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Semiclassical Resolvent Estimates and Wave Decay in Low Regularity

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GLOSSARY OF NOTATION

$B(x, R)$	Open ball of radius $R > 0$ in $\mathbb{R}^n$ about $x \in \mathbb{R}^n$
$D_z(R)$	Open disk of radius $R > 0$ in $\mathbb{C}$ about $z \in \mathbb{C}$
$\mathbb{R}_\pm$	$\{x \in \mathbb{R} : \pm x \geq 0\}$
$\mathbf{1}_{\geq R}$	Characteristic function of the set $\{x \in \mathbb{R}^n : \lvert x\rvert \geq R\}$
u' or $\partial_r u$	Derivative of $u : \mathbb{R}^n \to \mathbb{C}$ with respect to the radial variable

ABSTRACT

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In this thesis, we prove weighted resolvent upper bounds for semiclassical Schrödinger operators. These upper bounds hold in the semiclassical limit.

First, we consider operators in dimension two when the potential is Lipschitz with long range decay. We prove that the resolvent norm grows at most exponentially in the inverse semiclassical parameter, while near infinity it grows at most linearly. Both of these bounds are optimal.

Second, we work in any dimension and require that the potential belong to L^∞ and have compact support. Again, we find that the weighted resolvent norm grows at most exponentially, but this time with an additional loss in the exponent.

Finally, we apply the resolvent bounds to prove two logarithmic local energy decay rates for the wave equation, one when the wavespeed is a compactly supported Lipschitz perturbation of unity, and the other when the wavespeed is a compactly supported L^∞ perturbation of unity.

1. INTRODUCTION

Let

$$\Delta := \sum_{j=1}^{n} \partial_j^2 \leq 0$$

be the Laplacian on $\mathbb{R}^n$. The central object of study in this thesis is the semiclassical Schrödinger operator,

$$P = P(h) := -h^2\Delta + V - E, \tag{1.1}$$

where $E > 0$ is the energy and $0 < h \leq 1$ is the semiclassical parameter.

We suppose that

$$V \in L^\infty(\mathbb{R}^n) \text{ is real-valued}, \tag{1.2}$$

with some decay at infinity. We will make more precise assumptions about the behavior of V at infinity within Theorems 1 and 2 below. By the Kato-Rellich Theorem, if V satisfies (1.2), then P is self-adjoint $L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ with respect to the domain $H^2(\mathbb{R}^n)$. Therefore, the resolvent $(P - z)^{-1}$ is bounded $L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ for all $z \in \mathbb{C} \setminus \mathbb{R}$.

For energies E belonging to the spectrum of P, we have

$$\|(P - i\varepsilon)^{-1}\|_{L^2 \to L^2} \to \infty \text{ as } \varepsilon \to 0^+.$$

However, if we take $(P - i\varepsilon)^{-1}$ to act on suitable weighted L^2-spaces, we can instead achieve a uniform resolvent bound as $\varepsilon \to 0^+$. In particular, we want to display the h-dependence of this upper bound, as this feature is important for applications such as local energy energy decay for the wave equation. We explore in detail the connection between resolvent estimates and local energy decay in Chapter 5.

In Chapter 3, we consider long-range Lipschitz potentials in dimension two.

Theorem 1. *Let $n = 2$. In addition to (1.2), suppose that ∇V, defined in the sense of distributions, belongs to $(L^\infty(\mathbb{R}^2))^2$, and that there exist $\delta_0, c_0 > 0$, such that*

$$V(x) \leq c_0(1 + |x|)^{-\delta_0}, \qquad |\nabla V(x)| \leq c_0(1 + |x|)^{-1-\delta_0}, \tag{1.3}$$

for almost all $x \in \mathbb{R}^2$. Furthermore, let $[E_{\min}, E_{\max}] \subseteq (0, \infty)$ be a compact interval. Then, for any $s > 1/2$ there are $C, R, h_0 > 0$ such that

$$\left\| (1 + |x|)^{-s}(P(h) - i\varepsilon)^{-1}(1 + |x|)^{-s} \right\|_{L^2(\mathbb{R}^2) \to H^2(\mathbb{R}^2)} \leq e^{\frac{C}{h}}, \tag{1.4}$$

$$\left\| (1 + |x|)^{-s}\mathbf{1}_{\geq R}(P(h) - i\varepsilon)^{-1}\mathbf{1}_{\geq R}(1 + |x|)^{-s} \right\|_{L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)} \leq C/h, \tag{1.5}$$

for all $E \in [E_{\min}, E_{\max}]$, $0 < \varepsilon < 1$, and $h \in (0, h_0]$, where $\mathbf{1}_{\geq R}$ is the characteristic function of $\{x \in \mathbb{R}^2 \colon |x| \geq R\}$.

In Chapter 4, we study the case where the potential is bounded and compactly supported in arbitrary dimension.

Theorem 2. *Let $n \geq 1$, $V \in L^\infty_{\text{comp}}(\mathbb{R}^n)$, and $[E_{\min}, E_{\max}] \subseteq (0, \infty)$. For any $s > 1/2$, there exist $C, h_0 > 0$ such that*

$$\left\| (1 + |x|)^{-s}(P(h) - i\varepsilon)^{-1}(1 + |x|)^{-s} \right\|_{L^2(\mathbb{R}^n) \to H^2(\mathbb{R}^n)} \leq e^{Ch^{-4/3}\log(h^{-1})}, \tag{1.6}$$

for all $E \in [E_{\min}, E_{\max}]$, $0 < \varepsilon < 1$, and $h \in (0, h_0]$.

The proofs of Theorems 1 and 2 proceed in the following manner. First, we establish a certain Carleman estimate (see (3.11) and (4.2)) which holds for functions in $C_0^\infty(\mathbb{R}^n)$. Then, we apply a density argument, which uses the Carleman estimate, to show the resolvent has the stated bound(s) as operator $L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$. Finally, we convert the $L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ exponential bound into an $L^2(\mathbb{R}^n) \to H^2(\mathbb{R}^n)$ exponential bound. This last part of the argument is given in Appendix C.

Since the completion of the first draft of this thesis, the author has learned about the independent and parallel work of Klopp and Vogel [KlVo18]. They use a different Carleman estimate to show that, if the support of V is contained in the ball $B(0, R) :=$

$\{x \in \mathbb{R}^n : |x| < R\}$, and χ is a smooth cutoff supported near $B(0,R)$, then for any compact interval $I \subseteq \mathbb{R} \setminus \{0\}$, there exist constants $C > 0$ and $h_0 \in (0,1]$ so that

$$\lim_{\varepsilon \to 0^+} \|(-h^2\Delta + V - \lambda^2 - i\varepsilon)^{-1} v\|_{H^1(B(0,R))} \leq e^{Ch^{-4/3}\log(h^{-1})} \|v\|_{L^2(B(0,R))}, \tag{1.7}$$

for all $h \in (0, h_0]$ and all $v \in L^2_{\text{comp}}(B(0,R))$.

Exponential resolvent bounds have been studied under a wide range of geometric, regularity, and decay assumptions. Burq was the first to prove an $O(e^{Ch^{-1}})$ resolvent bound. In [Bu98], he showed such a bound holds for smooth, compactly supported perturbations of the Laplacian outside an obstacle. He later established the same bound for smooth, long-range perturbations [Bu02]. Cardoso and Vodev [CaVo02] extended Burq's estimate in [Bu02] to infinite volume Riemannian manifolds which may contain cusps.

In lower regularity, Datchev [Da14] proved (1.4) and (1.5) for dimension $n \neq 2$, while only needing a decay condition on the radial derivative $\partial_r V$, rather than on ∇V. Vodev [Vod14] showed an $O(e^{Ch^{-\ell}})$ bound, $0 < \ell < 1$, for potentials on $\mathbb{R}^n$, $n \geq 3$, that are Hölder continuous, h-dependent, and have decay depending on ℓ. Rodnianski and Tao [RT15] considered short-range, L^∞ potentials on asymptotically conic manifolds of dimension $n \geq 3$, and proved a non-semiclassical version of (1.6) in which the h-dependence of the right side is contained in an unspecified constant.

The novel aspect of Theorem 1, then, is that (1.4) and (1.5) are now established in dimension two, while maintaining low regularity and mild decay assumptions on V and its derivatives. The novel aspect of Theorem 2 and (1.7) is that, when $n \geq 3$, they are the first explicit h-dependent bound on the weighted resolvent for L^∞ potentials. When $n \leq 2$, they are the first weighted resolvent bound of any kind for general $V \in L^\infty(\mathbb{R}^n)$.

We should also mention Theorem 2.29 in [DyZw], which is a related result for compactly supported L^∞ potentials in dimension one. It says that, given $V \in L^\infty_{\text{comp}}(\mathbb{R})$

and $[E_{\min}, E_{\max}] \subseteq (0, \infty)$, there exists a constant $c > 0$ depending on $E_{\min}$, $E_{\max}$, and V such that the meromorphic continuation of the cutoff resolvent

$$\chi(-h^2\partial_x + V - z)^{-1}\chi, \qquad \chi \in C_0^\infty(\mathbb{R})$$

from $\operatorname{Im} z > 0$, $\operatorname{Re} z > 0$ to $\operatorname{Im} z \leq 0$, $\operatorname{Re} z > 0$ has the property that

$$z \text{ is a resonance of the continuation, } \operatorname{Re} z \in [E_{\min}, E_{\max}] \implies \operatorname{Im} z \geq -e^{-ch^{-1}}. \tag{1.8}$$

See Section 2.3 for the definition of a resonance for a related operator, and also [DyZw, Section 2.8] for further details.

Because resonance free regions are closely related to resolvent bounds, see, for instance, the arguments in Section 5.3, (1.8) strongly suggests that, in dimension one, the right side of (1.6) can be improved to $e^{C/h}$. The author consider this improvement shortly.

The $O(e^{Ch^{-1}})$ resolvent bound appearing in Theorem 1, as well as in [Bu98, Bu02, Da14, CaVo02], is well-known to be optimal. In particular, Datchev, Dyatlov, and Zworski [DDZ15] established the lower bound

$$\left\| (1 + |x|)^{-s}\mathbf{1}_{\geq R}(P(h) - i\varepsilon)^{-1}(1 + |x|)^{-s} \right\|_{L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)} \geq e^{ch^{-1}},$$

for a suitable smooth potential and an h-dependent energy $E(h) > 0$. See their paper for more background and references. However, it is still an open problem to determine the optimal resolvent bound for general $V \in L^\infty$. Relatively few results are known in this setting, and the specific examples studied thus far [Be03, DdeH16] have a $O(e^{Ch^{-1}})$ resolvent upper bound. Therefore, it would be very interesting to know how much low regularity problems can deviate from the known estimates, and this will be the subject of future work by the author.

Better upper bounds are known when V is smooth and conditions are placed on the Hamilton flow $\Phi(t) := \exp t(2\xi \cdot \nabla_x - \nabla_x V \cdot \nabla_\xi)$. The key dynamical object is the trapped set $\mathcal{K}(E)$ at energy E, defined by

$$\begin{aligned} \mathcal{K}(E) := \{(x_0, \xi_0) \in T^*(\mathbb{R}^n) : & |\xi_0|^2 + V(x_0) = E, \\ & \text{and } |\Phi(t)(x_0, \xi_0)| \text{ is bounded as } |t| \to \infty\}. \end{aligned}$$

For instance, if $V \in C_0^\infty(\mathbb{R}^n)$, $s > 1/2$, and $\mathcal{K}(E) = \emptyset$, then there exists $C, h_0 > 0$ such that

$$\left\| (1+|x|)^{-s}(P(h) - i\varepsilon)^{-1}(1+|x|)^{-s} \right\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} \leq C/h, \tag{1.9}$$

for all $\varepsilon > 0$ and $h \in (0, h_0]$.

This nontrapping estimate is due to Robert and Tamurua [RoTa87]. Proofs of such a result typically rely on propagation of singularities. For more recent results and references, see [BoBuRa10, Vod14, HiZw17b, Zw17].

In view of (1.9), one interpretation of the exterior estimate (1.5) is that applying cutoffs supported far away from zero removes the losses from (1.4) due to trapping. Although, in the setting of Theorems 1 and 2, $\Phi(t)$ may be undefined because ∇V may not be continuous.

Resolvent estimates up to the spectrum have important applications to regularity and decay results for operators involving P. In Chapter 5, we will consider the wave equation,

$$\begin{cases} (\partial_t^2 - c^2(x)\Delta)u(x,t) = 0, \quad (x,t) \in \mathbb{R}^n \times (0,\infty), \\ u(x,0) = u_0(x), \\ \partial_t u(x,0) = u_1(x). \end{cases} \tag{1.10}$$

To establish the connection between (1.10) and semiclassical resolvent estimates, we first take the Fourier transform of the wave operator $-\partial_t^2 - c^2\Delta$ with respect to time variable, replacing each instance of ∂_t with $i\lambda$. By doing so, we arrive at the operator $-c^2\Delta - \lambda^2$, $\lambda \in \mathbb{R}$. Then, we identify $h = |\lambda|^{-1}$, $|\lambda| >> 1$, and $V = 1 - c^{-2} \in L^\infty_{\text{comp}}$, to get

$$\chi(-c^2\Delta - \lambda^2)^{-1}\chi = h^2\chi(-h^2\Delta + V - 1)^{-1}\chi c^{-2}, \qquad \chi \in C_0^\infty(\mathbb{R}^n). \tag{1.11}$$

Therefore, we can use (1.4), (1.5), and (1.6), along with an appropriate χ, to obtain λ-dependent estimates for the cutoff resolvent $\chi(-c^2\Delta - \lambda^2)^{-1}\chi$ when $|\lambda| >> 1$. A formal and complete argument appears in Section 5.3.

In Chapter 5, we prove a general theorem, Theorem 5, which shows that exponential semiclassical resolvent bounds imply logarithmic local energy decay for the solution to (1.10). Two special cases of Theorem 5, which correspond to the resolvent bounds in (1.4) and (1.6), respectively, are the following.

Theorem 3. *Assume that*

$$c = c(x) > 0, \quad c, c^{-1} \in L^\infty(\mathbb{R}^n), \quad , \quad \mathrm{supp}(c-1) \text{ is compact.} \tag{1.12}$$

In addition, assume $\nabla c \in (L^\infty(\mathbb{R}^n))^n$. *Suppose the supports of* u_0 *and* u_1 *are contained in* $B(0, R_1)$, *and that* $\nabla u_0 \in (H^1(\mathbb{R}^n))^n$ *and* $u_1 \in H^1(\mathbb{R}^n)$. *Then for any* $R_2 > 0$, *there exists* $C > 0$ *such that the solution* u *to* (1.10) *satisfies for* $t \geq 0$,

$$\left(\int_{B(0,R_2)} |\nabla u|^2 + c^{-2}|\partial_t u|^2 dx\right)^{\frac{1}{2}} \leq \frac{C}{\log(2+t)} \left(\|\nabla u_0\|_{(H^1(\mathbb{R}^n))^n} + \|u_1\|_{H^1(\mathbb{R}^n)}\right). \tag{1.13}$$

Theorem 4. *Consider the same setting* (1.12) *as in Theorem 3, but without the assumption that the distributional derivatives of* c *are bounded. Then for any* $\varepsilon > 0$ *and* $R_2 > 0$, *there exists* $C > 0$ *such that the solution* u *to* (1.10) *satisfies for* $t \geq 0$,

$$\left(\int_{B(0,R_2)} |\nabla u|^2 + c^{-2}|\partial_t u|^2 dx\right)^{\frac{1}{2}} \leq \frac{C}{\log^{\frac{3}{4+\varepsilon}}(2+t)} \left(\|\nabla u_0\|_{(H^1(\mathbb{R}^n))^n} + \|u_1\|_{H^1(\mathbb{R}^n)}\right). \tag{1.14}$$

In Theorem 5, we will even see that one can put additional powers of $\log(2+t)$ in the denominators of (1.13) and (1.14) if u_0 and u_1 possess greater regularity with respect to the differential operator $-c^2(x)\Delta$.

In contrast with the local energy decay in Theorems 3 and 4, the global energy of the solution to (1.10) is conserved because the wave propagator is unitary. See (5.6) in Section 5.1.

The decay rate (1.13) was first obtained by Burq [Bu98,Bu02] for smooth perturbations of the Laplacian outside an obstacle. Similar decay rates have been established established on $\mathbb{R}^n$ when the Laplacian is defined by an asymptotically Euclidean metric, see [Bo11, CaVo04]. Therefore, the novel aspect of Theorem 1.13, is that (1.13) now with a weaker regularity condition on the wavespeed.

For logarithmic decay rates in transmission problems and general relativity, see [Be03, Ga17, Mo16].

Logarithmic decay rates are well-known to be optimal when resonances are exponentially close to the real axis. This connection was observed by Ralston [Ra69]. He later showed [Ra71] that such resonances exist for a certain class of smooth wavespeeds. See [HoSm14] for a related construction in general relativity.

The study of local energy decay more broadly has a long history which we will not review here. Additional papers that use techniques similar to those in this thesis include [PoVo99] and [Ch09]. See also [HiZw17a] for more historical background and references.

Semiclassical resolvent bounds have proven useful in several other contexts. Exterior bounds such as (1.5) are known to imply Kato local smoothing [DyZw, Theorem 7.2]:

$$\int_{\mathbb{R}} \|\langle x\rangle^{-s}\mathbf{1}_{\geq R}\varphi(P)e^{(-itP/h)}u\|^2_{L^2(\mathbb{R}^n)} \leq C\|u\|^2_{L^2(\mathbb{R}^n)}, \qquad \varphi \in C_0^\infty(0,\infty).$$

Christiansen [Chr15] used a resolvent bound of the form (1.5) to find a lower bound on the resonance counting function on even-dimensional Riemannian manifolds that are flat near infinity and contain a compactly supported perturbation. Furthermore, resolvent estimates are related to integrated local energy [RT15] and Strichartz estimates for the wave equation [SmSo00, Bu03, Me04, BoTz07, MMT08]. See also [St02, Mi04, GuHaSi13] for further applications of exterior resolvent estimates like (1.5).

Apart from the appendices, most of the presentation in this thesis also appears in the preprints [Sh16, Sh17, Sh18].

2. PRELIMINARIES

In this chapter, we recall several facts about the analytic continuation of the cutoff resolvent for the free Laplacian. We also explain why, when the resolvent is perturbed to include a suitable wavespeed, the cutoff resolvent still continues meromorphically, and that there are no real poles, away from zero. Finally, we introduce a certain homogeneous Sobolev space and decribe how the homogeneous Sobolev space of a ball behaves with respect to the perturbed resolvent. Several proofs in Chapter 5 rely on these facts. The interested reader can consult Appendices A and B for more details.

2.1 Continuation of the free resolvent

If $\operatorname{Im}\lambda > 0$, then $\lambda^2 \notin \mathbb{R}_+$. Therefore,

$$R_0(\lambda) := (-\Delta - \lambda^2)^{-1} : L^2(\mathbb{R}^n) \to H^2(\mathbb{R}^n)$$

is well-defined as a bounded operator, as standard elliptic estimates like those in Appendix C show. Moreover, for $\chi \in C_0^\infty(\mathbb{R}^n)$, the cutoff resolvent is bounded

$$\chi R_0(\lambda)\chi = \chi(-\Delta - \lambda^2)^{-1}\chi : L^2(\mathbb{R}^n) \to H^2(\mathbb{R}^n),$$

and continues analytically from $\operatorname{Im}\lambda > 0$ to $\mathbb{C}$ when $n \geq 2$ is odd and to $\mathbb{C} \setminus i\mathbb{R}_-$ when $n \geq 2$ is even. In fact, in even dimensions, the continuation can be made to the logarithmic cover of $\mathbb{C} \setminus \{0\}$, although we will not need this stronger fact.

Furthermore, the continuation of $\chi R_0(\lambda)\chi$ has the expansion

$$\chi R_0(\lambda)\chi = E_1(\lambda) + \lambda^{n-2} \log \lambda E_2(\lambda), \tag{2.1}$$

for $\lambda \in \mathbb{C} \setminus i\mathbb{R}_-$. Here, $E_1(\lambda)$ and $E_2(\lambda)$ are entire operator-valued functions, and $E_2 \equiv 0$ when n is odd.

In Appendix A, we establish the meromorphic continuation of $\chi R_0(\lambda)\chi : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ and justify (2.1) by computing the integral kernel of $\chi R_0(\lambda)\chi$. Further details can be found in [DyZw, Chapters 2 and 3], [Sj, Section 2.1] and [Vo01, Section 1.1].

2.2 Estimates for the continued free resolvent

Next, we recall well-known $L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ estimates for the cutoff resolvent away from the origin. In section 5.3, we will use these estimates to establish a bound on the perturbed resolvent at high energy.

$$\forall M > 0, \quad \exists C_M > 0 : \quad \text{if } |\operatorname{Re}\lambda| \geq M, \operatorname{Im}\lambda \geq -M, \text{ and } |\alpha_1| + |\alpha_2| \leq 2, \text{ then}$$
$$\|\partial^{\alpha_1}\chi R_0(\lambda)\chi\partial^{\alpha_2}\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} \leq C_M|\lambda|^{|\alpha_1|+|\alpha_2|-1}. \tag{2.2}$$

Using the Cauchy formula with (2.2) implies, for a different constant $\tilde{C}_M > 0$,

$$\left\| \frac{d}{d\lambda}\partial^{\alpha_1}\chi R_0(\lambda)\chi\partial^{\alpha_2} \right\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} \leq \tilde{C}_M|\lambda|^{|\alpha_1|+|\alpha_2|-1}, \tag{2.3}$$
$$|\operatorname{Re}\lambda| \geq M, \ \operatorname{Im}\lambda > -M, \ |\alpha_1| + |\alpha_2| \leq 2.$$

We omit the justification of (2.2), but note that one way to obtain it is to apply Schur's estimate [DyZw, Section A.5] to the integral kernel that we compute in Section A.2 of Appendix A. See also [DyZw, Section 3.1] for the odd dimensional case as well as [Vod14, Section 5].

2.3 Continuation of the perturbed resolvent

Consider a wavespeed c as in (1.12). We show in Appendix B that the operator $-c^2(x)$ is self-adjoint and nonnegative on the Hilbert space $L^2_c(\mathbb{R}^n) \coloneqq L^2(\mathbb{R}^n, c^{-2}dx)$ with respect to the domain $H^2(\mathbb{R}^n)$. Set $R(\lambda) \coloneqq (-c^2\Delta - \lambda^2)^{-1}$, $\operatorname{Im}\lambda > 0$. For $\chi \in C_0^\infty(\mathbb{R}^n)$, the cutoff resolvent $\chi R(\lambda)\chi$ satisfies the assumptions of the black box scattering framework introduced in [SjZw91] and also presented in [DyZw, Sj]. This implies that $\chi R(\lambda)\chi$ continues meromorphically $L^2(\mathbb{R}^n) \to H^2(\mathbb{R}^n)$ from $\operatorname{Im}\lambda > 0$ to

$\mathbb{C} \setminus \{0\}$ when $n \geq 2$ is odd, and to $\mathbb{C} \setminus i\mathbb{R}_-$ when $n \geq 2$ is even. As in the case of the free resolvent, the continuation in even dimensions can be made to the logarithmic cover of $\mathbb{C} \setminus \{0\}$, although this stronger result is not needed for our purposes. The poles of this meromorphic continuation are referred to as resonances of $\chi R(\lambda)\chi$

It is also follows that if $\lambda \in \mathbb{R} \setminus \{0\}$ is a resonance, then there must exist an embedded eigenvalue corresponding to λ. That is, there exists a nonzero function $u \in H^2_{\text{comp}}(\mathbb{R}^n)$ such that $(-c^2\Delta - \lambda^2)u = 0$. For more details, see Theorems 4.17 and 4.18 in [DyZw]. However, a Carleman estimate [DyZw, Lemma 3.31] rules out the possibility of embedded eigenvalues on $\mathbb{R} \setminus \{0\}$. Therefore, the continuation of $\chi R(\lambda)\chi$ has no poles there.

We give some additional details in Appendix B.

2.4 The homogeneous Sobolev space

Let $\Omega \subseteq \mathbb{R}^n$ be open, and let $\dot{H}^1(\Omega)$ denote the homogeneous Sobolev space of order one, defined as the Hilbert completion of $C_0^\infty(\Omega)$ with respect to the norm

$$\|\varphi\|_1^2 := \int_\Omega |\nabla\varphi(x)|^2 dx.$$

Thus, elements of $\dot{H}^1(\Omega)$ are equivalence classes $[\varphi_m]$ of sequences $\{\varphi_m\} \subseteq C_0^\infty(\Omega)$ which are Cauchy with respect to the $\|\cdot\|_1$-norm. For an element $u = [\varphi_m] \in \dot{H}^1(\Omega)$, we denote by ∇u the vector which is the limit in $(L^2(\Omega))^n$ of the vectors $\nabla\varphi_m$.

Because the non-homogeneous Sobolev space $H^1(\mathbb{R}^n)$ is the completion of $C_0^\infty(\mathbb{R}^n)$ with respect to a stronger norm, by inclusion we may regard $H^1(\mathbb{R}^n)$ as a closed subspace of $\dot{H}^1(\mathbb{R}^n)$. Also, for any $\Omega \subseteq \mathbb{R}^n$, the inclusion map $C_0^\infty(\Omega) \to C_0^\infty(\mathbb{R}^n)$ induces an isometry $\dot{H}^1(\Omega) \to \dot{H}^1(\mathbb{R}^n)$. So we may also regard $\dot{H}^1(\Omega)$ as a closed subspace of $\dot{H}^1(\mathbb{R}^n)$.

We note, for the sake of completeness, that, for $n \geq 2$, $\dot{H}^1(\mathbb{R}^n)$ may be regarded as a set of translation classes of functions $u \in H^1_{loc}(\mathbb{R}^n)$ such that $\nabla u \in L^2(\mathbb{R}^n)$, equipped with the inner product

$$(u, v) \mapsto \langle \nabla u, \nabla v \rangle_{(L^2(\mathbb{R}^n))^n}.$$

See [OrSu12] for further details.

2.5 Operators on the homogeneous Sobolev space of a ball

If $\lambda^2 \notin \mathbb{R}_+$, there is a constant C_λ depending on λ such that

$$\|R(\lambda)\varphi\|_{H^2(\mathbb{R}^n)} \leq C_\lambda \|\varphi\|_{L^2(\mathbb{R}^n)}, \qquad \varphi \in C_0^\infty(\mathbb{R}^n). \tag{2.4}$$

This follows from integration by parts and an elementary ellipticity argument, which we present in Appendix B.

Furthermore, if the support of φ is required to lie in a fixed ball $B(0,R)$, there is a Poincaré-type inequality for all $n \geq 2$,

$$\|\varphi\|_{L^2(\mathbb{R}^n)} \leq C_R \|\nabla\varphi\|_{L^2(\mathbb{R}^n)}, \qquad \varphi \in C_0^\infty(B(0,R)), \tag{2.5}$$

where $C_R \to \infty$ as $R \to \infty$. We prove (2.5) in Appendix C.

Having (2.4) and (2.5) allow us to extend $R(\lambda) : L^2(\mathbb{R}^n) \to H^2(\mathbb{R}^n)$ to a bounded operator $\dot{H}^1(B(0,R)) \to H^2(\mathbb{R}^n)$ by setting

$$R(\lambda)[\varphi_m] := R(\lambda)\left(L^2\text{-}\lim \varphi_m\right)\Bigg(\qquad [\varphi_m] \in \dot{H}^1(B(0,R)), \qquad \lambda^2 \notin \mathbb{R}_+, \tag{2.6}$$

where L^2-$\lim \varphi_m$ denotes the $L^2(\mathbb{R}^n)$-limit of $\{\varphi_m\}$, which exists on account of (2.5).

3. SEMICLASSICAL RESOLVENT BOUNDS FOR LIPSCHITZ POTENTIALS IN DIMENSION TWO

In this chapter we prove Theorem 1. Our proof hinges on a Carleman estimate (3.11) that is similar to the Carleman estimates in [CaVo02, Da14]. The strategy to produce the Carleman estimate is to construct two radial weight functions, $\varphi(r)$ and $w(r)$ that interact favorably with the conjugated operator $r^{-(n-1)/2}Pr^{(n-1)/2}$. This conjugation gives rise to the so-called effective potential term, which takes the form $(n-1)(n-3)(2r)^{-2}$. In dimension $n \geq 3$, the effective potential is positive and decreasing, and can be discarded in the ensuing estimates. But in dimension $n = 2$ only, the effective potential has a negative pole at the origin. Therefore, the additional challenge in our setting is that w needs to decay sufficiently at the origin to counteract this negative blow-up. As a result, our Carleman estimate in dimension two comes with a loss at the origin, because w is weak there. In Section 3.3 we make a resolvent gluing argument that removes the loss and allows us to establish Theorem 1.

By making C larger and h_0 smaller in Theorem 1, we can assume without loss of generality that $c_0 = 1/2$. That is, we may assume

$$V(x) \leq \frac{1}{2}(1+|x|)^{-\delta_0}, \qquad |\nabla V(x)| \leq \frac{1}{2}(1+|x|)^{-1-\delta_0}. \tag{3.1}$$

We may also assume without loss of generality that

$$0 < 2s - 1 < \delta_0 < 1/2. \tag{3.2}$$

This is because decreasing δ_0 only weakens the decay of V and ∇V, and increasing s only decreases the weighted resolvent norm. To simplify notation, we set $\delta \coloneqq 2s-1 > 0$ throughout this chapter, we use the same notation in Chapter 4).

The outline of this chapter is as follows. In Section 3.1 we justify the existence of the weight function φ by citing lemmas from [Da14] and [DdeH16]. In Section 3.2, we prove the Carleman estimate. In Section 3.3 we glue two versions of the Carleman estimate together and then prove Theorem 1 via a density argument.

3.1 Construction of the Carleman weight

We use the usual polar coordinates $(r, \theta) \in (0, \infty) \times [0, 2\pi)$ to denote a point $(x, y) = (r\cos\theta, r\sin\theta) \in \mathbb{R}^2 \setminus \{0\}$. Let $\partial_r V$ denote the radial distributional derivative of V. That is,

$$\partial_r V(r, \theta) := \partial_x V(r, \theta)\cos\theta + \partial_y V(r, \theta)\sin\theta. \tag{3.3}$$

Throughout this section, we need only assume that

$$V \leq (1 + r)^{-\delta_0}, \qquad |\partial_r V| \leq (1 + r)^{-1-\delta_0} \tag{3.4}$$

for almost all $(r, \theta) \in \mathbb{R}^2 \setminus \{0\}$. Note that (3.1) implies (3.4).

The following two lemmas, Lemmas 3.1 and 3.2, we state without proof. They establish the existence of the radial weight function $\varphi(r)$ we use in the Carleman estimate (3.11). Lemma 3.1 is a version of [Da14, Lemma 2.1]. For each δ sufficiently small, it constructs a continuous function $\psi_\delta(r)$ that obeys a crucial inequality with V and $\partial_r V$ and $E_{\min}$.

Lemma 3.2 is an adaptation of [DdeH16, Proposition 3.1], and it constructs φ as a solution to an ordinary differential equation with right hand side ψ. Although we do not prove Lemma 3.2, we remark that its proof is similar to the one for Lemma 4.3 in Chapter 4, and there we do supply all the details.

Lemma 3.1. *For $\delta > 0$ sufficiently small, there exist constants $B, R_0, R_1 > 0$ depending on δ and $E_{\min}$ so that the function*

$$\psi = \psi_\delta(r) := \begin{cases} \delta_0^{-1}, & r \leq R_0, \\ \frac{B}{1-(1+r)^{-\delta}} - \frac{E_{\min}}{4}, & R_0 < r < R_1, \\ 0, & r \geq R_1, \end{cases}$$

is continuous and satisfies the inequality

$$-\frac{E_{\min}}{2} \le \psi - V - (\partial_r V - \psi')\frac{1-(1+r)^{-\delta}}{\delta(1+r)^{-1-\delta}} \tag{3.5}$$

for almost all points $(r,\theta) \in \mathbb{R}^2 \setminus \{0\}$.

Lemma 3.2. *For any* $h > 0$, *there is a solution* $\varphi = \varphi_h(r) \in C^2(0,\infty)$ *to the equation*

$$(\varphi'(r))^2 - h\varphi''(r) = \psi(r), \tag{3.6}$$

with the properties that $\varphi' \ge 0$ *and the support of* φ' *is contained in* $[0, R_0]$ *and independent of* h.

Note that, because $\varphi'' = ((\varphi')^2 - \psi)/h$, it follows that $\varphi'''(r)$ exists for almost all $r \in [0,\infty)$.

3.2 Proof of the Carleman estimate

We continue to assume only that (3.4) holds throughout this section (that is, we do not need to assume (3.1)). Before establishing the Carleman estimate, which is Lemma 3.4, we need to prove a preliminary inequality.

Define the weight $w(r)$ to be

$$w = w_\delta(r) := \begin{cases} \tilde{c}r^2, & r \le R_0, \\ 1-(1+r)^{-\delta}, & r > R_0, \end{cases}$$

where we set $\tilde{c} := (1-(1+R_0)^{-\delta})/R_0^2$ to make w continuous on $[0,\infty)$. Note that

$$\frac{w}{w'} = \frac{1-(1+r)^{-\delta}}{\delta(1+r)^{-1-\delta}}, \qquad r > R_0. \tag{3.7}$$

Therefore, (3.5) shows that

$$-\frac{E_{\min}}{2} w' \le (\psi - V)w' + (-\partial_r V + \psi')w, \qquad r > R_0. \tag{3.8}$$

Set

$$V_\varphi := V - (\varphi')^2 + h\varphi'' - h^2/4r^2.$$

The inequality we need is as follows.

Lemma 3.3. *If $\delta > 0$ is small enough, then there exists $h_0 \in (0, 1]$ so that if $h \in (0, h_0]$ and $E \geq E_{\min}$.*

$$\partial_r \left(w(r)(E - V_\varphi(r, \theta))\right) \geq \frac{E_{min}}{4} w'(r), \tag{3.9}$$

for almost all $(r, \theta) \in \mathbb{R}^2 \setminus \{0\}$.

Proof. First, we expand $\partial_r(w(E - V_\varphi))$, making use of (3.6),

$$\partial_r(w(E - V_\varphi)) = (E - V + \psi)w' + (-\partial_r V + \psi')w + \frac{h^2}{4r^2}\left(w' - \frac{2w}{r}\right).$$

We use this expansion to investigate two cases separately: first when $r \in (0, R_0)$, and second, when $r \in (R_0, \infty)$. In the case $r \in (0, R_0)$, $\psi = \delta_0^{-1}$, and hence $\psi' = 0$. We also have, $w' - 2w/r = 0$. Using these facts, along with the bounds on V and $\partial_r V$ from (3.4), we arrive at the following inequality for $\partial_r(w(E - V_\varphi))$ when $r \in (0, R_0)$.

$$\begin{aligned}
\partial_r(w(E - V_\varphi)) &= \left(E + \delta_0^{-1} - V - \frac{r\partial_r V}{2}\right)w' \\
&\geq \left(E + \delta_0^{-1} - \frac{1}{(1+r)^{\delta_0}} - \frac{r}{2(1+r)^{1+\delta_0}}\right)w' \\
&\geq \left(E + \delta_0^{-1} - \frac{3}{2}\right)w' \\
&\geq \frac{E_{\min}}{4} w'.
\end{aligned}$$

The last inequality follows because $\delta_0 < 1/2$.

It remains to establish (3.9) in the case where $r \in (R_0, \infty)$. According to (3.8), we have

$$(E - V + \psi)w' + (-\partial_r V + \psi')w \geq \frac{E_{\min}}{2} w', \qquad r > R_0.$$

And so, to establish (3.9) when $r > R_0$, it suffices to show that, for h small enough, we can achieve

$$\frac{h^2}{4r^2}\left(w' - \frac{2w}{r}\right) \geq -\frac{E_{\min}}{4} w', \qquad r > R_0. \tag{3.10}$$

To this end, define $g(r)$ to be the function

$$g := g_\delta(r) = \frac{1}{4r^2}\left(1 - \frac{2}{\delta}\frac{(1+r)^{1+\delta} - (1+r)}{r}\right), \qquad r > 0.$$

Observe that g is bounded on the interval $[R_0, \infty)$, and that for $r \in (R_0, \infty)$, we have

$$\left(\frac{h^2}{4r^2}\right)\left(w' - \frac{2w}{r}\right) = h^2 g(r) w'.$$

If we set $h_0 = (E_{\min}/4 \sup_{[R_0,\infty)} |g|))^{\frac{1}{2}} > 0$, then (3.10) holds for $h \in (0, h_0]$.

□

Define m to be the function $m = m_\delta(r) := (1 + r^2)^{(1+\delta)/4}$. We now establish the Carleman estimate.

Lemma 3.4 (Carleman estimate)**.** *Let δ, h_0, and φ and be as in Lemma 3.3. There is a $C > 0$ such that*

$$\|(w')^{\frac{1}{2}} e^{\varphi h^{-1}} v\|^2_{L^2(\mathbb{R}^2)} \leq \frac{C}{h^2} \|m e^{\varphi h^{-1}} (P - i\varepsilon) v\|^2_{L^2(\mathbb{R}^2)} + \frac{C\varepsilon}{h} \|e^{\varphi h^{-1}} v\|^2_{L^2(\mathbb{R}^2)} \qquad (3.11)$$

for all, $v \in C_0^\infty(\mathbb{R}^2)$, $E \in [E_{\min}, E_{\max}]$, $\varepsilon \geq 0$, and $h \in (0, h_0]$.

Proof. We consider the conjugated operator

$$\begin{aligned} P_\varphi &:= e^{\varphi h^{-1}} r^{\frac{1}{2}} (P - i\varepsilon) r^{-\frac{1}{2}} e^{-\varphi h^{-1}} \\ &= -h^2 \partial_r^2 + 2h\varphi' \partial_r + \Lambda + V_\varphi - E - i\varepsilon, \end{aligned}$$

where

$$0 \leq \Lambda := -\frac{h^2}{r^2} \Delta_{\mathbb{S}^1},$$

and $\Delta_{\mathbb{S}^1}$ is the spherical Laplacian on the unit circle $\mathbb{S}^1$.

Next, let $\int_{r,\theta}$ denote the integral over $(0, \infty) \times \mathbb{S}^1$ with respect to $dr d\theta$, where $d\theta$ is the usual arclength measure on $\mathbb{S}^1$. Throughout the remainder of this chapter, $C > 0$ denotes a constant depending possibly on w, φ, E, and δ, but not on h or u. Its precise value will change from line to line, but it will always remain independent of h and u.

To show (3.11) it suffices to prove that

$$\int_{r,\theta} \partial_r (w(E - V_\varphi)) |u|^2 \leq \frac{C}{h^2} \int_{r,\theta} \frac{w}{w'} |P_\varphi u|^2 + \frac{C\varepsilon}{h} \int_{r,\theta} |u|^2, \qquad u \in e^{\varphi h^{-1}} r^{\frac{1}{2}} C_0^\infty(\mathbb{R}^2). \qquad (3.12)$$

This is because we can apply (3.9) along with the fact that $w/w' \leq \max\{2/\delta, R_0/2\}m^2$ for all $r \in [0,\infty)$. Additionally, because w' is bounded, we may assume without loss of generality that $\varepsilon \leq h \in (0, h_0]$. Our first step in showing (3.12) is to define the following spherical energy functional

$$F(r) := \|h\partial_r u(r,\theta)\|_S^2 - \langle(\Lambda + V_\varphi(r,\theta) - E)u(r,\theta), u(r,\theta)\rangle_S, \qquad r > 0.$$

Here, $\|\cdot\|_S$ and $\langle\cdot,\cdot\rangle_S$ denote the norm and inner product on $L^2(\mathbb{S}^1)$, respectively. This functional was used by Cardoso and Vodev [CaVo02] and by Datchev [Da14] to prove their own Carleman estimates.

We compute the derivative of F, which exists for almost all $r > 0$,

$$\begin{aligned} F' &= 2\operatorname{Re}\langle h^2 u'', u'\rangle_S - 2\operatorname{Re}\langle(\Lambda + V_\varphi - E)u, u'\rangle_S + 2r^{-1}\langle \Lambda u, u\rangle_S - \langle V_\varphi' u, u\rangle_S \\ &= -2\operatorname{Re}\langle P_\varphi u, u'\rangle_S + 4h\varphi'\|u'\|_S^2 + 2\varepsilon\operatorname{Im}\langle u, u'\rangle_S + 2r^{-1}\langle\Lambda u, u\rangle - \langle V_\varphi' u, u\rangle_S. \end{aligned}$$

The calculation of F' is straightforward, but it relies on fact that we can apply the dominated convergence theorem to get,

$$\lim_{t\to 0}\int_{\mathbb{S}^1} \frac{V(r+t,\theta) - V(r,\theta)}{t}|u(r,\theta)|^2 d\theta = \int_{\mathbb{S}^1} \partial_r V(r,\theta)|u(r,\theta)|^2 d\theta \tag{3.13}$$

for almost all $r > 0$. This is a consequence of Fubini's theorem. For now, we continue with the proof of (3.12) and postpone the proof of (3.13) until the very end . The formula for F' allows us to compute $wF' + w'F$,

$$\begin{aligned} wF' + w'F = &-2w\operatorname{Re}\langle P_\varphi u, u'\rangle_S + \left(4h^{-1}w\varphi' + w'\right)\|hu'\|_S^2 + 2w\varepsilon\operatorname{Im}\langle u, u'\rangle_S \\ &+ \left(2wr^{-1} - w'\right)\langle\Lambda u, u\rangle_S + \langle(w(E - V_\varphi))' u, u\rangle_S. \end{aligned}$$

If we now use the facts $w\varphi' \geq 0$, $w' > 0$, $\Lambda \geq 0$, $2wr^{-1} - w' \geq 0$, and $-2\operatorname{Re}\langle a, b\rangle + \|b\|^2 \geq -\|a\|^2$, then the preceding inequality implies

$$wF' + w'F \geq -\frac{w^2}{h^2 w'}\|P_\varphi u\|_S^2 + 2w\varepsilon\operatorname{Im}\langle u, u'\rangle_S + \langle(w(E - V_\varphi))' u, u\rangle_S. \tag{3.14}$$

In addition, Fatou's lemma, along with the fundamental theorem of calculus, show that

$$\int_0^\infty (w(r)F(r))' \leq -\liminf_{r\to 0} w(r)F(r) = 0. \tag{3.15}$$

Integrating (3.14) with respect to dr and using (3.15), we arrive at

$$\int_{r,\theta} (w(E - V_\varphi))'|u|^2 \leq \frac{1}{h^2} \iint_{r,\theta} \frac{w^2}{w'}|P_\varphi u|^2 + 2\varepsilon \iint_{r,\theta} w|uu'|. \tag{3.16}$$

We focus on the last term in (3.16). Our goal is to show

$$2\int_{r,\theta} w|uu'| \leq \frac{C}{h} \iint_{r,\theta} w^2|P_\varphi u|^2 + \frac{C}{h} \iint_{r,\theta} |u|^2. \tag{3.17}$$

If we have shown (3.17), we can substitute it into (3.16) to get

$$\int_{r,\theta} (w(E - V_\varphi))'|u|^2 \leq \frac{1}{h^2} \iint_{r,\theta} \frac{w^2}{w'}|P_\varphi u|^2 + \frac{C\varepsilon}{h} \int_{r,\theta} w^2|P_\varphi u|^2 + \frac{C\varepsilon}{h} \iint_{r,\theta} |u|^2, \tag{3.18}$$

If we use the assumptions $\varepsilon \leq h$, $h \leq h_0 \leq 1$, along with the fact $(w^2/w' + w^2) \leq (1+\delta)w/w'$, we see that (3.18) implies (3.12).

To show (3.17), we first write.

$$2\iint_{r,\theta} w|uu'| \leq \frac{1}{h} \int_{r,\theta} |u|^2 + \frac{1}{h} \iint_{r,\theta} w^2|hu'|^2.$$

We will now show that

$$\frac{1}{h}\int_{r,\theta} w^2|hu'|^2 \leq \frac{C}{h} \int_{r,\theta} w^2|P_\varphi u|^2 + \frac{C}{h} \iint_{r,\theta} |u|^2, \tag{3.19}$$

which will complete the proof of the Lemma. To show this, we use integration by parts, along with the facts that $h \leq 1$ and $ab \leq \gamma a^2/2 + b^2/2\gamma$ for any $\gamma > 0$.

$$\begin{aligned}\frac{1}{h}\iint_{r,\theta} w^2|hu'|^2 &= \frac{1}{h}\operatorname{Re}\left[\iint_{r,\theta} \bar{u}(-2h^2ww'u') + \int_{r,\theta} \bar{u}(-h^2w^2u'')\right]\\ &\leq \frac{(\max w')^2}{\gamma h} \int_{r,\theta} |u|^2 + \frac{\gamma}{h}\iint_{r,\theta} w^2|hu'|^2 + \frac{1}{h}\operatorname{Re}\left[\int_{r,\theta} \bar{u}(-h^2w^2u'')\right]\end{aligned}$$

Furthermore, for any $\eta > 0$,

$$\begin{aligned}&\frac{1}{h}\operatorname{Re}\left[\iint_{r,\theta} \bar{u}(-h^2w^2u'')\right]\\ &\quad= \frac{1}{h}\operatorname{Re}\left[\iint_{r,\theta} w^2\bar{u}(P_\varphi - 2h\varphi'\partial_r - \Lambda - V_\varphi + E + i\varepsilon)u\right]\\ &\quad\leq \frac{1}{h}\iint_{r,\theta} w^2|P_\varphi u||u| + \frac{2}{h}\int_{r,\theta} w^2\varphi'|hu'||u| + \frac{1}{h}\iint_{r,\theta} w^2|E - V_\varphi||u|^2\\ &\quad\leq \frac{1}{2h}\int_{r,\theta} w^2|P_\varphi u|^2 + \frac{\eta \max\varphi'}{h}\iint_{r,\theta} w^2|hu'|^2\\ &\quad\quad+ \frac{1}{h}\left(\frac{\max(\varphi' w^2)}{\eta} + \max(w^2(E_{\max} + |V_\varphi|) + \frac{\max(w^2)}{2}\right)\iint_{r,\theta} |u|^2.\end{aligned}$$

Now, take $\gamma = 1/4, \eta = 1/(4 \max \varphi')$. and combine the previous two estimates to get

$$\frac{1}{h} \int_{r,\theta} w^2 |hu'|^2 \leq \frac{1}{2h} \int_{r,\theta} w^2 |P_\varphi u|^2 + \frac{C}{h} \int_{r,\theta} |u|^2 + \frac{1}{2h} \int_{r,\theta} w^2 |hu'|^2.$$

If we subtract the last term to the left side of this inequality, and multiply through by 2, we arrive at (3.19).

To finish, we now prove (3.13). Because $\nabla V \in L^\infty(\mathbb{R}^2)$, V has a Lipschitz representative, and the Lipschitz constant for V is bounded by $\|\nabla V\|_{L^\infty(\mathbb{R}^2)}$. Furthermore, both $\partial_x V$ and $\partial_y V$ exist in the strong sense at almost all points $(x, y) \in \mathbb{R}^2$. See, for instance, [Ev, Section 5.8]. If $(x_0, y_0) = (r_0 \cos\theta_0, r_0 \sin\theta_0) \in \mathbb{R}^2 \setminus \{0\}$ is a point at which both $\partial_x V$ and $\partial_y V$ exist, then the chain rule implies that the map $(r, \theta) \mapsto V(r, \theta)$ is differentiable with respect to r and the point (r_0, θ_0), and the formula for $\partial_r V(r_0, \theta_0)$ is given by (3.3).

Our goal is to show that, for almost all $r > 0$, the complement in $\mathbb{S}^1$ of the set

$$W_r := \left\{ \theta \in \mathbb{S}^1 : \lim_{t \to 0} \frac{V(r + t, \theta) - V(r, \theta)}{h} = \partial_r V(r, \theta) \right\}$$

has measure zero with respect to the arclength measure on $\mathbb{S}^1$. If we combine this with the fact that

$$\frac{V(r + t, \theta) - V(r, \theta)}{t} \leq \|\nabla V\|_{L^\infty(\mathbb{R}^n)}, \qquad t \neq 0,$$

then (3.13) follows from Lebesgue's dominated convergence theorem. To show W_r has zero arclength, let $N \subseteq \mathbb{R}^2$ be a set of Lebesgue measure zero such that $\partial_x V, \partial_y V$ exist in the strong sense at all points $(x, y) \in \mathbb{R}^2 \setminus N$. And set χ_N to be the characteristic function of N. The integral of χ_N over $\mathbb{R}^2$ equals the integral of χ_N over the product space $(0, \infty) \times \mathbb{S}^1$,

$$0 = \int_{\mathbb{R}^2} \chi_N(x, y) dx dy = \int_0^\infty \int_{\mathbb{S}^1} \chi_N(r \cos\theta, r \sin\theta) d\theta r dr. \tag{3.20}$$

Fubini's Theorem then says that the inner integral

$$f(r) := \int_{\mathbb{S}^1} \chi_N(r \cos\theta, r \sin\theta) d\theta.$$

is itself a measurable function of r. Since $f \geq 0$ and

$$\int_0^\infty f(r) r dr = 0,$$

we must have $f(r) = 0$ for almost all $r > 0$. That is, the set

$$N_r := \{\theta \in \mathbb{S}^1 : (r\cos\theta, r\sin\theta) \in N\}$$

has measure zero with respect to the arclength measure on $\mathbb{S}^1$. A straightforward set containment argument shows that $\mathbb{S}^1 \setminus W_r \subseteq N_r$, and this finishes the proof of (3.13). □

3.3 Proof of Theorem 1

Set $C_0 = 2\max\varphi$. Our strategy is to take the Carleman estimate (3.11) and show that there exist constants $C > 0, R > 0$ so that

$$e^{-C_0 h^{-1}} \|m^{-1}\mathbf{1}_{\leq R} v\|^2 + \|m^{-1}\mathbf{1}_{\geq R} v\|^2 \leq \frac{C}{h^2}\|m(P - i\varepsilon)v\|^2 + \frac{C\varepsilon}{h}\|v\|^2, \qquad (3.21)$$

for all $v \in C_0^\infty(\mathbb{R}^2)$. Then (3.21) allows us to prove (1.4) and (1.5) using the same density argument given by Datchev in in [Da14], which is independent of dimension. We cannot obtain (3.21) directly from our Carleman estimate (3.11), because the estimate is weak near the origin. However, the decay assumption on ∇V from (1.3) allows us to make a small shift of coordinates and still maintain (3.4). We obtain the same Carleman estimate as (3.11) with respect to a new origin. We add the two estimates together and recover (3.21).

Lemma 3.5. *Suppose $V \in L^\infty(\mathbb{R}^2)$, $\nabla V \in (L^\infty(\mathbb{R}^2))^2$ and that V, ∇V satisfy* (3.1). *If $x_0 \in \mathbb{R}^2$ is chosen so that*

$$|x_0| \leq 2^{(1+\delta_0)^{-1}} - 1, \qquad (3.22)$$

then the functions $V(\cdot - x_0), \partial_r V(\cdot - x_0)$ obey (3.4).

Proof. Observe that

$$V(x - x_0) \leq \frac{1}{2}(1 + |x - x_0|)^{-\delta_0} = \frac{1}{2}\left[\frac{1 + |x|}{1 + |x - x_0|}\right]^{\delta_0}(1 + |x|)^{-\delta_0},$$

$$\partial_r V(x-x_0) \leq |\nabla V(x-x_0)| \leq \frac{1}{2}(1+|x-x_0|)^{-1-\delta_0}$$
$$= \frac{1}{2}\left[\frac{1+|x|}{1+|x-x_0|}\right]^{1+\delta_0}(1+|x|)^{-1-\delta_0}.$$

Because $(1+|x|)/(1+|x-x_0|) \leq 1+|x_0|$, it suffices to choose $|x_0|$ small enough so that

$$\max\{(1+|x_0|)^{\delta_0}, (1+|x_0|)^{1+\delta_0}\} = (1+|x_0|)^{1+\delta_0} \leq 2.$$

And this is achieved if we pick x_0 to satisfy (3.22).

□

Proof of Theorem 1. Let $L^2 = L^2(\mathbb{R}^2)$, $H^2 = H^2(\mathbb{R}^2)$, $C_0^\infty(\mathbb{R}^2) = C_0^\infty$. Pick $x_0 \in \mathbb{R}^2$ so that

$$0 < |x_0| < 2^{(1+\delta_0)^{-1}} - 1.$$

Then $V(\cdot + x_0)$ satisfies the bounds (3.4), according to Lemma 3.5. Therefore, the Carleman estimate (3.11) can be applied to the operator

$$P_0 := -h^2\Delta + V(\cdot + x_0) - E.$$

We shift coordinates, apply (3.11) with P_0 in place of P, and then shift back.

$$\begin{aligned}\left\|(w'(|\cdot - x_0|))^{\frac{1}{2}} e^{\varphi(|\cdot - x_0|)h^{-1}} v\right\|_{L^2}^2 &= \left\|(w')^{\frac{1}{2}} e^{\varphi h^{-1}} v(\cdot + x_0)\right\|_{L^2}^2 \\ &\leq \frac{C}{h^2}\left\|m e^{\varphi h^{-1}}(P_0 - i\varepsilon)v(\cdot + x_0)\right\|_{L^2}^2 \\ &\quad + \frac{C\varepsilon}{h}\left\|e^{\varphi h^{-1}} v(\cdot + x_0)\right\|_{L^2}^2 \\ &= \frac{C}{h^2}\left\|m(|\cdot - x_0|) e^{\varphi(|\cdot - x_0|)h^{-1}}(P - i\varepsilon)v\right\|_{L^2}^2 \\ &\quad + \frac{C\varepsilon}{h}\left\|e^{\varphi(|\cdot - x_0|)h^{-1}} v\right\|_{L^2}^2\end{aligned}$$

Summarizing this estimate, we have

$$\begin{aligned}\left\|(w'(|\cdot - x_0|))^{\frac{1}{2}} e^{\varphi(|\cdot - x_0|)h^{-1}} v\right\|_{L^2} &\leq \frac{C}{h^2}\left\|m(|\cdot - x_0|) e^{\varphi(|\cdot - x_0|)h^{-1}}(P - i\varepsilon)v\right\|_{L^2} \\ &\quad + \frac{C\varepsilon}{h}\left\|e^{\varphi(|\cdot - x_0|)h^{-1}} v\right\|_{L^2}.\end{aligned} \tag{3.23}$$

To proceed, choose $R > 0$ large enough so that $|x| \geq R$ implies that $\varphi(|x|) = \varphi(|x - x_0|) = \max \varphi$. Multiply both (3.11) and (3.23) through by $e^{-C_0 h^{-1}}$ to obtain

$$e^{-C_0 h^{-1}}\|w' \mathbf{1}_{\leq R} v\|_{L^2}^2 + \|w' \mathbf{1}_{\geq R} v\|_{L^2}^2 \leq \frac{C}{h^2}\|m(P - i\varepsilon)v\|_{L^2}^2 + \frac{C\varepsilon}{h}\|v\|_{L^2}^2, \tag{3.24}$$

$$e^{-C_0h^{-1}}\|w'(|\cdot - x_0|)\mathbf{1}_{\leq R}v\|_{L^2}^2 + \|w'(|\cdot - x_0|)\mathbf{1}_{\geq R}v\|_{L^2}^2 \\ \leq \frac{C}{h^2}\|m(|x-x_0|)(P-i\varepsilon)v\|_{L^2}^2 + \frac{C\varepsilon}{h}\|v\|_{L^2}^2. \tag{3.25}$$

Next, note that there exists some constant $K > 0$, depending on x_0 and δ_0, so that

$$(m^{-1})^2 \leq K((w')^2 + (w'(|\cdot - x_0|))^2), \qquad m^2 + (m(|\cdot - x_0|))^2 \leq Km^2. \tag{3.26}$$

If we then add (3.24) and (3.25) and apply (3.26) to both sides of the inequality, we arrive at

$$e^{-C_0h^{-1}}\|m^{-1}\mathbf{1}_{\leq R}v\|_{L^2}^2 + \|m^{-1}\mathbf{1}_{\geq R}v\|_{L^2}^2 \leq \frac{C}{h^2}\|m(P-i\varepsilon)v\|_{L^2}^2 + \frac{C\varepsilon}{h}\|v\|_{L^2}^2,$$

which is (3.21).

From this point, we follow reasoning from the proof of the Theorem in [Da14]. For any $\gamma, \eta > 0$, we have

$$\begin{aligned} 2\varepsilon\|v\|_{L^2}^2 &= -2\operatorname{Im}\langle (P-i\varepsilon)v, v\rangle_{L^2} \\ &\leq \gamma^{-1}\|m\mathbf{1}_{\geq R}(P-i\varepsilon)v\|_{L^2}^2 + \gamma\|m^{-1}\mathbf{1}_{\geq R}v\|_{L^2}^2 \\ &+ \eta^{-1}\|m\mathbf{1}_{\leq R}(P-i\varepsilon)v\|_{L^2}^2 + \eta\|m^{-1}\mathbf{1}_{\leq R}v\|_{L^2}^2. \end{aligned}$$

Setting $\gamma = h/C$ and $\eta = e^{-2C_0h^{-1}}$, we estimate $\varepsilon\|v\|_{L^2}^2$ from above in (3.21) and find that, for h sufficiently small

$$e^{-Ch^{-1}}\|m^{-1}\mathbf{1}_{\leq R}v\|_{L^2}^2 + \|m^{-1}\mathbf{1}_{\geq R}v\|_{L^2}^2 \\ \leq e^{Ch^{-1}}\|m\mathbf{1}_{\leq R}(P-i\varepsilon)v\|_{L^2}^2 + \frac{C}{h^2}\|m\mathbf{1}_{\geq R}(P-i\varepsilon)v\|_{L^2}^2. \tag{3.27}$$

The final task is to use (3.27) to show that for any $f \in L^2$,

$$e^{-Ch^{-1}}\|\mathbf{1}_{\leq R}(P-i\varepsilon)^{-1}m^{-1}f\|_{L^2}^2 + \|m^{-1}\mathbf{1}_{\geq R}(P-i\varepsilon)^{-1}m^{-1}f\|_{L^2}^2 \\ \leq e^{Ch^{-1}}\|\mathbf{1}_{\leq R}f\|_{L^2}^2 + \frac{C}{h^2}\|\mathbf{1}_{\geq R}f\|_{L^2}^2, \tag{3.28}$$

from which (1.4) and (1.5) follow. To establish (3.28), we prove a simple estimate and then apply a density argument which relies on (3.27).

In what follows, we use $a \lesssim b$ to denote $a \leq C_{\varepsilon,h} b$ for $C_{\varepsilon,h}$ depending on ε and h, but not on v. By the Kato-Rellich Theorem, $(P - i\varepsilon)^{-1} : L^2 \to H^2$ is bounded. In addition, the commutator $[P, m] = -h^2 \Delta m + 2h^2 \nabla m \cdot \nabla : H^2 \to L^2$ is bounded. So for $v \in H^2$ such that $mv \in H^2$, we have

$$\begin{aligned} \|m(P - i\varepsilon)v\|_{L^2} &\lesssim \|(P - i\varepsilon)mv\|_{L^2} + \|[P, m]v\|_{L^2} \\ &\lesssim \|mv\|_{H^2} + \|v\|_{H^2} \\ &\lesssim \|mv\|_{H^2}. \end{aligned}$$

Thus we have shown

$$\|m(P - i\varepsilon)v\|_{L^2} \leq C_{\varepsilon,h} \|mv\|_{H^2}, \qquad v \in H^2 \text{ such that } mv \in H^2. \tag{3.29}$$

For fixed $f \in L^2$, the function $m(P - i\varepsilon)^{-1} m^{-1} f \in H^2$ because

$$\begin{aligned} m(P - i\varepsilon)^{-1} m^{-1} f &= (P - i\varepsilon)^{-1} f + [m, (P - i\varepsilon)^{-1}] m^{-1} f \\ &= (P - i\varepsilon)^{-1} f + (P - i\varepsilon)^{-1} [P, m] (P - i\varepsilon)^{-1} m^{-1} f. \end{aligned}$$

Now, choose a sequence $v_k \in C_0^\infty$ such that $v_k \to m(P - i\varepsilon)^{-1} m^{-1} f$ in H^2. Define $\tilde{v}_k := m^{-1} v_k$. Then, as $k \to \infty$

$$\|m^{-1} \tilde{v}_k - m^{-1}(P - i\varepsilon)^{-1} m^{-1} f\|_{L^2} \leq \|v_k - m(P - i\varepsilon)^{-1} m^{-1} f\|_{H^2} \to 0.$$

Also, applying (3.29)

$$\|m(P - i\varepsilon)\tilde{v}_k - f\|_{L^2} \lesssim \|v_k - m(P - i\varepsilon)^{-1} m^{-1} f\|_{H^2} \to 0.$$

We then achieve (3.28) by replacing v by $\tilde{v}_k$ in (3.27) and sending $k \to \infty$.

$\square$

4. SEMICLASSICAL RESOLVENT BOUND FOR COMPACTLY SUPPORTED L^∞ POTENTIALS

The key to proving Theorem 2 is to establish a slightly different Carleman estimate from (3.11).

Lemma 4.1 (Carleman estimate)**.** *Let* $R_0 > 3$ *so that* $\operatorname{supp} V \subseteq B(0, R_0/2)$*. There exist* $K, C > 0$*,* $h_0 \in (0, 1]$*, and* $\varphi = \varphi_h \in C^2(0, \infty)$ *depending on* $E_{\min}$*,* $E_{\max}$*,* $\|V\|_\infty$*,* R_0*,* n*, and* s *such that*

$$\max \varphi = K \log(h^{-1}), \qquad h \in (0, h_0], \tag{4.1}$$

and

$$\left\| \left(\mathbf{1}_{\leq 1}|x|^{1/2} + \mathbf{1}_{\geq 1}(1 + |x|)^{-s}\right) e^{\varphi/h^{4/3}} v \right\|^2_{L^2(\mathbb{R}^n)} \leq \frac{C}{h^{10/3}} \left\| (1 + |x|)^s e^{\varphi/h^{4/3}} (P(h) - i\varepsilon) v \right\|^2_{L^2(\mathbb{R}^n)} + \frac{C\varepsilon}{h^{10/3}} \left\| e^{\varphi/h^{4/3}} v \right\|^2_{L^2(\mathbb{R}^n)}, \tag{4.2}$$

for all $E \in [E_{\min}, E_{\max}]$*,* $\varepsilon > 0$*,* $h \in (0, h_0]$*, and* $v \in C_0^\infty(\mathbb{R}^n)$*.*

In addition to (4.1), a key property of the Carleman weight φ is that $\partial_r \varphi$ is large on $\operatorname{supp} V$. We construct φ to have these properties in Lemma 4.3.

To prove Lemma 4.1, we adapt the strategy appearing in the previous chapter and in [CaVo02,Da14,RT15]. As before, we start with a certain spherical energy functional $F : (0, \infty) \to \mathbb{R}$ that includes φ, see (4.28). As before, we intend to differentiate the product wF, where $w : (0, \infty) \to \mathbb{R}$ is a second weight function defined by (4.7). But since we cannot necessarily differentiate $V \in L^\infty$, this time we leave V out of F, and add it back only after differentiation. By doing so, we recover the terms needed to prove (4.2), at the cost of introducing a remainder term that may be large on the support of V. We control the remainder with two innovations that go beyond the techniques used in the previous chapter and in [CaVo02, Da14, RT15]. First we increase the h-dependence of the exponent in (4.2) to from h^{-1} to $h^{-4/3}$ (compare with (3.11)). Second, we require that $\partial_r \varphi \geq c$ on $\operatorname{supp} V$, where c is chosen large enough to satisfy (4.3) and (4.4) below.

A previous version of Lemma 4.1 asserted only that $\max \varphi \leq Kh^{-1/3}$, resulting in an larger $h^{-5/3}$ exponent on the right side of (1.6). Not until seeing the estimate (1.7) of Klopp and Vogel did the author realize that, without changing the construction, φ' could be estimated more sharply outside the support of V (see (4.25)). As a result, the author was able to improve the exponent in (1.6) from $h^{-5/3}$ to $h^{-4/3}\log(h^{-1})$ where it currently stands. The author is very grateful to Klopp and Vogel for helping to bring about this improvement.

The outline of this chapter is as follows. In Section 4.1, we construct the weights w and φ and prove their key properties. In Section 4.2, we prove the Carleman estimate. In Section 4.3, we first glue two versions of the Carleman estimate togther to remove the loss at the origin. Then we prove the Theorem via a density argument. The density argument is straightforward and closely follows the one from the previous chapter and the ones in [Da14, DyZw], but we recall it for the reader's convenience.

4.1 Construction of the Carleman weight

As in the previous chapter, we put

$$\delta := 2s - 1.$$

Without loss of generality, we assume $0 < \delta < 1$. Fix $R_0 > 3$ large enough so that

$$\operatorname{supp} V \subseteq B(0, R_0/2).$$

Next, choose $c > 1$ large enough so that

$$c > \|V\|_\infty R_0/4, \tag{4.3}$$

$$\sqrt{c}\tanh(\sqrt{c}/2) > \max\{\|V\|_\infty/4, 1\}. \tag{4.4}$$

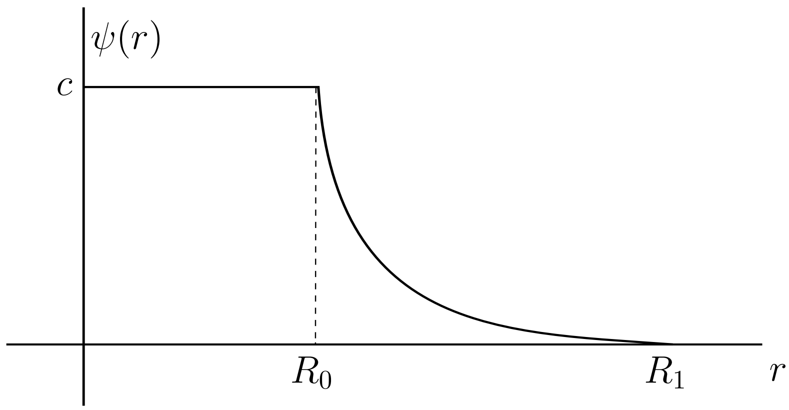

Fig. 4.1. The graph of ψ.

Set

$$\psi = \psi_h(r) := \begin{cases} c & 0 < r \leq R_0, \\ \frac{B}{r^2} - \frac{h^{2/3}E_{\min}}{4} & R_0 < r < R_1, \\ 0 & r \geq R_1, \end{cases} \tag{4.5}$$

where we put

$$\begin{gathered} B = B(h) := \left(c + \frac{h^{2/3}E_{\min}}{4}\right)R_0^2, \\ R_1^2 = R_1^2(h) := \frac{4B}{h^{2/3}E_{\min}} = \left(1 + \frac{4c}{h^{2/3}E_{\min}}\right)R_0^2, \end{gathered} \tag{4.6}$$

so that ψ is continuous. In Lemma 4.3, we will construct the Carleman weight φ so that $(\varphi')^2$ is approximately equal to ψ for h small. From this relationship, we will deduce the properties of φ needed to prove the Carleman estimate.

To continue, define

$$w = w_{h,\delta}(r) := \begin{cases} r^2 & 0 < r < R_1, \\ R_1^2 + 1 - (1 + (r - R_1))^{-\delta} & r \geq R_1. \end{cases} \tag{4.7}$$

According to (4.5), ψ and w satisfy the inequality

$$h^{-2/3}(w\psi)' \geq -\frac{E_{\min}}{4}w', \qquad r > 0,\ r \neq R_0,\ r \neq R_1. \tag{4.8}$$

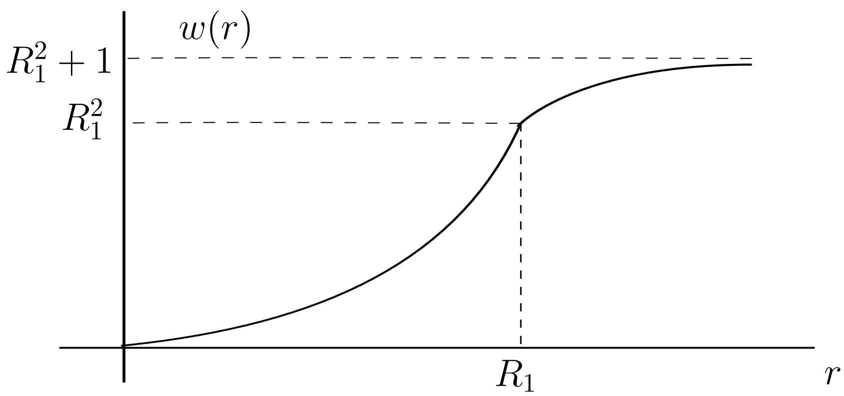

Fig. 4.2. The graph of w.

We use (4.8) in the proof of the Carleman estimate to ensure that a group of remainder terms is not too negative, see (4.31).

The next lemma proves elementary estimates involving w and w'. We use them in the proof of Lemma 4.1 to bring intermediate steps closer to (4.2), note in particular (4.35).

Lemma 4.2. *Suppose $h \in (0,1]$. There exists $C > 1$ depending on $E_{\min}$, R_0, c, and δ so that for each $r \neq R_1$, it holds that*

$$2wr^{-1} - w' \geq 0, \tag{4.9}$$

$$w(r) \leq Ch^{-2/3}, \tag{4.10}$$

$$w^2(r)/w'(r) \leq Ch^{-4/3}(1+r)^{1+\delta}, \tag{4.11}$$

$$w'(r) \geq C^{-1}\left(\mathbf{1}_{\leq 1} r + \mathbf{1}_{\geq 1}(1+r)^{-1-\delta}\right). \tag{4.12}$$

Proof. When $r < R_1$, $2wr^{-1} - w' = 0$. If $r > R_1$, then

$$2wr^{-1} - w' = 2r^{-1}\left(R_1^2 + 1 - (1 + (r - R_1))^{-\delta}\right) - \delta(1 + (r - R_1))^{-1-\delta}.$$

So to finish proving (4.9), it is enough to show,

$$2R_1^2 \geq 2(1 + r - R_1)^{-\delta} + \delta r(1 + r - R_1)^{-1-\delta}, \qquad r > R_1.$$

Using $0 < \delta < 1$ and $R_1 > R_0 > 3$, we estimate,

$$\begin{aligned} 2(1 + (r - R_1))^{-\delta} + \delta r\,(1 + r - R_1)^{-1-\delta} &\leq 2 + \delta + \delta R_1 \\ &\leq 2R_1^2. \end{aligned}$$

To show (4.10), simply note that

$$\begin{aligned} w(r) &\leq R_1^2 + 1 \\ &\leq 2R_1^2 \\ &= 8h^{-2/3}BE_{\min}^{-1}. \end{aligned}$$

For (4.11), when $0 < r \leq R_1$,

$$\begin{aligned} w^2(r)/\left(w'(r)(1+r)^{1+\delta}\right) &\leq 2^{-1}r^2 \\ &\leq 2^{-1}R_1^2 \\ &= 2h^{-2/3}BE_{\min}^{-1}. \end{aligned}$$

If $r \geq R_1$, then we use the bound $w(r) \leq 2R_1^2$,

$$\begin{aligned} w^2(r)/\left(w'(r)(1+r)^{1+\delta}\right) &\leq \delta^{-1} w^2(r) \left(\frac{1+r-R_1}{1+r}\right)^{1+\delta} \\ &\leq 4\delta^{-1} R_1^4 \\ &= 64 h^{-4/3} \delta^{-1} B^2 E_{\min}^{-2}. \end{aligned}$$

As for (4.12), observe that when $1 < r \leq R_1$,

$$\begin{aligned} w'(r)(1+r)^{1+\delta} &= 2r(1+r)^{1+\delta} \\ &\geq 2, \end{aligned}$$

and when $r > R_1$,

$$\begin{aligned} w'(r)(1+r)^{1+\delta} &= \delta \left(\frac{1+r}{1+r-R_1}\right)^{1+\delta}. \\ &\geq \delta \end{aligned}$$

□

We now construct the Carleman weight $\varphi \in C^2(0,\infty)$ as a solution to an ODE with right hand side equal to ψ. The argument is modeled after Proposition 3.1 [DdeH16].

Lemma 4.3. *Let $h \in (0,1]$. There exists $\varphi = \varphi_h \in C^2(0,\infty)$ with the properties that*

$$(\varphi')^2 - h^{4/3}\varphi'' = \psi, \qquad r > 0, \tag{4.13}$$

$$0 \leq \varphi'(r) \leq \sqrt{c}, \qquad r > 0, \tag{4.14}$$

$$0 \leq \varphi'(r) \leq K r^{-1}, \qquad R_0 < r < R_1 \tag{4.15}$$

$$1 \leq \max \varphi = K \log(h^{-1}), \tag{4.16}$$

$$\varphi'(r) \geq \sqrt{c}\tanh(\sqrt{c}/2), \qquad 0 < r < R_0/2, \tag{4.17}$$

where $K > 0$ depends on $\|V\|_\infty$, R_0 and $E_{\min}$ but not on h.

Once we construct φ according to (4.13), it holds that $\varphi' \approx \sqrt{\psi}$ for h small, and so (4.14) through (4.17) follow naturally from the definition of ψ.

Proof. To begin, consider the solution to the initial value problem

$$y' = h^{-4/3}(y^2 - \psi), \qquad y(R_1) = 0. \tag{4.18}$$

According to Theorem 1.2 in Chapter 1 of [CoLe], there exists an open interval I containing R_1 and a solution $y \in C^1(I)$ to (4.18). In fact, this solution is unique on I. For if y_1, y_2 are two solutions to (4.18), then $\tilde{y} := y_1 - y_2$ solves $\tilde{y}' = (y_1 + y_2)\tilde{y}$, $\tilde{y}(R_1) = 0$, and hence is identically zero.

We take

$$\varphi(r) := \int_0^r y(s)ds. \tag{4.19}$$

Hence φ satisfies (4.13). We now analyze y to establish (4.14), (4.16) and (4.17).

First, we show that $y(r) = 0$ for $r \geq R_1$, $r \in I$, and therefore y extends to be identically zero on $[R_1, \infty)$. Because $y(R_1) = 0$, there exists $\varepsilon \in (0, h^{4/3})$ so that $[R_1, R_1 + \varepsilon) \subseteq I$ and $|y(r)| \leq 1/2$ on $[R_1, R_1 + \varepsilon)$. Therefore, using (4.18), we see that $|y'(r)| = h^{-4/3}|y(r)|^2 \leq (4h^{4/3})^{-1}$ on $[R_1, R_1 + \varepsilon)$. Hence

$$\begin{aligned} |y(r)| &\leq \int_{R_1}^r |y'(s)|ds \leq \frac{\varepsilon}{4h^{4/3}} \\ &\leq \frac{1}{4}, \qquad r \in [R_1, R_1 + \varepsilon). \end{aligned}$$

Applying $|y'(r)| = h^{-4/3}|y(r)|^2$ on $[R_1, R_1 + \varepsilon)$ another time, we then get $|y'(r)| \leq (16h^{4/3})^{-1}$ and use it to show that $|y(r)| \leq 16^{-1}$, $r \in [R_1, R_1 + \varepsilon)$. Continuing in this fashion, we see that $y(r) = 0$ for $r \in [R_1, R_1 + \varepsilon)$. Therefore y extends to be identically zero on $[R_1, \infty)$.

Moving on, we now show that

$$0 \leq y \leq \sqrt{\psi(R_0)} = \sqrt{c} \tag{4.20}$$

where it is defined on $(0, R_1]$. To show $y \geq 0$, assume for contradiction that there exists $0 < r_0 < R_1$ with $y(r_0) < 0$. Then, because $y' = h^{-4/3}(y^2 - \psi) \leq h^{-4/3}y^2$, we have $y'(r)/(y(r))^2 \leq h^{-4/3}$, for r near r_0. This implies

$$\begin{aligned} y(r_0)^{-1} - y(r)^{-1} &= \int_{r_0}^r \frac{y'(s)}{(y(s))^2}ds \\ &\leq \frac{r - r_0}{h^{4/3}}, \qquad r \geq r_0,\ r \text{ near } r_0. \end{aligned} \tag{4.21}$$

As r approaches $\inf\{r \in [r_0, \infty) : y(r) = 0\} \leq R_1$, (4.21) must hold. But this is a contradiction because the left side becomes arbitrarily large, while the right side remains bounded. So $y(r) \geq 0$ where it is defined on $(0, R_1]$.

To show $y \leq \sqrt{\psi(R_0)}$, we compare y to the solution of the initial value problem

$$\begin{aligned} z' &= h^{-4/3}(z^2 - \psi(R_0)) \\ &= h^{-4/3}(z^2 - c), \qquad z(R_1) = 0, \end{aligned}$$

This solution exists for all $r > 0$ and is given by

$$\begin{aligned} z(r) &= \sqrt{c}\frac{1 - \exp\left(-2h^{-4/3}\sqrt{c}(R_1 - r)\right)}{1 + \exp\left(-2h^{-4/3}\sqrt{c}(R_1 - r)\right)} \\ &= \sqrt{c}\tanh\left(h^{-4/3}\sqrt{c}(R_1 - r)\right). \end{aligned}$$

Suppose for contradiction that there exists $r_0 < R_1$ such that $y(r_0) > z(r_0)$. Set $\zeta := y - z$. Then $\zeta' \geq h^{-4/3}(y + z)\zeta$, $\zeta(r_0) > 0$, and $\zeta(R_1) = 0$.

Put $r_1 := \inf\{r \in (r_0, R_1] : \zeta(r) = 0\}$. By the mean value theorem, there exists $\tilde{r} \in (r_0, r_1)$ so that

$$\begin{aligned} \zeta'(\tilde{r}) &= \frac{\zeta(r_1) - \zeta(r_0)}{r_1 - r_0} \\ &= \frac{-\zeta(r_0)}{r_1 - r_0} \\ &< 0. \end{aligned}$$

In addition, $\zeta(\tilde{r}) > 0$ by the definition of r_1. But this contradicts $\zeta'(\tilde{r}) \geq h^{-4/3}\zeta(\tilde{r})(y(\tilde{r}) + z(\tilde{r}))$ since $y + z \geq 0$ where y is defined on $(0, R_1)$.

So we have shown that $0 \leq y \leq z \leq \sqrt{c}$ where it is defined on $(0, R_1)$.It then follows by Theorem 1.3 in Chapter 2 of [CoLe] that y extends to all of $(0, R_1)$, where it obeys the same bounds.

We omit the proof of (4.15). However, we remark that one can show

$$y \leq \xi(r) := \tilde{B}/r \qquad \text{on } (R_0, R_1), \tag{4.22}$$

where

$$\tilde{B} := \left(\sqrt{4B + h^{8/3}} - h^{4/3}\right)/2,$$

by first noting that ξ solves

$$\xi' = h^{-4/3}(\xi^2 - (B/r^2)), \quad \xi(R_1) = \tilde{B}/r, \tag{4.23}$$

and then comparing y and ξ by the same method as in the preceding paragraph.

Lastly, we show that

$$y(r) \geq \sqrt{c}\tanh\left(\sqrt{c}/2\right) \qquad r \in (0, R_0/2]. \tag{4.24}$$

To see this, let $\tilde{z}$ solve the initial value problem

$$\tilde{z}' = h^{-4/3}(\tilde{z}^2 - \psi), \qquad \tilde{z}(R_0) = 0.$$

Then $\tilde{z}$ is given by

$$\tilde{z}(r) = \sqrt{c}\tanh\left(h^{-4/3}\sqrt{c}\,(R_0 - r)\right)$$

Set $\tilde{\zeta} := y - \tilde{z}$. To show (4.24), it is enough to see that $\tilde{\zeta} \geq 0$ on $(0, R_0)$, and we give an argument similar to the one in the preceding paragraph. For contradiction, suppose there exists $0 < r_2 \leq R_0$ such that $\tilde{\zeta}(r_2) < 0$. Put $r_3 := \inf\{r \in (r_2, R_0] : \tilde{\zeta}(r) = 0\}$. Such an r_3 exists because $\tilde{\zeta}(R_0) = y(R_0) \geq 0$. By the mean value theorem, there is some $r^* \in (r_2, r_3)$ so that $\tilde{\zeta}'(r^*) = -(r_3 - r_2)^{-1}\tilde{\zeta}(r_2) > 0$, and furthermore $\tilde{\zeta}(r^*) < 0$ by the definition of r_3 . But also $\tilde{\zeta}'(r^*) = h^{-4/3}\tilde{\zeta}(r^*)(y(r^*) + \tilde{z}(r^*)) \leq 0$, and so we have contradiction.

We now have enough properties of y to finish the proof. With φ defined by (4.19), we observe that (4.14) follows from (4.20), and (4.17) from (4.24).

Lastly, we use (4.6), (4.4), (4.22), $R_0 > 3$, and $h \in (0, 1]$ to see

$$\max\varphi \geq \sqrt{c}\int_0^{R_0/2} \tanh\left(\sqrt{c}/2\right) ds \geq 1,$$

$$\max\varphi \leq \int_0^{R_0} \sqrt{c}ds + \int_{R_0}^{R_1} \tilde{B}/s ds \leq \sqrt{c}R_0 + \tilde{B}\log(R_1/R_0) \leq K\log(h^{-1}), \tag{4.25}$$

where $K > 0$ depends on $\|V\|_\infty$, R_0 and $E_{\min}$ but not on h. This shows (4.16) and completes the proof.

□

4.2 Proof of the Carleman estimate

In this section, we use the weight functions w and φ constructed in the previous section to prove Lemma 4.1. We make integral estimates using polar coordinates $(r,\theta) \in (0,\infty) \times \mathbb{S}^{n-1}$ on $\mathbb{R}^n$. As in the previous chapter, the starting point is a conveniently chosen conjugation

$$\begin{aligned} P_\varphi &:= e^{\varphi/h^{4/3}} r^{(n-1)/2}(P(h) - E - i\varepsilon) r^{-(n-1)/2} e^{-\varphi/h^{4/3}} \\ &= -h^2\partial_r^2 + 2h^{2/3}\varphi'\partial_r + \Lambda + \rho + V - h^{-2/3}\psi - E - i\varepsilon, \end{aligned}$$

where

$$0 \le \Lambda = \Lambda_h(r) := -h^2 r^{-2}\Delta_{\mathbb{S}^{n-1}}, \qquad \rho = \rho_h(r) := h^2(2r)^{-2}(n-1)(n-3).$$

To prove the Carleman estimate, we need another simple estimate, this time involving involving w, w' and ρ.

Lemma 4.4. *There exists $h_0 \in (0,1]$ depending on $E_{\min}$ and n so that*

$$\left(2w(r)r^{-1} - w'(r)\right)\rho(r) \ge -\frac{E_{\min}}{4} w'(r), \tag{4.26}$$

for all $E \ge E_{\min}$, $r \ne R_1$, and $h \in (0,h_0]$.

Proof. If $r < R_1$, then $2wr^{-1} - w' = 0$ and (4.26) follows immediately. On the other hand, if $r > R_1$, we use $R_1 > 3$ to see that

$$\begin{aligned} \left(2w(r)r^{-1} - w'(r)\right)\rho(r) &\ge -h^2(2r)^{-2}|n-1||n-3|w'(r) \\ &\ge -h^2|n-1||n-3|w'(r)/36. \end{aligned}$$

So we obtain (4.26) for $r > R_1$ by taking h_0 sufficiently small.

□

Proof of Lemma 4.1. Let $\int_{r,\theta}$ denote the integral over $(0,\infty) \times \mathbb{S}^{n-1}$ with respect to $drd\theta$, where $d\theta$ is the usual surface measure on $\mathbb{S}^{n-1}$.

To show (4.2), it suffices to prove that

$$\iint_{r\,\theta} (\mathbf{1}_{\leq 1} r + \mathbf{1}_{\geq 1}(1+r)^{-1-\delta})|u|^2 \leq \frac{C}{h^{10/3}} \left(\iint_{r\,\theta} (1+r)^{1+\delta}|P_\varphi u|^2 + \varepsilon \int_{r,\theta} |u|^2 \right), \quad u \in r^{(n-1)/2} e^{\varphi/h^{4/3}} C_0^\infty(\mathbb{R}^n). \tag{4.27}$$

Without loss of generality, we may assume $\varepsilon \leq h^{10/3}$. To show (4.27), we proceed in the spirit of the previous chapter and of [CaVo02, Da14, RT15] and define the functional F by

$$F(r) := \|hu'\|_S^2 - \langle (\Lambda + \rho - h^{-2/3}\psi - E)u, u\rangle_S, \qquad r > 0, \tag{4.28}$$

where $\|\cdot\|_S$ and $\langle \cdot, \cdot \rangle_S$ denote the norm and inner product on $\mathbb{S}^{n-1}$, respectively.

We compute the derivative of F, which exists for all $r \neq R_0$, $r \neq R_1$,

$$\begin{aligned} F'(r) &= 2\operatorname{Re}\langle h^2 u'', u'\rangle_S - 2\operatorname{Re}\langle (\Lambda + \rho - h^{-2/3}\psi - E)u, u'\rangle_S \\ &\quad + 2r^{-1}\langle (\Lambda + \rho)u, u\rangle + \langle h^{-2/3}\psi' u, u\rangle_S. \end{aligned}$$

Next, we calculate, for $r \neq R_0$, $r \neq R_1$,

$$\begin{aligned} wF' + w'F &= 2w\operatorname{Re}\langle h^2 u'', u'\rangle_S - 2w\operatorname{Re}\langle (\Lambda + \rho - h^{-2/3}\psi - E)u, u'\rangle_S \\ &\quad + 2wr^{-1}\langle (\Lambda + \rho)u, u\rangle_S + h^{-2/3} w\psi' \|u\|_S^2 \\ &\quad + w'\|hu'\|_S^2 - w'\langle (\Lambda + \rho)u, u\rangle_S + w'\langle (h^{-2/3}\psi + E)u, u\rangle_S \\ &= -2w\operatorname{Re}\langle P_\varphi u, u'\rangle_S + 2w\varepsilon \operatorname{Im}\langle u, u'\rangle_S \\ &\quad + h^2 w'\|u'\|_S^2 + (2wr^{-1} - w')\langle (\Lambda + \rho)u, u\rangle_S \\ &\quad + Ew'\|u\|_S^2 + 4h^{2/3} w\varphi'\|u'\|_S^2 + h^{-2/3}(w\psi)'\|u\|_S^2 + 2w\operatorname{Re}\langle Vu, u'\rangle_S. \end{aligned}$$

Note that we have have added and subtracted $2w\operatorname{Re}\langle Vu, u'\rangle_S$, $4h^{2/3}w\varphi'\|u\|_S^2$, and $2w\varepsilon\operatorname{Im}\langle u, u'\rangle_S$ in order to recover $P_\varphi u$ in line four. Using $w' > 0$, $2wr^{-1} - w' \geq 0$, $\Lambda \geq 0$ and $-2\operatorname{Re}\langle a, b\rangle + \|b\|^2 \geq -\|a\|^2$, we estimate, for $r \neq R_0$, $r \neq R_1$,

$$\begin{aligned} wF' + w'F &\geq -\frac{w^2}{h^2 w'}\|P_\varphi u\|_S^2 + 2w\varepsilon\operatorname{Im}\langle u, u'\rangle_S \\ &\quad + Ew'\|u\|_S^2 + (2wr^{-1} - w')\rho\|u\|_S^2 \\ &\quad + 4h^{2/3} w\varphi'\|u'\|_S^2 + h^{-2/3}(w\psi)'\|u\|_S^2 + 2w\operatorname{Re}\langle Vu, u'\rangle_S. \end{aligned} \tag{4.29}$$

To continue, let $\mathbf{1}_{B(0,R_0/2)}$ denote the characteristic function of $B(0,R_0/2)$. We bound $2w\operatorname{Re}\langle Vu,u'\rangle_S$ from below by

$$\begin{aligned}2w\operatorname{Re}\langle Vu,u'\rangle_S &\geq -2w(r)\int_\theta |V(r,\theta)u(r,\theta)u'(r,\theta)|d\theta\\ &\geq -\gamma\|V\|_\infty \mathbf{1}_{B(0,R_0/2)}(r)w(r)\|u'(r,\theta)\|_S^2\\ &-\gamma^{-1}\|V\|_\infty \mathbf{1}_{B(0,R_0/2)}(r)w(r)\|u(r,\theta)\|_S^2, \qquad \gamma>0.\end{aligned}$$

Plugging this lower bound into (4.29), we get for $r\neq R_0, r\neq R_1$.

$$\begin{aligned}wF'+w'F\geq &-\frac{w^2}{h^2w'}\|P_\varphi u\|_S^2+2w\varepsilon\operatorname{Im}\langle u,u'\rangle_S\\ &+\left(4h^{2/3}\varphi'-\gamma\|V\|_\infty\mathbf{1}_{B(0,R_0/2)}\right)w\|u'\|_S^2\\ &+\left(Ew'+(2wr^{-1}-w')\rho+h^{-2/3}(w\psi)'-\gamma^{-1}\|V\|_\infty\mathbf{1}_{B(0,R_0/2)}w\right)\|u\|_S^2.\end{aligned} \tag{4.30}$$

Now, fix $\gamma=h^{2/3}$ (the author is grateful to Jeff Galkowski for the suggestion to use an h-dependent γ). Then, use $\psi=c$ on $(0,R_0]$ along with (4.3) to get

$$\begin{aligned}(w\psi)'-\|V\|_\infty\mathbf{1}_{B(0,R_0/2)}w &\geq r\left(2c-\|V\|_\infty R_0/2\right)\\ &\geq 0, \qquad r\in(0,R_0/2].\end{aligned}$$

Combining this with (4.8) and (4.26), we have

$$\left(Ew'+(2wr^{-1}-w')\rho+h^{-2/3}(w\psi)'-\gamma^{-1}\|V\|_\infty\mathbf{1}_{B(0,R_0/2)}w\right)\|u\|_S^2\geq\frac{E_{\min}}{2}w'\|u\|_S^2. \tag{4.31}$$

for all $r>0$, $r\neq R_0$, $r\neq R_1$, and all $h\in(0,h_0]$, where h_0 is as given in Lemma 4.4.

On the other hand, according to (4.4), (4.14), and (4.17), we have

$$4\varphi'-\|V\|_\infty\mathbf{1}_{B(0,R_0/2)}\geq 0, \qquad r>0.$$

Updating (4.30) with these lower bounds, we get

$$\begin{aligned}wF'+w'F\geq &-\frac{w^2}{h^2w'}\|P_\varphi\|_S^2+2w\varepsilon\operatorname{Im}\langle u,u'\rangle_S\\ &+\frac{E_{\min}}{2}w'\|u\|_S^2, \qquad r\neq R_0,\ R_1.\end{aligned} \tag{4.32}$$

Next, we apply Fatou's lemma, along with the fundamental theorem of calculus to get

$$\int_0^\infty (w(r)F(r))' \leq -\liminf_{r\to 0} w(r)F(r) = 0. \tag{4.33}$$

Integrating (4.32) with respect to dr and using (4.33), we arrive at

$$\frac{E_{\min}}{2}\int_{r,\theta} w'|u|^2 \leq \frac{1}{h^2}\int_{r,\theta}\frac{w^2}{w'}|P_\varphi u|^2 + 2\varepsilon\int_{r,\theta} w|uu'|. \tag{4.34}$$

Combining (4.34) with, (4.11) and (4.12) gives for $h \in (0, h_0]$

$$\int_{r,\theta}\left(\mathbf{1}_{\leq 1}r + \mathbf{1}_{\leq 1}(1+r)^{-1-\delta}\right)|u|^2 \leq \frac{C}{h^{10/3}}\int_{r,\theta}(1+r)^{1+\delta}|P_\varphi u|^2 + 2\varepsilon\int_{r,\theta} w|uu'|, \tag{4.35}$$

where $C > 1$ is a constant that depends on $E_{\min}$, R_0, n, c and δ, but is independent of h and u. We will reuse C is the ensuing estimates, but its precise value will change from line to line.

We focus on the last term in (4.35). Our goal is to show

$$2\int_{r,\theta} w|uu'| \leq \frac{C}{h^2}\left(\int_{r,\theta} w^2|P_\varphi u|^2 + \int_{r,\theta}\left(1 + w^2 + \rho w^2\right)|u|^2\right) \qquad h \in (0, h_0]. \tag{4.36}$$

If we have shown (4.36), we can substitute it into (4.35) and use (4.10) along with

$$|\rho w^2| \leq Ch^{2/3}, \qquad r > 0$$

to get

$$\begin{aligned}\int_{r,\theta}\left(\mathbf{1}_{\leq 1}r + \mathbf{1}_{\leq 1}(1+r)^{-1-\delta}\right)|u|^2 \leq \frac{C}{h^{10/3}}\int_{r,\theta}(1+r)^{1+\delta}|P_\varphi u|^2 + \frac{C\varepsilon}{h^{10/3}}\int_{r,\theta}|P_\varphi u|^2 \\ + \frac{C\varepsilon}{h^{10/3}}\int_{r,\theta}|u|^2, \qquad h \in (0, h_0].\end{aligned}$$

Using $\varepsilon \leq h^{10/3}$ then gives (4.27).

To show (4.36), we first write

$$2\int_{r,\theta} w|uu'| \leq \frac{1}{h^2}\int_{r,\theta}|u|^2 + \int_{r,\theta} w^2|hu'|^2. \tag{4.37}$$

We will now show that

$$\int_{r,\theta} w^2|hu'|^2 \leq C\int_{r,\theta} w^2|P_\varphi u|^2 + \frac{C}{h^{2/3}}\int_{r,\theta}(w^2 + |\rho w^2|)|u|^2, \qquad h \in (0, h_0], \tag{4.38}$$

which will complete the proof of the Lemma. To show (4.38), we first integrate by parts,

$$\int_{r,\theta} w^2|hu'|^2 = \mathrm{Re}\left(\int_{r,\theta} w^2\bar{u}(-h^2u'') - 2h^2ww'\bar{u}u'\right)$$

and then estimate,

$$\mathrm{Re}\int_{r,\theta} -2h^2ww'\bar{u}u' \leq \frac{h^2}{\eta_1}\int_{r,\theta}(w')^2|u|^2 + \eta_1\int_{r,\theta} w^2|hu'|^2, \qquad \eta_1 > 0, \tag{4.39}$$

$$\begin{aligned}
&\mathrm{Re}\int_{r,\theta} w^2\bar{u}(-h^2u'')\\
&\quad = \mathrm{Re}\int_{r,\theta} w^2\bar{u}(P_\varphi - 2h^{2/3}\varphi'\partial_r - \Lambda - \rho - V + h^{-2/3}\psi + E + i\varepsilon)u\\
&\quad \leq \int_{r,\theta} w^2|P_\varphi u||u| + 2\int_{r,\theta} w^2\varphi'|h^{2/3}u'||u|\\
&\quad + \int_{r,\theta} w^2|E - \rho - V + h^{-2/3}\psi||u|^2\\
&\quad \leq \frac{1}{2}\int_{r,\theta} w^2|P_\varphi u|^2 + \eta_2\sqrt{c}\int_{r,\theta} w^2|hu'|^2 + \int_{r,\theta}|\rho w^2||u|^2\\
&\quad + \left(\frac{\sqrt{c}}{h^{2/3}\eta_2} + E_{\max} + \|V\|_\infty + \frac{c}{h^{2/3}} + \frac{1}{2}\right)\int_{r,\theta} w^2|u|^2, \qquad \eta_2 > 0.
\end{aligned} \tag{4.40}$$

Now, take $\eta_1 = 1/4$, $\eta_2 = 1/(4\sqrt{c})$, and bound $\int_{r,\theta} w^2|hu'|^2$ from above in (4.37) using (4.39) and (4.40). We get, for $h \in (0, h_0]$,

$$\int_{r,\theta} w^2|hu'|^2 \leq C\int_{r,\theta} w^2|P_\varphi u|^2 + \frac{C}{h^{2/3}}\int_{r,\theta}(w^2 + \rho w^2)|u|^2 + \frac{1}{2}\int_{r,\theta} w^2|hu'|^2.$$

Subtracting the last term to the left side and multiplying through by 2, we arrive at (4.38).

□

4.3 Proof of Theorem 2

In this final section, we use Lemma 4.1 to prove Theorem 2. We condense notation by setting $L^2 = L^2(\mathbb{R}^n)$, $H^2 = H^2(\mathbb{R}^n)$, and by renaming the weight appearing on the left side of (4.2),

$$b(r) := \mathbf{1}_{\leq 1}r^{1/2} + \mathbf{1}_{\geq 1}(1 + r)^{-s}.$$

We also employ of a smooth version of the weight $(1+r)^s$, as we did in the last chapter,

$$m = m_\delta(r) \coloneqq (1+r^2)^{(1+\delta)/4}.$$

Proof of Theorem 2. Let $\tilde{R}_0 > 3$ be large enough so that $\operatorname{supp} V \subseteq B(0, \tilde{R}_0/4)$. Pick $x_0 \in \mathbb{R}^n$ with $1/2 < |x_0| < 3/4$, which implies

$$\operatorname{supp} V_0(\cdot + x_0) \subseteq B(0, \tilde{R}_0/2).$$

We shift coordinates, apply (4.2) to the operator $P_0 = P_0(h) \coloneqq -h^2\Delta + V(\cdot + x_0) - E$ in place of P, and then shift back.

$$\begin{aligned} \left\| b(|\cdot - x_0|)e^{\varphi(|\cdot - x_0|)h^{-4/3}} v \right\|_{L^2}^2 &= \left\| b e^{\varphi h^{-4/3}} v(\cdot + x_0) \right\|_{L^2}^2 \\ &\leq \frac{C}{h^{10/3}} \left\| m e^{\varphi h^{-4/3}} (P_0 - i\varepsilon) v(\cdot + x_0) \right\|_{L^2}^2 \\ &+ \frac{C\varepsilon}{h^{10/3}} \left\| e^{\varphi h^{-4/3}} v(\cdot + x_0) \right\|_{L^2}^2 \\ &= \frac{C}{h^{10/3}} \left\| m(|\cdot - x_0|) e^{\varphi(|\cdot - x_0|)h^{-4/3}} (P - i\varepsilon) v \right\|_{L^2}^2 \\ &+ \frac{C\varepsilon}{h^{10/3}} \left\| e^{\varphi(|\cdot - x_0|)h^{-4/3}} v \right\|_{L^2}^2, \qquad h \in (0, h_0]. \end{aligned}$$

For $C > 1$ depending on E, $\|V\|_\infty$, $\tilde{R}_0$, n, and s. Summarizing in a single inequality, we have

$$\begin{aligned} \left\| b(|\cdot - x_0|)e^{\varphi(|\cdot - x_0|)h^{-4/3}} v \right\|_{L^2} &\leq \frac{C}{h^{10/3}} \left\| m(|\cdot - x_0|) e^{\varphi(|\cdot - x_0|)h^{-4/3}} (P - i\varepsilon) v \right\|_{L^2} \\ &+ \frac{C\varepsilon}{h^{10/3}} \left\| e^{\varphi(|\cdot - x_0|)h^{-4/3}} v \right\|_{L^2}, \qquad h \in (0, h_0] \end{aligned} \tag{4.41}$$

Set $C_\varphi = C_\varphi(h) \coloneqq 2 \max \varphi$. Recall that by (4.16),

$$1 \leq C_\varphi \leq K \log(h^{-1}), \tag{4.42}$$

for $K > 0$ depending on $\tilde{R}_0$, $\|V\|_\infty$, and $E_{\min}$, but not on h. Multiply (4.2) and (4.41) through by $e^{-C_\varphi h^{-4/3}}$ to obtain for $h \in (0, h_0]$,

$$e^{-C_\varphi h^{-4/3}} \|bv\|_{L^2}^2 \leq \frac{C}{h^{10/3}} \|m(P - i\varepsilon)v\|_{L^2}^2 + \frac{C\varepsilon}{h^{10/3}} \|v\|_{L^2}^2, \tag{4.43}$$

$$e^{-C_\varphi h^{-4/3}} \|b(|\cdot - x_0|)v\|_{L^2}^2 \leq \frac{C}{h^{10/3}} \|m(|\cdot - x_0|)(P - i\varepsilon)v\|_{L^2}^2 + \frac{C\varepsilon}{h^{10/3}} \|v\|_{L^2}^2. \tag{4.44}$$

It is straightforward to show that

$$4^{-1}m^{-2} \leq b^2 + b^2(|\cdot - x_0|), \qquad m^2 + m^2(|\cdot - x_0|)) \leq 17m^2, \tag{4.45}$$

We add (4.44) and (4.43) and apply (4.45) to arrive at

$$e^{-C_\varphi h^{-4/3}} \|m^{-1}v\|_{L^2}^2 \leq \frac{C}{h^{10/3}} \|m(P - i\varepsilon)v\|_{L^2}^2 + \frac{C\varepsilon}{h^{10/3}} \|v\|_{L^2}^2.$$

For any $\eta > 0$,

$$\begin{aligned} 2\varepsilon\|v\|_{L^2}^2 &= -2\,\mathrm{Im}\langle (P - i\varepsilon)v, v\rangle_{L^2} \\ &\leq \eta^{-1}\|m(P - i\varepsilon)v\|_{L^2}^2 + \eta\|m^{-1}v\|_{L^2}^2. \end{aligned}$$

Setting $\eta = h^{10/3}(2C)^{-1}e^{-C_\varphi h^{-4/3}}$ and applying (4.42), we estimate $\varepsilon\|v\|_{L^2}^2$ from above and find that

$$\|m^{-1}v\|_{L^2}^2 \leq e^{Ch^{-4/3}\log(h^{-1})}\|m(P - i\varepsilon)v\|_{L^2}^2, \qquad h \in (0, h_0]. \tag{4.46}$$

The final task is to use (4.46) to show that for any $f \in L^2$,

$$\|m^{-1}(P - i\varepsilon)^{-1}m^{-1}f\|_{L^2}^2 \leq e^{Ch^{-4/3}\log(h^{-1})}\|f\|_{L^2}^2, \qquad h \in (0, h_0]. \tag{4.47}$$

from which (1.6) follows.

To establish (4.47), we prove a simple Sobolev space estimate, (4.48), and then apply a density argument which uses (4.46). The proof of (4.48) is identical to the proof of (3.29), and the density argument matches the one given at the end of the proof of Theorem 1. However, we repeat both arguments for the reader's convenience.

We use $a \lesssim b$ to denote $a \leq C_{\varepsilon,h}b$ for $C_{\varepsilon,h}$ depending on ε and h, but not on $v \in H^2$. The commutator $[P, m] = -h^2\Delta m + 2h^2\nabla m \cdot \nabla : H^2 \to L^2$ is bounded. So for $v \in H^2$ such that $mv \in H^2$, we have

$$\begin{aligned} \|m(P - i\varepsilon)v\|_{L^2} &\leq \|(P - i\varepsilon)mv\|_{L^2} + \|[P, m]v\|_{L^2} \\ &\lesssim \|mv\|_{H^2} + \|v\|_{H^2} \\ &\lesssim \|mv\|_{H^2}. \end{aligned}$$

Thus we have shown

$$\|m(P - i\varepsilon)v\|_{L^2} \leq C_{\varepsilon,h}\|mv\|_{H^2}, \qquad v \in H^2 \text{ such that } mv \in H^2. \tag{4.48}$$

For fixed $f \in L^2$, the function $m(P - i\varepsilon)^{-1}m^{-1}f \in H^2$ because

$$\begin{aligned} m(P - i\varepsilon)^{-1}m^{-1}f &= (P - i\varepsilon)^{-1}f + [m, (P - i\varepsilon)^{-1}]m^{-1}f \\ &= (P - i\varepsilon)^{-1}f + (P - i\varepsilon)^{-1}[P, m](P - i\varepsilon)^{-1}m^{-1}f. \end{aligned}$$

Now, choose a sequence $v_k \in C_0^\infty$ such that $v_k \to m(P - i\varepsilon)^{-1}m^{-1}f$ in H^2. Define $\tilde{v}_k := m^{-1}v_k$. Then, as $k \to \infty$

$$\|m^{-1}\tilde{v}_k - m^{-1}(P - i\varepsilon)^{-1}m^{-1}f\|_{L^2} \leq \|v_k - m(P - i\varepsilon)^{-1}m^{-1}f\|_{H^2} \to 0.$$

Also, applying (4.48)

$$\|m(P - i\varepsilon)\tilde{v}_k - f\|_{L^2} \lesssim \|v_k - m(P - i\varepsilon)^{-1}m^{-1}f\|_{H^2} \to 0.$$

We then achieve (4.47) by replacing v by $\tilde{v}_k$ in (4.46) and sending $k \to \infty$.

□

5. LOCAL ENERGY DECAY FOR L^∞ WAVESPEEDS

Theorems 3 and 4 follow from the more general Theorem 5, which we prove in this chapter. To prove Theorem 5, the key is to establish suitable Sobolev space estimates at high and low energy on the norm of the meromorphic continuation of the cutoff resolvent $\chi R(\lambda)\chi \coloneqq \chi(-c^2\Delta - \lambda^2)^{-1}\chi$, where $\chi \in C_0^\infty(\mathbb{R}^n)$ and $\lambda \in \mathbb{R} \setminus \{0\}$. Here, the relevant spaces are $L^2(\mathbb{R}^n)$ and $\dot{H}^1(\mathbb{R}^n)$. They correspond to the second and first terms on the left side of (1.13), respectively.

Throughout this chapter, we suppose the dimension $n \geq 2$, and, as before, that our wavespeed has the properties

$$c = c(x) > 0, \quad c, c^{-1} \in L^\infty(\mathbb{R}^n), \qquad \text{supp}(c-1) \text{ is compact.}$$

Our other standing assumption is that we have an exponential semiclassical resolvent bound depending on a parameter $\ell \geq 1$.

Assumption. *Let $n \geq 2$. Suppose that $V \in L^\infty_{comp}(\mathbb{R}^n)$ is real-valued. Let $[E_{\min}, E_{\max}] \subseteq (0,\infty)$ and $\chi \in C_0^\infty(\mathbb{R}^n)$. There exist constants $\ell, C, h_0 > 0$ so that*

$$\|\chi(P(h) - i\varepsilon)^{-1}\chi\|_{L^2 \to L^2} \leq e^{Ch^{-\ell}} \tag{5.1}$$

for all $E \in [E_{\min}, E_{\max}]$, $0 < \varepsilon < 1$, $h \in (0, h_0]$.

For instance, Theorem 1, along with the Theorem in [Da14], show that (5.1) holds for $\ell = 1$ when $\nabla V \in L^\infty(\mathbb{R}^n)$. Theorem 2 shows that (5.1) holds, at worst, for $\ell = 4/3 + \eta$, any $\eta > 0$. As discussed in Chapter 1, the sharp value of ℓ for general $V \in L^\infty$ is still an open problem.

We use (5.1) and the connection suggested by (1.11) to show that, at high energy (Proposition 5.2):

$$\|\chi R(\lambda)\chi\|_{L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)} \leq e^{C_1|\lambda|^\ell}, \qquad \lambda \in \mathbb{R} \setminus [-M, M], \text{ some } M > 1. \tag{5.2}$$

At low energy, we use properties of the free resolvent to find (Proposition 5.1):

$$\|\chi R(\lambda)\chi\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} \leq C_1(1+|\lambda|^{n-2}|\log\lambda|), \tag{5.3}$$
$$\lambda \in [-\varepsilon_0, \varepsilon_0] \setminus \{0\}, \text{ some } 0 < \varepsilon_0 < 1.$$

One technical innovation presented in this chapter is the careful distinction between $\dot{H}^1(\mathbb{R}^n)$ and $\dot{H}^1(B(0,R))$ in order to deduce from (5.2) and (5.3) analogous estimates for the homogeneous space.

If $n \geq 3$, one can extend the continuation of $\chi R(\lambda)\chi$ as bounded operator $\dot{H}^1(\mathbb{R}^n) \to \dot{H}^1(\mathbb{R}^n)$ using that for any $\chi \in C_0^\infty(\mathbb{R}^n)$, there exists $C_\chi > 0$ such that

$$\|\chi\varphi\|_{L^2(\mathbb{R}^n)} \leq C_\chi\|\nabla\varphi\|_{L^2(\mathbb{R}^n)}, \qquad \text{all } \varphi \in C_0^\infty(\mathbb{R}^n). \tag{5.4}$$

This estimate follows, for instance, from the Gagliardo-Nirenberg-Sobolev (GNS) inequality [Ev, Theorem 1, Section 5.6.1]. In Appendix D, we use the GNS inequality to prove (5.4).

On the other hand, (5.4) fails when $n = 2$, also see Appendix D for a counterexample. However, for any $R_1 > 0$ as in Theorem 3, restricting to $C_0^\infty(B(0,R_1))$ restores access to (5.4), with C_χ now also depending on R_1. Then, for any dimension $n \geq 2$, the continuation of $\chi R(\lambda)\chi$ extends as a bounded operator $\dot{H}^1(B(0,R_1)) \to \dot{H}^1(\mathbb{R}^n)$ with norm estimates similar to (5.2) and (5.3), which are sufficient to prove Theorem 5.

The logarithmic singularity appearing in (5.3) when $n = 2$ differs from the case of an obstacle, where the resolvent is bounded near zero in all dimensions. Although, this singularity is still weak enough to allow integral estimates via Stone's formula, similar to the those appearing in [PoVo99]. From these estimates (Section 5.5) we conclude Theorem 5.

The outline of this chapter is as follows. In Section 5.1, we set up the energy space H in which we work, and give the more general statement of the local energy decay (Theorem 5). In Sections 5.2 and 5.3, we prove the $L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ cutoff resolvent estimates at high and low energy, respectively. In Section 5.4 we convert these estimates on $L^2(\mathbb{R}^n)$ into estimates on H. Finally, in Section 5.5, we combine these estimates with Stone's formula to prove the local energy decay rate.

We remind the reader that Appendix B contains various facts about the operators $-c^2\Delta$ and $(-c^2\Delta - \lambda^2)^{-1}$ to which we refer throughout this chapter.

5.1 Functional analytic statement of the local energy decay

For $\Omega \subseteq \mathbb{R}^n$ open, set $L^2_c(\Omega) := L^2(\Omega, c^{-2}dx)$. We work in the Hilbert space $H := \dot{H}^1(\mathbb{R}^n) \oplus L^2_c(\mathbb{R}^n)$. For $R > 0$, let

$$H_R := \{(u_0, u_1) \in H : u_0 \in \dot{H}^1(B(0,R)),\ \operatorname{supp} u_1 \subseteq B(0,R)\}. \tag{5.5}$$

This is a closed subspace of H, and is the space on which our logarithmic decay rate holds.

Set $L := -c^2(x)\Delta : L^2_c(\mathbb{R}^n) \to L^2_c(\mathbb{R}^n)$, which is nonnegative and self-adjoint with respect to the domain $D(L) = H^2(\mathbb{R}^n)$. Define the operator B by the matrix

$$B := \begin{bmatrix} 0 & iI \\ -iL & 0 \end{bmatrix} : H \to H,$$

which is self-adjoint with respect to the domain

$$D(B) := \{(u_0, u_1) \in H : \Delta u_0 \in L^2(\mathbb{R}^n), u_1 \in H^1(\mathbb{R}^n)\}.$$

The proofs that L and B are self-adjoint are in Appendix B.

Another fact we will deploy in Section 5.5 is that if $(u_0, u_1) = ([\varphi_m], u_1) \in D(B) \cap H_R$, then $B(u_0, u_1) = (iu_1, -iLu_0) \in H_{R'}$ for some $R' > R$. To show this, first observe that since $u_1 \in H^1(\mathbb{R}^n)$ and $\operatorname{supp} u_1 \subseteq B(0,R)$, u_1 may be approximated in $H^1(\mathbb{R}^n)$ by $C_0^\infty(\mathbb{R})$-functions with supports contained in a slightly larger ball $B(0,R') \supset B(0,R)$. Therefore $u_1 \in \dot{H}^1(B(0,R'))$. To see that $\operatorname{supp} \Delta u_0 \subseteq B(0,R')$, we integrate against

$\varphi \in C_0^\infty(\mathbb{R}^n \setminus \overline{B(0,R)})$ and apply integration by parts twice. We may then take advantage of the fact that each $\operatorname{supp}\varphi_m \subseteq B(0,R)$,

$$\begin{aligned}\int_{\mathbb{R}^n\setminus\overline{B(0,R)}} \Delta u_0 \varphi &= -\int_{\mathbb{R}^n\setminus\overline{B(0,R)}} \nabla u_0 \cdot \nabla\varphi \\ &= -\lim_{m\to\infty}\int_{\mathbb{R}^n\setminus\overline{B(0,R)}} \nabla\varphi_m\cdot\nabla\varphi \\ &= \lim_{m\to\infty}\int_{\mathbb{R}^n\setminus\overline{B(0,R)}} \varphi_m\Delta\varphi \\ &= 0.\end{aligned}$$

For $k \in \mathbb{N}$, let $\|\cdot\|_{D(B^k)}$ be the graph norm associated to B^k:

$$\|(u_0,u_1)\|_{D(B^k)} := \|(u_0,u_1)\|_H + \|B^k(u_0,u_1)\|_H, \qquad (u_0,u_1)\in D(B^k).$$

The operator B allows us to write the wave equation as a first order system. That is, given $(u_0,u_1) \in H$,

$$U(t) := (U_0(t), U_1(t)) = e^{-itB}(u_0,u_1), \tag{5.6}$$

is the unique solution in H to the wave equation

$$\begin{cases} \partial_t U + iBU = 0, & \text{in } \mathbb{R}^n\times(0,\infty), \\ U(0) = (u_0,u_1). \end{cases} \tag{5.7}$$

We now state the local energy decay rate for the solution of (5.7).

Theorem 5. *Assume* (5.1) *holds and suppose* $(u_1,u_0) \in D(B^k)\cap H_{R_1}$ *for some* $k \in \mathbb{N}$ *and* $R_1 > 0$*. Then for any* $R_2 > 0$*, there exists* $C > 0$ *depending on* ℓ*,* R_1*, and* R_2 *such that for* $t \geq 0$*,*

$$\left(\int_{B(0,R_2)} |\nabla U_0|^2(t) + |U_1|^2(t)dx\right)^{\frac{1}{2}} \leq \frac{C}{(\log(2+t))^{k/\ell}}\|(u_0,u_1)\|_{D(B^k)}. \tag{5.8}$$

5.2 Resolvent estimate at low energy

The purpose of this section is to combine the expansion for the free cutoff resolvent with a remainder argument to establish the following low energy bound for the perturbed cutoff resolvent.

Proposition 5.1. *Suppose that $\chi \in C_0^\infty(\mathbb{R}^n)$. Then there exists an $0 < \varepsilon_0 < 1$ so that*

$$\chi R(\lambda)\chi : L^2(\mathbb{R}^n) \to H^2(\mathbb{R}^n),$$

is analytic in $Q_{\varepsilon_0} := \{\lambda \in \mathbb{C} : |\operatorname{Re}\lambda|,\ |\operatorname{Im}\lambda| \le \varepsilon_0\} \setminus i\mathbb{R}_-$. Furthermore, there exists $C > 0$ such that

$$\|\chi R(\lambda)\chi\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} \le C(1 + |\lambda|^{n-2}|\log\lambda|), \qquad \lambda \in Q_{\varepsilon_0}. \tag{5.9}$$

Proof. It suffices to take $\chi = 1$ on the support of $c - 1$. Initially, for $\operatorname{Im}\lambda > 0$, define $K(\lambda) : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ by

$$\begin{aligned} K(\lambda) &:= (1 - c^{-2})\lambda^2(-\Delta - \lambda^2)^{-1} \\ &= (1 - c^{-2})\lambda^2\chi(-\Delta - \lambda^2)^{-1}. \end{aligned}$$

The continuation of $\chi R_0(\lambda)\chi$ then provides a continuation for $K(\lambda)\chi$ to $\mathbb{C} \setminus i\mathbb{R}_-$. From (2.1), we see that

$$K(\lambda)\chi = (1 - c^{-2})\lambda^2(E_1(\lambda) + \lambda^{n-2}\log\lambda E_2(\lambda)).$$

This implies that there exists $0 < \varepsilon_0 < 1$ sufficiently small so that

$$\lambda \in Q_{\varepsilon_0} \implies \|K(\lambda)\chi\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} < \frac{1}{2}. \tag{5.10}$$

Therefore, $I + K(\lambda)\chi$ can be inverted by a Neumann series for $\lambda \in Q_{\varepsilon_0}$,

$$(I + K(\lambda)\chi)^{-1} = \sum_{n=0}^{\infty}(-1)^n(K(\lambda)\chi)^n : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n).$$

Furthermore, $(I + K(\lambda)\chi)^{-1}$ is analytic in Q_{ε_0} because the series converges locally uniformly there.

To proceed, notice that $(1-\chi)K(\lambda) \equiv 0$ for $\operatorname{Im}\lambda > 0$ because $(1-\chi)(1-c^{-2}) \equiv 0$. From this, it follows that, when $\operatorname{Im}\lambda > 0$, $(I - K(\lambda)(1-\chi))$ is both a left and right inverse for $(I + K(\lambda)(1-\chi))$. Additionally, observe that

$$I + K(\lambda) = (I + K(\lambda)(1-\chi))(I + K(\lambda)\chi), \qquad \operatorname{Im}\lambda > 0.$$

Putting the two facts together, we get a left and right inverse for $I + K(\lambda)$

$$(I + K(\lambda))^{-1} = (I + K(\lambda)\chi)^{-1}(I - K(\lambda)(1-\chi)), \qquad \operatorname{Im}\lambda > 0,\ \lambda \in Q_{\varepsilon_0},$$

We can now write for, $\operatorname{Im}\lambda > 0$, $\lambda \in Q_{\varepsilon_0}$.

$$\begin{aligned}
\chi(-\Delta - c^{-2}\lambda^2)^{-1}\chi &= \chi(-\Delta - \lambda^2)^{-1}(I + K(\lambda))^{-1}\chi \\
&= \chi(-\Delta - \lambda^2)^{-1}(I + K(\lambda)\chi)^{-1}(I - K(\lambda)(1-\chi))\chi \\
&= \chi(-\Delta - \lambda^2)^{-1}(I + K(\lambda)\chi)^{-1}((I + K(\lambda)\chi) - K(\lambda))\chi \\
&= \chi R_0(\lambda)\chi - \chi(-\Delta - \lambda^2)^{-1}(I + K(\lambda)\chi)^{-1}K(\lambda)\chi \\
&= \chi R_0(\lambda)\chi - \chi(-\Delta - \lambda^2)^{-1}\sum_{n=0}^{\infty}(-1)^n \left(K(\lambda)\chi\right)^{n+1} \\
&= \chi R_0(\lambda)\chi - \chi(-\Delta - \lambda^2)^{-1}K(\lambda)\chi\sum_{n=0}^{\infty}(-1)^n \left(K(\lambda)\chi\right)^{n} \\
&= \chi R_0(\lambda)\chi - \chi R_0(\lambda)\chi K(\lambda)\chi\sum_{n=0}^{\infty}(-1)^n \left(K(\lambda)\chi\right)^{n} \\
&= \chi R_0(\lambda)\chi \left(I - \sum_{n=0}^{\infty}(-1)^n \left(K(\lambda)\chi\right)^{n+1} \right)
\end{aligned}$$

For the second-to-last equality, we use $K(\lambda) = \chi K(\lambda)$. We see that the left side continues analytically to Q_{ε_0} because the right side does.

To finish the proof, observe that

$$\|\chi R_0(\lambda)\chi\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} \le C(1 + |\lambda|^{n-2}|\log\lambda|), \qquad \lambda \in Q_{\varepsilon_0},$$

according to (2.1). It also follows from (5.10) that

$$\left\| I - \sum_{n=0}^{\infty}(-1)^n \left(K(\lambda)\chi\right)^{n+1} \right\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} \le 3, \qquad \lambda \in Q_{\varepsilon_0}.$$

We now conclude (5.9) because

$$\chi(-\Delta - c^{-2}\lambda^2)^{-1}\chi = \chi R(\lambda)\chi c^2.$$

$\square$

5.3 Resolvent estimate at high energy

The goal of this section is to establish an exponential bound on the perturbed cutoff resolvent when $|\operatorname{Re}\lambda|$ is large. Specifically, we prove the following.

Proposition 5.2. *For each $\chi \in C_0^\infty(\mathbb{R}^n)$, there exist constants $C_1, C_2 > 0$, $M > 1$ such that the cutoff resolvent $\chi R(\lambda)\chi$ continues analytically from $\operatorname{Im}\lambda > 0$ into the set $\{\lambda \in \mathbb{C} : |\operatorname{Re}\lambda| > M, |\operatorname{Im}\lambda| < e^{-C_2|\operatorname{Re}\lambda|^\ell}\}$, where it satisfies the bound*

$$\|\chi R(\lambda)\chi\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} \leq e^{C_1|\operatorname{Re}\lambda|^\ell}. \tag{5.11}$$

To prove Proposition 5.2, we need two Lemmas. Essentially, these lemmas convert (5.1) into suitable statements about $\chi R(\lambda)\chi$, from which we conclude (5.11). For $z \in \mathbb{C}$ and $R > 0$, let $D_R(z)$ denote the disk $\{w \in \mathbb{C} : |w - z| < R\}$. The first lemma is a non-semiclassical version of a continuation argument due to Vodev [Vod14, Theorem 1.5].

Lemma 5.1. *Let $\chi \in C_0^\infty(\mathbb{R}^n)$. Suppose that there exist $C > 0$ and $M > 1$ such that whenever $\lambda_0 \in \mathbb{R} \setminus [-M, M]$, the continuation of $\chi R(\lambda)\chi$ from $\operatorname{Im}\lambda > 0$ to $\mathbb{C} \setminus i\mathbb{R}_-$ satisfies*

$$\|\chi R(\lambda_0)\chi\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} \leq e^{C|\lambda_0|^\ell}. \tag{5.12}$$

Then there exist $C_1, C_2 > 0$ such that for each $\lambda_0 \in \mathbb{R} \setminus [-M, M]$, the continued cutoff resolvent is analytic in the disk $D_{\lambda_0}(e^{-C_2|\lambda_0|^\ell})$, where it has the estimate

$$\|\chi R(\lambda)\chi\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} \leq e^{C_1|\lambda_0|^\ell}. \tag{5.13}$$

Proof. Let $\chi_1 \in C_0^\infty(\mathbb{R}^n)$ have the property that $\chi_1 \equiv 1$ on the support of $c - 1$. Without loss of generality, we may assume that $\chi \equiv 1$ on the support of χ_1. For $\operatorname{Im}\lambda$, $\operatorname{Im}\mu > 0$, we have the resolvent identity

$$R(\lambda) - R(\mu) = (\lambda^2 - \mu^2)R(\lambda)R(\mu) \implies$$
$$R(\lambda) - R(\mu) = (\lambda^2 - \mu^2)R(\lambda)\chi_1(2 - \chi)R(\mu) + (\lambda^2 - \mu^2)R(\lambda)(1 - \chi_1)^2 R(\mu), \tag{5.14}$$

The first equality implies the second because $(1 - \chi_1)^2 + \chi_1(2 - \chi_1) = 1$.

We also compute

$$R(\lambda)(1-\chi_1) - (1-\chi_1)R_0(\lambda) = R(\lambda)[\chi_1, \Delta]R_0(\lambda), \qquad \operatorname{Im}\lambda > 0, \tag{5.15}$$

$$(1-\chi_1)R(\mu) - R_0(\mu)(1-\chi_1) = R_0(\mu)[\Delta, \chi_1]R(\mu), \qquad \operatorname{Im}\mu > 0. \tag{5.16}$$

Using (5.14), (5.15), and (5.16), we express $\chi R(\lambda)\chi - \chi R(\mu)\chi$ as a sum of five operators which we denote by $T_k(\lambda, \mu)$, $k = 1, \dots, 5$.

$$\begin{aligned}
\chi R(\lambda)\chi - \chi R(\mu)\chi &= (\lambda^2 - \mu^2)(\chi R(\lambda)\chi)(\chi_1(2-\chi_1))(\chi R(\mu)\chi) \\
&\quad + (1-\chi_1)\left[\chi R_0(\lambda)\chi - \chi R_0(\mu)\chi\right](1-\chi_1) \\
&\quad + (1-\chi_1)\left[\chi R_0(\lambda)\chi - \chi R_0(\mu)\chi\right][\Delta, \chi_1](\chi R(\mu)\chi) \\
&\quad - (\chi R(\lambda)\chi)[\Delta, \chi_1]\left[\chi R_0(\lambda)\chi - \chi R_0(\mu)\chi\right](1-\chi_1) \\
&\quad - (\chi R(\lambda)\chi)[\Delta, \chi_1]\left[\chi R_0(\lambda)\chi - \chi R_0(\mu)\chi\right][\Delta, \chi_1](\chi R(\mu)\chi). \\
&= \sum_{k=1}^{5} T_k(\lambda, \mu).
\end{aligned} \tag{5.17}$$

This formula continues to hold after continuing both λ and μ to $\mathbb{C} \setminus i\mathbb{R}$.

To proceed, take $\mu = \lambda_0$. We bound the $L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ norm of each $T_k(\lambda, \lambda_0)$ for $\lambda \in D_{\lambda_0}(e^{-C_2|\lambda_0|^\ell})$, where the precise value of $C_2 > 0$ will be determined later. Suppose that $\lambda \in D_{\lambda_0}(e^{-C_2|\lambda_0|^\ell})$ is not a pole of $\chi R(\lambda)\chi$. Using (2.3) along with the fundamental theorem of calculus for line integrals, we have, for $|\alpha_1| + |\alpha_2| \leq 2$,

$$\begin{aligned}
&\|\partial_x^{\alpha_1}\chi R_0(\lambda)\chi\partial_x^{\alpha_2} - \partial_x^{\alpha_1}\chi R_0(\lambda_0)\chi\partial_x^{\alpha_2}\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} \leq \\
&\qquad C_M|\lambda - \lambda_0| \sup_{|\lambda-\lambda_0|<e^{-C_2|\lambda_0|}} |\lambda|^{\alpha_1+\alpha_2-1}, \quad \lambda \in D_{\lambda_0}(e^{-C_2|\lambda_0|^\ell}).
\end{aligned}$$

Therefore, for some $K > 0$ large enough,

$$\begin{aligned}
&\|\partial_x^{\alpha_1}\chi R_0(\lambda)\chi\partial_x^{\alpha_2} - \partial_x^{\alpha_1}\chi R_0(\lambda_0)\chi\partial_x^{\alpha_2}\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} \\
&\qquad \leq |\lambda - \lambda_0| e^{K|\lambda_0|}, \qquad \lambda \in D_{\lambda_0}(e^{-C_2|\lambda_0|^\ell}).
\end{aligned} \tag{5.18}$$

Using (2.2), (5.18), and further increasing $K > 0$ if necessary, we conclude that for $\lambda \in D_{\lambda_0}(e^{-C_2|\lambda_0|^\ell})$

$$\|T_k(\lambda, \lambda_0)\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} \leq |\lambda - \lambda_0| e^{K|\lambda_0|^\ell}, \quad k = 2, 3,$$

$$\|T_k(\lambda, \lambda_0)\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} \leq |\lambda - \lambda_0| e^{K|\lambda_0|^\ell} \|\chi R(\lambda)\chi\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)}, \quad k = 1, 4, 5.$$

Hence, by (5.17) we arrive at

$$\|\chi R(\lambda)\chi\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} \le 3|\lambda-\lambda_0|e^{K|\lambda_0|^\ell}\|\chi R(\lambda)\chi\|_{L^2\to L^2} + 2e^{K|\lambda_0|^\ell}.$$

Now, require C_2 to be large enough so that

$$3|\lambda-\lambda_0|e^{K|\lambda_0|^\ell} < \frac{1}{2},$$

in which case there is a $C_1 > 0$ so that

$$\|\chi R(\lambda)\chi\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} < e^{C_1|\lambda_0|^\ell}, \qquad \lambda \in D_{\lambda_0}(e^{-C_2|\lambda_0|^\ell}).$$

We have shown then, that $\chi R(\lambda)\chi$ is uniformly bounded in $D_{\lambda_0}(e^{-C_2|\lambda_0|^\ell})$ when λ is not a pole. Therefore, we conclude that $\chi R(\lambda)\chi$ has no poles in $D_{\lambda_0}(e^{-C_2|\lambda_0|^\ell})$.

□

With Lemma 5.1 now in hand, we just need to show (5.12), which will complete the proof of Proposition 5.2. To establish (5.12), we apply the exponential resolvent estimate (5.1). By setting $V_c := 1 - c^{-2} \in L^\infty_{\text{comp}}(\mathbb{R}^n)$ and identifying $h = |\operatorname{Re}\lambda|^{-1}$, we can translate (5.1) into estimates for $\chi R(\lambda)\chi$ when $|\operatorname{Re}\lambda|$ is large.

Proof of (5.12). Set $V_c := 1 - c^{-2}$ and $\mathcal{O} := \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \neq 0,\ \operatorname{Im}\lambda > 0\}$. Without loss of generality, take $\chi \equiv 1$ on $\operatorname{supp} V_c$. Define on $\mathcal{O}$ the following families of operators $L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ with domain $H^2(\mathbb{R}^n)$,

$$\begin{aligned} A(\lambda) &:= -(\operatorname{Re}\lambda)^{-2}\Delta + V_c + (\operatorname{Im}\lambda)^2(\operatorname{Re}\lambda)^{-2}c^{-2} - i2\operatorname{Im}\lambda(\operatorname{Re}\lambda)^{-1}c^{-2} - 1, \\ B(\lambda) &:= -(\operatorname{Re}\lambda)^{-2}\Delta + V_c - i2\operatorname{Im}\lambda(\operatorname{Re}\lambda)^{-1} + (\operatorname{Im}\lambda)^2(\operatorname{Re}\lambda)^{-2} - 1, \end{aligned} \tag{5.19}$$

Furthermore, define on $\mathcal{O}$ the family $L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$,

$$D(\lambda) := (\operatorname{Im}\lambda)^2(\operatorname{Re}\lambda)^{-2}V_c - i2\operatorname{Im}\lambda(\operatorname{Re}\lambda)^{-1}V_c.$$

We first subtract,

$$B(\lambda) - A(\lambda) = D(\lambda).$$

Composing with inverses, we get

$$A(\lambda)^{-1} - B(\lambda)^{-1} = B(\lambda)^{-1}D(\lambda)A(\lambda)^{-1} \implies (I - B(\lambda)^{-1}D(\lambda))A(\lambda)^{-1} = B(\lambda)^{-1},$$

Multiplying on the left and right by χ and noticing that $D(\lambda) = \chi D(\lambda)\chi$, we arrive at

$$(I - \chi B(\lambda)^{-1}\chi D(\lambda))\chi A(\lambda)^{-1}\chi = \chi B(\lambda)^{-1}\chi, \qquad \lambda \in \mathcal{O}. \tag{5.20}$$

Setting $E = 1$ and $h = |\operatorname{Re}\lambda|^{-1}$, we apply (5.1) to $B(\lambda)^{-1}$. This gives $M, C > 0$ so that

$$\|\chi B(\lambda)^{-1}\chi\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} \leq e^{C|\operatorname{Re}\lambda|^\ell}, \qquad |\operatorname{Re}\lambda| > M, \qquad \lambda \in \mathcal{O}. \tag{5.21}$$

There exists a constant $L_\lambda > 0$ depending on $\operatorname{Re}\lambda$ so that

$$\|D(\lambda)\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} < \frac{1}{2}e^{-C|\operatorname{Re}\lambda|^\ell}, \qquad |\operatorname{Re}\lambda| > M,\ 0 < \operatorname{Im}\lambda < L_\lambda. \tag{5.22}$$

Therefore, if $|\operatorname{Re}\lambda| > 0$ and $0 < \operatorname{Im}\lambda < L_\lambda$, we can invert $(I - \chi B(\lambda)^{-1}\chi D(\lambda))$ by a Neumann series. From (5.20) we get

$$\chi A(\lambda)^{-1}\chi = \quad \sum_{k=0}^{\infty}\left(\chi B(\lambda)^{-1}\chi D(\lambda))^k\right)\left(\chi B(\lambda)^{-1}\chi. \right. \tag{5.23}$$

Next, we notice that

$$\chi R(\lambda)\chi = (\operatorname{Re}\lambda)^{-2}\chi A(\lambda)^{-1}c^{-2}\chi, \qquad \lambda \in \mathcal{O}.$$

Then (5.12) follows from the estimates (5.21) and (5.22) as well as the identity (5.23).

□

5.4 Resolvent estimate on the energy space

The objective in this section is to prove Proposition 5.3. It states that when the resolvent $R_B(\lambda)$ acts on initial data in H_R, it continues analytically from $\operatorname{Im}\lambda > 0$ to a region in the lower half plane with estimates on the norm there. These properties follow from the resolvent estimates proved for $\chi R(\lambda)\chi : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ in the previous two sections.

To keep our notation manageable, we set

$$(L^2_R)^{n+1} := (L^2(B(0,R))^n \oplus L^2_c(B(0,R)).$$

Define $S_R : H \to (L^2_R)^{n+1}$ by $S_R(u_0, u_1) = (\nabla u_0, u_1)$. Note that $\|S_R\|_{H\to(L^2_R)^{n+1}} = 1$. Also, throughout this section, $a \lesssim b$ means that $a \leq Cb$ for some $C > 0$ that does not depend on λ.

Proposition 5.3. *Let $R_1, R_2 > 0$. There exist $C_1, C_2 > 0$, $M > 1$, and $0 < \varepsilon_0 < 1$ so that for all $(u_0, u_1) \in H_{R_1}$, $S_{R_2} R_B(\lambda)(u_0, u_1)$ continues analytically from $\operatorname{Im}\lambda > 0$ to the region*

$$\begin{aligned}\Theta :=&\{\lambda \in \mathbb{C} : |\operatorname{Re}\lambda| > M,\ \operatorname{Im}\lambda > -e^{-C_2|\operatorname{Re}\lambda|^\ell}\}\cup \\ &\{\lambda \in \mathbb{C} : 0 < |\operatorname{Re}\lambda| \leq M,\ \operatorname{Im}\lambda > -e^{C_2 M^\ell}\}.\end{aligned} \tag{5.24}$$

One possible Θ is depicted in Figure 5.1. Furthermore, $S_{R_2} R_B(\lambda)(u_0, u_1)$ obeys the estimate

$$\|S_{R_2} R_B(\lambda)(u_0, u_1)\|_{(L^2_R)^{n+1}} \lesssim \begin{cases} e^{C_1|\operatorname{Re}\lambda|^\ell} & \lambda \in \Theta \cap \{|\operatorname{Re}\lambda| > \varepsilon_0\}, \\ 1 + |\lambda|^{n-2}|\log\lambda| & \lambda \in \Theta \cap \{0 < |\operatorname{Re}\lambda| \leq \varepsilon_0\}. \end{cases} \tag{5.25}$$

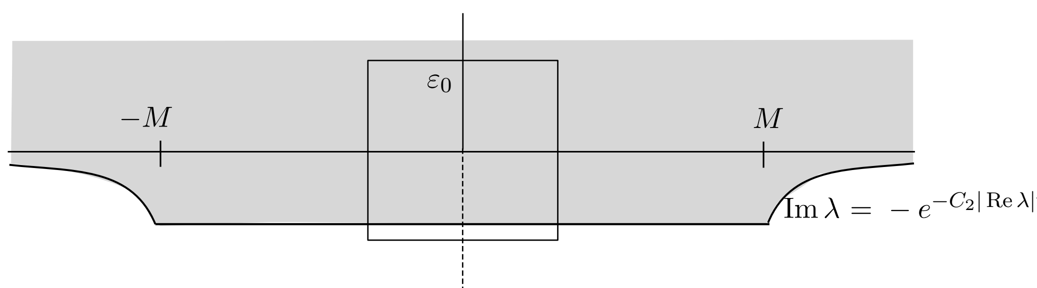

Fig. 5.1. One possible region Θ.

To prove Proposition 5.3, we first make a compactness argument to show that we may combine the resolvent estimates of Propositions 5.1 and 5.2 to obtain an estimate for $\chi R(\lambda)\chi : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ resembling (5.25) in a region of the form (5.24).

Lemma 5.2. *There exist* $C_1, C_2 > 0$*,* $M > 1$*, and* $0 < \varepsilon_0 < 1$ *such that the meromorphic continuation of the cutoff resolvent* $\chi R(\lambda)\chi : L^2 \to L^2$ *has no poles in the region* Θ *of* (5.24)*, where it obeys*

$$\|\chi R(\lambda)\chi\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} \lesssim \begin{cases} e^{C_1|\operatorname{Re}\lambda|^\ell} & \lambda \in \Theta \cap \{|\operatorname{Re}\lambda| > \varepsilon_0\}, \\ 1 + |\lambda|^{n-2}|\log\lambda| & \lambda \in \Theta \cap \{0 < |\operatorname{Re}\lambda| \le \varepsilon_0\}. \end{cases} \tag{5.26}$$

Proof. Let ε_0 be as in the statement of Proposition 5.1. Let C_1, C_2, and M be as in the statement of Proposition 5.11.

Set $L \coloneqq \min\{\varepsilon_0, e^{-C_2M^\ell}\}$. There exist only finitely many poles of $\chi R(\lambda)\chi$ in the compact set $\{\lambda \in \mathbb{C} : \varepsilon_0 \le |\operatorname{Re}\lambda| \le M, -L \le \operatorname{Im}\lambda \le 0\}$. Furthermore, as discussed in Section 2.3, there are no poles of $\chi R(\lambda)\chi$ on the strips $\{\lambda \in \mathbb{R} : \varepsilon_0 \le |\operatorname{Re}\lambda| \le M\}$. Therefore, there exists $0 < L' \le L$ so that $\{\lambda \in \mathbb{C} : \varepsilon_0 \le |\operatorname{Re}\lambda| \le M, -L' \le \operatorname{Im}\lambda \le 0\}$ contains no poles of $\chi R(\lambda)\chi$. If we redefine $M^\ell = -(\log L')/C_2$ (so that $L' = e^{-C_2M^\ell}$), then $\chi R(\lambda)\chi$ has no poles in (5.24)

Using (5.9), (5.11), and the continuity of $\chi R(\lambda)\chi$ on the rectangles $\{\lambda \in \mathbb{C} : \varepsilon_0 \le |\operatorname{Re}\lambda| \le M,\ e^{-C_2M^\ell} \le -\operatorname{Im}\lambda \le 1\}$, we get

$$\begin{aligned} &\|\chi R(\lambda)\chi\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} \\ &\lesssim \begin{cases} e^{C_1|\operatorname{Re}\lambda|^\ell} & |\operatorname{Re}\lambda| > M,\ |\operatorname{Im}\lambda| \le e^{-C_2|\operatorname{Re}\lambda|^\ell}, \\ e^{C_1|\operatorname{Re}\lambda|^\ell} & \varepsilon_0 < |\operatorname{Re}\lambda| \le M,\ \ -e^{-C_2M^\ell} \le \operatorname{Im}\lambda \le 1, \\ 1 + |\lambda|^{n-2}|\log\lambda| & 0 < |\operatorname{Re}\lambda| \le \varepsilon_0,\ \ -e^{-C_2M^\ell} \le \operatorname{Im}\lambda \le 1, \end{cases} \end{aligned} \tag{5.27}$$

where it may be necessary to increase the value of C_1.

To finish showing (5.26), we invoke the spectral theorem, which says that for $\operatorname{Im}\lambda > 0$,

$$\|\chi R(\lambda)\chi\|_{H\to H} \lesssim \frac{1}{\operatorname{dist}(\lambda^2, \mathbb{R}_+)}.$$

The above bound implies, for instance,

$$\|\chi R(\lambda)\chi\|_{H\to H} \lesssim \begin{cases} (\operatorname{Im}\lambda)^{-1} \le e^{C_2|\operatorname{Re}\lambda|^\ell}, & |\operatorname{Re}\lambda| > M,\ \operatorname{Im}\lambda > e^{-C_2|\operatorname{Re}\lambda|^\ell} \\ 1 & |\operatorname{Re}\lambda| \le M,\ \operatorname{Im}\lambda > 1. \end{cases} \tag{5.28}$$

Piecing together (5.27) and (5.28), and further increasing C_1 if needed, we arrive at (5.26).

□

Recall in section 2.5 we extended $R(\lambda)$ for $\lambda^2 \notin \mathbb{R}_+$ to a bounded operator $\dot{H}^1(B(0,R)) \to H^2(\mathbb{R}^n)$ using (2.6). Along with this, we now define bounded $I_1 : \dot{H}^1(B(0,R)) \to H^1(\mathbb{R}^n)$ by

$$I_1[\varphi_m] := L^2\text{-}\lim \varphi_m, \qquad [\varphi_m] \in \dot{H}^1(B(0,R)).$$

The estimate (2.5) shows that the above limit function exists and belongs to $H^1(\mathbb{R}^n)$.

Using these operators, we build the bounded matrix operator $\mathcal{M}_R(\lambda) : H_R \to H^2(\mathbb{R}^n) \oplus H^1(\mathbb{R}^n) \subseteq H$,

$$\mathcal{M}_R(\lambda)\begin{bmatrix} u_0 \\ u_1 \end{bmatrix} := \begin{bmatrix} \lambda R(\lambda) & iR(\lambda) \\ -i\lambda^2 R(\lambda) - iI_1 & \lambda R(\lambda) \end{bmatrix}\begin{bmatrix} u_0 \\ u_1 \end{bmatrix}, \qquad \operatorname{Im}\lambda > 0, \tag{5.29}$$

where $R(\lambda)$ acts on u_1 as the usual resolvent sending $L^2(\mathbb{R}^n) \to H^2(\mathbb{R}^n)$. A brief calculation shows that for all $(u_0, u_1) \in H_R$

$$\mathcal{M}_R(\lambda)\begin{bmatrix} u_0 \\ u_1 \end{bmatrix} \in D(B) \quad \text{and} \quad (B-\lambda)\mathcal{M}_R(\lambda)\begin{bmatrix} u_0 \\ u_1 \end{bmatrix} = \begin{bmatrix} u_0 \\ u_1 \end{bmatrix}, \qquad \operatorname{Im}\lambda > 0.$$

Therefore we conclude

$$R_B(\lambda)(u_0, u_1) = \mathcal{M}_R(\lambda)\begin{bmatrix} u_0 \\ u_1 \end{bmatrix} \qquad (u_0, u_1) \in H_R, \qquad \operatorname{Im}\lambda > 0. \tag{5.30}$$

Now that we have the estimate (5.26) and the identity (5.30), we can prove Proposition 5.3.

Proof of Proposition 5.3. Let $\chi \in C_0^\infty(\mathbb{R}^n)$ with $\chi \equiv 1$ on $\overline{B(0,R_1)} \cup \overline{B(0,R_2)}$. For $(u_0, u_1) = ([\varphi_m], u_1) \in H_{R_1}$, set $\varphi = L^2\text{-}\lim \varphi_m$. Note that $\chi\varphi = \varphi$. Also, for any function $u \in H^1(\mathbb{R}^n)$, $\nabla u = \nabla(\chi u)$ as vectors in $(L^2(B(0,R_2)))^n$. Combining these observations with (5.29) and (5.30), we get

$$S_{R_2}R_B(\lambda)(u_0, u_1) = \begin{bmatrix} \lambda\nabla\chi R(\lambda)\chi\varphi + i\nabla\chi R(\lambda)\chi u_1 \\ -i\lambda^2\chi R(\lambda)\chi\varphi - i\varphi + \lambda\chi R(\lambda)\chi u_1 \end{bmatrix}, \qquad \operatorname{Im}\lambda > 0. \tag{5.31}$$

By Lemma 5.2, the entries in the second component of the right side of (5.31) continue analytically from $\operatorname{Im}\lambda > 0$ to (5.24). Their $L^2(B(0,R_2))$-norms have estimates of the form (5.26) for a possibly larger constant C_1, to account for the factors of λ that appear.

The terms in the first component continue analytically to (5.24) by the identity

$$\nabla\chi R(\lambda)\chi = \nabla\chi R_0(\lambda)\chi + \nabla\chi\tilde{\chi}R_0(\lambda)\tilde{\chi}(1-c^{-2})(\chi + \tilde{\chi}R(\lambda)\tilde{\chi}\chi). \tag{5.32}$$

where $\tilde{\chi} \in C_0^\infty(\mathbb{R}^n)$ is identically one on $\operatorname{supp}\chi_{R_1} \cup \operatorname{supp}(1-c^{-2})$. The bounds (2.2) and (5.26) imply that $\|\nabla\chi R(\lambda)\chi\|_{L^2\to(L^2)^n}$ also has a bound of the form (5.26), where again we may need to increase C_1. Because we have shown each component of $S_{R_2}\mathcal{M}_{R_1}(\lambda)(u_0,u_1)$ obeys an estimate of the form (5.26), the triangle inequality ensures that (5.25) holds.

□

We collect one additional fact before proving the local energy decay in the next section. By the spectral theorem, $R(\lambda)^* = R(\overline{\lambda})$, $\lambda^2 \notin \mathbb{R}_+$. Therefore, when $\operatorname{Im}\lambda < 0$, we have the identities

$$\begin{aligned}\|\chi R(\lambda)\chi\|_{L^2\to L^2} &= \|(\chi R(\lambda)\chi)^*\|_{L^2\to L^2}\\ &= \|\chi R(\overline{\lambda})\chi\|_{L^2\to L^2},\\ \|\partial_x^\alpha\chi R_0(\lambda)\chi\|_{L^2\to L^2} &= \|(\partial_x^\alpha\chi R(\lambda)\chi)^*\|_{L^2\to L^2}\\ &= \|\chi R(\overline{\lambda})\chi\partial_x^\alpha\|_{L^2\to L^2}, \qquad |\alpha| = 1.\end{aligned} \tag{5.33}$$

Noting that we can make the same definition (5.29) for $\operatorname{Im}\lambda < 0$, and then using (5.26), (5.33), and the proof strategy of Proposition 5.3, we get

$$S_{R_2}R_B(\lambda)(u_0,u_1) \lesssim \begin{cases} e^{C_1|\operatorname{Re}\lambda|^\ell} & |\operatorname{Re}\lambda| > \varepsilon_0,\ \operatorname{Im}\lambda < 0,\\ 1 + |\lambda|^{n-2}|\log\lambda| & 0 < |\operatorname{Re}\lambda| \le \varepsilon_0,\ \operatorname{Im}\lambda < 0.\end{cases} \tag{5.34}$$

5.5 Proof of Theorem 5

We now give the proof of Theorem 5.1, our local energy decay. The proof proceeds in the spirit of [PoVo99, Proposition 1.4]. The idea is to rewrite the wave propagator using the spectral theorem and Stone's formula, and then make an appropriate contour deformation which is made possible by Proposition 5.3.

Proof of Theorem 5.1. Throughout the proof, we use $a \lesssim b$ to denote $a \leq Cb$, where $C > 0$ is a constant that does not depend on t or the initial data (u_0, u_1). If a norm appears without a subscript, it denotes the norm on $(L^2_R)^{n+1}$.

It it enough to show that

$$\|S_{R_2} e^{itB}(u_0, u_1)\| \lesssim \frac{1}{(\log(2+t))^k} \|(u_0, u_1)\|_{D(B^k)}, \qquad t \geq 0.$$

Moreover, we can replace $\|(u_0, u_1)\|_{D(B^k)}$ by $\|(B-i)^k(u_0, u_1)\|_H$ on the right side because the spectral theorem shows that the operators B^k and $(B-i)^k$ have the same domain and that the norms $\|(u_0, u_1)\|_{D(B^k)}$ and $\|(B-i)^k(u_0, u_1)\|_H$ are equivalent.

Let E denote the spectral measure associated to B, and let $X = X(t)$ be a parameter which depends on t. In the last step of the proof, we give the explicit dependence of X on t.

To keep our notation concise, set $F(\lambda) = e^{-it\lambda}(\lambda - i)^{-k}$. The wave propagator may be rewritten as

$$\begin{aligned} e^{-itB}(u_0, u_1) &= e^{-itB}(B-i)^{-k}(B-i)^k(u_0, u_1) \\ &= \int_{-\infty}^{\infty} F(\lambda) dE(\lambda)(B-i)^{-k}(u_0, u_1) \\ &= \left(\int_{-X}^{X} F(\lambda) dE(\lambda) + \int_{|\lambda| \geq X} F(\lambda) dE(\lambda) \right) (B-i)^k(u_0, u_1) \\ &= (I_{|\lambda|<X} + I_{|\lambda| \geq X})(B-i)^k(u_0, u_1) \end{aligned} \tag{5.35}$$

We apply S_{R_2} to each of the two integrals and estimate them by separate methods.

To handle the term $S_{R_2} I_{|\lambda| \geq X}(B-i)^k(u_0, u_1)$, let $\mathbf{1}_{\mathbb{R} \setminus [-X,X]}$ denote the indicator function of the set $\mathbb{R} \setminus [-X, X]$. Then, by properties of the spectral measure,

$$\begin{aligned} \|S_{R_2} I_{|\lambda| \geq X}\|_{H \to (L^2_R)^{n+1}} &\leq \|I_{|\lambda| \geq X}\|_{H \to H} \\ &\leq \sup_{|\lambda| \geq X} F(\lambda) \mathbf{1}_{\mathbb{R} \setminus [-X,X]}(\lambda) \\ &\lesssim X^{-k}. \end{aligned} \tag{5.36}$$

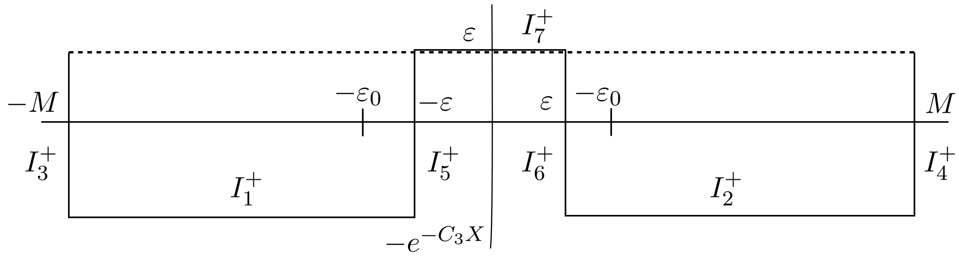

Fig. 5.2. The contour deformation for $I^+(\varepsilon)$.

To estimate the term $S_{R_2} I_{|\lambda|<X}(B-i)^k(u_0,u_1)$, we use Stone's formula, which says that, with respect to strong convergence, the spectral measure may be expressed as

$$dE(\lambda) = \lim_{\varepsilon\to 0^+}(2\pi i)^{-1}(R_B(\lambda+i\varepsilon) - R_B(\lambda-i\varepsilon))d\lambda.$$

For each $\varepsilon > 0$, we can move $(B-i)^k(u_0,u_1)$ inside the integral. In addition, the boundedness of S_{R_2} allows us to commute it through this strong limit. We get

$$\begin{aligned} 2\pi i S_{R_2} I_{|\lambda|\geq X}\langle B\rangle^k(u_0,u_1) &= \lim_{\varepsilon\to 0^+}\Bigg(\int_{-X}^{X} F(\lambda) S_{R_2} R_B(\lambda+i\varepsilon)(B-i)^k(u_0,u_1)d\lambda \\ &+ \int_{-X}^{X} F(\lambda) S_{R_2} R_B(\lambda-i\varepsilon)(B-i)^k(u_0,u_1)d\lambda\Bigg) \\ &= \lim_{\varepsilon\to 0^+}\Bigg(\int_{-X+i\varepsilon}^{X+i\varepsilon} F(\lambda-i\varepsilon) S_{R_2} R_B(\lambda)(B-i)^k(u_0,u_1)d\lambda \\ &+ \int_{-X-i\varepsilon}^{X-i\varepsilon} F(\lambda+i\varepsilon) S_{R_2} R_B(\lambda)(B-i)^k(u_0,u_1)d\lambda\Bigg) \\ &= \lim_{\varepsilon\to 0^+}\left(I^+(\varepsilon) + I^-(\varepsilon)\right). \end{aligned} \tag{5.37}$$

The endpoints for the final two integrals indicate that we integrate over the line segments $\{\lambda \pm i\varepsilon : \lambda \in [-X, X]\}$.

As discussed in section 2.6, the operator B sends H_{R_1} into $H_{R'}$ for some $R' > R$, hence $(B-i)^k(u_0,u_1) \in H_{R'}$. Therefore, Proposition 5.3 applies to $S_{R_2}R_B(\lambda)(B-i)^k(u_0,u_1)$. Setting $C_3 := \max\{2C_1, C_2\}$, we perform a contour deformation for $I^+(\varepsilon)$ which has seven segments, $I_k^+ = I_k^+(\varepsilon)$, $1 \leq k \leq 7$. See Figure 5.2.

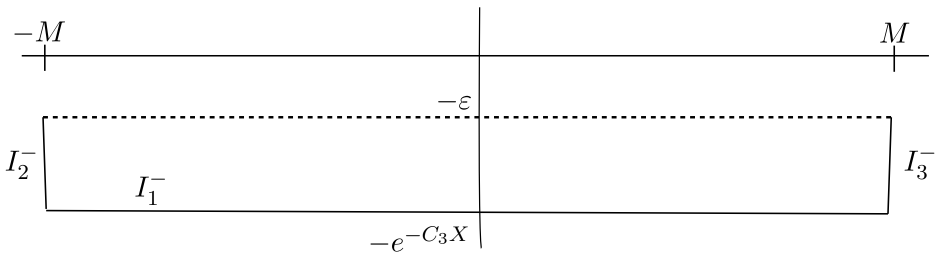

Fig. 5.3. The contour deformation for $I^-(\varepsilon)$.

We use (5.25) to estimate the integral over each segment, and omit the factor $\|(B-i)^k(u_0,u_1)\|_H$ that should appear on the right side of each inequality:

$$\begin{aligned} \|I_1^+(\varepsilon)\|, \|I_2^+(\varepsilon)\| &\lesssim Xe^{-te^{-C_3X^\ell}+C_1X^\ell}, \\ \|I_3^+(\varepsilon)\|, \|I_4^+(\varepsilon)\| &\lesssim (\varepsilon + e^{-C_3X})X^{-k}e^{C_1X^\ell} \\ \|I_5^+(\varepsilon)\|, \|I_6^+(\varepsilon)\| &\lesssim \int_{-e^{-C_3X}}^{\varepsilon} e^{\varepsilon t}|\log|r||dr, \\ \|I_7^+(\varepsilon)\| &\lesssim \int_{-\varepsilon}^{\varepsilon} e^{\varepsilon t}|\log|r||dr. \end{aligned} \tag{5.38}$$

To handle $I^-(\varepsilon)$, we deform it into three segments, $I_k^- = I_k^-(\varepsilon)$, $1 \le k \le 3$. Using (5.34), and again omitting the factor $\|(B-i)^k(u_0,u)\|_H$, we have

$$\begin{aligned} \|I_1^-(\varepsilon)\| &\lesssim Xe^{-te^{C_3X^\ell}+C_1X^\ell}, \\ \|I_2^-(\varepsilon)\|, \|I_3^-(\varepsilon)\| &\lesssim X^{-k}e^{(C_1-C_3)X^\ell}, \end{aligned} \tag{5.39}$$

Taking $\varepsilon \to 0^+$ and using the bounds from (5.36), (5.38), and (5.39), we get

$$\|S_{R_2}e^{itB}(u_0,u_1)\| \lesssim \left(Xe^{-te^{-C_3X^\ell}+C_1X^\ell} + X^{-k}\right)\|(B-i)^k(u_0,u_1)\|_{D(B^k)}. \tag{5.40}$$

To finish the proof, set

$$X(t) = \left(\frac{\log(2+t)}{2C_3}\right)^{1/\ell}.$$

We have,

$$\begin{aligned} (\log(2+t))^{1/\ell} &\lesssim t(t+2)^{-1/2} - C_1(2C_3)^{-1}\log(2+t) \\ &= te^{-C_3X(t)^\ell} - C_1X(t)^\ell, \qquad t \to \infty. \end{aligned} \tag{5.41}$$

Furthermore, for any $C > 0$,

$$xe^{-Cx} \lesssim x^{-k}, \qquad x > 0. \tag{5.42}$$

Plugging the expression for $X(t)$ into (5.40) and estimating using (5.41) and (5.42) completes the proof.

□

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APPENDICES

A. THE FREE RESOLVENT

In this Appendix, we first give a short review of Bessel functions in Section A.1. Then in Section A.2, we use spectral theory and special functions to compute the integral kernel for the free cutoff resolvent

$$\chi R_0(\lambda)\chi := \chi(-\Delta - \lambda^2)^{-1}\chi : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n),$$

where $\chi \in C_0^\infty(\mathbb{R}^n)$ and $\operatorname{Im}\lambda > 0$. Throughout, we assume $n \geq 2$. Note once again that $\operatorname{Im}\lambda > 0$ implies $\lambda^2 \notin \mathbb{R}_+$.

Our calculations yield the following formula for the integral kernel:

$$\begin{aligned}(\chi R_0(\lambda)\chi)\varphi(x) &= \\ &\frac{1}{4} i e^{\frac{i\nu\pi}{2}} \chi(x) \int_{\mathbb{R}^n} \frac{1}{2\pi} \left(\frac{\lambda}{2\pi|x-y|} \right)^\nu H_\nu^{(1)}(i\lambda|x-y|)\chi(y)\varphi(y)dy,\end{aligned} \tag{A.1}$$

for $\varphi \in C_0^\infty(\mathbb{R}^n)$ and $\operatorname{Im}\lambda > 0$, and where $H_\nu^{(1)}$ is the Hankel function of the first kind of order ν, and ν is related to the dimension by

$$\nu = \frac{n}{2} - 1.$$

Since both sides of (A.1) are analytic in $\{\lambda : \operatorname{Im}\lambda > 0\}$, it suffices to show (A.1) holds on the positive imaginary axis, that is when $\lambda = i\mu$, $\mu > 0$. Therefore, when we calculate (A.1) in Section A.2, we work exclusively with λ of this form.

In section A.3, we use (A.1), along with the expression for $H_\nu^{(1)}$ in terms of the Bessel functions J_ν and Y_ν, to deduce that $\chi R_0(\lambda)\chi$ continues meromorphically to $\mathbb{C}$ when n is odd, and to the logarithmic cover of $\mathbb{C} \setminus \{0\}$ when n is even. The series for J_ν and Y_ν yield the expansion (2.1) for $\chi R_0(\lambda)\chi$, which we recall for the reader's convenience

$$\chi R_0(\lambda)\chi = E_1(\lambda) + \lambda^{n-2}\log\lambda E_2(\lambda),$$

for $\lambda \in \mathbb{C} \setminus i\mathbb{R}_-$ and $E_1(\lambda)$, $E_2(\lambda)$ entire operator-valued functions, $E_2 \equiv 0$ when n is odd.

One can apply Schur's test to the kernel in (A.1) to prove, for the continuation of $\chi R_0(\lambda)\chi$, the λ-dependent bounds (2.2) for $\|\partial^{\alpha_1}\chi R_0(\lambda)\chi\partial^{\alpha_2}\|_{L^2\to L^2}$. However, we omit the details of this argument.

The reader who is interested in the one dimensional case may consult [DyZw, Chapter 2].

A.1 Review of Bessel functions

Our notational conventions for the various special functions match those of [Ol]. We begin with the Bessel function of the first kind of order ν. It is denoted by $J_\nu(z)$ and defined by the series [Ol, page 57]:

$$J_\nu(z) := \left(\frac{1}{2}z\right)^\nu \sum_{k=0}^{\infty} \frac{(-1)^k(\frac{1}{4}z^2)^k}{k!\Gamma(\nu+k+1)},$$

where Γ is the Gamma function [Ol, page 31]:

$$\Gamma(z) = \int_0^\infty e^{-t}t^{z-1}dt, \qquad \operatorname{Re} z > 0.$$

Next, we consider the Bessel function of the second kind of order ν, denoted by $Y_\nu(z)$. It has a different formula depending on whether ν is an integer. If $\nu \notin \mathbb{N}$ (that is, if n is odd), then [Ol, page 243]:

$$Y_\nu(z) := \frac{J_\nu(z)\cos(\nu\pi) - J_{-\nu}(z)}{\sin(\nu\pi)}.$$

On the other hand, if $\nu \in \mathbb{N}$ (that is, if n is even) [Ol, equation (5.07)]:

$$\begin{aligned} Y_\nu(z) := \frac{\left(\frac{1}{2}z\right)^{-\nu}}{\pi} \sum_{k=0}^{n-1} \frac{(n-s-1)!}{s!}\left(\frac{1}{4}z^2\right)^k + \frac{2}{\pi}\log\left(\frac{1}{2}z\right)J_\nu(z) \\ - \frac{\left(\frac{1}{2}z\right)^{\nu}}{\pi} \sum_{k=0}^{\infty} \{\psi(s+1)+\psi(n+s+1)\}\frac{(-1)^s}{s!(n+s)!}\left(\frac{1}{4}z^2\right)^k, \end{aligned} \tag{A.2}$$

where $\psi = \Gamma'/\Gamma$ is the logarithmic derivative of the Gamma function.

The Hankel function $H_\nu^{(1)}(z)$ of the first kind of order ν is given by [Ol, page 241]:

$$H_\nu^{(1)}(z) := J_\nu(z) + iY_\nu(z).$$

In turn, the modified Bessel function $K_\nu(z)$ of order ν, also known as Macdonald's function, is defined in terms of $H_\nu^{(1)}$ [Ol, page 250]:

$$K_\nu(z) := \frac{1}{2}\pi i e^{\frac{\nu\pi i}{2}} H_\nu^{(1)}(iz). \tag{A.3}$$

The asymptotics of $K_\nu(z)$ are [Te, page 202],

$$K_\nu(z) = \begin{cases} \frac{\Gamma(\nu)}{2}\left(\frac{z}{2}\right)^{-\nu} + O(z^{-\nu+2}) & \nu > 0, \\ -\log\left(\frac{z}{2}\right) + O(1) & \nu = 0, \end{cases} \tag{A.4}$$

as $|z| \to 0$, and

$$K_\nu(z) = \sqrt{\frac{\pi}{2z}}\, e^{-z}\left(1 + O(z^{-1})\right)$$

for $|z| \to \infty$. The small-z asymptotics follow simply by writing K_ν in terms of the series that define J_ν and Y_ν. See also [Ol, page 250] for a more explicit version of the large-z asymptotic for K_ν, as well as its derivation.

A.2 Computation of integral kernel in the upper half-plane

As mentioned above, in this section we work exclusively on the positive imaginary axis, that is, with λ of the form $\lambda = i\mu$, $\mu > 0$. The principle of analytic continuation then ensures that (A.1) holds throughout $\{\lambda \in \mathbb{C} : \operatorname{Im}\lambda > 0\}$.

We start with the following two facts from [Te, Section 7.5]. Their proofs rely on the spectral theorem for unbounded self-adjoint operators on a Hilbert space. The particular operator we are concerned with is $-\Delta : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, which is nonnegative and self-adjoint with respect to the domain $H^2(\mathbb{R}^n)$. The reader can find a thorough introduction to the spectral theorem in [Te, Chapter 3].

Proposition A.1. *For all* $\mu > 0$,

$$R_0(i\mu) = \int_0^\infty e^{-\mu^2 t} e^{-t(-\Delta)} dt : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n).$$

Proposition A.2. *For $t > 0$ and $\varphi \in C_0^\infty(\mathbb{R}^n)$,*

$$e^{-t(-\Delta)}\psi(x) = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{\frac{-|x-y|^2}{4t}} \psi(y)dy.$$

Combining the two previous propositions with Fubini's Theorem we get, for $\varphi \in C_0^\infty(\mathbb{R}^n)$,

$$\begin{aligned}
R_0(\lambda)\varphi &= \int_0^\infty e^{-\mu^2 t} e^{-t(-\Delta)} dt \varphi \\
&= \int_0^\infty e^{-\mu^2 t} e^{-t(-\Delta)} \varphi dt \\
&= \int_0^\infty e^{-\mu^2 t} \left[\frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{\frac{-|x-y|^2}{4t}} \varphi(y) dy \right] dt \\
&= \int_{\mathbb{R}^n} \left[\int_0^\infty \frac{1}{(4\pi t)^{n/2}} e^{-\mu^2 t + \frac{-|x-y|^2}{4t}} dt \right] \varphi(y) dy.
\end{aligned} \tag{A.5}$$

Our goal is to compute the final bracketed expression, which in the integral kernel. For $|x-y| > 0$, we make the change of variables $t = |x-y|e^s/(2\mu)$ to find

$$\begin{aligned}
\int_0^\infty \frac{1}{(4\pi t)^{n/2}} e^{-\mu^2 t + \frac{-|x-y|^2}{4t}} dt &= \frac{|x-y|}{2\mu} \left(\frac{\mu}{2\pi |x-y|} \right)^{\frac{n}{2}} \cdot \\
&\quad \int_{-\infty}^\infty e^{-\left(\frac{n}{2}-1\right)s} e^{-\frac{\mu|x-y|e^s}{2} - \frac{\mu|x-y|e^{-s}}{2}} ds \\
&= \frac{1}{4\pi} \left(\frac{\mu}{2\pi |x-y|} \right)^{\frac{n}{2}-1} \cdot \\
&\quad \int_{-\infty}^\infty e^{-\left(\frac{n}{2}-1\right)s} e^{-\mu|x-y|\cosh(s)} ds \\
&= \frac{1}{2\pi} \left(\frac{\mu}{2\pi |y|} \right)^{\frac{n}{2}-1} \cdot \\
&\quad \int_0^\infty \cosh\left(\left(\frac{n}{2} - 1 \right) s \right) e^{-\mu|x-y|\cosh(s)} ds.
\end{aligned} \tag{A.6}$$

To continue, we express the last integral in terms of special functions. In terms of the modified Bessel function $K_\nu(z)$, $\nu = n/2 - 1$ [Ol, page 250]:

$$K_\nu(\mu|x-y|) = \int_0^\infty \cosh\left(\left(\frac{n}{2} - 1 \right) s \right) e^{-\mu|x-y|\cosh(s)} ds. \tag{A.7}$$

We now combine (A.3), (A.5), (A.6), and (A.7) to achieve (A.1) for $\lambda = i\mu$, $\mu > 0$. Note in particular that Note in particular that the integral kernel of (A.1) is locally integrable on account of (A.4). Then, by analytic continuation, (A.1) holds in all of $\{\lambda \in \mathbb{C} : \operatorname{Im} \lambda > 0\}$.

A.3 Meromorphic continuation of the free resolvent

We now meromorphically continue beyond the upper half-plane. First, suppose that ν is not an integer (that is, suppose n is odd). Then both $z^\nu J_\nu(z)$ and $z^\nu J_{-\nu}(z)$ are entire in z. Because, in this case, $z^\nu H_\nu^{(1)}$ is a linear combination of the functions $z^\nu J_\nu$ and $z^\nu J_{-\nu}$, we conclude that $z^\nu H_\nu^{(1)}$ is entire when ν is not an integer. Therefore, (A.1) may be extended to all of $\mathbb{C}$ and we may write

$$(\chi R_0(\lambda)\chi)\varphi(x) = \int_{\mathbb{R}^n} E_1(\lambda, x, y)\varphi(y)dy,$$

where E_1 is analytic in λ, compactly supported in x and y, as well as locally integrable in $|x-y|$. This justifies (2.1) when n is odd.

If ν is an integer (that is, if n is even), then again $z^\nu H_\nu^{(1)}(z)$ is entire in z except for the logarithmic factor that now appears in the formula (A.2) for $Y_\nu(z)$. Hence, in this case, (A.1) continues to the logarithmic cover of $\mathbb{C}\setminus\{0\}$.

Extracting the λ-dependence from the z^ν factors multiplying the logarithmic term in (A.2), we find

$$(\chi R_0(\lambda)\chi)\varphi(x) = \int_{\mathbb{R}^n} \left(\tilde{E}_1(\lambda, x, y) + \lambda^{n-2}\log\lambda E_2(\lambda, x, y)\right)\varphi(y)dy,$$

where $\tilde{E}_1$ and E_2 have the same properties as E_1 above. This justifies (2.1) when n is even.

B. THE RESOLVENT OF THE LAPLACIAN WITH A ROUGH WAVESPEED

In this appendix, we first prove the elliptic estimate (2.4) for the perturbed resolvent $R(\lambda) = (-c^2\Delta - \lambda^2)^{-1}$, which we recall:

$$\|R(\lambda)\|_{H^2(\mathbb{R}^n)} \leq C_\lambda \|\varphi\|_{L^2(\mathbb{R}^n)}, \quad \varphi \in C_0^\infty(\mathbb{R}^n),\ \lambda^2 \notin \mathbb{R}_+.$$

Then, we show that the operators $L = -c^2(x)\Delta$ and

$$B = \begin{bmatrix} 0 & iI \\ -iL & 0 \end{bmatrix},$$

as defined in Section 5.1, are self-adjoint with respect to their given domains. Finally, we present some details concerning the meromorphic continuation of black box scatterers, and explain how the cutoff resolvent $\chi R(\lambda)\chi$ fits into this framework. The theory of black box scatterers was first developed in [SjZw91]. More recent presentations can be found in [Sj, Chapter 2] and [DyZw, Chapter 4].

As in Appendix A, we work in dimension $n \geq 2$. We abbreviate $L^2 = L^2(\mathbb{R}^n)$, $L^2_c = L^2(\mathbb{R}^n, c^{-2}dx)$, $H^2 = H^2(\mathbb{R}^n)$, and $\dot{H}^1 = \dot{H}^1(\mathbb{R}^n)$.

B.1 Elliptic estimates for the perturbed resolvent

Proposition B.1. *Let $\lambda^2 \notin \mathbb{R}_+$. The resolvent $R(\lambda)$ is bounded $L^2_c(\mathbb{R}^n) \to H^2(\mathbb{R}^n)$.*

Proof. Let $\varphi \in C_0^\infty(\mathbb{R}^n)$. Integration by parts yields

$$\begin{aligned}\operatorname{Re}\langle (L-\lambda^2)\varphi, \varphi\rangle_{L^2_c} &= \operatorname{Re}\left(\iint\left((-\Delta\varphi - c^{-2}\lambda^2\varphi)\overline{\varphi}\right)\right(\\ &= \iint\Big(|\nabla\varphi|^2 - c^{-2}\operatorname{Re}(\lambda^2)|\varphi|^2.\end{aligned}$$

Rearranging terms, estimating with Cauchy-Schwarz, and using $ab \leq a^2/2 + b^2/2$, we get

$$\|\nabla \varphi\|_{L^2} \leq C_\lambda \left(\|(L-\lambda^2)\varphi\|_{L^2_c} + \|\varphi\|_{L^2_c} \right) \tag{B.1}$$

where $C_\lambda > 0$ depends on λ.

By density, (B.1) holds for any $\varphi \in H^2(\mathbb{R}^n)$, yielding

$$\|\nabla R(\lambda)\varphi\|_{L^2} \leq C_\lambda \left(\|\varphi\|_{L^2_c} + \|R(\lambda)\varphi\|_{L^2_c} \right) \qquad \varphi \in C_0^\infty(\mathbb{R}^n).$$

We then calculate

$$\begin{aligned} \|R(\lambda)\varphi\|_{L^2} + \|\nabla R(\lambda)\varphi\|_{L^2} + \|\Delta R(\lambda)\varphi\|_{L^2} &\leq \|c\|_\infty \|\varphi\|_{L^2_c} \\ &\quad + C_\lambda \left(\|\varphi\|_{L^2_c} + \|R(\lambda)\varphi\|_{L^2_c} \right) \\ &\quad + \|c^{-2}\varphi\|_{L^2} + \|\lambda^2 c^{-2} R(\lambda)\|_{L^2} \\ &\leq C_{\lambda,c} \|\varphi\|_{L^2_c}, \end{aligned} \tag{B.2}$$

where $C_{\lambda,c} > 0$ depends λ and c. Note that we have used the identity

$$\Delta R(\lambda) = c^{-2} + c^{-2}\lambda^2 R(\lambda)$$

as well as the fact that $R(\lambda)$ is bounded $L^2_c(\mathbb{R}^n) \to L^2_c(\mathbb{R}^n)$. By properties of the Fourier transform,

$$\sum_{|\alpha|=2} \left\| \partial^\alpha u \right\|_{L^2} \leq 2^n \left\| \Delta R(\lambda)\varphi \right\|_{L^2}.$$

Therefore $\|R(\lambda)\varphi\|_{H^2}$ is bounded by a constant, depending on n, times the left side of (B.2). This shows $R(\lambda)$ is bounded $L^2_c(\mathbb{R}^n) \to H^2(\mathbb{R}^n)$ as desired.

□

B.2 Self-adjointness of the perturbed Laplacian

Proposition B.2. *The operator L is self-adjoint $L^2_c(\mathbb{R}^n) \to L^2_c(\mathbb{R}^n)$.*

Proof. We need to show that $D(L^*) = H^2(\mathbb{R}^n)$, and that

$$\langle Lu, v\rangle_{L^2_c} = \langle u, Lv\rangle_{L^2_c}, \qquad u, v \in H^2(\mathbb{R}^n). \tag{B.3}$$

First, we show that $H^2(\mathbb{R}^n) \subseteq D(L^*)$, and that (B.3) holds. Let $u, v \in H^2(\mathbb{R}^n)$. We use the fact that integration by parts holds for functions $u, v \in H^2(\mathbb{R}^n)$.

$$\begin{aligned}
\langle u, Lv \rangle_{L^2_c} &:= \iint u(\overline{-c^2 \Delta v}) c^{-2} \\
&= -\iint u \Delta \overline{v} \\
&= -\iint \Delta u \overline{v} \\
&= \iint -c^2 \Delta u \overline{v} c^{-2} \\
&= \langle Lu, v \rangle_{L^2_c}.
\end{aligned}$$

To see that $D(L^*) \subseteq H^2(\mathbb{R}^n)$, suppose $u \in D(L^*)$. Then, by definition of the domain of the adjoint, there exists a unique $\tilde{u} \in L^2_c(\mathbb{R}^n)$ so that for all $v \in H^2(\mathbb{R}^n)$

$$\langle u, Lv \rangle_{L^2_c} = \langle \tilde{u}, v \rangle_{L^2_c}. \tag{B.4}$$

Let $\mathcal{F}$ denote the Fourier transform. Using the Fourier transform characterization of $u \in H^2(\mathbb{R}^n)$, it suffices to show there exists $C > 0$ so that for all $\varphi \in C_0^\infty(\mathbb{R}^n)$

$$\langle (1 + |\cdot|^2) \mathcal{F} u, \varphi \rangle_{L^2} \ \leq C \|\varphi\|_{L^2}, \tag{B.5}$$

By properties of $\mathcal{F}$,

$$\begin{aligned}
\langle (1 + |\cdot|^2) \mathcal{F} u, \varphi \rangle_{L^2} &= \langle u, \mathcal{F}^{-1}(1 + |\cdot|^2) \varphi \rangle_{L^2} \\
&= \langle u, \mathcal{F}^{-1} \varphi \rangle_{L^2} + \langle u, L \mathcal{F}^{-1} \varphi \rangle_{L^2_c} \\
&= \langle u, \mathcal{F}^{-1} \varphi \rangle_{L^2} + \langle \tilde{u}, \mathcal{F}^{-1} \varphi \rangle_{L^2_c} \\
&\leq \|u\|_{L^2} \|\mathcal{F}^{-1} \varphi\|_{L^2} + \|c^2\|_\infty \|\tilde{u}\|_{L^2} \|\mathcal{F}^{-1} \varphi\|_{L^2}. \\
&\leq C \|\varphi\|_{L^2},
\end{aligned}$$

where $C > 0$ depends on u, $\tilde{u}$, c and $\|\mathcal{F}^{-1}\|_{L^2 \to L^2}$. This establishes (B.5) and completes the proof.

□

Proposition B.3. *The operator B is self adjoint $H \to H$.*

Proof. Suppose that $(u_1, v_1), (u_2, v_2) \in D(B)$. We compute

$$\begin{aligned}\langle (u_1, v_1), B(u_2, v_2)\rangle_H &= \langle \nabla u_1, i\nabla v_2\rangle_{L^2} + \langle v_1, -iLu_2\rangle_{L^2_c} \\ &= -\langle \Delta u_1, iv_2\rangle_{L^2} + \langle \nabla v_1, i\nabla u_2\rangle_{L^2} \\ &= \langle -iLu_1, v_2\rangle_{L^2_c} + \langle i\nabla v_1, \nabla u_2\rangle_{L^2} \\ &= \langle (iv_1, -iLu_1), (u_2, v_2)\rangle_H \\ &= \langle B(u_1, v_1), (u_2, v_2)\rangle_H.\end{aligned}$$

It remains to show that $D(B^*) \subseteq D(B)$. To this end, suppose $(u, v) \in D(B^*)$. Then there exists unique $(\tilde{u}, \tilde{v}) \in H$ such that for all $(u_1, v_1) \in D(B)$,

$$\langle (u, v), B(u_1, v_1)\rangle_H = \langle (\tilde{u}, \tilde{v}), (u_1, v_1)\rangle_H. \tag{B.6}$$

This implies that

$$\langle v, -iLu_1\rangle_{L^2_c} = \langle \tilde{u}, u_1\rangle_{\dot{H}^1}, \qquad u_1 \in \dot{H}^1,\ \Delta u_1 \in L^2, \tag{B.7}$$

$$\langle u, iv_1\rangle_{\dot{H}^1} = \langle \tilde{v}, v_1\rangle_{L^2_c}, \qquad v_1 \in H^1. \tag{B.8}$$

To show that $(u, v) \in D(B)$, it suffices to show that

$$(1 + |\xi|^2)^{\frac{1}{2}} \mathcal{F}v \in L^2(\mathbb{R}^n), \tag{B.9}$$

$$\sum_{j=1}^{n} \xi_j \mathcal{F}(\partial_{x_j} u) \in L^2(\mathbb{R}^n). \tag{B.10}$$

Observe that (B.10) ensures that the distributional Laplacian of u belongs to $L^2(\mathbb{R}^n)$, according to the calculation

$$\begin{aligned}\int u(x)\Delta\overline{\varphi(x)}dx &= -\int \sum_{j=1}^{n} \partial_{x_j} u(x)\partial_{x_j}\overline{\varphi(x)}dx \\ &= \int \sum_{j=1}^{n} i\xi \mathcal{F}(\partial_{x_j} u)(\xi)\overline{\mathcal{F}\varphi(\xi)}d\xi, \qquad \varphi \in C_0^\infty(\mathbb{R}^n).\end{aligned}$$

To show (B.9), we first demonstrate that the subspace $\{|\xi|\mathcal{F}\varphi : \varphi \in \mathcal{S}(\mathbb{R}^n)\}$ is dense in $L^2(\mathbb{R}^n)$. Suppose that $u \in L^2(\mathbb{R}^n)$ has the property that

$$\int |\xi| u(\xi)\overline{\mathcal{F}\varphi(\xi)}d\xi = 0, \qquad \varphi \in \mathcal{S}(\mathbb{R}^n).$$

This implies that $|\cdot|u \in L^1_{loc}(\mathbb{R}^n)$ has the property that

$$\iint |\xi|u(\xi)\eta(\xi)d\xi = 0, \qquad \eta \in C_0^\infty(\mathbb{R}^n).$$

So, almost everywhere, we must have $|\xi|u(\xi) = 0$, which in turn requires that $u = 0$ in $L^2(\mathbb{R}^n)$.

Now, for all $\varphi \in \mathcal{S}(\mathbb{R}^n)$, (B.7) says that

$$\begin{aligned} |\langle|\cdot|\mathcal{F}v, |\cdot|\mathcal{F}\varphi\rangle_{L^2}| &= \iint v(x)\overline{-\Delta\varphi(x)}dx \\ &= \langle v, -iL(i\varphi)\rangle_{L^2_c} \\ &= |\langle \tilde{u}, i\varphi\rangle_{\dot{H}^1}| \\ &\le \|\tilde{u}\|_{\dot{H}^1}\|\varphi\|_{\dot{H}^1} \\ &= \|\tilde{u}\|_{\dot{H}^1}\||\cdot|\mathcal{F}\varphi\|_{L^2}. \end{aligned}$$

This shows that $|\cdot|\mathcal{F}v \in L^2(\mathbb{R}^n)$.

Next, we want to show (B.10). For $\varphi \in \mathcal{S}(\mathbb{R}^n)$, we use (B.8) to calculate

$$\begin{aligned} \left\langle \sum_{j=1}^n \xi_j\mathcal{F}(\partial_{x_j}u), i\mathcal{F}\varphi \right\rangle_{L^2} &= |\langle \nabla u, i\nabla\varphi\rangle_{L^2}| \\ &= |\langle u, i\varphi\rangle_{\dot{H}^1}| \\ &= \langle \tilde{v}, \varphi\rangle_{L^2_c} \\ &\le \|c^2\|_\infty\|\tilde{v}\|_{L^2}\|\varphi\|_{L^2}. \end{aligned}$$

This establishes (B.10) and completes the proof that B is self-adjoint.

□

B.3 Meromorphic continuation of the perturbed resolvent

B.3.1 General assumptions for black box Hamiltonians

Suppose $\mathcal{H}$ is a complex Hilbert space with the orthogonal decomposition

$$\mathcal{H} = \mathcal{H}_{R_0} \oplus L^2(\mathbb{R}^n \setminus B(0, R_0)), \qquad R_0 > 0,$$

where $\mathcal{H}_{R_0}$ is an arbitrary Hilbert space. Let $\mathbf{1}_{B(0,R_0)}$ and $\mathbf{1}_{\mathbb{R}^n \setminus B(0,R_0)}$ denote the orthogonal projections onto the first and second summands, respectively. To condense notation, we set

$$u|_{B(0,R_0)} := \mathbf{1}_{B(0,R_0)} u,$$
$$u|_{\mathbb{R}^n \setminus B(0,R_0)} := \mathbf{1}_{\mathbb{R}^n \setminus B(0,R_0)} u,$$

for each $u \in \mathcal{H}$. To go along with $\mathcal{H}$, we define the subspaces

$$\mathcal{H}_{\text{comp}} := \{u \in \mathcal{H} : u|_{\mathbb{R}^n \setminus B(0,R_0)} \in L^2_{\text{comp}}(\mathbb{R}^n \setminus B(0,R_0)\},$$
$$\mathcal{H}_{\text{loc}} := \mathcal{H}_{R_0} \oplus L^2_{\text{loc}}(\mathbb{R}^n \setminus B(0,R_0)).$$

If $\chi \in L^\infty(\mathbb{R}^n)$ and $\chi \equiv \beta \in \mathbb{C}$ on $B(0,R_0)$, then we can define χ as a bounded multiplication operator $\mathcal{H} \to \mathcal{H}$

$$\chi u := \beta u|_{B(0,R_0)} + \chi|_{\mathbb{R}^n \setminus B(0,R_0)} u|_{\mathbb{R}^n \setminus B(0,R_0)}.$$

We also assume we have an unbounded self-adjoint operator $P : \mathcal{H} \to \mathcal{H}$, with the domain $\mathcal{D} \subseteq \mathcal{H}$. The elements locally in $\mathcal{D}$ are defined by

$$\mathcal{D}_{\text{loc}} := \{u \in \mathcal{H}_{\text{loc}} : \chi \in C_0^\infty(\mathbb{R}^n),\ \chi|_{B(0,R_0)} \equiv 1 \implies \chi u \in \mathcal{D}\},$$

and the compactly supported elements of $\mathcal{D}$ are

$$\mathcal{D}_{\text{comp}} := \mathcal{D} \cap \mathcal{H}_{\text{comp}}.$$

In addition, we place several conditions on P and its domain:

1. $\mathbf{1}_{\mathbb{R}^n \setminus B(0,R_0)} \mathcal{D} \subseteq H^2(\mathbb{R}^n \setminus B(0,R_0))$,
2. $\mathbf{1}_{\mathbb{R}^n \setminus B(0,R_0)}(Pu) = -\Delta u|_{\mathbb{R}^n \setminus B(0,R_0)}, \quad u \in \mathcal{D}$,
3. $v \in H^2(\mathbb{R}^n)$, $v|_{B(0,R_0+\varepsilon)} \equiv 0$ for some $\varepsilon > 0$ implies $v|_{B(0,R_0+\varepsilon)} \in \mathcal{D}$,
4. $\mathbf{1}_{B(0,R_0)}(P - i)^{-1}$ is compact.

Each P satisfying these four properties is referred to as a black box Hamiltonian.

We pause briefly to notice how the operator $-c^2\Delta$, defined on $\mathcal{H} = L^2(\mathbb{R}^n, c^{-2}dx)$ with domain $\mathcal{D} = H^2(\mathbb{R}^n, dx)$, is a black box Hamiltonian. First, $L^2(\mathbb{R}^n, c^{-2}dx)$, may be decomposed as

$$L^2(\mathbb{R}^n, c^{-2}dx) = L^2(B(0, R_0), c^{-2}dx) \oplus L^2(\mathbb{R}^n \setminus B(0, R_0), dx)$$

for any $R_0 > 0$ such that $B(0, R_0) \supseteq \operatorname{supp}(1 - c)$. Properties 1, 2, and 3 follow immediately by the definition of $-c^2\Delta$ and the fact that the domain is $H^2(\mathbb{R}^n)$. The compactness of $\mathbf{1}_{B(0,R_0)}(P - i)^{-1}$ follows from the Rellich-Kondrachov Compactness Theorem, see for instance [Ev, Section 5.7].

B.3.2 Meromorphic continuation for black box Hamiltonians

The cutoff resolvent for each blackbox Hamiltonian has a meromorphic continuation to $\mathbb{C}$ when n is odd and to the logarithmic cover of $\mathbb{C} \setminus \{0\}$ when n is even. This continuation follows from the meromorphic continuation of the free cutoff resolvent, as developed in the previous Appendix A, along with the theory of analytic Fredholm operators. We omit the details, which can be found in [DyZw, Theorem 4.4] and [Sj, Theorem 2.2], and just state the result.

Theorem 6. *Suppose P is a black box Hamiltonian in the sense described in Section B.3.1.Then*

$$R(\lambda) := (P - \lambda^2)^{-1} : \mathcal{H} \to \mathcal{D}$$

is meromorphic for $\operatorname{Im} \lambda > 0$.

Moreover, when n is odd, the resolvent extends to a meromorphic family

$$R(\lambda) : \mathcal{H}_{comp} \to \mathcal{D}_{comp}, \qquad \lambda \in \mathbb{C}.$$

When n is even, the continuation still holds but with $\mathbb{C}$ by the logarithmic cover of $\mathbb{C} \setminus \{0\}$.

Having already checked that our operator $-c^2\Delta$ fits into the blackbox framework, Theorem (6) justifies the existence of the continuation of $\chi(-c^2\Delta - \lambda^2)^{-1}\chi$, which we make repeated use of in Chapter 5.

B.3.3 Absence of resonances on the real axis

Let P be a black box Hamiltonian. We say that $E > 0$ is an embedded eigenvalue of P if there exists $v \in \mathcal{D}$ such that $(P - E)v = 0$.

An important property of the meromophic continuation of $R(\lambda)$ is that any poles $\lambda \in \mathbb{R} \setminus \{0\}$ must also be embedded eigenvalues. This follows from [DyZw, Theorems 4.17 and 4.18].

However, a Carleman estimate [DyZw, Lemma 3.31], which can be successfully applied to our operator $-c^2\Delta$, rules out the possibility of embedded eigenvalues for $-c^2\Delta$. We omit further details about this Carleman estimate, but remark that related but more complicated Carleman estimates can be found in Chapters 3 and 4.

C. ELLIPTIC ESTIMATE FOR WEIGHTED RESOLVENTS

Suppose that $V \in L^\infty(\mathbb{R}^n)$. Let

$$P(h) := -h^2\Delta + V - E, \qquad E,\ h > 0,$$

be a semiclassical Schrödinger operator. As noted before, it is self-adjoint $L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ with respect to the domain $H^2(\mathbb{R}^n)$.

Our goal is to show the following semiclassical elliptic estimate. See [Zw, Theorem 7.1] for a related simpler version of such an estimate.

As we have done previously, in the estimates we abbreviate $L^2 = L^2(\mathbb{R}^n)$, $H^1 = H^1(\mathbb{R}^n)$, and $H^2 = H^2(\mathbb{R}^n)$

Theorem 7. *Suppose that there exists* $s > 1/2$, $0 < h_0 \le 1$ *and some function* $A(h) > 1$ *depending on* h *such that*

$$\langle x\rangle^{-s}(P(h) - E - i\varepsilon)^{-1}\langle x\rangle^{-s} \ {}_{L^2\to L^2} \le A(h), \tag{C.1}$$

for all $E \in [E_{\min}, E_{\max}] \subseteq (0,\infty)$, $\varepsilon > 0$ *and* $h \in (0, h_0]$. *Then there exists a constant* $C_{V,E_{\min},E_{\max},n,s} > 1$ *depending on* $E_{\min}$, $E_{\max}$, $\|V\|_\infty$, n *and* s *such that*

$$\langle x\rangle^{-s}(P(h) - E - i\varepsilon)^{-1}\langle x\rangle^{-s} \ {}_{L^2\to H^2} \le C_{V,E_{\min},E_{\max},n,s}h^{-2}A(h), \tag{C.2}$$

for all $0 < \varepsilon < 1$ *and* $h \in (0, h_0]$.

Theorem 6 is a standard result, but we prove it here for the reader's convenience and for the sake of completeness. It shows that one only needs to prove that (1.4) and (1.6) hold $L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$.

First, we prove some preliminary lemmas, which are all straightforward widely-known, and then use them to prove Theorem 7.

Lemma C.1. *For any $u \in L^2(\mathbb{R}^n)$ and $\varepsilon > 0$, it holds that*

$$-\Delta(P(h) - E - i\varepsilon)^{-1}u = \left(h^{-2} - h^{-2}\,(V - E - i\varepsilon)\,(P(h) - E - i\varepsilon)^{-1}\right) u.$$

Proof. We write

$$\begin{aligned} -\Delta &= h^{-2}\left((-h^2\Delta) + V - E - i\varepsilon\right) - h^{-2}\,(V - E - i\varepsilon) \\ &= h^{-2}\,(P(h) - E - i\varepsilon) - h^{-2}\,(V - E - i\varepsilon)\,. \end{aligned}$$

Therefore

$$-\Delta(P(h) - E - i\varepsilon)^{-1} = h^{-2} - h^{-2}\,(V - E - i\varepsilon)\,(P(h) - E - i\varepsilon)^{-1}.$$

This completes the proof of the lemma.

□

Corollary C.1. *For any $u \in L^2(\mathbb{R}^n)$ and $0 < \varepsilon < 1$, it holds that*

$$\langle x\rangle^{-s}\Delta(P(h) - E - i\varepsilon)^{-1}\langle x\rangle^{-s}u \;{}_{L^2} \le C_{V,E}h^{-2}A(h)\|u\|_{L^2}$$

for some constant $C_{V,E} > 1$ that depends on $\|V\|_\infty$ and $E_{\max}$.

Proof. By Lemma C.1,

$$\begin{aligned} &\langle x\rangle^{-s}\Delta(P(h) - E - i\varepsilon)^{-1}\langle x\rangle^{-s}u = \\ &\qquad h^{-2}\left((V - E - i\varepsilon)\langle x\rangle^{-s}(P(h) - E - i\varepsilon)^{-1}\langle x\rangle^{-s} - \langle x\rangle^{-2s}\right) u. \end{aligned}$$

Using $0 < \varepsilon < 1$, we then get

$$\begin{aligned} &\|\langle x\rangle^{-s}\Delta(P(h) - E - i\varepsilon)^{-1}\langle x\rangle^{-s}u\|_{L^2} \\ &\qquad \le h^{-2}\left((\|V\|_\infty + E_{\max} + 1)A(h) + 1\right)\|u\|_{L^2}, \end{aligned}$$

completing the proof.

□

Lemma C.2. *The operator $[\Delta, \langle x\rangle^{-s}]\langle x\rangle^s$ is bounded $H^1(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ with*

$$[\Delta, \langle x\rangle^{-s}]\langle x\rangle^s \;{}_{H^1 \to L^2} \le C_{n,s}, \tag{C.3}$$

for some constant $C_{n,s} > 1$ depending on n and s.

Proof. A quick computation shows

$$\Delta\langle x\rangle^s u = \left(\Delta\langle x\rangle^s\right)u + 2\left(\nabla\langle x\rangle^s\right)\cdot\nabla u + \langle x\rangle^s\Delta u,$$

for any $u \in C_0^\infty(\mathbb{R}^n)$. So, computing the commutator $[\Delta, \langle x\rangle^{-s}] = \Delta\langle x\rangle^{-s} - \langle x\rangle^{-s}\Delta$, we get

$$\begin{aligned}[\Delta, \langle x\rangle^{-s}]\langle x\rangle^s u &= \Delta\langle x\rangle^{-s}\langle x\rangle^s u - \langle x\rangle^{-s}\Delta\langle x\rangle^s u \\ &= \left(\langle x\rangle^{-s}\Delta\langle x\rangle^s\right)u + 2\langle x\rangle^{-s}\nabla\langle x\rangle^s\cdot\nabla u.\end{aligned}$$

We explicity compute the functions $\langle x\rangle^{-s}\Delta\langle x\rangle^s$ and $\langle x\rangle^{-s}\partial_j\langle x\rangle^s$, $j = 1, \dots n$:

$$\begin{aligned}\langle x\rangle^{-s}\Delta\langle x\rangle^s &= ns\langle x\rangle^{-2} + s\left(s-2\right)|x|^2\langle x\rangle^{-4} \\ \langle x\rangle^{-s}\partial_j\langle x\rangle^s &= sx_j\langle x\rangle^{-2}\end{aligned}$$

Both of these functions are bounded with

$$\begin{aligned}|\langle x\rangle^{-s}\Delta\langle x\rangle^s| &\le ns + s|s-2|, \\ |\langle x\rangle^{-s}\partial_j\langle x\rangle^s| &\le s.\end{aligned}$$

□

Lemma C.3. *There exists $C_n > 0$ depending on n such that for all $u \in H^2(\mathbb{R}^n)$*

$$\|u\|_{H^2} \le C\left(\|u\|_{L^2} + \|\Delta u\|_{L^2}\right) \tag{C.4}$$

Proof. By density of $C_0^\infty(\mathbb{R}^n)$ in $H^2(\mathbb{R}^n)$, it suffices to prove (C.4) for $u \in C_0^\infty(\mathbb{R}^n)$. For a single partial derivative $\partial_j u$, $1 \le j \le n$, we integrate by parts just once to see

$$\int |\partial_j u|^2 dx = -\int \left(\partial_j^2 u\right)\bar{u}dx \le \frac{1}{2}\left(\gamma\int |u|^2 dx + \gamma^{-1}\int |\partial_j^2 u|^2 dx\right), \qquad \gamma > 0. \tag{C.5}$$

For the mixed partial derivatives $\partial_j \partial_k u$, $1 \le j \ne k \le n$, we integrate by parts several times to get

$$\begin{aligned}\int |\partial_j \partial_k u|^2 dx &= \int (\partial_k \partial_j \partial_j \partial_k u)\, \overline{u} dx \\ &= \int (\partial_j^2 \partial_k^2 u)\, \overline{u} dx \\ &= \int (\partial_k^2 u)\, \partial_j^2 \overline{u} dx \\ &\le \frac{1}{2} \left(\int |\partial_k^2 u|^2 dx + \int |\partial_j^2 u|^2 dx \right) \end{aligned} \tag{C.6}$$

We now fix $\gamma = 1$ and combine (C.5) and (C.6) to find

$$\begin{aligned}\|u\|_{H^2}^2 &= \|u\|_{L^2}^2 + \sum_{j=1}^{n} \|\partial_j u\|_{L^2}^2 \\ &\quad + \sum_{1 \le j \ne k \le n}^{n} \|\partial_j \partial_k u\|_{L^2}^2 + \|\Delta u\|_{L^2}^2 \\ &\le \frac{n+2}{2} \|u\|_{L^2}^2 + C_n \|\Delta u\|_{L^2}^2, \end{aligned} \tag{C.7}$$

where the constant $C_n > 0$ depends on n but is independent of u.

□

If we do not fix γ in (C.5), we obtain the following.

Corollary C.2. *It holds that*

$$\|u\|_{H^1} \le \left(1 + \frac{\gamma}{\sqrt{2}}\right) \|u\|_{L^2} + \frac{1}{\sqrt{2}\gamma} \|\Delta u\|_{L^2}, \qquad \gamma > 0.$$

We are now ready to prove Theorem 7.

Proof of Theorem 7. By Lemma C.3, it suffices to show

$$\left\| \Delta \langle x \rangle^{-s} (P(h) - E - i\varepsilon)^{-1} \langle x \rangle^{-s} u \right\|_{L^2} \le C_{V,E,n,s} h^{-2} A(h) \|u\|_{L^2}, \tag{C.8}$$

for all $0 < \varepsilon < 1$ and $h \in (0, h_0]$, and $u \in L^2(\mathbb{R}^n)$. We have

$$\begin{aligned}\Delta \langle x \rangle^{-s} (P(h) - E - i\varepsilon)^{-1} \langle x \rangle^{-s} &= [\Delta, \langle x \rangle^{-s}] \langle x \rangle^{s} \langle x \rangle^{-s} (P(h) - E - i\varepsilon)^{-1} \langle x \rangle^{-s} \\ &\quad + \langle x \rangle^{-s} \Delta (P(h) - E - i\varepsilon)^{-1} \langle x \rangle^{-s}. \end{aligned}$$

The following estimates are made possible by Lemma C.2 and Corollaries C.1 and C.2. For $0 < \varepsilon < 1$, $h \in (0, h_0]$, and $u \in L^2$,

$$
\begin{aligned}
&\|\Delta\langle x\rangle^{-s}(P(h) - E - i\varepsilon)^{-1}\langle x\rangle^{-s}u\|_{L^2} \\
&\qquad \le \|[\Delta, \langle x\rangle^{-s}]\langle x\rangle^{s}\langle x\rangle^{-s}(P(h) - E - i\varepsilon)^{-1}\langle x\rangle^{-s}u\|_{L^2} \\
&\qquad + \|\langle x\rangle^{-s}\Delta(P(h) - E - i\varepsilon)^{-1}\langle x\rangle^{-s}u\|_{L^2} \\
&\qquad \le C_{n,s}\|\langle x\rangle^{-s}(P(h) - E - i\varepsilon)^{-1}\langle x\rangle^{-s}u\|_{H^1} \\
&\qquad + C_{V,E_{\max}}h^{-2}A(h)\|u\|_{L^2} \\
&\qquad \le C_{n,s}\frac{\sqrt{2} + \gamma}{\sqrt{2}}A(h)\|u\|_{L^2} \\
&\qquad + C_{n,s}\frac{1}{\gamma\sqrt{2}}\|\Delta\langle x\rangle^{-s}(P(h) - E - i\varepsilon)^{-1}\langle x\rangle^{-s}u\|_{L^2} \\
&\qquad + C_{V,E_{\max}}h^{-2}A(h)\|u\|_{L^2}
\end{aligned}
$$

If we set $\gamma = \sqrt{2}C_{n,s}$, then we can absorb the term in line seven on the right side into the left side. Then, after multiplying through by 2, the theorem is proved.

□

D. POINCARÉ INEQUALITIES

In this appendix, we review the proofs of the two Poincaré inequalities, (2.5):

$$\|\varphi\|_{L^2(\mathbb{R}^n)} \leq C_R \|\nabla\varphi\|_{L^2(\mathbb{R}^n)}, \qquad \varphi \in C_0^\infty(B(0,R)),\ n \geq 2,$$

and (5.4):

$$\|\chi\varphi\|_{L^2(\mathbb{R}^n)} \leq C_\chi \|\nabla\varphi\|_{L^2(\mathbb{R}^n)}, \qquad \chi \in C_0^\infty(\mathbb{R}^n),\ \varphi \in C_0^\infty(B(0,R)),\ n \geq 3,$$

that appear in Chapters 2 and 5, respectively. We also present a counterexample to (5.4) in dimension two. It is a compactly supported version of the function

$$f(x) = \begin{cases} 0 & |x| = 0, \\ \log(\log(1 + |x|^{-1})) & |x| > 0. \end{cases}$$

D.1 Poincaré inequality with support-dependent constant

We begin with the proof of (2.5).

Proposition D.1. *Let $\varphi \in C_0^\infty(\mathbb{R}^n)$, $n \geq 2$. Suppose that the support of φ is contained in the cube $(-c,c)^n$. Then it's true that*

$$\|\varphi\|_{L^2(\mathbb{R}^n)} \leq \sqrt{2}c\|\nabla\varphi\|_{L^2(\mathbb{R}^n)}.$$

Proof. Write $x = (x', x_n)$ (so $x' = (x_1, \dots, x_{n-1})$). We have:

$$\varphi(x) = \varphi(x', x_n) = \int_{-c}^{x_n} \frac{\partial}{\partial t}\varphi(x', t)dt, \qquad x_n \in (-c, c).$$

We can then apply Hölder's inequality to get, for $x_n \in (-c, c)$:

$$\begin{aligned} |\varphi(x)|^2 &\leq \left(\int_{-c}^{x_n} \frac{\partial}{\partial t}\varphi(x', t)\ dt\right)^2 \\ &\leq \int_{-c}^{x_n} dt \cdot \int_{-c}^{x_n} |\nabla\varphi(x', t)|^2 dt \\ &= (x_n + c)\int_{-c}^{x_n} |\nabla\varphi(x', t)|^2 dt. \end{aligned}$$

And so

$$
\begin{aligned}
\|\varphi\|_{L^2}^2 &= \int_{[-c,c]^{n-1}} \int_{-c}^{c} |\varphi(x)|^2 dx_n dx' \\
&\leq \int_{[-c,c]^{n-1}} \int_{-c}^{c} \left((x_n + c) \int_{-c}^{x_n} |\nabla\varphi(x', t)|^2 dt \right) dx_n dx' \\
&\leq \int_{-c}^{c} (x_n + c) \int_{[-c,c]^{n-1}} \int_{-c}^{c} |\nabla\varphi(x', t)|^2 dt dx' dx_n \\
&= 2c^2 \|\nabla\varphi\|_{L^2}.
\end{aligned}
$$

□

D.2 Global Poincaré inequality in dimension three and higher

To prove (5.4), we first recall the Gagliardo-Nirenberg-Sobolev (GNS) inequality, as proved in [Ev, Section 5.6].

Theorem 8 (Gagliardo-Nirenberg-Sobolev inequality)**.** *Assume* $1 \leq p < n$. *There exists a constant* $C > 0$, *depending only on* p *and* n, *such that*

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \leq C\|\nabla u\|_{L^p(\mathbb{R}^n)} \tag{D.1}$$

for all $u \in C_0^1(\mathbb{R}^n)$. *Here* p^* *is the Sobolev conjugate of* p,

$$p^* := \frac{np}{n-p}.$$

We now apply the GNS inequality along with Hölder's inequality to prove (5.4).

Proposition D.2. *Let* $\chi \in C_0^\infty(\mathbb{R}^n)$, $n \geq 3$. *There exists a constant* $C_\chi > 0$ *depending on* χ *and the dimension* n *such that*

$$\|\chi u\|_{L^2} \leq C_\chi \|\nabla u\|_{L^2},$$

for all $u \in C_0^1(\mathbb{R}^n)$.

Proof. We begin by applying the GNS inequality for

$$p^* = 2, \qquad 1 \leq p = 1 + \frac{n-2}{n+2} = \frac{2n}{n+2} < n.$$

We then get a constant C depending on n such that

$$\|\chi u\|_{L^2} \leq C\|\nabla(\chi u)\|_{L^{2n/n+2}}.$$

We then majorize the right hand side using the triangle inequality and two applications of the generalized Hölder inequality.

$$\begin{aligned}\|\nabla(\chi u)\|_{L^{2n/n+2}} &\leq C\|\chi\nabla u\|_{L^{2n/n+2}} + C\|(\nabla\chi)u\|_{L^{2n/n+2}} \\ &\leq C\|\chi\|_{L^n}\|\nabla u\|_{L^2} + C\|\nabla\chi\|_{L^{n/2}}\|u\|_{L^{2n/n-2}}.\end{aligned}$$

To the term $\|u\|_{L^{2n/n-2}}$, we apply the GNS inequality an additional time, with

$$p^* = \frac{2n}{n-2}, \qquad p = 2,$$

to arrive at

$$\|u\|_{L^{2n/n-2}} \leq C_\chi\|\nabla u\|_{L^2}.$$

□

D.3 Counterexample to global Poincaré inequality in dimension two

To finish this appendix, we use the function f as defined above to construct a counterexample to (5.4) in dimension two. First, we multiply f by a cutoff $\tilde{\chi}$ and show $\tilde{\chi}f \in H^1(\mathbb{R}^2)$. Then, we show that no estimate of the form (5.4) can hold for $\tilde{\chi}f$.

Proposition D.3. *Let $\tilde{\chi} \in C_0^\infty(\mathbb{R}^2)$ such that $\tilde{\chi} \equiv 1$ near $0 \in \mathbb{R}^2$. Let*

$$f(x) = \begin{cases} 0 & |x| = 0, \\ \log(\log(1 + |x|^{-1})) & |x| > 0. \end{cases}$$

Then $\tilde{\chi}f \in H^1(\mathbb{R}^2)$.

Proof. Step 1: We have $\tilde{\chi}f \in L^2(\mathbb{R}^2)$ because, for $\varepsilon > 0$ sufficiently small, $\tilde{\chi}f = f$ on $B(0, \varepsilon)$ and:

$$\int_0^\varepsilon (\log(\log(1 + r^{-1}))^2 r\,dr < \infty.$$

The finiteness of the integrals follows by combining L'Hospital's rule with the estimate $\log(1+x) \le x$, all $x \ge 0$:

$$\begin{aligned}\lim_{r\to 0^+} r^{1/2}\log(\log(1+r^{-1})) &\le \lim_{r\to 0^+} -r^{1/2}\log(r)\\ &= 0.\end{aligned}$$

Step 2: The weak derivative $\partial_j(\tilde{\chi} f)$ belongs to $L^2(\mathbb{R}^n)$ and is given by

$$\partial_j(\tilde{\chi} f) = \begin{cases} 0 & |x| = 0,\\ (\partial_j\tilde{\chi})f - \tilde{\chi}\left(\frac{x_j}{\log(1+|x|^{-1})(1+|x|)|x|}\right) & |x| > 0.\end{cases}$$

First, L^2-integrability of the above function follows because for $|x| > 0$,

$$|\partial_j f|^2 = \left|\frac{x_j}{\log(1+|x|^{-1})(1+|x|)|x|}\right|^2 \le \frac{1}{\log^2(1+|x|^{-1})|x|^2}, \tag{D.2}$$

and, for $0 < \varepsilon < 1$,

$$\begin{aligned}\int_{B(0,\varepsilon)} \frac{1}{\log^2(1+|x|^{-1})|x|^2}dx &= 2\pi\int_0^{\varepsilon} \frac{1}{\log^2(1+r^{-1})r}dr\\ &= 2\pi\int_{\varepsilon^{-1}}^{\infty} \frac{1}{\log^2(1+r)r}dr\\ &< \infty.\end{aligned}$$

Second, for $\varphi \in C_0^\infty(\mathbb{R}^n)$, Lebesgue's dominated convergence theorem and the divergence theorem say that

$$\begin{aligned}\int_{\mathbb{R}^2}\tilde{\chi} f\partial_j\varphi dx &= \lim_{\varepsilon\to 0}\int_{\mathbb{R}^2\setminus B(0,\varepsilon)}\tilde{\chi} f\partial_j\varphi dx\\ &= \lim_{\varepsilon\to 0} -\int_{\mathbb{R}^n\setminus B(0,\varepsilon)}(\partial_j\tilde{\chi})f\varphi + \tilde{\chi}(\partial_j f)\varphi dx + \int_{\partial B(0,\varepsilon)}\tilde{\chi} f\varphi\frac{x_j}{|x|}dS(x)\\ &= -\int_{\mathbb{R}^n}(\partial_j\tilde{\chi})f\varphi + \tilde{\chi}\left(\left(\frac{x_j}{\log(1+|x|^{-1})(1+|x|)|x|}\right)\right)\varphi dx\end{aligned}$$

where dS denotes surface measure on the circle of radius ε in $\mathbb{R}^2$, and we have used that

$$\left|\int_{\partial B(0,\varepsilon)}\tilde{\chi} f\varphi\frac{x_j}{|x|}dS(x)\right| \le C_{\tilde{\chi},\varphi}\varepsilon^2\log(\log(1+\varepsilon^{-1}) \le C_{\tilde{\chi},\varphi}\varepsilon \to 0$$

as $\varepsilon \to 0^+$.

□

Proposition D.4. *Let* $\chi \in C_0^\infty(\mathbb{R}^2)$*,* $\chi \not\equiv 0$*. There does not exist a constant* $C_\chi > 0$ *such that, for all* $\varphi \in C_0^\infty(\mathbb{R}^2)$*,*

$$\|\chi\varphi\|_{L^2(\mathbb{R}^n)} \leq C_\chi \|\nabla\varphi\|_{L^2(\mathbb{R}^2)}.$$

Proof. Let $\tilde{\chi}$ and f be as in Proposition D.3. Take $\{\varphi_m\} \subseteq C_0^\infty(\mathbb{R}^2)$ to be a sequence such that $\varphi_m \to \tilde{\chi}f$ in $H^1(\mathbb{R}^2)$. Let $\{\varphi_{m_k}\}_{k=1}^\infty$ be a subsequence that converges to $\tilde{\chi}f$ pointwise almost everywhere with respect to Lebesgue measure.

If the proposed estimate holds for a certain C_χ, then for any $\varphi \in C_0^\infty(\mathbb{R}^2)$ and $x_0 \in \mathbb{R}^2$,

$$\begin{aligned} C_\chi^2 \|\nabla\varphi\|_{L^2(\mathbb{R}^2)}^2 &= C_\chi^2 \|\nabla(\varphi(x_0 + \varepsilon x))\|_{L^2(\mathbb{R}^2)}^2 \\ &\geq \iint_{\mathbb{R}^2} |\chi(x)\varphi(x_0 + \varepsilon x)|^2 dx \\ &\to |\varphi(x_0)|^2 \iint_{\mathbb{R}^2} |\chi(x)|^2 dx \end{aligned} \tag{D.3}$$

as $\varepsilon \to 0^+$. Setting $\varphi = \varphi_{m_k}$ and letting $k \to \infty$ in (D.3) says that $|\tilde{\chi}f| \leq C_\chi \|\nabla(\tilde{\chi}f\|_{L^2(\mathbb{R}^2)}$ almost everywhere, which is a contradiction. Note that the first line of (D.3) is the crucial step where we use the dimension two assumption. It ensures that scaling the integration variable by ε does not change the L^2-norm of the gradient.

□