflecting a circle seemed to encourage Linda to think about the interior points of the circle. In response to the question, "is there any point outside the figure reflected?" Linda indicated "no." This is still evidence of action structure of the domain for geometric reflection since she reflected all of the points on the figure and inside the figure, but she did not show that she mapped all of the points outside of the figure.

Suggesting one of the reasons for her rationale seems to be related to her definition of the plane. Even though Linda gave a correct definition of a plane as a "surface [on] which points and lines can be drawn" (Line 235; 09/14/2017, Initial Interview) and that it is also "unlimited," she still operated metaphorically on a point without attending to all points in a plane because she might consider the plane to be an empty space and did not consider anything else except the given points, figures, and interior points. Linda's definition of the plane was inconsistent when she performed a geometric reflection.

In hopes of provoking a consideration of the exterior points, I posed a task using an open figure (see Figure 20). She had no difficulty reflecting a given figure by using equidistance and perpendicularity properties. I asked her what points were being reflected. She said, "I reflected the three standing points [referring to points A, B, C] and then the end points [referring to points D, E] of this arc and then the top end point [referring to the point F] of this parabola, if you think about it that way I guess. And then all of the points on this arc as well" (Line155-157; 09/21/2017, Interview 2). Her explanation was consistent with what she did previously with other tasks. Then I asked, "Are there any other points reflected beside you mentioned?" Linda elaborated:

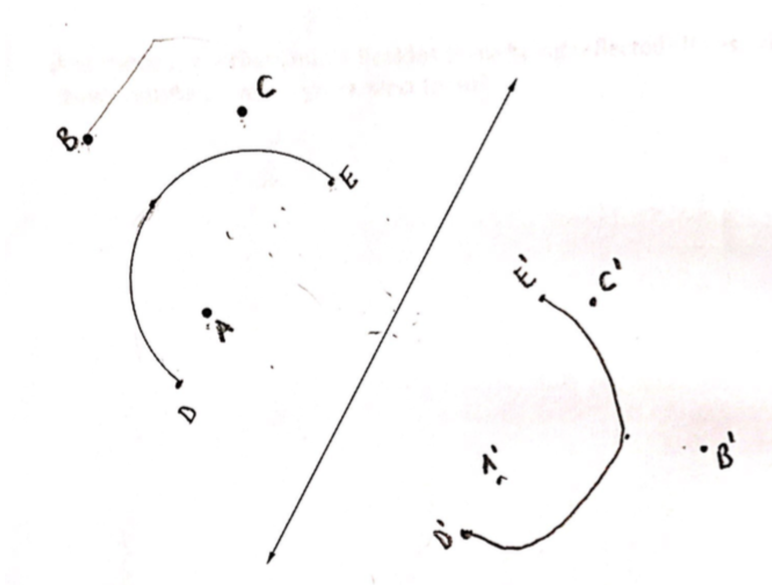


Figure 20. Linda's drawing of a reflection on an arc task.

L: So when it was a circle, if this arc were complete then all the points inside the arc will be reflected but ... since this is an open figure ... so ... I would say yeah. So then I reflected all of the points from even I guess this point down [referring to the point B]. And this area [referring to point B and all points below B] (see Figure 21a), I reflected across the line of reflection as well. Okay. I think ... Yeah.

R: Can you say again what points did you reflect?

L: Okay. I think I am predicting that I reflected all of the points in this area [referring to point B and all points below B]. So from this highest point [referring to point B] ... down, to the line of reflection, and then I reflected them across this line [referring to the reflection line].

R: Okay. How about the other side [referring to the up part of the point B] (see Figure 21b)?

L: So yeah ... Then yeah, all of these will be reflected as well, across the line of reflection then. Because this is an entire plane. That is a good way to think about it. And you are reflecting the plane onto the other side of the line of reflection.

R: How do you start to think that you need to reflect entire plane?

L: Because, well I saw this point [referring to point A] and it was inside my arc, and so I knew that I reflected this point [referring to the point A] to this side [referring to image figure plane], and so then I started to think about all the other points inside this arc [referring to pre-image], and then thought about how this point [referring to point A] could be over here [referring to image figure plane]. I will still reflect it [referring to point A] because it [referring to point A] was on this side [referring to pre-image figure plane] of the line of reflection. Then I saw that these points [referring to the points B, C] that were outside of the arc and how they were enclosed, and how they were reflected onto the other side [referring to image figure plane] of the line of reflection as well. (Line159-181; 09/21/2017, Interview 2).

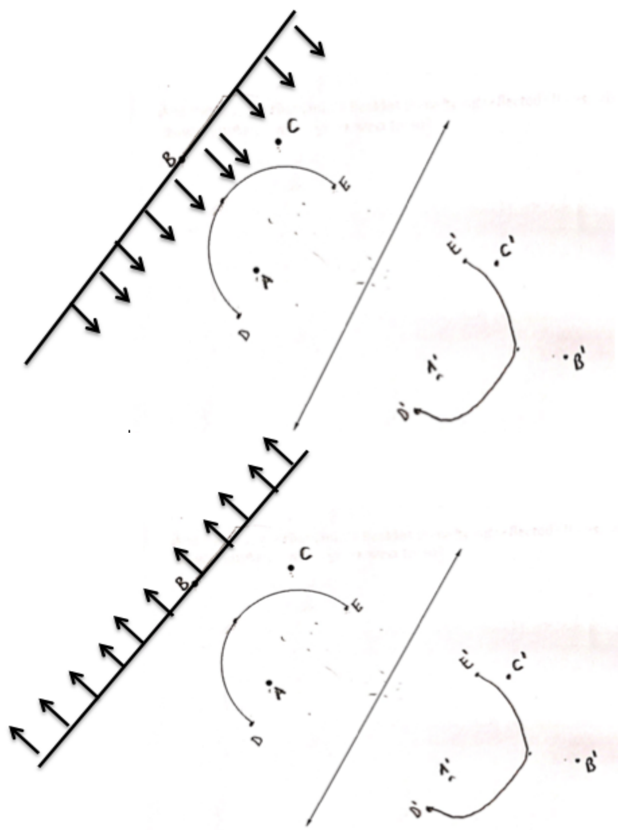


Figure 21 a,b. Linda's drawing of a reflection on a semi-circle task.

In this excerpt, Linda considers more points in the plane because the figures interior and exterior were not clearly defined for her. She reflected points she considered. When she saw the two points "outside" of the arc figure (B, C), Linda started to think about the half plane, which I infer to mean that because Linda imagined that there were infinitely many points inside and outside of the arc figure on which to perform a reflection. In other words, after the arc task, she used her definition of a plane in geometric reflection in a way that aligned with her mathematical definition of a plane because, following this episode, when she performed a reflection she described reflecting all the points in the pre-image plane (such as points on the perimeter, inside points and outside points of the figure). Hence, I interpreted her shift in explanations, to indicate that she started to think of the half of the plane as a non-empty space consisting of an infinite number of points when performing a geometric reflection.

In hopes of provoking a consideration of the all points in entire plane, I posed a task using multiple figures for both plane to perform geometric reflection (see Figure 22) since geometric reflection reflects all points from pre-image plane to the image plane, and image plane to pre-image plane. She had no difficulty reflecting given multiple figures for both plane. Then, I asked her to explain what points she reflected. She explained, "I reflected all the points in this plane [referring to pre-image plane] over to this side [referring to image plane] of the plane, and then reflected all the points from this side [referring to image plane] of the plane onto this side [referring to pre-image plane] of the plane." (Line 121-122; 09/26/2017, Interview 3).

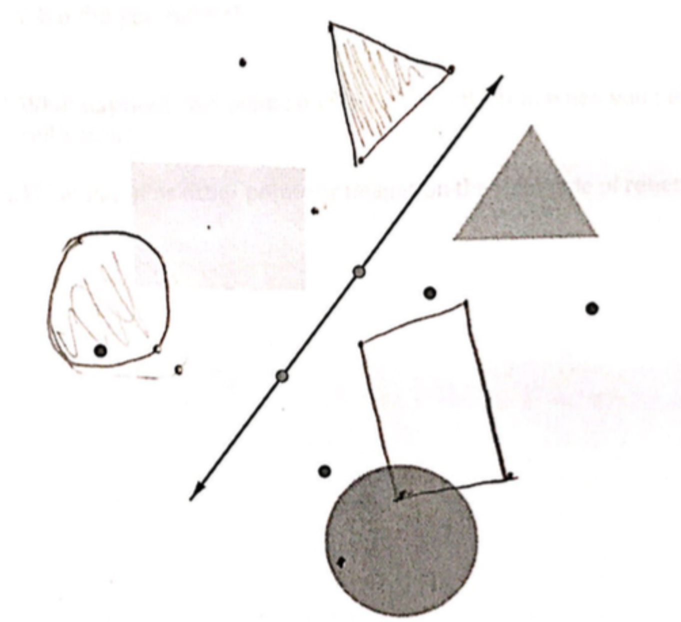


Figure 22. Linda's drawing of geometric reflection for both sides of the plane.

This task seemed helpful for Linda because it caused her to consider both sides of the plane when performing a geometric reflection: she previously did not consider all of the points from the image plane reflecting onto the pre-image plane. In other words, to perform a geometric reflection, Linda only reflected one part of the plane, which was the pre-image figure part, rather than entire plane when performing a reflection.

Linda started to think about all points in the plane when performing a reflection. Although she did not consider reflecting unlabeled points for geometric reflection physically, she understood that performing a reflection reflects labeled and unlabeled points, which means the all points were in the entire plane. The tasks with circle, arc figures, and multiple figures helped Linda to consider "interior" and "exterior" points when performing a geometric reflection. Linda's operational definition of the plane in geometric reflection aligned with her mathematical definition of a plane. I inferred her explanations to mean that she had reached the process structure for the concept of the domain in geometric reflection. In the following figure, I describe how she reached process structure of domain by unpacking her mental structures through the lens of APOS theory (see Figure 23).

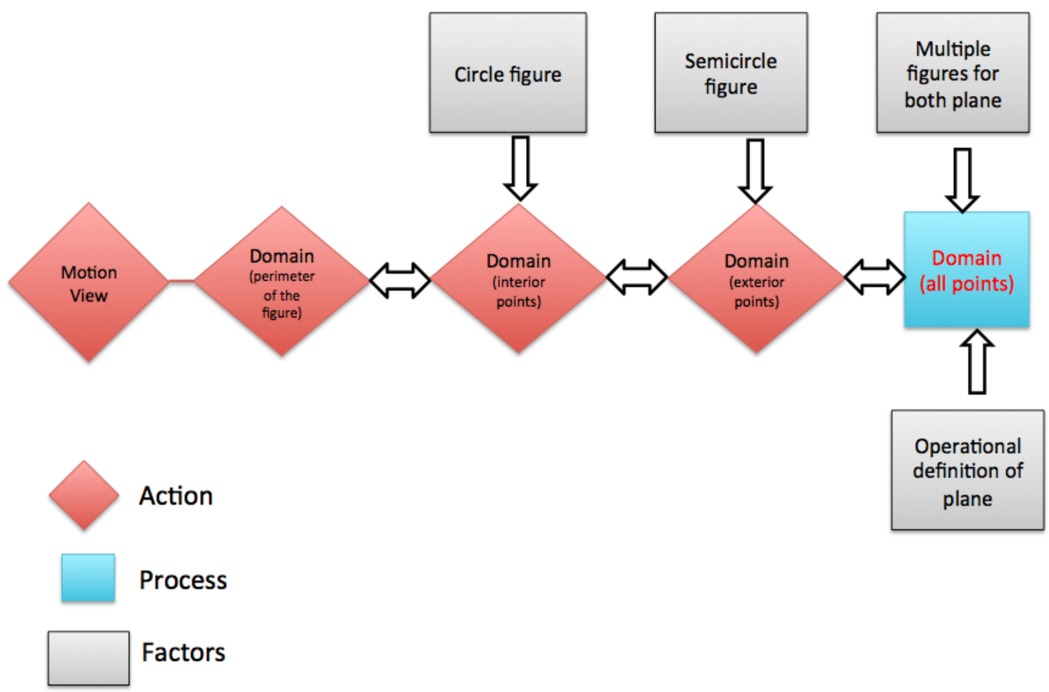


Figure 23. Linda's mental structures of domain

As previously mentioned in my PGD, I hypothesized that PTs begin with a motion view of domain as an action structure to develop a mental structure for domain as a process structure. Figure 23 shows Linda's development of mental structures of domain throughout first, second, and third interviews, as well as showing the impact of the various tasks on Linda's operational definition / consideration of domain. Linda's model revealed that she reached the process structure of domain at the end of the third interview.

During the first interview, Linda considered the domain as the perimeter of the figure when she performed a geometric reflection. This implies that she had an action structure of domain in geometric reflection. In the second interview, when she worked on the circle and semicircle tasks, she began to consider all interior and exterior points of the figures. Working with the circle and semicircle task encouraged Linda to consider more points in the plane. Linda still had an action structure of domain in geometric reflections because she considered all points in the pre-image plane (half of the plane) rather than all points in the entire plane. During the third interview, when she worked with multiple figures in both the pre-image and image planes, Linda started thinking of the image plane as well as the pre-image plane having an infinite number of unlabeled points. Therefore, she began to think about all points in the entire plane for performing geometric reflection. Her operational definition of the plane in geometric reflections aligned with her mathematical definition of a plane in geometric reflections. The tasks with circles, semicircles, and multiple figures for both the pre-image and image planes helped Linda to consider all points in the plane.

Linda's Understanding of the Plane

To examine the PT's mental structures in terms of the concept of the plane, I analyzed her reactions and explanations on tasks when performing geometric reflection. The second set of interviews revealed that Linda had an action structure of plane for geometric reflection because she seemed to think of the geometric reflection as "a movement of points or figures on a plane rather than a mapping of the plane onto itself" (Yanik & Flores, p. 46). For instance, when performing a reflection with a circle task, Linda viewed points as "separate from the plane" (Yanik & Flores, p. 55), which meant that she could move points or figures to perform reflection (see Figure 24). The following episode illustrates our conversation:

> R: Okay. How about the outside points? Are there any outside points that you reflected?
>
> L: I do not think so. The reason I guess I do not do that is because ... this point [referring to a point outside of the circle she drew herself] out here, well, you [did] not get reflected over here [referring to another side of the reflection line] because ... I guess I didn't have to do anything to these points [referring to outside points of the circle] in putting the points inside the circle I had to move the surrounding [referring to other points inside the circle] to the other side of the line, I guess is what I am trying to say. (Line 131-135; 09/21/2017, Interview 2).

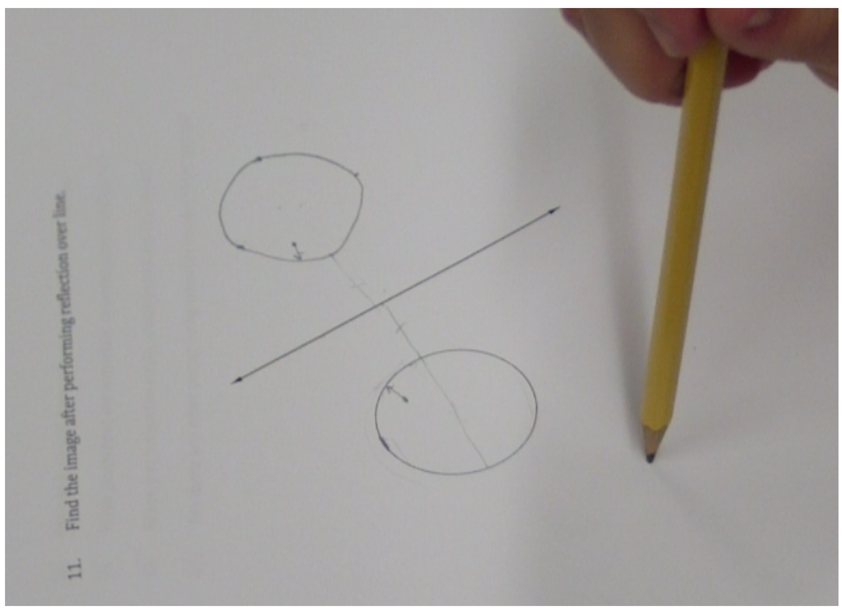


Figure 24. Linda's execution of reflection on a circle task

Linda seemed to think that points were located on the plane, which means that points were "separated from the plane" (Yanik & Flores, p. 55) rather than part of the plane. Hence, she used the word "move" to perform geometric reflection because she might think that when performing a reflection, the points or figures are relocated to a new position relative to other points in the other plane.

In the next task, she reflected a trapezoid over an oblique reflection line (see Figure 25). Linda explained, “I reflected all of the points in this plane [referring to pre-image plane] over here [referring to image plane], and then all of the points over here [referring to image plane] would be reflected onto there [referring to pre-image plane].” (Line 143-144; 09/26/2017, Interview 3).

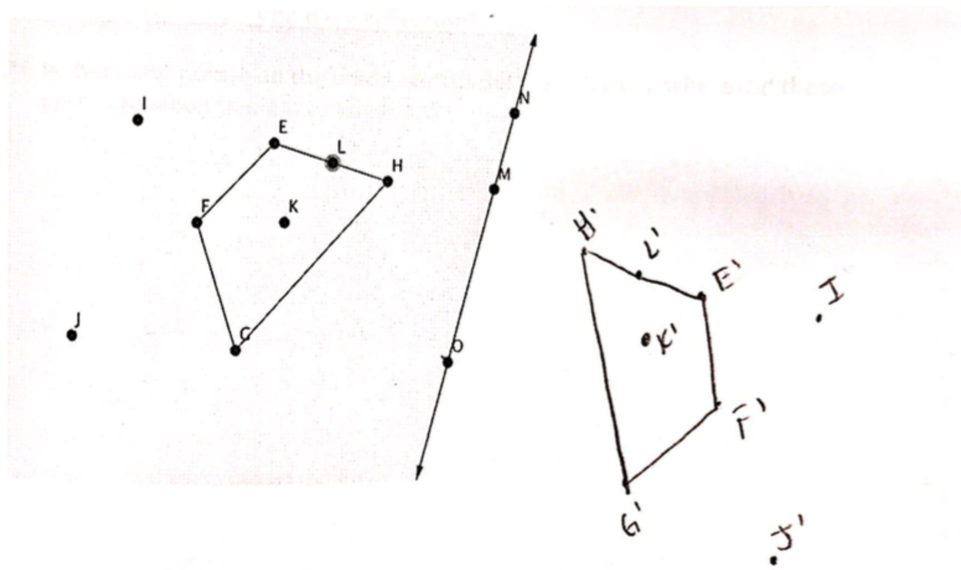


Figure 25. Linda’s drawing of reflection over an oblique reflection line

After this task, Linda was asked about the relationship between points/figures and the plane. Specifically, she was asked, “When you perform a reflection, is there any movement of the points or figures from half of the plane to another half of the plane?”, “when you reflected the points, what is left here [referring to the plane that constructed the pre-image figure]?” She explained her reasoning as follows:

> I believe that it stays on this side of the line of reflection because ... Well, it is still there now. But, it still existed. It still exists on this plane; I just reflected it over to this line. I did not pick it up and move it, but I like copied it. (Line 157-159; 09/21/2017, Interview 3).

I inferred from this explanation that Linda considered the points or figures to be part of the plane rather than "separate from the plane" (Yanik & Flores, p. 55) because she thought that there was no movement of the points or figures from one half of the plane to the other half of the plane. This is because Linda began to use her operational definition of plane in geometric reflections. Since Linda understood the relationship between points or figures and the plane, she seemed to have reached the process structure of the plane (see Figure 26).

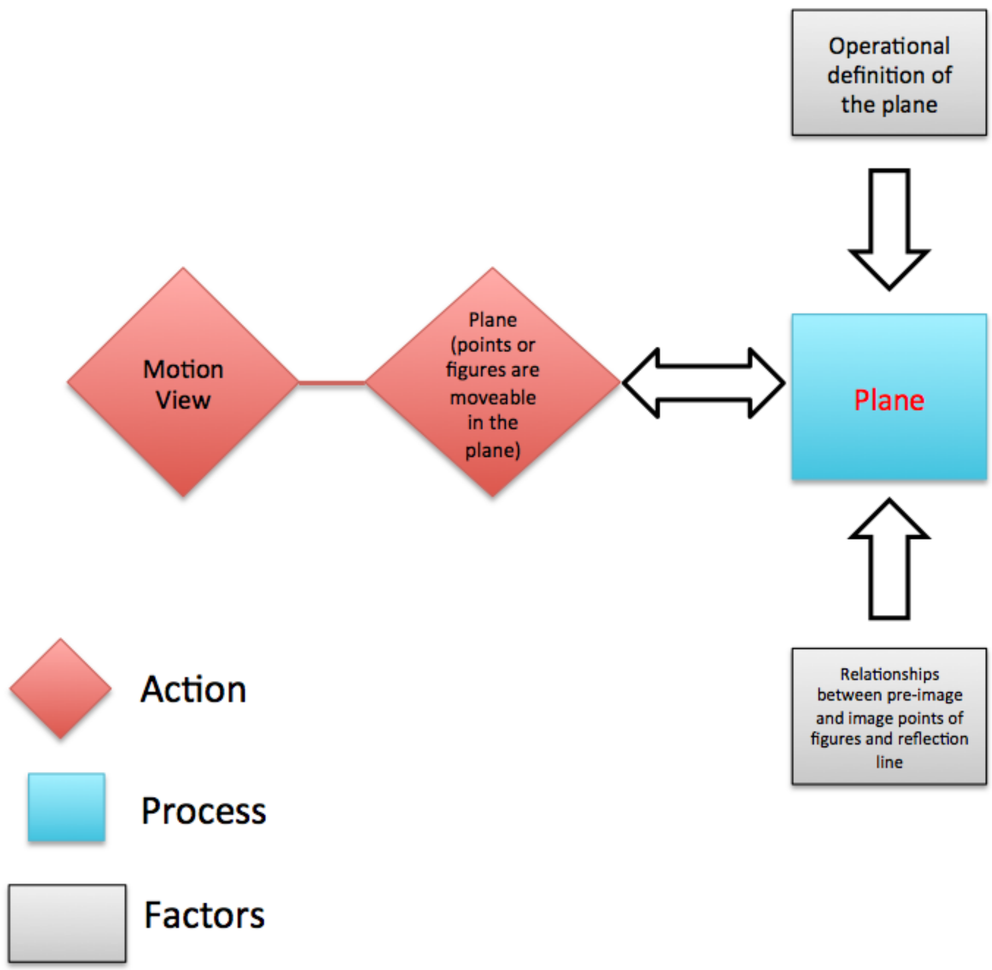


Figure 26. Linda's mental structures of plane

As previously mentioned in my (PGD), I hypothesized that PTs begin with a motion view of plane as an action structure to develop a mental structure for plane as a process structure. Figure 26 shows Linda's mental structures of plane throughout the first, second, and third interviews. Linda's model revealed that she had the process structure of plane at the end of the third interviews. During the first and second interviews, Linda consistently used the word "move" to describe how she performed geometric reflection. This is because I interpreted that when she perform a geometric reflection, the points or figures were not part of the plane. This implies that Linda had an action structure of plane. In the third interview, her explanations revealed that her mathematical understanding of the relationship between the figures or points and the plane was accurate; that is, she knew that the points (and hence the figure) were embedded in the plane. Therefore, Linda had a process structure of the plane by having an operational definition of plane and the relationship between the points of the figure and the plane.

Linda's Mental Structures of Mapping View

Briefly, Linda reached the process structure in terms of reflection line, domain, and plane. The findings show that Linda knew that a reflection line is necessary for performing reflections to position where to position the image of the reflection. She also knew that the reflection line maps every point in the plane onto itself for a geometric reflection. Additionally, the types of figures (e.g., circle and arc tasks, multiple figures for both plane) were crucial factors that helped Linda to consider all of the points in the domain rather than just a single figure. The use of definitions and understanding the relationships between points/figures and the plane helped Linda to reach process structure of plane (see Figure 27).

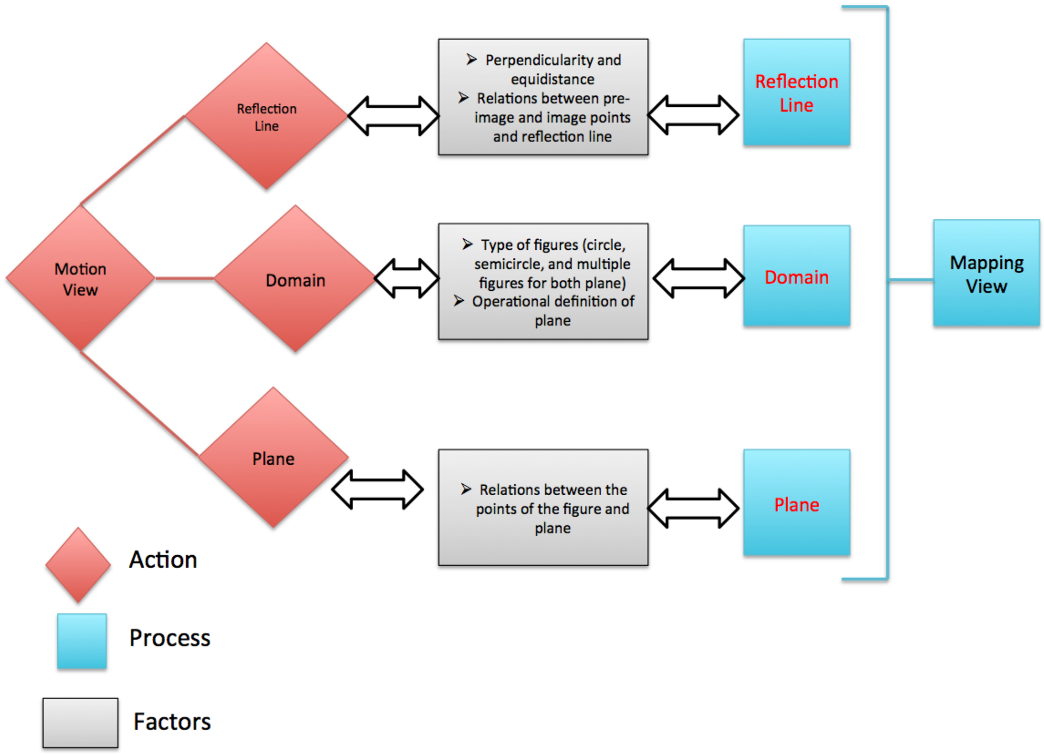


Figure 27. Linda's mental structures of mapping view

Emily's Understanding of Geometric Reflections

Introduction

Emily was a 21-year-old student, majoring in mathematics and mathematics education. In the fall of 2015, she took a geometry course which focused on how to perform proofs using theorems and making logical connections between them. Prior to this geometry course, she had taken several theoretical mathematics courses (e.g., Real Analysis, Calculus I, Calculus II, Calculus III, Abstract Algebra, Linear Algebra, and Differential Equations). In general, she described herself as a hardworking mathematics student; she earned As in most of her mathematics and mathematics education courses.

According to Emily, her background experiences with mathematics were based on "learning and proving about why things are true." Memorizing formulas to apply to problems was not a good strategy for her to learn mathematics. She wanted to understand what she was doing and what the results meant. She described mathematical understanding as "you would need to be able to apply it and see how it connects to other things, too." (Line 72-73; 09/15/2017, Initial Interview).

Emily recalled being exposed to the concept of geometric reflection in her high school geometry course. She did not recall any concepts related to geometric reflection in her college geometry course. She described geometric reflection as "taking the shape and keeping everything the same about it (e.g., length and area), but just kind of flipping it over." (Line 140-141; 09/15/2017, Initial Interview) (see Figure 28). Emily described the role of the reflection line, as "it is just like the line that you are going to flip it over and it is going to be the same distance from the original shape to the reflected one" (Line 162-163; 09/15/2017, Initial Interview). When she was asked about the properties of geometric reflection, she stated that geometric reflection preserve the length, areas, angles and the distance between pre-image and image "shapes."

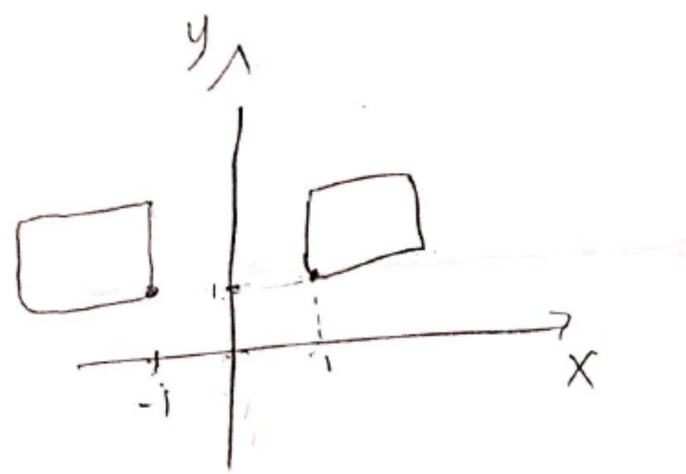


Figure 28. Emily's drawing of a reflection.

Emily’s Understanding of the Reflection Line

I administered several tasks to investigate Emily’s mental structures about reflection line and inferred that Emily had action and process structures of the reflection line. My analysis of her understanding of the reflection line is based on two tasks from the first interview. She was given a task with two figures without a reflection line and asked: “Is this reflection? Why or why not?” (see Figure 29). First, Emily used an index card to measure all three segments of both triangles, and stated that all segments of both triangles were the same size. Then, she drew an oblique reflection line between two triangles, and stated that both triangles should be same distance from the reflection line. She stated, “This point [A] needs to be the same distance from the line of reflection. So, no, I do not think that this is a reflection.” (Line 40-41; 09/20/2017, Interview 1). I inferred that she labeled A and A' points on the triangles as corresponding points and found these were not the same distance from the reflection line so the picture did not represent a reflection. Emily reflected the left triangle over the reflection line to prove her answer.

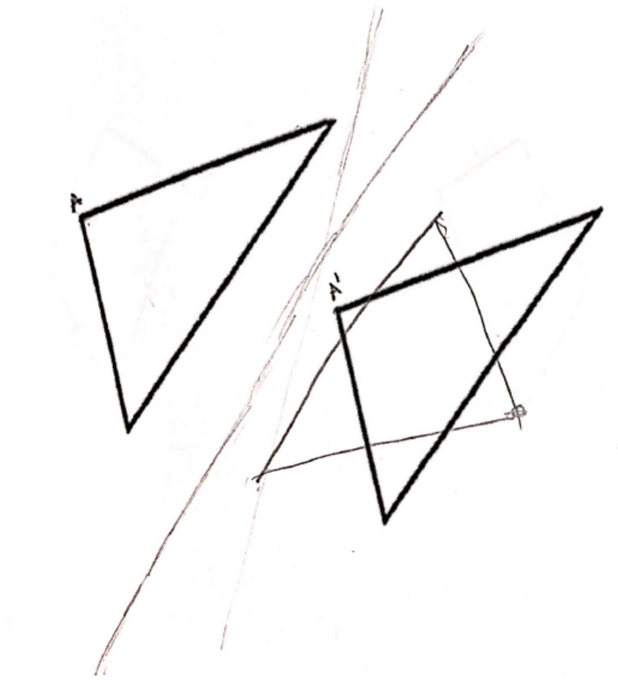


Figure 29. Emily’s drawing of the reflection line between given figures.

Her drawing of the reflection line suggested that her understanding of a geometric reflection included a reflection line and was evidence that she had an action structure for the reflection line. It was not clear whether Emily had a process structure of reflection line or not because she did not use the property of perpendicularity and whether she had a role of reflection line or not to reflect points.

Later in the same interview, Emily was given a triangle with a vertical reflection line and asked to perform a geometric reflection (see Figure 30). She selected the closest point of the triangle (referring to point A) to the reflection line to measure its distance to the reflection line (see Figure 31a), and from the reflection line to the corresponding point on A' (see Figure 31b). She stated that the distance between A to the reflection line and the reflection line to A' would be the “same” and “perpendicular.” Then, she reflected the remaining vertices (B and C) and connected them to make a triangle.

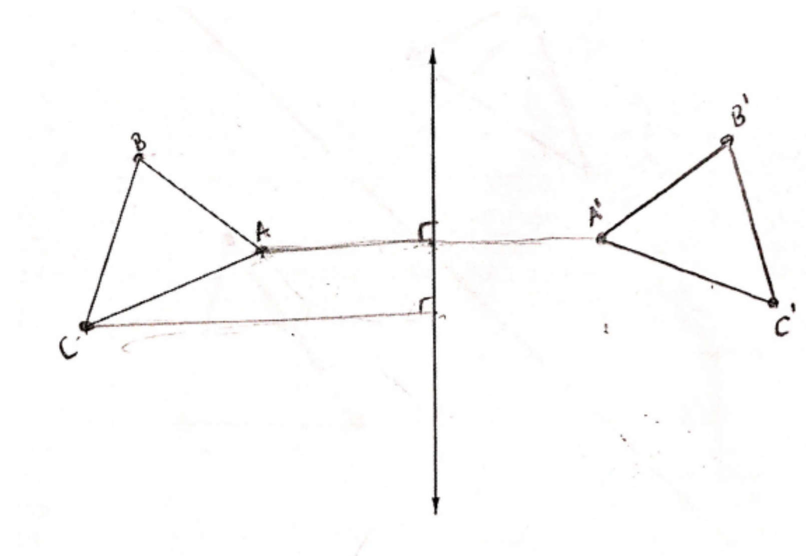


Figure 30. Emily's drawing of a reflection on a reflection line.

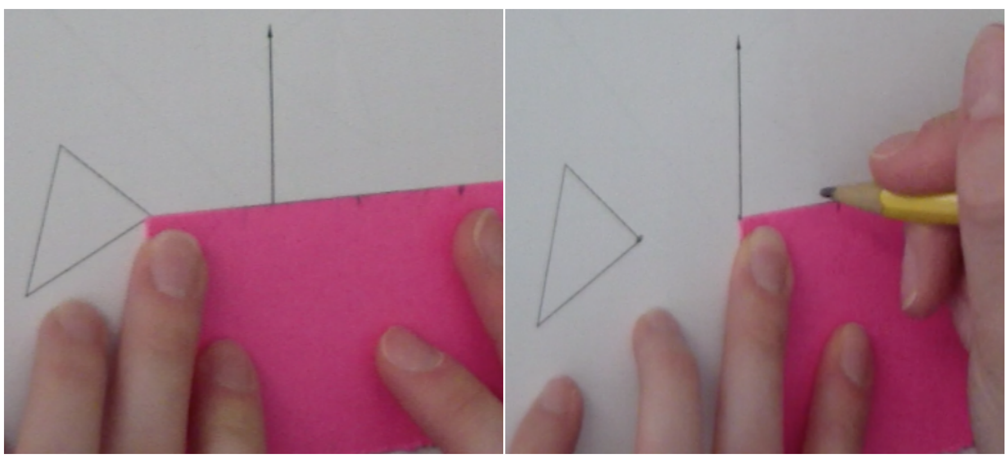


Figure 31 a, b. Emily measuring the distance between points on a triangle task.

To understand how her mental structures related to how she used the reflection line, I asked her to explain how she determined where to place the figure. Emily explained:

> I just look to add the three points [A, B, C] of the triangle on this side [referring to pre-image plane], and I know that if I am [going to] reflect the triangle on this side [referring to image plane], then this point [A] needs to be the same [distance] but then the reflected point [A'] for this one needs to be the same distance from the line of reflection. So I need to

> make sure that this distance [referring to distance from point A to the reflection line] is the same as this distance [referring to distance from point A` to the reflection line]. (Line 102-106; 09/20/2017, Interview 1).

Emily's explanations were evidence that her understanding of the reflection line included a role in mapping points on a figure across the reflection line. To find the image point, A', she drew a perpendicular line from pre-image point A to the reflection line using an index card. She then used the index card to determine where to place A' by using the equidistance property. Emily seemed to understand that the reflection line has a role in determining to where to position the points. I interpreted her drawing as meaning that Emily determined the relationships between pre-image and image points of the figures and the reflection line by using the properties of equidistance and perpendicularity. Her drawing of mapping the points and using the properties of equidistance and perpendicularity are evidence that she had a process structure of the reflection line.

To summarize, during the first interview, Emily performed geometric reflections with vertical, horizontal and oblique reflection lines. She knew that reflection line is an essential sub-concept for geometric reflections. She consistently used equidistance and perpendicularity properties when she performed or explained a geometric reflection. She knew how to perform a reflection by measuring the distance between a point of the figure and the reflection line, and she also knew that each point of the figure would be perpendicular to the reflection line. At the end of the first interview, she described, "A geometric reflection takes every point of what you want to reflect ... and makes a new point on the opposite side that is the same distance and perpendicular from the line of reflection. It is just on the opposite side of the line." (Line 309-311; 09/20/2017, Interview 1). With this definition, Emily started to describe geometric reflections by referencing the role of the reflection line and the properties of equidistance and perpendicularity. Compared to the definition of geometric reflections she gave in the initial interview, she revealed her mental structures by using the reflection line and properties of perpendicularity and equidistance of geometric reflections in the first interview tasks. Therefore, my analysis of the first interview indicated that Emily had action and process structures of the reflection line. In the following figure, I describe how she had an action and process structure of the reflection line by unpacking her mental structures through the lens of APOS theory (see Figure 32).

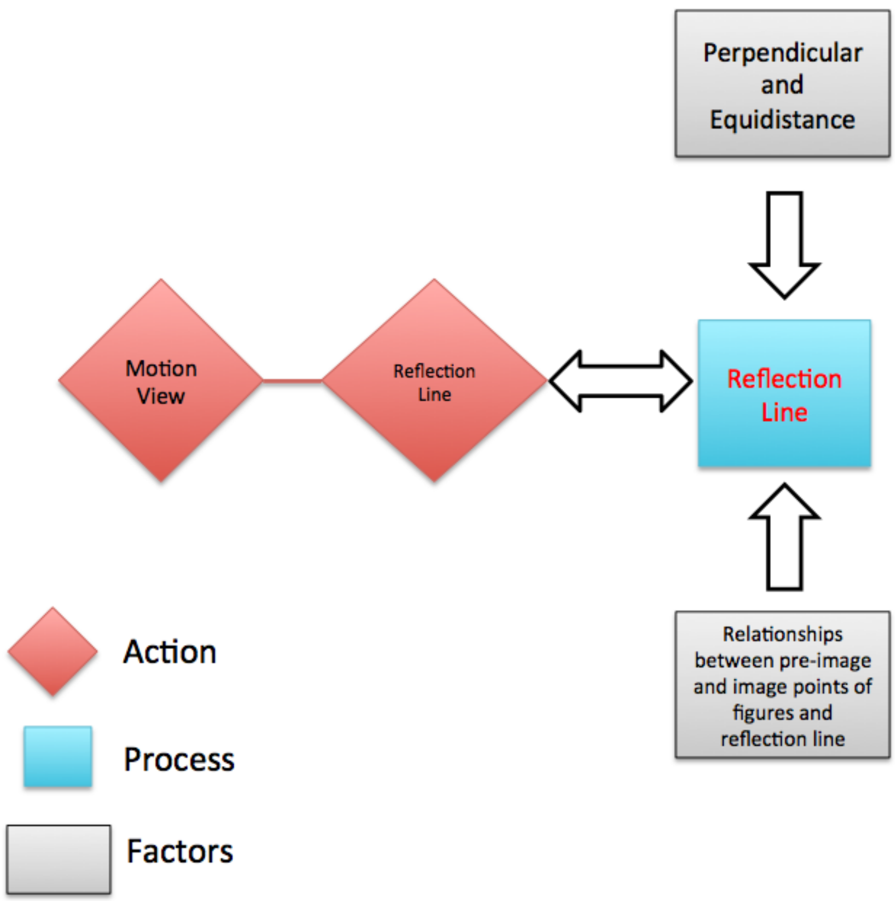


Figure 32. Emily's mental structures of the reflection line.

As I hypothesized in my preliminary geometric decomposition (PGD) that PTs begin with a motion view of reflection line as an action to develop a mental structure for reflection line as a process structure. An action structure of the reflection line means that PTs knew the reflection line is essential for geometric reflection. To have a process structure of reflection line, PTs need to know two critical factors involved in the role of the reflection line and the relationship between the pre-image and image points and the reflection line. These two factors are sufficient to have a process structure of reflection line in geometric reflections. The process structure of reflection line involved action structure of reflection line as well. I investigated how Emily coordinated her mental structures of the role of the reflection line and the relationship between the pre-image and image points and the reflection line to perform geometric reflection. During the first interview, Emily had mental structures for the properties of equidistance and perpendicularity and the relationship between pre-image and image points and the reflection line. Therefore, it is clear that Emily had an action and a process structure of reflection line for performing geometric reflection.

Emily's Understanding of Domain

Several tasks were used to examine Emily's mental structures in terms of the concept of domain in relation to reflection. I interpreted that Emily had an action structure of domain for geometric reflections. My analysis of her understanding of domain is based on four tasks from the first, second, and third interviews. In the first interview, she was given a trapezoid with an oblique reflection line and was asked to "find the figure after performing a reflection across the line" (see Figure 33). Emily reflected the four vertices of the trapezoid by measuring the distance of each vertex to the reflection line using an index card. She then connected the vertices in the image to make the trapezoid. When I asked her to explain what points she reflected, she elaborated her ideas in the following conversation:

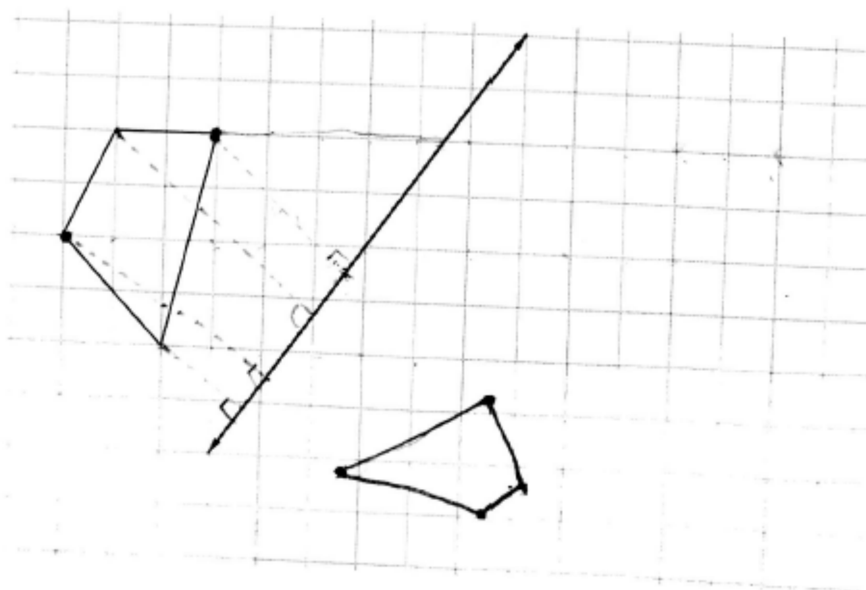


Figure 33. Emily's drawing of a reflection on an oblique line.

> E: I reflected these [referring to four vertices of trapezoid] and then all the points that are on the line here [referring to all points on the perimeter].
>
> R: Are there any other points being reflected beside perimeter of the figure?
>
> E: I do not think so. (Line 244-249; 09/12/2017, Interview 1).

I interpreted from her explanation that Emily considered the vertices and the perimeter of the figure to perform this geometric reflection. This is evidence of an action structure of the domain since she did not consider all points in the plane, including the interior and exterior points of the figure. During the first interview, Emily's explanations were consistent with other tasks to reflect the perimeter of the figure.

Further evidence of the domain as an action structure was gathered during the second interview. I asked Emily to reflect a circle without labeling the center point of the circle with an oblique reflection line (see Figure 34). She first reflected point A using an index card to measure the distance from point A to the reflection line, and from the reflection line to A'. She then measured the diameter of the circle to reflect point B using an index card. She then reflected points C and D in a similar fashion, and connected all four points (A, B, C, D) to create the reflection of the circle. Emily further explained that reflecting more points on the circle would help her to draw a more accurate circle. In the following excerpt, you can see how Emily thought about this task when asked to explain her approach.

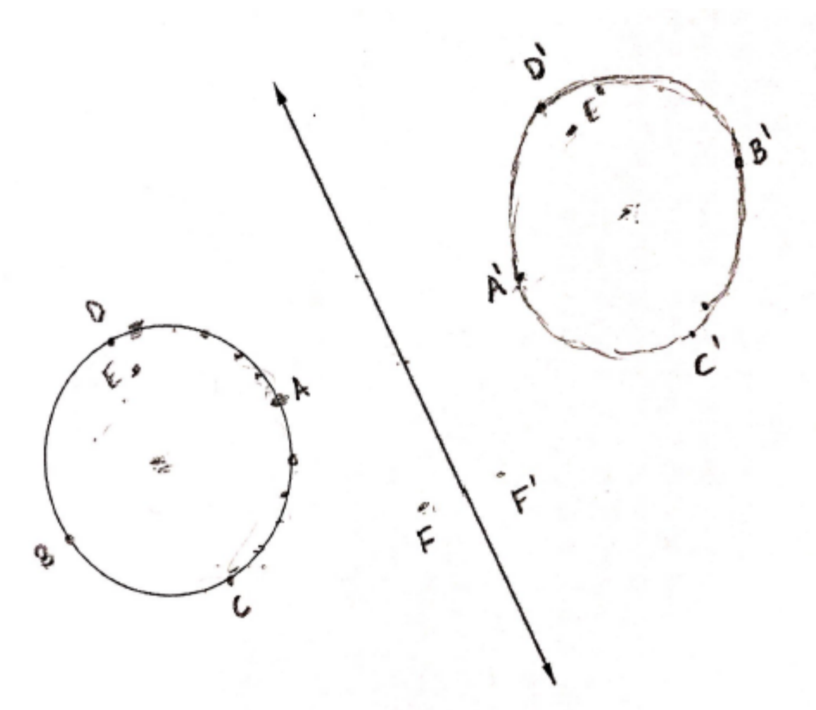


Figure 34. Emily’s drawing of a reflection on a circle task.

> R: What points did you reflect?
>
> E: All the points in the circle and on the circle here [referring to perimeter of the pre-image figure].
>
> R: All points in the circle, you stated. Why do you think that all the points inside circle reflected, as well?
>
> E: I was talking about the center. If you were to put the center over here [referring to the center of the pre-image figure], then it should be the same distance from this [referring to the reflection line]. This distance from the center here [referring to the center point of the pre-image figure to the reflection line] should be the same distance as this [referring to the reflection line] to this [referring to the center point of the image figure], so I think it [referring to the pre-image center point] is reflected, too.
>
> R: Can you say then what points did you reflect?
>
> E: All the points on the circle here [referring to perimeter of the pre-image figure]. There is lots of different points. Then, just the points on the inside of the circle, too, because it is in a plane, so there is points here [referring to all points inside the pre-image figure] but we are just not seeing them. Whatever this point is [referring to point E] is the same as whatever it is, over here [referring to point E`].
>
> R: Are there any other points being reflected besides on the circle or inside the circle?
>
> E: I have said no for all the other ones [referring to exterior points of the figure], but I guess if you think about whatever point is here [referring to point F], it is over there [referring to point F`], too, I guess? I do not know. But we are just looking at this [referring to pre-image circle figure], here, so I do not think we really care about other points [referring to exterior points]. (Line 91-113; 09/26/2017, Interview 2).

From this we see that Emily is aware that when a figure is visible on a plane that the points on the perimeter and interior of the figure are contained within the plane. In addition she is aware that points that are not explicit, such as the interior points of a figure, are also in the plane; however she does not consider the exterior (additionally not explicit) points that are also in the plane. So, Emily's performance on geometric reflections now include interior points of the figure, at least when the figure is a circle. I inferred from her explanations that she knew that there are infinitely many points in the plane. When performing a geometric reflection, she started to use her operational definition of the plane for reflecting interior points rather than all points in the entire plane. In her final statement she explained that there are many exterior points outside of the figure, although she would not consider reflecting them. This is because she thinks that exterior points are not to be considered (or relevant) for performing geometric reflection. Reflecting a circle seemed to encourage Emily to think about the interior points of the circle by reflecting the center point of the circle. In response to the question, "is there any point outside the figure reflected?" Emily indicated "no." This is still evidence of an action structure of the domain for geometric reflection since she reflected all of the points on the figure and the interior of the figure rather than all points in the entire plane. She did not show that she reflected all of the exterior points of the figure, even though she did explicitly refer to the possibility of considering these points when performing a geometric reflection.

In hopes of provoking a consideration of the exterior points, I posed a task using an open figure (see Figure 35). She first reflected point A, then she then drew the diameter of the "semicircle," in order to find the center of the semicircle. This was a similar strategy when compared to the one that she used to perform a geometric reflection with a circle in the previous task. She reflected the center of the semicircle (point M), then measured the radius of the semicircle by using an index card to find the length of the radius (see Figure 36a). She used an index card to measure the length of the radius from M' to position some additional points to draw the semicircle (see Figure 36b). Finally, she reflected points B and C. She had no difficulty reflecting a given figure by using the properties of equidistance and perpendicularity. When Emily completed the reflection, I asked her to explain her approach.

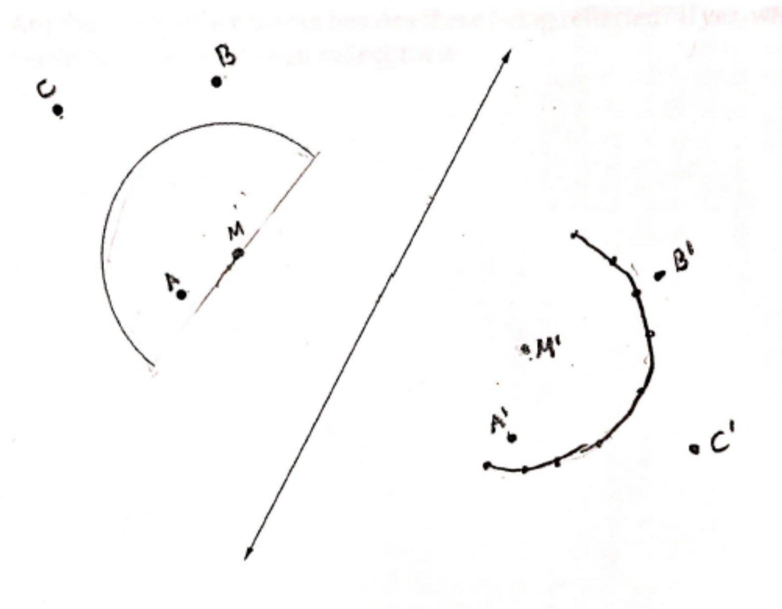


Figure 35. Emily's drawing of a reflection on a semicircle task.

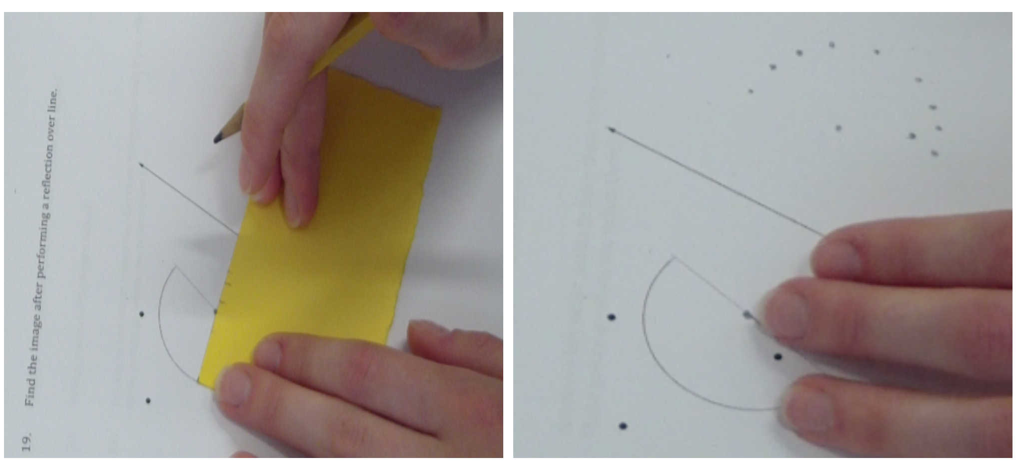


Figure 36 a,b. Emily's drawing of a reflection on a semicircle task.

R: Which points did you reflect?

E: All the points here [referring to perimeter of the circle], and this point [A], and this point [B], and this point [C]. I guess, technically, the ones that we cannot see [referring to all unlabeled points in the pre-image plane], like in the last one [referring to the circle task that she worked on previously] we reflected ... So how I made the circle, I tried to find the center of the circle. We put it here [referring to the image plane], and then measured the radius and put points all along that so I could outline that (see Figure 36b). The center was not part of the picture that you provided [referring to the pre-image figure], but I still could find it, and reflect it over here [referring to the image plane]. But it was not part of the original picture.

R: So, which points did you reflect?

E: I guess all of them [referring to all points in pre-image plane].

R: What do you mean, all of them?

E: All the points. There is lots of other points on the other side of this line [referring to all points in the pre-image plane], but they can all be reflected to the same distance to this side of the line [referring to image plane]. But the only ones that we marked were the ones that were part of the picture [referring to perimeter of the semicircle], because I put the center there [referring to point M`] but then I erased it because I was actually drawing the ones that are in [referring to points A, B, C], like actually marked on this side. (Line 180-201; 09/26/2017, Interview 2).

After the semicircle task, Emily considered more points in the plane because the figure's interior and exterior were not clearly defined for her. When she worked with the circle task, she thought that the center point was part of the circle (and so all interior points were also part of the circle) when performing geometric reflection. When she worked with the semicircle task, she used the center point as a reference point to position the perimeter of the semicircle. However, she noticed that the center point of the semicircle was not part of the figure (not an interior point). Although the center point is not part of the semicircle, she thought that she needed to reflect the center point as well. Then, Emily started to think about all points in the pre-image plane for performing geometric reflection. I infer this to mean that she started to use the definition of the plane for reflecting the perimeter, the interior, and the exterior points of the figure in the half plane. The reflecting a semicircle task seemed to encourage Emily to think about the exterior points of the semicircle in the pre-image plane.

In hopes of provoking a consideration of pre-image and image planes (all points in the entire plane) I posed a task using multiple figures in the plane in interview 3 (see Figure 37). She had no difficulty reflecting the given figures for both planes by using the properties of equidistance and perpendicularity. When asked to explain which points she reflected, Emily stated, “I just reflected all points that is on this side [referring to the pre-image plane], I reflected it over here [referring to the image plane], and then I reflected all points on this side [referring to the image plane], over to this side [referring to the pre-image plane].” (Line 97-98; 09/28/2017, Interview 3). Requesting that Emily reflect several figures located in both planes might encourage her to consider all points in the plane when performing a geometric reflection.

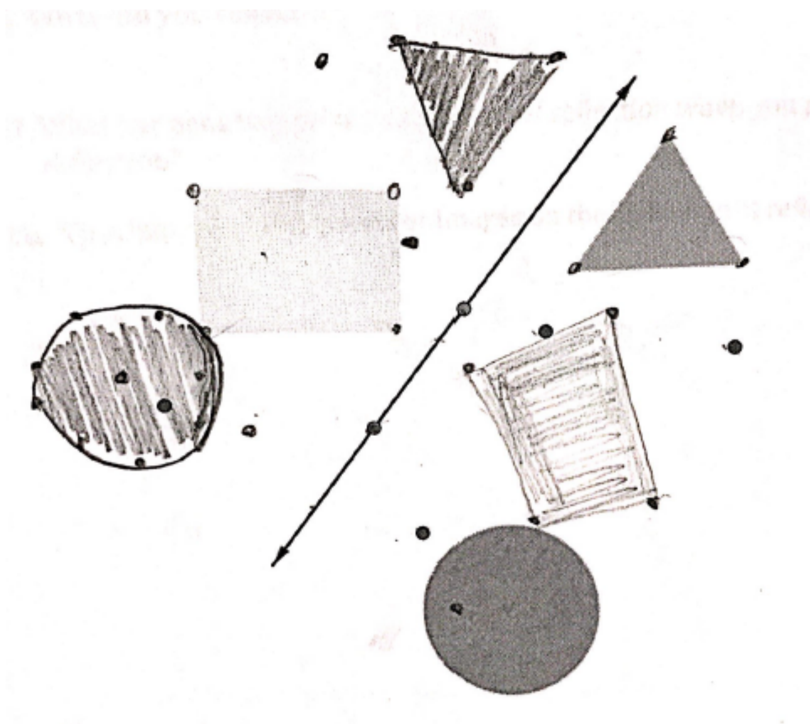


Figure 37. Emily's drawing of a reflection on several figures.

Emily started to think about all points in the entire plane when performing a reflection. Although she did not consider reflecting unlabeled points for geometric reflections physically, she understood that performing a reflection reflects labeled and unlabeled points (which means the all points were in the plane). The tasks with the circle, and semicircle figures helped Emily to consider "interior" and "exterior" points in the pre-image plane when performing a geometric reflection. Providing multiple figures in both half planes helped her to consider all points in the entire plane. Emily's operational definition of the plane in geometric reflections aligned with her mathematical definition of a plane. I inferred her explanations to mean that she had reached the process structure for the concept of the domain in geometric reflection (see Figure 38).

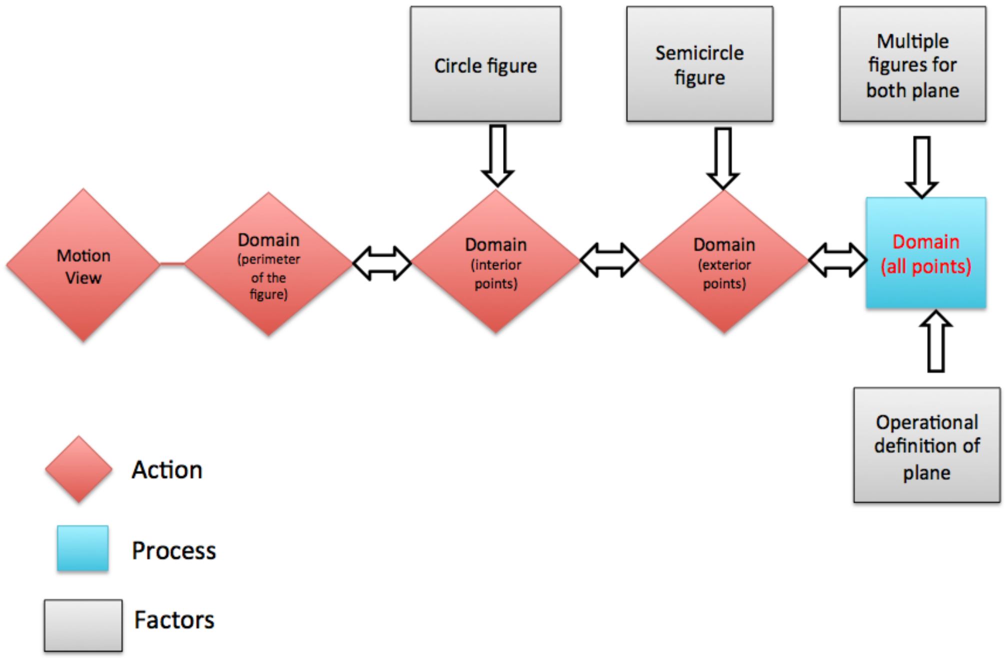


Figure 38. Emily's mental structures of domain

As I hypothesized in my PGD, PTs begin with a motion view of domain as an action structure to develop a mental structure for domain as a process structure. During the first interview, Emily considered the domain as the perimeter of the given figure when she performed a geometric reflection. This implies that she had an action structure of domain in geometric reflection. In the second interview, when she worked on the circle and semicircle tasks, she began to consider all interior and exterior points of the figure. Working with circle and semicircle task helped Emily to consider more points in the plane. Emily still had an action structure of domain in geometric reflection because she considered all points in the pre-image plane (half of the plane), rather than all points in the entire plane. During the third interview, when she worked with multiple figures in both the pre-image and image planes, Emily started thinking about the image plane as well as the pre-image plane. She started to realize that the plane has an infinite number of unlabeled points, and therefore considered that all points in the plane are used for performing geometric reflection. Her operational definition of the plane in geometric reflection aligned with their mathematical definition of a plane. The tasks with circles, semicircles, and multiple figures for both the pre-image and image planes helped Emily to consider all points in the entire plane.

Emily's Understanding of the Plane

To examine Emily's mental structures in terms of the concept of the plane, I analyzed her reactions and explanations on tasks in which she performed geometric reflection. It was unclear from the first and second interviews how she thought about the relationship between points or figures and the plane. The third interviews revealed that Emily had a process structure of plane. She consistently used the word "reflected" during the all three interviews to explain how she reflected the points for performing geometric reflection. In the third interview, I specifically asked, when you perform a reflection, is there any movement of points or figures from half of the plane to another half of the plane to learn how she was thinking of the relationship between the points or figures and the plane. The third interview revealed that Emily had a process structure of plane for geometric reflection, because she thought of the points or figures as a subset of the plane rather than "separated from the plane" (Yanik & Flores, p. 55). For instance, I asked her to reflect a shaded trapezoid located in the pre-image half plane (see Figure 39). She had no difficulty reflecting the given figure. I then asked her, is this a reflection, and if so, why? She elaborated her ideas in the following conversation:

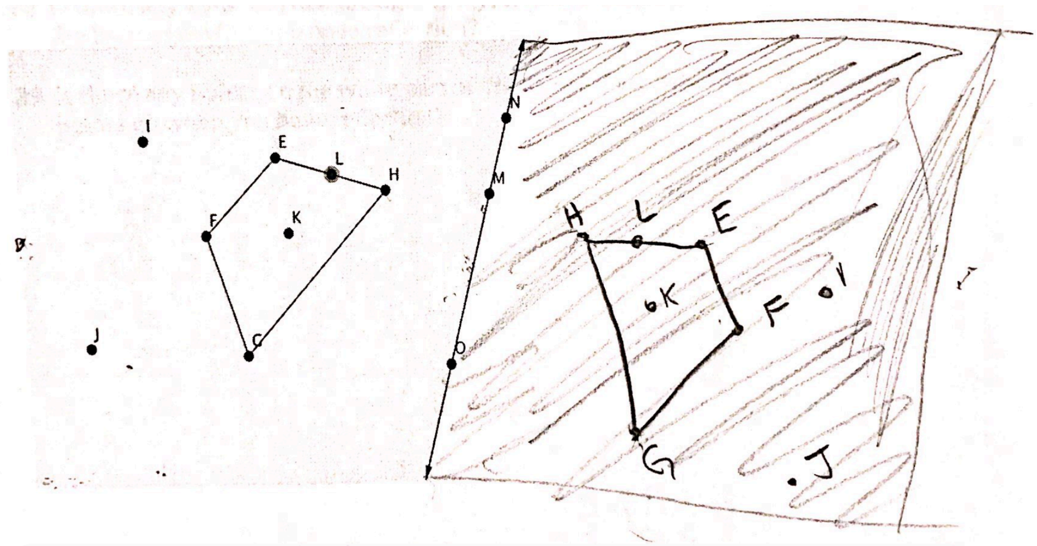


Figure 39. Emily's drawing of a reflection on oblique reflection line

> E: Yeah. So I just reflected all the points. I just made the points be the same distance from one line of reflection on the opposite side of the line.
>
> R: Is there any point inside of figure?
>
> E: So there are lots of points here.
>
> R: Where did those points go, when you performed a reflection?
>
> E: So they just ... I mean, we are just marking the points on this side [referring to the image plane] that are the same distance away from the line [referring to the reflection line].
>
> R: Do you move the points?
>
> E: No. So these [referring to all points in the pre-image plane] are staying where they are. But these [referring to all points in the image plane] are ... I am just marking ... like, there is other points over here [referring to unlabeled points in the image plane] and we are just marking the ones that are on the opposite side of the reflection line that are the same distance but in the opposite direction. (Line 160-167; 09/28/2017, Interview 3).

I interpreted from her explanation that Emily considered the points or figures to be part of the plane rather than "separate from the plane" (Yanik & Flores, p. 55). This is because her explanation indicated that she mapped each point in the pre-image plane to the image plane rather than moving the points or figures. This is because Emily knew the operational definition of the plane for geometric reflection. Since she knows the relationship between points or figures and the plane, Emily seemed to have a process structure of the plane (see Figure 40).

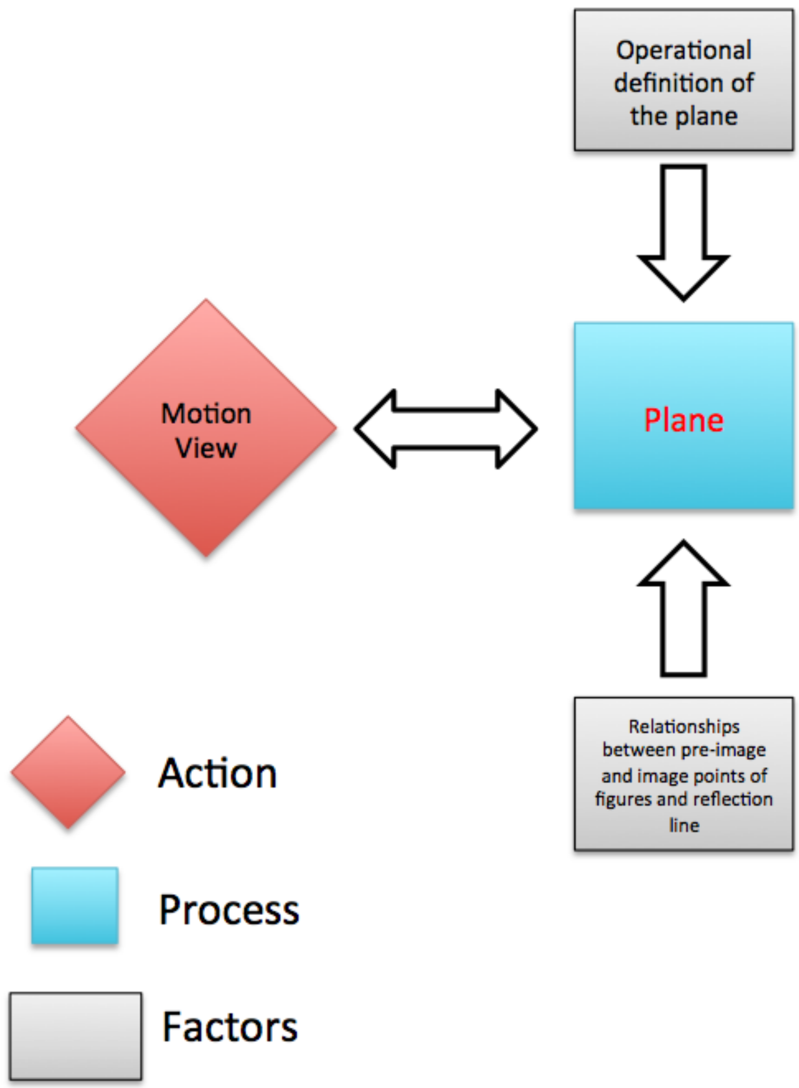


Figure 40. Emily's mental structures of plane

As previously mentioned in my PGD, I hypothesized that PTs begin with a motion view of plane as an action structure to develop a mental structure for plane as a process structure. Figure 40 shows Emily's mental structures of plane throughout the third interview. Emily's model revealed that she had the process structure of plane at the end of the third interview. Her explanations revealed that she had an operational definition of plane and the relationship between the points of the figure and the plane.

Emily's Mental Structures of Mapping View

To summarize, Emily reached the process structure in terms of the reflection line, domain, and plane for geometric reflections. The findings revealed that her understanding of the geometric reflection include a reflection line. She also knows the role of the reflection line, which offers a reference marker for the mapping points of the figures from the pre-image plane to the image plane by using the properties of equidistance and perpendicularity. The tasks with the circle and semicircle figures encouraged Emily to consider the interior and exterior points in the pre-image plane for performing geometric reflection. Providing multiple figures on both half-planes also supported her to consider all points in the entire plane. Emily's operational definition of the plane with respect to geometric reflection aligned with her mathematical definition of the plane. Therefore, she knew that the points or figures are a subset of the plane. Understanding this relation helped Emily to reach a mapping view of geometric reflections (see Figure 41).

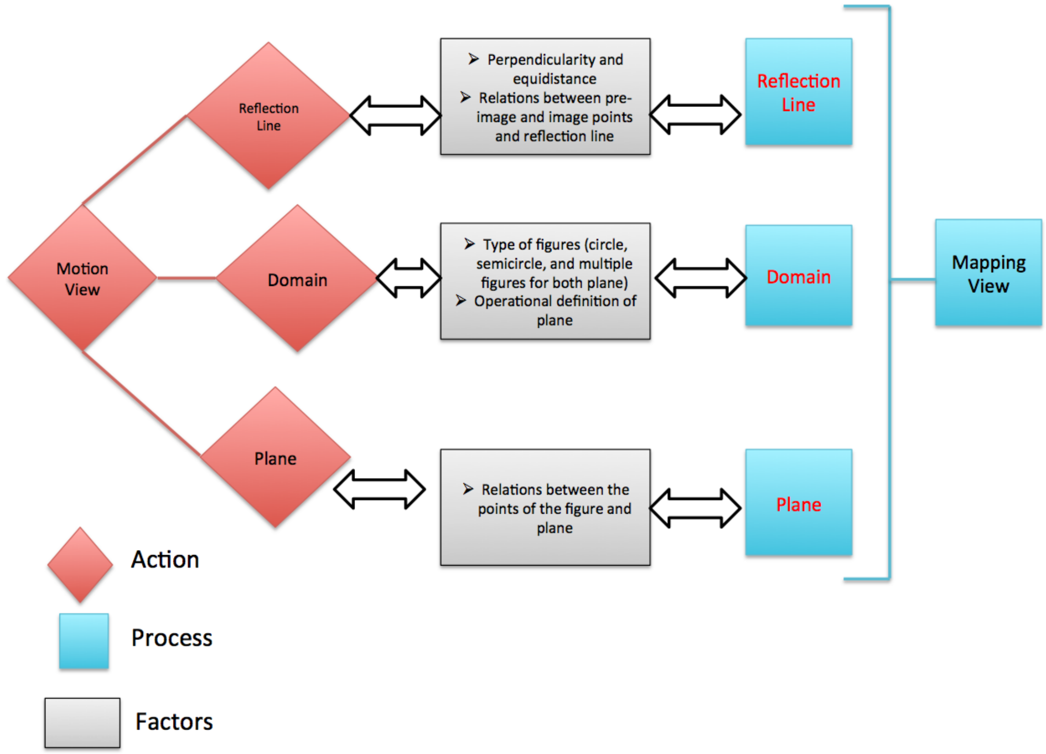


Figure 41. Emily's mental structures of mapping view

Michael's Understanding of Geometric Reflection

Introduction

Michael (pseudonym) was a 25-year-old student, majoring in mathematics education. He had taken one geometry course one in high school and one in the university, both mostly focused on the structure of the proofs. He had also taken several theoretical mathematics courses (e.g., calculus, linear algebra, abstract algebra, geometry, and differential equations), and in general, he described himself as an average math student; he had received high Bs in most mathematics and mathematics education courses and low As in a few.

According to Michael, his background experiences with mathematics involved learning the basics on which to build further knowledge instead of memorizing formulas for solving problems. He described mathematical understanding as "you can have more in-depth, immediate knowledge of it, and you can have mastery of it" (Line 194-195; 09/14/2017, Initial Interview). He explained that a good instructional strategy is to teach a topic to someone in a way so that he/she can understand well enough to achieve mastery of it.

Michael had last briefly worked on geometric reflection at the university. He also said he had learned much more deeply in his high school geometry course. Michael described geometric reflection as "you pick a line and then each point of the original image has to be the same distance from the line [referring to reflection line] on the opposite side of the line." (Line 325-327; 09/14/2017, Initial Interview). Interestingly, he further stated that not only each point of the figure but also everything inside the figure is reflected (see Figure 42). He explained his reasoning as follows:

> The idea of just the edges or just the vertices transformed and then you redraw the thing. That is how it was taught to me. When in actuality, that is the easy way to draw it. That is not actually what is happening. What is happening in there is that every point, not just the vertices, but the point here [referring to a point on the segment of the triangle], [a] point here [referring to another segment of the triangle], the continuous line [referring to a segment of the triangle] and the continuous shape [referring to whole triangle], that is the triangle that you end up with [that] is transformed over. It is just when you are drawing it to students; it is just easiest to say, "All right, you know it is a triangle. Here are the three points. Move the three points over and the redraw the triangle." That is how it was taught to me. (Line 339-348; 09/14/2017, Initial Interview).

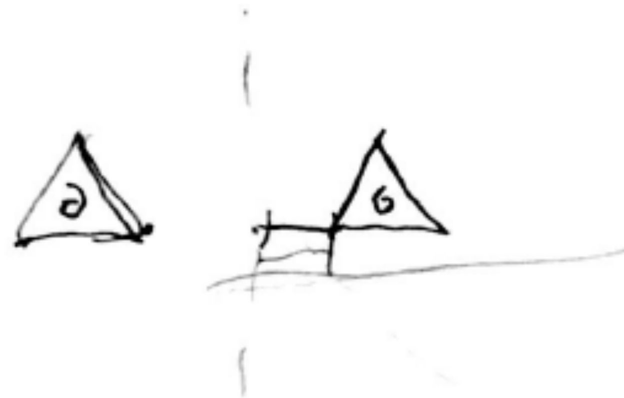


Figure 42. Michael's drawing of a reflection.

Michael described the role of the reflection line as "It is just something that you are reflecting over" (Line 360-361; 09/14/2017, Initial Interview). He also defined a plane as "infinitely many lines next to each other" (Line 383; 09/14/2017, Initial Interview). When he was asked about the properties of geometric reflections, he stated that each corresponding pair of pre-image and image points should be at equal distance from the reflection line, and pre-image and image figures should be congruent (all the side-lengths will be same length, all the vertices have the same angle, and the area is the same).

Michael's Understanding of Reflection Line

I administered several tasks to examine Michael's mental structures in terms of the concept of the reflection line and, based on two tasks from the first interview. I inferred that Michael had action and process structures of the reflection line. He was given a task with two figures without a reflection line and asked: "Is this reflection? Why or why not?" Michael stated that it did not represent a reflection " because I do not think that there would be any line you could draw to reflect it over and get this image [referring to pre-image figure]" (Line 17-18; 09/19/2017, Interview 1). Next, he drew a reflection line between the given two figures and labeled pre-image vertices (A, B, C) (see Figure 43). Then, he reflected vertices A, B, C over the line of reflection by using equidistance property. Finally, he concluded that pre-image vertices did not match the image figure vertices.

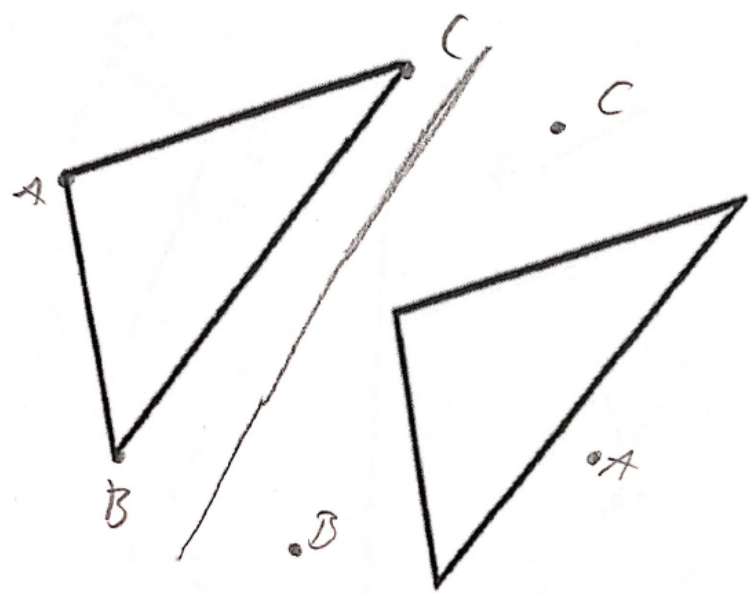


Figure 43. Michael's drawing of reflection line between given figures.

His drawing of a reflection line suggested his understanding that a geometric reflection must involve a reflection line and was evidence that he had an action structure for the reflection line.

Later in same interview, Michael reflected a triangle over an oblique reflection line (see Figure 44). First, he selected the triangle's closest point (point A) to the reflection line, and drew perpendicular line from point A to the reflection line. Then, he reflected point A over the reflection line estimating the same distance between A and A` by using the equidistance property. He did the same process for the remaining vertices (point B and point C) and connected all three vertices to make a triangle.

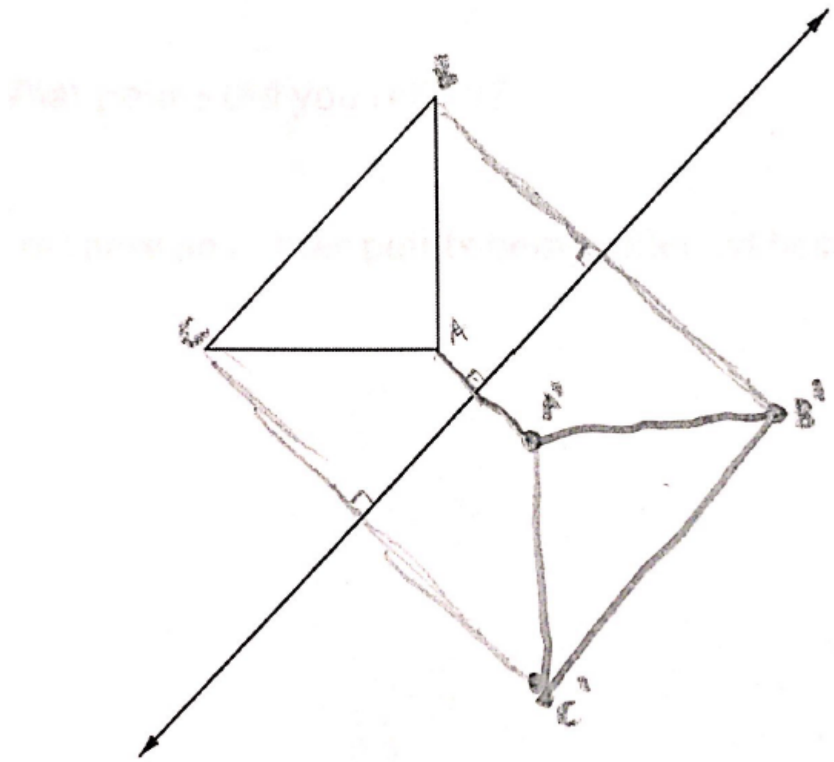


Figure 44. Michael's drawing of a reflection on an oblique reflection line.

I inferred from Michael's performance of the triangle task that his understanding of the reflection line included a role in mapping points on a figure across the reflection line. He knew the relationships between pre-image and image points of the figures and the reflection line using the geometric reflection properties of equidistance and perpendicularity. His action of mapping the points and use of the properties of equidistance and perpendicularity are evidence that he had a process structure of the reflection line. I interpreted his drawing as meaning that Michael had an accurate understanding of the reflection line. At the end of the first interview, he described geometric reflection as “A reflection would be the drawing of each significant point at equal distance perpendicular to the line of reflection.” (Line 304-305; 09/19/2017, Interview 1). Based on this definition, Michael started to describe geometric reflection by referencing the role of the reflection line and the properties of equidistance, and perpendicularity. Compared to the definition of geometric reflections he gave in the initial interview, he revealed additional mental structures by using the perpendicularity property of geometric reflections in the first interview tasks. Hence, the analysis of the first interview indicated that Michael had action and process structures of the reflection line (see Figure 45). The role of the reflection line has significant sub-concepts for the concept of geometric reflection that are essential to move from a motion view to a mapping view. In the following figure, I describe how he had an action and process structure of the reflection line by unpacking his mental structures through the lens of APOS theory.

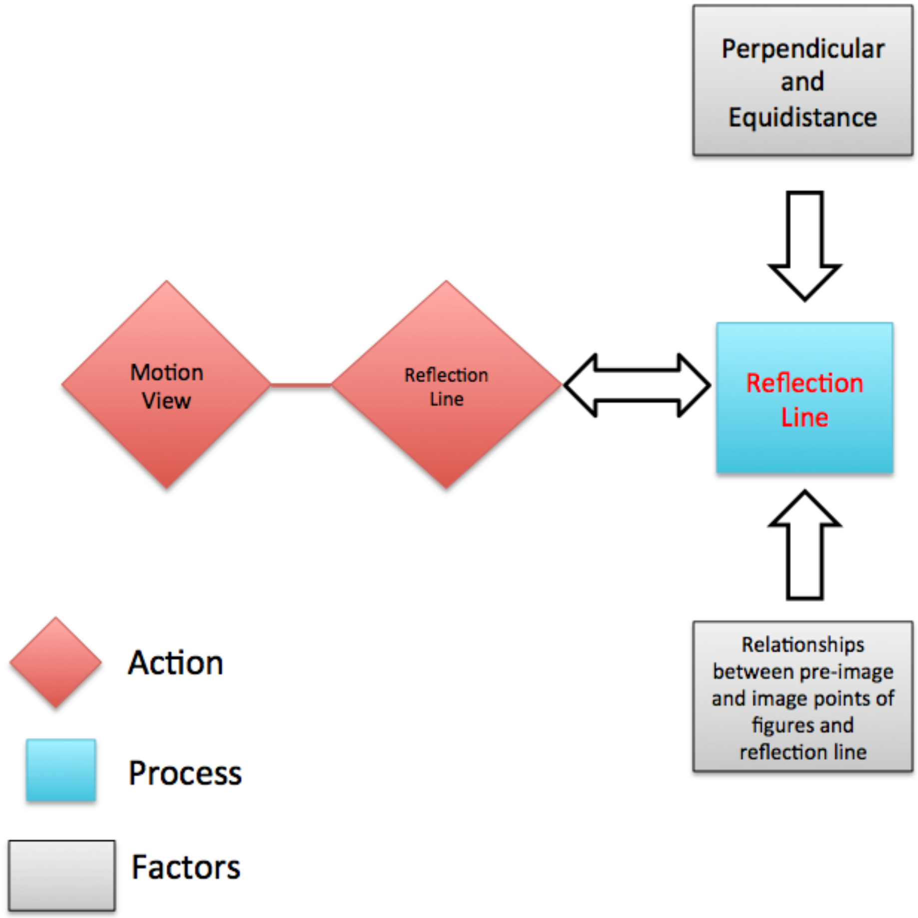


Figure 45. Michael's mental structures of reflection line

I hypothesized in my PGD that PTs begin with a motion view of reflection line as an action structure to develop a mental structure for reflection line as a process structure. Michael with an action structure knew that the reflection line defines the geometric reflection. To have a process structure of the reflection line, PTs need to know two critical factors involved in the role of the reflection line and the relationship between the pre-image and image points and the reflection line. The process structure of reflection line involved action structure of reflection line as well. During the first interviews, Michael had mental structures of the properties of equidistance and perpendicularity, and the role of reflection line. Hence, Michael had an action and process structure of reflection line in geometric reflections.

Michael's Understanding of Domain

Several tasks were used to examine Michael's mental structures in terms of the concept of domain. I interpreted that Michael had and action and a process structure of domain with respect to geometric reflections. My analysis of Michael's understanding of the notion of domain is based on three tasks from the first, second and third interviews. In the first interview, he was given a triangle with an oblique reflection line and was asked to "find the figure after performing a reflection across the line" (see Figure 46). Michael reflected three vertices of the triangle by measuring the distance of each vertex to the reflection line using an index card. He then connected the vertices in the image to make the triangle. When I asked him to explain what points he reflected, he elaborated on his ideas in the following conversation:

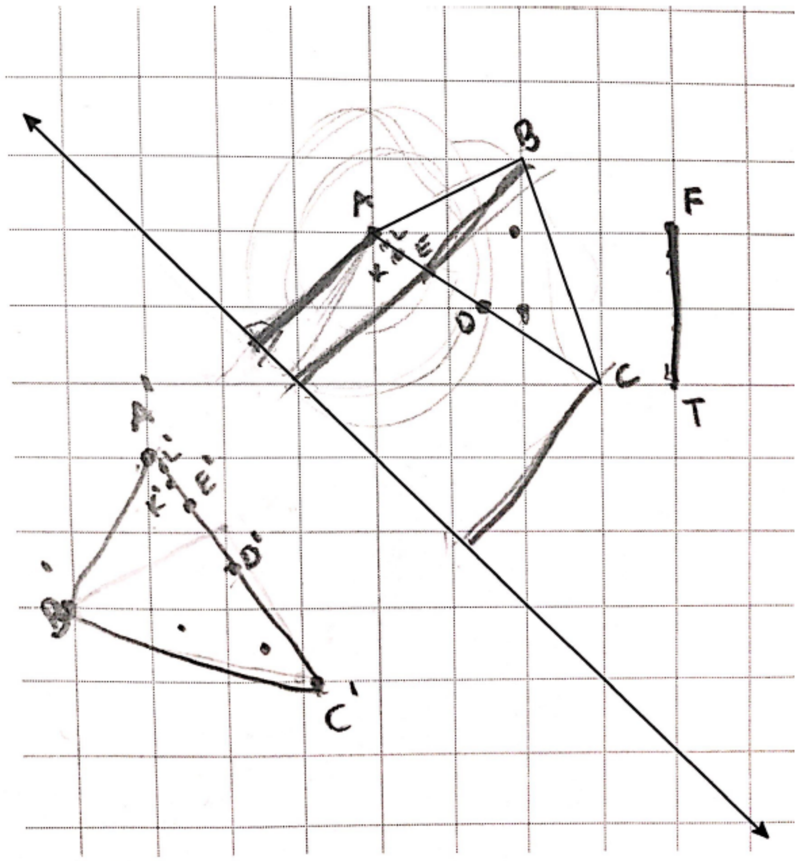


Figure 46. Michael's drawing of a reflection on an oblique line.

M: I reflected the vertices.

R: Okay. Are there any other points being reflected besides vertices?

M: So the way I drew it, no. But in actuality, there are infinitely many points being reflected.

R: Where are those infinitely many points? Can you show it to me? What do you mean infinitely many points?

M: Sure. So say I can draw the, say, this is the medium of this [referring to point D]. So this point is being reflected as well [referring to point D], the meeting of that is being reflected as well [referring to point E], meeting of that is being reflected [referring to point K], meeting of that is being reflected [referring to point L], and the same on the other side [referring to other segments of the triangle]. Same on all these sides [referring to AB, and BC segment]. So even if the shape is hollow [referring to inside the triangle], there is not an area in-between it, it is just three lines, there would still be infinitely many points. I am reflecting this line [referring to

> FT line], ... here [referring to pre-image plane] to here [referring to image plane] it would still be infinitely many points. (Line 138-152; 09/19/2017, Interview 1).

Michael's explanation and drawing demonstrate that although he did not consider reflecting unlabeled points in the pre-image plane physically, he knew that performing a reflection affects both labeled and unlabeled points, which means that all of the points were in the pre-image plane. Michael still had an action structure of the domain because he considered all points in the pre-image plane rather than all points in the entire plane. From his explanations, I inferred that Michael's operational definition of the plane with regard to geometric reflections aligned with his mathematical definition of a plane.

Later in the second interview, Michael was given a semicircle with an oblique reflection line and asked to perform a geometric reflection (see Figure 47). He had no difficulty reflecting the semicircle. I asked him what points were being reflected. He said that "All of the plane" [referring to all points in the pre-image plane]. Then, I asked him, what did you reflect for this plane [referring to image plane]? He said that "I sort of think of them as being overwritten. I suppose B over here, or the B that was over here is replaced by a non-empty point, B prime." (Line 166-167; 09/29/2017, Interview 1) (see Figure 47). I interpreted his explanations to mean that he still considered all points in the pre-image plane rather than all points in entire plane for performing geometric reflection. His explanations revealed that he reflects point B to point B` but he did not consider reflecting point B` to B point. While he knows that pre-image points map to the image, he sees the half planes as somehow separate. This is still evidence of action structure of domain.

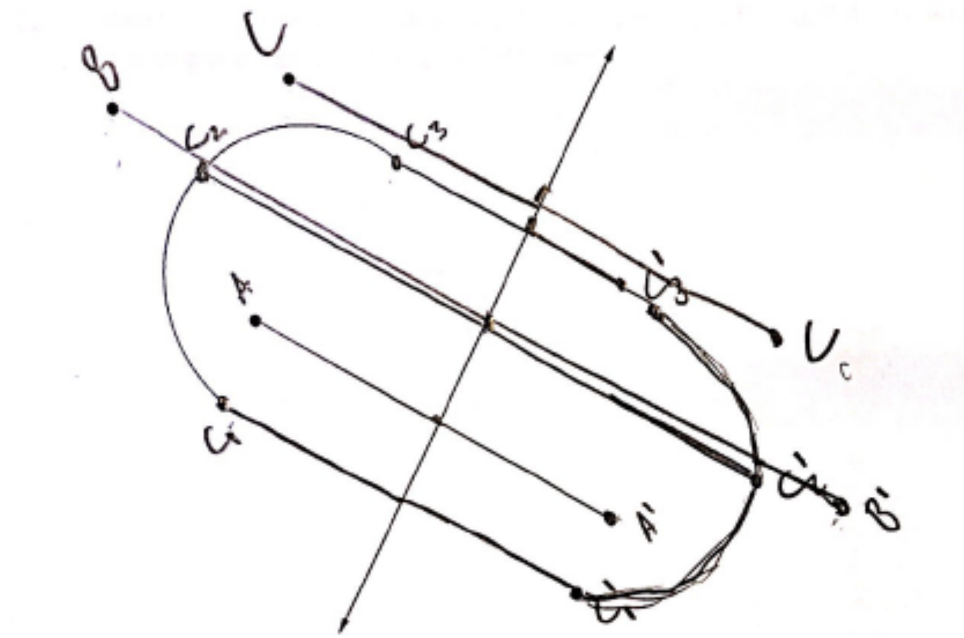


Figure 47.Michael’s drawing of a reflection on an oblique line.

In hopes of provoking a consideration of the all points in pre-image and image plane (i.e., all points in the entire plane), I posed a task using multiple figures for both plane to perform geometric reflection in the third interview (see Figure 48). He had no difficulty reflecting given multiple figures for both plane. Then, I asked him to explain what points he reflected. Michael stated, “I just reflected all points that is on this side [referring to pre-image plane], I reflected it over here [referring to image plane], and then I reflected all points on this side [referring to image plane], over to this side [referring to pre-image plane].” (Line 97-98; 09/28/2017, Interview 3). Asking several figures for both planes might encourage Michael to consider all points in the entire plane to perform a geometric reflection. I asked him to explain how he thought. He elaborated:

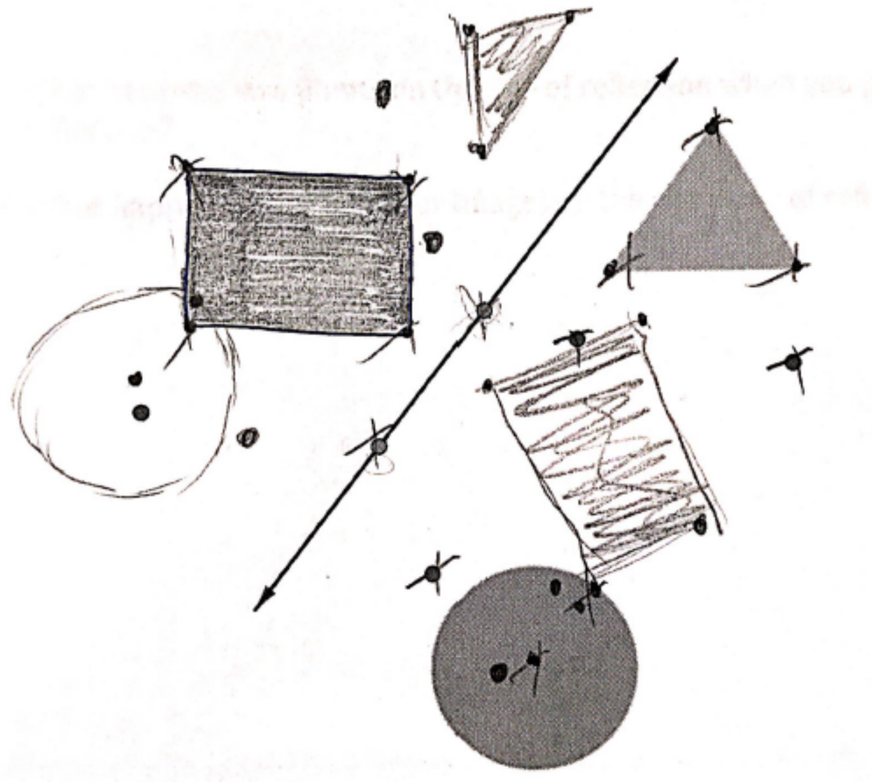


Figure 48. Michael's drawing of a reflection on an oblique line.

> M: So in previous images what I have been reflecting was just one side onto the other but ... what is really happening is two sides are switching it is just the other side was empty at the time and so it is just easier to think of it as flipping over. So I kind of see why in the beginning when I was using the analogy of it flipping onto here, it seemed like you are kind of leading me away from that and that is a good way of thinking of it because it is not actually flipping over here it is kind of rotating, I am going to access the whole plane is reflected.
>
> R: okay, what did you reflect?
>
> M: Okay so what I reflected was all points on the left side to the right side and all points on the right side to the left side (Line 137-146; 09/28/2017, Interview 3).

During the first and second interviews, Michael consistently succeeded in performing geometric reflection when posed with a variety of tasks featuring varying figures (e.g., circle, triangle, trapezoid). Initially he thought of these reflections as reflecting half of the plane onto the other half of the plane. In the third interview, when he was tasked with performing geometric reflections with multiple figures located on both half-planes, Michael started to consider all points in the plane when performing a geometric reflection. The task with multiple figures on both half-planes might have encouraged Michael to consider all points in the plane as the pre-image. Also, his operational definition of the plane with respect to the idea of geometric reflection aligned with his mathematical definition of a plane. I inferred his explanations to mean that he had reached the process structure for the concept of domain in regard to geometric reflections (see Figure 49).

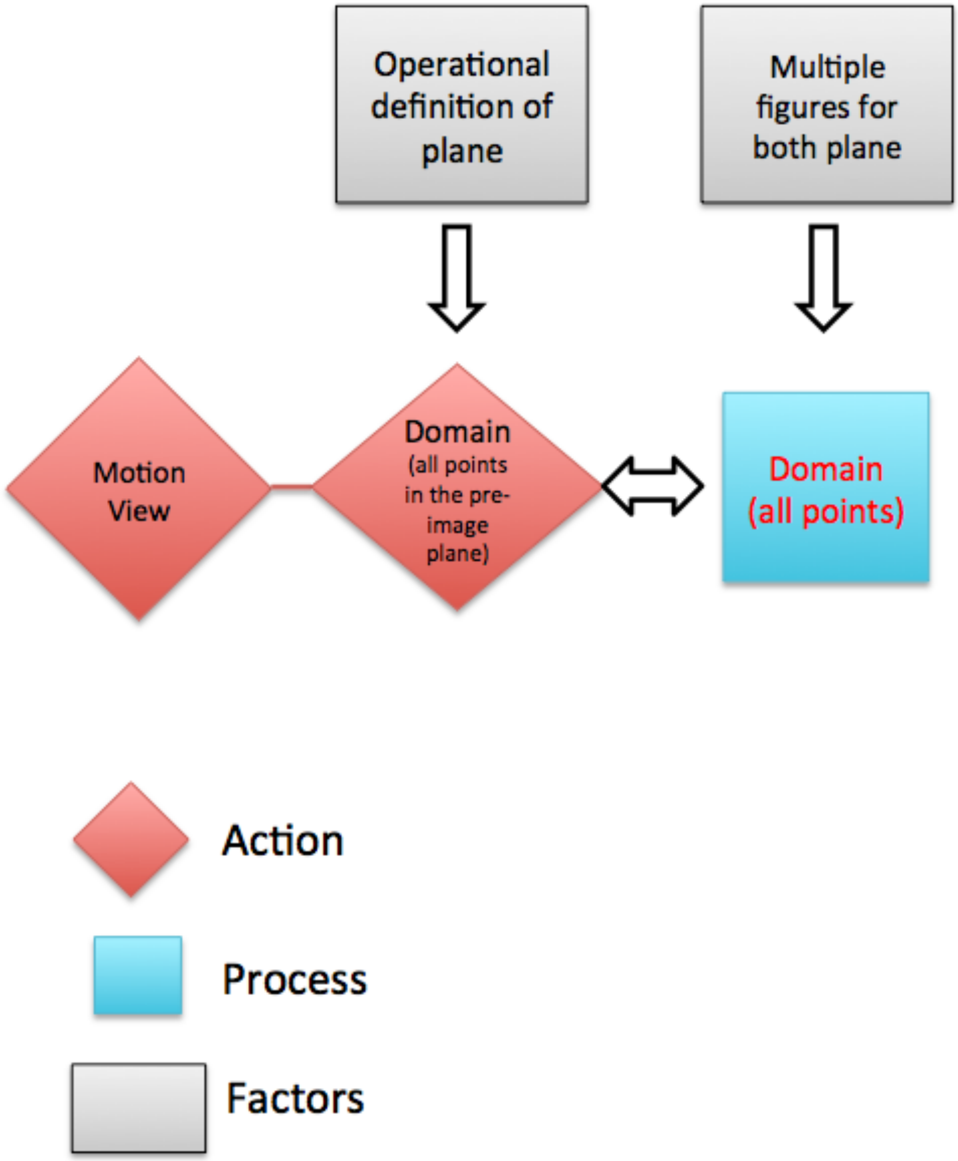


Figure 49. Michael's mental structures of domain

As I hypothesized in my PGD, PTs begin with a motion view of domain as an action structure to develop a mental structure for domain as a process structure. Figure 49 shows Michael's development of mental structures of domain throughout first, second, and third interviews. Michael's model revealed that he reached the process structure for domain at the end of the third interview. During the first and second interviews, Michael considered all points in the pre-image plane for performing geometric reflection. His operational definition of the plane might encourage him to think about all points in the pre-image plane. During the third interview, when he worked with multiple figures on both planes, Michael started to consider all points in the plane. Hence, Michael reached the process structure of domain in geometric reflection.

Michael's Understanding of the Plane

To examine Michael's mental structures in terms of the concept of the plane, I analyzed his reactions and explanations on tasks when performing geometric reflection. The first interview revealed that Michael seemed to have an action structure of the plane for geometric reflections because when he performed a geometric reflection, he thought of the points as "separate from the plane" (Yanik & Flores, p. 55). For instance, when he was asked to explain how he positioned the triangle after he reflected it over the reflection line, he stated: "I picked this point [referring a vertex of the triangle] and move it to here [referring to the image plane] using equidistance property" (Line 98-99; 09/19/2017, Interview 1). Then, I asked him, how and where did you learn this information? He explained, "I picked up at my geometry class here, at university. The ideas that there are infinitely many points being moved over, it is just thinking about it; introspection" (Line 101-103; 09/19/2017, Interview 1). I inferred from this explanation that he thinks of points as "separated from the plane" (Yanik & Flores, p. 55), rather than a subset of the plane, because he used "move" as the word to describe how he reflected the points of the figure from the pre-image plane to the image plane. He might have difficulty understanding the relationship between points or figures and the plane. He also knew that the plane has infinitely many points but he could not use his operational definition of the plane in geometric reflections.

During the second and third interviews, Michael consistently used "move" as a word to describe his actions when he performed geometric reflections. To understand his reasoning in the third interview, I asked him, "when you perform a reflection, is there any movement of points or figures from half of the plane to other half of the plane?" He elaborated his reasoning in the following conversation:

> M: No I guess they are not moving; you are kind of creating a new point on the opposite side.
>
> R: Okay why do you think like that?
>
> M: because you are performing a transformation, you are kind of fundamentally changing the points. Really, the only thing that defines a point is its location and so if it changes location it is kind of fundamentally a different point. So I guess in mathematics you cannot actually ever move a point it would just be your creating a new point. (Line 32-38; 10/04/2017, Interview 3).

I interpreted from this explanation that Michael considered the points of figures to be part of the plane because when he performed a geometric reflection, Michael thought there was no movement of points from the pre-image plane to the image plane. Then, I asked him to explain why he used "move" as a word to describe the geometric reflections. He elaborated:

> So it is just kind of the tactile way that we grow up and stuff it is we are much more familiar as people with physical 3D objects and the concept of moving and stuff like that. So some things that exist in the mathematic universe like creation of a new point, does not really track well with real life so it is easy to kind of make the mistake of putting, or saying that something is similar to real life is happening here even though what we are doing here is something that we cannot actually do perfectly in real life. So what I would do in real life I would rotate this or move it like move the physical points but what I am actually doing mathematically on here is, I am not actually moving them I am creating a new image. (Line 176-183; 10/04/2017, Interview 3).

Michael's explanation demonstrated that his mathematical understanding of the relationship between the figures or points and the plane was accurate; that is he knew that the points (and hence the figure) were embedded in the plane. Further, in practice he did understand that the reflection generated a new image, although he had difficulty explaining this idea. This difficulty arose primarily because Michael saw it as easier to describe his thinking in colloquial terms that are easily relatable to how we operate in the real-world, versus using technical mathematical language to describe his thought process in performing reflections. Hence, he had a process structure of plane in geometric reflection (see Figure 50).

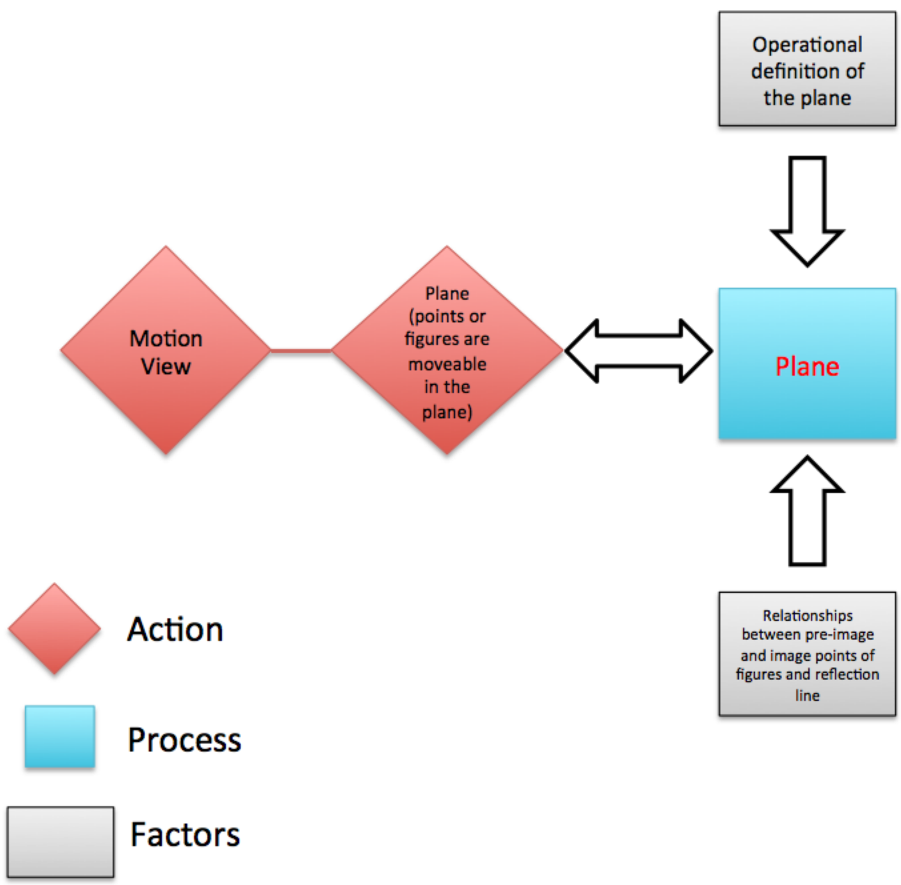


Figure 50. Michael's mental structures of plane

As I hypothesized in my PGD that PTs begin with a motion view plane as an action structure to develop a mental structure for plane as a process structure. Figure 50 shows Michael's mental structures of plane throughout the first, second, and third interviews. Michael's model revealed that he had the process structure of plane at the end of the third interview. During the first and second interviews, Michael consistently used "move" as a way to describe how he performed geometric reflection, I infer that because he thought that when performing a reflection, the points or figures were not a subset of the plane. This is evidence of an action structure of plane; however, when he explained why he used "move" to describe geometric reflection, his explanation demonstrated that his mathematical understanding of the relationship between the figures or points and the plane was accurate (that is he knew that the points, and hence the figure, were embedded in the plane). Hence, Michael had a process structure of plane having operational definition of plane and the relationship between the points of the figure and the plane.

Michael's Mental Structures of Mapping View

Briefly, Michael reached the process structure in terms of the reflection line, domain and plane. The findings revealed that Michael knew that reflection line is necessary and has a role in the mapping of pre-image points to image points using the properties of equidistance and perpendicularity. The tasks with multiple figures for both planes encouraged him to consider all points in the plane for performing geometric reflections. He also knew how to use operational definition of the plane in geometric reflections, which highlights the relations between the points or figures and plane. I interpreted his explanations to mean that he had reached the process structure for the concept of the reflection line, domain and plane for geometric reflection. Therefore, Michael has a mapping view of geometric reflection (see Figure 51).

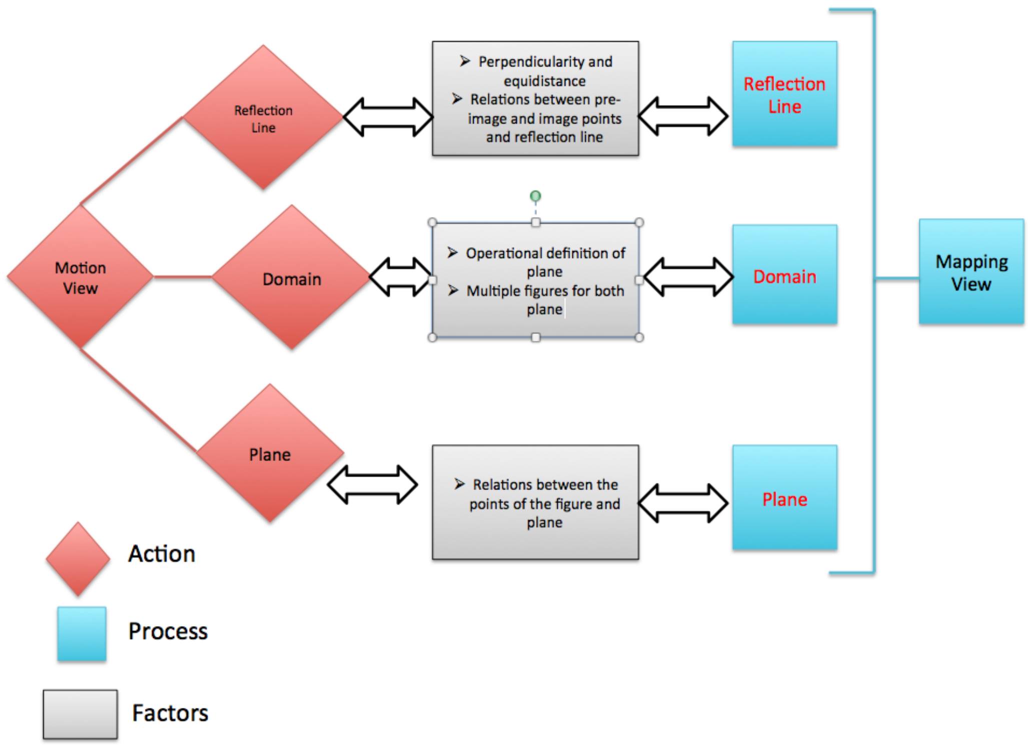


Figure 51. Michael's mental structures of mapping view.

John's Understanding of Geometric Reflections

Introduction

John (a pseudonym) was a 21-year-old undergraduate, majoring in mathematics education. He had taken one geometry course in high school and one in his current program, both of which focused on geometric proofs. He had also taken several theoretical mathematics courses (e.g., abstract algebra, linear algebra, geometry, and Calculus I, II, III). In general, he described himself as an average math student and explained that he was not concerned about his grades but was concerned about having sufficient understanding of mathematics to teach it to others. According to John, although he had lost confidence in his mathematics ability when he failed a test in elementary school, he had reached his present level of knowledge and confidence through hard work and determination to grow as a mathematician.

When asked what it means to understand a mathematical concept, John stated, "it means to be able to provide examples of where it works and what does not work and also to be able to communicate that with mathematics and not just being able to do a problem" (Line 94-96; 09/15/2017, Initial Interview). He explained that solving a problem is not sufficient for him to fully understand the concept involved; he also needs to be able to explain why he is solving it in a particular way. John recalled being exposed to geometric reflection in his high school geometry course, but not in his college course. He described geometric reflections as "if there was a mirror in between the two objects and then just copy it down like that" (Line 139-140; 09/15/2017, Initial Interview) (see Figure 52). He also noted some properties of geometric reflection such as different orientation, equidistance, same shape, same length, same angles, and both shapes are congruent.

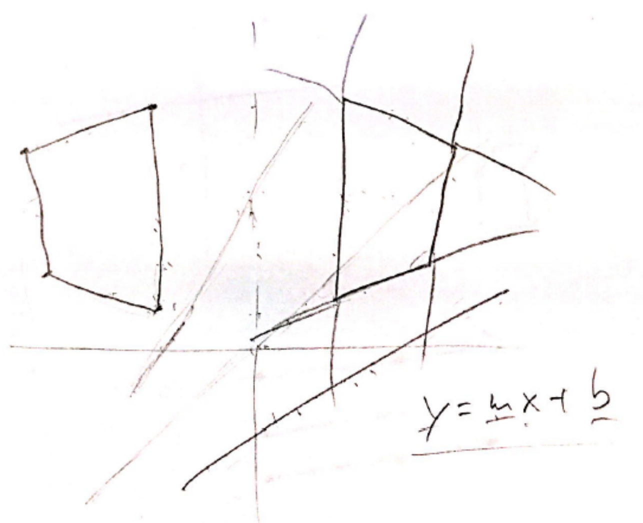


Figure 52. John's drawing of a reflection.

John explained the role of reflection line as "if you take any point or any position on what you are reflecting, it will be the same on both sides of that line [referring to reflection line]. It is our compass." (Line 172-173; 09/15/2017, Initial Interview). From this explanation I inferred that he understood the role of the reflection line in positioning points or figures. John also defined the plane as "a plane has a bunch of lines that are all on the same plane, and then it also has a bunch of points." (Line 208-209; 09/15/2017, Initial Interview).

John's Understanding of Reflection Line

I administered several tasks to investigate John's mental structures about geometric reflections. Evidence from John's performance on two of the tasks was consistent with him having action and process structures of the reflection line. For the first task, he was given two figures without a reflection line and was asked: "Is this a reflection? Why or why not?" (see Figure 53). John said that it did not represent a reflection. He drew several reflection lines between the two figures to examine whether or not they were the same. For instance, he drew an oblique reflection line between two triangles and stated, "these vertices [referring to A and A` points] would have to match" (Line 14-15; 09/18/2017, Interview 1). His action of drawing indicated that the distance from point A to the reflection line and the distance from A` to the reflection line were not the same. His drawing of several reflection lines suggested that his understanding of geometric reflection included a reflection line and that he had an action structure of the reflection line.

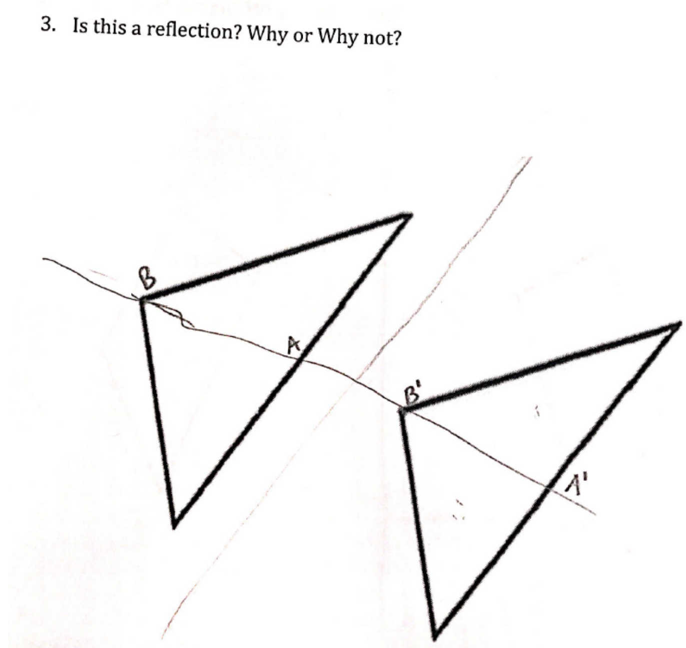


Figure 53. John drawing of a reflection

Later in the same interview, he was given a triangle with an oblique reflection line and was asked to perform a geometric reflection (see Figure 54). John selected three vertices of the triangle and drew three lines perpendicular to the reflection line. He used an index card to measure the distance between pre-image and image vertices (see Figure 55). Then, he connected the three reflected vertices to draw a reflected triangle.

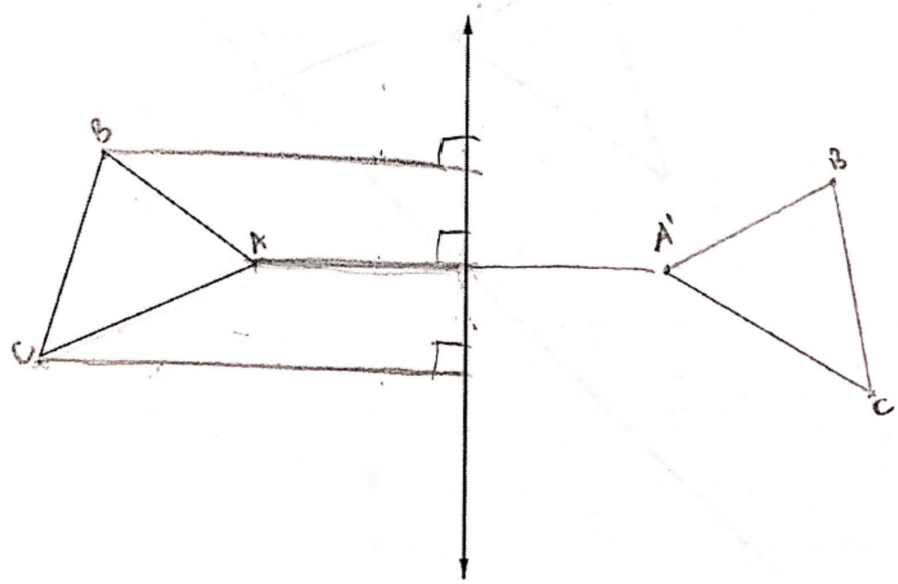


Figure 54. John's drawing of a reflection on a triangle task

To understand his mental structures related to performing geometric reflection, I asked John to explain how he determined where to place the figure. John stated, "I determined where to put this figure [referring to the pre-image figure] by making the vertices equidistance from this line [referring to the reflection line]. Then also every point here [referring to pre-image plane] is the same distance as every point here [referring to image plane]" (Line 224-225; 09/18/2017, Interview 1).

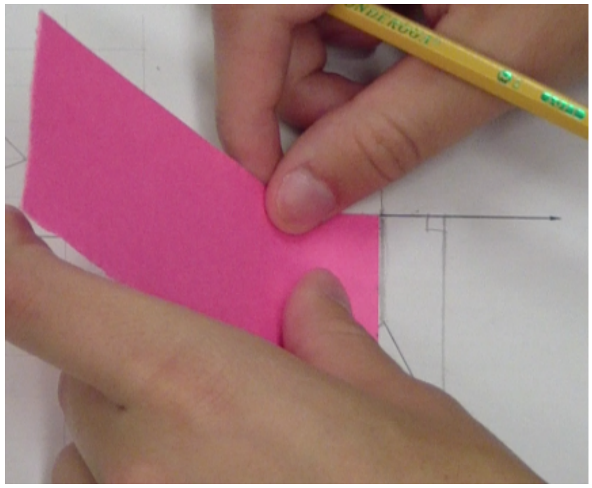


Figure 55. John's use of index card to measure distance

John's actions and explanations were evidence that he had a process structure of the reflection line because his understanding of reflection line included a method to map points, vertices, or sides on a figure across the reflection line. Also, based on his drawing and explanations, I determined that John knew the relationship between pre-image and image points of the figures and reflection line through his use of equidistance and perpendicular properties to perform a geometric reflection. He had progressed beyond just identifying the reflection line as an essential component of a geometric reflection towards the idea that he needed particular points, mapped using the properties of equidistance and perpendicularity.

During the first clinical interview, John performed geometric reflections with vertical, horizontal and oblique reflection lines. He consistently used equidistance and perpendicularity properties when he performed geometric reflection. He knew that a reflection line is a necessary tool to decide where to place the figure. He knew how to perform a reflection by measuring distance between a point of the figure and the reflection line, and he also knew that each point of the figure would be perpendicular to the reflection line. Therefore, my analysis of the first interview demonstrated that John had action and process structures of the reflection line (see Figure 56). The role of the reflection line is an important sub-concept to move from a motion view to a mapping view. In the following figure, I describe how John had an action and process structure of reflection line by unpacking his mental structures the lens of APOS theory.

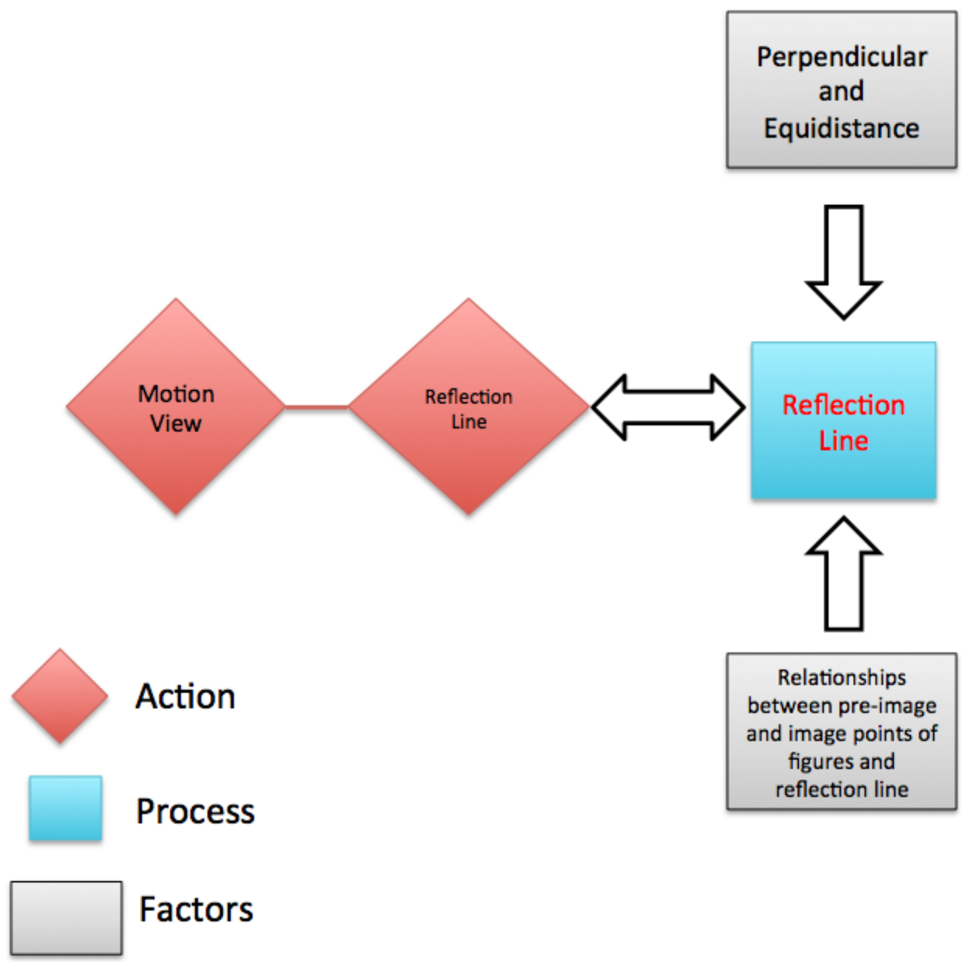


Figure 56. John's mental structures of reflection line

I hypothesized in my PGD that PTs begin with a motion view of the reflection line as an action structure to develop a mental structure for the reflection line as a process structure. Action structure of reflection line means that a reflection line is necessary for performing geometric reflections. To have a process structure of reflection line, PTs need to know two critical factors involved with the role of a reflection line and the relationship between the pre-image and image points and the reflection line. These two factors are sufficient to have a process structure of reflection line in geometric reflections. During the first interview, John consistently used the properties of equidistance and perpendicularity for performing geometric reflection. John also has relations between pre-image and image points and reflection line for performing geometric reflection. Hence, John had an action and process structure of reflection line for geometric reflection.

John's Understanding of Domain

Several tasks were used to examine John's mental structures in terms of the concept of a domain. Evidence from the tasks demonstrated that John had an action and process structure of the domain of geometric reflection. My analysis of his understanding of the domain is based on two tasks from the first and second interviews. In the first interview, he was given a triangle with an oblique reflection line and was asked to "find the figure after performing a reflection across the line" (see Figure57). John selected three vertices of a triangle and made a line perpendicular to the reflection line. Then, he used an index card to measure the distance between each of the vertices to the reflection line in order to position the three vertices of the triangle's reflection. He then connected the vertices of the reflection to make the triangle. He correctly reflected the triangle by using the properties of equidistance and perpendicularity.

When I asked him to explain what points he reflected, he stated, "I only plotted three points (referring to vertices of triangle), and then I made lines (referring to segment of the triangle), but every line (referring to segment of triangle) has an infinite amount of points, so I technically reflected an infinite amount of points." (Line 168-169; 09/18/2017, Interview 1). John explained that he reflected three vertices of the triangle. When I asked him about other points being reflected beside the perimeter of the triangle points, he elaborated his ideas:

> J: We have all the center points [referring to all points inside triangle]. If we take the perpendicular bisectors of all three [referring to three vertices of triangle] and then find where that point is, and then if we take the midpoint bisectors and just, if we find all the points inside, or just a random point like that [referring any point inside triangle], that would be the same distance to this line [referring to reflection line]. I only use three [referring to three vertices of triangle] to do it, but I can find many more that would be a reflection of each other on both. (Line 196-202; 09/18/2017, Interview 1).

John's explanations suggest that he started to think of reflecting points interior to the triangle for performing geometric reflection. Then, I asked, are there any points reflected outside of the figure? He elaborated:

> J: If you gave me the point here [referring a point outside of the figure], its reflection would be right there (see Figure 57). However, all these points [referring to all points in the pre-image plane] ... I mean, well, I guess all these points [referring to all points in the pre-image plane] would be reflected across here [referring to all points image plane]. It is just you do not really visually see that from reflecting across just a triangle. But yeah, there is a infinite amount of points here [referring to all points in the pre-image plane], and there is a infinite amount of points this way [referring to all points image plane], and they all, like this random point right here [referring a point outside of the figure] has a point somewhere over here [referring to image plane]. That is the same distance away from this line [referring to the reflection line]. (Line 210-216; 09/18/2017, Interview 1).

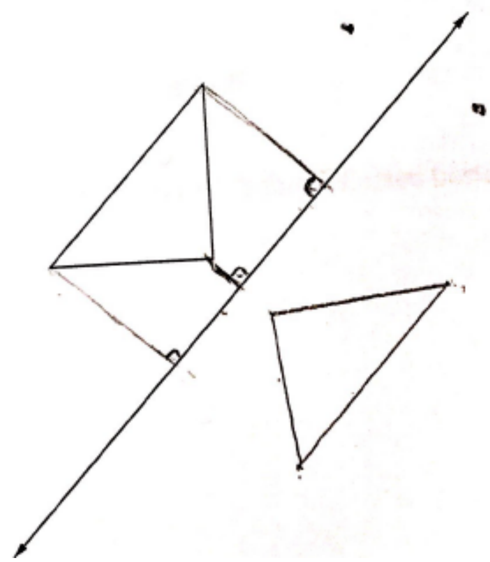


Figure 57. John's drawing of a reflection on an oblique line.

John's conclusion now includes points interior and exterior to the figure reflected in the pre-image plane to image plane for geometric reflections. He started to think about all points in the half plane. When he performed a geometric reflection, he described reflecting all points in the half plane including interior, exterior, and perimeter of the figure points. John used his definition of a plane in geometric reflections in a way that aligned with his mathematical definition of a plane. Based on his responses during the interviews, there is evidence that he thinks of the plane as a non-empty space consisting of an infinite number of points and all of these points are part of the geometric reflection.

Further evidence of the domain as a process structure was gathered during the second interview. I asked John to reflect a circle with an oblique reflection line (see Figure 58). He first reflected labeled two points (A and M) across the reflection line by using an index card to measure the distance to the reflection line. Then, he labeled a point (B) on the perimeter of the circle to get diameter of the circle and reflected across the reflection line. Finally, he drew the circle. I asked him what did you reflect? He explained, "I reflected every single point in this plane [referring to pre-image plane] to this plane [referring to image plane]" (Line 94-95; 09/29/2017, Interview 2). Then, I asked, what did you reflect for this plane [referring to image plane]? He elaborated:

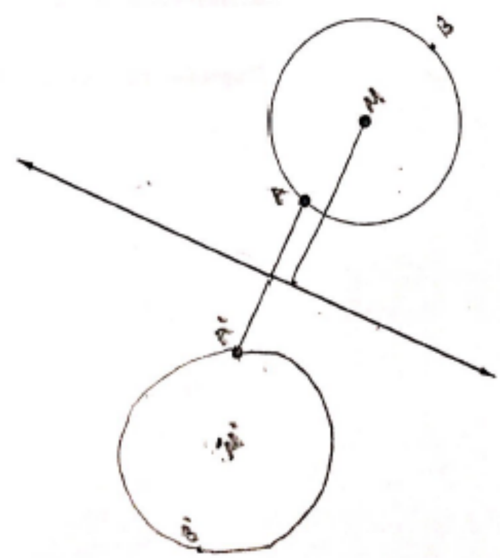


Figure 58. John's drawing of a reflection on an oblique line.

> I reflected this plane [referring to all points in the pre-image plane] onto here [referring to all points in the pre-image plane] and this plane [referring to all points in the pre-image plane] onto here [referring to all points in the pre-image plane]. You cannot really see the reflection onto this plane [referring to image plane]. Because there is no points visible or no points highlighted, I guess, on this plane [referring to image plane] needing a reflection (Line 157-159; 09/29/2017, Interview 2).

Then, I asked him to explain how he started to think about all of the points in the pre-image plane being reflected to the image plane, and all points in the image plane being reflected to pre-image plane. I interpreted from his explanations that providing interior and exterior points with the figure encouraged him to consider not only the figure being reflected, but also that the other points in the plane were reflected for performing geometric reflections. In the following figure, I describe how John reached a process structure of domain by unpacking his mental structures through the lens of APOS theory (see Figure 59).

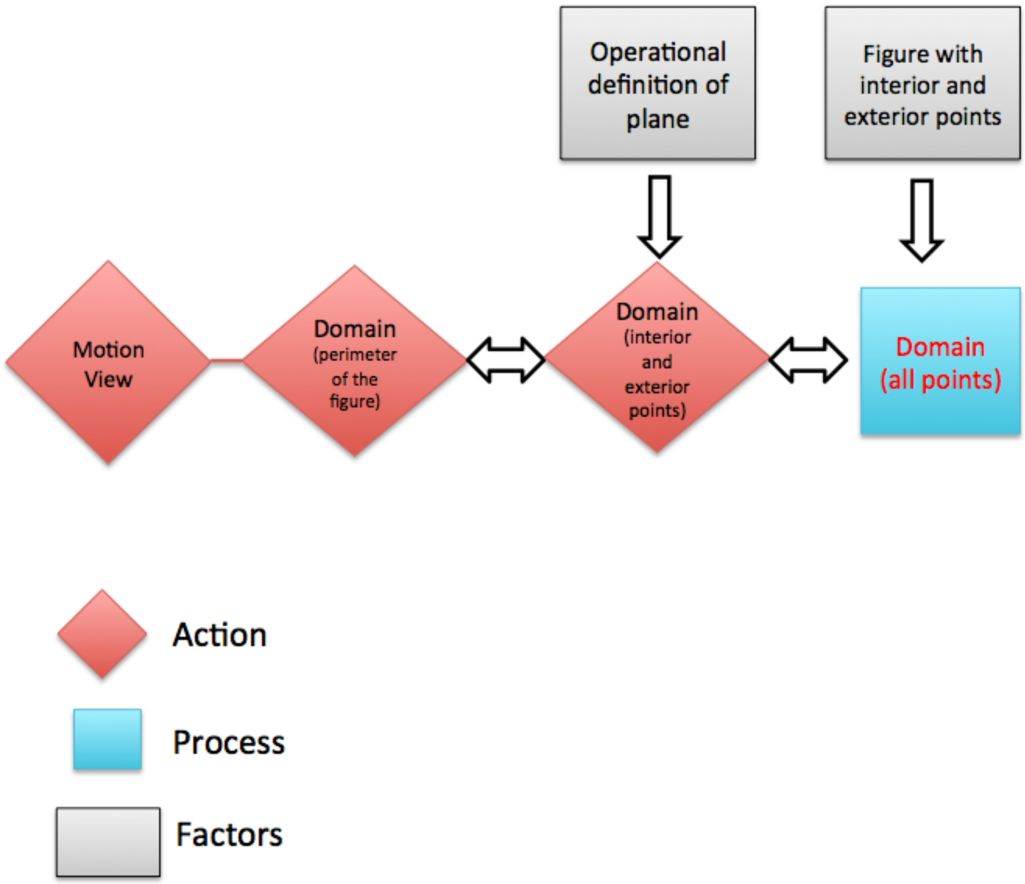


Figure 59. John's mental structures of domain

I hypothesized in my PGD that PTs begin with a motion view of domain as an action structure, then develop a mental structure to see domain as a process structure. John began with a motion view of domain, as he considered only the points in the pre-image plane. At the end of the second interview, John started to think about all points in the plane as he performed a geometric reflection. He used his operational definition of the plane in geometric reflections (i.e. there are infinitely many points in the plane for performing geometric reflection). His explanations further demonstrated that providing a task with interior and exterior points helped him to support his ideas to consider all of the points in the plane, rather than just a single figure. This evidence supports that he reached the process structure for the concept of domain in geometric reflection.

John's Understanding of the Plane

To examine the John's mental structures in terms of the concept of the plane, I analyzed his reactions and explanations on tasks when performing geometric reflection. The second interviews revealed that John seemed to have an action structure of plane for geometric reflections because he thought of the geometric reflections as a movement of points or figures on a plane rather than subset of the plane. For instance, when performing a geometric reflection with a trapezoid task, John viewed points as "separated from the plane" (Yanik & Flores, p. 55). After he reflected a trapezoid, I asked him, when you perform a reflection, is there any movement of points or figures from half of the plane to another half of the plane when you perform a reflection? He elaborated:

> The reflection is not one side reflecting onto another, it is both sides coming together and then displayed in two different angles. Everything over here [referring to pre-image plane] gets moved to here [referring image plane] in the reflection format of the closest point to this becomes the closest point to this line [referring to the reflection line]. It is not a shift. And then everything over here [referring to image plane] comes over here [referring to pre-image plane] in the same manner. (Line 155-158; 09/29/2017, Interview 2].

John seemed to think that points are located on the "top of the plane" (Edwards, 2003, p. 8) rather than part of the plane. He used the word "move" to describe the activity of relocating points in the pre-image plane to another plane rather than mapping the points for performing geometric reflection. Before making a decision his mental structures of plane, I asked one more question. I asked, when you reflected the points or figures, what is left here [referring to pre-image plane]? He explained:

> J: If I reflect this across this line onto the other half of the table, everything would be reflected onto the other side but really it is being duplicated. This pencil here, there will be an exact same pencil on the other side. Whereas the pencil stays the same and it duplicates onto the other side.
>
> R: Is there any movement?
>
> J: This figure does not move. I am not shifting this or rotating it, I am just duplicating it onto the other side [referring to image plane]. And that side [referring to image plane] is not moving either because I am placing them. Yeah. I do not think that there is any movement. Everything on this will remain on this side [referring to pre-image plane], it just will also be on the other side [referring to image plane]. (Line 193-203; 09/29/2017, Interview 2].

John's first explanation provides evidence that he did not think about the relations between points or figures and plane when he answered "is there any movement of points or figures from half of the plane to another half of the plane?" When I asked, when you reflected the points or figures, what is left here [referring to pre-image plane], he began to see the relationship between the points of the figure and the plane. He knew that the plane has infinitely many points, and that points and figures are a subset of the plane. When he performed geometric reflections, John mapped points rather than moving the points in geometric reflection. John seemed to know the relationship between points or figures and the plane. This is evidence of process structure of plane for geometric reflection (see Figure 60).

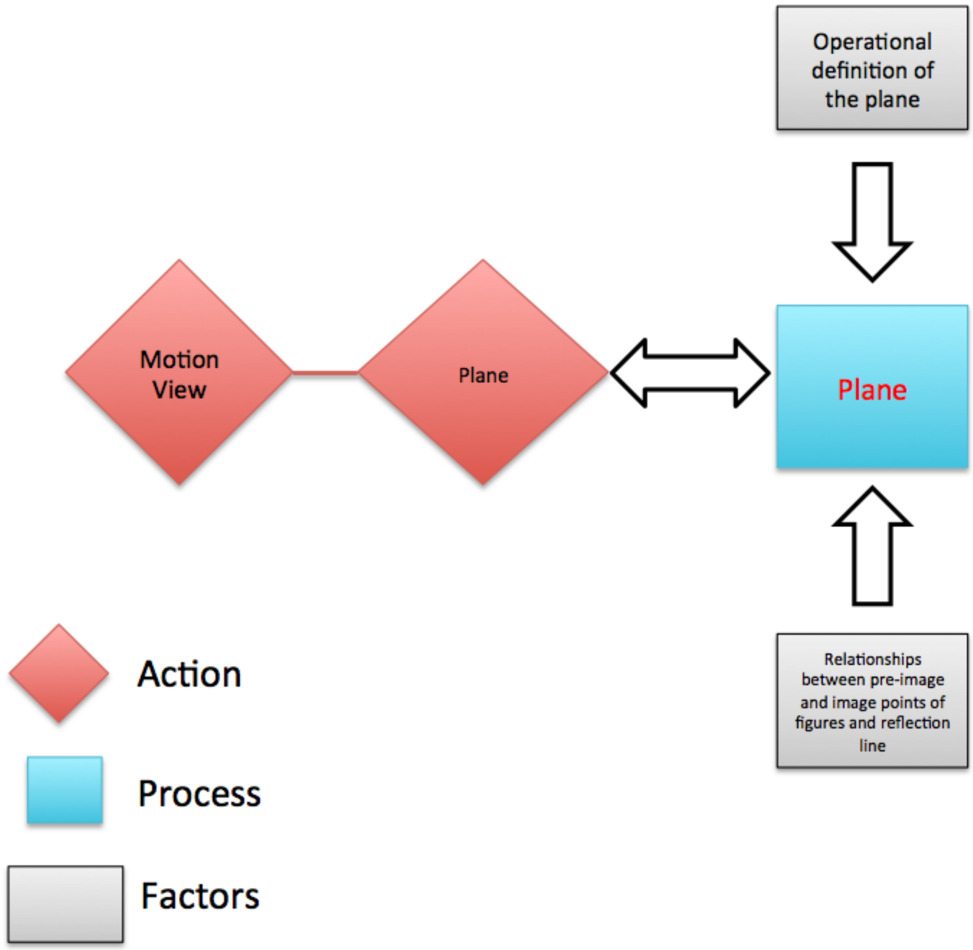


Figure 60. John's mental structures of plane

I hypothesized in my PGD that PTs begin with a motion view of plane as an action structure to develop a mental structure for the plane as a process structure. John seemed to have a motion view of the plane as an action structure throughout the second interview, because he seemed to consider geometric figures as "separated from the plane" (Yanik & Flores, p. 55) rather than as a part of the plane. To deeply investigate his mental structures of plane, I asked more questions, which encouraged him to think about his operational definition of the plane and the relationship between the points or figure and plane. I interpreted from his explanations that John had a process structure of the plane in geometric reflection.

John's Mental Structures of Mapping View

Briefly, John reached the process structure in terms of reflection line, domain, and plane. The findings revealed that John knew that a reflection line is necessary, and had a role to map points from the pre-image plane to the image plane for performing geometric reflection by using the properties of equidistance and perpendicularity. He also considered all points in both the pre-image and image planes for performing geometric reflection. The type of tasks presented and his operational definition of the plane are two important factors for him to consider all points in the plane. Understanding the relationship between points or figures and the plane helped him to reach a process structure of the plane (see Figure 61).

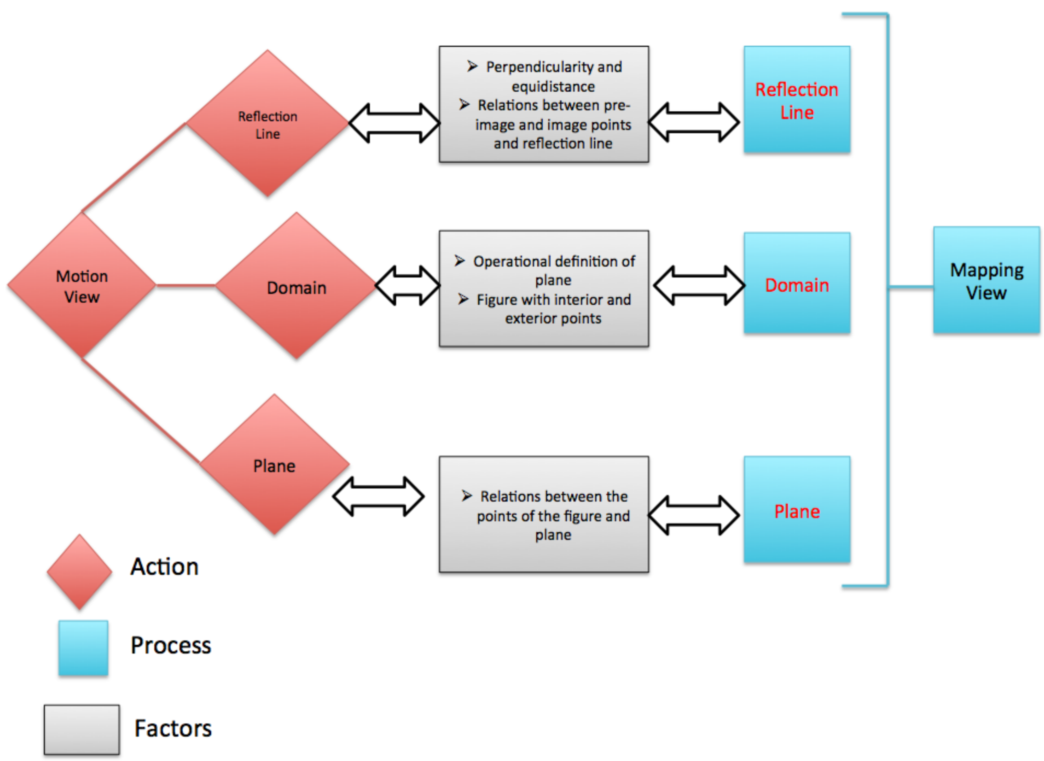


Figure 61. John's mental structures of mapping view

Revised Genetic Decomposition

The preliminary genetic decomposition (PGD) of geometric reflection was described in Chapter 3, based on the literature and inferences that I drew from it. During my analysis of the data, the PGD evolved, and my final genetic decomposition (GD) is presented in figure 62. The GD was created based on my analysis of four cases. Throughout this analysis, I looked for similarities and differences in each PT's mental structures of three sub-concepts of geometric reflections. In the PGD, I hypothesized that to develop a mental structure for geometric reflection as a mapping view, PTs would begin with a motion view. I hypothesized that acquiring a process structure of reflection line, domain and plane is sufficient to move from a motion view to a mapping view. My hypotheses were confirmed as my findings demonstrated that to move from a motion view of geometric reflections to a mapping view, having the process structure of reflection line, domain, and plane is sufficient.

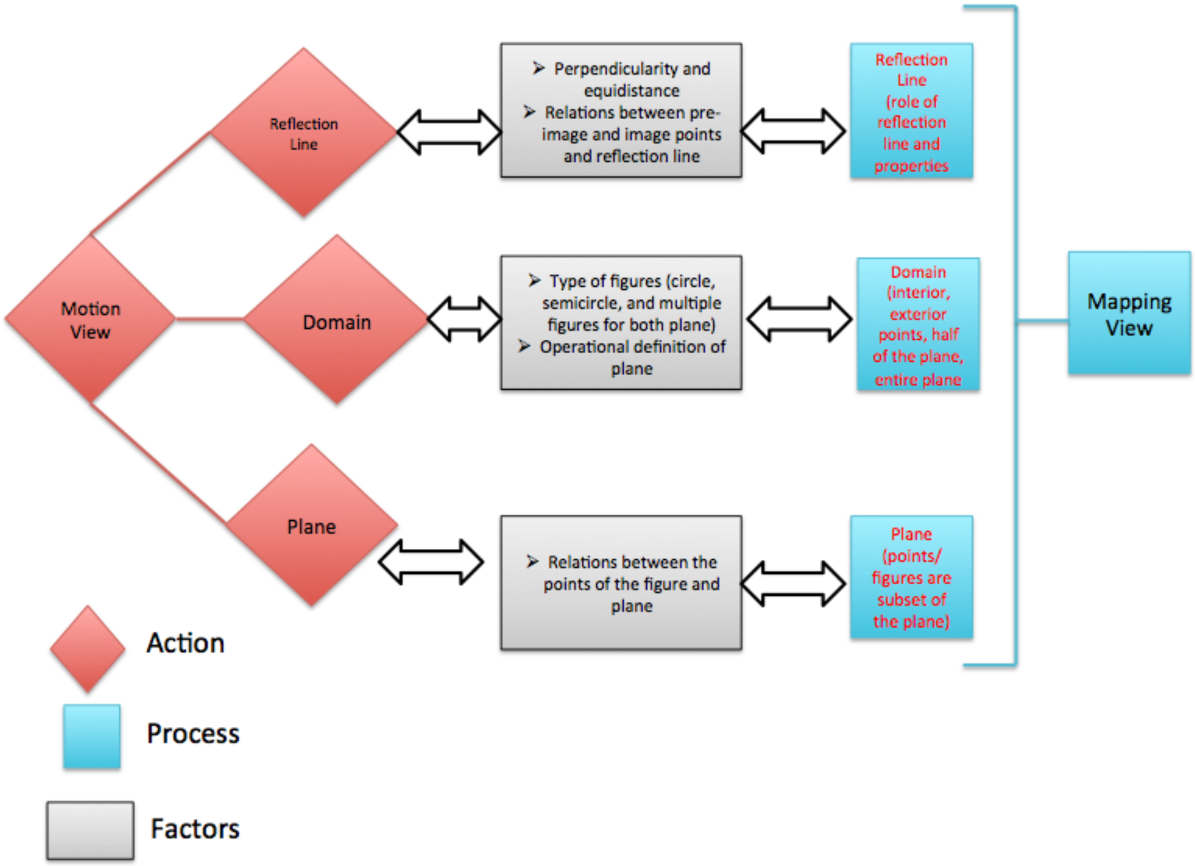


Figure 62. Genetic Decomposition of Geometric Reflections

By the end of the third interviews, all four PTs had similar mental structures of reflection line, domain and plane. In articular, I initially hypothesized in my PGD that to achieve a process structure PTs must know that the reflection line is an essential sub-concept for performing geometric reflections because the reflection line defines geometric reflections. For a process structure of the reflection line, PTs must have mental structures of equidistance and perpendicularity properties, and understand relations between pre-image and image points and reflection line. All four PTs provided evidence of action and process structures of the reflection line during the first interviews. Therefore, my findings confirmed my initial hypothesis in the PGD.

In my PGD, I hypothesized that under the motion view, a PT with an action structure considers the domain to be a single target figure; whereas under the mapping view, a PT with a process structure considers the entire plane as the domain. This latter feature is a different mental structure for domain, an action mental structure for domain must evolve towards the mapping view. My data analysis revealed that type of figures (e.g., circle, semicircle, and multiple figures to represent the plane) and the PT's operational definition of a plane are two significant factors that facilitate the evolution of a PT's action structure into a process structure of domain. In my PGD, I hypothesized that PTs would consider the plane to be an empty space and consider only the given points or figures. The type of figures used to illustrate a plane encouraged PTs to consider the domain as all points in the plane. When PTs started to consider the domain as all points in the plane, they begin using the mathematical definition of a plane in geometric reflection.

In my PGD, I hypothesized that under the motion view, a PT with an action structure of plane considers the plane as a background that is separate from geometric points or figures, which can be manipulated. Under the mapping view, a PT with a process structure considers the points or figure as a subset of the plane rather than "separate from the plane" (Yanik & Flores, p. 55). My data analysis revealed that understanding the relations between the points or figures and plane is an important factor that facilitates the evolution of a PT's action structure into a process structure of plane.

In summary, this study investigated four PTs' understanding of geometric reflections in terms of motion and mapping views. The findings suggest that by the end of three task-based interviews, all four PTs had reached a mapping view of geometric reflections based on the reflection line, domain, and plane. In moving from a motion view to a mapping view, a variety of factors must be understood, such as the role of the reflection line; the properties of perpendicularity and equidistance; relationships between pre-image and image points of the focal figure; the relationship between figure and plane; and mathematical definitions of reflection line, domain, and plane. Also the types of figures used to represent plane may facilitate understanding.

CHAPTER 5. DISCUSSION

In this study, I focused on two views of geometric reflections: the motion and the mapping views. The purpose of this study was to describe how a motion view can develop into a mapping view and what factors can facilitate this development. The findings from the task-based interviews with pre-service (PTs) teachers of secondary mathematics revealed that, based on their conceptions of the three sub-concepts of reflection line, domain, and plane, all four participants initially had a motion view of geometric reflections. After three interviews, the motion view of geometric reflections of all four participants had evolved into a mapping view facilitated by the properties of equidistance and perpendicularity, the role of the reflection line, the types of figures reflected, the relation between points or figures and the plane, and the operational definition of the plane. I will begin by discussing the results of my interviews using the sub-concepts involved (reflection line, domain and plane) under a motion or a mapping view. I will further identify the factors that facilitate the development of mental structures from a motion view to a mapping view. Finally, I will continue to discuss the limitations of the study, its implications, and suggestions for future mathematics education research on the learning and teaching of geometric reflection.

Understanding of Reflection Line

Previous studies have found that both students and pre-service teachers (PTs) have an action structure of the reflection line (Boulter & Kirby, 1994; Flanagan, 2001; Yanik, 2006). This is in contrast with my study, in which all PTs exhibited both an action and a process structure of the reflection line. More specifically, Boulter and Kirby (1994), Flanagan (2001), and Yanik (2006) found that students and PTs had difficulty with knowing how to use a reflection line to position the points or figures in a reflection, and they did not know whether or not a reflection line was necessary for performing a geometric reflection. When they reflected points or figures, they did not use the properties of equidistance and perpendicularity. However in my study, all four participants demonstrated understanding of the roles that the reflection line and the properties of equidistance and perpendicularity played in performing geometric reflections during the first, second and third interviews. Specifically, when the PTs had an action structure, they still knew that the reflection line is an essential sub-concept for performing geometric reflections. When the PTs had a process structure, they understood the role of the reflection line in terms of the relationship between the pre-image and image points of figures, and they used the properties of equidistance and perpendicularity for performing geometric reflections. Possible reasons for this difference from previous studies might be the level of the participants, the nature of the tasks, and the purpose of the initial interviews. Boulter and Kirby worked with seventh and eight grade students, Flanagan with tenth grade students, and Yanik with pre-service elementary teachers (PETs). However, I worked with pre-service secondary mathematics teachers (PTs), who can be expected to have more elaborated knowledge of geometric reflections than PETs and K-12 students. Another possible reason might be the nature of the tasks. For example, I asked the PTs to identify whether two figures constituted a reflection without giving them the reflection line and to justify their stance. This type of task was important to see whether or not the PTs knew that the reflection line is necessary for performing geometric reflections. Instead of using these kinds of tasks, Boulter and Kirby, Flanagan and Yanik provided reflection lines for all the tasks they used, which might have limited the collection of evidence of the role of reflection line. Another reason might be the purpose of the initial interviews. I used these interviews for the selection of my participants. Even though I conducted initial interviews with six PTs, I selected only four based on their knowledge of geometric reflections and willingness to explain their thinking. However, Boulter and Kirby, Flanagan and Yanik did not use initial interviews to screen their selection of participants. Therefore, my PTs might have had better knowledge of geometric reflections than their participants.

To summarize, all four PTs knew of the role of the reflection line and the properties of equidistance and perpendicularity in geometric reflections. My findings generated two significant commonalities as factors that facilitated their performance of geometric reflections and the progression of their understanding from an action structure to a process structure of the reflection line: the role of the reflection line and the properties of equidistance and perpendicularity. Therefore, the reflection line is shown to be a significant sub-concept for the concept of geometric reflection and essential in the move from a motion view to a mapping view.

Understanding of Domain

Hollebrands (2003) and Yanik (2006) found that students and PTs have an action structure of domain as a single figure, and they identified considering the domain as all points in the plane as the most challenging sub-concept to grasp in performing geometric reflections. In alignment with these two authors, this study also found that considering the domain as all points in the plane was a challenging sub-concept for the participating PTs, all of whom demonstrated an action structure of domain for geometric reflections in the first interview by considering the domain as a single figure (or a point or a line) rather than all points in the plane. Whereas all four PTs had an action structure of domain, however, there were some differences in their underlying mental structures. For instance, Linda and Emily considered that geometric reflections were applied to vertices or the perimeter of the figure. On the other hand, John and Michael thought that geometric reflections were applied to all points in the pre-image plane (half of the plane). One reason for this difference might be their operational definitions of the plane. Although all participants gave a correct formal definition of a plane in the initial interview, they seemed to not use this as their operational definition when performing geometric reflections. Linda and Emily appeared to consider the plane as an empty space and did not attend to anything except the given points or figures. On the other hand, it is unclear how Michael and John viewed the plane in geometric reflections because they considered all points in the half of the plane, including both interior and exterior points to the figure, for performing geometric reflections.

Another reason why all four PTs performed a geometric reflection as a single figure might be the way information is presented in textbooks. Many educators have reported that they are dissatisfied with the content emphases in textbooks (Ball, 1993; Jones, 2004, Ma, 1999; Zorin, 2011). My CMP textbook analysis showed that they support a motion view by implying that given points or figures are reflected rather than all points in the plane. Another likelihood is that the description “a reflection line maps the figure” may result in students’ understanding the reflection line as reflecting a figure as a whole rather than the points that constitute the figure. I hypothesize from the literature that the perception that a figure is being reflected as a whole supports a motion rather than a mapping view because to have a mapping view of performing a geometric reflection, students need to consider all points in the plane rather than only given figures (Boulter & Kirby, 1994; Flanagan, 2001; Yanik, 2006). A close analysis of the teacher's guide for CMP textbooks found no example or explanation of performing a geometric reflection that emphasizes reflecting all points in the plane rather than a single figure. I contend that asking, "Do any points remain unchanged under the reflection" would prompt students to consider all points in the plane when performing a geometric reflection. Before being asked this kind of question, students need practice with specific tasks (e.g., circle tasks, inside and outside colored tasks, etc.) to develop a mental construction of domain and plane. Hence, I suggest that all four PTs had never learned about reflecting all points in the plane when performing a reflection in their previous geometry classes. By the end of the first interview, all four participants still had an action structure of domain because they seemed to consider domain as a perimeter of the figure or as all points in the pre-image plane rather than all points in the entire plane.

The second interviews demonstrated the importance of the type of figure for encouraging consideration of more points for performing geometric reflections. In this interview, Linda, Emily, and John showed progress toward a process structure for the domain, but Michael did not. For example, while working on circle and semicircle tasks, Linda and Emily progressed from reflecting the vertices or the perimeter of the figure to reflecting all points in the half plane. As they were thinking about the center point of the circle, they started to consider interior points of the figure. Also, the semicircle (open figure) task encouraged Linda and Emily to consider all points in the half plane because the figure's interior and exterior were not clearly defined for them. Labeling points inside and outside the semicircle helped them to consider interior and exterior points of the figure (all points in the half plane) while performing geometric reflections. I infer that Linda and Emily speculated that there was an infinite number of points inside and outside of the semicircle figure in the half plane on which to perform a reflection.

Similarly, John showed progress toward a process concept of the domain when he worked on a circle task, during which he progressed from reflecting all points in half of the plane to all points in the plane. Because it was not clear how he started to think about all points in the plane for performing geometric reflections, I asked him to explain his reasoning. His explanation showed that thinking about interior and exterior points of the figure encouraged him to consider reflecting not only the figure but also the other points in the plane. John was the only one who had a process structure of geometric reflection at the end of the second interviews. Michael, on the other hand, was still considering all points in the half plane rather than all points in the plane at this point. Linda, Emily, and Michael were all still progressing toward regarding the domain as all points in the plane, signifying that they still had an action structure of domain for geometric reflections. Hence, it may be inferred that type of figure (here, half, or full circle) is important for encouraging consideration of more points for performing geometric reflections.

By the end of the third interviews, Linda, Emily, and Michael were, like John, considering that geometric reflections applied to all points in the plane. Working with multiple figures in both the pre-image and image planes encouraged them to start thinking that geometric reflections not only reflect all points from the pre-image plane to the image plane but also reflect all points from image plane to the pre-image plane.

Working with multiple figures in both pre-image and image planes helped all four participants to consider that the image plane as well as the pre-image plane has an infinite number of unlabeled points and therefore to think about all points in the plane for performing geometric reflections.

After all three interviews, all four participants had started to think about all points in the entire plane when performing a reflection. Although they did not physically demonstrate reflecting unlabeled points for geometric reflection, they understood that performing a reflection involves both labeled and unlabeled points, that is, all points in the plane. The tasks with circles, semicircles, and multiple figures for both the pre-image and image planes helped Linda, Emily and Michael to consider all points in the plane. On the other hand, providing interior and exterior points in the figure and drawing an analogy with a mirror encouraged John to consider all points in the plane. Their operational definition of the plane in geometric reflections thus became aligned with their mathematical definition of a plane. I inferred their explanations to mean that they had achieved a process structure for the concept of the domain in geometric reflections. Hence, considering the domain as all points in the plane is another significant sub-concept for the concept of geometric reflection and essential to move from a motion view to a mapping view.

Understanding of the Plane

Previous researchers have found that students and PTs have an action structure of plane (Flanagan, 2001; Yanik, 2006). In my study, in contrast, all PTs had a process structure of plane. Specifically, Flanagan (2001), and Yanik (2006) found that students and PTs considered geometric reflection as a movement of points or figures, implying that when they performed geometric reflections, they considered the points or figures as "separate from the plane" (Yanik & Flores, p. 55) rather than as a subset of the plane. During the first and second interviews, Michael, John, and Linda consistently used the verb "move" when performing geometric reflections, suggesting that they considered geometric points or figures as moveable on the plane rather than as a part of it. On the other hand, during all three interviews, Emily consistently used "reflected" as the verb for performing geometric reflections, although there was not enough evidence to investigate how she thought about the relationship between the points or figures and the plane. To understand all four participants' mental structures concerning the relationship between points or figures and the plane, I posed direct questions during the third interview. For instance, I asked whether there was any movement of the points or figures from the pre-image to the image plane when they performed a reflection. All four participants indicated that they considered the points or figures as part of the plane rather than separate from it by explaining that there is no movement of points or figures when performing geometric reflections (Edwards, 2003); instead they said, "we are duplicating the figure," "mapping the points," "copying the figure," and "creating a new point or figure." All four PTs' explanations demonstrated that their mathematical understanding of the relationship between the figures or points and the plane was accurate; that is, they knew that the points (and hence the figure) were embedded in the plane. Therefore, after all three interviews I interpreted from their descriptions that all four PTs had a process structure of the plane.

One source of students' and PTs' difficulties with developing an understanding of the relationship between the points or figures and the plane might be textbooks. My CMP curriculum analysis indicated that the relationships between the points or figures and plane were not emphasized explicitly in the geometric reflection unit. There is no clear explanation or example that shows all points and figures as embedded in the plane, rather than "separated from the plane" (Yanik & Flores, p. 55).

Briefly, all four participants reached the process structure in terms of the reflection line, domain, and plane. These findings show that they understood that, in performing geometric reflection, a reflection line is necessary to position the transformed figure and that every point in the plane is mapped onto itself. Additionally, the types of figures (e.g., circle and arc tasks) were crucial components of the tasks and eventually helped all four participants to consider all the points in the domain, rather than just the points of a single figure. The use of definitions and understanding the relationships between points/figures and the plane helped them to reach a process structure of plane.

The findings of this study offered some insight into a variety of factors involved as all the participants moved from a motion view to a mapping view, such as the role of reflection line, the properties of perpendicularity and equidistance, the relationships between pre-image and image points of the figures and the reflection line, the meaning of domain, types of figures used for solving problems, the nature of the plane, and the relation between figures plane, and the definitions of all terms.

Implications

Previous studies (Flanagan, 2001; Yanik, 2006) found that students and pre-service elementary teachers have a motion view of geometric reflections, and no one has explicated a mapping view of geometric transformations in general or of geometric reflections specifically. Also, there has been no clear evidence documenting how a learner's motion view evolves into a mapping view. This study outlined the progression of PTs' conceptualizations from a motion view into a mapping view of geometric reflections, and the factors that facilitated this change (see Figure 62). This study also identified the reflection line, domain, and plane as important sub-concepts and demonstrated their role in facilitating PTs' motion view to a mapping view. Therefore teachers need to consider teaching these three sub-concepts to prepare students to accurately understand geometric reflections.

The findings of this study indicated that considering the domain as all points in the plane is a challenging concept for PTs. Practicing with circle, semicircle and multiple figures encouraged PTs to consider more points in the plane when they were performing geometric reflections. Therefore, textbooks should have provide examples of and exercises with these types of figures to help learners consider the domain as all points in the plane rather than as a single figure. In addition, having an understanding of the points or figures and the plane is important for developing a mapping view. In order for learners to develop a mapping view for geometric reflections, teachers should emphasize the plane and its relationship to points and figures. This study provides useful insights that can be utilized to support PT's understanding of reflections and thus prepare them to teach this topic more effectively. There has been limited research attention to geometric reflections, and this study was the first to document how PTs move from a motion view to a mapping view.

Limitations of the Study

This dissertation study has three limitations. First, because there is lack of tasks related to domain and plane for the geometric reflections in the United States mathematics curriculum, I adapted several tasks from previous studies to emphasize the concepts of the domain and plane. Therefore the findings were limited to the tasks that I used. Second, there were six participants for the initial interviews, and I selected four based on their willingness and ability to explain their thought processes to use for the subsequent interviews. The findings are limited to these four participants and may serve as informative examples with the caveat that they might be different with another sample. The third limitation is that the data consisted not only of verbal and but also of nonverbal behaviors, idiosyncratic speech characteristics, gestures, and incomplete utterances. It is possible that the participants may not have fully verbalized their thinking, and I had to draw inferences from communications that often were not conventional, complete, and clear. Hence, my evidence is based on my interpretations of the collected data, and the extent to which I had to infer meaning should be taken into account.

Direction for Future Studies

The findings of this study suggest three directions for future studies. First, this study focused only on pre-service secondary mathematics teachers' (PTs) understanding of geometric reflections in terms of motion versus mapping views. One could use the same tasks with different participants (e.g., students, in-service teachers) to investigate their understanding of geometric reflections. Second, this study demonstrated that a process structure of reflection line, domain, and plane is sufficient to move from a motion view to a mapping view of reflections. One could extend this result and consider the object and schema structures of reflection line, domain, and plane in geometric reflections. Third, Flanagan's (2001), Yanik's (2006) and my studies investigated three important sub-concepts of geometric reflections, which were reflection line, domain, and plane. One could extend this investigation to determine whether or not there is a preferred order for studying reflection line, domain, and plane for learning and teaching this concept.

Conclusion

The existing literature (Boulter & Kirby, 1994; Edwards & Zaskis, 1993; Flanagan, 2001; Yanik, 2006) and standards (NCTM, 1989, 2000; CCSSM, 2010) emphasized the importance of geometric reflections in mathematics. However, there has been little research on PTs' understanding of geometric reflections, and there is a lack of research to investigate how a motion view evolves into a mapping view of geometric reflections. The findings of this study documented how PTs' motion view can develop into a mapping view of geometric reflections, and what factors can facilitate this development. Consistent with other researchers, I found that reflection line, domain, and plane are three important sub-concepts in conceiving of geometric reflections as a mapping view. As mentioned above, a variety of factors may facilitate progression from a motion view to a mapping view, such as the role of the reflection line, perpendicularity and equidistance properties, relationships between pre-image and image points of the figures and the reflection line, accurate understanding of domain, type of figures used in practice, the nature of the plane, the relationship between figures and plane and definitions of these terms. Findings further indicated that the use of the circle and open circle (e.g., semicircle) tasks play a significant role for PTs to consider more points in the plane when they perform geometric reflection. These tasks utilize many of the elements mentioned above into account, and pushed the boundaries on how the PTs were conceptualizing the key notions (e.g. reflection line, plane, domain). Therefore, circle and open circle tasks should be used in teaching geometric reflections for eliciting the domain as all points in the plane. In addition, when teaching geometric reflections, teachers should change their language of classroom. For example, they should use "map" instead of "move" as a word to describe geometric reflections. We saw the lasting impact that the word "move" had on the four PTs in this study, and how it took several varied tasks to begin to shift their understanding and language of geometric reflections. In textbooks we see this omitted as well, even when geometric reflections are discussed as mappings in the teacher commentary. The types of questions that teachers pose also can play a significant role in how their students understand geometric reflections. For instance, my questioning strategies helped PTs to consider more points in the plane. When teaching geometric reflection, teachers can use questions such as, "which points did you reflect when you perform geometric reflection", "are there another points being reflected besides vertices or perimeter of the figure" etc. However, there is still much more that needs investigated with regards to geometric reflections in terms of motion and mapping views, so that additional factors affecting the development from a motion view to a mapping may be identified.

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APPENDIX A. RECRUITMENT LETTER

If you are an undergraduate student who is going to have a bachelor's degree in mathematics education, (**specifically pre-service secondary mathematics teachers**) you are invited to participate in a research study to discuss your experiences in mathematics. Participants will be interviewed in person and all information given by the participant will be confidential. If you or someone you know is interested, please contact Murat Akarsu at **makarsu@purdue.edu**. The main purpose of this study is to identify pre-service secondary mathematics teachers' (PTs) understanding of the concept of geometric reflections. The results of the study will describe how a motion view of geometric reflections develop into a mapping view for secondary mathematics PTs, and what factors facilitate PTs' development of a motion view of geometric reflections into a mapping view. If you choose to participate, you will be paid $60 ($20 per interview) at the end of all fours interviews. Because I only need five participants, priority is going to be given to do person who emails first. Your responses will be kept strictly confidential and all data will be identified using pseudonyms, thus any reported responses will be unidentifiable.

We appreciate your participation in this important study that has been reviewed and approved by the Purdue University Human Subjects Research Committee.

Thank you for your participation,

Murat Akarsu
Graduate Student
Mathematics Education
Curriculum and Instruction
College of Education
Purdue University
West Lafayette, IN
makarsu@purdue.edu

Participants' Name:

Email:

APPENDIX B. INITIAL INTERVIEW PROTOCOL

1) Tell me about yourself.
2) How did you decide to be a teacher?
3) What are your educational goals (for yourself, and for your students)?
4) Tell me a little about your mathematics background
    a. What mathematics courses have you taken in college?
    b. Please describe yourself as a mathematics student (Ex: Tell me about the grades you typically receive in mathematics. Would you say that they accurately reflect your abilities in mathematics?)
    c. How would you rate your confidence in mathematics? Why? (What experiences have affected your confidence in understanding mathematics?)
5) Explain what it means to "understand a mathematics concept".
6) Please tell me about a mathematics concept and when you felt that you understood it.
    a) How did you feel about it?
    b) What helped you to understand?
7) Please tell me about a time you felt that you did not understand a mathematics concept.
    a) How did you feel about that?
    b) What did you do to try and understand?
8) Have you taken any geometry course? Do you remember geometric reflection that you learn in Geometry course? If yes, when was the last time you saw the concept of geometric reflection? What was your experience like? What do you remember for the concept of geometric reflection?

9) Using your own words and any pictures or diagrams you need to express your ideas, define the term the

   i) Geometric reflection

   ii) Reflection line

   iii) Point, line, plane

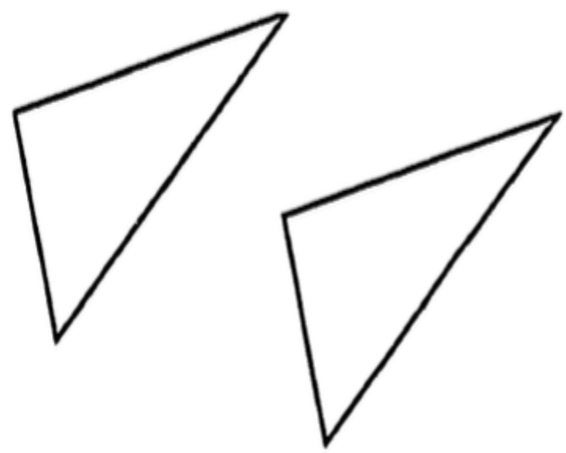
10) Could you list all the properties of the concept of geometric reflection (or rules about geometric reflection) that you can recall?

APPENDIX C. INTERVIEW 1

1. Tell me how you were taught about reflection in high school.

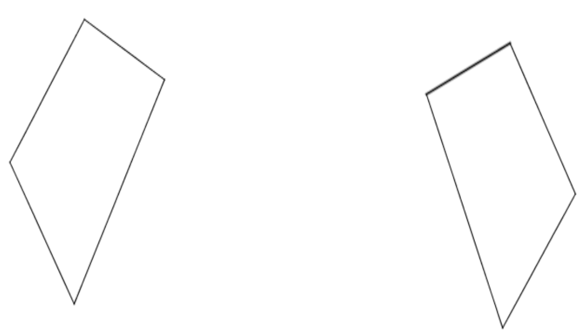
2. Can you give me a few examples?

3. Is this a reflection? Why or Why not?



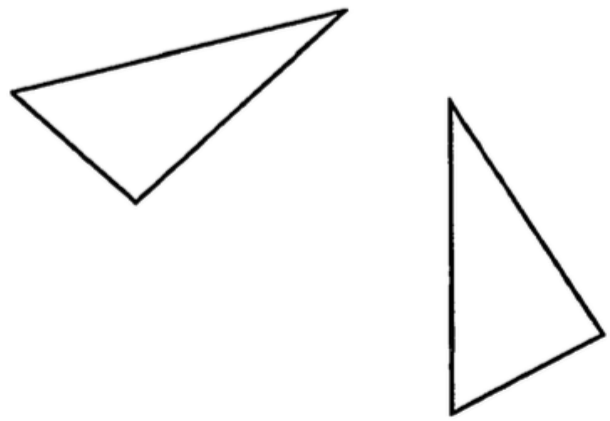
(Molina, 1992, p. 122)

4. Is this a reflection? Why or Why not?



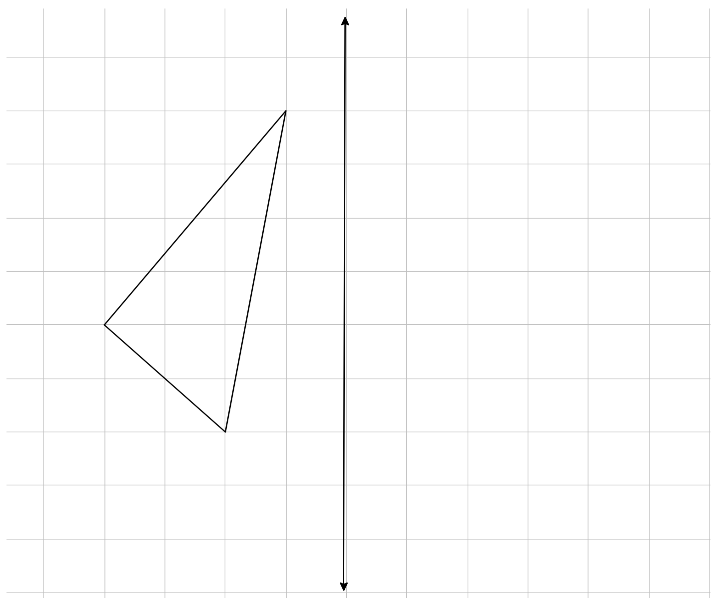
(Molina, 1992, p. 122)

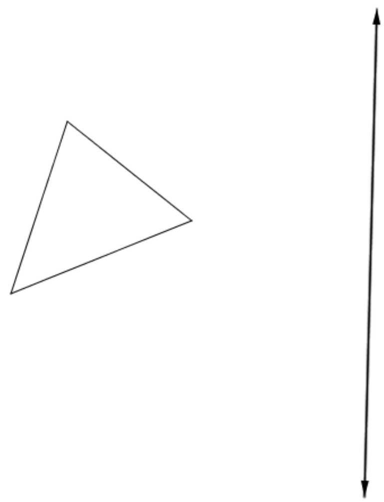
5. Is this a reflection? Why or Why not?



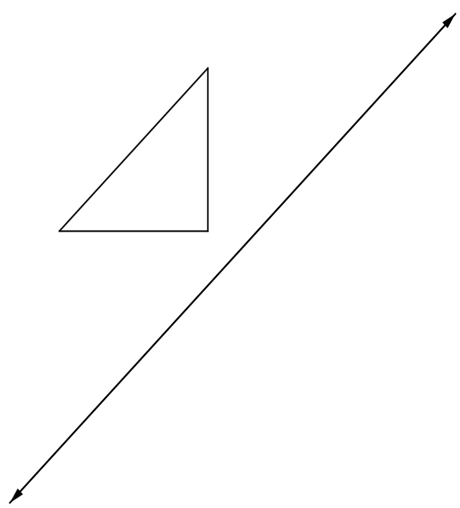
(Molina, 1992, p. 124)

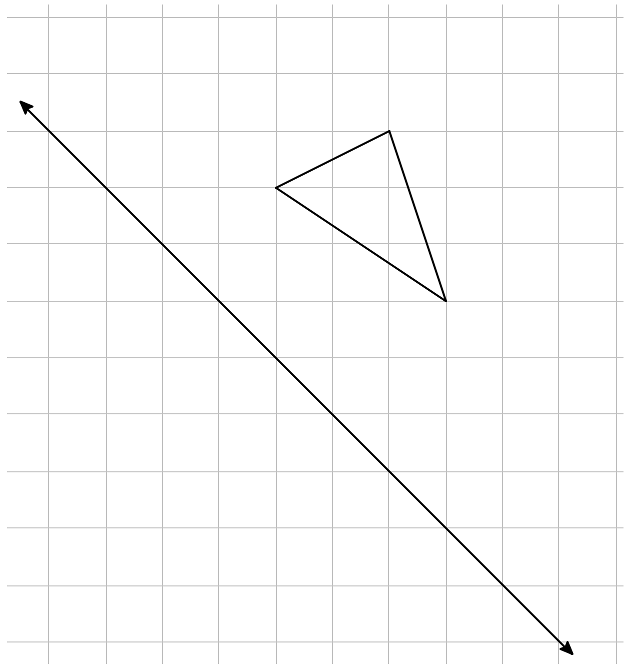
6. Find the image after performing reflection across the line.





7. Find the image after performing reflection across the line.





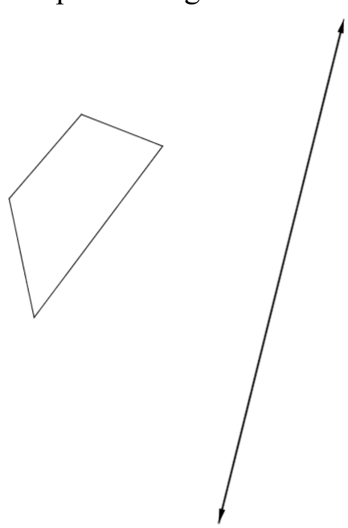
8. Explain how you determined where to place the figure.

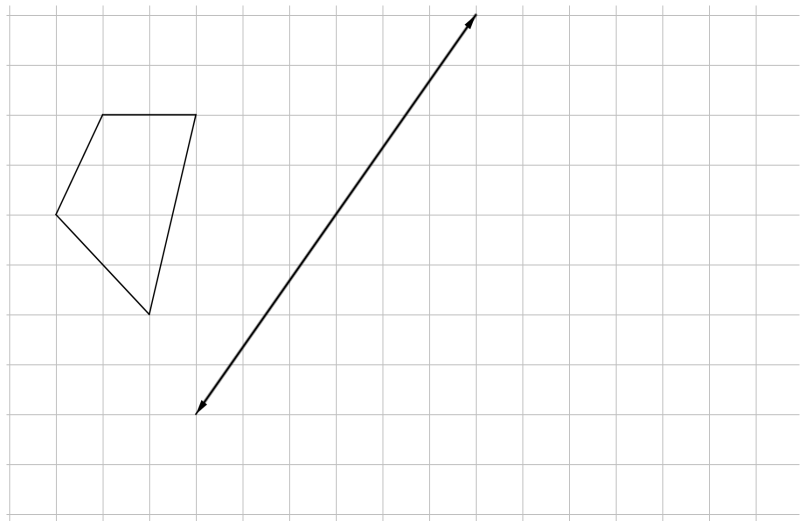
9. What properties are important in performing a reflection?

10. What points did you reflect?

11. Are there any other points being reflected besides these points?

12. Find the image after performing reflections across the given lines.





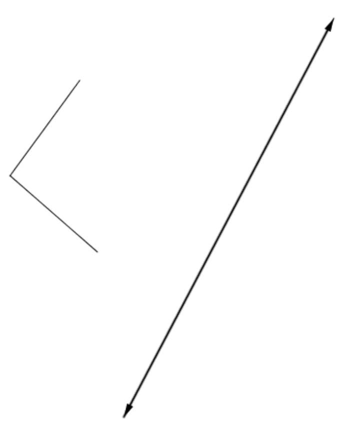
13. Explain how you determined where to place the figure.

14. What properties are important in performing a reflection?

15. What points did you reflect?

16. Are there any other points being reflected besides these points?

17. Find the image after performing reflection across the line.



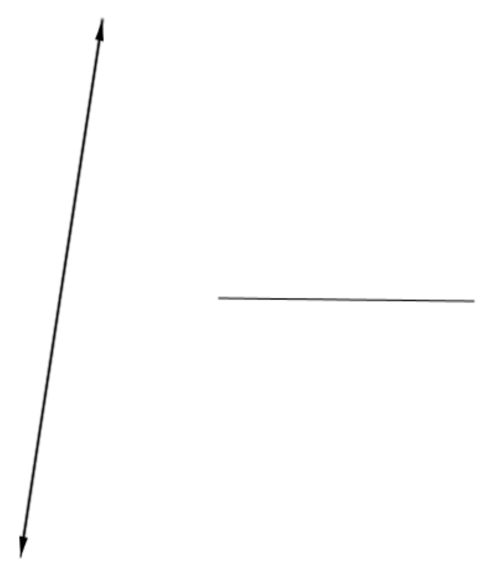
18. Explain how you determined where to place the figure.

19. What properties are important in performing a reflection?

20. What points did you reflect?

21. Are there any other points being reflected besides these points?

22. Find the image after performing reflection across the line.



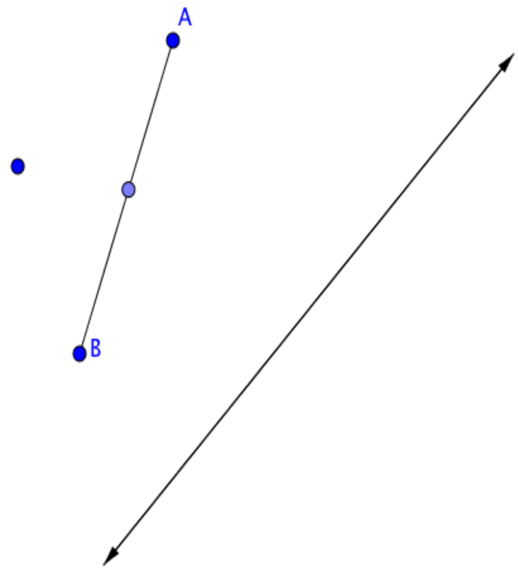
23. Explain how you determined where to place the figure.

24. What properties are important in performing a reflection?

25. What points did you reflect?

26. Are there any other points being reflected besides these points?

27. Find the image after performing reflection across the line.



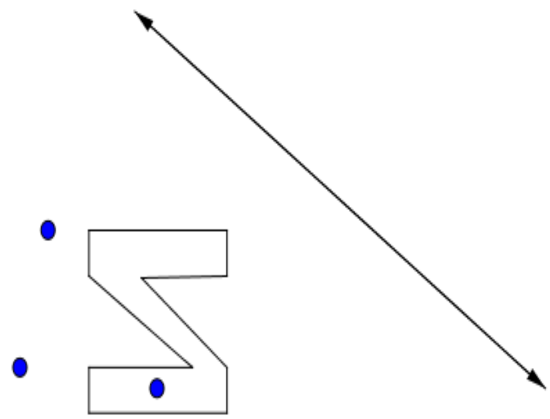
28. Explain how you determined where to place the figure.

29. What properties are important in performing a reflection?

30. What points did you reflect?

31. Are there any other points being reflected besides these points?

32. Find the image after performing reflection across the line



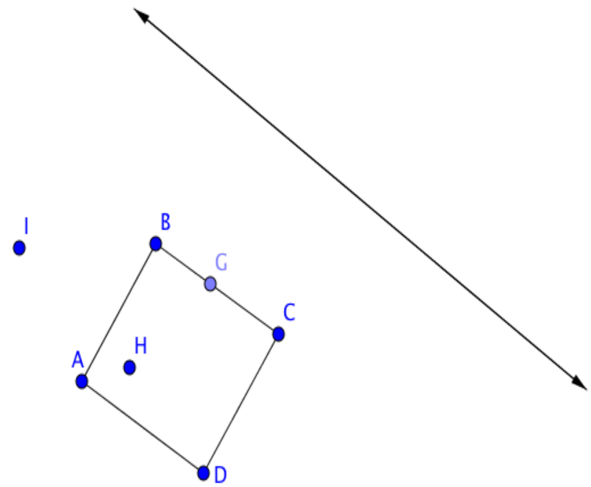
33. Explain how you determined where to place the figure.

34. What properties are important in performing a reflection?

35. What points did you reflect?

36. Are there any other points being reflected besides these points?

37. Find the image after performing reflection across the line.



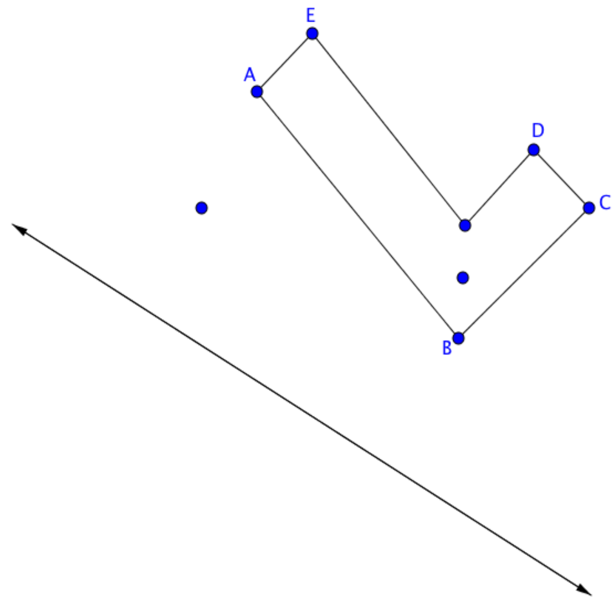
38. Explain how you determined where to place the figure.

39. What properties are important in performing a reflection?

40. What points did you reflect?

41. Are there any other points being reflected besides these points?

42. Find the image after performing reflection across the line.



43. Explain how you determined where to place the figure.

44. What properties are important in performing a reflection?

45. What points did you reflect?

46. Are there any other points being reflected besides these points?

47. What information do you need to perform a reflection?

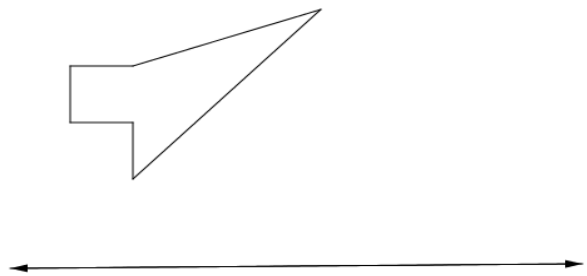
48. How would you define a geometric reflection in terms of this information?

49. Where have you seen something like this before?

50. [In your high school geometry class] did you ever talk about reflecting points inside or outside of a figure or all points in the plane?

APPENDIX D. INTERVIEW 2

1. Find the image after performing reflection over line.



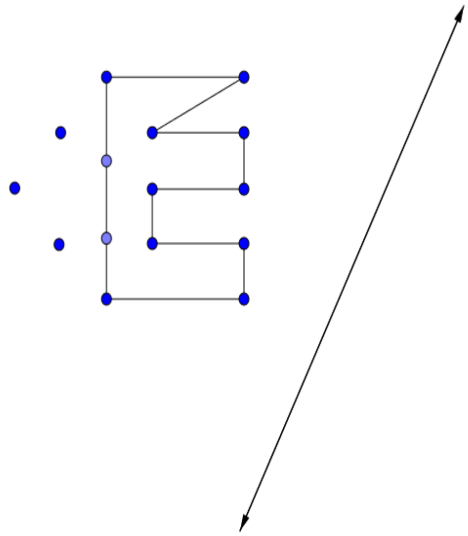
2. Explain how you determined where to place the image.

3. What properties are important in performing a reflection?

4. What points did you reflect?

5. Are there any other points being reflected besides these points?

6. Find the image after performing reflection over line.



(Glass, 2001, p. 201)

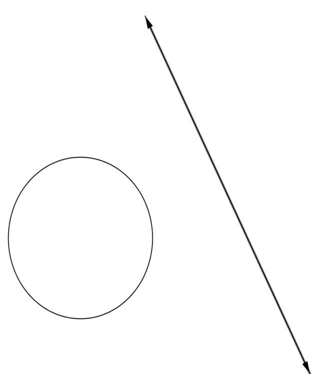
7. Explain how you determined where to place the image.

8. What properties are important in performing a reflection?

9. What points did you reflect?

10. Are there any other points being reflected besides these points

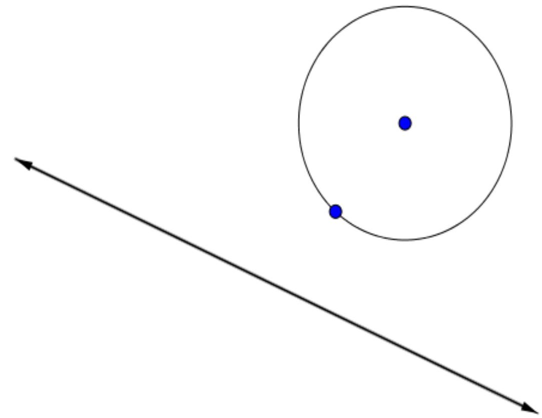
11. Find the image after performing reflection over line.



12. When you reflect the circle, which points did you reflect?

13. Are there any other points being reflected besides these points?

14. Find the image after performing reflection over line.



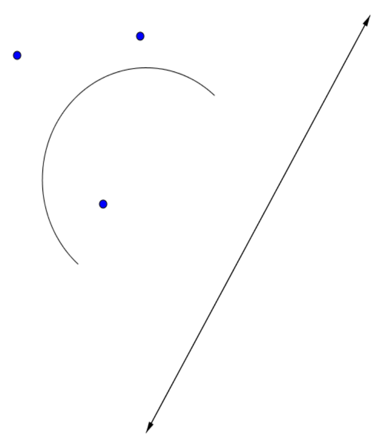
15. Explain how you determined where to place the image.

16. Are there any points inside the image? If yes, where do the points inside the image go when you perform a reflection?

17. Are there any points outside the image? If yes, where do the points outside the image go when you perform a reflection?

18. What changes? What remains unchanged?

19. Find the image after performing a reflection over line.

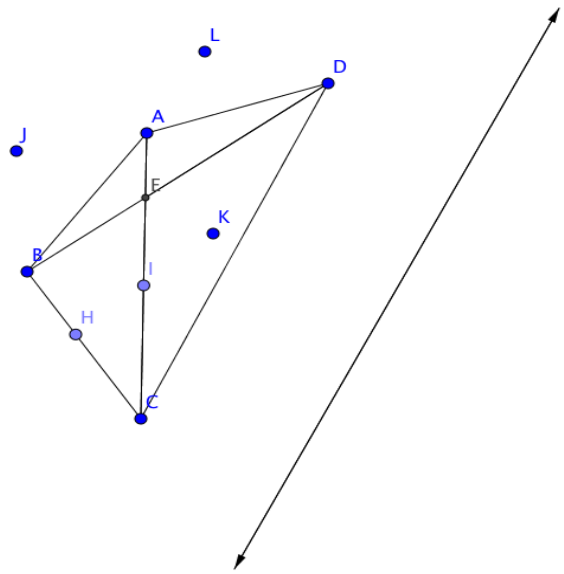


20. Explain how you determined where to place the figure?

21. Which points of the figure reflect?

22. Are there any other points besides these being reflected? If yes, where do these points go when you reflect them?

23. Find the image after performing reflection over line.



24. Explain how you determined where to place the image.

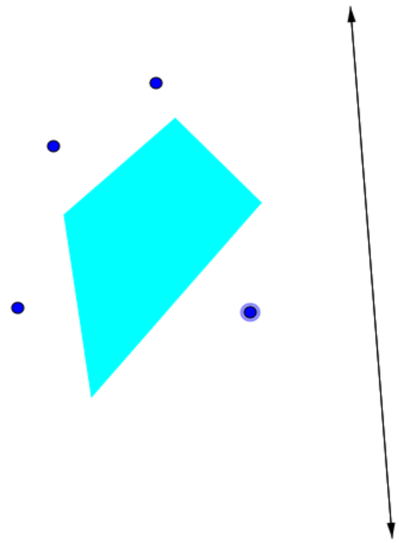
25. Are there any points inside the image? If yes, where do the points inside the image go when you perform a reflection?

26. Are there any points outside the image? If yes, where do the points outside the image go when you perform a reflection?

27. Explain how you determined where to place any image point?

28. What changes? What remains unchanged?

29. Find the image after performing reflection over line.



30. Explain how you determined where to place the image.

31. Are there any points inside the image? If yes, where do the points inside the image go when you perform a reflection?

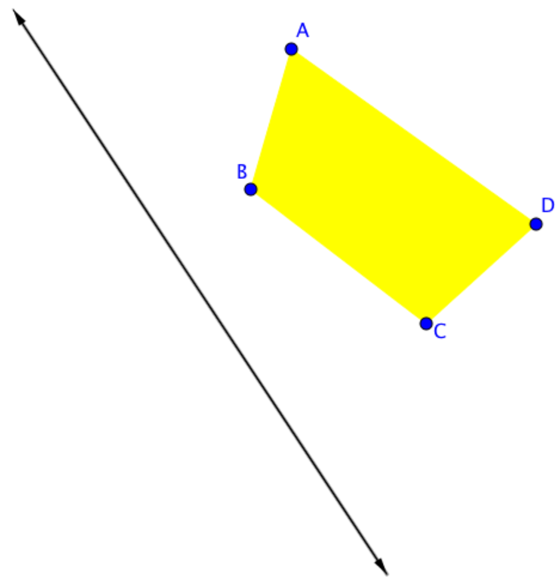
32. Are there any points outside the image? If yes, where do the points outside the image go when you perform a reflection?

33. Explain how you determined where to place any image point?

34. What changes? What remains unchanged?

35. How is #23 different from #29?

36. Find the image after performing a reflection over the given line.



37. Explain how you determined where to place the figure.

38. Are there any points inside the figure? If yes, where do the points inside the figure go when you perform a reflection?

39. Are there any points outside the figure? If yes, where do the points outside the figure go when you perform a reflection?

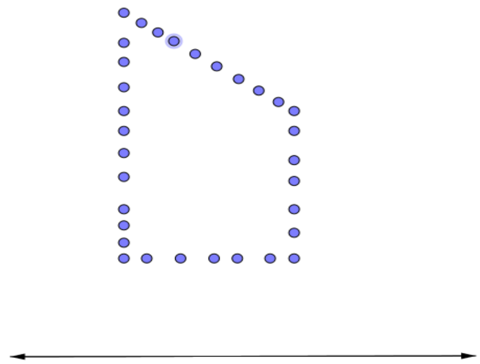
40. What changes? What remains unchanged?

41. What would happen if you move the reflection line horizontally to the left? What is constant? What is changing?

42. What would happen if you move the reflection line vertically up or down? What is constant? What is changing?

43. What would happen if you rotate the reflection line? What is constant? What is changing?

44. Find the image after performing reflection over line.



45. Is this a reflection? Why or why not?

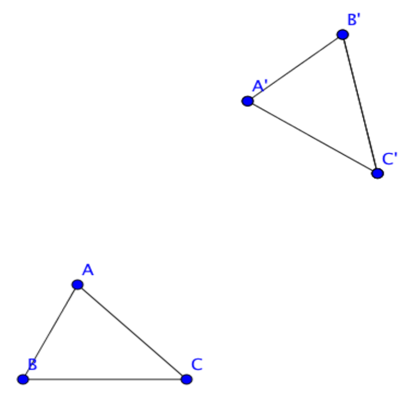
46. Which points of the figure reflect?

47. Are there any points inside the figure? If yes, where do the points inside the figure go when you reflect them?

48. Are there any points outside the figure? If yes, where do the points outside the figure go when reflect them?

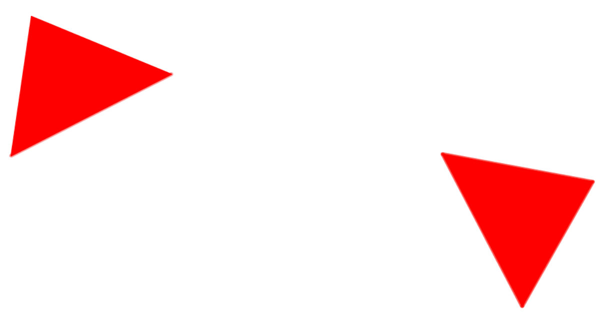
49. How is #36 different from #45?

50. Find the line of reflection given pre-image and image figures.



(Molina, 1992, p. 137)

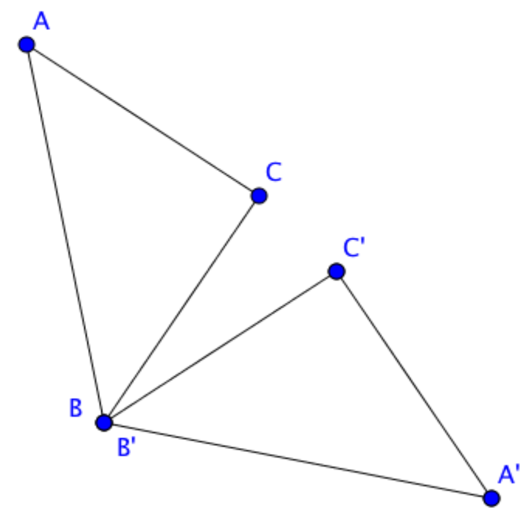
51. Find the line of reflection by given pre-image and image figures? How did you decide where to place the reflection line?



(Molina, 1992, p. 137)

52. How is #51 different from #52? (If the student focuses on only on the attribute of orientation or the attribute of size, ask: Can you find some other way to explain any differences)

53. Find the line of reflection by given a pre-image and image figures.



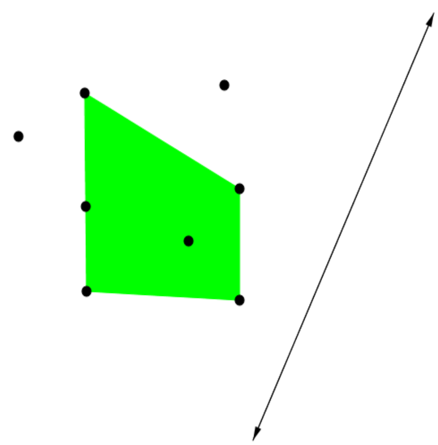
52. How did you decide where to place the reflection line?

54. What information do you need to perform a reflection?

55. How would you define a geometric reflection in terms of this information?

APPENDIX E. INTERVIEW 3

1. Find the image after performing reflection over line.



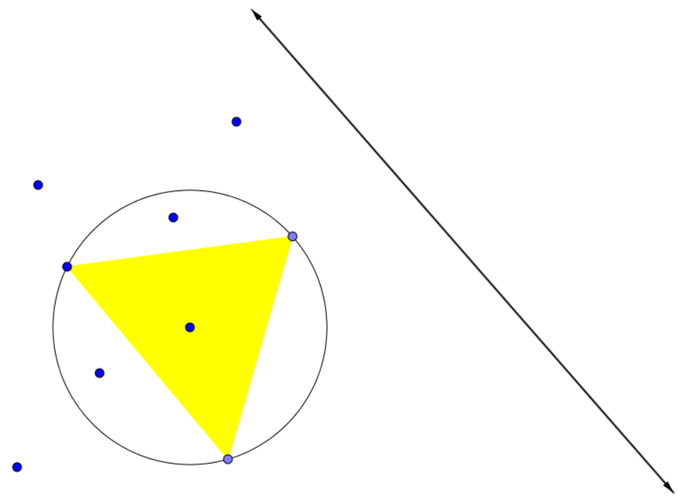
2. Explain how you determined where to place the figure.

3. Is there any point inside the figure? If yes, where do the points inside the figure go when you have reflection?

4. Is there any point outside the figure? If yes, where do the points outside the figure go when you have reflection?

5. What is constant? What is changing?

6. Find the image after performing reflection over line.



7. Explain how you determined where to place the image.
8. Is there any point on the image? If yes, where do the points on the image go when you have reflection?

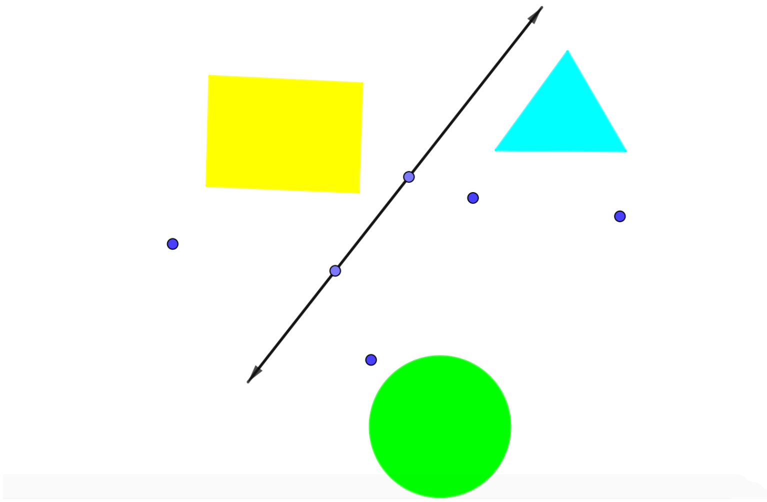
9. Is there any other point on the image? If yes, where do these points on the image go when you have reflection?

10. Is there any point inside the image? If yes, where do the points inside the image go when you have reflection?

11. Is there any point outside the image? If yes, where do the points outside the image go when you have reflection?

12. What did you reflected (points, vertices, sides, circle)? Do the points on the figure reflect? Do the points inside the figure reflect? Do the points outside the figure reflect?

13. Find the image over the line of reflection.



14. Is this reflection? Why? Why not?

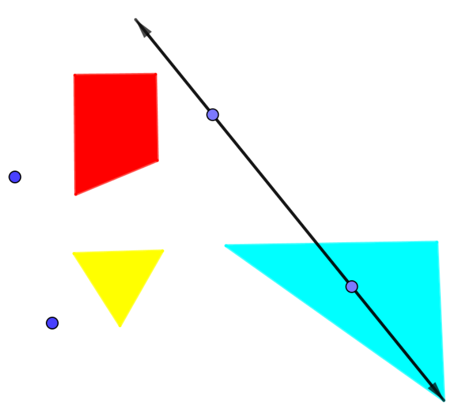
15. Explain how you determined where to place the image.

16. What did you reflect?

17. What happens two points on the line of reflection when you perform reflection?

18. What happens other points or images on the right side of reflection?

19. Find the image under the reflection over line of reflection.



20. Is this reflection? Why or Why not?

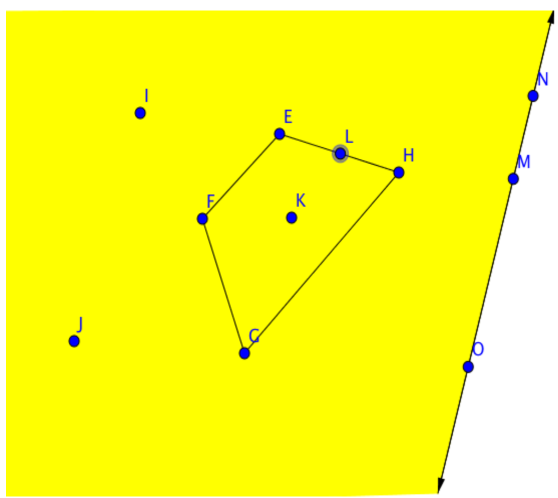
21. Explain how you determined where to place the image.

22. What did you reflect?

23. What happens two points on the line of reflection when you perform reflection?

24. What happens other points or images on the right side of reflection?

25. Find the image over the line of reflection.



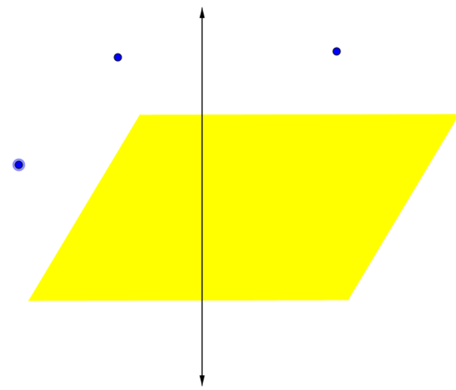
26. Is this reflection? Why or Why not?

27. Is there any point inside the figure? If yes, where do the points inside the figure go when you have reflection?

28. Is there any point outside the figure? If yes, where do the points outside the figure go when you have reflection?

29. Is there any points on the white part of the plane? If yes, where did those points go when you have reflection?

30. Find the image over line of reflection.



(Yanik, 2006, p. 102)

31. Explain how you determined where to place the image.

32. What did you reflect? (Asking probing questions for both side of the plane)

33. Is there any points on the white part of the plane? If yes, where did those points go when you have reflection?

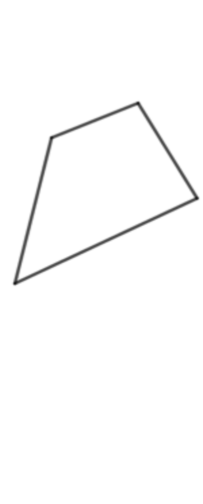
34. Find the image after performing reflection over line.

35. Is this reflection? Why or Why not?

36. What did you reflect?

37. Is there any points on the white part of the plane? If yes, where did those points go when you have reflection?

38. Find the image after performing a reflection first across line



(Molina, 1992, p. 144)

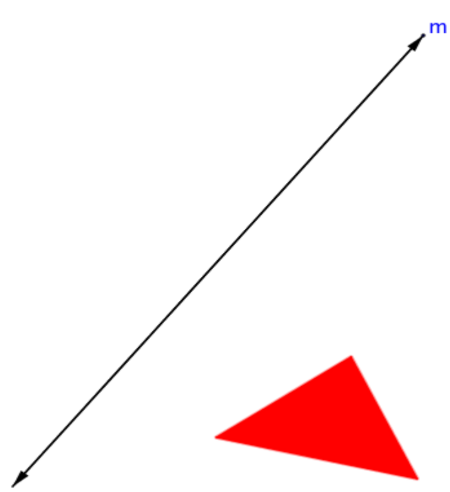
39. What did you reflect when you perform the first reflection?

40. What did you reflect when you perform the second reflection?

41. What is the relationship between first images and final images?

42. When the order of reflections is changed, what will be the results? What changes? What does not changes?

43. Find the image after performing a reflection in line "m"



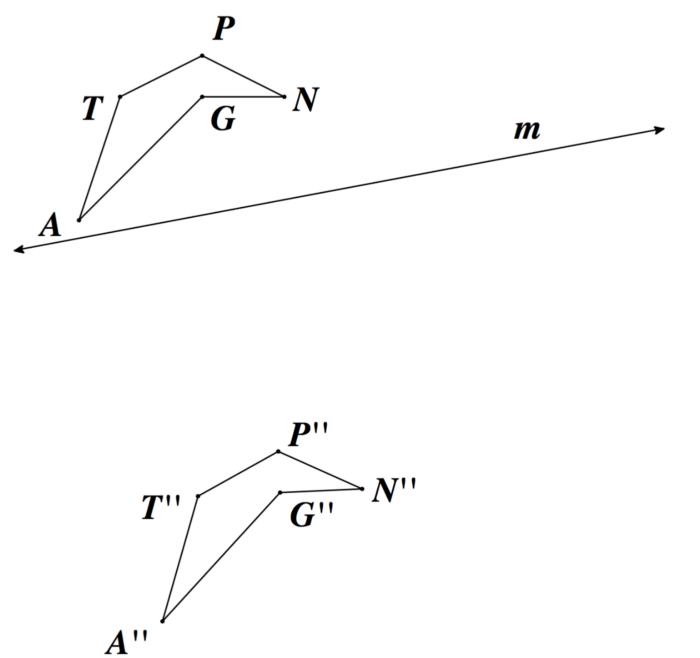
44. What did you reflect when you perform the first reflection?

45. What did you reflect when you perform the second reflection?

46. What is the relationship between first images and final images?

47. When the order of reflections is changed, what will be the results? What changes? What does not changes?

48. *PTAGN* is translated to image $P''T''A''G''N''$ by reflection first across line *m* and then across line *n* that is parallel to line *m*. Use your ruler and pencil to construct the lines needed to accurately locate line *n*.



47. Explain how you determined where to place the reflection line.