12-2017

Consensus algorithms for networks of systems with second- and higher-order dynamics

Michael Fruhnert
Purdue University

CONSENSUS ALGORITHMS FOR NETWORKS OF SYSTEMS WITH SECOND- AND HIGHER-ORDER DYNAMICS

A Dissertation

Submitted to the Faculty

of

Purdue University

by

Michael Fruhnert

In Partial Fulfillment of the

Requirements for the Degree

of

Doctor of Philosophy

December 2017

Purdue University

West Lafayette, Indiana

THE PURDUE UNIVERSITY GRADUATE SCHOOL
STATEMENT OF DISSERTATION APPROVAL

Dr. Martin J. Corless, Chair
School of Aeronautics and Astronautics
Dr. Inseok Hwang
School of Aeronautics and Astronautics
Dr. Dengfeng Sun
School of Aeronautics and Astronautics
Dr. Jianghai Hu
School of Electrical and Computer Engineering

Approved by:

Dr. Weinong W. Chen
Head of the School Graduate Program

TABLE OF CONTENTS

Page

LIST OF TABLES

LIST OF FIGURES

Figure Page

ABSTRACT

Fruhnert, Michael Ph.D., Purdue University, December 2017. Consensus Algorithms for Networks of Systems with Second- and Higher-Order Dynamics. Major Professor: Martin J. Corless.

This thesis considers homogeneous networks of linear systems. We consider linear feedback controllers and require that the directed graph associated with the network contains a spanning tree and systems are stabilizable. We show that, in continuous-time, consensus with a guaranteed rate of convergence can always be achieved using linear state feedback.

For networks of continuous-time second-order systems, we provide a new and simple derivation of the conditions for a second-order polynomials with complex coefficients to be Hurwitz. We apply this result to obtain necessary and sufficient conditions to achieve consensus with networks whose graph Laplacian matrix may have complex eigenvalues. Based on the conditions found, methods to compute feedback gains are proposed. We show that gains can be chosen such that consensus is achieved robustly over a variety of communication structures and system dynamics. We also consider the use of static output feedback.

For networks of discrete-time second-order systems, we provide a new and simple derivation of the conditions for a second-order polynomials with complex coefficients to be Schur. We apply this result to obtain necessary and sufficient conditions to achieve consensus with networks whose graph Laplacian matrix may have complex eigenvalues. We show that consensus can always be achieved for marginally stable systems and discretized systems. Simple conditions for consensus achieving controllers are obtained when the Laplacian eigenvalues are all real.

For networks of continuous-time time-variant higher-order systems, we show that uniform consensus can always be achieved if systems are quadratically stabilizable. In this case, we provide a simple condition to obtain a linear feedback control.

For networks of discrete-time higher-order systems, we show that constant gains can be chosen such that consensus is achieved for a variety of network topologies. First, we develop simple results for networks of time-invariant systems and networks of time-variant systems that are given in controllable canonical form. Second, we formulate the problem in terms of Linear Matrix Inequalities (LMIs). The condition found simplifies the design process and avoids the parallel solution of multiple LMIs. The result yields a modified Algebraic Riccati Equation (ARE) for which we present an equivalent LMI condition.

1. INTRODUCTION

Consensus algorithms have been the focus of research over decades. Motivated by animal scientists that studied flocking behavior in schools of fish since the 1950s, [1] proposed an algorithm to model this flocking behavior with local interactions only. This work became very popular and boosted the research on consensus algorithms. These algorithms have in common that there is a group of individuals that try to agree on a common value, that is, to achieve consensus. Like the school of fish that moves as a flock where each fish keeps a certain distance to their neighbor - not too close and not too far - forming a structure, the flock. Another key element of consensus algorithms is their distributed communication. Their is no leading fish telling everybody else where to swim. Moreover each fish communicates to its closest neighbors only. A similar behavior can be observed in social groups forming a certain opinion or defining the next fashion trend.
Some of the early work by [2–4] and [5] motivated applications in different fields. The area of control in particular is driven by applications in synchronization, distributed averaging, autonomous formation flight, and the cooperative control of unmanned vehicles, to name a few (see [6–12] and the references therein). Their distributed nature, the development of parallel computing, and the advancement in communication technologies made consensus algorithms popular in many fields. The work in [13] for example presents a new approach to solve the resource allocation problem in a distributed fashion.

Consensus for networks of second-order systems

Many systems can be approximated by linear second-order dynamics. In the initial work, individual systems were modeled by double integrators in continuous-time, e.g.

[7, 14–18] and discrete-time where individual systems are given as double-integrators and the results were proven by applying a bilinear transformation (see [19–25]). In this thesis and in [26], we present a more general form of system dynamics and the direct use of Schur conditions.

Modifying each system to act like a double integrator neglects the natural dynamics and changes the steady-state behavior. Synchronization of oscillators or the tracking of a system with a significant drag component are examples of cases where more general second-order dynamics must be considered. The general second-order case is also important for digital control (see [11, 27]).

To achieve consensus for a homogenous network of linear systems applying linear control, it is necessary and sufficient to ensure the stability of a bunch of polynomials associated with the closed-loop network (see [28–30]). The coefficients of these polynomials can be complex since they depend on the possibly complex eigenvalues of the Laplacian matrix associated with the graph. Hence, in continuous-time, we need conditions which guarantee that a polynomial with complex coefficients is Hurwitz, that is, its roots have negative real part, and for the discrete-time case, we need conditions which guarantee that a polynomial with complex coefficients is Schur, that is, its roots have magnitude less than one.

For the continuous-time case, [31], [32] and [33] presented necessary and sufficient conditions for consensus, where [31] for example limits the control analysis to systems with a sufficiently large and stable open-loop pole. In this thesis, we shift the focus of the control design and obtain simple controllers for the general case of homogeneous networks of linear second-order systems to achieve consensus by adjusting two gains only. The Laplacian eigenvalues can be complex, and we only require that the associated graph contains a spanning tree and systems are stabilizable. Further, conditions to achieve consensus with a guaranteed rate of convergence are derived. These conditions generalize some of the existing results in [34], and they are useful for networks of double integrators as well. They are also important for many other applications, e.g. clock synchronization, power grid control, etc. [35].

[36], [37], [38] and [39] already established necessary and sufficient conditions for Schur polynomials, which are required to obtain necessary and sufficient conditions for consensus in the discrete-time case.

In this thesis, we present new and simple conditions for second-order polynomial with complex coefficients to be Hurwitz or Schur. Our proofs are independent of Routh-Hurwitz and Schur-Cohn criteria. We apply the new results to develop necessary and sufficient conditions to achieve consensus with guaranteed rate of convergence for the continuous- and the discrete-time case.

For the continuous-time case, we show results for static output feedback and present simple controllers that are capable of robustly achieving consensus over changing communication structures and system dynamics. The result is useful if systems frequently enter or leave the network (changing the graph), parts of the network are unknown, or system dynamics change, e.g. if non-linear systems are approximated and the point of linearization changes.

For the discrete-time case, we show that linear control is always sufficient to achieve consensus if systems are marginally stable. This is important if we wish to synchronize networks of oscillators or try to achieve consensus using digital control. The double-integrator model is a special case of marginally stable systems. Therefore, our results extend previous findings and model a broader class of systems. The general result is also useful to guarantee a desired rate of convergence for e.g. double-integrators.

Consensus for networks of higher-order systems

Modeling the dynamics of the individual systems by double integrators is only a first approximation. The linearized model of a satellite in an orbit around the earth is an example of a system with at least third-order dynamics (a high-fidelity model as introduced in [40] is already of sixth-order). A more accurate linearization of aircraft dynamics leads to a system with fourth-order dynamics.

Consensus algorithms for systems with dynamics of order $n > 2$ have been studied for systems modeled by n integrators. Early work allowed the communication of the full state (e.g. [7], [41] or [42]), afterwards the more complicated problem of using the system output information only was discussed (e.g. [43] or [44]) or switching topologies, time-delays and other non-linearities were introduced (e.g. [16], [45] or [46]).

Another approach considered the use of Linear Matrix Inequalities (LMIs) to provide sufficient conditions to achieve consensus. [28] requires the solution of N LMIs in parallel where N is the number of systems. The results in [47] are rather complicated and specifically trimmed to time-delays. [30] provided necessary and sufficient conditions for linear controllers to exist so that the network achieves consensus. However, no specific controller design was given.

A fundamental result is the existence of a spanning tree, which is a necessary condition to achieve consensus if systems are not asymptotically stable (see [48]). In continuous time, it can be shown that this condition is also sufficient if individual systems are stabilizable. Then, a control can always be found using a Riccati desgin for the feedback gain (e.g. [49], [50]). In discrete-time however, the interplay between the communication structure, the individual systems, and the control is more complex. It can be shown that communication structures containing a spanning tree cannot always achieve consensus using linear control only (see [51], [26]). This phenomena is related to the concept of the synchronizing region (e.g. [52]), and it is necessary to formulate conditions with respect to this interplay. A result for undirected graphs and single input systems was presented in [51]. An extension to directed graphs and multivariate systems is given in [53]. Both works were inspired by earlier work on mean square stabilization for quantized control (see [54]).

In this thesis, we obtain LMI conditions that give rise to a modified type of Algebraic Riccati Equation (ARE). Similar types of AREs were observed by several other researchers (e.g. [54], [55] [51] and [53]). However, obtaining solutions for these modified AREs is challenging.

The LMI approach proposed by [54] solves a slightly different problem, which introduces a non-zero weighting parameter. In our case, this parameter has to be zero, which yields a singularity. The LMI propsed by [55] obtains a solution by solving a related problem. In this thesis, we propose a small modification to the LMI propsed by [55] in order to yield a control for the discrete-time consensus problem with a specific rate of convergence.

Nonlinear, adaptive, and robust consensus control

Consensus algorithms are not limited to linear systems and controllers. Moreover, nonlinear consensus algorithms play an important practical role, for example, as actuators saturate. Previous results in the literature like [56], [57], [58] and [59] focused on nonlinear controllers and/or nonlinear system dynamics and provide a sound foundation for Lyapunov function candidates. Different nonlinear effects can be studied and lay the ground for future research in this area.

Switching systems are commonly studied nonlinearities and were already mentioned above. Another field of research analyzes heterogeneous, possibly uncertain, network structures (e.g. [17] or [60]) or heterogeneous and possibly unknown system dynamics which are mostly approached with an adaptive control design (e.g. [61], [62], [59] or [58]).

Thesis organization

The thesis is organized as follows. In Chapter 2, we provide preliminary results on graph theory and formulate the general problem in continuous-time and in discrete-time. Then, we identify a transformation in Chapter 3 that reduces the consensus problem to the simultaneous stabilization of a bunch of systems. This will be the basis for all other results in this thesis. Next, we present our work for linear second-order continuous-time systems ([63] and [64]) in Chapter 4. A brief outlook to the higher-order continuous-time case is given in Chapter 5. In Chapter 6, we present our work for linear second-order discrete-time systems ([26] and [65]), which we extend to the hgiher-order case in Chapter 7.

2. PROBLEM FORMULATION

2.1 Continuous-time systems

Consider a homogeneous network of linear continuous-time systems described by

$$\boxed{\dot{\mathbf{x}}_i(t) = \mathbf{A}(t)\,\mathbf{x}_i(t) + \mathbf{B}(t)\,\mathbf{u}_i(t) + \mathbf{u}_0(t)} \tag{2.1}$$

where $\mathbf{x}_i(t) \in \mathbb{R}^n$ is the state of system i at time $t \in \mathbb{R}$ for $i = 1, \cdots, N$. Systems are of order n, and matrices $\mathbf{A}(t) \in \mathbb{R}^{n\times n}$ and $\mathbf{B}(t) \in \mathbb{R}^{n\times q}$ are time-varying. The control input $\mathbf{u}_i(t) \in \mathbb{R}^q$ is applied to system i only, and input $\mathbf{u}_0(t) \in \mathbb{R}^n$ is common to all systems. The control input $\mathbf{u}_i$ provides feedback and can be used to achieve consensus while input $\mathbf{u}_0$ can be used to provide the systems with an arbitrary, nominal trajectory (feedforward control).

The communication structure of the network is described by a weighted, directed graph $G = (V, E, W)$. The set $V = \{1, 2, \cdots, N\}$ is called a vertex set, and we have a one-to-one correspondence between the vertices (elements of V) and the systems in the network. The edge set E is a subset of $V \times V$. An edge (j, i) is in E if system j can send information to system i; in this case, we say that j is an in-neighbor of i. We let $N_i = \{j \,|\, (j, i) \in E\}$ be the set of all in-neighbors of i. Each system i assigns a weight $w_{ij} > 0$ to each of its in-neighbors j. Letting $w_{ij} = 0$ when j is not an in-neighbor of i yields the weighting matrix $W = \{w_{ij}\}$.

Input $\mathbf{u}_i$ can only depend on the information available to system i. Initially we assume that each system has access to its own state; hence $\mathbf{u}_i$ can only be based on the state of system i and its in-neighbors. *We want to obtain feedback controllers for each* $\mathbf{u}_i$ *so that the closed-loop network achieves global uniform asymptotic consensus (GUAC) according to the following definition.*

2.1.1 Consensus definitions

Definition 2.1.1 *A network of N systems (2.1) achieves* global uniform asymptotic consensus *(GUAC) if*

(a) For each $\epsilon > 0$, there exists $\delta > 0$ such that, for any initial time t_0, if

$$\|\mathbf{x}_i(t_0) - \mathbf{x}_j(t_0)\| < \delta, \qquad i, j = 1, \cdots, N$$

then

$$\|\mathbf{x}_i(t) - \mathbf{x}_j(t)\| < \epsilon, \qquad i, j = 1, \cdots, N \quad t \geq t_0 \,.$$

(b) For each initial state $\mathbf{x}_{10}, \cdots, \mathbf{x}_{N0}$ there exists a bound β such that, for any initial time t_0, if

$$\mathbf{x}_i(t_0) = \mathbf{x}_{i0}, \qquad i = 1, \cdots, N$$

then

$$\|\mathbf{x}_i(t) - \mathbf{x}_j(t)\| \leq \beta \qquad i, j = 1, \cdots, N \quad t \geq t_0 \,.$$

(c) For each $\epsilon > 0$ and each initial state $\mathbf{x}_{10}, \cdots, \mathbf{x}_{N0}$ there exists a time $T > 0$ such that, for any initial time t_0, if

$$\mathbf{x}_i(t_0) = \mathbf{x}_{i0}, \qquad i = 1, \ldots, N$$

then

$$\|\mathbf{x}_i(t) - \mathbf{x}_j(t)\| < \epsilon, \qquad i, j = 1, \ldots, N \quad t \geq t_0 + T \,.$$

where $\|\mathbf{x}\| = \sqrt{\mathbf{x}^T \mathbf{x}}$.

Definition 2.1.2 *A network of N systems (2.1) achieved* consensus *if*

$$\mathbf{x}_i - \mathbf{x}_j = \mathbf{0}\,, \qquad i, j = 1, \cdots, N$$

Sometimes we are interested not only in convergence, but convergence with a specified exponential rate, which is defined as follows.

Definition 2.1.3 *A network of N systems (2.1) achieves* global uniform exponential consensus with convergence rate $\alpha_0 > 0$ *(GUEC with rate $\alpha_0 > 0$) if there exists a $c \geq 0$ such that for any initial time t_0, every initial state $\mathbf{x}_1(t_0), \cdots, \mathbf{x}_N(t_0)$, all $i, j = 1, \cdots, N$, and all $t \geq t_0$,*

$$\|\mathbf{x}_j(t) - \mathbf{x}_i(t)\| \leq c\,\|\mathbf{x}_j(t_0) - \mathbf{x}_i(t_0)\|\, e^{-\alpha_0\,(t-t_0)}$$

where $\|\mathbf{x}\| = \sqrt{\mathbf{x}^T\,\mathbf{x}}$.

For each system i, we consider the following linear controller

$$\boxed{\mathbf{u}_i(t) = \mathbf{K}(t) \sum_{j \in N_i} w_{ij} \left[\mathbf{x}_j(t) - \mathbf{x}_i(t)\right]} \tag{2.2}$$

where the gain matrix $\mathbf{K}(t) \in \mathbb{R}^{q \times n}$ is common to all systems. We will call (2.1)-(2.2) the closed-loop network.

Remark 2.1.1 *If the network achieved consensus, then $\mathbf{u}_i(t) = \mathbf{0}$ for all i, and the behavior of each system is identical and governed by*

$$\dot{\mathbf{x}}_i(t) = \mathbf{A}(t)\,\mathbf{x}_i(t) + \mathbf{u}_0(t)\,.$$

2.1.2 The network of transformed systems

We can use the conditions for GUAC to achieve GUEC with rate α_0. To illustrate this, we use the following transformed systems

$$\dot{\tilde{\mathbf{x}}}_i(t) = \tilde{\mathbf{A}}(t)\,\tilde{\mathbf{x}}_i(t) + \mathbf{B}(t)\,\tilde{\mathbf{u}}_i(t) + \tilde{\mathbf{u}}_0(t) \tag{2.3}$$

with inputs $\tilde{\mathbf{u}}_i(t) = e^{\alpha_0 t}\mathbf{u}_i(t)$ and $\tilde{\mathbf{u}}_0(t) = e^{\alpha_0 t}\mathbf{u}_0(t)$, and $\tilde{\mathbf{A}}(t)$ given by

$$\tilde{\mathbf{A}}(t) := \mathbf{A}(t) + \alpha_0\,\mathbf{I}\,. \tag{2.4}$$

First, we present the following result.

Lemma 2.1.1 *Network (2.1) achieves GUEC with rate α_0 if the associated transformed network (2.3) achieves GUAC.*

Proof Let $\tilde{\mathbf{x}}_i(t) := e^{\alpha_0 t}\mathbf{x}_i(t)$ for all i. Then $\mathbf{x}_i(t) = \tilde{\mathbf{x}}_i(t)e^{-\alpha_0 t}$ and

$$\|\mathbf{x}_j(t) - \mathbf{x}_i(t)\| = \|\tilde{\mathbf{x}}_j(t) - \tilde{\mathbf{x}}_i(t)\|\, e^{-\alpha_0 t}\,.$$

Hence, $\mathbf{x}_j(t) - \mathbf{x}_i(t)$ exponentially converges to zero with guaranteed rate α_0 if $\tilde{\mathbf{x}}_j(t) - \tilde{\mathbf{x}}_i(t)$ converges to zero. The result is obtained by noting that when the behavior of $\mathbf{x}_i$ is governed by (2.1), the behavior of $\tilde{\mathbf{x}}_i$ is described by (2.3). ■

Remark 2.1.2 *Lemma 2.1.1 holds for any type of controller. Here we consider linear controllers similar to (2.2) for network (2.3), that is,*

$$\tilde{\mathbf{u}}_i(t) = \mathbf{K}(t)\sum_{j\in N_i} w_{ij}\left[\tilde{\mathbf{x}}_j(t) - \tilde{\mathbf{x}}_i(t)\right].$$

Since $\tilde{\mathbf{u}}_i(t) = e^{\alpha_0 t}\mathbf{u}_i(t)$ and $\tilde{\mathbf{x}}_i(t) = e^{\alpha_0 t}\mathbf{x}_i(t)$ this is equivalent to controller (2.2).

2.2 Discrete-time systems

Consider a homogeneous network of linear discrete-time systems described by

$$\boxed{\mathbf{x}_i(k+1) = \mathbf{A}(k)\,\mathbf{x}_i(k) + \mathbf{B}(k)\,\mathbf{u}_i(k) + \mathbf{u}_0(k)} \tag{2.5}$$

where $\mathbf{x}_i(k) \in \mathbb{R}^n$ is the state of system i at step k for $i = 1, \cdots, N$. Systems are of order n and matrices $\mathbf{A}(k) \in \mathbb{R}^{n\times n}$ and $\mathbf{B}(k) \in \mathbb{R}^{n\times q}$ are time-varying. The control input $\mathbf{u}_i(k) \in \mathbb{R}^q$ is applied to system i only, and input $\mathbf{u}_0(k) \in \mathbb{R}^n$ is common to all systems. The control input $\mathbf{u}_i$ provides feedback and can be used to achieve consensus while input $\mathbf{u}_0$ can be used to provide the systems with an arbitrary, nominal trajectory (feedforward control).

Analog to Section 2.1, the communication structure of the network is described by a weighted, directed graph $G = (V, E, W)$, and we have the following definitions.

2.2.1 Consensus definitions

Definition 2.2.1 *A network of N systems* (2.5) *achieves* global uniform asymptotic consensus *(GUAC) if*

(a) For each $\epsilon > 0$, there exists $\delta > 0$ such that, for any initial time k_0, if

$$\|\mathbf{x}_i(k_0) - \mathbf{x}_j(k_0)\| < \delta, \qquad i, j = 1, \cdots, N$$

then

$$\|\mathbf{x}_i(k) - \mathbf{x}_j(k)\| < \epsilon, \qquad i, j = 1, \cdots, N \quad k \geq k_0 \,.$$

(b) For each initial state $\mathbf{x}_{10}, \cdots, \mathbf{x}_{N0}$ there exists a bound β such that, for any initial time k_0, if

$$\mathbf{x}_i(k_0) = \mathbf{x}_{i0}, \qquad i = 1, \cdots, N$$

then

$$\|\mathbf{x}_i(k) - \mathbf{x}_j(k)\| \leq \beta \qquad i, j = 1, \cdots, N \quad k \geq k_0 \,.$$

(c) For each $\epsilon > 0$ and each initial state $\mathbf{x}_{10}, \cdots, \mathbf{x}_{N0}$ there exists a time $T > 0$ such that, for any initial time k_0, if

$$\mathbf{x}_i(k_0) = \mathbf{x}_{i0}, \qquad i = 1, \ldots, N$$

then

$$\|\mathbf{x}_i(k) - \mathbf{x}_j(k)\| < \epsilon, \qquad i, j = 1, \ldots, N \quad k \geq k_0 + T \,.$$

where $\|\mathbf{x}\| = \sqrt{\mathbf{x}^T \mathbf{x}}$.

Definition 2.2.2 *A network of N systems* (2.5) *achieved* consensus *if*

$$\mathbf{x}_i - \mathbf{x}_j = \mathbf{0} \,, \qquad i, j = 1, \cdots, N$$

Definition 2.2.3 *A network of* N *systems* (2.5) *achieves* global uniform exponential consensus with convergence rate $0 < \rho < 1$ *(GUEC with rate* $0 < \rho < 1$*) if there exists a* $c \geq 0$ *such that for any initial time* k_0*, every initial state* $\mathbf{x}_1(k_0), \cdots, \mathbf{x}_N(k_0)$*, all* $i, j = 1, \cdots, N$*, and all* $k \geq k_0$*,*

$$\|\mathbf{x}_j(k) - \mathbf{x}_i(k)\| \leq c \, \|\mathbf{x}_j(k_0) - \mathbf{x}_i(k_0)\| \, \rho^{k-k_0}$$

where $\|\mathbf{x}\| = \sqrt{\mathbf{x}^T \mathbf{x}}$*.*

For each system i, we consider the following linear controller

$$\boxed{\mathbf{u}_i(k) = \mathbf{K}(k) \sum_{j \in N_i} w_{ij} \left[\mathbf{x}_j(k) - \mathbf{x}_i(k)\right]} \tag{2.6}$$

where the gain matrix $\mathbf{K}(k) \in \mathbb{R}^{q \times n}$ is common to all systems. We will call (2.5)-(2.6) the closed-loop network.

Remark 2.2.1 *If the network achieved consensus, then* $\mathbf{u}_i(k) = \mathbf{0}$ *for all* i*, and the behavior of each system is identical and governed by*

$$\mathbf{x}_i(k+1) = \mathbf{A}(k) \, \mathbf{x}_i(k) + \mathbf{u}_0(k) \, .$$

2.2.2 The network of transformed systems

We can use the conditions for GUAC to achieve GUEC with rate α_0. To illustrate this, we use the following transformed systems

$$\tilde{\mathbf{x}}_i(k+1) = \tilde{\mathbf{A}}(k) \, \tilde{\mathbf{x}}_i(k) + \tilde{\mathbf{B}}(k) \, \tilde{\mathbf{u}}_i(k) + \tilde{\mathbf{u}}_0(k) \tag{2.7}$$

with inputs $\tilde{\mathbf{u}}_i(k) = \mathbf{u}_i(k)/\rho^k$ and $\tilde{\mathbf{u}}_0(k) = \mathbf{u}_0(k)/\rho^{k+1}$, and $\tilde{\mathbf{A}}(k)$, $\tilde{\mathbf{B}}(k)$ given by

$$\tilde{\mathbf{A}}(k) := \rho^{-1} \, \mathbf{A}(k) \, , \qquad \tilde{\mathbf{B}}(k) := \rho^{-1} \, \mathbf{B}(k) \tag{2.8}$$

First, we present the following result.

Lemma 2.2.1 *Network* (2.5) *achieves GUEC with rate* ρ *if the associated transformed network* (2.7) *achieves GUAC.*

Figure 2.1.. A graph with (left) and without (right) a spanning tree

Proof Consider any $0 < \rho < 1$. Let $\tilde{\mathbf{x}}_i(k) := \rho^{-k}\mathbf{x}_i(k)$ for all i. Then $\mathbf{x}_i(k) = \tilde{\mathbf{x}}_i(k)\,\rho^k$ and

$$\|\mathbf{x}_j(k) - \mathbf{x}_i(k)\| = \|\tilde{\mathbf{x}}_j(k) - \tilde{\mathbf{x}}_i(k)\|\,\rho^k\,.$$

Hence, $\mathbf{x}_j(k) - \mathbf{x}_i(k)$ exponentially converges to zero with guaranteed rate ρ if $\tilde{\mathbf{x}}_j(k) - \tilde{\mathbf{x}}_i(k)$ converges to zero. The result is obtained by noting that when the behavior of $\mathbf{x}_i(k)$ is governed by 2.5, the behavior of $\tilde{\mathbf{x}}_i(k)$ is described by 2.7. ■

Remark 2.2.2 *Lemma 2.2.1 holds for any type of controller. Here we consider linear controllers similar to (2.6) for network (2.7), that is,*

$$\tilde{\mathbf{u}}_i(k) = \mathbf{K}(k)\sum_{j\in N_i} w_{ij}\left[\tilde{\mathbf{x}}_j(k) - \tilde{\mathbf{x}}_i(k)\right]$$

Since $\tilde{\mathbf{u}}_i(k) = \rho^{-k}\,\mathbf{u}_i(k)$ and $\tilde{\mathbf{x}}_i(k) = \rho^{-k}\mathbf{x}_i(t)$ this is equivalent to controller (2.2).

2.3 Graph theory

To present the results in this thesis, we require some concepts and results from graph theory.

Associated with a weighted graph $G = (V, E, W)$ is its Laplacian matrix $\mathbf{L} = \{l_{ij}\}$, defined by

$$l_{ii} = \sum_{j\neq i} w_{ij} \qquad \text{and} \qquad l_{ij} = -w_{ij} \quad \text{for} \quad i \neq j\,.$$

Note that, in terms of the Laplacian, control (2.2) or (2.6) can be expressed as

$$\mathbf{u}_i = -\mathbf{K}\sum_{j=1}^{N} l_{ij}\,\mathbf{x}_j\,.$$

A directed path in G from a vertex j to a vertex i is a sequence $(i_1, i_2, \cdots, i_m)$ in V with $i_1 = j$, $i_m = i$, and $(i_k, i_{k+1}) \in E$ for $k = 1, \cdots, m-1$. A graph G is said to contain a spanning tree if at least one vertex j^* in V has the following property: for every i in V there is a directed path from j^* to i. An example for a graph that contains a spanning tree and one that does not is given in Figure 2.1. One can show that, if the systems are not asymptotically stable, then a spanning tree is always necessary for consensus [48].

If $\mathbf{1}$ is the vector of all ones, then any nonzero multiple of this vector is an eigenvector of $\mathbf{L}$ corresponding to eigenvalue zero. If the graph G has a spanning tree, then there are no other eigenvectors corresponding to zero.

Fact 2.3.1 ([48]) *A graph G contains a spanning tree if and only if the associated Laplacian matrix has one zero eigenvalue and all its other eigenvalues have a positive real part.*

If the systems are asymptotically stable, then consensus can be trivially achieved with $\mathbf{K} = 0$. Therefore, throughout this thesis it is assumed that

G contains a spanning tree.

Next, we identify bounds on the real parts of the non-zero Laplacian eigenvalues:

$$\alpha_m = \min\{\Re(\mu) \,|\, \mu \in \Lambda_L\}\,, \quad \alpha_M = \max\{\Re(\mu) \,|\, \mu \in \Lambda_L\} \tag{2.9}$$

where Λ_L is the set of all non-zero eigenvalues of the graph Laplacian. We note that $\alpha_m, \alpha_M > 0$ since G contains a spanning tree.

If the graph contains a vertex with no in-neighbors, then the behavior of the system associated with this vertex is unaffected by the other systems. We will call this system a leader and when consensus is achieved, then the states of all the systems will equal that of the leader.

2.4 Complex valued state variables and system matrices

Remark 2.4.1 *Systems (2.1) and (2.5) are introduced for real valued state vectors $\mathbf{x}_i$ and real valued system matrices $\mathbf{A}$ and $\mathbf{B}$ only. However, results for the closed-loop* *networks* (2.1)-(2.2) *and* (2.5)-(2.6) *can easily be extended to complex valued* $\mathbf{x}_i$*,* $\mathbf{A}$*,* $\mathbf{B}$*, and* $\mathbf{K}$*.*

2.5 Matrix inequalities

To present results for networks of higher-order systems, we will make use of matrix inequalities and the following definitions.

Definition 2.5.1 *A matrix* $\mathbf{Q} \in \mathbb{C}^{n\times n}$ *is* hermitian *if and only if* $\mathbf{Q} = \mathbf{Q}'$ *where* $()'$ *denotes the conjugate transpose of a matrix.*

Definition 2.5.2 *A matrix* $\mathbf{Q} \in \mathbb{C}^{n\times n}$ *is* skew-hermitian *if and only if* $\mathbf{Q} = -\mathbf{Q}'$*.*

Definition 2.5.3 *A hermitian matrix* $\mathbf{Q}$ *is* positive definite *(*$\mathbf{Q} > \mathbf{0}$*) if and only if* $\mathbf{z}'\,\mathbf{Q}\,\mathbf{z} > 0$ *for all* $\mathbf{z} \neq 0$ *where* $\mathbf{Q} \in \mathbb{C}^{n\times n}$ *and* $\mathbf{z} \in \mathbb{C}^n$*.*

Definition 2.5.4 *A hermitian matrix* $\mathbf{Q}$ *is* positive semi-definite *(*$\mathbf{Q} \geq \mathbf{0}$*) if and only if* $\mathbf{z}'\,\mathbf{Q}\,\mathbf{z} \geq 0$ *for all* $\mathbf{z} \neq 0$ *where* $\mathbf{Q} \in \mathbb{C}^{n\times n}$ *and* $\mathbf{z} \in \mathbb{C}^n$*.*

Definition 2.5.5 *A hermitian matrix* $\mathbf{Q}$ *is* negative definite *(*$\mathbf{Q} < \mathbf{0}$*) if and only if* $-\mathbf{Q}$ *is positive definite.*

Definition 2.5.6 *A hermitian matrix* $\mathbf{Q}$ *is* negative semi-definite *(*$\mathbf{Q} \leq \mathbf{0}$*) if and only if* $-\mathbf{Q}$ *is positive semi-definite.*

3. THE CLOSED-LOOP NETWORK

The closed-loop networks (2.1)-(2.2) and (2.5)-(2.6) are linear systems. If graph G is fixed, then the dynamics due to the networks' communication structure can be decomposed into the different modes. [29] illustrated this strategy for time-invariant higher-order continuous-time systems. The technique extends to the time-invariant discrete-time case (see [30]), and we will review it here to present results for time-variant systems in the presence of a time-invariant communication structure.

3.1 Separation of the closed-loop dynamics

3.1.1 Transforming the graph Laplacian into Jordan normal form

First, we apply a similarity transformation to $\mathbf{L}$, which preserves the eigenvalues of the Laplacian. For any matrix, we can always identify a transformation such that the transformed matrix is in Jordan normal form. Let $\mathbf{T}$ be such that $\hat{\mathbf{L}} = \mathbf{T}^{-1}\,\mathbf{L}\,\mathbf{T}$ is in Jordan normal form. From the definition of the graph Laplacian, $\mathbf{1}$ is an eigenvector of $\mathbf{L}$ corresponding to eigenvalue zero. Therefore, we can choose $\mathbf{T}$ so that each element of the first column of $\mathbf{T}$ is one.

Lemma 3.1.1 *Suppose* $\mathbf{T}$ *is a transformation matrix such that* $\hat{\mathbf{L}} = \mathbf{T}^{-1}\,\mathbf{L}\,\mathbf{T}$ *is in Jordan normal form and each element of the first column of* $\mathbf{T}$ *is one. Then,*

$$\sum_{k=1}^{N} s_{ik} = \begin{cases} 1 & \textit{if} \quad i = 1 \\ 0 & \textit{otherwise} \end{cases} \tag{3.1}$$

$$\sum_{k=1}^{N} s_{ik}\, l_{kj} = \sum_{k=1}^{N} \hat{l}_{ik}\, s_{kj}\,, \qquad i, j = 1, \cdots, N \tag{3.2}$$

where $\mathbf{S} = \mathbf{T}^{-1}$*, and a new set of variables given by*

$$\mathbf{z}_i = \sum_{j=1}^{N} s_{ij}\,\mathbf{x}_j\,, \qquad i = 1, \cdots, N \tag{3.3}$$

will have properties

$$\mathbf{z}_i = \sum_{j=1}^{N} s_{ij}\,(\mathbf{x}_j - \mathbf{x}_i)\,, \qquad i = 2, \cdots, N \tag{3.4}$$

and

$$\mathbf{x}_i - \mathbf{x}_j = \sum_{k=2}^{N} (t_{ik} - t_{jk})\,\mathbf{z}_k\,, \qquad i, j = 1, \cdots, N\,. \tag{3.5}$$

Proof Let $\mathbf{T}$ be a transformation matrix such that $\hat{\mathbf{L}} = \mathbf{T}^{-1}\,\mathbf{L}\,\mathbf{T}$ is in Jordan normal form and each element of the first column of $\mathbf{T}$ is one. The algebraic multiplicity of zero as an eigenvalue of $\mathbf{L}$ associated with eigenvector $\mathbf{1}$ is one (Fact 2.3.1). Hence,

$$\hat{\mathbf{L}} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & \mu_2 & 0 & \cdots & 0 \\ 0 & * & \ddots & & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & * & \mu_N \end{bmatrix}$$

where $\mu_2, \cdots, \mu_N$ are the non-zero eigenvalues of $\mathbf{L}$ and $* = 0$ or $* = 1$. Let $\mathbf{S} = \mathbf{T}^{-1}$; then for $i, j = 1, \cdots, N$:

$$\sum_{k=1}^{N} t_{ik}\,s_{kj} = \sum_{k=1}^{N} s_{ik}\,t_{kj} = \delta_{ij} = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases}$$

In particular, (3.1) is obtained by noting that $t_{i1} = 1$ for $i = 1, \cdots, N$. Thus,

$$\sum_{k=1}^{N} s_{ik} = \sum_{k=1}^{N} s_{ik}\,t_{k1} = \delta_{i1}\,.$$

We also have $\mathbf{S}\,\mathbf{L} = \hat{\mathbf{L}}\,\mathbf{S}$ since $\hat{\mathbf{L}} = \mathbf{T}^{-1}\,\mathbf{L}\,\mathbf{T}$ and $\mathbf{S} = \mathbf{T}^{-1}$. Hence, (3.2) holds. Note that, as a consequence of (3.1), we have $\sum_{j=1}^{N} s_{ij}\,\mathbf{x}_i = \delta_{i1}\,\mathbf{x}_i$, which is $\mathbf{0}$ for $i > 1$.

If we introduce new state variables (3.3), then we can add this $\mathbf{0}$ to (3.3) and obtain (3.4) for $i > 1$. Also, for $i = 1, \cdots, N$:

$$\sum_{j=1}^{N} t_{ij}\,\mathbf{z}_j = \sum_{k=1}^{N}\sum_{j=1}^{N} t_{ij}\, s_{jk}\,\mathbf{x}_k = \sum_{k=1}^{N} \delta_{ik}\,\mathbf{x}_k = \mathbf{x}_i$$

Thus, since $t_{i1} = 1$,

$$\mathbf{x}_i = \sum_{j=1}^{N} t_{ij}\,\mathbf{z}_j = \mathbf{z}_1 + \sum_{j=2}^{N} t_{ij}\,\mathbf{z}_j\,, \qquad i = 1, \cdots, N$$

and (3.5) is recovered. ■

Clearly, if consensus is achieved, then $\mathbf{z}_i \to 0$ for all $i > 1$ from (3.4).

Remark 3.1.1 *Since graph G contains a spanning tree, the Laplacian matrix has one zero eigenvalue (Fact 2.3.1). The dynamics associated with this eigenvalue are the consensus dynamics of the closed-loop network. Therefore, if we are interested in consensus, then we are looking at the dynamics of $\mathbf{z}_i$ for $i = 2, \cdots, N$ only.*

3.1.2 Transforming the system dynamics

Next, we apply the transformation to the closed-loop network. The approach applies to continuous-time and discrete-time systems, and we will make use of functions f_i to keep the approach generic.

Lemma 3.1.2 *Suppose we are given N functions f_i and N vectors $\mathbf{x}_i$ such that*

$$f_i(t, \mathbf{x}) = \mathbf{A}(t)\,\mathbf{x}_i - \mathbf{B}(t)\,\mathbf{K}(t) \sum_{j=1}^{N} l_{ij}\,\mathbf{x}_j + \mathbf{u}_0(t) \tag{3.6}$$

for all $i = 1, \cdots, N$ where $\mathbf{x}$ is the stacked vector of all $\mathbf{x}_i$. Let $\mathbf{T}$ and $\mathbf{S} = \mathbf{T}^{-1}$ be matrices that satisfy the conditions in Lemma 3.1.1, and functions $\hat{f}_i$ be given by

$$\hat{f}_i(t, \mathbf{z}) = \sum_{j=1}^{N} s_{ij}\, f_i(t, \mathbf{x}) \tag{3.7}$$

where $\mathbf{z}$ *is the stacked vector of all* $\mathbf{z}_i$ *given by* (3.3). *Then,* $\hat{\mathbf{L}} = \mathbf{T}^{-1}\,\mathbf{L}\,\mathbf{T}$ *has* H *Jordan blocks of length* $l_1, \cdots, l_H$ *and we obtain*

$$\hat{f}_1(t, \mathbf{z}) = \mathbf{A}(t)\,\mathbf{z}_1 + \mathbf{u}_0(t) \tag{3.8}$$

and the blocks of all the non-zero eigenvalues μ_h *of the graph Laplacian yield cascades*

$$\hat{f}_i(t, \mathbf{z}) = [\mathbf{A}(t) - \mu_h\,\mathbf{B}(t)\,\mathbf{K}(t)]\,\mathbf{z}_i\,, \qquad i = j_h \tag{3.9}$$

$$\hat{f}_i(t, \mathbf{z}) = [\mathbf{A}(t) - \mu_h\,\mathbf{B}(t)\,\mathbf{K}(t)]\,\mathbf{z}_i - \mathbf{B}(t)\,\mathbf{K}(t)\,\mathbf{z}_{i-1}\,, \qquad j_h < i < j_{h+1} \tag{3.10}$$

where $h = 2, \cdots, H$, $j_h = j_{h-1} + l_{h-1}$, $j_1 = l_1 = 1$, *and* $j_H + l_H = N$.

Proof Let the conditions of the lemma be satisfied. Then,

$$\hat{f}_i(t, \mathbf{z}) = \sum_{j=1}^{N} s_{ij}\, f_i(t, \mathbf{x}) \tag{3.11}$$

$$= \mathbf{A}(t) \sum_{j=1}^{N} s_{ij}\,\mathbf{x}_j - \mathbf{B}(t)\,\mathbf{K}(t) \sum_{j=1}^{N}\sum_{k=1}^{N} s_{ij}\, l_{jk}\,\mathbf{x}_k + \sum_{j=1}^{N} s_{ij}\,\mathbf{u}_0(t) \tag{3.12}$$

for $i = 1, \cdots, N$. It follows from (3.2) and (3.3) that

$$\sum_{j=1}^{N}\sum_{k=1}^{N} s_{ij}\, l_{jk}\,\mathbf{x}_k = \sum_{k=1}^{N} \left(\sum_{j=1}^{N} s_{ij}\, l_{jk} \right) \mathbf{x}_k = \sum_{k=1}^{N} \left(\sum_{j=1}^{N} \hat{l}_{ij}\, s_{ij} \right) \mathbf{x}_k$$

$$= \sum_{j=1}^{N} \hat{l}_{ij} \sum_{k=1}^{N} s_{ij}\,\mathbf{x}_k = \sum_{j=1}^{N} \hat{l}_{ij}\,\mathbf{z}_j\,. \tag{3.13}$$

It follows from (3.1) that

$$\sum_{j=1}^{N} s_{ij}\,\mathbf{u}_0(t) = \delta_{1i}\,\mathbf{u}_0(t) \tag{3.14}$$

Now, substituting (3.3), (3.13), and (3.14) into (3.12) yields

$$\hat{f}_i(t, \mathbf{z}) = \mathbf{A}(t)\,\mathbf{z}_i - \mathbf{B}(t)\,\mathbf{K}(t) \sum_{j=1}^{N} \hat{l}_{ij}\,\mathbf{z}_j + \delta_{1i}\,\mathbf{u}_0(t)$$

where $\hat{\mathbf{L}} = \mathbf{T}^{-1}\,\mathbf{L}\,\mathbf{T}$ is in Jordan normal form. Thus, we can identify H Jordan blocks of length l_h to obtain (3.8)-(3.10). ■

3.1.3 Stability concepts

The Laplacian matrix $\mathbf{L}$ is constant. Thus, the similarity transformation $\mathbf{L} \rightarrow \hat{\mathbf{L}}$ stays constant, and we can apply Lemma 3.1.2 and Remark 3.1.1 to reduce consensus to the stability of a bunch of systems where we define stability as follows.

Definition 3.1.1 *System* $\dot{\mathbf{x}}(t) = \mathbf{A}(t)\,\mathbf{x}(t)$ *is* globally uniformly asymptotically stable *(GUAS) if*

(a) For each $\epsilon > 0$*, there exists* $\delta > 0$ *such that, for any initial time* t_0*, if* $\|\mathbf{x}(t_0)\| < \delta$*, then* $\|\mathbf{x}(t)\| < \epsilon$ *for all* $t \geq t_0$*.*

(b) For each initial state $\mathbf{x}_0$ *there exists a bound* β *such that, for any initial time* t_0*, if* $\mathbf{x}(t_0) = \mathbf{x}_0$*, then* $\|\mathbf{x}(t)\| \leq \beta$ *for all* $t \geq t_0$*.*

(c) For each $\epsilon > 0$ *and each initial state* $\mathbf{x}_0$ *there exists a time* $T > 0$ *such that, for any initial time* t_0*, if* $\mathbf{x}(t_0) = \mathbf{x}_0$*, then* $\|\mathbf{x}(t)\| < \epsilon$ *for all* $t \geq t_0 + T$*.*

where $\|\mathbf{x}\| = \sqrt{\mathbf{x}^T\,\mathbf{x}}$*.*

Definition 3.1.2 *System* $\dot{\mathbf{x}}(t) = \mathbf{A}(t)\,\mathbf{x}(t)$ *is* globally uniformly exponentially stable with convergence rate $\alpha_0 > 0$ *(GUES with rate* $\alpha_0 > 0$*) if there exists a* $c \geq 0$ *such that for any initial time* t_0*, every initial state* $\mathbf{x}(t_0)$*, and all* $t \geq t_0$*,* $\|\mathbf{x}(t)\| \leq c\,\|\mathbf{x}(t_0)\|\, e^{-\alpha_0\,(t-t_0)}$ *where* $\|\mathbf{x}\| = \sqrt{\mathbf{x}^T\,\mathbf{x}}$*.*

Definition 3.1.3 *System* $\mathbf{x}(k+1) = \mathbf{A}(k)\,\mathbf{x}(k)$ *is* globally uniformly asymptotically stable *(GUAS) if*

(a) For each $\epsilon > 0$*, there exists* $\delta > 0$ *such that, for any initial time* k_0*, if* $\|\mathbf{x}(k_0)\| < \delta$*, then* $\|\mathbf{x}(k)\| < \epsilon$ *for all* $k \geq k_0$*.*

(b) For each initial state $\mathbf{x}_0$ *there exists a bound* β *such that, for any initial time* k_0*, if* $\mathbf{x}(k_0) = \mathbf{x}_0$*, then* $\|\mathbf{x}(k)\| \leq \beta$ *for all* $k \geq k_0$*.*

(c) For each $\epsilon > 0$ *and each initial state* $\mathbf{x}_0$ *there exists a time* $T > 0$ *such that, for any initial time* k_0*, if* $\mathbf{x}(k_0) = \mathbf{x}_0$*, then* $\|\mathbf{x}(k)\| < \epsilon$ *for all* $k \geq k_0 + T$*.*

where $\|\mathbf{x}\| = \sqrt{\mathbf{x}^T\mathbf{x}}$.

Definition 3.1.4 *System* $\mathbf{x}(k+1) = \mathbf{A}(k)\,\mathbf{x}(k)$ *is* globally uniformly exponentially stable with convergence rate $0 < \rho < 1$ *(GUES with rate* $0 < \rho < 1$*) if there exists a* $c \geq 0$ *such that for any initial time* k_0*, every initial state* $\mathbf{x}(k_0)$*, and all* $k \geq k_0$*,* $\|\mathbf{x}(k)\| \leq c\,\|\mathbf{x}(k_0)\|\,\rho^{k-k_0}$ *where* $\|\mathbf{x}\| = \sqrt{\mathbf{x}^T\mathbf{x}}$*.*

3.1.4 Formulating consensus as a stabilization problem

From Lemma 3.1.2 and Remark 3.1.1, we immediately obtain the following two results.

Corollary 3.1.1 *The closed-loop network* (2.1)-(2.2) *achieves GUAS (GUEC with rate* $\alpha_0 > 0$*) if and only if the cascaded systems*

$$\dot{\mathbf{z}}_1 = [\mathbf{A}(t) - \mu\,\mathbf{B}(t)\,\mathbf{K}(t)]\,\mathbf{z}_1\,, \tag{3.15}$$

$$\dot{\mathbf{z}}_i = [\mathbf{A}(t) - \mu\,\mathbf{B}(t)\,\mathbf{K}(t)]\,\mathbf{z}_i - \mathbf{B}(t)\,\mathbf{K}(t)\,\mathbf{z}_{i-1}\,, \qquad i > 1 \tag{3.16}$$

are GUAS (GUES with rate $\alpha_0 > 0$*) for each non-zero eigenvalue* μ *of the graph Laplacian where* $i \leq l_{M\mu}$ *and* $l_{M\mu}$ *is the maximum length of a Jordan block associated with eigenvalue* μ*.*

Corollary 3.1.2 *The closed-loop network* (2.5)-(2.6) *achieves GUAS (GUEC with rate* $0 < \rho < 1$*) if and only if the cascaded systems*

$$\mathbf{z}_1(k+1) = [\mathbf{A}(k) - \mu\,\mathbf{B}(k)\,\mathbf{K}(k)]\,\mathbf{z}_1(k)\,, \tag{3.17}$$

$$\mathbf{z}_i(k+1) = [\mathbf{A}(k) - \mu\,\mathbf{B}(k)\,\mathbf{K}(k)]\,\mathbf{z}_i(k) - \mathbf{B}(k)\,\mathbf{K}(k)\,\mathbf{z}_{i-1}(k)\,, \qquad i > 1 \tag{3.18}$$

are GUAS (GUES with rate $0 < \rho < 1$*) for each non-zero eigenvalue* μ *of the graph Laplacian where* $i \leq l_{M\mu}$ *and* $l_{M\mu}$ *is the maximum length of a Jordan block associated with eigenvalue* μ*.*

One can show that under certain assumptions on the system matrices **A**, **B**, and **K**, the stability properties of the cascade systems (3.15)-(3.16) or (3.17)-(3.18) are determined by the stability property of systems (3.15) or (3.17) [66–70].

Lemma 3.1.3 ([70]) *Consider a cascade system*

$$\begin{pmatrix} \dot{\mathbf{z}}_1(t) \\ \dot{\mathbf{z}}_2(t) \end{pmatrix} = \begin{bmatrix} \mathbf{A}_{cl}(t) & \mathbf{0} \\ \mathbf{A}_{couple}(t) & \mathbf{A}_{cl}(t) \end{bmatrix} \begin{pmatrix} \mathbf{z}_1(t) \\ \mathbf{z}_2(t) \end{pmatrix} \tag{3.19}$$

where $\mathbf{A}_{cl}(\cdot)$ *and* $\mathbf{A}_{couple}(\cdot)$ *are piece-wise continuous and bounded. Then, system* (3.19) *is GUES if and only if* $\dot{\mathbf{z}}_1(t) = \mathbf{A}_{cl}(t)\,\mathbf{z}_1(t)$ *is GUES.*

Lemma 3.1.4 *Suppose* $\mathbf{A}(\cdot)$ *and* $\mathbf{B}(\cdot)\mathbf{K}(\cdot)$ *are piece-wise continuous and bounded. Then, the closed-loop network* (2.1)-(2.2) *achieves GUAC if and only if systems*

$$\dot{\mathbf{z}} = [\mathbf{A}(t) - \mu\,\mathbf{B}(t)\mathbf{K}(t)]\;\mathbf{z} \tag{3.20}$$

are GUAS for all non-zero eigenvalues μ *of the graph Laplacian.*

Proof From Corollary 3.1.1, we have to show stability of (3.15)-(3.16) for all $\mu \neq 0$. In our case (bounded linear systems), it is well known that the notion of GUAS and GUES are equivalent [77, 78]. Thus, we conclude from repeated application of Lemma 3.1.3 that systems (3.15)-(3.16) are GUAS if and only if (3.20) is GUAS. ■

Lemma 3.1.5 ([69]) *Consider a cascade system*

$$\begin{pmatrix} \mathbf{z}_1(k+1) \\ \mathbf{z}_2(k+1) \end{pmatrix} = \begin{bmatrix} \mathbf{A}_{cl}(k) & \mathbf{0} \\ \mathbf{A}_{couple}(k) & \mathbf{A}_{cl}(k) \end{bmatrix} \begin{pmatrix} \mathbf{z}_1(k) \\ \mathbf{z}_2(k) \end{pmatrix} \tag{3.21}$$

where $\mathbf{A}_{cl}(\cdot)$ *and* $\mathbf{A}_{couple}(\cdot)$ *are bounded. Then, system* (3.21) *is GUAS if and only if* $\mathbf{z}_1(k+1) = \mathbf{A}_{cl}(k)\,\mathbf{z}_1(k)$ *is GUAS.*

Lemma 3.1.6 *Suppose* $\mathbf{A}(\cdot)$ *and* $\mathbf{B}(\cdot)\mathbf{K}(\cdot)$ *are bounded. Then, the closed-loop network* (2.5)-(2.6) *achieves GUAC if and only if systems*

$$\mathbf{z}(k+1) = [\mathbf{A}(k) - \mu\,\mathbf{B}(k)\mathbf{K}(k)]\;\mathbf{z}(k) \tag{3.22}$$

are GUAS for all non-zero eigenvalues μ *of the graph Laplacian.*

Proof From Corollary 3.1.2, we have to show stability of (3.17)-(3.18) for all $\mu \neq 0$. Systems are bounded and we conclude from repeated application of Lemma 3.1.5 that the cascade (3.17)-(3.18) is GUAS if and only if (3.22) is GUAS. ■

3.1.5 First-order systems

We will see later that GUEC can always be achieved for a homogeneous network of continuous-time systems. For discrete-time systems, GUAC can always be achieved if systems are at least marginally stable. However, GUAC cannot always be achieved for networks of discrete-time systems. We will illustrate this limitation for a network of first-order systems. We present conditions which are necessary and sufficient for time-invariant systems.

A closed-loop network of first order contintuous-time or discrete-time systems achieves GUAC if the systems

$$\dot{z}_i(t) = (a - \mu\, K)\, z_i(t) \qquad \text{or} \qquad z_i(k+1) = (a - \mu\, K)\, z_i(k)$$

respectively, are GUAS for all $\mu \in \Lambda_L$ and Λ_L is the set of all non-zero Laplacian eigenvalues. We wish to identify $K \in \mathbb{R}$ such that the closed-loop network achieves GUAC.

In continuous-time, GUAC is achieved if $Re(a - \mu\, K) < 0$ for all $\mu \in \Lambda_L$. Since $\alpha > 0$ for all $\mu \in \Lambda_L$ (Fact 2.3.1), it is clear that such a K always exists and, recalling α_m from (2.9), K will be given by

$$K \geq 0\,, \qquad K > a/\alpha_m\,.$$

In discrete-time, GUAC is achieved if $|a - \mu\, K|^2 < 1$ for all $\mu \in \Lambda_L$ where

$$|a - \mu\, K|^2 = a^2 - 2\,\alpha\, a\, K + |\mu|^2\, K^2 = |\mu|^2 \left(K - \frac{\alpha\, a}{|\mu|^2}\right)^2 + a^2 \left(1 - \frac{\alpha^2}{|\mu|^2}\right) \geq \frac{\omega^2\, a^2}{|\mu|^2}\,.$$

Thus, if GUAC is achieved, then $a^2\omega^2/|\mu|^2 < 1$, that is,

$$(a^2 - 1)\, \omega^2 < \alpha^2\,.$$

Suppose systems are unstable, that is $|a| > 1$, then GUAC cannot be achieved if

$$\left(\frac{\omega}{\alpha}\right)^2 \geq \frac{1}{a^2 - 1}$$

for some $\mu \in \Lambda_L$.

Remark 3.1.2 *The simple first-order example already shows that there are some* $\mathbf{A}$, $\mathbf{B}$, *and* $\mu \in \mathbb{C}$ *for which* $\mathbf{A} - \mu\,\mathbf{BK}$ *cannot be stabilized with a real valued* $\mathbf{K}$.

3.2 Linear time-invariant systems

In this section, we will present pre-liminary results for networks of linear time-invariant systems where we make use of the following definitions.

Definition 3.2.1 *A polynomial is* Hurwitz *(continuous-time) or* Schur *(discrete-time) if and only if all its roots have negative real part (Hurwitz) or magnitude less than one (Schur), respectively.*

Definition 3.2.2 *A matrix is* Hurwitz *(continuous-time) or* Schur *(discrete-time) if and only if all its eigenvalues have negative real part (Hurwitz) or magnitude less than one (Schur), respectively.*

Definition 3.2.3 *In the time-invariant case, a polynomial is* asymptotically stable *if and only if it is Hurwitz (continuous-time) or Schur (discrete-time), respectively.*

Definition 3.2.4 *In the time-invariant case, a matrix is* asymptotically stable *if and only if it is Hurwitz (continuous-time) or Schur (discrete-time), respectively.*

Corollary 3.2.1 *If systems are time-invariant, then the closed-loop network* (2.1)-(2.2) *or* (2.5)-(2.6) *achieves GUAC if and only if for each non-zero eigenvalue μ of the graph Laplacian,*

- *Continuous-time:* $\mathbf{A} - \mu\,\mathbf{B}\,\mathbf{K}$ *is Hurwitz [29].*
- *Discrete-time:* $\mathbf{A} - \mu\,\mathbf{B}\,\mathbf{K}$ *is Schur [30].*

3.2.1 Stabilizable linear time-invariant systems

Suppose (A, B) is not controllable but stabilizable. Then, there is a nonsingular matrix $\mathbf{T}$ such that

$$\hat{\mathbf{A}} := \mathbf{T}^{-1}\,\mathbf{A}\,\mathbf{T} = \begin{bmatrix} \mathbf{A}_c & \mathbf{A}_{cu} \\ \mathbf{0} & \mathbf{A}_u \end{bmatrix} \qquad \text{and} \qquad \hat{\mathbf{B}} := \mathbf{T}^{-1}\,\mathbf{B} = \begin{bmatrix} \mathbf{B}_c \\ \mathbf{0} \end{bmatrix}.$$

The pair (A_c, B_c) is controllable and A_u is asymptotically stable. Let

$$\mathbf{K} = \hat{\mathbf{K}}\,\mathbf{T}^{-1} \qquad \text{where} \qquad \hat{\mathbf{K}} = \begin{bmatrix} \mathbf{K}_c & \mathbf{0} \end{bmatrix}. \tag{3.23}$$

Then

$$\mathbf{A} - \mu\,\mathbf{B}\mathbf{K} = \mathbf{T}\left(\hat{\mathbf{A}} - \mu\,\hat{\mathbf{B}}\hat{\mathbf{K}}\right)\mathbf{T}^{-1}$$

and

$$\hat{\mathbf{A}} - \mu\,\hat{\mathbf{B}}\hat{\mathbf{K}} = \begin{bmatrix} \mathbf{A}_c - \mu\,\mathbf{B}_c\mathbf{K}_c & \mathbf{A}_{cu} \\ \mathbf{0} & \mathbf{A}_u \end{bmatrix}.$$

Since $\mathbf{A} - \mu\,\mathbf{B}\mathbf{K}$ and $\hat{\mathbf{A}} - \mu\,\hat{\mathbf{B}}\hat{\mathbf{K}}$ are similar, they have the same characteristic polynomial; hence

$$\begin{aligned} \det(s\mathbf{I} - \mathbf{A} + \mu\,\mathbf{B}\mathbf{K}) &= \det(s\mathbf{I} - \hat{\mathbf{A}} + \mu\,\hat{\mathbf{B}}\hat{\mathbf{K}}) \\ &= \det(s\mathbf{I} - \mathbf{A}_c + \mu\,\mathbf{B}_c\mathbf{K}_c)\,c_u(s) \end{aligned} \tag{3.24}$$

where $c_u(s) = \det(s\mathbf{I} - \mathbf{A}_u)$ is asymptotically stable. One can now use the results on controllable systems to obtain a matrix $\mathbf{K}_c$ such that $\mathbf{A}_c - \mu\,\mathbf{B}_c\mathbf{K}_c$ is asymptotically stable for all μ. Then, with $\mathbf{K}$ given by (3.23) the matrix $\mathbf{A} - \mu\,\mathbf{B}\mathbf{K}$ is asymptotically stable for all μ.

3.2.2 Single-input linear time-invariant systems

For GUAC, it is necessary and sufficient that $\mathbf{A} - \mu\,\mathbf{B}\mathbf{K}$ be asymptotically stable for all non-zero eigenvalues μ of the graph Laplacian. This is equivalent to the requirement that, for each non-zero eigenvalue μ of the graph Laplacian, the characteristic polynomial of $\mathbf{A} - \mu\,\mathbf{B}\mathbf{K}$,

$$d_\mu(s) = \det(s\mathbf{I} - \mathbf{A} + \mu\,\mathbf{B}\mathbf{K})$$

is Hurwitz (continuous-time) or Schur (discrete-time).

Our first result tells us that, in the single-input case, d_μ depends in a linear affine fashion on μ.

Lemma 3.2.1 *Suppose* $\mathbf{A} \in \mathbb{C}^{n\times n}$, $\mathbf{B} \in \mathbb{C}^{n\times 1}$, *and* $\mathbf{K} \in \mathbb{C}^{1\times n}$. *Then*

$$\det(s\mathbf{I} - \mathbf{A} + \mu\,\mathbf{BK}) = c(s) + \mu\,\gamma(s) \tag{3.25}$$

where $c(s) = \det(s\mathbf{I} - \mathbf{A})$ *and* γ *is a polynomial whose degree is less than* n.

Proof When s is not an eigenvalue of $\mathbf{A}$, we use Sylvester's determinant identity to obtain that

$$\begin{aligned}\det(s\mathbf{I} - \mathbf{A} + \mu\mathbf{BK}) &= \det(s\mathbf{I} - \mathbf{A})\,\det\left(\mathbf{I} + \mu\,(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{BK}\right) \\ &= \det(s\mathbf{I} - \mathbf{A})\,\det\left(1 + \mu\,\mathbf{K}\,(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\right) \\ &= \det(s\mathbf{I} - \mathbf{A})\,\left(1 + \mu\,\mathbf{K}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\right)\end{aligned}$$

Hence (3.25) holds with

$$\gamma(s) = \det(s\mathbf{I} - \mathbf{A})\,\mathbf{K}\,(s\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{B}\,. \tag{3.26}$$

It follows from (3.26) that γ is a polynomial whose degree is at most $n - 1$. ∎

From the above result, we see that it may be possible to reduce the problem of obtaining $\mathbf{K}$ such that $\mathbf{A} - \mu\,\mathbf{BK}$ is Hurwitz or Schur for every non-zero μ to the problem of obtaining a polynomial γ of degree less than n such that $c + \mu\,\gamma$ is Hurwitz or Schur, respectively, for every non-zero μ. Having found such a polynomial how does one obtain $\mathbf{K}$? It follows from (3.25) that

$$\det(s\mathbf{I} - \mathbf{A} + \mathbf{BK}) = c(s) + \gamma(s)\,.$$

If (A, B) is controllable, then one could use pole placement techniques to obtain $\mathbf{K}$ given c and γ. First, we identify the relation between the parameters.

Lemma 3.2.2 *Suppose* $\mathbf{A} \in \mathbb{C}^{n\times n}$, $\mathbf{B} \in \mathbb{C}^{n\times 1}$, *and* $\mathbf{K} \in \mathbb{C}^{1\times n}$. *Then*

$$\det(s\mathbf{I} - \mathbf{A} + \mathbf{B}\mathbf{K}) = c(s) + \gamma(s) \tag{3.27}$$

where

$$c(s) = \det(s\mathbf{I} - \mathbf{A}) = \sum_{i=0}^{n} c_i \, s^i \,, \qquad c_n = 1 \tag{3.28}$$

and

$$\gamma(s) = \sum_{i=0}^{n-1} \gamma_i \, s^i \,, \qquad \gamma_i = \mathbf{K} \left[\sum_{j=i+1}^{n} c_j \, \mathbf{A}^{j-i-1} \right] \mathbf{B} \,. \tag{3.29}$$

Proof From (3.26),

$$\gamma(s) = \det(s\mathbf{I} - \mathbf{A}) \, \mathbf{K} \, (s\mathbf{I} - \mathbf{A})^{-1} \, \mathbf{B} \,.$$

The matrix $(s\mathbf{I} - \mathbf{A})^{-1}$ can be expressed by the power series

$$(s\mathbf{I} - \mathbf{A})^{-1} = \sum_{k=0}^{\infty} s^{-(k+1)} \, \mathbf{A}^k$$

and recalling that $\det(s\mathbf{I} - \mathbf{A})$ is given by (3.28) yields

$$\gamma(s) = \mathbf{K} \sum_{j=0}^{n} c_j \left(\sum_{k=0}^{\infty} s^{j-k-1} \, \mathbf{A}^k \, \mathbf{B} \right) .$$

Comparing the coefficients of s^i, for $i = 0, \cdots, n-1$, on both sides of the above equation yields

$$\gamma_i = \mathbf{K} \left[\sum_{j=i+1}^{n} c_j \, \mathbf{A}^{j-i-1} \right] \mathbf{B} \,.$$

∎

Remark 3.2.1 *The open-loop dynamics of the transformed network* (2.3) *are given by the characteristic polynomial of* $\mathbf{A} + \alpha_0 \mathbf{I}$, *that is,*

$$\tilde{c}(s) = \det(s\mathbf{I} - \mathbf{A} - \alpha_0 \mathbf{I}) = \det\left((s - \alpha_0)\mathbf{I} - \mathbf{A} \right) = c(s - \alpha_0)$$

Hence, $\tilde{c}(s)$ *can be obtained by replacing* s *in* (3.28) *by* $s - \alpha_0$.

Remark 3.2.2 *The open-loop dynamics of the transformed network (2.7) are given by the characteristic polynomial of* $\rho^{-1}\mathbf{A}$*, that is,*

$$\tilde{c}(s) = \det\left(s\mathbf{I} - \rho^{-1}\mathbf{A}\right) = \rho^{-n} \det\left(s\rho\,\mathbf{I} - \mathbf{A}\right) = \rho^{-n}\, c\,(s\rho)\ .$$

Hence, $\tilde{c}(s)$ *can be obtained by replacing* s *in (3.28) by* $s\rho$ *and multiplying the resulting polynomial by* ρ^{-n}*.*

Now, if c and γ are given, then the following corollary provides an explicit expression for $\mathbf{K}$.

Corollary 3.2.2 *Suppose* (A, B) *is controllable with* $\mathbf{A} \in \mathbb{C}^{n\times n}$*,* $\mathbf{B} \in \mathbb{C}^{n\times 1}$*. Then,* $\mathbf{K} \in \mathbb{C}^{1\times n}$ *satisfies (3.27) if and only if, recalling* c *and* γ *from (3.28)-(3.29),*

$$\boxed{\mathbf{K} = \begin{bmatrix} \gamma_0 & \gamma_1 & \cdots & \gamma_{n-1} \end{bmatrix} \left(\mathbf{Q}_c \boldsymbol{\Upsilon}\right)^{-1}} \tag{3.30}$$

where the invertible matrix $\boldsymbol{\Upsilon}$ *is given by*

$$\boldsymbol{\Upsilon} = \begin{bmatrix} c_1 & c_2 & \cdots & c_{n-2} & c_{n-1} & 1 \\ c_2 & c_3 & \cdots & c_{n-1} & 1 & 0 \\ c_3 & c_4 & \cdots & 1 & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ c_{n-1} & 1 & \cdots & 0 & 0 & 0 \\ 1 & 0 & \ldots & 0 & 0 & 0 \end{bmatrix} \tag{3.31}$$

and $\mathbf{Q}_c$ *is the controllability matrix associated with* (A, B)*, that is,*

$$\mathbf{Q}_c = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \cdots & \mathbf{A}^{n-1}\mathbf{B} \end{bmatrix}.$$

Proof From (3.29), we note that

$$\begin{bmatrix} \mathbf{KB} & \mathbf{KAB} & \cdots & \mathbf{KA}^{n-1}\mathbf{B} \end{bmatrix} \boldsymbol{\Upsilon} = \begin{bmatrix} \gamma_0 & \gamma_1 & \cdots & \gamma_{n-1} \end{bmatrix} \tag{3.32}$$

which is equivalent to

$$\mathbf{K}\mathbf{Q}_c\boldsymbol{\Upsilon} = \begin{bmatrix} \gamma_0 & \gamma_1 & \cdots & \gamma_{n-1} \end{bmatrix}$$

where $\boldsymbol{\Upsilon}$ is the invertible matrix given in (3.31) and $\mathbf{Q}_c$ is the controllability matrix for (A, B) that is invertible since the pair (A, B) is controllable. This readily implies that the gain matrix $\mathbf{K}$ is uniquely given by (3.30). ■

If systems are in controllable canonical form, then one can easily show that $\gamma_i = K_i$.

4. CONSENSUS FOR CONTINUOUS-TIME SECOND-ORDER SYSTEMS

Here we consider a homogeneous network of linear time-invariant second-order systems. In this case, (2.1) is given by

$$\dot{\mathbf{x}}_i(t) = \mathbf{A}\,\mathbf{x}_i(t) + \mathbf{B}\,u_i(t) + \mathbf{u}_0(t) \tag{4.1}$$

where $\mathbf{x}_i(t) \in \mathbb{R}^2$, $\mathbf{A} \in \mathbb{R}^{2\times 2}$, $\mathbf{B} \in \mathbb{R}^{2\times 1}$, a linear control $u_i(t) \in \mathbb{R}$ is given by

$$u_i(t) = \mathbf{K} \sum_{j \in N_i} w_{ij} \left[\mathbf{x}_j(t) - \mathbf{x}_i(t)\right] \tag{4.2}$$

and the special system parameter (3.28)-(3.29) reduce to

$$c(s) = \det\left(s\,\mathbf{I} - \mathbf{A}\right) = s^2 + c_1\,s + c_0 \tag{4.3}$$

and

$$[\gamma_0 \quad \gamma_1] = \mathbf{K}\left[(\mathbf{A} + c_1\,\mathbf{I})\,\mathbf{B} \quad \mathbf{B}\right]. \tag{4.4}$$

If (A, B) is in controllable canonical form, then one can easily show that $[\gamma_0\ \gamma_1] = \mathbf{K}$.

4.1 Necessary and sufficient conditions for consensus

4.1.1 Conditions for consensus

The following lemma provides conditions that are necessary and sufficient for the closed-loop network (4.1)-(4.2) to achieve GUAC.

Lemma 4.1.1 *The closed-loop network (4.1)-(4.2) achieves GUAC if and only if, for each non-zero eigenvalue $\mu_l = \alpha_l + \jmath\,\omega_l$ of the graph Laplacian matrix,*

$$\delta_l := c_1 + \alpha_l\,\gamma_1 > 0 \tag{4.5}$$

$$(c_0\,\alpha_l + |\mu_l|^2\,\gamma_0)\,\delta_l^2 - c_1\,\omega_l^2\,\gamma_0\,\delta_l - \alpha_l\,\omega_l^2\,\gamma_0^2 > 0 \tag{4.6}$$

where c_0, c_1, γ_0, and γ_1 are given by (4.3)-(4.4).

[31] developed a similar result. Here we treat c_0, c_1 as fixed parameters. A simple and new proof of Lemma 4.1.1 is given in Section 4.4.

Remark 4.1.1 *If a Laplacian eigenvalue μ_l is real, then $\mu_l = \alpha_l$, $\omega_l = 0$, and conditions (4.5)-(4.6) in Lemma 4.1.1 simplify to*

$$c_1 + \mu_l \gamma_1 > 0 \quad \text{and} \quad c_0 + \mu_l \gamma_0 > 0\,. \tag{4.7}$$

Lemma 4.1.1 provides conditions on the gains γ_0 and γ_1. If (A, B) is controllable, then the matrix $[(\mathbf{A} + c_1\,\mathbf{I})\,\mathbf{B} \quad \mathbf{B}]$ is invertible and $\mathbf{K}$ is given by

$$\mathbf{K} = [(\mathbf{A} + c_1\,\mathbf{I})\,\mathbf{B} \quad \mathbf{B}]^{-1}\,[\gamma_0 \quad \gamma_1]\,.$$

Section 4.2.3 will show that for stabilizable systems, provided the graph has a spanning tree, *the inequalities in Lemma 4.1.1 can always be satisfied, regardless of the network under consideration, by choosing* $\mathbf{K}$ *appropriately. In particular, the results are not restricted to undirected networks.*

4.1.2 Guaranteed convergence rate

Recalling Lemma 2.1.1, we can use Lemma 4.1.1 above to obtain a guaranteed convergence rate α_0. From Remark 3.2.1, the characteristic polynomial of $\tilde{\mathbf{A}} = \mathbf{A} + \alpha_0\,\mathbf{I}$ is given by $\tilde{c}(s) = s^2 + \tilde{c}_1\,s + \tilde{c}_0$ where

$$\boxed{\tilde{c}_0 = \alpha_0^2 - c_1\,\alpha_0 + c_0, \quad \tilde{c}_1 = c_1 - 2\,\alpha_0\,.} \tag{4.8}$$

Recalling Remark 2.1.2 and noting that $\tilde{\mathbf{A}} + \tilde{c}_1\mathbf{I} = \mathbf{A} + (c_1 - \alpha_0)\,\mathbf{I}$, we proceed as in (4.4) and define

$$\boxed{[\tilde{\gamma}_0 \quad \tilde{\gamma}_1] = \mathbf{K}\,\big[(\mathbf{A} + (c_1 - \alpha_0)\;\mathbf{I})\mathbf{B} \quad \mathbf{B}\big]\,.} \tag{4.9}$$

Now, Lemmas 2.1.1 and 4.1.1 yield the following corollary.

Corollary 4.1.1 *The closed-loop network (4.1)-(4.2) achieves GUEC with rate* $\alpha_0 > 0$ *if for each non-zero eigenvalue* $\mu_l = \alpha_l + \jmath\,\omega_l$ *of the graph Laplacian,*

$$\tilde{\delta}_l := \tilde{c}_1 + \alpha_l\,\tilde{\gamma}_1 > 0 \tag{4.10}$$

$$(\tilde{c}_0\,\alpha_l + |\mu_l|^2\tilde{\gamma}_0)\,\tilde{\delta}_l^2 - \tilde{c}_1\omega_l^2\tilde{\gamma}_0\,\tilde{\delta}_l - \alpha_l\omega_l^2\tilde{\gamma}_0^2 > 0 \tag{4.11}$$

where $\tilde{c}_0$, $\tilde{c}_1$, $\tilde{\gamma}_0$, *and* $\tilde{\gamma}_1$ *are given by (4.8)-(4.9).*

If (A, B) is controllable, then, provided the graph has a spanning tree, any guaranteed rate of convergence α_0 can be achieved by appropriate choice of $\mathbf{K}$. This will be shown in Section 4.2.

Remark 4.1.2 *If* (A, B) *is in controllable canonical form, then one can easily show that*

$$\mathbf{K} = [\tilde{\gamma}_0 + \tilde{\gamma}_1\,\alpha_0 \quad \tilde{\gamma}_1]\,.$$

4.1.3 Networks of double integrators

A network of double integrators is a special case of (4.1) with $c_0 = c_1 = 0$. In this case, conditions (4.5)-(4.6) in Lemma 4.1.1 simplify to

$$\alpha_l\,\gamma_1 > 0 \qquad \text{and} \qquad \alpha_l^2\,|\mu_l|^2\,\gamma_0\,\gamma_1^2 - \alpha_l\,\omega_l^2\,\gamma_0^2 > 0\,.$$

For non-zero μ_l, we have $\alpha_l > 0$ from Fact 2.3.1. In this case, $\gamma_0 > 0$ from the second inequality; and in addition to a spanning tree,

$$\frac{\gamma_1^2}{\gamma_0} > \frac{\omega_l^2}{\alpha_l\,|\mu_l|^2} \qquad \text{and} \qquad \gamma_1 > 0$$

for all non-zero μ_l provide necessary and sufficient conditions to achieve GUAC. [28] and [15] obtained a similar result but assumed that $\gamma_0, \gamma_1 > 0$. Here we did not assume $\gamma_0, \gamma_1 > 0$. However, we showed that it is necessary.

Remark 4.1.3 *If we wish to achieve GUEC with $\alpha_0 > 0$, then K should be chosen according to (4.9) where $\tilde{\gamma}_0$ and $\tilde{\gamma}_1$ satisfy (4.10)-(4.11) with*

$$\tilde{c}_0 = \alpha_0^2 \quad \textit{and} \quad \tilde{c}_1 = -2\alpha_0 \,.$$

4.2 Obtaining gains

Lemma 4.1.1 provides necessary and sufficient conditions on the gains γ_0, γ_1 for a closed-loop network (4.1)-(4.2) to achieve GUAC. Satisfying inequalities (4.5)-(4.6) for all $\mu_l \neq 0$ is equivalent to simultaneously stabilizing $N-1$ polynomials, where inequality (4.6) is nonlinear in γ_0, γ_1 and needs to be analyzed numerically, especially if the number of systems N is large.

This motivates different methods for choosing gains. We start with a high-gain approach, which is easy to implement. Next, we develop lower bounds on the gains, which guarantee that all $N-1$ polynomials are stable. Then, we show that GUAC can always be achieved if the pair (A, B) is stabilizable. Finally, we consider output feedback.

In the following, sufficient conditions to achieve GUAC are established by simplifying conditions (4.5)-(4.6) in Lemma 4.1.1. All eigenvalues $\mu_l = \alpha_l + \jmath\omega_l$ of the Laplacian L are labeled so that $\mu_1 = 0$ and $\alpha_l > 0$ for $l > 1$, which can be done since it is assumed that G contains a spanning tree (Fact 2.3.1).

4.2.1 High gain

A very simple method to choose gains is to fix their relationship and make them large.

Corollary 4.2.1 *Consider the closed-loop network (4.1)-(4.2) where, for some $\kappa > 0$, the gains in (4.4) are given by*

$$\gamma_1 = \kappa\,\gamma_0 \,.$$

The network achieves GUAC if γ_0 is sufficiently large.

Proof Suppose gains are related by $\gamma_1 = \kappa\,\gamma_0$ for any fixed $\kappa > 0$; then the choice of gains reduces to a single design variable γ_0. Condition (4.5) is rearranged to

$$\gamma_0 > -c_1/(\kappa\,\alpha_l)$$

for all $l > 1$ and is satisfied for large γ_0. Furthermore, the left-hand side of inequality (4.6) becomes cubic in γ_0 where the coefficient of γ_0^3 is $|\mu_l|^2\alpha_l^2\kappa^2$, which is positive. Hence, condition (4.6) is satisfied for large γ_0 as well. It follows from Lemma 4.1.1 that GUAC is achieved when γ_0 is sufficiently large. ∎

4.2.2 Simpler gain conditions

The next method simplifies the process of choosing gains by providing explicit lower bounds on γ_0 and γ_1.

Theorem 4.2.1 *The closed-loop network (4.1)-(4.2) achieves GUAC if gains γ_0, γ_1 given by (4.4) satisfy*

$$\gamma_0 > -\frac{\alpha_l\,c_0}{|\mu_l|^2}, \qquad \gamma_1 > g_l - \frac{c_1}{\alpha_l} \tag{4.12}$$

for all $l > 1$ where

$$g_l = \frac{c_1\,\gamma_0\,\tau_l}{2\,\alpha_l} + \sqrt{\left(\frac{c_1\,\gamma_0\,\tau_l}{2\,\alpha_l}\right)^2 + \frac{\tau_l\,\gamma_0^2}{\alpha_l}} \tag{4.13}$$

$$\tau_l = \frac{\omega_l^2}{c_0\,\alpha_l + |\mu_l|^2\,\gamma_0}\,. \tag{4.14}$$

Proof The first inequality in (4.12) implies that

$$c_0\,\alpha_l + |\mu_l|^2\,\gamma_0 > 0\,. \tag{4.15}$$

Now, with $\tau_l := \frac{\omega_l^2}{c_0\,\alpha_l+|\mu_l|^2\,\gamma_0} \geq 0$, inequality (4.6) of Lemma 4.1.1 is equivalent to

$$f_l(\delta_l) := \delta_l^2 - (c_1\,\gamma_0\,\tau_l)\,\delta_l - \alpha_l\,\tau_l\,\gamma_0^2 > 0\,.$$

The second-order polynomial f_l has two real roots; the biggest root being

$$r_l := \frac{c_1\,\gamma_0\,\tau_l}{2} + \sqrt{\left(\frac{c_1\,\gamma_0\,\tau_l}{2}\right)^2 + \alpha_l\,\tau_l\,\gamma_0^2} \geq 0\,.$$

Since $\lim_{\delta_l \to \infty} f_l(\delta_l) = \infty$, we have $f_l(\delta_l) > 0$ for $\delta_l > r_l$. Thus, (4.5) and (4.6) hold provided $\delta_l > r_l$, that is $\gamma_1 > g_l - c_1/\alpha_l$. ■

[31] presented Theorem 4.2.1 for the special case of $c_1 \geq 0$ and $\gamma_0 \geq 0$. In this case, the conditions in Theorem 4.2.1 are not only sufficient but also necessary to achieve GUAC. We will show in Section 4.5 that the necessity of the conditions in Theorem 4.2.1 is actually very common. Some of the cases are indicated in the remark below.

Remark 4.2.1 *The conditions in Theorem 4.2.1 are necessary if*

1. *all μ_l are real. If $\omega_l = 0$, then τ_l and g_l are 0, and Remark 4.1.1 is recovered.*
2. *$c_0 = c_1 = 0$. This case reduces to the double-integrator, that is, $\gamma_1^2 > \frac{\omega_l^2}{\alpha_l\,|\mu_l|^2}\,\gamma_0$.*
3. *$c_0 \geq 0$ and $c_1 < 0$. If $c_1 < 0$, then from (4.6), either $\gamma_0 > 0$ or (4.15) has to hold. $\gamma_0 > 0$ and $c_0 \geq 0$ imply (4.15). Hence, (4.15) holds regardless.*

4.2.3 Stabilizable systems

Corollary 4.2.1 demonstrates the sufficiency of a simple linear controller to achieve GUAC. However, in order to obtain $\mathbf{K}$ for arbitrary γ_0 and γ_1, relation (4.4) has to be invertible, which is the case if (A, B) is controllable. Here we will show that GUAC can always be achieved if the systems are only stabilizable and the graph G contains a spanning tree.

Remark 4.2.2 *If (A, B) is uncontrollable, then rank$[\mathbf{AB} \quad \mathbf{B}] < 2$. Hence, either $\mathbf{B} = \mathbf{0}$ or $\mathbf{B}$ is an eigenvector of $\mathbf{A}$. If $\mathbf{B} = \mathbf{0}$, then GUAC is achieved if and only if $\mathbf{A}$ is stable, that is, (A, B) is stabilizable. If $\mathbf{B}$ is an eigenvector of $\mathbf{A}$ and λ_c is the controllable eigenvalue, then $\mathbf{AB} = \lambda_c\,\mathbf{B}$. Since the matrices $\mathbf{A}$ and $\mathbf{B}$ are real, λ_c and therefore the uncontrollable eigenvalue λ_u are real. Furthermore, $c_1 = -(\lambda_u + \lambda_c)$ and (4.4) reduces to*

$$\gamma_0 = -\lambda_u\,\mathbf{KB}\,, \quad \gamma_1 = \mathbf{KB}\,. \tag{4.16}$$

If (A, B) is stabilizable, then $\lambda_u < 0$ and (4.16) reduces to $\gamma_1 = \kappa\,\gamma_0$ where $\kappa = -1/\lambda_u > 0$. From Corollary 4.2.1, GUAC is achieved if γ_0 is sufficiently large. $\mathbf{K}$ is chosen such that $\mathbf{KB} = \gamma_1$.

Remark 4.2.3 *Applying the reasoning of Remark 4.2.2 to (4.9) yields*

$$\tilde{\gamma}_1 = \kappa\tilde{\gamma}_0\,, \qquad \kappa = -1/(\lambda_u + \alpha_0)\ .$$

Then, GUEC with rate α_0 can be achieved if $\lambda_u < -\alpha_0$.

4.2.4 Output feedback

Controllers of the form (4.2) with no restrictions on the structure of $\mathbf{K}$ assume that each agent has knowledge of the full state of all its neighbors. Sometimes we wish to reduce the amount of data transmitted over the network, or only a part of the state of each agent is available. So we consider the situation in which the information available from each agent i is an output

$$y_i = \mathbf{C}\,\mathbf{x}_i \tag{4.17}$$

where $\mathbf{C} \in \mathbb{R}^{2\times 1}$. Considering static output feedback, control u_i is given by

$$\boxed{u_i = k \sum_{j\in N_i} w_{ij}\left[y_j - y_i\right]} \tag{4.18}$$

where $k \in \mathbb{R}$. These controllers are in the form of (4.2) with

$$\mathbf{K} = k\,\mathbf{C}\,. \tag{4.19}$$

Let

$$G(s) = \frac{b_1\,s + b_0}{s^2 + c_1\,s + c_0} \tag{4.20}$$

be the transfer function associated with each agent. Let (A, B, C) be a realization of G with (A, B) in controllable canonical form; thus $G(s) = \mathbf{C}\,(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}$. Then $\mathbf{C} = [b_0 \quad b_1]$ and $\mathbf{K} = [\gamma_0 \quad \gamma_1]$. Hence,

$$\boxed{\gamma_0 = k\,b_0 \quad \text{and} \quad \gamma_1 = k\,b_1\,.} \tag{4.21}$$

If $b_1 \neq 0$, then the transfer function has a single zero at $z = -b_0/b_1$. Otherwise it has no zeros. We need to consider the following cases:

1. ($z < 0$) Here $\gamma_1 = \kappa\,\gamma_0$ where $\kappa = b_1/b_0 > 0$, and a controller can be obtained by using Corollary 4.2.1 to obtain γ_0; then k is given by $k = \gamma_0/b_0$.

2. ($z = 0$, $b_1 \neq 0$) Here $b_0 = 0$; hence $\gamma_0 = 0$ and conditions (4.5)-(4.6) reduce to

$$c_0 > 0 \quad \text{and} \quad \gamma_1 > -c_1/\alpha_l$$

   Hence, provided $c_0 > 0$, one can obtain γ_1 to satisfy (4.5)-(4.6); then $k = \gamma_1/b_1$.

3. ($z > 0$) Here one needs to substitute the expressions for γ_0 and γ_1 into (4.5)-(4.6) to obtain two set of inequalites in k; a controller exists if these inequalities have a solution for k.

4. ($b_1 = 0$, $b_0 > 0$) Here $\gamma_1 = 0$ and conditions (4.5)-(4.6) reduce to

$$c_1 > 0 \quad \text{and} \quad (c_0 + \alpha_l\, b_0\, k)\, c_1^2 - \omega_l^2\, b_0^2\, k^2 > 0 \tag{4.22}$$

   Let Λ be the set of all non-zero Laplacian eigenvalues and for each $\mu \in \Lambda$ with $\omega \neq 0$ let $\nu = \alpha/\omega$ where $\mu = \alpha + \jmath\omega$. Then, the second inequality in (4.22) is satisfied for all $l > 1$ if and only if

$$\nu^2 \geq -4\, c_0\, /\, c_1^2 \;\; \text{for all } \mu \in \Lambda \text{ with } \omega > 0 \quad \text{and} \quad \underline{k} < \overline{k} \tag{4.23}$$

   where

$$\overline{k} = \inf\left\{ \left(\nu + \sqrt{\nu^2 + 4\,\frac{c_0}{c_1^2}}\right) \frac{c_1^2}{2\,\omega\, b_0} \;\; \mu \in \Lambda,\, \omega > 0 \right\} \tag{4.24}$$

$$\underline{k} = \max\{\underline{k}_1,\, \underline{k}_2\}\,, \quad \underline{k}_1 = \sup\left\{ \frac{-c_0}{\alpha\, b_0} \;\; \mu \in \Lambda,\, \omega = 0 \right\}, \tag{4.25}$$

$$\underline{k}_2 = \sup\left\{ \left(\nu - \sqrt{\nu^2 + 4\,\frac{c_0}{c_1^2}}\right) \frac{c_1^2}{2\,\omega\, b_0} \;\; \mu \in \Lambda,\, \omega > 0 \right\} \tag{4.26}$$

   The control gains yielding desired behavior are given by k where

$$\underline{k} < k < \overline{k}\,. \tag{4.27}$$

5. ($b_1 = 0$, $b_0 < 0$) Here we can use the design for $b_1 = 0, b_0 > 0$ if we multiply b_0 and k by -1.

Remark 4.2.4 *If GUAC cannot be achieved via static output feedback control, then assuming that (C, A) is detectable, it is always possible to use an observer based design. Each agent i employs an observer to obtain an estimate $\hat{\mathbf{x}}_i$ of its state $\mathbf{x}_i$ and supplies this to its neighbors. Then, u_i is given by*

$$u_i = \mathbf{K} \sum_{j \in N_i} w_{ij} \left[\hat{\mathbf{x}}_j - \hat{\mathbf{x}}_i \right] \tag{4.28}$$

where $\mathbf{K}$ is designed as before for state feedback. The error dynamics of the observers decouple from the overall system dynamics, and GUAC is achieved with dynamic output feedback controllers.

4.3 Robust consensus control

Necessary and sufficient conditions for the closed-loop network (4.1)-(4.2) to achieve GUAC are given by (4.5)-(4.6). These conditions are based on the eigenvalues of the Laplacian matrix associated with the network graph. In practice it is possible that the communication structure changes, e.g. when systems enter or leave the network, that is, vertices get added or removed from the graph. Further, the system matrices $\mathbf{A}$ and $\mathbf{B}$ could be approximated, their true value may be unknown, or we may wish that the controller work for a range of $\mathbf{A}$ and $\mathbf{B}$ matrices. Then, GUAC must be achieved in a robust way.

4.3.1 Robustness with respect to changes in graph

First, we assume that the system matrices $\mathbf{A}$ and $\mathbf{B}$ are known and that the only uncertainty lies in the communication structure, which is fixed. For a fixed $\alpha_m > 0$, we consider the set of weighted graphs G whose Laplacians L satisfy

$$\boxed{\mu_1 = 0 \quad \text{and} \quad 0 < \alpha_m \leq \alpha_l} \tag{4.29}$$

for all $l > 1$, i.e., the Laplacian has one zero eigenvalue and the real part of all its other eigenvalues are bounded below by α_m. One can estimate α_m without global knowledge of the directed and weighted graph G using a distributed algorithm, e.g. [71].

Then, conditions to robustly control a network of second-order systems are as follows.

Theorem 4.3.1 *The closed-loop network (4.1)-(4.2) achieves GUAC for any graph satisfying (4.29) if and only if*

$$\gamma_0,\, \gamma_1 \geq 0 \tag{4.30}$$

$$\gamma_0 > -c_0/\alpha_m \tag{4.31}$$

$$\begin{aligned} \gamma_1 &\geq \tfrac{-c_1+\sqrt{c_1^2+4\,\alpha_m\,\gamma_0}}{2\,\alpha_m} \quad &\text{if} \quad \gamma_0 > 0 \\ \gamma_1 &> -c_1/\alpha_m \quad &\text{if} \quad \gamma_0 = 0 \end{aligned} \tag{4.32}$$

where c_0, c_1, γ_0 and γ_1 are given by (4.3)-(4.4).

Proof We prove this theorem using Lemma 4.1.1. First, we note that (4.29) requires that $\mu_1 = 0$ and $0 < \alpha_l$ for all $l > 1$. This is equivalent to requiring that the graph G contains a spanning tree (Fact 2.3.1). From Lemma 4.1.1, we see that GUAC is achieved for all graphs satisfying (4.29) if and only if conditions (4.5)-(4.6) hold for all $\omega_l \in \mathbb{R}$ and $\alpha_l \geq \alpha_m$. Using the definition of δ_l in (4.5) and $\alpha_l > 0$, condition (4.6) is equivalent to

$$\gamma_0\,(\delta_l\,\gamma_1 - \gamma_0)\,\omega_l^2 + (c_0 + \alpha_l\,\gamma_0)\,\delta_l^2 > 0\,. \tag{4.33}$$

With $\delta_l > 0$, condition (4.33) holds for all ω_l if and only if

$$c_0 + \alpha_l\,\gamma_0 > 0 \tag{4.34}$$

$$\gamma_0\,(\delta_l\,\gamma_1 - \gamma_0) \geq 0\,. \tag{4.35}$$

Conditions (4.5) and (4.34) hold for all $\alpha_l \geq \alpha_m$ if and only if

$$\gamma_0 \geq 0 \quad \text{and} \quad \gamma_0 > -c_0/\alpha_m \tag{4.36}$$

$$\gamma_1 \geq 0 \quad \text{and} \quad \gamma_1 > -c_1/\alpha_m\,. \tag{4.37}$$

With $\gamma_0 \geq 0$, inequality (4.35) holds for all $\alpha \geq \alpha_m$ if and only if

$$\gamma_0(\alpha_m \gamma_1^2 + c_1 \gamma_1 - \gamma_0) \geq 0\,,$$

that is, $\gamma_0 = 0$ or $\gamma_0 > 0$ and

$$\alpha_m \gamma_1^2 + c_1 \gamma_1 - \gamma_0 \geq 0\,. \tag{4.38}$$

With $\gamma_1 \geq 0$, this last condition is equivalent to

$$\gamma_1 \geq \frac{-c_1 + \sqrt{c_1^2 + 4\,\alpha_m \gamma_0}}{2\,\alpha_m}\,. \tag{4.39}$$

Note that (4.39) implies (4.37) when $\gamma_0 > 0$. ■

4.3.2 Robustness with respect to changes in graph and model uncertainties

Theorem 4.3.1 provides necessary and sufficient conditions on gains γ_0, γ_1 to achieve GUAC over a range of uncertain, but fixed graphs. The controller gain matrix $\mathbf{K}$ is obtained from (4.4), which is possible if (A, B) is controllable and is fixed. Now we consider a range of systems described by

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ b_2 \end{bmatrix} \quad \text{where} \quad a_{12} > 0,\ b_2 > 0\,. \tag{4.40}$$

Uncertain systems satisfying (4.40) are said to satisfy generalized matching conditions [72]. Suppose there exist $\beta_0, \cdots, \beta_3$ such that

$$\boxed{\frac{b_2}{a_{12}} \geq \beta_0, \quad \frac{a_{11}}{a_{12}} \leq \beta_1, \quad \frac{a_{22}}{a_{12}} \leq \beta_2 \quad \text{and} \quad \frac{a_{12}a_{21} - a_{11}a_{22}}{a_{12}\, b_2} \leq \beta_3} \tag{4.41}$$

holds, which is the case if (A, B) lie in a compact set.

Then, we obtain the following result.

Theorem 4.3.2 *Consider any network of systems (4.1) satisfying (4.40)-(4.41) and whose graph satisfies (4.29). Then, there exists k_1, k_2 satisfying*

$$k_2 > \frac{\beta_1 + \beta_2 + \sqrt{(\beta_1 + \beta_2)^2 + \max\{0,\, 4\,\beta_0\,\beta_3\}}}{2\,\alpha_m\,\beta_0} \tag{4.42}$$

$$k_1 > \beta_1\, k_2 + \max\{0,\, \beta_3/\alpha_m\} \tag{4.43}$$

$$k_1 \leq -\beta_2\, k_2 + \alpha_m \beta_0\, k_2^2 \tag{4.44}$$

and, with $\mathbf{K} = \begin{bmatrix} k_1 & k_2 \end{bmatrix}$, *the closed-loop network (4.1)-(4.2) achieves GUAC.*

Proof The existence of k_1 and k_2 satisfying (4.43)-(4.44) is equivalent to the existence of k_2 satisfying

$$\alpha_m\,\beta_0\,k_2^2 - (\beta_1 + \beta_2)\,k_2 - \max\{0,\, \beta_3/\alpha_m\} > 0$$

which is guaranteed by k_2 satisfying (4.42). Hence, there exists k_1, k_2 satisfying (4.42)-(4.44).

We now use Theorem 4.3.1 to show that if k_1, k_2 satisfy (4.42)-(4.44), then $\mathbf{K} = \begin{bmatrix} k_1 & k_2 \end{bmatrix}$ results in GUAC for any network of systems (4.1) satisfying (4.40)-(4.41) and whose graph satisfies (4.29). When $\mathbf{B}$ has the structure given in (4.40) equality (4.4) reduces to

$$\gamma_0 = b_2\,(a_{12}\,k_1 - a_{11}\,k_2) \quad \text{and} \quad \gamma_1 = b_2\,k_2\,. \tag{4.45}$$

Also, with $\mathbf{A}$ given by (4.40) we have

$$c_0 = a_{11}\,a_{22} - a_{12}\,a_{21} \qquad \text{and} \qquad c_1 = -(a_{11} + a_{22})\,.$$

Recalling the proof of Theorem 4.3.1 and (4.38), we see that the hypotheses of that theorem are satisfied if

$$\gamma_1 \geq 0 \tag{4.46}$$

$$\gamma_0 > \max\{0, -c_0/\alpha_m\} \tag{4.47}$$

$$\alpha_m \gamma_1^2 + c_1 \gamma_1 - \gamma_0 \geq 0 \tag{4.48}$$

Substituting for c_0, c_1, γ_0 and γ_1 yields

$$
\begin{aligned}
& b_2\, k_2 \geq 0 \\
& a_{12}\, b_2\, k_1 - a_{11}\, b_2\, k_2 > \max\left\{0, \frac{a_{12}\, a_{21} - a_{11}\, a_{22}}{\alpha_m}\right\} \\
& \alpha_m\, b_2^2\, k_2^2 - a_{22}\, b_2\, k_2 - a_{12}\, b_2\, k_1 \geq 0
\end{aligned}
$$

Since $a_{12},\ b_2 > 0$, the above inequalities are equivalent to

$$k_2 \geq 0 \tag{4.49}$$

$$k_1 > \frac{a_{11}}{a_{12}}\, k_2 + \max\left\{0, \frac{a_{12}\, a_{21} - a_{11}\, a_{22}}{a_{12}\, b_2\, \alpha_m}\right\} \tag{4.50}$$

$$k_1 \leq -\frac{a_{22}}{a_{12}}\, k_2 + \frac{\alpha_m\, b_2}{a_{12}}\, k_2^2 \tag{4.51}$$

and these inequalities are guaranteed by (4.42)-(4.44). ■

Remark 4.3.1 *If* $a_{11} = 0$ *(e.g. if systems are in controllable canonical form), then* γ_0 *is independent of* k_2 *and* (4.42)-(4.44) *are equivalent to*

$$k_1 > \max\left\{0, \frac{\beta_3}{\alpha_m}\right\} \quad \text{and} \quad k_2 > \frac{\beta_2 + \sqrt{\beta_2^2 + 4\, \alpha_m\, \beta_0\, k_1}}{2\, \alpha_m\, \beta_0}.$$

Remark 4.3.2 *In this section, systems are assumed to be uncertain, but fixed and homogeneous. Thus, all systems have to change at the same time and the results do not extend to a heterogeneous network of systems. Higher-order systems can be considered, but applying Routh-Hurwitz to the resulting higher-order polynomials yields more nonlinear inequalities to solve. The methodology presented here can be used. However, it is not guaranteed that the nonlinear inequalities simplify to nice and explicit conditions on the gains. An alternative control design based on LMIs results in sufficient, but not always necessary conditions.*

Networks of marginally stable systems

If systems (4.1) are stable, then $c_0, c_1 > 0$ and GUAC is achieved if $\mathbf{K} = \mathbf{0}$.

If these systems are marginally stable, then either $c_0 = 0$ or $c_1 = 0$. If $c_0 \geq 0$, then $\beta_3 \leq 0$ and conditions (4.42)-(4.44) in Theorem 4.3.2 are satisfied if

$$k_2 > \frac{\beta_4 + |\beta_4|}{2\,\alpha_m\,\beta_0} \quad \text{and} \quad \beta_1\,k_2 < k_1 \leq \alpha_m\,\beta_0\,k_2^2 - \beta_2\,k_2 \tag{4.52}$$

where $\beta_4 := \beta_1 + \beta_2$.

If, in addition, $\beta_4 \leq 0$, then the first condition in (4.52) reduces to $k_2 > 0$. In this case, stabilizing gains k_1, k_2 can be chosen arbitrary small.

If $\beta_3 \leq 0$ and β_1, $\beta_2 \leq 0$, then (4.52) is satisfied if

$$k_1 > 0 \quad \text{and} \quad k_2 \geq \sqrt{k_1/(\alpha_m\,\beta_0)}\,. \tag{4.53}$$

[73] investigated special types of networks for which $a_{11} = a_{21} = 0$, $a_{22} \leq 0$, $a_{12} = b_2 = 1$ and $k_1 = 1$. Their Theorem 4.3 states that GUAC is achieved if the graph G contains a spanning tree and $k_2 > \sqrt{2\left[|\mu_l|\,\cos\left(\frac{\pi}{2} - \tan^{-1}\frac{\alpha_l}{\omega_l}\right)\right]^{-1}}$ for all $l > 1$. If the angular relations among α_l, ω_l, and $|\mu_l|$ are analyzed in the complex plane, then it can be shown that this is equivalent to $k_2 > \sqrt{2/\alpha_l}$. Their result can be strengthened by using (4.53), which yields $k_2 \geq \sqrt{1/\alpha_m}$ for $k_1 = \beta_0 = 1$. Furthermore, (4.53) also applies to systems for which $c_0 > 0$ and $c_1 = 0$. Therefore, it extends to the class of undamped oscillators, which are important for applications like clock synchronization, power grid control and others [35].

4.3.3 Robust consensus with rate $\alpha_0 > 0$

To achieve GUEC with rate α_0 we suppose there exists $\tilde{\beta}_0, \ldots, \tilde{\beta}_3$ such that, similar to (4.41),

$$\boxed{\frac{b_2}{a_{12}} \geq \tilde{\beta}_0\,, \quad \frac{a_{11}+\alpha_0}{a_{12}} \leq \tilde{\beta}_1\,, \quad \frac{a_{22}+\alpha_0}{a_{12}} \leq \tilde{\beta}_2 \quad \text{and} \quad \frac{a_{12}\,a_{21} - (a_{11}+\alpha_0)(a_{22}+\alpha_0)}{a_{12}\,b_2} \leq \tilde{\beta}_3} \tag{4.54}$$

and we obtain the following corollary.

Corollary 4.3.1 *Consider any network of systems* (4.1) *satisfying* (4.40), (4.54), *and whose graph satisfies (4.29). Then, there exists* k_1, k_2 *satisfying*

$$k_2 > \frac{\tilde{\beta}_1 + \tilde{\beta}_2 + \sqrt{(\tilde{\beta}_1 + \tilde{\beta}_2)^2 + \max\{0,\, 4\,\tilde{\beta}_0\,\tilde{\beta}_3\}}}{2\,\alpha_m\,\tilde{\beta}_0} \tag{4.55}$$

$$k_1 > \tilde{\beta}_1\,k_2 + \max\left\{0,\, \tilde{\beta}_3/\alpha_m\right\} \tag{4.56}$$

$$k_1 \leq -\tilde{\beta}_2\,k_2 + \alpha_m\,\tilde{\beta}_0\,k_2^2 \tag{4.57}$$

and, with $\mathbf{K} = \begin{bmatrix} k_1 & k_2 \end{bmatrix}$, *the closed-loop network (4.1)-(4.2) achieves GUEC with rate* $\alpha_0 > 0$.

Proof This follows from Lemma 2.1.1 and Theorem 4.3.2. ■

Remark 4.3.3 *The results presented in this section assume uncertain, but fixed graphs and system dynamics. Therefore, they cannot be applied to switching topologies that are continuously changing. Instead, we consider networks that vary slowly relative to the rate* $\alpha_0 > 0$ *at which GUEC is achieved. If for example a large sensor array is deployed underwater, then communication links might break over time, changing the communication structure. If the control is robust, then the system reconverges after any type of disruption, which avoids immediate repair.*
Alternatively, a customer might want to use devices in different setups. However, it is not desired to reprogram the devices all the time. Then, the control chosen has to work for alternating scenarios.

4.3.4 Robust consensus using output feedback

Similar to Section 4.2.4, we consider output feedback controllers described by (4.18). If the uncertain system transfer functions are given by (4.20) then, γ_0 and γ_1 are given by (4.21). Suppose b_0, b_1, c_0 and c_1 are unknown but lie in a compact set. We consider the following cases:

1. $(b_0,\, b_1 > 0)$ It follows from Theorem 4.3.1 that controller (4.18) yields GUAC if

$$b_0\, k > \max\left\{0,\, \frac{-c_0}{\alpha_m}\right\}\Bigg(\quad \text{and} \quad b_1\, k > \frac{-c_1 + \sqrt{c_1^2 + 4\,\alpha_m\, b_0\, k}}{2\,\alpha_m}\,. \tag{4.58}$$

   Rearranging, squaring, and dividing the second condition in (4.58) by $b_1\, k$ shows that this condition is equivalent to $\alpha_m\, b_1\, k + c_1 > b_0/b_1$ and $2\,\alpha_m\, b_1\, k + c_1 > 0$. Since $\alpha_m, b_0, b_1, k > 0$ these conditions are equivalent to $\alpha_m\, b_1\, k + c_1 > b_0/b_1$. Now, (4.58) is equivalent to

$$k > \max\left\{0,\, \frac{\beta_5}{\alpha_m},\, \frac{\beta_6}{\alpha_m}\right\}\Bigg(\text{where} \quad -\frac{c_0}{b_0} \le \beta_5 \quad \text{and} \quad \frac{b_0 - b_1\, c_1}{b_1^2} \le \beta_6\,. \tag{4.59}$$

2. $(b_0 = 0,\, b_1 > 0)$ Here $\gamma_0 = 0$ and, using Theorem 4.3.1, controller (4.18) yields GUAC if $k \ge 0$ and

$$c_0 > 0 \qquad \text{and} \qquad k > -\frac{c_1}{b_0\,\alpha_m}$$

3. $(b_0 > 0,\, b_1 = 0)$ Here $\gamma_1 = 0$ and from (4.22) we see that we have to restrict ω_l if $\gamma_0 \neq 0$. Thus, we assume $\mu \in \Lambda$, where Λ is a compact set, and a controller (4.27) exists if and only if (4.23)-(4.24) are satisfied for all eigenvalues and parameters.

4. $(b_0,\, b_1 \le 0)$ Here we can use the design for $b_0,\, b_1 \ge 0$ if we multiply $b_0,\, b_1$ and k by -1.

If a static output controller does not exist, then an observer based controller can be used when the graph is unknown but the system parameters are known.

4.3.5 Laplacian eigenvalues in a disc

Theorem 4.3.1 and following provide conditions such that (4.5)-(4.6) hold for all Laplacian eigenvalues contained inside the half-plane given by $0 < \alpha_m \le \alpha_l$. In this section, we investigate wether or not we can indentify conditions on γ_0, γ_1 that are less conservative if the Laplacian eigenvalues are not just restricted to a half-plane but also to a disc, that is, the eigenvalues of the Laplacian L satisfy (4.29) and there exists $R_c \in \mathbb{R}$ such that

$$\boxed{l_{ii} \leq R_c} \tag{4.60}$$

for all $i = 1, \cdots, N$. Hence, the range of graphs G will be characterized by constants α_m and R_c. From the proof of Theorem 4.3.1, bound α_m is necessary to obtain gains that are finite. Constant R_c on the other hand is a restriction that we place on the graphs to obtain controllers that are less conservative. R_c is motivated by the idea of limited communication bandwidth and the fact that systems should impact their neighbors equally. If l_{ii} is large for some i only, then these systems receive a lot of information from their neighbors, while others do not. Placing a limit on R_c will force the graph to be more distributive in that manner. If l_{ii} is large in general, then the graph could be rescaled to decrease R_c. Though, α_m will be scaled as well and the overall control u_i will not be impacted.

Theorem 4.3.3 *The closed-loop network (4.1)-(4.2) achieves GUAC for any graph satisfying* (4.29) *and* (4.60) *if*

$$\gamma_0 > r_0 \tag{4.61}$$

$$\gamma_1 > r_1 + r_2 + \sqrt{r_2^2 + \gamma_0^2\,\eta} \tag{4.62}$$

where

$$r_i = \max\left\{-\frac{c_i}{2R_c}, -\frac{c_i}{\alpha_m}\right\}, \quad i = 0, 1, \quad r_2 = \frac{c_1\,\eta}{2}\gamma_0,$$

$$\eta = \frac{1}{c_0 + 2\,R_c\,\gamma_0}\left(\frac{2\,R_c}{\alpha_m} - 1\right).$$

Remark 4.3.4 *Conditions in Theorem 4.3.3 are given in terms of constant R_c, which depends on the structure of graph G. Mainly, an upper limit is placed on the weighted in-degree l_{ii} of each vertex i. It is reasonable to assume that such information is globally available. However, if for some reason this information cannot be obtained, then results of the previous sections will be obtained if we take the limit $R \to \infty$.*

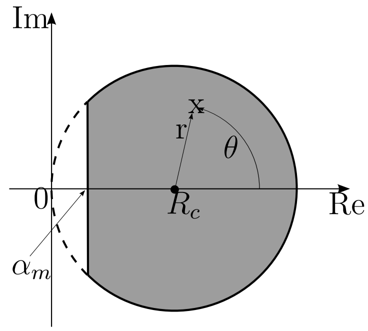


Figure 4.1.. Location of the non-zero eigenvalues for a graph G containing a spanning tree.

Proof of Theorem 4.3.3

Theorem 4.3.3 is a result of Theorem 4.2.1 and the following lemmas. First, we identify bounds on the Laplacian eigenvalues.

Lemma 4.3.1 *Suppose G contains a spanning tree. Then all non-zero Laplacian eigenvalues $\mu = \alpha + \jmath\omega$ satisfy*

$$\frac{1}{2R_c} \leq \frac{\alpha}{|\mu|^2} \leq \frac{1}{\alpha_m}$$

where $0 < \alpha_m \leq \alpha$ and $l_{ii} \leq R_c$ for all i.

Proof If we apply the Gershgorin disc theorem to the special structure of L, then eigenvalues will be restricted to a disc and they can be parametrized by

$$\alpha = R_c + r\,\cos\theta\,, \quad \omega = r\,\sin\theta \tag{4.63}$$

where $0 \leq r \leq R_c$. Figure 4.1 illustrates this relation. Now, the upper bound is obtained by noting that

$$\alpha^2 \leq \alpha^2 + \omega^2 = |\mu|^2 \quad \stackrel{\alpha > 0}{\Rightarrow} \quad \frac{\alpha}{|\mu|^2} \leq \frac{1}{\alpha_m}$$

where $\alpha > 0$ from Fact 2.3.1. Similarly, the lower bound is obtained by observing that for any given α, $1/|\mu|^2$ is minimized for $r = R_c$ (maximum ω^2), that is,

$$\frac{\alpha}{|\mu|^2} \geq \frac{R_c\,(1+\cos\theta)}{2\,R_c^2\,(1+\cos\theta)} = \frac{1}{2\,R_c}$$

where R_c is the radius of the largest Gershgorin disc. ■

Next, we obtain a bound on an expressions that is similar to (4.13)

Lemma 4.3.2 *If* $f(\eta) = a\,\eta + \sqrt{a^2\,\eta^2 + \gamma_0^2\,\eta}$, *then* $f'(\eta) \geq 0$ *for all* $\eta \geq 0$

Proof First, we note that $f'(\eta) \geq 0$ if

$$a\sqrt{a^2\,\eta^2 + \gamma_0^2\,\eta} \quad \geq \quad -a^2\,\eta - 0.5\,\gamma_0^2\,.$$

The right-hand side is always strictly negative and the inequality clearly holds for $a \geq 0$. Suppose $a < 0$, then

$$\sqrt{a^2\,\eta^2 + \gamma_0^2\,\eta} \quad \leq \quad -a\,\eta - 0.5\,\gamma_0^2/a$$

has to hold. Now, the right-hand side is strictly positive and both sides can be squared. This yields $0 \leq 0.25\,\gamma_0^2/a^2$. Therefore, $f'(\eta) \geq 0$ for all $\eta \geq 0$. ■

Finally, we prove Theorem 4.3.3 by applying Theorem 4.2.1. From Lemma 4.3.1, if condition (4.61) holds, then the first inequality in Theorem 4.2.1 is satisfied. Further, $c_0 + \alpha\,\gamma_0 > 0$ and $\frac{d}{d\omega^2}\tau(\omega^2) > 0$. Therefore, if α is fixed, then τ is largest if ω is largest, that is, $r = R_c$ in (4.63). We define $\eta := \tau/\alpha \geq 0$ and obtain

$$\eta = \frac{1}{c_0 + 2R_c\,\gamma_0}\,\frac{\sin^2\theta}{(1+\cos\theta)^2} \tag{4.64}$$

where $\theta \in [0, \pi)$. From Lemma 4.3.2, we want to maximize η. Taking the first derivative with respect to θ yields

$$\left(\frac{\sin^2\theta}{(1+\cos\theta)^2}\right)' = \left(\frac{1-\cos\theta}{1+\cos\theta}\right)' = \frac{2\sin\theta}{(1+\cos\theta)^2} > 0\,.$$

Therefore, η is largest if $\cos\theta$ is smallest. From (4.63),

$$\alpha = R_c\,(1+\cos\theta) \geq \alpha_m \quad \Rightarrow \quad \cos\theta \geq \frac{\alpha_m}{R_c} - 1\,.$$

Substituting the expression back into η and yields

$$\eta = \frac{2\,R_c - \alpha_m}{c_0\,\alpha_m + 2\,R_c\,\alpha_m\,\gamma_0}\,.$$

Finally, r_1 is obtained by maximizing $-c_1/\alpha$ over all α.

Estimating the conservatism of the bound on γ_1

In the proof of Theorem 4.3.3, the maximum over g and $-\frac{c_1}{\alpha}$ is taken independently. This approach is accurate if $c_1 \leq 0$. However, if $c_1 > 0$, then $-\frac{c_1}{\alpha}$ will not reach its maximum for α_m. Therefore a smaller lower bound on γ_1 will be given for some α that is in between α_m and $2R_c$. Let $\underline{\gamma}_{1,best}$ be this smaller lower bound, $\underline{\gamma}_{1,approx}$ the lower bound given by (4.62). Then,

$$\Delta\underline{\gamma}_1 := \underline{\gamma}_{1,approx} - \underline{\gamma}_{1,best} \leq \begin{cases} c_1 \left(\frac{1}{\alpha_m} - \frac{1}{2R_c}\right) & , c_1 > 0 \\ 0 & , c_1 \leq 0 \end{cases}$$

where $\Delta\underline{\gamma}_1$ describes the conservatism of conditions in Theorem 4.3.3 with respect to the conditions in Theorem 4.2.1, which themselves are sufficient only. However, it can be shown that we have necessity for Theorem 4.2.1 and 4.3.3 if $c_0 \geq 0$ and $c_1 \leq 0$.

4.4 Proof of main result

4.4.1 The characteristic polynomial of the closed-loop network

To prove Lemma 4.1.1, we apply Corollary 3.2.1, which we restate here for convenience.

Lemma 4.4.1 ([29]) *The closed-loop network (4.1)-(4.2) achieves GUAC if and only if for each non-zero eigenvalue μ_l of the graph Laplacian L, $\mathbf{A}-\mu_l\mathbf{BK}$ is Hurwitz.*

A matrix is Hurwitz if and only if its characteristic polynomial is Hurwitz. From Lemma 3.2.2, the characteristic polynomial of matrix $\mathbf{A} - \mu_l\mathbf{BK}$ is given by

$$\boxed{p_l(s) = s^2 + (c_1 + \gamma_1\,\mu_l)\,s + (c_0 + \gamma_0\,\mu_l)}$$

where c_0, c_1, γ_0, and γ_1 are given by (4.3)-(4.4). The coefficients of polynomials p_l are not necessarily real, as the eigenvalues of the Laplacian can be complex.

4.4.2 A simple characterization of second-order Hurwitz polynomials

The conditions for second-order Hurwitz polynomials are a direct consequence of the work in [74] (see also [75]). [31] redeveloped and applied these conditions to consensus algorithms. Here we provide a simple new proof, which does not depend on the Routh-Hurwitz criterion. $Re(z)$ and $Im(z)$ refer to the real and imaginary part of a complex number z.

Lemma 4.4.2 *A polynomial $p(\lambda) = \lambda^2 + d_1\,\lambda + d_0$ with $d_0,\, d_1 \in \mathbb{C}$ is Hurwitz if and only if*

$$u_1 > 0 \qquad \textit{and} \qquad u_0\,u_1^2 + u_1 w_0\,w_1 - w_0^2 > 0$$

where $d_1 := u_1 + \jmath\,w_1$ and $d_0 := u_0 + \jmath\,w_0$.

Proof First, we note that p is Hurwitz if and only if $\tilde{p}$ is Hurwitz where $\tilde{p}(\tilde{\lambda}) := p(\tilde{\lambda} + \lambda_0)$ for any imaginary number λ_0, that is $Re(\lambda_0) = 0$. With $\lambda_0 := -\jmath\,w_1/2$,

$$\tilde{p}(\tilde{\lambda}) = \tilde{\lambda}^2 + \tilde{d}_1\,\tilde{\lambda} + \tilde{d}_0 \tag{4.65}$$

where

$$\tilde{d}_1 = u_1\,, \qquad \tilde{d}_0 = \tilde{u}_0 + \jmath\,\tilde{w}_0 \tag{4.66}$$

$$\tilde{u}_0 := u_0 + w_1^2/4, \quad \tilde{w}_0 := w_0 - u_1 w_1/2\,. \tag{4.67}$$

If z_1 and z_2 are the roots of $\tilde{p}$, then

$$\tilde{p}(\tilde{\lambda}) = \tilde{\lambda}^2 - (z_1 + z_2)\,\tilde{\lambda} + z_1 z_2\,. \tag{4.68}$$

Comparing (4.65) and (4.68), we see that $z_1 + z_2 = -\tilde{d}_1 = -u_1$; hence $z_1 + z_2$ is real. This implies that $Im(z_1) = -Im(z_2) := b$. Therefore, the roots can be expressed as $z_1 := a_1 + \jmath\,b$ and $z_2 := a_2 - \jmath\,b$ where a_1, a_2 and b are real. Equating coefficients in (4.65) and (4.68) while using (4.66)-(4.67) results in

$$u_1 = -(a_1 + a_2) \tag{4.69}$$

$$\tilde{u}_0 = a_1 a_2 + b^2 \tag{4.70}$$

$$\tilde{w}_0 = b\,(a_2 - a_1)\,. \tag{4.71}$$

The polynomial $\tilde{p}$ is Hurwitz if and only if its roots have negative real parts, that is $a_1, a_2 < 0$, which is equivalent to

$$a_1 + a_2 < 0 \tag{4.72}$$

$$a_1 a_2 > 0\,. \tag{4.73}$$

It follows from (4.69) that inequality (4.72) is equivalent to

$$u_1 > 0\,. \tag{4.74}$$

From (4.69) and (4.71), we obtain

$$4b^2\, a_1 a_2 = b^2\, u_1^2 - \tilde{w}_0^2\,.$$

Substituting $b^2 = \tilde{u}_0 - a_1 a_2$ from (4.70) into the above equation and rearranging yields

$$(4b^2 + u_1^2)\, a_1 a_2 = \tilde{u}_0 u_1^2 - \tilde{w}_0^2\,.$$

$u_1 > 0$ from (4.74). Hence, $4b^2 + u_1^2 > 0$ and

$$a_1 a_2 > 0 \quad \Leftrightarrow \quad \tilde{u}_0 u_1^2 - \tilde{w}_0^2 > 0\,.$$

The proof is completed by substituting the expressions in (4.67) for $\tilde{u}_0$ and $\tilde{w}_0$. ■

4.4.3 Proof of Lemma 4.1.1

Recalling Lemma 4.4.1, we see that we can prove Lemma 4.1.1 by showing that inequalities (4.5)-(4.6) are equivalent to the polynomial

$$p_l(s) = s^2 + (c_1 + \gamma_1\, \mu_l)\, s + (c_0 + \gamma_0\, \mu_l)$$

being Hurwitz. Applying Lemma 4.4.2 to p_l, we have

$$u_0 = c_0 + \gamma_0\, \alpha_l, \quad u_1 = c_1 + \gamma_1\, \alpha_l, \quad w_0 = \gamma_0\, \omega_l, \quad w_1 = \gamma_1\, \omega_l\,.$$

and p_l is Hurwitz if and only if

$$\delta_l := u_1 = c_1 + \gamma_1\, \alpha_l > 0,$$

$$\delta_l\, \left[(c_0 + \gamma_0\, \alpha_l)\, \delta_l + \gamma_0\, \gamma_1\, \omega_l^2\right] - \omega_l^2\, \gamma_0^2 > 0\,.$$

Multiplying the second inequality by $\alpha_l > 0$ (Fact 2.3.1) and utilizing $\alpha_l\,\gamma_1 = \delta_l - c_1$ and $|\mu_l|^2 = \alpha_l^2 + \omega_l^2$, we obtain

$$(\alpha_l\,c_0 + |\mu_l|^2\,\gamma_0)\,\delta_l^2 - c_1\,\omega_l^2\,\gamma_0\,\delta_l - \alpha_l\,\omega_l^2\,\gamma_0^2 > 0$$

that is, inequality (4.6).

4.5 Necessity of Theorem 4.2.1 conditions

The conditions in Lemma 4.1.1 yield a larger set of feasible gains than those of Theorem 4.2.1. However, the conditions in Lemma 4.1.1 and Theorem 4.2.1 are equivalent if the additional constraint

$$c_0\,\alpha_l + |\mu_l|^2\,\gamma_0 > 0 \tag{4.75}$$

is satisfied for all $l > 1$ (see proof of Theorem 4.2.1). This section presents cases in which (4.75) has to hold. The first two cases involve the system parameters and are discussed in Remark 4.2.1. The third case focuses on the properties of the graph.

Case I: $c_1 = 0$

Since $\alpha_l > 0$ from Fact 2.3.1, inequality (4.75) must hold to guarantee (4.6).

Case II: $c_0 \geq 0$, $c_1 < 0$

If $c_1 < 0$, then from (4.6), either $\gamma_0 > 0$ or (4.15) has to hold. $\gamma_0 > 0$ and $c_0 \geq 0$ imply (4.15). Hence, (4.15) holds regardless.

Case III: Conditions on the non-zero Laplacian eigenvalues

Here we show that satisfaction of one of the following two conditions implies that (4.75) will hold.

Condition 4.5.1 $c_0 \geq 0$ *and the Laplacian has a real eigenvalue* μ_{l^*} *for which*

$$\mu_{l^*} = \alpha_{l^*} \geq |\mu_l|^2/\alpha_l$$

for all $l > 1$.

Condition 4.5.2 *$c_0 \leq 0$ and the Laplacian has a real eigenvalue $\mu_{l^*} \neq 0$ for which*

$$\mu_{l^*} = \alpha_{l^*} \leq |\mu_l|^2/\alpha_l$$

for all $l > 1$.

To prove that satisfaction of Condition 4.5.1 or Condition 4.5.2 implies that (4.75) holds for all $l > 1$, we first note that, since $\mu_l \neq 0$, inequality (4.75) is equivalent to

$$\gamma_0 > -c_0\,\alpha_l/|\mu_l|^2\,. \tag{4.76}$$

When μ_l is real, inequality (4.6) of Lemma 4.1.1 is equivalent to the second inequality in (4.7) which, since $\alpha_l > 0$, is equivalent to

$$\gamma_0 > -c_0/\alpha_l = -c_0\,\alpha_l/|\mu_l|^2\,. \tag{4.77}$$

Thus, when inequality (4.6) and Condition 4.5.1 or 4.5.2 hold,

$$\gamma_0 > -c_0/\alpha_{l^*} \geq -c_0\,\alpha_l/|\mu_l|^2$$

and (4.76) holds for all $l > 1$.

Remark 4.5.1 *The conditions in Theorem 4.2.1 are necessary if networks of double integrators are considered. In such a network, if we wish to achieve GUAC or GUEC with rate $\alpha_0 > 0$, then the constants $\tilde{c}_0 = \alpha_0^2$ and $\tilde{c}_1 = -2\,\alpha_0$ of the transformed systems result in either Case I or II.*

Statistical analysis of Conditions 4.5.1 and 4.5.2

Conditions 4.5.1 and 4.5.2 seem to hold for many Laplacians as demonstrated by the statistical analysis below. It should be noted that the system parameter c_0 is ignored in order to study the properties of the graph independently. A summary of the results is presented in Table 4.1.

We analyzed $10,000$ networks with varying size N where N ranges from 5 to 50.

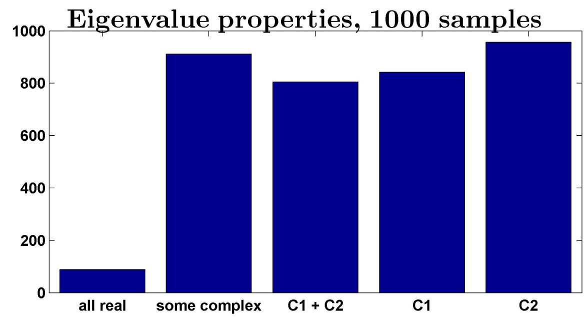


Figure 4.2.. Laplacian eigenvalue properties of $1,000$ randomly generated graphs containing a spanning tree

Table 4.1.. Laplacian eigenvalue properties of $10,000$ randomly generated graphs containing a spanning tree

	% of the cases
All eigenvalues are real	9 %
Some eigenvalues are complex	91 %
Condition 4.5.1 holds	83 %
Condition 4.5.2 holds	97 %
Condition 4.5.1 and 4.5.2 holds	80 %
Condition 4.5.1 holds, 4.5.2 does not	2.5 %
Condition 4.5.2 holds, 4.5.1 does not	17 %

Each graph G was generated by starting out with N unconnected systems. Edges with positive, uniformly distributed weights were randomly added in an iterative process. The iteration was stopped once G contained a spanning tree.

Table 4.1 shows that the Laplacian matrices of most graphs have at least one complex eigenvalue. About 80 % of the generated graphs have eigenvalues that satisfy both Conditions 4.5.1 and 4.5.2. These conditions are always satisfied if all eigenvalues are real, which is about 9 % of the cases. Condition 4.5.2 is more likely to hold

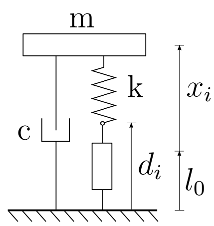


Figure 4.3.. Example: distributed platforms

than Condition 4.5.1, and there are cases where only one or neither of them holds. However, overall Table 4.1 illustrates that Condition 4.5.1 or 4.5.2 are likely to hold.

4.6 Simulations

In this section, we illustrate the effectiveness of our results by different simulations. First, we show the process of selecting gains that achieve GUEC with different rates of convergence. Then, we show how to select gains that achieve GUAC for a range of system parameters and communication structures. Finally, we illustrate the conservatism of the robust control result.

4.6.1 Example: Distributed platforms - oscillator synchronization

We apply our results to a network of distributed platforms (Figure 4.3) whose motions are described by

$$m\,\ddot{x}_i + c\,\dot{x}_i + k\,x_i = k\,d_i\,.$$

Here d_i, the displacement of the actuator, is the physical input to the system. The parameters m, c and k are the common mass, damping coefficient, and spring constant, respectively. With $c_0 = k/m$ and $c_1 = c/m$, we note that these systems are described by (4.1) where $u_i := \frac{k}{m}\,d_i$ and $u_0 = 0$. Controller (4.2) is used to close the loop so that GUAC on x_i and $\dot{x}_i$ is achieved. This control structure emphasizes the importance of system alignment rather than an individual position x_i. This could be

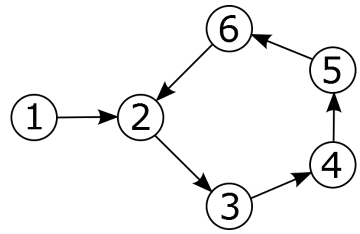


Figure 4.4.. communication network

important if, for example, heavy weights are lifted in a distributed fashion. The same considerations apply to any other type of active damping or suspension system, e.g. building control.

We consider a communication structure as given by the graph in Figure 4.4. In this network, System 1 is a leader, and when GUAC is achieved, the positions of the other systems will equal that of System 1. Systems 2-6 assign a weighting of 1 to their in-neighbors, except 2, which assigns a higher weighting of 3 to the leader. The Laplacian matrix for the weighted graph is

$$L = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -3 & 4 & 0 & 0 & 0 & -1 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}$$

which has the following eigenvalues

$$\mu_1 = 0, \quad \mu_2 = 0.28, \quad \mu_{3/4} = 0.95 \pm .75\jmath, \quad \mu_5 = 1.82, \quad \mu_6 = 3.99\,.$$

For purposes of simulation, we considered $c_0 = 4$ and $c_1 = 0.1$. We chose gains γ_0, γ_1 so that GUEC was achieved with rate α_0 and the quantity $\gamma_0^2 + \gamma_1^2$ was minimized. Three different desired convergence rates for GUEC were considered and the resulting parameters are presented in Table 4.2.

Table 4.2.. Gains γ_0, γ_1 to achieve GUEC - $c_0 = 4, c_1 = 0.1$

	α_0	γ_0	γ_1	$\tilde{c}_0$	$\tilde{c}_1$
case I	0.25	0.00	1.43	4.04	−0.40
case II	0.50	0.56	3.21	4.20	−0.90
case III	1.00	5.56	6.79	4.90	−1.90

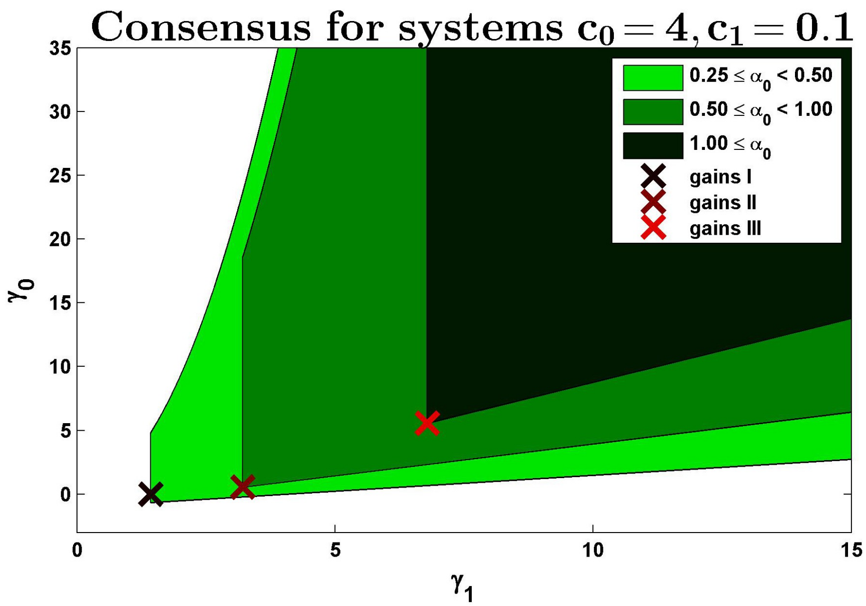
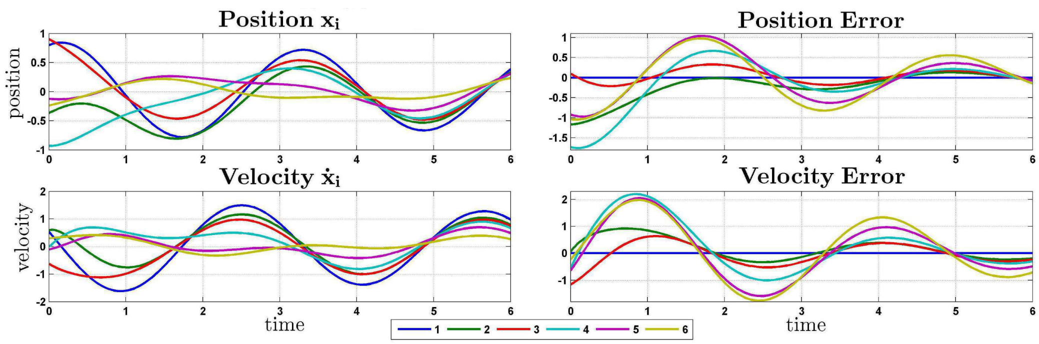
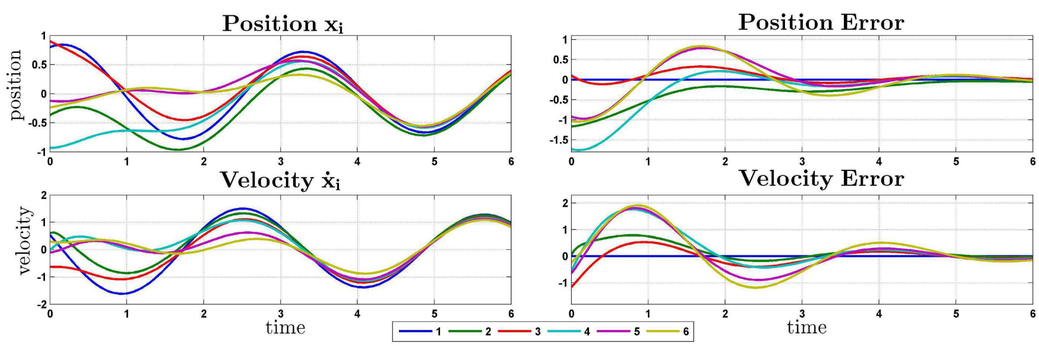


Figure 4.5.. Gains γ_0, γ_1 satisfying necessary and sufficient conditions to achieve GUEC with rate α_0 where $c_0 = 4, c_1 = 0.1$.

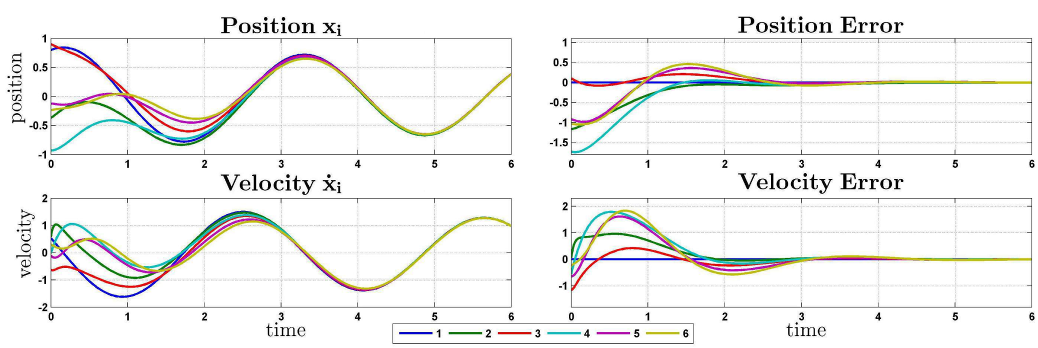
The individual systems were stable without control. With $\gamma_0 = 0$, GUAC can be achieved for Case I. Figure 4.5 shows the regions of feasible gains corresponding to the three desired convergence rates. It should be noted that the eigenvalues of the Laplacian satisfy Condition 4.5.1. Hence, the conditions in Theorem 4.2.1 are necessary and sufficient to achieve GUAC. This is further justified by the fact that all cases presented in Table 4.2 satisfy $\tilde{c}_0 \geq 0$ and $\tilde{c}_1 < 0$.



(a) Case I - GUEC with rate $\alpha_0 = 0.25$



(b) Case II - GUEC with rate $\alpha_0 = 0.50$



(c) Case III - GUEC with rate $\alpha_0 = 1.00$

Figure 4.6.. Synchronization of systems with $c_0 = 4$, $c_1 = 0.1$ and different convergence rates

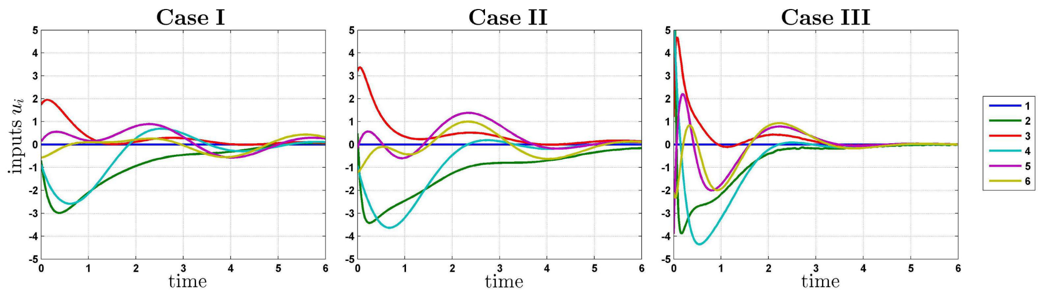


Figure 4.7.. Inputs u_i for different convergence rates

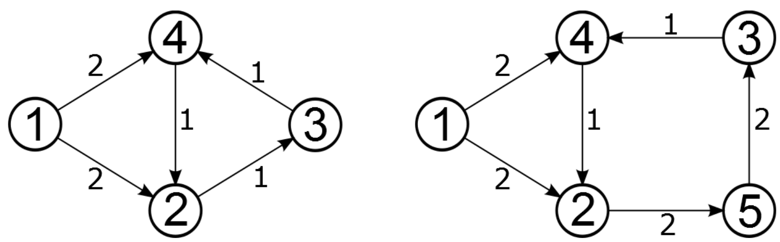


Figure 4.8.. Communication network with assigned edge weights for time $t < 5$ (left) and $t \geq 5$ (right).

All three cases were simulated with the same randomly generated, uniformly distributed initial conditions. The gains γ_0, γ_1 were set to the values shown in Table 4.2. The plots in Figure 4.6 show the time history of the position and velocity of each oscillator on the left. Though all systems were initially unsynchronized, they achieved GUEC over time. The right-hand side of Figure 4.6 shows the error that each oscillator possesses relative to the leader, which is indicated by the blue line. Clearly, as expected, synchronization was achieved more quickly with a larger α_0.

A faster rate of convergence results in larger inputs u_i. Figure 4.7 shows the inputs that must be applied to each oscillator in order to achieve GUEC as illustrated in Figure 4.6.

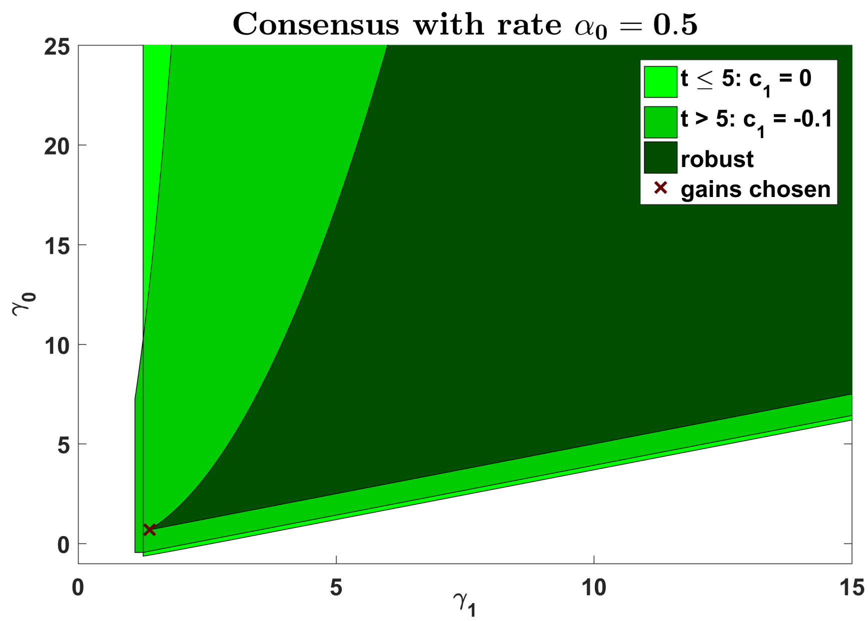


Figure 4.9.. Gains γ_0, γ_1 satisfying necessary and sufficient conditions to achieve GUEC with rate $\alpha_0 = 0.5$, $c_0 = 4$.

4.6.2 Example: Robust consensus

The chosen example demonstrates how gains γ_0, γ_1 are designed robustly such that the closed-loop network achieves GUEC with rate $\alpha_0 = 0.5$ over different communication structures and system dynamics.

First, we simulated the synchronization of four undamped oscillators wth $u_0 = 0$. The matrices in (4.40) had values $a_{11} = 0$, $a_{12} = 1$, $a_{21} = -4$, $a_{22} = 0$ and $b_2 = 1$. At time $t = 5$, System 5 was added to the network, some damping $a_{22} = 0.1$ was introduced, and systems were externally driven by $u_0(t) = \cos 5t$.

The communication structure for $t < 5$ was described by the graph on the left in

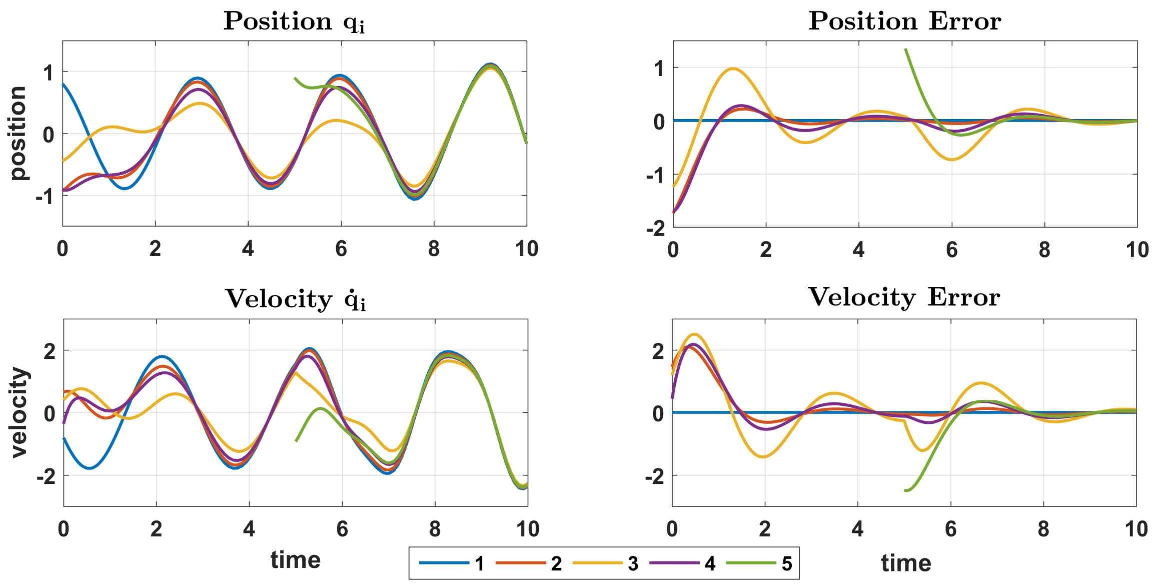


Figure 4.10.. Simulation of five linear second-order systems achieving GUEC on System 1 (blue line)

Figure 4.8. The graph on the right shows the communication structure for $t \geq 5$. The associated Laplacian matrices

$$L_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -2 & 3 & 0 & -1 \\ 0 & -1 & 1 & 0 \\ -2 & 0 & -1 & 3 \end{bmatrix}, \quad L_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ -2 & 3 & 0 & -1 & 0 \\ 0 & 0 & 2 & 0 & -2 \\ -2 & 0 & -1 & 3 & 0 \\ 0 & -2 & 0 & 0 & 2 \end{bmatrix}$$

have eigenvalues

$$\mu_l \text{ for } L_1: \quad \mu_1 = 0,\ \mu_2 = 0.80,\ \mu_{3/4} = 3.1 \pm .67\jmath\,,$$
$$\mu_l \text{ for } L_2: \quad \mu_1 = 0,\ \mu_2 = 1,\ \mu_{3/4} = 2.5 \pm \jmath\sqrt{7}/2,\ \mu_5 = 4\,.$$

In these networks, System 1 is a leader, and when GUAC is achieved, the positions of the other systems will equal that of System 1.

Considering a rate of convergence $\alpha_0 = 0.5$, we applied Corollary 4.1.1 to both setups and noted that $\tilde{c}_0 \geq 4.25$ and $-1.1 \leq \tilde{c}_1 \leq -1.0$. Therefore, conditions in Theorem 4.2.1 are necessary and sufficient based on Remark 4.2.1. Figure 4.9 illustrates the regions of feasible gains γ_0, γ_1 to achieve GUEC with rate $\alpha_0 = 0.5$. System dynamics and communication structures were very similar, and we observed plenty of overlap. Additionally, Figure 4.9 shows gains γ_0, γ_1 satisfying Corollary 4.3.1 ($\alpha_m = 0.80$) and therefore ensuring GUEC with rate $\alpha_0 = 0.5$ robustly. Finally, we chose gains $\gamma_0 = k_1 = 0.69$ and $\gamma_1 = k_2 = 1.39$, such that $k_1^2 + k_2^2$ is minimal and Corollary 4.3.1 is satisfied.

The network achieved GUEC as shown in Figure 4.10. q_i is the position of oscillator i, and $\dot{q}_i$ is its velocity. States were initialized randomly over a uniform distribution. For each system, we introduced a position error $e_i = q_i - q_1$ and a velocity error $\dot{e}_i = \dot{q}_i - \dot{q}_1$. GUEC was achieved, and the steady state dynamics were governed by oscillations according to the initial conditions of System 1 (blue line) as seen on the left side of Figure 4.10. The process of achieving GUEC is illustrated on the right side of Figure 4.10 where position and velocity errors are plotted. All errors decayed and every root λ_l of p_l satisfied $Re(\lambda_l) \leq -0.55$ for $t < 5$ and $Re(\lambda_l) \leq -0.64$ for $t \geq 5$. Therefore, GUEC was achieved with rate $\alpha_0 \geq 0.55$.

4.6.3 Example: Formation control with veloctiy tracking

Here we analyze a network of double-integrators, e.g. trucks on a highway, that want to stay in formation, but also track a desired velocity v_0. The dynamics are given by

$$\ddot{q}_i = -k_d\left(\dot{q}_i - v_0\right) + \sum_{j \in N_i} w_{ij}\left[\gamma_0\left(q_j - q_i\right) + \gamma_1(\dot{q}_j - \dot{q}_i)\right]$$

where k_d, γ_0 and γ_1 are control gains to be designed, and q_i is describing system i's position relative to the formation. It should be noted that we only introduce v_0 and no q_0. Therefore, a structure with a virtual leader or reference model will not suffice. However, we can rewrite the problem as

$$\ddot{q}_i + c_1\,\dot{q}_i = u_i + u_0$$

where $c_1 = k_d$, $u_0 = k_d\,v_0$ and u_i is given by (4.2). If we want to track v_0, then we choose $k_d > 0$. The tracking performance will improve if k_d is larger. Gains γ_0 and γ_1 are chosen once k_d is fixed. Mainly, achieving GUAC and tracking v_0 are competing goals, and if it is desired to change k_d based on the current network state, then gains γ_0, γ_1 have to be chosen for a range of $c_1 = k_d$.

As an example, we tracked $v_0 = 0.1$ and systems achieved GUEC with rate $\alpha_0 = 1$. We used the same changing communication structure as in the previous example (see Figure 4.8) and chose $k_d = 0.5$.

Figure 4.11 illustrates the regions of feasible gains γ_0, γ_1 that are necessary and sufficient to achieve GUEC with rate $\alpha_0 = 1$ for both system setups. System dynamics and communication structures were very similar, and we observed plenty of overlap. Additionally, Figure 4.11 shows gains γ_0, γ_1 that were obtained using Theorem 4.3.3 and a network of transformed systems where $\tilde{c}_0 = 0$, $\tilde{c}_1 = -0.5$, $\alpha_m = 0.80$, and $R_c = 3$ ($robust_{disc}$). Results are also compared to gains that were obtained using

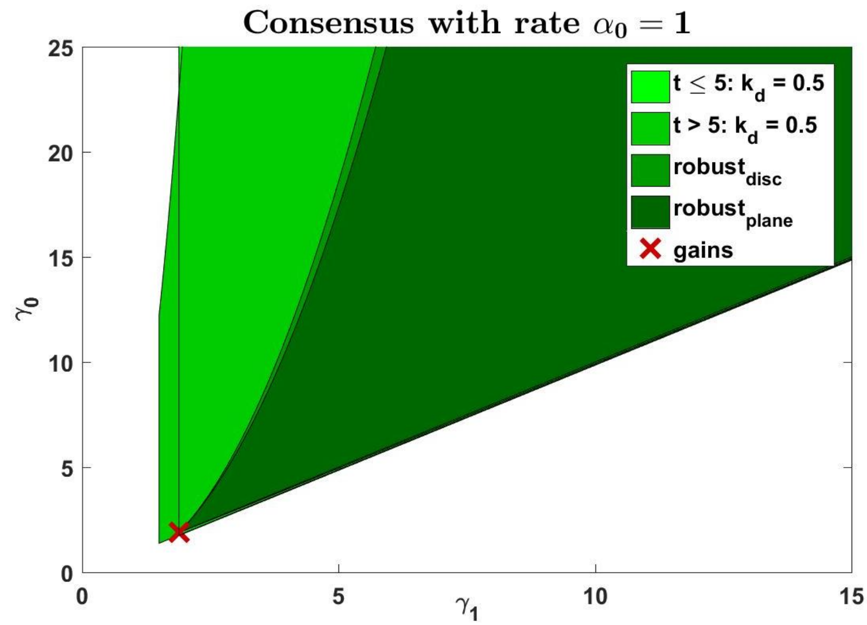


Figure 4.11.. Gains γ_0, γ_1 to achieve GUEC with rate $\alpha_0 = 1$.

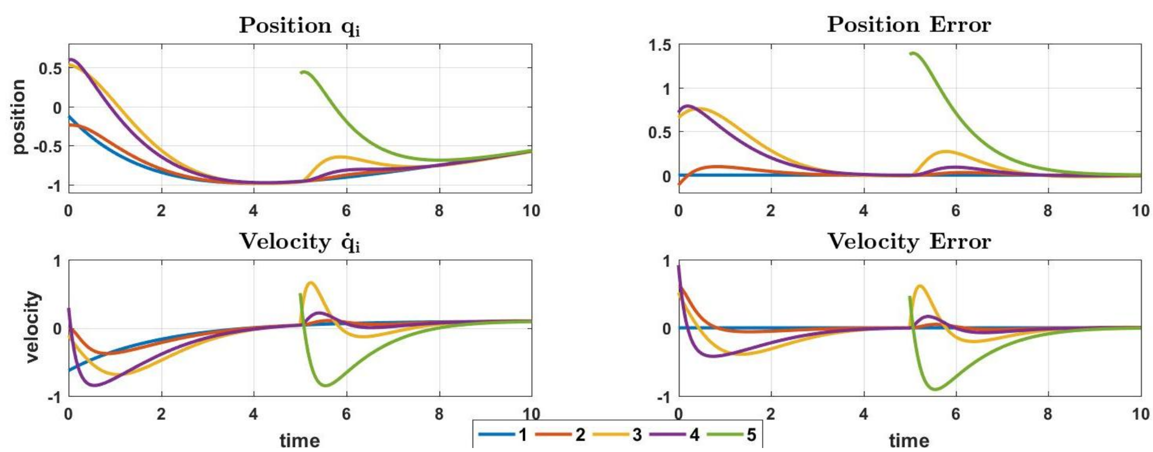


Figure 4.12.. Simulation of five double integrators achieving GUEC on System 1 (blue line)

Corollary 4.3.1 ($robust_{plane}$). Finally, we chose gains $\gamma_0 = 1.90$ and $\gamma_1 = 1.89$ such that $\gamma_0^2 + \gamma_1^2$ is minimal and Theorem 4.3.3 is satisfied for the transformed network.

The network achieved GUEC as shown in Figure 4.12. q_i is the position of system i, and $\dot{q}_i$ is its velocity. States were initialized randomly over a uniform distribution. For each system, we introduced a position error $e_i = q_i - q_1$ and a velocity error $\dot{e}_i = \dot{q}_i - \dot{q}_1$. GUEC was achieved and the steady state dynamics were governed by the behavior of System 1 (blue line) as seen on the left side of Figure 4.12. The process of achieving GUEC is illustrated on the right side of Figure 4.12 where position and velocity errors are plotted. All errors decayed, and the objective of achieving GUEC with a guaranteed rate of convergence while tracking velocity v_0 was achieved.

From Figure 4.11, we notice that the feasible regions for γ_0, γ_1 that are covered by $robust_{disc}$ (Theorem 4.3.3) and $robust_{plane}$ (Theorem 4.3.1) are almost identical. Especially for small $\gamma_0^2 + \gamma_1^2$, the difference is very small. Both robust results however are very conservative compared to the gains that would be obtained if graphs L_1 and L_2 are considered only. Therefore, we conclude that α_m is the main driver for the conservatism of the robust results. If the non-zero Laplacian eigenvalues are restricted to a half-plane and a disc, then we slightly increase the feasible region. However, we require knowledge of R_c. Hence, from a practical perspective, adding the additional restriction in terms of a disc is not improving the results for robust GUAC in continuous-time.

5. CONSENSUS FOR CONTINUOUS-TIME HIGHER-ORDER SYSTEMS

The description of the closed-loop network (2.1)-(2.2) is not restricted to second-order systems, and some of the ideas presented in Chapter 4, e.g. the use of a network of transformed systems, apply to the higher-order case as well. Here we summarize the approach for linear higher-order systems that other researchers observed as well (e.g. [49], [32], or [76]).

Conceptionally, the closed-loop network is transformed such that the Laplacian matrix appears in Jordan normal form (see Section 3.1). Then, controllers can be indentified for systems under full state feedback [49]. The transition to an observer based control [32], [76] is the result of the fact that the convergence of the observer states are independent of the consensus state. Therefore, the separation principle can be applied, and controllers and observers can be designed indepently.

In this thesis, we extend the work to a homogeneous network of time-varying systems

$$\dot{\mathbf{x}}_i(t) = \mathbf{A}(t)\,\mathbf{x}_i(t) + \mathbf{B}(t)\,\mathbf{K}(t)\sum_{j\in N_i} w_{ij}\left[\mathbf{x}_j(t) - \mathbf{x}_i(t)\right] + \mathbf{u}_0(t) \tag{5.1}$$

where we assume that

$\mathbf{A}(\cdot)$ *and* $\mathbf{B}(\cdot)\,\mathbf{K}(\cdot)$ *are piece-wise continuous and bounded.*

First, we present a simple result for linear time-invariant single-input systems.

5.1 Linear time-invariant single-input systems

Here we show that GUAC can be achieved for any homogeneous network of linear time-invariant single-input stabilizable sytems described by

$$\dot{\mathbf{x}}_i = \mathbf{A}\,\mathbf{x}_i + \mathbf{B}\,\mathbf{K}\sum_{j\in\mathcal{N}_i}\Big(w_{ij}\,(\mathbf{x}_i - \mathbf{x}_j) + \mathbf{u}_0, \qquad i = 1,\ldots,N\,.$$

We will derive the consensus properties of the closed-loop network from the characteristic polynomial d_μ where we recall from (3.24) that if systems are stabilizable, then there exists a controllable subspace of systems (A_c, B_c) and control $\mathbf{K}_c$ such that

$$\begin{aligned} d_\mu(s) &= \det(s\mathbf{I} - \mathbf{A} + \mu\,\mathbf{B}\mathbf{K}) \\ &= \det(s\mathbf{I} - \mathbf{A}_c + \mu\,\mathbf{B}_c\mathbf{K}_c)\,c_u(s) \\ &= [c_c(s) + \mu\,\gamma_c(s)]\;c_u(s) \end{aligned} \tag{5.2}$$

where c_u is Hurwitz. If systems (A, B) are controllable, then $c_u = 1$.

5.1.1 Achieving GUAC

For GUAC, it is necessary and sufficient that d_μ be Hurwitz for all non-zero eigenvalues μ of the graph Laplacian. From Lemma 3.2.2, we conclude that

$$d_\mu(s) = c(s) + \mu\,\gamma(s) \tag{5.3}$$

where

$$c(s) = \det(s\mathbf{I} - \mathbf{A}) \qquad \text{and} \qquad \gamma(s) = \det(s\mathbf{I} - \mathbf{A})\,\mathbf{K}\,(s\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{B}\,.$$

Polynomial c is of degree n and γ is a polynomial whose degree is at most $n-1$. If (A, B) is stabilizable, then either c is Hurwitz and GUAC is achieved with $\mathbf{K} = \mathbf{0}$ or there exists a $\mathbf{K}$ such that γ is Hurwitz. One can use pole-placement techniques to obtain a $\mathbf{K}_c$ such that γ_c and therefore $\gamma(s) = \gamma_c(s)\,c_u(s)$ are Hurwitz. Then, GUAC will be achieved if the associated $\mathbf{K}$ is multiplied by a sufficiently large scalar κ.

Lemma 5.1.1 *Suppose c and m_0 are monic polynomials of degree n and $n-1$ respectively and m_0 is Hurwitz. Then, $c + \mu\,\kappa\, m_0$ is Hurwitz for every non-zero eigenvalue of the graph Laplacian if κ is sufficiently large.*

Proof Since c and m_0 are monic polynomials of degree n and $n-1$ respectively,

$$c(s) = s\, m_0(s) + r(s)$$

where r is a polynomial whose degree is at most $n-1$. Hence,

$$d_\mu(s) := c(s) + \mu\,\kappa\, m_0(s) = (s + \mu\,\kappa)\, m_0(s) + r(s)\,.$$

Since m_0 is Hurwitz, $m_0(s) \neq 0$ when $Re(s) \geq 0$. Since, in addition, r/m_0 is a proper rational function, there exists β such that

$$\frac{r(s)}{m_0(s)} \geq \beta \qquad \text{for} \qquad Re(s) \geq 0\,.$$

Recall α_m from (2.9) and consider any κ satisfying

$$\kappa > \frac{\beta}{\alpha_m}\,.$$

Now consider any $s \in \mathbb{C}$ with $Re(s) \geq 0$. Then,

$$\frac{d_\mu(s)}{m_0(s)} = s + \mu\,\kappa + \frac{r(s)}{m_0(s)}$$

Hence,

$$Re\left(\frac{d_\mu(s)}{m_0(s)}\right)\Bigg(= Re(s) + \alpha\,\kappa + Re\left(\frac{r(s)}{m_0(s)}\right)\Bigg(\;\; \geq \alpha_m\,\kappa + \beta > 0\,.$$

This implies that $d_\mu(s)/m_0(s)$ is non-zero; hence $d_\mu(s)$ is non-zero. Since $d_\mu(s)$ is non-zero for all s with $Re(s) \geq 0$, d_μ is Hurwitz. ■

5.1.2 Achieving GUEC

For GUEC with rate $\alpha_0 > 0$, it is sufficient that $\tilde{d}_\mu(s) = d_\mu(s - \alpha_0)$ be Hurwitz for all non-zero eigenvalues μ of the graph Laplacian. From (5.2), it is necessary that $\tilde{c}_u(s) = c_u(s - \alpha_0)$ is Hurwitz.

Corollary 5.1.1 *Suppose c and m_0 are monic polynomials of degree n and $n-1$ respectively and $\tilde{m}_0 = m_0(s-\alpha_0)$ is Hurwitz. Then, $\tilde{d}_\mu = \tilde{c} + \mu\,\kappa\,\tilde{m}_0$ is Hurwitz for every non-zero eigenvalue of the graph Laplacian if κ is sufficiently large where $\tilde{c}(s) = c(s-\alpha_0)$.*

Proof The result is a consequence of Lemma 5.1.1. ■

Remark 5.1.1 *A Hurwitz polynomial $\tilde{m}_0$ can be obtained if $\tilde{c}_u(s) = c_u(s-\alpha_0)$ is Hurwitz. Then, one can use pole-placement techniques to obtain a $\mathbf{K}_c$ such that $\tilde{\gamma}_c(s) = \gamma_c(s-\alpha_0)$ and therefore $\tilde{\gamma}(s) = \gamma(s-\alpha_0)$ are Hurwitz.*

5.2 Sufficient conditions for consensus - Quadratic stability

For time-invariant systems, Qu [29] developed necessary and sufficient conditions for the closed-loop network (2.1)-(2.2) to achieve GUAC. The result extends to the time-variant case, which we will show next.

We wish to identify a control $\mathbf{K}(t)$ that can stabilize (3.20) for a variety of complex valued μ.

Remark 5.2.1 *Lemma 3.1.4 reduced the consensus problem to the simultaneous stabilization of a bunch of systems, which is similar to solving a robust control problem.*

There are different techniques to obtain controllers that simultaneously stabilize a bunch of (single) uncertain systems. It should be noted though that we have to stabilize (3.20) for different $\mu \in \mathbb{C}$ at the same time. Therefore, adaptive control techniques that adjust an estimate of μ will not be applicable in this case.

We now focus on control designs that make use of quadratic stability. If the systems are quadratically stable, then they are GUES and therefore GUAS.

Definition 5.2.1 *The system $\dot{\mathbf{x}} = \mathbf{A}(t)\,\mathbf{x}$ is quadratically stable with common Lyapunov matrix $\mathbf{P} = \mathbf{P}' > \mathbf{0}$ if and only if there exists an $\epsilon > 0$ such that for all t:*

$$\mathbf{P}\,\mathbf{A}(t) + \mathbf{A}'(t)\,\mathbf{P} + \epsilon\,\mathbf{P} \leq \mathbf{0}\,.$$

Next, we will present results where either $\mathbf{B}(t)$ is known and we can use a time-varying $\mathbf{K}(t)$ (e.g. a network of switching systems where the switching patterns are known), or $\mathbf{K}$ is fixed and $\mathbf{B}(t)$ can be bounded (e.g. a network of uncertain systems). Both results are based on the LMI condition developed for the general case where $\mathbf{K}(t)$ is time-varying.

5.2.1 Time-varying $\mathbf{K}(t)$ - general case

First, we have the following preliminary result, which allows us to formulate LMI conditions such that all matrix entries are real.

Lemma 5.2.1 *Suppose $\mathbf{Q}_{real} \in \mathbb{R}^{n\times n}$ is hermitian and $\mathbf{Q}_{img} \in \mathbb{R}^{n\times n}$ is skew-hermitian. Then, $\mathbf{Q}_{real} + \jmath\,\mathbf{Q}_{img} \leq \mathbf{0}$ if and only if*

$$\begin{bmatrix} \mathbf{Q}_{real} & \mathbf{Q}'_{img} \\ \mathbf{Q}_{img} & \mathbf{Q}_{real} \end{bmatrix} \leq \mathbf{0}\,. \tag{5.4}$$

Proof We recall that $\mathbf{Q}_{real} + \jmath\,\mathbf{Q}_{img} \leq \mathbf{0}$ if and only if for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$,

$$(\mathbf{x} - \jmath\,\mathbf{y})^T\ [\mathbf{Q}_{real} + \jmath\,\mathbf{Q}_{img}]\ (\mathbf{x} + \jmath\,\mathbf{y}) \leq 0$$

which is equivalent to

$$\mathbf{x}'\,\mathbf{Q}_{real}\,\mathbf{x} - \mathbf{x}'\,\mathbf{Q}_{img}\,\mathbf{y} + \mathbf{y}'\,\mathbf{Q}_{img}\,\mathbf{x} + \mathbf{y}'\,\mathbf{Q}_{real}\,\mathbf{y} \leq 0\,. \tag{5.5}$$

This can be seen by noting that $\mathbf{Q}_{real}$ is hermitian and $\mathbf{Q}_{img}$ is skew-hermitian. Thus, for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$: $\mathbf{x}'\,\mathbf{Q}_{real}\,\mathbf{y} = \mathbf{y}'\,\mathbf{Q}_{real}\,\mathbf{x}$, $\mathbf{x}'\,\mathbf{Q}_{img}\,\mathbf{y} = -\mathbf{y}'\,\mathbf{Q}_{img}\,\mathbf{x}$, and $\mathbf{x}'\,\mathbf{Q}_{img}\,\mathbf{x} = 0$. Finally, writing (5.5) in matrix form and noting that $\mathbf{Q}_{img} = -\mathbf{Q}'_{img}$ yields (5.4). ■

Now applying Lemma 3.1.4 yields the following result.

Theorem 5.2.1 *The closed-loop network (5.1) achieves GUAC if there exists a matrix* $\mathbf{S} = \mathbf{S}' > \mathbf{0}$*, an* $\epsilon > 0$*, and for each* t*, there is a matrix* $\mathbf{X}(t)$ *such that, for all non-zero eigenvalues* $\mu = \alpha + \jmath\,\omega$ *of the graph Laplacian,*

$$\begin{bmatrix} \mathbf{Y}(t) + \mathbf{Y}'(t) + \epsilon\,\mathbf{S} & \omega\,[\mathbf{B}(t)\,\mathbf{X}(t) - \mathbf{X}'(t)\,\mathbf{B}'(t)] \\ \omega\,[\mathbf{X}'(t)\,\mathbf{B}'(t) - \mathbf{B}(t)\,\mathbf{X}(t)] & \mathbf{Y}(t) + \mathbf{Y}'(t) + \epsilon\,\mathbf{S} \end{bmatrix} \le \mathbf{0}\,, \qquad (5.6)$$

where $\mathbf{Y}(t) = \mathbf{A}(t)\,\mathbf{S} - \alpha\,\mathbf{B}(t)\,\mathbf{X}(t)$ *and*

$$\mathbf{K}(t) = \mathbf{X}(t)\,\mathbf{S}^{-1}. \qquad (5.7)$$

Proof From Lemma 3.1.4, we prove Theorem 5.2.1 by showing that systems

$$\dot{\mathbf{z}} = \mathbf{A}_{cl}(t)\,\mathbf{z}\,, \qquad \mathbf{A}_{cl}(t) = \mathbf{A}(t) - \mu\,\mathbf{B}(t)\mathbf{K}(t) \qquad (5.8)$$

are GUAS for all $\mu \neq 0$. From Definition 5.2.1, system (5.8) is quadratically stable (and therefore GUAS) if there exists a $\mathbf{P} = \mathbf{P}' > \mathbf{0}$ and an $\epsilon > 0$ such that for all t:

$$\mathbf{P}\,\mathbf{A}_{cl}(t) + \mathbf{A}'_{cl}(t)\,\mathbf{P} + \epsilon\,\mathbf{P} \le \mathbf{0}\,.$$

Let $\mathbf{S} = \mathbf{P}^{-1}$, then pre- and post-multiplying by $\mathbf{S}$ yields

$$\mathbf{A}_{cl}(t)\,\mathbf{S} + \mathbf{S}\,\mathbf{A}'_{cl}(t) + \epsilon\,\mathbf{S} \le \mathbf{0}\,. \qquad (5.9)$$

Substituting for $\mathbf{A}_{cl}(t)$ and $\mathbf{K}(t)$, we note that (5.9) is equivalent to

$$\mathbf{A}(t)\,\mathbf{S} + \mathbf{S}\,\mathbf{A}'(t) - \mu\,\mathbf{B}(t)\,\mathbf{X}(t) - \bar{\mu}\,\mathbf{X}'(t)\,\mathbf{B}'(t) + \epsilon\,\mathbf{S} \le \mathbf{0} \qquad (5.10)$$

Since $\mu = \alpha + \jmath\,\omega$, (5.10) can be written as $\mathbf{Q}(t) = \mathbf{Q}_{real}(t) + \jmath\,\mathbf{Q}_{img}(t) \le \mathbf{0}$ where

$$\mathbf{Q}_{real}(t) = \mathbf{Y}(t) + \mathbf{Y}'(t) + \epsilon\,\mathbf{S}\,, \qquad \mathbf{Q}_{img}(t) = \omega\,[\mathbf{X}'(t)\,\mathbf{B}'(t) - \mathbf{B}(t)\,\mathbf{X}(t)]$$

and $\mathbf{Y}(t) = \mathbf{A}(t)\,\mathbf{S} - \alpha\,\mathbf{B}(t)\,\mathbf{X}(t)$. From Lemma 5.2.1, $\mathbf{Q}(t) \le \mathbf{0}$ is equivalent to (5.6) and guarantees that systems (5.8) are GUAS for all $\mu \neq 0$. ■

Remark 5.2.2 *If a Laplacian eigenvalue μ is real, then $\mu = \alpha$, $\omega = 0$, and condition (5.6) simplifies to*

$$\mathbf{Y}(t) + \mathbf{Y}'(t) + \epsilon\, \mathbf{S} \leq \mathbf{0}\,.$$

Remark 5.2.3 *We note the symmetry of (5.6) with respect to ω. Thus, if (5.6) holds for some ω, then it also holds for $\tilde{\omega} = -\omega$. If (5.6) has to hold for all ω, then from (5.5) we require that for all ω and all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$,*

$$\omega\, [\mathbf{x}'\, (\mathbf{B}\,\mathbf{X} - \mathbf{X}'\,\mathbf{B}')\, \mathbf{y} + \mathbf{y}'\, (\mathbf{X}'\,\mathbf{B}' - \mathbf{B}\,\mathbf{X})\, \mathbf{x}] \leq 0$$

which can only be satisfied if $\mathbf{B}\,\mathbf{X} = \mathbf{X}'\,\mathbf{B}'$ since ω can be positive or negative.

LMI (5.6) was formed such that all matrix entries can be chosen to be real. However, this form is less compact.

Remark 5.2.4 *From the proof of Theorem 5.2.1, LMI (5.6) is equivalent to*

$$\mathbf{A}(t)\,\mathbf{S} + \mathbf{S}\,\mathbf{A}'(t) - \mu\,\mathbf{B}(t)\,\mathbf{X}(t) - \bar{\mu}\,\mathbf{X}'(t)\,\mathbf{B}'(t) + \epsilon\,\mathbf{S} \leq \mathbf{0}\,. \tag{5.11}$$

From Lemma 2.1.1, we can achieve GUAC for a network of time-varying transformed systems $(\tilde{A}(t), B(t))$. Recalling (2.4), we note that

$$\mathbf{A}(t) = \tilde{\mathbf{A}}(t) - \alpha_0\,\mathbf{I} \tag{5.12}$$

and from Remark 2.1.2 we know that the gain matrix $\mathbf{K}(t)$ for the original and the transformed network are equivalent.

Lemma 5.2.2 *The closed-loop network (5.1) achieves GUEC with rate $\alpha_0 > 0$ if conditions in Theorem 5.2.1 hold for some $\epsilon > 2\,\alpha_0$.*

Proof From Remark 5.2.4, LMI (5.6) is equivalent to (5.11). If (5.11) holds for some $\epsilon > 2\,\alpha_0$, then $\tilde{\epsilon} := \epsilon - 2\,\alpha_0 > 0$ and substituting $\mathbf{A}(t)$ from (5.12) yields

$$\tilde{\mathbf{A}}(t)\,\mathbf{S} + \mathbf{S}\,\tilde{\mathbf{A}}'(t) - \mu\,\mathbf{B}(t)\,\mathbf{X}(t) - \bar{\mu}\,\mathbf{X}'(t)\,\mathbf{B}'(t) + \tilde{\epsilon}\,\mathbf{S} \leq \mathbf{0}\,.$$

Hence, the network of transformed systems $(\tilde{A}(t), B(t))$ achieves GUAC with $\mathbf{K}(t) = \mathbf{X}(t)\,\mathbf{S}^{-1}$, and the closed-loop network (5.1) achieves GUEC with rate α_0. ∎

5.2.2 Time-varying $\mathbf{K}(t)$ - $\mathbf{B}(t)$ known

We note that condition (5.6) simplifies for $\mathbf{X}(t) = \mathbf{B}'(t)$, which was observed by Tuna [49] for the case of time-invariant systems.

Theorem 5.2.2 *The closed-loop network (5.1) achieves GUEC with rate $\alpha_0 > 0$ if there exists a matrix $\mathbf{S} = \mathbf{S}' > \mathbf{0}$ and an $\epsilon > 2\,\alpha_0$ such that for all t,*

$$\mathbf{A}(t)\,\mathbf{S} + \mathbf{S}\,\mathbf{A}'(t) - \mathbf{B}(t)\,\mathbf{B}'(t) + \epsilon\,\mathbf{S} \leq \mathbf{0}\,. \tag{5.13}$$

and

$$\mathbf{K}(t) = \frac{1}{2\,\alpha_m}\mathbf{B}'(t)\,\mathbf{S}^{-1} \tag{5.14}$$

where α_m is given by (2.9).

Proof Suppose (5.13) is satisfied for some $\mathbf{S} = \mathbf{S}' > \mathbf{0}$. If (5.13) is multiplied by $2\,\alpha_m$ and $\tilde{\mathbf{S}} := 2\,\alpha_m\,\mathbf{S}$ is substituted, then this yields

$$\mathbf{A}(t)\,\tilde{\mathbf{S}} + \tilde{\mathbf{S}}\,\mathbf{A}'(t) - 2\,\alpha_m\,\mathbf{B}(t)\,\mathbf{B}'(t) + \epsilon\,\tilde{\mathbf{S}} \leq \mathbf{0}\,.$$

We observe that $\mathbf{B}(t)\,\mathbf{B}'(t) \geq \mathbf{0}$. Thus,

$$\mathbf{A}(t)\,\tilde{\mathbf{S}} + \tilde{\mathbf{S}}\,\mathbf{A}'(t) - 2\,\alpha\,\mathbf{B}(t)\,\mathbf{B}'(t) + \epsilon\,\tilde{\mathbf{S}} \leq \mathbf{0}$$

for all $\alpha \geq \alpha_m$, or equivalently

$$\mathbf{A}(t)\,\tilde{\mathbf{S}} + \tilde{\mathbf{S}}\,\mathbf{A}'(t) - (\mu + \bar{\mu})\,\mathbf{B}(t)\,\mathbf{B}'(t) + \epsilon\,\tilde{\mathbf{S}} \leq \mathbf{0}$$

for all non-zero eigenvalues $\mu = \alpha + \jmath\,\omega$ of the graph Laplacian. The proof is completed by identifying $\mathbf{X}(t) = \mathbf{B}'(t)$ in (5.11), choosing $\mathbf{K}(t) = \mathbf{X}(t)\,\tilde{\mathbf{S}}^{-1}$ from (5.7), and applying Lemma 5.2.2 and Remark 5.2.4. ∎

In principle, conditions (5.6) and (5.13) should only be applied to a finite number of systems, e.g. a network of switching systems with known switching pattern. Otherwise, unless restrictions on $\mathbf{A}(t)$ and $\mathbf{B}(t)$ are made, applying Theorem 5.2.1 or Theorem 5.2.2 will yield an infinite number of LMIs and a solution will not be feasible. An infinite number of LMIs can be avoided if the set of $\mathbf{A}(t)$ and $\mathbf{B}(t)$ lies inside a polytope for example. Then, we wish to identify a time-invariant $\mathbf{X}$ such that the conditions hold for all vertices of this polytope. Alternatively, one could use results for quadratically stabilizable systems, which are discussed next.

5.2.3 Quadratically stabilizable systems - fixed K

Here we assume that we do not know the exact value of $\mathbf{B}(t)$ at time t, and we wish to identify a static gain matrix $\mathbf{K}$ such that the time-varying closed-loop network achieves GUEC.

Theorem 5.2.3 *Let $\mathbf{B}(t)$ be given by $\mathbf{B}(t) = \mathbf{\Delta}(t)\,\mathbf{B}_0$ where $\mathbf{\Delta}(t) \geq \beta_m\,\mathbf{I} > \mathbf{0}$ and $\mathbf{B}(t)\,\mathbf{B}_0$ is symmetric. Then, the closed-loop network (5.1) achieves GUEC with rate $\alpha_0 > 0$ if there exists a matrix $\mathbf{S} = \mathbf{S}' > \mathbf{0}$ and an $\epsilon > 2\,\alpha_0$ such that for all t,*

$$\mathbf{A}(t)\,\mathbf{S} + \mathbf{S}\,\mathbf{A}'(t) - \mathbf{B}_0\,\mathbf{B}_0' + \epsilon\,\mathbf{S} \leq \mathbf{0} \tag{5.15}$$

and

$$\mathbf{K} = \frac{1}{2\,\alpha_m\,\beta_m}\mathbf{B}_0'\,\mathbf{S}^{-1} \tag{5.16}$$

where α_m is given by (2.9).

Proof Suppose (5.15) is satisfied for some $\mathbf{S} = \mathbf{S}' > \mathbf{0}$ and $\epsilon > 0$. If (5.15) is multiplied by $2\,\alpha_m\,\beta_m$ and $\tilde{\mathbf{S}} := 2\,\alpha_m\,\beta_m\,\mathbf{S}$ is substituted, then this yields

$$\mathbf{A}(t)\,\tilde{\mathbf{S}} + \tilde{\mathbf{S}}\,\mathbf{A}'(t) - 2\,\alpha_m\,\beta_m\,\mathbf{B}_0\,\mathbf{B}_0' + \epsilon\,\tilde{\mathbf{S}} \leq \mathbf{0}\,.$$

We observe that $\mathbf{\Delta}(t)\,\mathbf{B}_0\,\mathbf{B}_0' = \mathbf{B}(t)\,\mathbf{B}_0' \geq \beta_m\,\mathbf{B}_0\,\mathbf{B}_0' \geq \mathbf{0}$. Thus,

$$\mathbf{A}(t)\,\tilde{\mathbf{S}} + \tilde{\mathbf{S}}\,\mathbf{A}'(t) - 2\,\alpha\,\mathbf{B}(t)\,\mathbf{B}_0' + \epsilon\,\tilde{\mathbf{S}} \leq \mathbf{0} \tag{5.17}$$

for all $\alpha \geq \alpha_m$, or equivalently

$$\mathbf{A}(t)\,\tilde{\mathbf{S}} + \tilde{\mathbf{S}}\,\mathbf{A}'(t) - (\mu + \bar{\mu})\,\mathbf{B}(t)\,\mathbf{B}_0' + \epsilon\,\tilde{\mathbf{S}} \leq \mathbf{0} \tag{5.18}$$

for all $\mu = \alpha + \jmath\omega$. Since (5.17) holds for all $\alpha \geq \alpha_m$, (5.18) holds for all non-zero eigenvalues of the graph Laplacian. Matrix $\mathbf{B}(t)\,\mathbf{B}_0'$ is symmetric. Thus, the proof is completed by identifying $\mathbf{X} = \mathbf{B}_0'$ in (5.11), choosing $\mathbf{K} = \mathbf{X}\,\tilde{\mathbf{S}}^{-1}$ from (5.7), and applying Lemma 5.2.2 and Remark 5.2.4. ■

Remark 5.2.5 *If systems (A, B) are controllable, then (5.13) or (5.15) can be satisfied for any $\epsilon > 0$. Thus, the rate $\alpha_0 > 0$ can be chosen arbitrarily large.*

Remark 5.2.6 *In Theorem 5.2.1, the number of LMIs that have to be satisfied depends on the number of non-zero Laplacian eigenvalues. The LMI condition in Theorems 5.2.2 and 5.2.3 is independent of the number of eigenvalues.*

Remark 5.2.7 *Theorem 5.2.3 is based on (5.11) where $\mathbf{B}(t)$ and $\mathbf{K}(t)$ always appear as a product. Therefore, scaling $\mathbf{B}(t)$ is equivalent to scaling $\mathbf{K}(t)$. Since Theorem 5.2.3 requires a lower bound on $\mathbf{B}(t)$ only, we conclude that the closed-loop network (5.1) achieves GUEC with rate α_0 for any control $\tilde{\mathbf{K}} = \kappa\,\mathbf{K}$ where $\kappa \geq 1$ and $\mathbf{K}$ is given by (5.16).*

To strengthen Theorem 5.2.3, we make the following observation.

Lemma 5.2.3 *There exist a $\mathbf{P} = \mathbf{P}' > \mathbf{0}$, an $\tilde{\epsilon} > 0$, and a $\mathbf{K}_0$ such that for all t,*

$$\mathbf{P}\,\mathbf{A}(t) + \mathbf{A}'(t)\,\mathbf{P} - \mathbf{P}\,\mathbf{B}_0\,\mathbf{K}_0 - \mathbf{K}_0'\,\mathbf{B}_0'\,\mathbf{P} + \tilde{\epsilon}\,\mathbf{P} \leq \mathbf{0}\,. \tag{5.19}$$

if and only if there exists a $\mathbf{S} = \mathbf{S}' > \mathbf{0}$ and an $\epsilon > 0$ such that (5.15) holds for all t, where $\mathbf{S} = \delta\,\mathbf{P}^{-1}$ for some $\delta > 0$.

Proof To show that (5.15) implies (5.19), we choose $\mathbf{P} = \mathbf{S}^{-1}$, $\tilde{\epsilon} = \epsilon$, and $\mathbf{K}_0 = \mathbf{B}_0'\,\mathbf{P}$. Now, suppose that (5.19) holds for some $\mathbf{P} = \mathbf{P}' > \mathbf{0}$ and some $\mathbf{K}_0$. First, we note that $\delta^{-1}\,(a - \delta\,b)^2 \geq 0 \;\Leftrightarrow\; 2\,a\,b \leq \delta^{-1}\,a^2 + \delta\,b^2$. Therefore, for all $\mathbf{x} \neq 0$ and $\delta > 0$,

$$2\,\mathbf{x}'\,\mathbf{P}\,\mathbf{B}_0\,\mathbf{K}_0\,\mathbf{x} = 2\,\mathbf{x}'\,\mathbf{K}_0'\,\mathbf{B}_0'\,\mathbf{P}\,\mathbf{x} \leq \delta^{-1}\,\mathbf{x}'\,\mathbf{P}\,\mathbf{B}_0\,\mathbf{B}_0'\,\mathbf{P}\,\mathbf{x} + \delta\,\mathbf{x}'\,\mathbf{K}_0'\,\mathbf{K}_0\,\mathbf{x}\,.$$

Thus,

$$-\mathbf{P}\,\mathbf{B}_0\,\mathbf{K}_0 - \mathbf{K}_0'\,\mathbf{B}_0'\,\mathbf{P} \geq -\delta^{-1}\mathbf{P}\,\mathbf{B}_0\,\mathbf{B}_0'\,\mathbf{P} - \delta\,\mathbf{K}_0'\,\mathbf{K}_0 \tag{5.20}$$

for any $\delta > 0$. Substituting $\mathbf{P} = \delta\,\mathbf{S}^{-1}$ in (5.19), using (5.20), and re-arranging yields

$$\mathbf{A}(t)\,\mathbf{S} + \mathbf{S}\,\mathbf{A}'(t) - \mathbf{B}_0\,\mathbf{B}_0' - \delta^2\,\mathbf{S}\,\mathbf{K}_0'\,\mathbf{K}_0\,\mathbf{S} + \tilde{\epsilon}\,\mathbf{S} \leq \mathbf{0} \tag{5.21}$$

for any $\delta > 0$. Since δ can be arbitrarily small and $\tilde{\epsilon} > 0$, we can always choose an $\epsilon > 0$ and $\delta > 0$ such that $\tilde{\epsilon}\,\mathbf{S} - \delta^2\,\mathbf{S}\,\mathbf{K}_0'\,\mathbf{K}_0\,\mathbf{S} \geq \epsilon\,\mathbf{S}$, which if substituted into (5.21) yields (5.18). ■

Identifying matrices $\mathbf{P} = \mathbf{P}' > 0$ and $\mathbf{K}$ such that (5.19) holds for all t is a well studied problem and its (numerical) solution simplifies for special cases of $\mathbf{A}(t)$ [79–84]. Once $\mathbf{P}$ is obtained, then it is only a matter of adjusting the scalar parameter δ to obtain an $\mathbf{S} = \delta\,\mathbf{P}^{-1}$ that satisfies (5.15). In principle, if $\mathbf{P}$ is known, then a control will be given by $\mathbf{K} = \kappa\,\mathbf{B}_0'\,\mathbf{P}$ where $\kappa > 0$ is a single, linear, and scalar parameter that has to be tuned. Thus, Theorem 5.2.3 and Lemma 5.2.3 yield the following remark.

Remark 5.2.8 *If systems are quadratically stabilizable, that is,* (5.19) *can be satisfied, then there exists a linear control such that the closed-loop network* (5.1) *achieves GUEC.*

5.3 Necessary conditions to achieve consensus for time-invariant (A, B)

The conditions above are sufficient only since we identify a single matrix $\mathbf{S} = \mathbf{S}' > \mathbf{0}$ that guarantees stability for a bunch of systems. If systems are time-invariant and quadratically stabilizable, then they are always stabilizable with a linear control, and we recover the following corollary from Theorem 5.2.3.

Corollary 5.3.1 ([30]) *If systems are time-invariant, then there exists a matrix* $\mathbf{K}$ *for which the closed-loop network* (2.1)-(2.2) *achieves GUAC if and only if the pair* $(\mathbf{A}, \mathbf{B})$ *is stabilizable.*

5.4 Simulation

The chosen example demonstrates how gain matrix $\mathbf{K}$ is designed robustly such that the closed-loop network achieves GUEC with rate $\alpha_0 = 0.1$ for a network of arbitrary switching systems (A_1, B) and (A_2, B) in controllable canonical form where

$$\mathbf{A}_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -4.04 & -0.8 & -5.01 & -0.2 \end{bmatrix}, \quad \mathbf{A}_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -9 & 0 & -10 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

and the open-loop poles of $\mathbf{A}_1$ and $\mathbf{A}_2$ were given by

$$\mathbf{A}_1 : \quad -0.1 \pm 1\jmath, \pm 2\jmath \quad \text{and} \quad \mathbf{A}_2 : \quad \pm 1\jmath, \pm 3\jmath\,.$$

The communication structure was described by the graph shown in Figure 5.1. The associated Laplacian matrix

$$L_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -2 & 3 & 0 & -1 \\ 0 & -1 & 1 & 0 \\ -2 & 0 & -1 & 3 \end{bmatrix}, \quad L_2 = \begin{bmatrix} 1 & 0 & 0 & -1 \\ -1 & 3 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & -2 & 3 \end{bmatrix}$$

had eigenvalues $\mu_1 = 0$, $\mu_2 = 0.80$, and $\mu_{3/4} = 3.1 \pm .67\jmath$. In this network, System 1 is a leader, and when GUEC is achieved, the positions of the other systems will equal that of System 1.

Considering a rate of convergence $\alpha_0 = 0.1$ and the fact that $\mathbf{B}$ was constant, we applied Theorem 5.2.2 ($\alpha_m = 0.8$) and obtained the constant gain matrix

$$\mathbf{K} = \begin{bmatrix} 5.48 & 11.3 & 5.82 & 3.86 \end{bmatrix}.$$

The network achieved GUEC as shown in Figure 5.2. Systems were given in controllable canonical form. Therefore, the figure shows the trajectory of the last element of each state vector $\mathbf{x}_i(t)$ only. States were initialized randomly over a uniform distribution, and we switched between the two configurations every second. For each system,

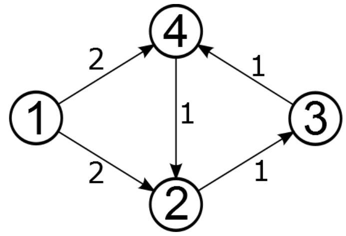


Figure 5.1.. Communication network with assigned edge weights.

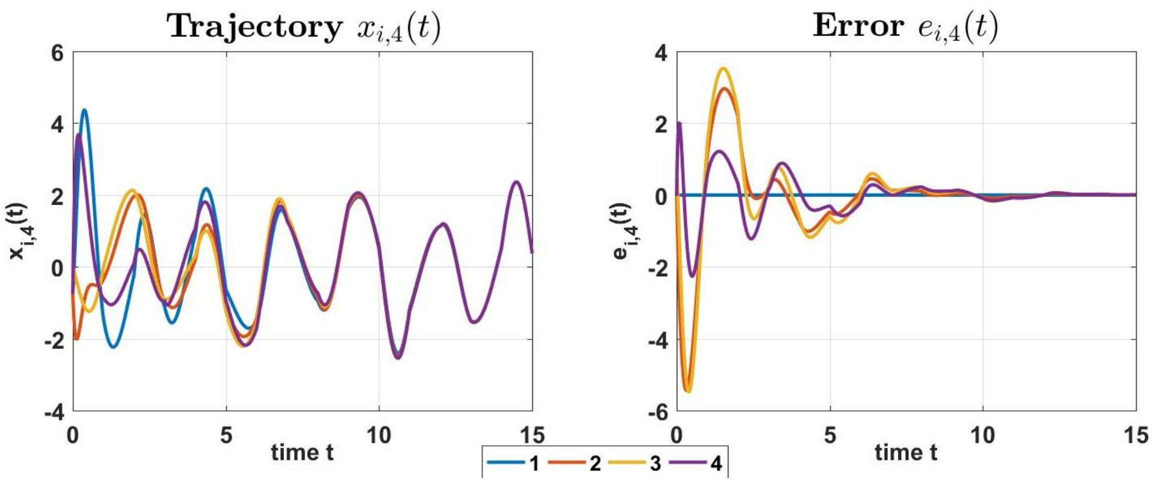


Figure 5.2.. Simulation of four time-variant switching systems achieving GUEC. Note that only $x_{i,4}(t)$ and $e_{i,4}(t)$ are shown since systems are in controllable canonical form.

we introduced an error $\mathbf{e}_i(t) = \mathbf{x}_i(t) - \mathbf{x}_1(t)$. GUEC was achieved and the steady state dynamics were determined by the initial conditions of System 1 (blue line) as seen on the left side of Figure 5.2. The process of achieving GUEC is illustrated on the right side of Figure 5.2 where the last element of each error $\mathbf{e}_i(t)$ is plotted.

6. CONSENSUS FOR DISCRETE-TIME SECOND-ORDER SYSTEMS

Here we consider a homogeneous network of linear time-invariant second-order systems. In this case, (2.5) is given by

$$\mathbf{x}_i(k+1) = \mathbf{A}\,\mathbf{x}_i(k) + \mathbf{B}\,u_i(k) + \mathbf{u}_0(k) \tag{6.1}$$

where $\mathbf{x}_i(k) \in \mathbb{R}^2$, $\mathbf{A} \in \mathbb{R}^{2\times 2}$, $\mathbf{B} \in \mathbb{R}^{2\times 1}$, a linear control $u_i(k) \in \mathbb{R}$ is given by

$$u_i(k) = \mathbf{K} \sum_{j \in N_i} w_{ij} \left[\mathbf{x}_j(k) - \mathbf{x}_i(k) \right] \tag{6.2}$$

and the special system parameter (3.28)-(3.29) reduce to

$$c(s) = \det\left(s\,\mathbf{I} - \mathbf{A}\right) = s^2 + c_1\,s + c_0 \tag{6.3}$$

and

$$[\gamma_0 \quad \gamma_1] = \mathbf{K}\left[(\mathbf{A} + c_1\,\mathbf{I})\,\mathbf{B} \quad \mathbf{B}\right]. \tag{6.4}$$

If (A, B) is in controllable canonical form, then one can easily show that $[\gamma_0\ \gamma_1] = \mathbf{K}$.

Throughout this chapter, it is assumed that

(A, B) *is controllable.*

6.1 Necessary and sufficient conditions for consensus

6.1.1 Conditions for consensus

The following theorem summarizes the necessary and sufficient conditions for the closed-loop network (6.1)-(6.2) to achieve GUAC.

Theorem 6.1.1 *The closed-loop network (6.1)-(6.2) achieves GUAC if and only if for each non-zero eigenvalue μ of the graph Laplacian matrix,*

$$|d_0|^2 + |d_1 - d_0\,\bar{d}_1| < 1 \tag{6.5}$$

where

$$d_0 = c_0 + \mu\,\gamma_0\,, \qquad d_1 = c_1 + \mu\,\gamma_1 \tag{6.6}$$

and c_0, c_1 and γ_0, γ_1 are given by (6.3) and (6.4).

Proof A proof of Theorem 6.1.1 is provided in Section 6.4. ■

For a real Laplacian eigenvalue, condition (6.5) simplifies as described in the following remark.

Remark 6.1.1 *If μ is real, then condition (6.5) simplifies to*

$$d_0 < 1 \quad \textit{and} \quad |d_1| < 1 + d_0\,. \tag{6.7}$$

Theorem 6.1.1 provides a necessary and sufficient condition on the gains γ_0 and γ_1 for GUAC, namely (6.5). When (6.5) is satisfied, a consensus achieving gain matrix $\mathbf{K}$ is given by (3.30). We now discuss condition (6.5).

6.1.2 Condition (6.5)

Inequality (6.5) is equivalent to

$$|d_2| < 1 - |d_0|^2 \quad \text{where} \qquad d_2 := d_1 - d_0\,\bar{d}_1\,. \tag{6.8}$$

For this to be satisfied, we must have

$$|d_0| < 1\,. \tag{6.9}$$

If we let

$$\mu = \alpha + \jmath\,\omega \tag{6.10}$$

where α and ω are real, then one may readily compute that

$$d_2 = (1 - c_0)\, c_1 - (|\mu|^2 \gamma_0 + 2\, c_0\, \alpha)\, \gamma_1 + \mu\, \gamma_2 \tag{6.11}$$

where $\gamma_2 = (1 + c_0)\, \gamma_1 - c_1\, \gamma_0$.

If $c_0 \neq -1$, then

$$d_2 = \frac{c_1\, \delta}{1 + c_0} + \frac{\tilde{a}}{1 + c_0}\, \gamma_2 + \jmath\, \omega\, \gamma_2 \tag{6.12}$$

$$\gamma_1 = \frac{c_1\, \gamma_0 + \gamma_2}{1 + c_0} \tag{6.13}$$

where

$$\tilde{a} = (1 - c_0)\, \alpha - |\mu|^2\, \gamma_0\,, \quad \delta = 1 - |d_0|^2 > 0\,. \tag{6.14}$$

If $c_0 = -1$, then $\gamma_2 = -c_1\, \gamma_0$ and

$$d_2 = (2 - \alpha\, \gamma_0)\, c_1 + (2\, \alpha - |\mu|^2 \gamma_0)\, \gamma_1 - \jmath\, \omega\, c_1\, \gamma_0 \tag{6.15}$$

$$\delta = (2\, \alpha - |\mu|^2 \gamma_0)\, \gamma_0 \tag{6.16}$$

Remark 6.1.2 *Inequalities (6.8) and (6.9) are of the form*

$$|c + \eta\, \gamma| < b\,, \qquad \eta = a + \jmath\, v \tag{6.17}$$

which is equivalent to $b > 0$ and

$$|\eta|^2\, \gamma^2 + 2\, a\, c\, \gamma + c^2 - b^2 < 0 \tag{6.18}$$

or $p(\gamma) < 0$, where $p(\gamma) = |\eta|^2\, \gamma^2 + 2\, a\, c\, \gamma + c^2 - b^2$.

Since $\lim_{|\gamma| \to \infty} p(\gamma) = \infty$, p must have two distinct real roots $\underline{\gamma} < \bar{\gamma}$ for $p(\gamma) < 0$ to be satisfiable; these roots are

$$\underline{\gamma}, \bar{\gamma} = \frac{-a\, c \pm \sqrt{a^2\, b^2 + (b^2 - c^2)\, v^2}}{|\eta|^2}\,. \tag{6.19}$$

Thus, a necessary condition for the existence of a γ satisfying inequality (6.17) is

$$(c^2 - b^2)\, v^2 < a^2\, b^2\,, \tag{6.20}$$

and inequality (6.17) holds if and only if

$$\underline{\gamma} < \gamma < \bar{\gamma}\,. \tag{6.21}$$

Special case: η real. If η is real, then $\eta = a$ and $v = 0$; hence

$$\underline{\gamma}, \bar{\gamma} = (-c \pm b)/\eta\,, \quad b > 0\,. \tag{6.22}$$

Remark 6.1.3 *If $|c| < b$, then (6.17) holds with γ small. If $|c| = b$, then*

$$\underline{\gamma}, \bar{\gamma} = \frac{-a\,c \pm |a\,c|}{|\eta|^2}\,. \tag{6.23}$$

Hence, either $\underline{\gamma} = 0$ or $\bar{\gamma} = 0$ and inequality (6.17) can be satisfied by an arbitrarily small non-zero γ whose sign is opposite to the sign of ac. Thus, if one has a collection of inequalities of the form (6.17) for which each ac has the same sign, then these inequalities can be simultaneously satisfied by an arbitrarily small non-zero γ whose sign is the opposite of the sign of the ac's.

Remark 6.1.4 *Based on Remark 6.1.2, we note that, in order for (6.9) to hold, we must have*

$$(c_0^2 - 1)\,\frac{\omega^2}{\alpha^2} < 1 \quad \textit{and} \quad \underline{\gamma}_0 < \gamma_0 < \bar{\gamma}_0 \tag{6.24}$$

where

$$\underline{\gamma}_0, \bar{\gamma}_0 = \frac{-c_0\,\alpha \pm \sqrt{\alpha^2 + (1 - c_0^2)\,\omega^2}}{|\mu|^2}\,. \tag{6.25}$$

If μ is real, then $\mu = \alpha > 0$ and

$$\underline{\gamma}_0, \bar{\gamma}_0 = (-c_0 \pm 1)/\mu\,. \tag{6.26}$$

If $|c_0| < 1$, then Remark 6.1.3 tells us that (6.21) holds for all non-zero μ if $\gamma_0 = 0$. If $|c_0| = 1$, then it follows from Remark 6.1.3 that (6.21) can be satisfied by all non-zero μ if γ_0 is sufficiently small and its sign equals that of $-c_0$.

Remark 6.1.5 *Based on Remark 6.1.2, when* $c_0 \neq -1$*, then in order for (6.8) to hold, we must have*

$$\left[c_1^2 - (1+c_0)^2\right] \omega^2 < \tilde{a}^2 \quad \textit{and} \quad \underline{\gamma}_2 < \gamma_2 < \bar{\gamma}_2 \tag{6.27}$$

where

$$\underline{\gamma}_2, \bar{\gamma}_2 = \frac{-c_1\, \tilde{a} \pm (1+c_0)\sqrt{\tilde{a}^2 + [(1+c_0)^2 - c_1^2)]\,\omega^2}}{\tilde{a}^2 + (1+c_0)^2\,\omega^2}\,\delta \tag{6.28}$$

If μ *is real, then* $\underline{\gamma}_2, \bar{\gamma}_2 = [-c_1 \pm (1+c_0)]\,\delta/\tilde{a}$*, that is,*

$$\underline{\gamma}_2, \bar{\gamma}_2 = [-c_1 \pm (1+c_0)]\,(1 + c_0 + \mu\,\gamma_0)/\mu$$

Hence

$$\underline{\gamma}_1 < \gamma_1 < \bar{\gamma}_1$$

where

$$\underline{\gamma}_1, \bar{\gamma}_1 = [-c_1 \pm (1 + c_0 + \mu\,\gamma_0)]\,/\mu\,. \tag{6.29}$$

If $|c_1| < |1+c_0|$*, then Remark 6.1.3 tells us that (6.27) holds for all non-zero* μ *if* γ_2 *is sufficiently small. If* $|c_1| = |1+c_0|$ *and* $\tilde{a}$ *has the same sign for all* μ*, then it follows from Remark 6.1.3 that (6.27) can be satisfied by all non-zero* μ *if* γ_2 *is sufficiently small and its sign equals that of* $-c_1\tilde{a}$*. If* $|c_0| \leq 1$*, then* γ_0 *can be chosen arbitrarily small; hence it can be chosen so that the sign of* $\tilde{a}$ *is the same as that of* $1 - c_0$ *for all non-zero* μ*. In this case* γ_1 *can be chosen arbitrarily small.*

Remark 6.1.6 *If* $c_0 = -1$*, then recalling (6.16), inequality (6.9) is equivalent to*

$$g\,\gamma_0 > 0 \qquad \textit{where} \qquad g = 2\,\alpha - |\mu|^2\,\gamma_0\,. \tag{6.30}$$

Using arguments similar to Remark 6.1.2, one can show that when $c_0 = -1$*, (6.8) is equivalent to*

$$\omega^2\, c_1^2 < g^2 \quad \textit{and} \quad \underline{\gamma}_1 < \gamma_1 < \bar{\gamma}_1 \tag{6.31}$$

where

$$\underline{\gamma}_1, \bar{\gamma}_1 = \frac{(\alpha\,\gamma_0 - 2)\,c_1 \pm \gamma_0\,\sqrt{g^2 - \omega^2\,c_1^2}}{g}\,.$$

If μ *is real, then* $g = \alpha\,(2 - \alpha\,\gamma_0)$ *and*

$$\underline{\gamma}_1, \bar{\gamma}_1 = -c_1/\mu \pm \gamma_0\,. \tag{6.32}$$

If $c_1 = 0$*, then*

$$\underline{\gamma}_1, \bar{\gamma}_1 = \pm\gamma_0\,. \tag{6.33}$$

With $\gamma_0 \neq 0$ *this can be satisfied by sufficiently small* γ_1*.*

Remark 6.1.7 *If systems (6.1) are asymptotically stable, then GUAC will be achieved even if no feedback is applied. For* $\gamma_0 = \gamma_1 = 0$*, condition (6.5) reduces to*

$$c_0 < 1 \qquad \text{and} \qquad |c_1| < 1 + c_0\,, \tag{6.34}$$

which are the conditions for open-loop stability.

6.1.3 Limitations

In the continuous-time analog of the problem considered here, which we presented in Chapter 4, GUAC can always be achieved by appropriate choice of K. However, that is not the case in discrete-time. We have already seen limitations on the imaginary parts of μ in (6.21), (6.27) and (6.30). The next result puts limitations on the real parts of μ.

Lemma 6.1.1 *If there exists a matrix* K *such that the closed-loop network* (6.1)-(6.2) *achieves GUAC, then*

$$\kappa\,|c_0| < 1 \quad \text{and} \quad \kappa\,|c_1| < 2 \tag{6.35}$$

where, recalling α_m *and* α_M *from* (2.9)*,*

$$\kappa = \frac{\alpha_M - \alpha_m}{\alpha_M + \alpha_m}\,. \tag{6.36}$$

Proof A proof of this result is given in Section 6.4. ■

In the next section, we look at some important classes of networked systems and provide easilly verifiable conditions for consensus.

6.2 Special cases

Identifying gains that satisfy (6.5) for all $\mu \neq 0$ can be challenging. Here we consider some special cases and obtain simpler conditions for the existence of feasible parameters γ_0 and γ_1.

6.2.1 Marginally stable systems

Here we claim that GUAC can always be achieved for networks of marginally stable systems. If the systems are marginally stable, then $c_0 \leq 1$ and $|c_1| \leq 1 + c_0$ and equality holds for at least one of the inequalities. This is a special case of

$$|c_0| \leq 1 \qquad \text{and} \qquad |c_1| \leq |1 + c_0| \,. \tag{6.37}$$

Lemma 6.2.1 *If (6.37) holds, then GUAC for the closed-loop network (6.1)-(6.2) can always be achieved with arbitrarily small γ_0 and γ_1.*

Proof If $c_0 \neq -1$, then the result follows from comments made in Remarks 6.1.4 and 6.1.5. If $c_0 = -1$, then we must have $c_1 = 0$, and the result follows from comments made in Remarks 6.1.4 and 6.1.6. ■

6.2.2 Laplacian eigenvalues in a disc

From Lemma 6.1.1 we see that if either $|c_0| > 1$ or $|c_1| > 2$, then the range of the eigenvalues of the Laplacian matrix L must be restricted to achieve GUAC. Here we consider the non-zero eigenvalues of the Laplacian to be constrained to a disc of radius R with center $C > 0$. First, we have the following result, which provides simpler sufficient conditions for achieving GUAC; Section 6.4 contains a proof.

Lemma 6.2.2 *The closed-loop network (6.1)-(6.2) achieves GUAC if, for each non-zero eigenvalue μ of the graph Laplacian matrix,*

$$|c_0 + \mu\,\gamma_0| + |c_1 + \mu\,\gamma_1| < 1\,. \tag{6.38}$$

Lemma 6.2.3 *Suppose that*

$$|\mu - C| \leq R \tag{6.39}$$

for all non-zero eigenvalues μ of the Laplacian. Then, the closed-loop network (6.1)-(6.2) achieves GUAC if

$$|c_0{+}\gamma_0\,C| + |c_1{+}\gamma_1\,C| + (|\gamma_0| + |\gamma_1|)\;R < 1\,. \tag{6.40}$$

There exist γ_0, γ_1 satisfying (6.40) if and only if:

1) $R/C \geq 1$:

$$|c_0| + |c_1| < 1\,. \tag{6.41}$$

In this case, (6.40) holds with $\gamma_0 = \gamma_1 = 0$.

2) $R/C < 1$:

$$(|c_0| + |c_1|)\;R/C < 1\,. \tag{6.42}$$

In this case, (6.40) holds with

$$\gamma_0 = -c_0/C \qquad \textit{and} \qquad \gamma_1 = -c_1/C\,. \tag{6.43}$$

Proof From Lemma 6.2.2, GUAC is achieved if, for each non-zero eigenvalue μ of the Laplacian,

$$|d_0| + |d_1| < 1 \tag{6.44}$$

where $d_i = c_i + \mu\,\gamma_i$ for $i = 0, 1$. If $|\mu - C| \leq R$, then $\mu = C + r\,(\cos\psi + \jmath\,\sin\psi)$ where $0 \leq r \leq R$ and $\psi \in \mathbb{R}$. Thus,

$$|d_i|^2 = (c_i + \gamma_i\,C)^2 + 2\,(c_i + \gamma_i\,C)\,\gamma_i\,r\,\cos\psi + \gamma_i^2\,r^2$$

and

$$\max_{|\mu - C| \leq R} |d_i|^2 = (c_i + \gamma_i C)^2 + 2\,|c_i + \gamma_i C||\gamma_i| R + \gamma_i^2 R^2$$
$$= (|c_i + \gamma_i C| + |\gamma_i|\, R)^2 \,.$$

Hence,

$$\max_{|\mu - C| \leq R} |d_i| = |c_i + \gamma_i C| + |\gamma_i|\, R$$

and (6.40) implies (6.44).

We note that $f_i(\gamma_i) := |c_i + C\,\gamma_i| + |\gamma_i|\, R$ is continuous and piecewise linear. Therefore, its minimum will occur at one of the two switching points $\gamma_i = 0$ and $\gamma_i = -c_i/C$; hence the minimum is the smallest of

$$f_i(0) = |c_i| \quad \text{and} \quad f_i(-c_i/C) = |c_i|\, R \,/\, C \,.$$

If $R/C \geq 1$, then the minimum is $|c_i|$ and (6.40) can be satisfied if and only if (6.41) holds. If $R/C < 1$, then $\gamma_i = -c_i/C$ is the minimizer of f_i, and (6.40) can be satisfied if and only if (6.42) holds. ∎

Remark 6.2.1 (Robustness) *The conditions in Lemma 6.2.3 can be used to guarantee robustness with respect to changes in the graph; the only knowledge needed about the graph is that the non-zero eigenvalues of the graph Lapacian lie within a disc of radius R and center C. To obtain robustness with respect to plant parameters suppose that, for some $\bar{c}_0$, $\bar{c}_1$ and Δc, parameters satisfy*

$$|c_0 - \bar{c}_0| + |c_1 - \bar{c}_1| \leq \Delta c \,. \tag{6.45}$$

If $R/C \geq 1$, then (6.40) holds with $\gamma_0 = \gamma_1 = 0$ for all c_0 and c_1 satisfying (6.45) if and only if

$$|\bar{c}_0| + |\bar{c}_1| + \Delta c < 1 \,.$$

If $R/C < 1$, then (6.40) holds with

$$\gamma_0 = -\bar{c}_0/C \quad \textit{and} \quad \gamma_1 = -\bar{c}_1/C$$

for all c_0 and c_1 satisfying (6.45) if and only if

$$(|\bar{c}_0| + |\bar{c}_1|)\, R/C + \Delta c < 1 \,.$$

6.2.3 Real Laplacian eigenvalues

Here we consider weighted graphs whose Laplacian eigenvalues are all real. This occurs, for example, if the graph is undirected and the weighting matrix is symmetric or if the graph is a string. For this important special case, we can obtain easily verifiable necessary and sufficient conditions for consensus control as outlined in the following result.

Lemma 6.2.4 *If all the eigenvalues of the Laplacian matrix are real, then there exists a matrix K such that the closed-loop network (6.1)-(6.2) achieves GUAC if and only if*

$$\kappa\,|c_0| < 1 \quad \textit{and} \quad \kappa\,(|1-c_0| + |c_1|) < 2 \tag{6.46}$$

where κ is given by (6.36). Such K will be given by (3.30) where γ_0 and γ_1 satisfy

$$\gamma_{0m} < \gamma_0 < \gamma_{0M} \tag{6.47}$$

$$-\gamma_0 + \gamma_{1m} < \gamma_1 < \gamma_0 + \gamma_{1M} \tag{6.48}$$

with

$$\gamma_{0m} = (\gamma_{1m} - \gamma_{1M})/\,2 \tag{6.49}$$

$$\gamma_{0M} = \beta_1\,(1-c_0) - \beta_2\,|1-c_0| \tag{6.50}$$

$$\gamma_{1m} = -\beta_1\,(1+c_0+c_1) + \beta_2\,|1+c_0+c_1| \tag{6.51}$$

$$\gamma_{1M} = \quad \beta_1\,(1+c_0-c_1) - \beta_2\,|1+c_0-c_1| \tag{6.52}$$

and

$$\beta_1 = \frac{\alpha_M + \alpha_m}{2\,\alpha_M\,\alpha_m}\,, \qquad \beta_2 = \frac{\alpha_M - \alpha_m}{2\,\alpha_M\,\alpha_m} \tag{6.53}$$

where α_m and α_M are defined in (2.9).

Proof Using Theorem 6.1.1 and recalling Remarks 6.1.4, 6.1.5, and 6.1.6 we see that GUAC is achieved if and only if

$$(-c_0-1)/\mu < \gamma_0 < (-c_0+1)/\mu \tag{6.54}$$

$$(-c_1-1-c_0)/\mu - \gamma_0 < \gamma_1 < (-c_1+1+c_0)/\mu + \gamma_0 \tag{6.55}$$

for all non-zero eigenvalues μ of the graph Laplacian. Noting that $\beta_1+\beta_2 = 1/\alpha_m$ and $\beta_1-\beta_2 = 1/\alpha_M$ we see that, for any real number c,

$$\min_{\mu>0} c/\mu = \beta_1\, c - \beta_2\,|c|\,, \quad \max_{\mu>0} c/\mu = \beta_1\, c + \beta_2\,|c|$$

Hence, (6.54) holds for all non-zero μ if and only if $\gamma_{0m,2} < \gamma_0 < \gamma_{0M}$ where

$$\gamma_{0m,2} = \max_{\mu>0} \frac{-1-c_0}{\mu} = -\beta_1\,(1+c_0) + \beta_2\,|1+c_0| \tag{6.56}$$

$$\gamma_{0M} = \min_{\mu>0} \frac{1-c_0}{\mu} = \beta_1\,(1-c_0) - \beta_2\,|1-c_0| \tag{6.57}$$

Similarly, (6.55) holds for all non-zero μ if and only if (6.48) holds. Now, there exists γ_1 satisfying (6.48) if and only if $2\,\gamma_0 > \gamma_{1m} - \gamma_{1M} = 2\,\gamma_{0m}$, that is, $\gamma_0 > \gamma_{0m}$. For any two real numbers a and b,

$$|a+b| + |a-b| = 2\max\{|a|,\, |b|\}\,. \tag{6.58}$$

Hence,

$$\begin{aligned}\gamma_{0m} &= (\gamma_{1m}-\gamma_{1M})/2 \\ &= -\beta_1(1+c_0) + \beta_2\left(|1+c_0+c_1| + |1+c_0-c_1|\right)/2 \\ &= -\beta_1(1+c_0) + \beta_2 \max\{|1+c_0|, |c_1|\}\,. \end{aligned} \tag{6.59}$$

Recalling (6.56), we see that $\gamma_{0m} \geq \gamma_{0m,2}$. Hence, (6.7) holds for all non-zero μ if and only if (6.47) and (6.48) hold.

Finally, there exists γ_0 satisfying (6.47) if and only if $\gamma_{0m} - \gamma_{0M} < 0$, that is,

$$-2\,\beta_1 + \beta_2\left(\max\{|1-c_0|, |c_1|\} + |1+c_0|\right) < 0$$

or, noting that $\kappa = \beta_2/\beta_1$,

$$\kappa\left(|1-c_0| + \max\{|1+c_0|, |c_1|\}\right) < 2\,. \tag{6.60}$$

It follows from (6.58) that $|1-c_0| + |1+c_0| = 2\max\{|c_0|, 1\}$. Since $\kappa < 1$, inequality (6.60) is equivalent to (6.46). ■

Remark 6.2.2 *From the proof of the above lemma, we see that if γ_0 satisfies (6.47), then there exists γ_1 satisfying (6.48). Thus, provided (6.46) holds, one can simply obtain a controller K by first choosing γ_0 to satisfy (6.47), then choosing γ_1 to satisfy (6.48), and letting K be given by (3.30).*

6.2.4 Discretized systems

From Chapter 4, GUAC can always be achieved by appropriate choice of gains if linear control is applied to a similar setup in continuous-time where $\dot{\mathbf{x}}_{c,i} = \mathbf{A}_c\,\mathbf{x}_{c,i} + \mathbf{B}_c\,u_{c,i}$. On the other hand, Lemma 6.1.1 shows that this is not always the case in discrete-time. If systems (A_c, B_c) are discretized with time constant $h > 0$, then

$$\mathbf{A} = e^{\mathbf{A}_c h} \quad \text{and} \quad \mathbf{B} = \int_0^h e^{\mathbf{A}_c \tau} d\tau\, \mathbf{B}_c. \tag{6.61}$$

Lemma 6.2.5 *The closed-loop network (6.1)-(6.2) of discretized systems (6.61) can always achieve GUAC if (A_c, B_c) is controllable, the graph contains a spanning tree, the discretization time constant $h > 0$ is chosen small enough, and gains are chosen appropriately.*

Proof If $h \to 0$, then $\mathbf{A} \to \mathbf{I}$. Thus, in the limit, systems are marginally stable where $c_0 = 1$ and $c_1 = -2$, and from Theorem 6.2.1, we can always choose gains γ_0, γ_1 satisfying (6.5). Inequality (6.5) is continuous in c_0 and c_1. Therefore, if (6.5) holds in the limit, then it holds in a neighborhood of c_0 and c_1.
If $h \to 0$, then (A, B) is controllable if (A_c, B_c) is controllable. If (A, B) is controllable, then $\mathbf{K}$ can be obtained from (3.30), and GUAC can always be achieved. ■

6.3 Guaranteed rate of convergence

From Lemma 2.2.1, Theorem 6.1.1 can be used to guarantee GUEC with rate ρ. From Remark 3.2.2, the characteristic polynomial of $\tilde{\mathbf{A}} = \rho^{-1}\mathbf{A}$ is given by $\tilde{c}(s) = s^2 + \tilde{c}_1\, s + \tilde{c}_0$ where

$$\boxed{\tilde{c}_0 = c_0/\rho^2, \quad \tilde{c}_1 = c_1/\rho\,.} \tag{6.62}$$

Recalling Remark 2.2.2 and $\tilde{\mathbf{B}} = \rho^{-1}\mathbf{B}$, and noting that $\tilde{\mathbf{A}} + \tilde{c}_1\,\mathbf{I} = \rho^{-1}(\mathbf{A} + c_1\mathbf{I})$, we proceed as in (6.4) and define

$$\boxed{\begin{bmatrix}\tilde{\gamma}_0 & \tilde{\gamma}_1\end{bmatrix} = \mathbf{K}\begin{bmatrix}\rho^{-2}\,(\mathbf{A} + c_1\,\mathbf{I})\,\mathbf{B} & \rho^{-1}\,\mathbf{B}\end{bmatrix}.} \tag{6.63}$$

Now, Theorem 6.1.1 and Lemma 2.2.1 yield the following result.

Theorem 6.3.1 *The closed-loop network* (6.1)-(6.2) *achieves GUEC with rate* $0 < \rho < 1$ *if for each non-zero eigenvalue* μ *of the graph Laplacian,*

$$|\tilde{d}_0|^2 + |\tilde{d}_1 - \tilde{d}_0\,\bar{\tilde{d}}_1| < 1$$

where $\tilde{d}_0 = \tilde{c}_0 + \mu\,\tilde{\gamma}_0$, $\tilde{d}_1 = \tilde{c}_1 + \mu\,\tilde{\gamma}_1$, *and* $\tilde{c}_0$, $\tilde{c}_1$, $\tilde{\gamma}_0$, *and* $\tilde{\gamma}_1$ *are given by (6.62)-(6.63).*

Remark 6.3.1 *If systems (6.1) are in controllable canonical form, then one can easily show that*

$$\mathbf{K} = \begin{bmatrix}\rho^2\,\tilde{\gamma}_0 & \rho\,\tilde{\gamma}_1\end{bmatrix}$$

Remark 6.3.2 One cannot achieve GUEC with arbitrary small rate ρ using linear controller (6.2) except in the trivial case in which all the non-zero eigenvalues of L are the same. *To see this, suppose convergence with an arbitrary small rate* ρ *can be achieved and recall* (6.62)*; then remark* (6.1.4) *implies that* $\omega = 0$ *for every eigenvalue of* $\mathbf{L}$*; Lemma 6.1.1 implies that* $\alpha_m = \alpha_M$*. Thus all the non-zero eigenvalues of* $\mathbf{L}$ *are the same.*

6.4 Proof of main results

6.4.1 The characteristic polynomial of the closed-loop network

Corollary 3.2.1 provides a useful condition for consensus of the closed-loop sytem, and we restate it here for convenience.

Lemma 6.4.1 ([30]) *The closed-loop network (6.1)-(6.2) achieves GUAC if for each non-zero eigenvalue μ of graph Laplacian, the matrix $\mathbf{A} - \mu\mathbf{BK}$ is Schur.*

Lemma 6.4.2 *The matrix $\mathbf{A} - \mu\mathbf{BK}$ is Schur if and only if*

$$\boxed{p(s) = s^2 + (c_1 + \gamma_1\,\mu)\,s + (c_0 + \gamma_0\,\mu)} \tag{6.64}$$

is Schur, where c_0, c_1, γ_0, and γ_1 are given by (6.3)-(6.4).

Proof A matrix is Schur if and only if its characteristic polynomial is Schur. From Lemma 3.2.2, the characteristic polynomial of matrix $\mathbf{A} - \mu\mathbf{BK}$ is given by (6.64). ■

The coefficients of p in (6.64) are not necessarily real, since μ can be complex. Thus we need conditions which guarantee that a second order polynomial with complex coefficients is Schur.

6.4.2 A simple characterization of second-order Schur polynomials

This section develops necessary and sufficient conditions for a second-order polynomial with complex coefficients to be Schur. It should be noted that this thesis provides a new result and a simple new proof for second-order Schur polynomials.

Lemma 6.4.3 *Suppose $p(\lambda) = \lambda^2 + d_1\,\lambda + d_0$ where $d_0, d_1 \in \mathbb{C}$. Then p is Schur if and only if*

$$|d_0|^2 + |d_1 - d_0\,\bar{d}_1| < 1\,. \tag{6.65}$$

Proof First, we express d_0 and d_1 in polar form, that is,

$$d_0 = |d_0|\, e^{\jmath\,\delta_0}\,, \qquad d_1 = |d_1|\, e^{\jmath\,\delta_1}\,. \tag{6.66}$$

where δ_0 and δ_1 are real. The polynomial p is Schur if and only if $\tilde{p}$ is Schur where $\tilde{p}(\lambda) = e^{-\jmath\,\delta_0}\, p(\lambda\, e^{\jmath\,\delta_0/2})$. Note that

$$\tilde{p}(\lambda) = \lambda^2 + |d_1|\, e^{\jmath\phi}\,\lambda + |d_0| \tag{6.67}$$

where

$$\phi = \delta_1 - \delta_0/2\,. \tag{6.68}$$

If z_1 and z_2 are the roots of $\tilde{p}$, then

$$\tilde{p}(\lambda) = \lambda^2 - (z_1 + z_2)\,\lambda + z_1 z_2\,. \tag{6.69}$$

Comparing (6.67) and (6.69), we see that

$$z_1 z_2 = |d_0| \qquad \text{and} \qquad z_1 + z_2 = -|d_1|\, e^{\jmath\phi}\,. \tag{6.70}$$

If $z_1 = r_1\, e^{\jmath\theta_1}$ and $z_2 = r_2\, e^{\jmath\theta_2}$, then it now follows that $r_1 r_2\, e^{\jmath(\theta_1+\theta_2)} = |d_0|$. Hence , $\theta_1 = -\theta_2 := \theta$ and (6.70) results in

$$r_1\, r_2 = |d_0| \tag{6.71}$$

$$(r_1 + r_2)\cos\theta = -|d_1|\cos\phi \tag{6.72}$$

$$(r_1 - r_2)\sin\theta = -|d_1|\sin\phi\,. \tag{6.73}$$

The polynomial $\tilde{p}$ is Schur if and only if the magnitudes of its roots are less than one, that is, $r_1, r_2 < 1$, which is equivalent to (assuming w.l.o.g. $r_1 \geq r_2 \geq 0$)

$$r_1\, r_2 < 1 \tag{6.74}$$

$$(r_1 - 1)(r_2 - 1) > 0 \;\text{ or }\; (r_1 - 1)(-r_2 - 1) > 0\,. \tag{6.75}$$

It follows from (6.71) that inequality (6.74) is equivalent to

$$|d_0| < 1\,. \tag{6.76}$$

With $r_1 r_2 < 1$, (6.75) is equivalent to

$$\frac{r_1 + r_2}{1 + r_1 r_2} < 1 \quad \text{or} \quad \frac{r_1 - r_2}{1 - r_1 r_2} < 1\,. \tag{6.77}$$

Since $1 = \cos^2\theta + \sin^2\theta$, $r_1 r_2 < 1$, and $r_1 \geq r_2$, inequality (6.77) holds if and only if

$$\left(\frac{r_1 + r_2}{1 + r_1 r_2}\right)^2 \cos^2\theta + \left(\frac{r_1 - r_2}{1 - r_1 r_2}\right)^2 \sin^2\theta < 1$$

which, recalling (6.71)-(6.73), is the same as

$$\left(\frac{|d_1|}{1 + |d_0|}\right)^2 \cos^2\phi + \left(\frac{|d_1|}{1 - |d_0|}\right)^2 \sin^2\phi < 1$$

or

$$\left(1 + |d_0|^2\right) |d_1|^2 - 2\,|d_0|\,|d_1|^2 \cos 2\phi < \left(1 - |d_0|^2\right)^2\,.$$

It follows from (6.66) and (6.68) that $\bar{d}_0\, d_1^2 = |d_0|\,|d_1|^2\, e^{\jmath 2\phi}$; hence the above inequality can be rewritten as

$$\left(1 + |d_0|^2\right) |d_1|^2 - \bar{d}_0\, d_1^2 - d_0\, \bar{d}_1^2 < \left(1 - |d_0|^2\right)^2\,. \tag{6.78}$$

Noting that

$$\begin{aligned}
&(1 + |d_0|^2)\, |d_1|^2 - \bar{d}_0\, d_1^2 - d_0\, \bar{d}_1^2 \\
&= \bar{d}_1 d_1 - \bar{d}_1 (d_0\, \bar{d}_1) - (\bar{d}_0\, d_1)\, d_1 + (\bar{d}_0\, d_1)(d_0\, \bar{d}_1) \\
&= (\bar{d}_1 - \bar{d}_0\, d_1)(d_1 - d_0\, \bar{d}_1) \;= |d_1 - d_0\, \bar{d}_1|^2
\end{aligned}$$

(6.78) can be written as

$$|d_1 - d_0\, \bar{d}_1|^2 < (1 - |d_0|^2)^2\,.$$

The above inequality and (6.76) are equivalent to

$$|d_1 - d_0\, \bar{d}_1| < 1 - |d_0|^2$$

which is equivalent to (6.65). ■

Corollary 6.4.1 *Suppose $p(\lambda) = \lambda^2 + d_1\,\lambda + d_0$ where $d_0, d_1 \in \mathbb{R}$. Then p is Schur if and only if*

$$d_0 < 1 \qquad \textit{and} \qquad |d_1| < 1 + d_0\,. \tag{6.79}$$

Proof When d_0 and d_1 are real, (6.65) can be expressed as $d_0^2 + |1 - d_0||d_1| < 1$, that is,

$$|1 - d_0||d_1| < (1 - d_0)(1 + d_0)\,.$$

which is equivalent to (6.79). ■

Corollary 6.4.2 *Suppose $p(\lambda) = \lambda^2 + d_1\,\lambda + d_0$ where $d_0, d_1 \in \mathbb{C}$. Then p is Schur if*

$$|d_0| + |d_1| < 1\,. \tag{6.80}$$

Also, if p is Schur, then we must have

$$|d_0| < 1 \qquad \textit{and} \qquad |d_1| < 1 + |d_0|\,. \tag{6.81}$$

Proof Note that (6.80) implies that

$$|d_1 - d_0\,\bar{d}_1| < (1 + |d_0|)(1 - |d_0|) \tag{6.82}$$

where

$$|d_1|\,(1 - |d_0|) \leq |d_1 - d_0\,\bar{d}_1| \leq |d_1|\,(1 + |d_0|)\,. \tag{6.83}$$

Thus, if (6.80) holds, then (6.82) holds and p is Schur.
Now suppose p is Schur. Then, it follows from (6.82) and (6.83) that $|d_0| < 1$ and $|d_1|\,(1 - |d_0|) < (1 + |d_0|)(1 - |d_0|)$. This yields (6.81). ■

6.4.3 Proof of Theorem 6.1.1, Lemma 6.1.1, and Lemma 6.2.2

Theorem 6.1.1 is a consequence of Lemmas 6.4.1, 6.4.2, and 6.4.3. Lemma 6.2.2 is a consequence of Corollary 6.4.1. To prove Lemma 6.1.1, we introduce the following result.

Lemma 6.4.4 *There exists a γ satisfying*

$$|c + \mu\,\gamma| < \xi \tag{6.84}$$

for some $c \in \mathbb{R}$*, some* $\xi > 0$*, and all eigenvalues* $\mu = \alpha + \jmath\,\omega$ *satisfying* $0 < \alpha_m \leq \alpha \leq \alpha_M$ *for some* $\alpha_m, \alpha_M \in \mathbb{R}$ *if*

$$\kappa\,|c| < \xi \quad \textit{where} \quad \kappa = \frac{\alpha_M - \alpha_m}{\alpha_M + \alpha_m}\,.$$

Proof First, we note that condition (6.84) holds if and only if $|c_0 + \mu\,\gamma_0| < 1$ where $c_0 = c/\xi$ and $\gamma_0 = \gamma/\xi$. Second, we note that condition (6.84) holds for some $\omega > 0$ if and only if it holds for $\omega = 0$ since $|c + \mu\,\gamma|^2 = |c + \alpha\,\gamma|^2 + \omega^2\,\gamma^2$. From the proof of Lemma 6.2.4, $|c_0 + \mu\,\gamma_0| < 1$ holds for a range of real valued μ if and only if $\gamma_{0m,2} < \gamma_{0M}$ where $\gamma_{0m,2}$ and γ_{0M} are given by (6.56)-(6.57). Applying (6.58) and recalling that $\kappa = \beta_2/\beta_1$ yields

$$2\,\beta_2 \max\{|1|,\,|c_0|\} < 2\,\beta_1 \quad \Leftrightarrow \quad \kappa\,\max\{|1|,\,|c_0|\} < 1$$

Thus, $\kappa\,|c_0| < 1$ is a necessary condition, which recalling $c_0 = c/\xi$, is equivalent to $\kappa\,|c| < \xi$. ■

Now, Lemma 6.1.1 is a consequence of Corollary 6.4.2 and Lemma 6.4.4. From Lemma 6.4.4, $\kappa\,|c_0| < 1$ is a necessary condition. Corollary 6.4.2 implies that $|d_0| < 1$ and $|d_1| < 1 + |d_0|$; hence $|d_1| < 2$. Applying Lemma 6.4.4 again yields $\kappa\,|c_1| < 2$ as another necessary condition.

6.5 Simulations

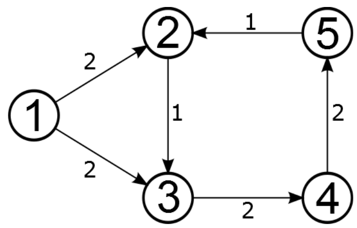
6.5.1 Example I

We illustrate our results by considering a network of discretized double integrators that are described by

$$q_i(k+1) = q_i(k) + T_k\,v_i(k) \tag{6.85}$$

$$v_i(k+1) = v_i^k + u_i(k) + u_0(k) \tag{6.86}$$

where $T_k > 0$ is the period between two samples, $q_i(k)$ describes the position and $v_i(k)$ describes the velocity of system i at step k. An application of this model would be the



$$\Rightarrow \qquad L = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ -2 & 3 & 0 & 0 & -1 \\ -2 & -1 & 3 & 0 & 0 \\ 0 & 0 & -2 & 2 & 0 \\ 0 & 0 & 0 & -2 & 2 \end{bmatrix}$$

Figure 6.1.. Example I: Communication network with assigned edge weights

impulsive control of satellites, whose continuous-time dynamics can be approximated by (6.85)-(6.86).

Let $T_k = 1$, then system dynamics are given by $c_0 = 1$, $c_1 = -2$, and

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

The systems can communicate over the graph shown in Figure 6.1. In this network, system 1 is a leader and when consensus is achieved, the positions of the other systems will equal that of the leader. Weights w_{ij} correspond to the numbers on the edges in Figure 6.1, e.g. $w_{31} = 2$. The Laplacian matrix is shown on the right in Figure 6.1 and has eigenvalues $\mu_1 = 0$, $\mu_2 = 1$, $\mu_{3/4} = 2.5 \pm \jmath, \mu_5 = \sqrt{7}/2$, and $\mu_6 = 4$. A common input $\mathbf{u}_0(k) = 0.04\, cos(0.2\, k)\, \mathbf{B}$ is assumed. We chose gains γ_0, γ_1 so that GUEC was achieved with rate ρ and the quantity $\gamma_0^2 + \gamma_1^2$ was minimized. Three different desired convergence rates for GUEC were considered, and the resulting parameters are summarized in Table 6.1.

Figure 6.2 shows the regions of feasible gains corresponding to the three desired convergence rates. Gains within these regions satisfy the conditions of Theorem 6.3.1.

All three cases were simulated with the same randomly generated, uniformly distributed initial conditions. The initial velocity of the leader was set to zero, that is $v_1(0) = 0$. The gains γ_0, γ_1 were set to the values shown in Table 4.2. The plots in Figure 6.4 show the time history of the position and velocity of each system on the

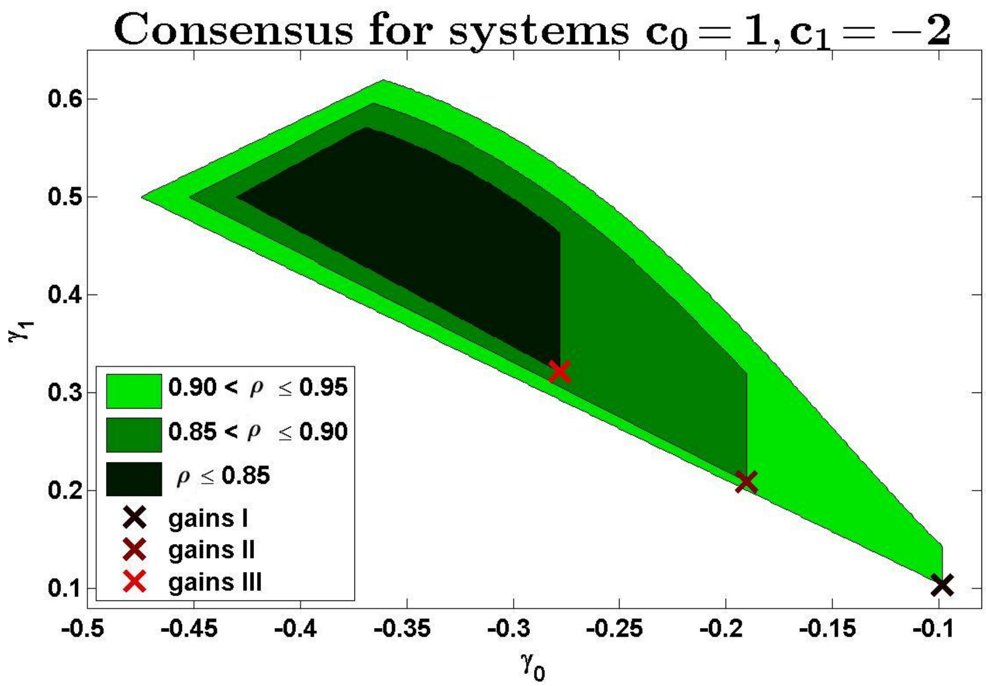


Figure 6.2.. Gains γ_0, γ_1 satisfying necessary and sufficient conditions to achieve GUEC with rate ρ with $c_0 = 1$ and $c_1 = -2$.

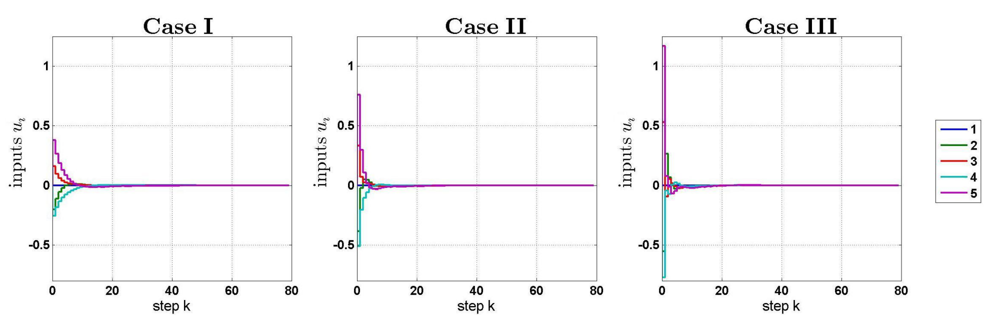
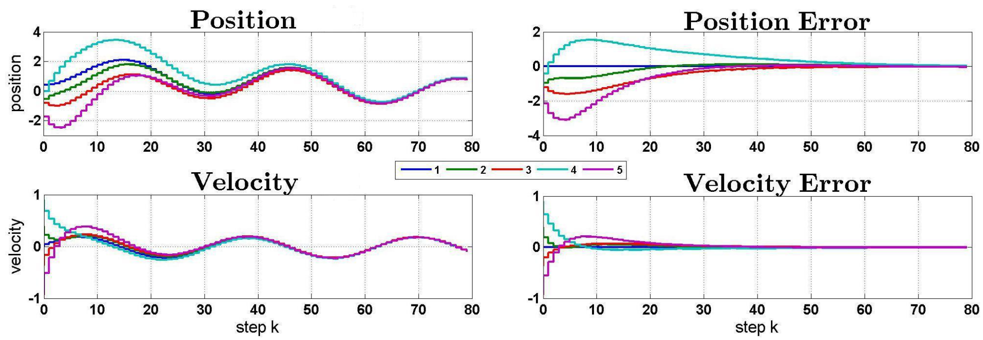
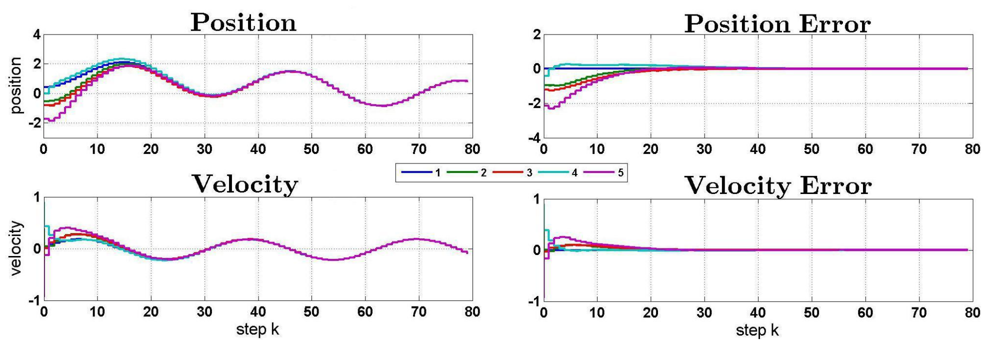


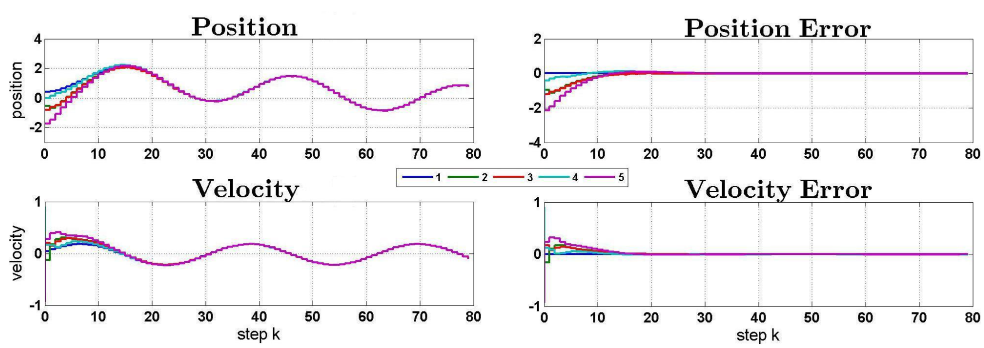
Figure 6.3.. Inputs $u_i(k)$ for different convergence rates - common input $\mathbf{u}_0(k) = 0.04\, cos(0.2\, k)\, \mathbf{B}$ not shown.



(a) Case I - GUEC with rate $\rho = 0.95$



(b) Case II - GUEC with rate $\rho = 0.90$

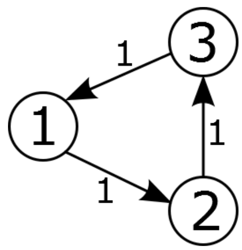


(c) Case III - GUEC with rate $\rho = 0.85$

Figure 6.4.. Synchronization of systems with $c_0 = 1$, $c_1 = -2$ and different convergence rates.

Table 6.1.. Gains γ_0, γ_1 to achieve GUEC - $c_0 = 1, c_1 = -2$

	ρ	γ_0	γ_1	$\gamma_0 + \gamma_1$	$\tilde{\mathbf{c}}_0$	$\tilde{\mathbf{c}}_1$
case I	0.95	-0.098	0.104	0.052	1.108	-2.105
case II	0.90	-0.190	0.209	0.018	1.235	-2.222
case III	0.85	-0.278	0.321	0.043	1.384	-2.353



$$\Rightarrow \qquad L = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

Figure 6.5.. Example II: Communication network with assigned edge weights

left. Though all systems were initially unsynchronized, they achieved GUEC over time. For each system, we introduced a position error $q_i - q_1$ and a velocity error $v_i - v_1$, which are relative to the leader. The right-hand side of Figure 6.4 shows these errors (leader - blue line). Clearly, as expected, synchronization was achieved more quickly with smaller ρ.

A faster rate of convergence results in larger inputs u_i. Figure 6.3 shows the inputs u_i that must be applied to each system in order to achieve GUEC as illustrated in Figure 6.4. It should be noted that the additional, commonly applied input $\mathbf{u}_0(k) = 0.04\, cos(0.2\, k)\, \mathbf{B}$ is not shown in these plots.

6.5.2 Example II

Consider a network of three systems with

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -0.16 & 1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}($$

and common input $u_0(k) = 0.5 \cos k$. Figure 6.5 describes the communication network and shows its associated Laplacian matrix L with non-zero eigenvalues $1.5 \pm \jmath\sqrt{3}/2$.

Since A has eigenvalues 0.2 and 0.8, the open loop systems are stable and GUEC with rate 0.8 can be achieved with $u_i = 0$. To achieve GUEC with rate smaller than 0.8 we use our consensus controllers. Controller gains satisfying the conditions in Theorem 6.3.1 are shown in Figure 6.6. We failed to identify gains for $\rho < 0.29$, and we picked $\gamma_0 = -0.1$ and $\gamma_1 = 0.5$ to guarantee convergence with at least $\rho \leq 0.6$. Figure 6.7 compares the behaviour of the open loop systems to the closed loop systems. The initial states are the same in each case and were chosen randomly with uniform distribution. We observe that GUEC was achieved more quickly for the closed loop systems.

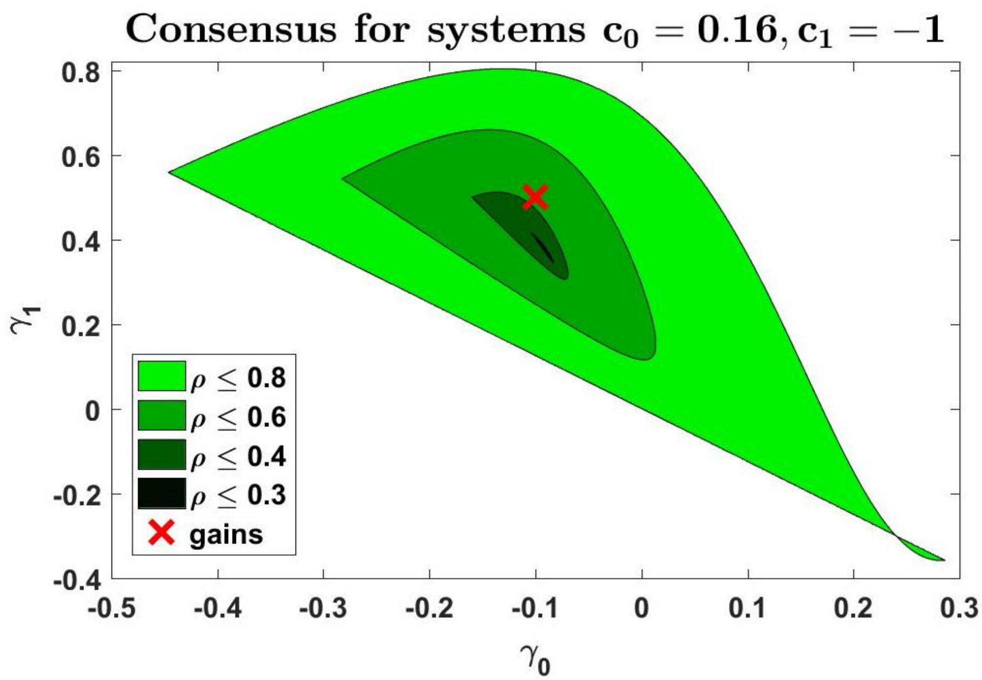


Figure 6.6.. Regions of gains achieving GUEC with guaranteed rate of convergence.

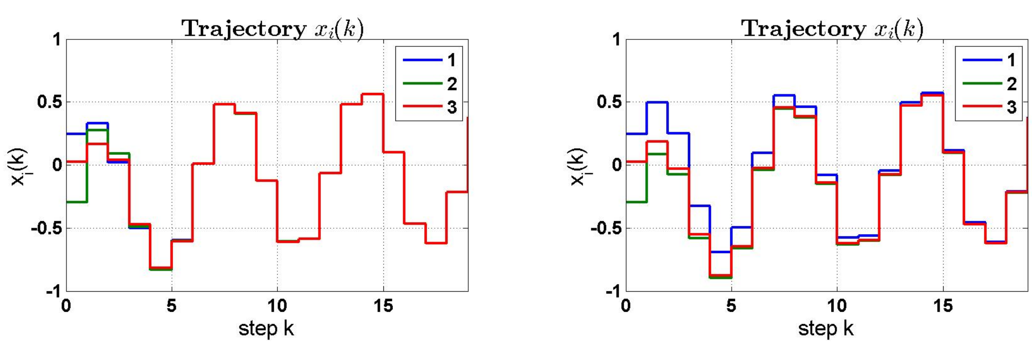


Figure 6.7.. Simulation of three systems achieving consensus with (left) and without (right) control u_i. Note that only $x_{i,1}(k)$ is shown since $x_{i,2}(k) = x_{i,1}(k+1)$.

7. CONSENSUS FOR DISCRETE-TIME HIGHER-ORDER SYSTEMS

In this chapter, we present conditions for higher-order discrete-time systems. First, we present simple results that are easily verifiable. Then, we develop and solve conditions that apply to the more general case.

7.1 A simple result for single-input time-invariant systems

Here we present conditions that depend on the system parameters (3.28). They apply directly to higher-order single-input systems given in controllable canonical form. If systems are not in controllable canonical form, then we can use the results of this section and apply relation (3.29), or we formulate the problem as a linear matrix inequality (see Section 7.4).

The following results are a generalization of the results presented in Section 6.2.2.

7.1.1 Conditions for consensus

We will present sufficient conditions for the closed-loop network (2.5)-(2.6) to achieve GUAC. Similar to Section 6.4, we will have to show that matrices $\mathbf{A} - \mu\mathbf{BK}$ are Schur. A matrix is Schur if and only if its characteristic polynomial is Schur. A polynomial is said to be Schur if all its roots have a magnitude less than one. From Lemma 3.2.2, the characteristic polynomial of matrix $\mathbf{A} - \mu\mathbf{BK}$ is given by

$$\boxed{p(s) = s^n + \sum_{i=0}^{n-1} d_i\, s^i\,, \qquad d_i := c_i + \mu\,\gamma_i} \tag{7.1}$$

where c_i and γ_i are given by (3.28)-(3.29).

First, we present the following preliminary result for a polynomial to be Schur.

Lemma 7.1.1 *Suppose* $p(s) = s^n + \sum_{i=0}^{n-1} d_i\, s^i$ *where* $d_i \in \mathbb{C}$ *and*

$$\sum_{i=0}^{n-1} |d_i| < 1\,. \tag{7.2}$$

Then p *is Schur.*

Proof We prove the result by contradiction. Suppose that λ is a root of p and $|\lambda| \geq 1$. Then,

$$\lambda^n + \sum_{i=0}^{n-1} d_i\, \lambda^i = 0\,.$$

Multiplying the equation by λ^{1-n} and rearranging yields

$$\lambda = -\sum_{i=0}^{n-1} d_i\, \lambda^{-(n-1-i)}\,.$$

Since $|\lambda| \geq 1$, we obtain that $|\lambda^{-(n-1-i)}| \leq 1$ for $0 \leq i \leq n-1$; hence

$$|\lambda| \leq \sum_{i=0}^{n-1} |d_i|\, |\lambda^{-(n-1-i)}| \leq \sum_{i=0}^{n-1} |d_i|\,.$$

Applying (7.2), we obtain the contradiction that $1 \leq |\lambda| < 1$. Hence, every root of p has magnitude less than one, that is, p is Schur. ■

Now the following corollary is a consequence of Lemma (7.1.1) and ensures that polynomials (7.1) are Schur.

Corollary 7.1.1 *The closed-loop network* (2.5)-(2.6) *achieves GUAC if, for each non-zero eigenvalue* μ *of the graph Laplacian matrix,*

$$\sum_{i=0}^{n-1} |c_i + \mu\, \gamma_i| < 1 \tag{7.3}$$

where c_i *and* γ_i *are given by* (3.28)-(3.29).

When (7.3) is satisfied and (A, B) is controllable, then a GUAC achieving gain matrix K is given by (3.30).

Remark 7.1.1 *The coefficients of polynomials* (7.1) *are not necessarily real, since the non-zero eigenvalues μ of the graph Laplacian can be complex. Corollary 7.1.1 is based on a conservative but simple result for such polynomials to be Schur. More complex but necessary and sufficient conditions for Schur polynomials can be obtained by applying the methods in [38].*

Next, we show that for some combinations of systems and graphs GUAC cannot be achieved. To state this result we need the binomial coefficient

$$\binom{n}{i}_b = \frac{n!}{i!\,(n-i)!}$$

where the factorial is defined by

$$n! = \begin{cases} 1 & , n = 0 \\ (n-1)! \cdot n & , n > 0 \end{cases}$$

and the following preliminary result.

Lemma 7.1.2 *Suppose $p(s) = s^n + \sum_{i=0}^{n-1} d_i\, s^i$ where $d_i \in \mathbb{C}$. If p is Schur, then*

$$|d_i| < \binom{n}{i}_b, \quad i = 0, \cdots, n-1\,. \tag{7.4}$$

Proof The lemma is proven by recursion, where the base case is given by $n = 1$. If $n = 1$, then the polynomial p is Schur if and only if $|d_0| < 1$. Hence, (7.4) holds for $n = 1$. Now, suppose z_n is the n-th root of the Schur polynomial p_n. Then,

$$p_n(s) = \sum_{i=0}^{n} d_i\, s^i = (s - z_n) \sum_{i=0}^{n-1} \tilde{d}_i\, s^i = (s - z_n)\, p_{n-1}(s)$$

where $d_n = 1$ and $\tilde{d}_{n-1} = 1$. Comparing coefficients on both sides yields

$$d_0 = -z_n\, \tilde{d}_0 \quad \text{and} \quad d_i = \tilde{d}_{i-1} - z_n\, \tilde{d}_i\,, \quad i = 1, \cdots, n-1\,.$$

The polynomial p_n is Schur. Thus, $|z_n| < 1$,

$$|d_0| < |\tilde{d}_0| \quad \text{and} \quad |d_i| < |\tilde{d}_{i-1}| + |\tilde{d}_i|\,, \quad i = 1, \cdots, n-1\,.$$

The binomial coefficient can be defined recursively:

$$\binom{n}{i}_b = \binom{n-1}{i-1}_b + \binom{n-1}{i}_b .$$

Therefore, if coefficients $\tilde{d}_i$ satisfy (7.4), then so do coefficients d_i. ■

Now we have the following result.

Lemma 7.1.3 *If the graph contains a spanning tree and the closed-loop network (2.5)-(2.6) achieves GUAC, then* $0 < \alpha_m \leq \alpha_M$ *and*

$$\kappa\,|c_i| < \binom{n}{i}_b, \quad i = 0, \cdots, n-1 \tag{7.5}$$

where κ *is defined in terms of* α_m, α_M *given by (2.9):*

$$\kappa = \frac{\alpha_M - \alpha_m}{\alpha_M + \alpha_m} \tag{7.6}$$

Proof From Corollary 3.2.1, if the closed-loop network (2.5)-(2.6) achieves GUAC, then matrices $\mathbf{A} - \mu\mathbf{BK}$ are Schur, that is, polynomials (7.1) are Schur and

$$|c_i + \mu\,\gamma_i| < \binom{n}{i}_b, \quad i = 0, \cdots, n-1 \tag{7.7}$$

from Lemma 7.1.2. Finally, we obtain (7.5) from (7.7) by applying Lemma 6.4.4. ■

7.1.2 Control design - Laplacian eigenvalues in a disc

From Lemma 7.1.3, we see that if any of the $|c_i|$ is large, then the range of the eigenvalues of the Laplacian matrix L must be restricted to achieve GUAC. Here we consider the non-zero eigenvalues of the Laplacian to be constrained to a disc of radius R with center $C > 0$.

Lemma 7.1.4 *Suppose that*

$$|\mu - C| \leq R \tag{7.8}$$

for all non-zero eigenvalues μ of the Laplacian. Then, the closed-loop network (2.5)-(2.6) achieves GUAC if

$$\sum_{i=0}^{n-1} |c_i + \gamma_i C| + |\gamma_i| R < 1 . \tag{7.9}$$

There exist gains γ_i satisfying (7.9) if and only if: 1) $R/C \geq 1$:

$$\sum_{i=0}^{n-1} |c_i| < 1 . \tag{7.10}$$

In this case, (7.9) holds with

$$\gamma_i = 0 , \qquad i = 0, \cdots , n-1 .$$

2) $R/C < 1$:

$$\frac{R}{C} \sum_{i=0}^{n-1} |c_i| < 1 . \tag{7.11}$$

In this case, (7.9) holds with

$$\gamma_i = -c_i/C , \qquad i = 0, \cdots , n-1 .$$

Proof From Corollary 7.1.1, GUAC is achieved if, for each nonzero eigenvalue μ of the Laplacian,

$$\sum_{i=0}^{n-1} |d_i| < 1 \quad \text{where} \quad d_i = c_i + \mu \, \gamma_i . \tag{7.12}$$

The rest of the proof is equivalent to the proof of Lemma 6.2.3. ∎

Remark 7.1.2 (Robustness with respect to plant parameters) *The conditions in Lemma 7.1.4 can be used to guarantee robustness with respect to changes in the graph; the only knowledge needed about the graph is that the non-zero eigenvalues of the graph Lapacian lie within a disc of radius R and center C.*

To obtain robustness with respect to plant parameters suppose that, for some $\bar{c}_0, \cdots , \bar{c}_{n-1}$, and Δc, the plant parameters $c_0, \cdots , c_{n-1}$ satisfy

$$\sum_{i=0}^{n-1} |c_i - \bar{c}_i| \leq \Delta c . \tag{7.13}$$

If $R/C \geq 1$, then f_i is minimized for $\gamma_i = 0$, and (7.9) holds with $\gamma_i = 0$ for all c_i satisfying (7.13) if and only if

$$\Delta c + \sum_{i=0}^{n-1} |\bar{c}_i| < 1\,.$$

If $R/C < 1$, then (7.9) holds with

$$\gamma_i = -\bar{c}_i / C\,, \quad i = 0, \cdots, n-1$$

for all c_i satisfying (7.13) if and only if

$$\frac{R}{C}\left(\sum_{i=0}^{n-1} |\bar{c}_i|\right) + \Delta c < 1\,.$$

7.1.3 Guaranteed rate of convergence

Using Remarks 2.2.2 and 3.2.2, we generalize the results presented in Section 6.3, which yields 1) the characteristic polynomial for the transformed systems as $\tilde{c}(s) = \sum_{i=0}^{n} \tilde{c}_i\, s^i$ where

$$\boxed{\tilde{c}_i = c_i\, \rho^{i-n}} \tag{7.14}$$

and 2) the transformed control

$$\tilde{\gamma}_i = \rho^{i-n}\, \mathbf{K} \left[\sum_{j=i+1}^{n} c_j\, \mathbf{A}^{j-i-1} \right] \mathbf{B} = \rho^{i-n}\, \gamma_i$$

where $i = 0, \cdots, n-1$. According to Corollary 7.1.1 the network of transformed systems achieves GUAC if,

$$\sum_{i=0}^{n-1} |\tilde{c}_i + \mu\, \tilde{\gamma}_i| < 1$$

for every non-zero eigenvalue μ of the graph Laplacian; this condition is equivalent to

$$\sum_{i=0}^{n-1} \rho^{i-n}\, |c_i + \mu\, \gamma_i| < 1\,.$$

Hence, Corollary 7.1.1 and Lemma 2.2.1 yield the following result.

Theorem 7.1.1 *The closed-loop network (2.5)-(2.6) achieves GUEC with rate ρ if for each non-zero eigenvalue μ of the graph Laplacian matrix,*

$$\sum_{i=0}^{n-1} \rho^i \, |c_i + \mu \, \gamma_i| < \rho^n$$

where c_i and γ_i are given by (3.28)-(3.29).

Remark 7.1.3 *From (7.14) and Lemma 7.1.3, we see that GUEC with an arbitrary small rate ρ cannot usually be achieved by network (2.5) subject to linear controller (2.6). In particular, ρ must satisfy the lower bound*

$$\rho > \max_{0 \le i \le n-1} \left\{ \left[\frac{i! \, (n-i)!}{n!} \cdot \frac{|c_i|}{\kappa} \right]^{\frac{1}{n-i}} \right\}.$$

This bound is non-zero, except in the trivial case of $c_i = 0$ for $i = 1, \cdots, n$.

7.2 Stability conditions for multi-input time-varying systems

Next, we consider a homogeneous network of higher-order, time-varying, discrete-time systems

$$\mathbf{x}_i(k+1) = \mathbf{A}(k) \, \mathbf{x}_i(k) + \mathbf{B}(k) \, \mathbf{K}(k) \sum_{j \in N_i} w_{ij} \left[\mathbf{x}_j(k) - \mathbf{x}_i(k) \right] + \mathbf{u}_0(k) \qquad (7.15)$$

where we assume that

$$\mathbf{A}(\cdot) \text{ and } \mathbf{B}(\cdot) \, \mathbf{K}(\cdot) \text{ are bounded.}$$

Remark 7.2.1 *Lemma 3.1.6 reduced the consensus problem to the simultaneous stabilization of a bunch of systems, which is similar to solving a robust control problem.*

From Lemma 2.2.1, the closed-loop network (7.15) achieves GUEC with rate $0 < \rho < 1$ if the network of transformed systems $(\tilde{A}(k), \tilde{B}(k))$ achieves GUAC. From

Remark 2.2.2, we know that the gain matrix for the original and the transformed network are equivalent, we recall from (2.8) that

$$\tilde{\mathbf{A}}(k) = \rho^{-1}\,\mathbf{A}(k) \quad \text{and} \quad \tilde{\mathbf{B}}(k) = \rho^{-1}\,\mathbf{B}(k)$$

and we immediately obtain the following corollary from Lemma 3.1.6.

Corollary 7.2.1 *The closed-loop network (7.15) achieves GUEC with rate $0 < \rho < 1$ if systems*

$$\tilde{\mathbf{z}}(k+1) = \rho^{-1}\left[\mathbf{A}(k) - \mu\,\mathbf{B}(k)\mathbf{K}(k)\right]\tilde{\mathbf{z}}(k) \tag{7.16}$$

are GUAS for all non-zero eigenvalues μ of the graph Laplacian.

Now, we use this result in the following two sections.

7.3 A simple result for single-input time-varying systems in controllable canonical form

Here we consider the closed-loop network (7.15) where systems (A, B) are given in controllable canonical form, that is,

$$\mathbf{A}(k) = \begin{bmatrix} 0 & 1 & 0 & & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -c_0(k) & -c_1(k) & -c_2(k) & & -c_{n-1}(k) \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \tag{7.17}$$

and

$$\mathbf{K}(k) = \begin{bmatrix} \gamma_0(k) & \gamma_1(k) & \cdots & \gamma_{n-1}(k) \end{bmatrix}. \tag{7.18}$$

Systems are of order $n \in \mathbb{N}$ and parameters $c_i(k) \in \mathbb{R}$ can vary with k. Input matrix $\mathbf{B}$ is fixed but the input gain matrix $\mathbf{K}(k)$ can vary with time. The graph Laplacian associated with the network is fixed and does not vary with k.

The following result summarizes sufficient conditions for the closed-loop network (7.15) to achieve GUEC.

Theorem 7.3.1 *If systems are given by* (7.17) *and* (7.18), *then the closed-loop network* (7.15) *achieves GUEC with rate* $0 < \rho < 1$ *if there exists a* $0 < \tilde{\rho} < \rho$ *such that for each non-zero eigenvalue* μ *of the graph Laplacian matrix and all steps* k,

$$\sum_{i=0}^{n-1} \tilde{\rho}^{i} \, |c_i(k) + \mu \, \gamma_i(k)| \leq \tilde{\rho}^{n} \, . \tag{7.19}$$

Proof From Corollary 7.2.1, we prove Theorem 7.3.1 by showing that systems (7.16) are GUAS for all $\mu \neq 0$. Since $\mathbf{A}(k)$ and $\mathbf{B}$ are given by (7.17), (7.16) reduces to

$$z_i(k+1) = \rho^{-1} \, z_{i+1}(k) \, , \quad i = 1, \cdots, n-1 \tag{7.20}$$

$$z_n(k+1) = - \, \rho^{-1} \sum_{i=0}^{n-1} d_i(k) \, z_{i+1}(k) \tag{7.21}$$

where $d_i(k) = c_i(k) + \mu \, \gamma_i(k)$, and inequality (7.19) implies that

$$\sum_{i=0}^{n-1} \tilde{\rho}^{i} \, |d_i(k)| \leq \tilde{\rho}^{n} < 1 \tag{7.22}$$

for all k and non-zero μ. Let

$$\epsilon := \tilde{\rho}^{-1} \tag{7.23}$$

and consider the candidate Lyapunov function V given by

$$V(\mathbf{z}) = \max \left\{ \epsilon \, |z_1|, \epsilon^2 \, |z_2|, \cdots, \epsilon^n \, |z_n| \right\} . \tag{7.24}$$

Then, $\epsilon\rho > 1$ and it follows from (7.20) that, for all k and $i = 1, \cdots, n-1$,

$$\epsilon^{i} \, |z_i(k+1)| = (\epsilon\rho)^{-1} \, \epsilon^{i+1} \, |z_{i+1}(k)| \leq (\epsilon\rho)^{-1} \, V(\mathbf{z}(k)) \, . \tag{7.25}$$

From (7.21), we obtain that

$$\begin{aligned} \epsilon^{n} \, |z_n(k+1)| &\leq \epsilon^{n} \rho^{-1} \sum_{i=0}^{n-1} |d_i(k)| \, |z_{i+1}(k)| \leq (\epsilon\rho)^{-1} \sum_{i=0}^{n-1} \epsilon^{n-i} \, |d_i(k)| \, \epsilon^{i+1} \, |z_{i+1}(k)| \\ &\leq (\epsilon\rho)^{-1} \, V(\mathbf{z}(k)) \sum_{i=0}^{n-1} \epsilon^{n-i} \, |d_i(k)| \end{aligned}$$

It follows from (7.22) and (7.23) that

$$\sum_{i=0}^{n-1} \epsilon^{n-i}\, |d_i(k)| = \epsilon^n \sum_{i=0}^{n-1} \tilde{\rho}^i |d_i(k)| \leq \epsilon^n\, \tilde{\rho}^n = 1\,.$$

Hence, recalling $\epsilon\rho > 1$,

$$\epsilon^n\, |z_n(k+1)| \leq (\epsilon\rho)^{-1}\, V(\mathbf{z}(k))\,. \tag{7.26}$$

It now follows from (7.25)-(7.26) and the definition of V that for all k and non-zero μ

$$V(\mathbf{z}(k+1)) \leq (\epsilon\rho)^{-1}\, V(\mathbf{z}(k))\,. \tag{7.27}$$

Since $(\epsilon\rho)^{-1} < 1$, it follows that systems (7.16) are GUAS for all $\mu \neq 0$. ■

Control designs that satisfy (7.19) for a range of c_i, γ_i and μ are given in Section 7.1.2.

7.4 An LMI approach

In this section, we will make use of the following Schur complement result to simplify matrix inequalities. Consider a hermitian 2×2 block matrix

$$\mathbf{Q} = \begin{bmatrix} \mathbf{Q}_{11} & \mathbf{Q}_{12} \\ \mathbf{Q}_{21} & \mathbf{Q}_{22} \end{bmatrix}$$

where $\mathbf{Q}_{22} < 0$. Then,

$$\mathbf{Q} \leq \mathbf{0} \qquad \Leftrightarrow \qquad \mathbf{Q}_{11} - \mathbf{Q}_{12}\, \mathbf{Q}_{22}^{-1}\, \mathbf{Q}_{21} \leq \mathbf{0}\,. \tag{7.28}$$

This can be seen by noting that

$$\begin{bmatrix} \mathbf{Q}_{11} & \mathbf{Q}_{12} \\ \mathbf{Q}_{21} & \mathbf{Q}_{22} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{Q}_{22}^{-1}\, \mathbf{Q}_{21} & \mathbf{I} \end{bmatrix}' \begin{bmatrix} \mathbf{Q}_{11} - \mathbf{Q}_{12}\, \mathbf{Q}_{22}^{-1}\, \mathbf{Q}_{21} & \mathbf{0} \\ \mathbf{0} & \mathbf{Q}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{Q}_{22}^{-1}\, \mathbf{Q}_{21} & \mathbf{I} \end{bmatrix}.$$

7.4.1 Sufficient conditions for consensus

We now focus on control designs that make use of quadratic stability.

Definition 7.4.1 *The system* $\mathbf{x}(k+1) = \mathbf{A}(k)\,\mathbf{x}(k)$ *is quadratically stable with common Lyapunov matrix* $\mathbf{P} = \mathbf{P}' > 0$ *if and only if there exists an* $\epsilon^2 < 1$ *such that for all* k*:*

$$\mathbf{A}'(k)\,\mathbf{P}\,\mathbf{A}(k) - \epsilon^2\,\mathbf{P} \leq \mathbf{0}\,. \tag{7.29}$$

If the systems are quadratically stable, then they are GUES and therefore GUAS. Applying this fact and Lemma 3.1.6 yields the following result.

Theorem 7.4.1 *The closed-loop network* (7.15) *achieves GUAC if there exists a matrix* $\mathbf{S} = \mathbf{S}' > \mathbf{0}$*, an* $\epsilon^2 < 1$*, and for each* k*, there is a matrix* $\mathbf{X}(k)$ *such that, for all non-zero eigenvalues* $\mu = \alpha + \jmath\,\omega$ *of the graph Laplacian,*

$$\begin{bmatrix} -\epsilon\,\mathbf{S} & \mathbf{Y}'(k) & \mathbf{0} & -\omega\,\mathbf{X}'(k)\,\mathbf{B}'(k) \\ \mathbf{Y}(k) & -\epsilon\,\mathbf{S} & \omega\,\mathbf{B}(k)\,\mathbf{X}(k) & \mathbf{0} \\ \mathbf{0} & \omega\,\mathbf{X}'(k)\,\mathbf{B}'(k) & -\epsilon\,\mathbf{S} & \mathbf{Y}'(k) \\ -\omega\,\mathbf{B}(k)\,\mathbf{X}(k) & \mathbf{0} & \mathbf{Y}(k) & -\epsilon\,\mathbf{S} \end{bmatrix} \leq \mathbf{0} \tag{7.30}$$

where $\mathbf{Y}(k) = \mathbf{A}(k)\,\mathbf{S} - \alpha\,\mathbf{B}(k)\,\mathbf{X}(k)$ *and*

$$\mathbf{K}(k) = \mathbf{X}(k)\,\mathbf{S}^{-1}\,. \tag{7.31}$$

Proof From Lemma 3.1.6, we prove Theorem 7.4.1 by showing that the systems

$$\mathbf{z}(k+1) = \mathbf{A}_{cl}(k)\,\mathbf{z}(k)\,, \qquad \mathbf{A}_{cl}(k) = \mathbf{A}(k) - \mu\,\mathbf{B}(k)\mathbf{K}(k) \tag{7.32}$$

are GUAS for all $\mu \neq 0$. From Definition 7.4.1, system (7.32) is quadratically stable (and therefore GUAS) if there exists a $\mathbf{P} = \mathbf{P}' > \mathbf{0}$ and an $\epsilon^2 < 1$ such that

$$\mathbf{A}_{cl}'(k)\,\mathbf{P}\,\mathbf{A}_{cl}(k) - \epsilon^2\,\mathbf{P} \leq \mathbf{0}\,. \tag{7.33}$$

Let $\mathbf{S} = \mathbf{P}^{-1}$, then pre- and post-multiplying by $\mathbf{S}$ yields

$$\mathbf{S}\,\mathbf{A}_{cl}'(k)\,\mathbf{S}^{-1}\,\mathbf{A}_{cl}(k)\,\mathbf{S} - \epsilon^2\,\mathbf{S} \leq \mathbf{0}\,. \tag{7.34}$$

Substituting for $\mathbf{A}_{cl}(k)$ and $\mathbf{K}(k)$, we note that (7.34) is equivalent to

$$[\mathbf{A}(k)\,\mathbf{S} - \mu\,\mathbf{B}(k)\,\mathbf{X}(k)]'\;\mathbf{S}^{-1}\;[\mathbf{A}(k)\,\mathbf{S} - \mu\,\mathbf{B}(k)\,\mathbf{X}(k)] - \epsilon^2\,\mathbf{S} \leq \mathbf{0} \tag{7.35}$$

and w.l.o.g. $\epsilon > 0$. Dividing by $\epsilon > 0$ and applying the Schur complement (7.28) yields that (7.35) is equivalent to

$$\begin{bmatrix} -\epsilon\,\mathbf{S} & \mathbf{S}\,\mathbf{A}'(k) - \bar{\mu}\,\mathbf{X}'(k)\,\mathbf{B}'(k) \\ \mathbf{A}(k)\,\mathbf{S} - \mu\,\mathbf{B}(k)\,\mathbf{X}(k) & -\epsilon\,\mathbf{S} \end{bmatrix} \leq \mathbf{0}\,. \tag{7.36}$$

Since $\mu = \alpha + \jmath\,\omega$, (7.36) can be written as $\mathbf{Q}(k) = \mathbf{Q}_{real}(k) + \jmath\,\mathbf{Q}_{img}(k) \leq \mathbf{0}$ where

$$\mathbf{Q}_{real}(k) = \begin{bmatrix} -\epsilon\,\mathbf{S} & \mathbf{Y}'(k) \\ \mathbf{Y}(k) & -\epsilon\,\mathbf{S} \end{bmatrix}, \qquad \mathbf{Q}_{img}(k) = \begin{bmatrix} \mathbf{0} & \omega\,\mathbf{X}'(k)\,\mathbf{B}'(k) \\ -\omega\,\mathbf{B}(k)\,\mathbf{X}(k) & \mathbf{0} \end{bmatrix}$$

and $\mathbf{Y}(k) = \mathbf{A}(k)\,\mathbf{S} - \alpha\,\mathbf{B}(k)\,\mathbf{X}(k)$. From Lemma 5.2.1, $\mathbf{Q}(k) \leq \mathbf{0}$ is equivalent to (7.30) and guarantees that systems (7.32) are GUAS for all $\mu \neq 0$. ■

Remark 7.4.1 *If a Laplacian eigenvalue μ is real, then $\mu = \alpha$, $\omega = 0$, and condition* (7.30) *simplifies to*

$$\begin{bmatrix} -\epsilon\,\mathbf{S} & \mathbf{S}\,\mathbf{A}'(k) - \alpha\,\mathbf{X}'(k)\mathbf{B}'(k) \\ \mathbf{A}(k)\,\mathbf{S} - \alpha\,\mathbf{B}(k)\mathbf{X}(k) & -\epsilon\,\mathbf{S} \end{bmatrix} \leq \mathbf{0}\,.$$

Remark 7.4.2 *In principle, condition* (7.30) *should only be applied to a finite number of systems $(A(k), B(k))$. Otherwise, unless restrictions on $\mathbf{A}(k)$ and $\mathbf{B}(k)$ are made, applying Theorem 7.4.1 will yield an infinite number of LMIs and a solution will not be feasible. An infinite number of LMIs can be avoided if the set of $\mathbf{A}(k)$ and $\mathbf{B}(k)$ lie inside a polytope for example. Then, we wish to identify a time-invariant $\mathbf{X}$ such that the conditions hold for all vertices of this polytope [85].*

7.4.2 Guaranteed rate of convergence

To achieve GUEC with rate ρ, we recall Corollary 7.2.1.

Lemma 7.4.1 *The closed-loop network (7.15) achieves GUEC with rate $0 < \rho$ if conditions in Theorem 7.4.1 hold for some $0 < \epsilon < \rho$.*

Proof From the proof of Theorem 7.4.1, LMI (7.30) is equivalent to (7.35). If (7.35) holds for some $0 < \epsilon < \rho$, then $\tilde{\epsilon} := \epsilon/\rho < 1$, and dividing (7.35) by $\rho^2 > 0$ and using (2.8) yields

$$\left[\tilde{\mathbf{A}}(k)\,\mathbf{S} - \mu\,\tilde{\mathbf{B}}(k)\,\mathbf{X}(k)\right]'\,\mathbf{S}^{-1}\,\left[\tilde{\mathbf{A}}(k)\,\mathbf{S} - \mu\,\tilde{\mathbf{B}}(k)\,\mathbf{X}(k)\right] - \tilde{\epsilon}^2\,\mathbf{S} \leq \mathbf{0}\,.$$

Hence, the network of transformed systems $(\tilde{A}(k), \tilde{B}(k))$ achieves GUAC with $\mathbf{K}(k) = \mathbf{X}(k)\,\mathbf{S}^{-1}$, and the closed-loop network (7.15) achieves GUEC with rate ρ. ■

Remark 7.4.3 *Lemma 7.4.1 relates the rate of convergence ρ to ϵ. Clearly, we can guarantee a small ρ if ϵ can be small, which yields the following optimization problem*

$$\epsilon_m = \inf_{\mathbf{S}=\mathbf{S}'>\mathbf{0},\mathbf{X}(k)} \epsilon \qquad s.t. \quad (7.30) \quad \textit{for all} \quad k \quad \textit{and} \quad \mu \in \Lambda_L$$

where Λ_L is the set of all non-zero eigenvalues of the graph Laplacian. It is easy to see that LMI constraint (7.30) can be rearranged to

$$\begin{bmatrix} \mathbf{0} & \mathbf{Y}'(k) & \mathbf{0} & -\omega\,\mathbf{X}'(k)\,\mathbf{B}'(k) \\ \mathbf{Y}(k) & \mathbf{0} & \omega\,\mathbf{B}(k)\,\mathbf{X}(k) & \mathbf{0} \\ \mathbf{0} & \omega\,\mathbf{X}'(k)\,\mathbf{B}'(k) & \mathbf{0} & \mathbf{Y}'(k) \\ -\omega\,\mathbf{B}(k)\,\mathbf{X}(k) & \mathbf{0} & \mathbf{Y}(k) & \mathbf{0} \end{bmatrix} \leq \epsilon \begin{bmatrix} \mathbf{S} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{S} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{S} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{S} \end{bmatrix}$$

where $\mathbf{Y}(k) = \mathbf{A}(k)\,\mathbf{S} - \alpha\,\mathbf{B}(k)\,\mathbf{X}(k)$. Hence, we are given a general eigenvalue minimization problem in standard form where the ϵ-dependent side is positive definite. Thus, a solution $\epsilon_m \geq 0$ always exists. However, we can only achieve GUEC if $\epsilon_m < 1$. It should be noted that $\mathbf{S} = \mathbf{S}' > \mathbf{0}$. Hence, solutions $\epsilon \leq 0$ are not feasible.

7.4.3 Robust conditions for consensus - $\mathbf{B}(k)$ bounded

LMI (7.30) depends in a linear affine fashion on the eigenvalues $\mu = \alpha + \jmath\,\omega$. Therefore, (7.30) will hold for all eigenvalues contained inside a polytope if it holds for all vertices describing this polytope. In the following, we will describe a convex polytope

$$\mathcal{P} = \left\{ \mu \quad : \quad \mu = \sum_{i=1}^{M} \phi_i \, \mu_i \, , \quad \sum_{i=1}^{M} \phi_i = 1 \, , \quad \phi_i \geq 0 \right\}$$

by its M vertices that are contained in the set $\mathcal{E} = \{\mu_1, \mu_2, \cdots, \mu_M\}$.

Corollary 7.4.1 *Suppose all non-zero eigenvalues μ of the graph Laplacian lie inside the polytope $\mathcal{P}$ with vertex set $\mathcal{E} = \{\mu_1, \mu_2, \cdots, \mu_M\}$. Then, the closed-loop network* (7.15) *achieves GUEC with rate $0 < \rho < 1$ and gain matrix* (7.31) *if there exists a matrix $\mathbf{S} = \mathbf{S}' > \mathbf{0}$, an $\epsilon^2 < \rho^2$, and for each k, there is a matrix $\mathbf{X}(k)$ such that* (7.30) *holds for all $\mu \in \mathcal{E}$.*

A simple polytope $\mathcal{P}$ is given by

$$\alpha_m \leq \alpha \leq \alpha_M \quad \text{and} \quad \omega_m \leq \omega \leq \omega_M$$

where α_m, α_M, ω_m, $\omega_M \in \mathbb{R}$. In this case, $\mathcal{E}$ is given by

$$\mathcal{E} = \{\alpha_m + \jmath\,\omega_m, \, \alpha_m + \jmath\,\omega_M, \, \alpha_M + \jmath\,\omega_m, \, \alpha_M + \jmath\,\omega_M\} \; .$$

Noting the special structure of (7.30), we conclude $\omega_m \geq 0$ from the following remark.

Remark 7.4.4 *By design, LMI* (7.30) *holds for some $\omega = \tilde{\omega}$ if and only if it holds for $\omega = -\tilde{\omega}$.*

For LMI (7.30), matrices $\mathbf{X}(k)$ and $\mathbf{B}(k)$ are multiplied by α and ω, which immediately yields the following result.

Remark 7.4.5 *Suppose $\mathbf{B}(k) = b\,\mathbf{B}_0$ for some $\mathbf{B}_0$ and, for all b and $\mu \neq 0$, $b\mu$ lies inside the polytope $\mathcal{P}$ with vertex set $\mathcal{E}$. Then, LMI*(7.30) *holds for all $\mathbf{B}(k)$ and all $\mu \neq 0$ if, replacing $\mathbf{B}(k)$ with $\mathbf{B}_0$ and μ with $b\mu$, it holds for all $b\mu \in \mathcal{E}$.*

7.4.4 Robust conditions for consensus - $\mathbf{B}(k)$ known

Next, we present a robust result to achieve GUEC for a variety of communication structures where we assume that $\mathbf{B}(k)$ is known for all k. Related results for time-invariant systems and quantized feedback control are presented in [53] and [55]. First, we have the following result.

Lemma 7.4.2 *Suppose* $\mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{B} \in \mathbb{R}^{n\times q}$ *has rank* q, $\mathbf{P}, \mathbf{Q} \in \mathbb{R}^{n\times n}$ *are symmetric with* $\mathbf{P}$ *positive definite, and* $\mu = \alpha + \jmath\,\omega \neq 0$.

(a) The following statements are equivalent

1. There exists $\mathbf{K} \in \mathbb{R}^{q\times n}$ *such that*

$$(\mathbf{A} - \mu\,\mathbf{BK})'\,\mathbf{P}\,(\mathbf{A} - \mu\,\mathbf{BK}) - \mathbf{Q} \leq \mathbf{0} \tag{7.37}$$

2. For $\xi = \alpha^2/|\mu|^2$,

$$\mathbf{A}'\mathbf{PA} - \xi\,\mathbf{A}'\mathbf{PB}\,(\mathbf{B}'\mathbf{PB})^{-1}\,\mathbf{B}'\mathbf{PA} - \mathbf{Q} \leq \mathbf{0}\,. \tag{7.38}$$

If, in addition, $\mathbf{Q} = \epsilon^2\mathbf{P}$ *for some* $\epsilon > 0$*, then the above statements are also equivalent to the existence of* $\mathbf{K}$, $\mathbf{X} \in \mathbb{R}^{q\times n}$ *such that for* $\mathbf{S} = \mathbf{P}^{-1}$,

$$\begin{bmatrix} -\epsilon\,\mathbf{S} & \mathbf{S}\,\mathbf{A}' - \alpha\,\mathbf{X}'\mathbf{B}' & \omega\,\mathbf{X}'\mathbf{B}' \\ \mathbf{A}\,\mathbf{S} - \alpha\,\mathbf{BX} & -\epsilon\,\mathbf{S} & \mathbf{0} \\ \omega\,\mathbf{BX} & \mathbf{0} & -\epsilon\,\mathbf{S} \end{bmatrix} \leq \mathbf{0}\,. \tag{7.39}$$

(b) Suppose (7.38) *holds with* $\xi \leq \alpha^2/|\mu|^2$ *and*

$$\mathbf{K} = \kappa\,(\mathbf{B}'\mathbf{PB})^{-1}\,\mathbf{B}'\mathbf{PA} \tag{7.40}$$

Then, (7.37) *holds if*

$$\kappa_m \leq \kappa \leq \kappa_M \tag{7.41}$$

where

$$\kappa_m = \frac{\alpha - \sqrt{\alpha^2 - \xi\,|\mu|^2}}{|\mu|^2} \qquad \text{and} \qquad \kappa_M = \frac{\alpha + \sqrt{\alpha^2 - \xi\,|\mu|^2}}{|\mu|^2}\,.$$

Proof Let

$$\mathbf{Y} = (\mathbf{A} - \mu\,\mathbf{BK})'\,\mathbf{P}\,(\mathbf{A} - \mu\,\mathbf{BK}) - \mathbf{Q} \tag{7.42}$$

Since $\mathbf{B}$ has full column rank and $\mathbf{P}$ is positive definite, $\mathbf{R} := \mathbf{B}'\mathbf{PB}$ is positive definite; hence invertible. Introducing

$$\mathbf{K}_0 = \mathbf{R}^{-1}\,\mathbf{B}'\mathbf{PA}, \qquad \mathbf{A}_0 = \mathbf{A} - \mathbf{BK}_0 \tag{7.43}$$

we have

$$\mathbf{A} = \mathbf{A}_0 + \mathbf{BK}_0 \tag{7.44}$$

and

$$\mathbf{B}'\mathbf{PA}_0 = \mathbf{0}\,. \tag{7.45}$$

Substituting (7.44) into (7.42) and using (7.45) yields

$$\mathbf{Y} = \mathbf{A}_0'\mathbf{PA}_0 + (\mathbf{K}_0 - \mu\,\mathbf{K})'\,\mathbf{R}\,(\mathbf{K}_0 - \mu\,\mathbf{K}) - \mathbf{Q}\,. \tag{7.46}$$

Now,

$$\begin{aligned}(\mathbf{K}_0 - \mu\mathbf{K})'\,\mathbf{R}\,(\mathbf{K}_0 - \mu\mathbf{K}) &= [\mathbf{K}_0 - (\alpha{+}\jmath\,\omega)\mathbf{K}]'\,\mathbf{R}\,[\mathbf{K}_0 - (\alpha{+}\jmath\,\omega)\mathbf{K}]\\ &= (\mathbf{K}_0 - \alpha\mathbf{K})'\,\mathbf{R}\,(\mathbf{K}_0 - \alpha\mathbf{K}) + \omega^2\,\mathbf{K}'\mathbf{RK}\\ &\quad + \jmath\,\omega[\mathbf{K}'\mathbf{R}(\mathbf{K}_0{-}\alpha\mathbf{K}) - (\mathbf{K}_0{-}\alpha\mathbf{K})'\mathbf{RK}]\end{aligned} \tag{7.47}$$

and

$$\begin{aligned}(\mathbf{K}_0{-}\alpha\mathbf{K})'\mathbf{R}(\mathbf{K}_0{-}\alpha\mathbf{K}) + \omega^2\,\mathbf{K}'\mathbf{RK} &= \mathbf{K}_0'\mathbf{RK}_0 - \alpha\mathbf{K}_0'\mathbf{RK} - \alpha\mathbf{K}'\mathbf{RK}_0 + |\mu|^2\,\mathbf{K}'\mathbf{RK}\\ &= \left(\frac{\alpha}{|\mu|}\mathbf{K}_0 - |\mu|\mathbf{K}\right)'\mathbf{R}\left(\frac{\alpha}{|\mu|}\mathbf{K}_0 - |\mu|\mathbf{K}\right)\Bigg(\\ &\quad + \left(1 - \frac{\alpha^2}{|\mu|^2}\right)\Bigg(\mathbf{K}_0'\mathbf{RK}_0\,.\end{aligned} \tag{7.48}$$

Note that

$$\mathbf{A}_0'\mathbf{PA}_0 = \mathbf{A}'\mathbf{PA} - \mathbf{K}_0'\mathbf{RK}_0\,. \tag{7.49}$$

It now follows from (7.46)-(7.49) that

$$\begin{aligned}\mathbf{Y} = \mathbf{A}'\mathbf{PA} - \frac{\alpha^2}{|\mu|^2}\,\mathbf{K}_0'\mathbf{RK}_0 - \mathbf{Q} \quad + \left(\frac{\alpha}{|\mu|}\mathbf{K}_0 - |\mu|\mathbf{K}\right)'\mathbf{R}\left(\frac{\alpha}{|\mu|}\mathbf{K}_0 - |\mu|\mathbf{K}\right)\Bigg(\\ + \jmath\,\omega\,[\mathbf{K}'\mathbf{R}(\mathbf{K}_0 - \alpha\mathbf{K}) - (\mathbf{K}_0 - \alpha\mathbf{K})'\mathbf{RK}]\;.\end{aligned} \tag{7.50}$$

To prove part (a), we first observe that (7.37) and (7.38) are equivalent to $\mathbf{Y} \leq \mathbf{0}$ and $\mathbf{Y}_0 \leq \mathbf{0}$, respectively where

$$\mathbf{Y}_0 := \mathbf{A}'\mathbf{P}\mathbf{A} - \frac{\alpha^2}{|\mu|^2}\mathbf{K}_0'\mathbf{R}\mathbf{K}_0 - \mathbf{Q} \tag{7.51}$$

If $\mathbf{Y} \leq \mathbf{0}$, then $Re(\mathbf{Y}) \leq \mathbf{0}$ and, since $Re(\mathbf{Y}) \geq \mathbf{Y}_0$, we obtain that $\mathbf{Y}_0 \leq \mathbf{0}$. If $\mathbf{Y}_0 \leq \mathbf{0}$, then letting $\mathbf{K} = \alpha/|\mu|^2\,\mathbf{K}_0$ we obtain that $\mathbf{Y} = Re(\mathbf{Y}) = \mathbf{Y}_0 \leq \mathbf{0}$.

To prove the equivalence of the third statement, we note from above that the existence of $\mathbf{K}$ such that $Re(\mathbf{Y}) \leq \mathbf{0}$ is equivalent to statements 1) and 2); also

$$Re(\mathbf{Y}) = (\mathbf{A} - \alpha\,\mathbf{B}\mathbf{K})'\,\mathbf{P}\,(\mathbf{A} - \alpha\,\mathbf{B}\mathbf{K}) + \omega^2\,\mathbf{K}'\mathbf{B}'\mathbf{P}\mathbf{B}\mathbf{K} - \mathbf{Q}\,.$$

Let $\mathbf{S} = \mathbf{P}^{-1}$ and $\mathbf{K} = \mathbf{X}\,\mathbf{S}^{-1}$, then pre- and post-multiplying $Re(\mathbf{Y}) \leq \mathbf{0}$ by $\mathbf{S}$, dividing by $\epsilon > 0$ and applying the Schur complement (7.28) twice yields (7.39).

For part (b), we substitute $\mathbf{K} = \kappa\,\mathbf{K}_0$ into (7.50) and obtain

$$\mathbf{Y} = \mathbf{A}'\mathbf{P}\mathbf{A} - \tilde{\xi}\,\mathbf{K}_0'\mathbf{R}\mathbf{K}_0 - \mathbf{Q}$$

where $\tilde{\xi} = -\kappa^2\,|\mu|^2 + 2\,\alpha\,\kappa$. If (7.38) holds, then $\mathbf{Y} \leq \mathbf{0}$ for all $\tilde{\xi} \geq \xi$, that is,

$$\kappa^2\,|\mu|^2 - 2\,\alpha\,\kappa + \xi \leq 0\,.$$

This inequality is satisfied by any κ satisfying (7.41). ■

Lemma 7.4.3 *Suppose* $\mathbf{B}(k)$ *has full column rank for all* k *and there exists a* $\mu_s = \alpha_s + \jmath\,\omega_s$ *such that the following conditiond hold.*

(a) $\kappa_m \leq \kappa_M$ *where*

$$\kappa_m = \max_{\mu\in\Lambda_L}\left\{\left(\frac{\alpha - \sqrt{\alpha^2 - \xi\,|\mu|^2}}{|\mu|^2}\right\}\right(\qquad \kappa_M = \min_{\mu\in\Lambda_L}\left\{\left(\frac{\alpha + \sqrt{\alpha^2 - \xi\,|\mu|^2}}{|\mu|^2}\right\}\right(\tag{7.52}$$

with

$$\xi := \frac{\alpha_s^2}{|\mu_s|^2} \quad \leq \min_{\mu\in\Lambda_L}\frac{\alpha^2}{|\mu|^2}\,, \qquad \alpha = Re(\mu)$$

and Λ_L *is the set of all non-zero eigenvalues of the graph Laplacian.*

(b) *There exists a matrix* $\mathbf{S} = \mathbf{S}' > \mathbf{0}$*, an* $\epsilon \in [0, 1)$*, and for each* k *there exists an* $\mathbf{X}(k)$ *such that*

$$\begin{bmatrix} -\epsilon\,\mathbf{S} & \mathbf{S}\mathbf{A}'(k) - \alpha_s\,\mathbf{X}'(k)\mathbf{B}'(k) & \omega_s\,\mathbf{X}'(k)\mathbf{B}'(k) \\ \mathbf{A}(k)\mathbf{S} - \alpha_s\,\mathbf{B}(k)\mathbf{X}(k) & -\epsilon\,\mathbf{S} & \mathbf{0} \\ \omega_s\,\mathbf{B}(k)\mathbf{X}(k) & \mathbf{0} & -\epsilon\,\mathbf{S} \end{bmatrix} \leq \mathbf{0} \tag{7.53}$$

Then, for any $\kappa \in [\kappa_m, \kappa_M]$ *and with*

$$\mathbf{K}(k) = \kappa\;[\mathbf{B}(k)'\mathbf{P}\mathbf{B}(k)]^{-1}\;\mathbf{B}(k)'\mathbf{P}\mathbf{A}(k) \tag{7.54}$$

the closed-loop network (7.15) *achieves GUEC for any rate satisfying* $\epsilon < \rho < 1$ *where* $\mathbf{P} = \mathbf{S}^{-1}$.

Proof From Lemma 7.4.1 and the proof of Theorem 7.4.1, we have to show that

$$[\mathbf{A}(k) - \mu\,\mathbf{B}(k)\mathbf{K}(k)]'\;\mathbf{P}\;[\mathbf{A}(k) - \mu\,\mathbf{B}(k)\mathbf{K}(k)] - \epsilon^2\,\mathbf{P} \leq \mathbf{0} \tag{7.55}$$

for all $\mu \in \Lambda_L$ and all k. The result follows from Lemma 7.4.2. If (7.53) holds, then (7.38) is satisfied for $\xi = \alpha_s^2/|\mu_s|^2$ and all k. If (7.38) holds, then (7.55) is satisfied for all $\kappa \in [\kappa_m, \kappa_M]$ where $\mathbf{K}(k)$ is given by (7.54). ■

Remark 7.4.6 *In Theorem 7.4.1, the number of LMIs that have to be satisfied depends on the number of non-zero Laplacian eigenvalues. The LMI condition in Lemma 7.4.3 is independent of the number of eigenvalues.*

Remark 7.4.7 *Similar to Theorem 7.4.1, condition* (7.53) *can yield an infinite number of LMIs. However, it is similar to* (7.30) *and can be solved for special cases (Remark 7.4.2).*

Remark 7.4.8 *Similar to Theorem 7.4.1, we can state the problem as a linear minimization of* ϵ *(Remark 7.4.3), and GUEC can be achieved with rate* $\rho > \epsilon_m$ *if* $\epsilon_m < 1$ *and* $\kappa_m \leq \kappa_M$ *where* κ_m *and* κ_M *are given by* (7.52).

Remark 7.4.9 *Lemma 7.4.3 separates the problem of identifying the gain matrix* $\mathbf{K}(k)$ *into determining a* μ_s *such that 1)* $\kappa_m \leq \kappa_M$*, and 2) an LMI can be satisfied. Note that the solution* $\mathbf{X}(k)$ *is not utilized to determine* $\mathbf{K}(k)$*. Only* $\mathbf{P} = \mathbf{S}^{-1}$ *is used.*

Remark 7.4.10 *One can show that* $\kappa_m \leq \kappa_M$ *is equivalent to the existence of a* κ *such that* $|1 - \kappa\,\mu|^2 \leq 1 - \xi$ *for all* $\mu \in \Lambda_L$*. It is desired to minimize* ξ *in order to support a large range of* μ*.*

Generally, we have to increase $\xi = \alpha_s^2/|\mu_s|^2$ in order to obtain smaller $\epsilon > 0$, that is, smaller ρ. However, Remark 7.4.10 shows that we want ξ to be small. Thus, for a fixed network, arbitrary rates of convergence are not possible. This is similar to an observation made in Section 6.3 and 7.1.3, where it was already shown that we cannot always identify $\mathbf{K}(k)$ such that the closed-loop network (2.5)-(2.6) achieves GUAC. However, if systems are time-invariant, marginally stable, and stabilizable, then we have the following results.

Lemma 7.4.4 ([53]) *If systems are time-invariant, marginally stable, and stabilizable and* $\mathbf{B}$ *has full column rank, then for all* $r^2 < 1$ *there exists a* $\mathbf{P} = \mathbf{P}' > \mathbf{0}$ *and an* $\epsilon^2 < 1$ *such that (7.38) is satisfied where* $\xi = 1 - r^2$*.*

Corollary 7.4.2 *If systems are time-invariant, marginally stable, and stabilizable and* $\mathbf{B}$ *has full column rank, then GUAC can always be achieved using linear control.*

Proof From Lemma 7.4.4, for all $\xi > 0$ there exists a $\mathbf{P} = \mathbf{P}' > \mathbf{0}$ and an $\epsilon^2 < 1$ such that (7.38) holds. If $\xi \to 0$, then from (7.52),

$$0 < \kappa < \frac{2\,\alpha}{|\mu|^2}$$

and we can always choose $\kappa > 0$ small enough such that $\kappa < \frac{2\,\alpha}{|\mu|^2}$ for all $\mu \neq 0$. ■

Clearly, if (7.38) holds for $\hat{\xi}$, then it holds for all $\xi \geq \hat{\xi}$. Since we are only interested in solutions $\xi \in [0, 1]$, we could check the bounds and apply a bisectioning procedure to determine ξ_{inf} for which LMI condition (7.53) has a solution. If $\xi = 0$, then solutions exists if and only if $\mathbf{A}$ is stable. If $\xi = 1$, then the problem reduces to the standard DARE.

In this thesis, since we also require $\kappa_m \leq \kappa_M$, we determine an upper bound on ξ based on the Laplacian eigenvalues first. This is shown in the next remark.

Remark 7.4.11 *A bisectioning scheme can be used to find ξ_{max} for which $\kappa_m \leq \kappa_M$ where κ_m and κ_M are given by (7.52). Then, Lemma (7.4.3) and Remark 7.4.8 can be used to check if there is a feasible $\mathbf{K}(k)$, which in addition would guarantee a specific rate of convergence.*

7.5 Simulations

We illustrate our results with two examples. First, we illustrate the LMI approach presented in Lemma 7.4.3 for linear time-invariant systems. Then, we compare results for linear time-variant systems using the simple control design and the LMI approach.

7.5.1 Linear time-invariant systems

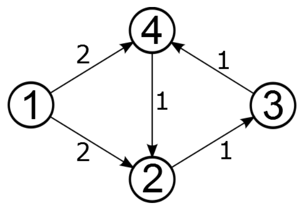
The control design presented in Lemma 7.4.3 separates the dependencies between the individual system dynamics and the graph associated with the network into condition (7.53) and, recalling κ_m and κ_M from (7.52), $\kappa_m \leq \kappa_M$ respectively. Therefore, first we solve condition (7.53) without specifying a particular network structure. Then, we add a communication network and show the effectiveness of the control.

As an example, we considered time-invariant systems with dynamics

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 1 & -0.5 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

and open-loop poles -1.281, 0.781. From Lemma 7.4.3 and Remark 7.4.8, we obtained the unscaled control gain,

$$\mathbf{K}_0 = (\mathbf{B}'\mathbf{P}\mathbf{B})^{-1}\mathbf{B}'\mathbf{P}\mathbf{A} = \begin{bmatrix} 1.000 & -1.281 \end{bmatrix}$$



$$\Rightarrow \qquad L = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -2 & 3 & 0 & -1 \\ 0 & -1 & 1 & 0 \\ -2 & 0 & -1 & 3 \end{bmatrix}$$

Figure 7.1.. Communication network with assigned edge weights

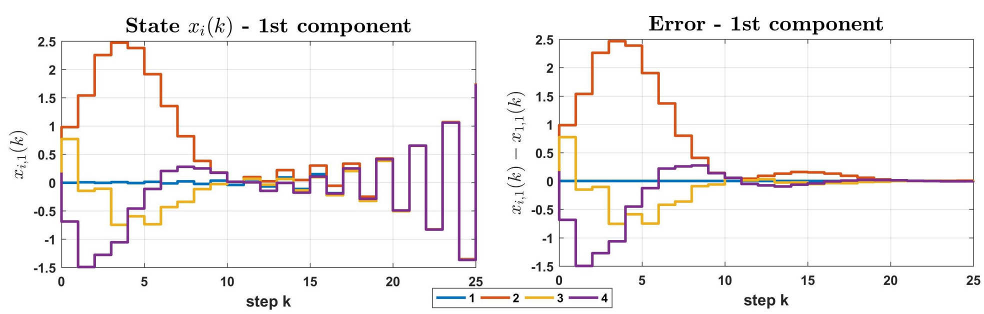


Figure 7.2.. Simulation of four linear second-order systems achieving GUEC on System 1 (blue line). Note that only $x_{i,1}(k)$ is shown since $x_{i,2}(k) = x_{i,1}(k+1)$.

where $\mathbf{K} = \kappa\,\mathbf{K}_0$ and $\xi_{inf} = 0.39$. [53] reported an exact bound on ξ_{inf} for single input time-invariant systems. Using the bisectioning scheme on the LMIs from Lemma 7.4.3, we approximated ξ_{inf} with an absolute error of about $1e-5$.

The network graph and its associated Laplacian matrix are given in Figure 7.1. The Laplacian eigenvalues are given by $\mu_1 = 0$, $\mu_2 = 0.80$, and $\mu_{3/4} = 3.1 \pm .67\jmath$. In this network, System 1 is a leader, and when GUEC is achieved, then the states of all the systems will equal that of System 1.
For $\xi = 0.39$, we obtain $\kappa_m = 0.276$ and $\kappa_M = 0.545$ from (7.52). If we use the scheme from Remark 7.4.11, then we obtain $\kappa_{opt} = 0.489$ and $\xi_{max} = 0.626$. The associated gain matrix $\mathbf{K}$ is given by (7.40),

$$\mathbf{K} = \begin{bmatrix} 0.489 & -0.626 \end{bmatrix} = 0.489\,\mathbf{K}_0$$

and guarantees a convergence rate of $\rho = 0.783$.
The simulation results for random initial conditions are shown in Figure 7.2. The results illustrate that GUEC is achieved. For each system, we introduced an error $e_i = x_i - x_1$ where System 1 (blue line) was the leader and its initial conditions governed the steady state dynamics as seen on the left side of Figure 7.2. The process of achieving GUEC is illustrated on the right side of Figure 7.2 where all errors decay.

7.5.2 Linear time-variant systems

Here, we illustrate our results for linear time-variant systems. As an example, we recalled the communication network and associated Laplacian matrix L with non-zero eigenvalues $1.5 \pm \jmath\sqrt{3}/2$ shown in Figure 6.5. We wished to achieve GUEC for a network of arbitrary switching systems (A_1, B) and (A_2, B) in controllable canonical form where

$$\mathbf{A}_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -.0075 & .01 & .76 & -1 \end{bmatrix}, \quad \mathbf{A}_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ .022 & .046 & -.06 & 1.1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

and the open-loop poles of $\mathbf{A}_1$ and $\mathbf{A}_2$ were given by

$$\mathbf{A}_1 : \quad -1.5,\ -1,\ 0.5,\ 1 \quad \text{and} \quad \mathbf{A}_2 : \quad -0.2,\ 0.1 \pm 0.3\jmath,\ 1.1$$

Simple control design

Although we had a fixed set of Laplacian eigenvalues, we made use of Lemma 7.1.4. First, we determined an appropriate R and C such that $|\mu - C| \leq R$ for all $\mu \in \Lambda_L$ where Λ_L is the set of all non-zero Laplacian eigenvalues. From (7.11), it is desired that R/C is small. Thus, we wanted to find

$$\frac{R_{opt}}{C_{opt}} = \min_{C \in \mathbb{R}} \max_{\mu \in \Lambda_L} \frac{\mu}{C} - 1 \ .$$

For a complex conjugate pair or single μ, it is easy to show that $C_{opt} = |\mu|^2/\alpha$. Hence,

$$\min_{\mu \in \Lambda_L} |\mu|^2/\alpha \leq C_{opt} \leq \max_{\mu \in \Lambda_L} |\mu|^2/\alpha$$

and even a brute-force optimization algorithm will yield sufficiently accurate results quickly. In our case, only one complex conjugate pair of μ was given, and we obtained

$$C_{opt} = 2 \quad \text{and} \quad R_{opt} = 1 \, .$$

Now, we chose control $\mathbf{K}(k)$ such that $\gamma_i(k) = -c_i(k)/C$, that is,

$$\mathbf{K}_{1,simple} = \begin{bmatrix} -0.00375 & 0.00500 & 0.380 & -0.500 \end{bmatrix}$$
$$\mathbf{K}_{2,simple} = \begin{bmatrix} 0.0110 & 0.0230 & -0.0300 & 0.550 \end{bmatrix}$$

From Theorem 7.1.1, we obtained the smallest rate of convergence for each system configuration ($\rho_1 = 0.923$ and $\rho_2 = 0.679$) by applying a bi-sectioning procedure. Thus, from Theorem 7.3.1, the closed-loop network of switching systems achieves GUEC with guaranteed rate $\rho = 0.923$.

LMI approach

We applied Lemma 7.4.3 and Remark 7.4.8 to obtain gain matrices

$$\mathbf{K}_{1,LMI} = \begin{bmatrix} -0.00372 & 0.00813 & 0.380 & -0.701 \end{bmatrix}$$
$$\mathbf{K}_{2,LMI} = \begin{bmatrix} 0.0110 & 0.0260 & -0.0312 & 0.350 \end{bmatrix}$$

that guarantee GUEC with rate $\rho = 0.777$.

Simulation

We simulated the closed-loop network for different switching patterns and random initial conditions. First, we switched between systems for each step k (Figure 7.3) and observed that the frequent switching results in a stable behaviour and GUAC could be achieved with $\mathbf{K} = \mathbf{0}$. Then, we simulated switching at every other step (Figure 7.4). GUEC is achieved in either case. The control obtained with the simple control design and the control obtained with the LMI approach show similar performance, which indicates that the simple control design is very conservative since it only guaranteed a rate of $\rho = 0.923$ but achieved GUEC much quicker. Finally, we note that systems were given in controllable canonical form. Therefore, the figures show the trajectory of the last element of each state vector $\mathbf{x}_i(k)$ only.

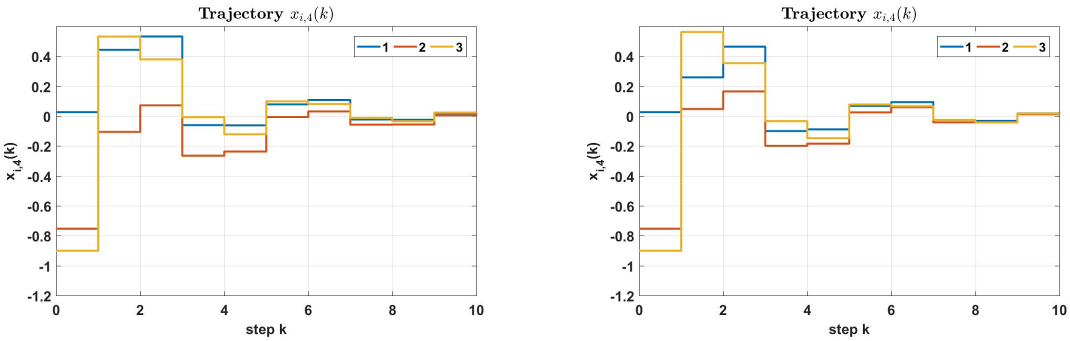


Figure 7.3.. Simulation of three time-variant switching systems achieving GUEC using the simple control design (left) and the LMI approach (right) while switching each step. Note that only $x_{i,4}(k)$ is shown since systems are in controllable canonical form.

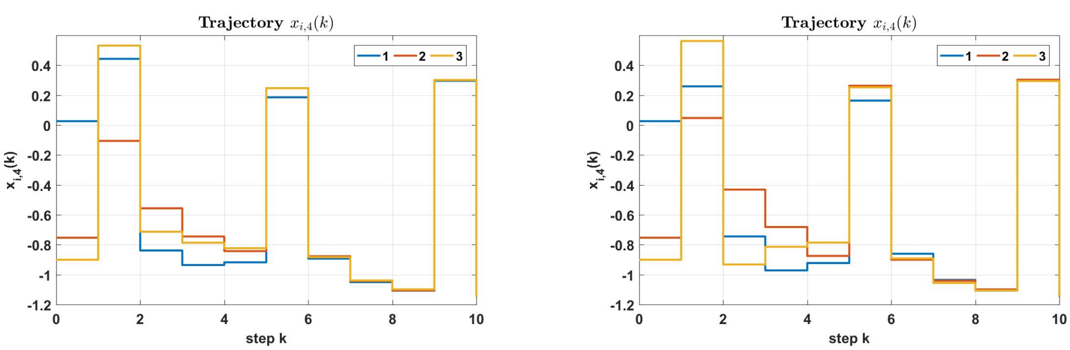


Figure 7.4.. Simulation of three time-variant switching systems achieving GUEC using the simple control design (left) and the LMI approach (right) while switching every other step. Note that only $x_{i,4}(k)$ is shown since systems are in controllable canonical form.

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