

# Model Reduction by Moment Matching for Convergent Lur'e-Type Models

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**Abstract**—This paper proposes an approach to model order reduction of convergent Lur'e-type models, which consist of a linear time-invariant (LTI) block and a static nonlinear block that is placed in feedback with the LTI block. In the proposed approach, we match a finite number of moments of the LTI block and keep the static nonlinear block to approximate the moments of the Lur'e-type model. The benefits of this approach are that the Lur'e-type structure is preserved after reduction, that the reduction method has an interpretation in terms of the frequency response function of the LTI block and that global exponential stability properties of the full-order model are preserved. The effectiveness of the approach is illustrated in a numerical example.

## I. INTRODUCTION

Model order reduction aims at reducing the complexity of dynamical models allowing for analysis, controller design and implementation. For linear time-invariant (LTI) models, many different methods, such as balanced truncation [7], Hankel-norm approximations [5] and the interpolation approach [4], have been proposed in the literature. The moment matching method, see [1], has the property that *moments* of reduced-order LTI model are equal to the *moments* of the full-order LTI model. Traditionally, the moment matching method is interpreted as a problem of interpolation of points on the complex plane. In particular, moments are the coefficients of the Laurent series expansion of the frequency response function (FRF) at a set of user-defined frequencies. Consequently, the moment matching property is particularly useful in applications where it is known at which frequencies the full-order LTI model is going to be excited and thus an accurate response of the reduced-order LTI model at those excitation frequencies is needed.

In [2], a time-domain interpretation of moment matching in terms of matching the steady-state response of the reduced-order LTI model to the steady-state response of the full-order LTI model for a specific class of input signals is given. This interpretation allows for a straightforward definition of moment matching for nonlinear models; namely, matching the steady-state response of the reduced-order nonlinear model to the steady-state response of the full-order nonlinear model for a specific class of input signals [3], [9].

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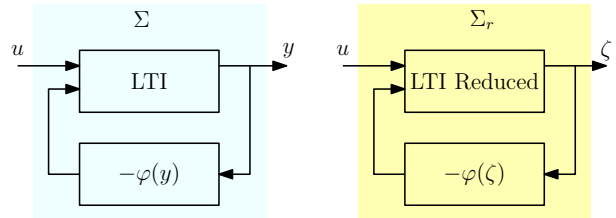


Fig. 1: Left: Full-order Lur'e-type model  $\Sigma$ . Right: Reduced-order Lur'e-type model  $\Sigma_r$  with the same nonlinear mapping  $\varphi$  as in  $\Sigma$ .

Although a solution to the moment matching problem for nonlinear models is given in [3], [9], there are still many open questions of which we list three here. The first open question is how to preserve the structure of the full-order nonlinear model. For example, if the full-order nonlinear model is of Lur'e-type form, then a straightforward application of the method proposed in [3], [9] produces a nonlinear reduced-order model that is not of Lur'e-type form. The second open question relates to an FRF interpretation of the reduction method. As mentioned earlier, traditional moment matching for LTI models is interpreted as an interpolation problem of points in the complex plane, which has a clear interpretation in terms of the FRFs of the full-order and reduced-order LTI models, see [4]. In the scope of moment matching for nonlinear models [3], a full characterization of nonlinear model reduction by moment matching in the frequency domain is still missing. The third open question is how to preserve *global* exponential stability properties of the full-order nonlinear model. There are moment matching methods that guarantee *global* exponential stability of reduced-order *LTI* models and *local* exponential stability of reduced-order *nonlinear* models, see [3], [9]. However, in the scope of moment matching for nonlinear models, it is an open question how to enforce a form of *global* exponential stability on the reduced-order nonlinear model, especially in the presence of time-varying inputs.

In this paper, we deal with the three open questions listed above for a practically relevant class of nonlinear models, namely Lur'e-type models, see [6], that are exponentially convergent, see [8]. Lur'e-type models contain LTI dynamics that are captured in an LTI block and nonlinearities that are captured in a static nonlinear block placed in feedback with the LTI block, see Figure 1. Exponentially convergent Lur'e-type models enjoy a global stability property that, loosely speaking, ensures that for any bounded input signal, the model response forgets its initial condition and converges

exponentially to a uniquely defined bounded steady-state solution [13]. Lur'e-type models naturally arise in, e.g., electronic circuits with local nonlinear elements, mechanical systems with nonlinear actuator/sensor characteristics and spatially discretized partial differential equations with local nonlinear elements.

The method that we propose consists of two steps. First, we construct a family of reduced-order LTI models in which the contributions of the full-order LTI block and of the nonlinear block are clearly partitioned. In particular, the nonlinear block can be interpreted as an approximation of the nonlinear moment of the Lur'e-type model by a finite number of linear moments of the LTI block. The first step allows preserving the Lur'e-type structure of the full-order Lur'e-type model, which is particularly important given that many analysis and synthesis results exist for this class of models [6]. Furthermore, the first step allows for an FRF interpretation of the LTI part of the full-order and reduced-order Lur'e-type model. Since this step provides a family of reduced-order models with a specific parametric freedom, we introduce a second step in which this freedom is exploited to preserve the desired stability property, i.e., exponential convergence, of the reduced-order Lur'e-type model by solving a constraint optimization problem. This optimization problem also fits the FRF optimally at prescribed frequencies, which is important in the scope of matching the moments of the nonlinear Lur'e-type model.

To summarize, the main contribution of this paper is a model order reduction technique for exponentially convergent Lur'e-type models that:

- preserves the Lur'e-type structure;
- matches the FRF of the LTI block at a set of frequencies and minimizes the mismatch between the FRFs of the LTI blocks of the full-order model and the reduced-order model at another set of frequencies; and
- guarantees that the reduced-order Lur'e-type model is also exponentially convergent.

The remainder of this paper is structured as follows. Section 2 introduces the considered class of Lur'e-type models, recalls sufficient conditions for exponential convergence and poses the model order reduction problem. Section 3 presents the proposed approach to solve the model order reduction problem. Section 4 describes the results of a numerical study that illustrates the application and benefits of our approach. Section 5 gives the concluding remarks.

## II. PROBLEM SETTING

### A. Convergent Lur'e-type models

Consider the class of Lur'e-type models described by the following state-space equations:

$$\Sigma : \quad \dot{x} = Ax + B_1u + B_2\varphi(y), \quad y = Cx, \quad (1)$$

where  $x(t) \in \mathbb{R}^n$  is the state,  $u(t) \in \mathbb{R}$  is the input,  $y(t) \in \mathbb{R}$  is the output,  $\varphi : \mathbb{R} \rightarrow \mathbb{R}$  is a nonlinear mapping and model matrices  $A \in \mathbb{R}^{n \times n}$ ,  $B_1, B_2 \in \mathbb{R}^{n \times 1}$ ,  $C \in \mathbb{R}^{1 \times n}$ . The

associated FRFs are defined as follows:

$$\mathcal{G}_i(j\omega) := C(j\omega I - A)^{-1}B_i, \quad i = 1, 2. \quad (2)$$

In many engineering applications, a form of model stability is desired. Hereto, we introduce the following notion of stability of models (1), where  $\mathcal{U}$  is the set of continuous and bounded functions.

*Definition 1 ([8]):* The model (1) is said to be globally exponentially convergent if for every input  $u \in \mathcal{U}$ , there exists a solution  $\bar{x}$  to (1) satisfying the following conditions:

- $\bar{x}$  is defined and bounded on  $t \in \mathbb{R}$ ,
- $\bar{x}$  is globally exponentially stable.

The solution  $\bar{x}$  is called the steady-state solution and depends on the applied input  $u$ . Independent of the initial conditions, all solutions of exponentially convergent models converge to the globally exponentially stable steady-state solution  $\bar{x}$ . The following theorem presents conditions that guarantee global exponential convergence of (1).

*Theorem 2 ([8]):* Consider model (1). Suppose that for some constant  $\gamma > 0$  the nonlinear function  $\varphi$  satisfies the following incremental sector condition:

$$\left| \frac{\varphi(y_2) - \varphi(y_1)}{y_2 - y_1} \right| \leq \gamma, \quad \forall y_1, y_2 \in \mathbb{R}. \quad (3)$$

Denote  $A_\gamma^- := A - \gamma B_2 C$  and  $A_\gamma^+ := A + \gamma B_2 C$ . If there exists a  $Q = Q^\top \succ 0$  such that

$$QA_\gamma^- + (A_\gamma^-)^\top Q \prec 0 \quad \text{and} \quad QA_\gamma^+ + (A_\gamma^+)^\top Q \prec 0 \quad (4)$$

hold, then model (1) is globally exponentially convergent according to Definition 1.

### B. Definition of a moment

Consider the signal generator

$$\dot{\tau} = S\tau, \quad u = L\tau \quad (5)$$

with state  $\tau(t) \in \mathbb{R}^v$ , output  $u(t) \in \mathbb{R}$  and matrices  $S \in \mathbb{R}^{v \times v}$ ,  $L \in \mathbb{R}^{1 \times v}$ . The input  $u$  to model (1) is generated by the output of the signal generator (5). The interconnected model reads as follows:

$$\begin{aligned} \dot{\tau} &= S\tau, \\ \dot{x} &= Ax + B_1L\tau + B_2\varphi(y), \\ y &= Cx. \end{aligned} \quad (6)$$

Next, we impose the following two assumptions on the interconnected model (6) consisting of (1) and (5).

*Assumption 1:* The model (1) satisfies the conditions of Theorem 2 for  $\gamma = \gamma^*$  for some  $\gamma^* > 0$  and is, therefore, exponentially convergent according to Definition 1.

*Assumption 2:* The matrix  $S$  of (5) has simple eigenvalues that are located on the imaginary axis. In addition, it is assumed that the pair  $(L, S)$  is observable.

These two assumptions guarantee the existence of a globally exponentially stable invariant manifold, as stated in the following lemma.

*Lemma 3 ([8]):* Under Assumptions 1 and 2, there exists a unique continuous mapping  $\Pi : \mathbb{R}^v \rightarrow \mathbb{R}^n : \tau \mapsto \Pi(\tau)$  that is invariant with respect to the interconnected model (6)

and such that  $\bar{x}(t) = \Pi(\tau(t))$  is the globally exponentially stable steady-state response of the interconnected model (6). Lemma 3 allows defining the moments for the Lur'e-type model (1) in accordance to the framework introduced in [3].

*Definition 4:* Consider the interconnected model (6) and suppose Assumptions 1 and 2 hold. The function  $C\Pi$ , with  $\Pi$  in Lemma 3, is the *moment of the model (1) at  $(S, L)$* .

### C. Moment matching problem

In the moment matching literature, the aim is to match the moments of the reduced-order nonlinear model to the moments of the full-order nonlinear model at  $(S, L)$  [3], [9]. However, in those approaches, the structure of the full-order nonlinear model is generally not preserved. The problem that we formulate below aims at preserving the Lur'e-type structure of the model. As we will show in the subsequent sections, preservation of the model structure comes at the expense of the reduced-order Lur'e-type model not matching the moments of the full-order Lur'e-type model (1) *exactly*, but rather *approximating* them.

The mismatch between the moments of the reduced-order model and the full-order model becomes clear if the steady-state output  $\bar{y}$  of the full-order Lur'e-type model is written in frequency domain as follows:

$$\bar{Y}(j\omega) = \mathcal{G}_1(j\omega)U(j\omega) + \mathcal{G}_2(j\omega)\bar{R}(j\omega), \quad \forall \omega \in \mathbb{R}, \quad (7)$$

where  $\mathcal{G}_1(j\omega)$  and  $\mathcal{G}_2(j\omega)$  are given in (2). The variables  $\bar{Y}(j\omega), U(j\omega), \bar{R}(j\omega)$  denote the Fourier transforms of  $\bar{y}(t), u(t), \varphi(\bar{y}(t))$ , respectively. Due to the nonlinearity  $\varphi$ , the variable  $\bar{R}(j\omega)$  is non-zero at an infinite number of frequencies, i.e., the signal  $u(t) = \varphi(y(t))$  contains an infinite number of frequencies. To solve the moment matching problem, the FRF  $\mathcal{G}_2$  should be matched at the infinite number of frequencies that are contained in  $\bar{R}(j\omega)$ , i.e., an infinite number of interpolation points in the moment matching problem. While some methods have been proposed to match moments at infinitely many interpolation points, see [10], it is unclear if and how these methods can be generalized to the current setting (because, e.g., these methods do not preserve the Lur'e-type structure of the full-order model).

Consider the reduced-order Lur'e-type model  $\Sigma_r$ :

$$\Sigma_r : \quad \dot{\xi} = F\xi + G_1u + G_2\varphi(\zeta), \quad \zeta = H\xi, \quad (8)$$

with state  $\xi(t) \in \mathbb{R}^m$ , input  $u(t) \in \mathbb{R}$ , output  $\zeta(t) \in \mathbb{R}$ , the same nonlinear mapping  $\varphi : \mathbb{R} \rightarrow \mathbb{R}$  as in (1) and model matrices  $F \in \mathbb{R}^{m \times m}, G_1, G_2 \in \mathbb{R}^{m \times 1}, H \in \mathbb{R}^{1 \times m}$ . Note that  $\Sigma_r$  in (8) preserves the structure of  $\Sigma$  in (1). The FRFs associated with the LTI part of (8) read as:

$$\Gamma_i(j\omega) := H(j\omega I - F)^{-1}G_i, \quad i = 1, 2. \quad (9)$$

Let us present some intuition behind the *approximate* moment matching problem, before we formally introduce the problem. Suppose that  $U(j\omega)$  in (7) excites  $\eta_1$  frequencies that are contained in  $\Omega_1^0 \in \mathbb{R}^{\eta_1}$ , which implies that at those frequencies, the FRF  $\Gamma_1$  of the reduced-order model should equal the FRF  $\mathcal{G}_1$  of the full-order model. Again from (7), we observe that the FRF  $\mathcal{G}_2$  is excited at the frequencies

contained in  $\bar{R}(j\omega)$ . To find out which frequencies are excited by  $\bar{R}(j\omega)$ , consider the following property.

*Property 5 ([8]):* Suppose model (1) is exponentially convergent according to Definition 1. If the input  $u$  is periodic with period  $T$ , then the corresponding steady-state output  $\bar{y}$  is also periodic with the same period  $T$ .

Property 5 ensures that if  $u$  is periodic with period  $T$ , then so is  $\bar{y}$ , which implies that  $\bar{r} = \varphi(\bar{y})$  is also periodic with period  $T$ . Thus,  $\bar{R}(j\omega)$  contains the same frequencies  $\Omega_1^0$  as  $U(j\omega)$ , but also an infinite number of higher harmonic frequencies (multiples of the frequencies in  $\Omega_1^0$ ) and an infinite number of intermodulation frequencies (frequencies combine to produce new frequencies). As mentioned above, only a finite number of frequencies of  $\mathcal{G}_2$  can be matched by  $\Gamma_2$  and these frequencies should be wisely chosen. Firstly, we can choose the frequencies in  $\Omega_0^1$ . Secondly, we can choose harmonics of the frequencies in  $\Omega_0^1$ . Thirdly, we can choose intermodulation frequencies, i.e., combinations of frequencies in  $\Omega_0^1$ . Lastly, we can choose frequencies associated to important model characteristics, such as, e.g., 0 Hz for equilibrium behavior and frequencies corresponding to resonance peaks. These frequencies are collected in  $\Omega_0^2 \in \mathbb{R}^{\eta_2}$  with the goal of matching the FRF  $\Gamma_2$  of the reduced-order model at those frequencies to the FRF  $\mathcal{G}_2$  of the full-order model. To obtain an accurate match between the moments of the reduced-order and full-order Lur'e-type model, it is important that the mismatch between the FRFs  $\Gamma_i$  and  $\mathcal{G}_i$ , for  $i = 1, 2$ , is also minimized at other frequencies, i.e., those not contained in  $\Omega_0^i, i = 1, 2$ . The mismatch between  $\Gamma_i$  and  $\mathcal{G}_i$  is minimized at the frequencies collected in  $\Omega_{M_i}^i \in \mathbb{R}^{M_i}, i = 1, 2$ .

Before formally presenting the problem statement, we define the mismatch in the FRFs as follows:

$$\Upsilon_i(j\omega) := \mathcal{G}_i(j\omega) - \Gamma_i(j\omega), \quad \text{for } i = 1, 2, \quad (10)$$

where  $\mathcal{G}_i(j\omega), \Gamma_i(j\omega)$  are defined in (2), (9), respectively.

*Problem 6:* Consider given sets of frequencies  $\Omega_0^i \in \mathbb{R}^{\eta_i}, \Omega_{M_i}^i \in \mathbb{R}^{M_i}$  for  $i = 1, 2$ . The model order reduction problem is to find  $F, G_1, G_2$  and  $H$  that define the reduced-order Lur'e-type model (8) with state dimension  $m = 2(\eta_1 + \eta_2)$ , i.e.,  $\xi(t) \in \mathbb{R}^m$ , such that

- 1) the reduced-order Lur'e-type model (8) is exponentially convergent according to Definition 1;
- 2) at the set of frequencies  $\omega^i \in \Omega_0^i$ , the mismatch

$$\Upsilon_i(j\omega^i) = 0, \quad \text{for } i = 1, 2. \quad (11)$$

- 3) at the set of frequencies  $\omega^i \in \Omega_{M_i}^i, i = 1, 2$ , the mismatch quantified by the following cost function:

$$J(F, G_1, G_2, H) := \sum_{i=1}^2 \sum_{k=1}^{M_i} |\Upsilon_i(j\omega_k^i)|^2 \quad (12)$$

is minimized, where  $\omega_k^i$  is the  $k$ -th element of  $\Omega_{M_i}^i$  for  $i = 1, 2$ .

Problem 6 has the following interpretation. *Item 1)* guarantees that the reduced-order Lur'e-type model (8) preserves the convergence property of the full-order Lur'e-type model (1). *Item 2)* ensures that the FRFs  $\mathcal{G}_i$  and  $\Gamma_i$  are equal at the

set of user-defined frequencies  $\Omega_0^i, i = 1, 2$ . *Item 3*) ensures an optimal fit between  $\Gamma_i$  and  $\mathcal{G}_i$  at user-defined frequencies  $\Omega_{M_i}^i$ , for  $i = 1, 2$ , for example, a logarithmic grid over a range of frequencies. The next section presents the proposed solution to Problem 6.

### III. MODEL ORDER REDUCTION APPROACH

#### A. Conceptual description

The reduction approach works as follows. First, the LTI part of the full-order Lur'e-type model is decoupled from the Lur'e-type model. Then, for the full-order LTI model, a family of reduced-order LTI models is given such that *Item 2*) of Problem 6 is satisfied. As will be shown, this step yields freedom in parameters  $G_1, G_2$ , which is exploited to solve a constrained optimization problem that minimizes the cost function (12), hence satisfying *Item 3*), and ensures that the conditions of Theorem 2 are satisfied, hence satisfying *Item 1*. The reduced-order Lur'e-type model consists of the reduced-order LTI model placed in feedback with the nonlinear mapping  $\varphi$  of the full-order Lur'e-type model.

#### B. Family of reduced-order LTI models satisfying *Item 2*) of Problem 6

The family of reduced-order models is found by applying moment matching for SISO LTI models, as described in [9], for each transfer function  $\mathcal{G}_i(j\omega), i = 1, 2$ , individually and then stacking the two reduced-order LTI models in a single state-space model. The result is summarized in the next theorem, where the notation  $\sigma(S_i) \simeq \Omega_0^i, i = 1, 2$ , means that the signal generator defined by  $(S_i, L_i)$  generates the frequencies in  $\Omega_0^i \in \mathbb{R}^{n_i}$ , i.e., if  $\alpha \in \Omega_0^i$  with  $\alpha \in \mathbb{R}$ , then  $\pm j\alpha \in \sigma(S_i)$ , where  $\sigma(S_i)$  denotes the spectrum of  $S_i \in \mathbb{R}^{v_i \times v_i}, v_i = 2n_i, i = 1, 2$  [3].

*Theorem 7:* Consider given sets of frequencies  $\Omega_0^i \in \mathbb{R}^{n_i}, i = 1, 2$ , and suppose that  $\sigma(S_i) \simeq \Omega_0^i, i = 1, 2$ . Furthermore, suppose Assumption 1 holds for the full-order Lur'e-type model (1) and Assumption 2 holds for both  $(S_i, L_i), i = 1, 2$ . Consider the following reduced-order LTI model:

$$\begin{aligned} \begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \end{bmatrix} &= \underbrace{\begin{bmatrix} F_1 & 0 \\ 0 & F_2 \end{bmatrix}}_F \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} + \underbrace{\begin{bmatrix} \mathcal{G}_1 \\ 0 \end{bmatrix}}_{G_1} u + \underbrace{\begin{bmatrix} 0 \\ \mathcal{G}_2 \end{bmatrix}}_{G_2} r, \\ \zeta &= \underbrace{\begin{bmatrix} H_1 & H_2 \end{bmatrix}}_H \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}, \end{aligned} \quad (13)$$

where  $\xi_1(t) \in \mathbb{R}^{v_1}, \xi_2(t) \in \mathbb{R}^{v_2}, u(t) \in \mathbb{R}, r(t) \in \mathbb{R}, \zeta(t) \in \mathbb{R}$  and with

$$F_i = S_i - \mathcal{G}_i L_i, \quad H_i = C \Pi_i, \quad (14)$$

where  $\Pi_i \in \mathbb{R}^{n_i \times v_i}$  is the unique solution of the Sylvester equation:

$$\Pi_i S_i = A \Pi_i + B_i L_i, \quad (15)$$

for  $i = 1, 2$ . For any  $\mathcal{G}_i \in \mathbb{R}^{v_i \times 1}$  that satisfies

$$\sigma(S_i - \mathcal{G}_i L_i) \cap \sigma(S_i) = 0, \quad (16)$$

the FRF  $\Gamma_i$  in (9) of the reduced-order LTI model (13) is equal to the FRF  $\mathcal{G}_i$  in (2) of the full-order model (1) at frequencies  $\Omega_0^i$ , for  $i = 1, 2$ , i.e.,  $\Upsilon_i(j\omega^i) = 0$  for  $\omega^i \in \Omega_0^i$ , for  $i = 1, 2$ , and, hence, *Item 2*) of Problem 6 is solved.

*Proof:* The proof is omitted for the sake of brevity. ■

The family of reduced-order LTI models is given in (13). We would like to note that even if  $\mathcal{G}_i$  is selected such that (16) is satisfied, the reduced-order Lur'e-type model  $\Sigma_r$  in (8) is not guaranteed to be exponentially convergent, as the conditions in Theorem 2 might not be satisfied. In the next section, we present a method to find a  $\mathcal{G}_1, \mathcal{G}_2$  such that *Items 1*) and *3*) of Problem 6 are also satisfied.

*Remark 8:* The specific parametrization in (13) leaves the  $\xi_1, \xi_2$  dynamics decoupled. Consequently, when the user wants to match the same frequencies in  $\mathcal{G}_1$  and  $\mathcal{G}_2$  by  $\Gamma_1$  and  $\Gamma_2$ , respectively, then these frequencies should be included in  $S_1$  and repeated in  $S_2$ , which raises the order of the reduced-order model. Future work aims at coupling the  $\xi_1, \xi_2$  dynamics, such that one, possibly, can eliminate repeating the frequencies of  $S_1$  in  $S_2$  and vice versa.

#### C. Constrained Optimization Satisfying *Items 1*) and *3*) of Problem 6

The family of reduced-order Lur'e-type models is given in (8) with  $F, G_1, G_2, H$  as in (13) in Theorem 7 and free parameters  $\mathcal{G}_1, \mathcal{G}_2$ . In [3], a similar freedom in  $\mathcal{G}_1, \mathcal{G}_2$  is exploited to, for example, place the poles of the reduced-order LTI model at desired locations or to enforce passivity of the reduced-order LTI model. In [11], a similar type of freedom, albeit with a different model parametrization than (14), is used to match the *transient* response in an optimal way by solving a nonlinear optimization problem. In this section, the freedom in  $\mathcal{G}_1, \mathcal{G}_2$  is exploited to (i) minimize the mismatch between FRFs  $\Gamma_i(j\omega^i)$  and  $\mathcal{G}_i(j\omega^i)$  at frequencies  $\omega^i \in \Omega_{M_i}^i$ , for  $i = 1, 2$ ; and (ii) enforce that the reduced-order Lur'e-type model in (8) satisfies the conditions of Theorem 2, which guarantees global exponential convergence.

*1) Constrained Gradient-Based Optimization:* The sets  $\Omega_{M_i}^i \in \mathbb{R}^{M_i}$  contain the frequencies at which an optimal fit between the FRFs  $\mathcal{G}_i$  and  $\Gamma_i$  is desired. For example, one can take for  $\Omega_{M_i}^i$  a grid of  $M_i$  points in a certain frequency range. Given the user-defined sets of frequencies  $\Omega_{M_i}^i$ , we present a method to minimize the cost function  $J(F, G_1, G_2, H)$  in (12). However, since the only free parameters in  $J$  are  $\mathcal{G}_1, \mathcal{G}_2$ , we can reparametrize  $J$  (with slight abuse of notation) as follows:

$$J(\mathcal{G}_1, \mathcal{G}_2) := \sum_{i=1}^2 \sum_{k=1}^{M_i} |\Upsilon_i(j\omega_k^i)|^2, \quad (17)$$

where  $\omega_k^i$  is the  $k$ -th element of  $\Omega_{M_i}^i$ , for  $i = 1, 2$ . Besides minimizing  $J$ , we would like to preserve the convergence property that the full-order Lur'e-type model enjoys. Therefore, we would like to choose  $\mathcal{G}_1, \mathcal{G}_2$  such that the conditions of Theorem 2 are satisfied. To this extent, we formulate

$$\tilde{\mathcal{G}}_1, \tilde{\mathcal{G}}_2 = \arg \min_{\mathcal{G}_1, \mathcal{G}_2 \in \mathbb{G}} J(\mathcal{G}_1, \mathcal{G}_2), \quad (18)$$

where  $\mathbb{G}$  is the set of  $\mathcal{G}_1, \mathcal{G}_2$  for which there exists a  $Q = Q^\top \succ 0$  such that the LMIs

$$\begin{aligned} QF(\mathcal{G}_1, \mathcal{G}_2)_{\gamma^*}^- + (F(\mathcal{G}_1, \mathcal{G}_2)_{\gamma^*}^-)^\top Q &< 0, \\ QF(\mathcal{G}_1, \mathcal{G}_2)_{\gamma^*}^+ + (F(\mathcal{G}_1, \mathcal{G}_2)_{\gamma^*}^+)^\top Q &< 0, \end{aligned} \quad (19)$$

are satisfied with  $F(\mathcal{G}_1, \mathcal{G}_2)_{\gamma^*}^\pm := F(\mathcal{G}_1, \mathcal{G}_2) \pm \gamma^* G_2(\mathcal{G}_2)H$  and  $\gamma^*$  as in Assumption 1. We would like to note that for any  $\mathcal{G}_1, \mathcal{G}_2 \in \mathbb{G}$ , condition (16) is satisfied, since satisfaction of (19) guarantees that  $F$  is Hurwitz.

The constraints (19) are derived from the statements in Theorem 2 and are linear in  $Q$  for fixed  $\mathcal{G}_1, \mathcal{G}_2$ , hence (19) are LMIs. Since  $J$  in (18) is nonlinear in  $\mathcal{G}_1, \mathcal{G}_2$ , by gradient-based optimization, a local minimum of  $J$  can be found, which solves *Item 3*) of Problem 6. Publicly available optimization packages, such as SeDuMi [12] for Matlab, can be employed to solve (19) in a gradient-descent fashion. The resulting  $\tilde{\mathcal{G}}_1, \tilde{\mathcal{G}}_2$  render the reduced-order Lur'e-type model exponentially convergent, hence solving *Item 1*) of Problem 6. In order to start a gradient-based search, initial  $\mathcal{G}_1^0, \mathcal{G}_2^0$  are required that satisfy the constraints (19), which is the topic of the next section.

2) *Exponentially Convergent Starting Point*: The following theorem presents an LMI-based method to find a  $\mathcal{G}_1^0, \mathcal{G}_2^0$  such that the conditions of Theorem 2 are satisfied.

*Theorem 9*: Suppose Assumption 1 holds for a certain  $\gamma^*$  and consider the reduced-order Lur'e-type model (8) with parametrization (13). If there exist positive definite matrices  $Q_1 = Q_1^\top \succ 0, Q_2 = Q_2^\top \succ 0$ , matrices  $X_1 \in \mathbb{R}^{\nu_1}, X_2 \in \mathbb{R}^{\nu_2}$ , such that the following two LMIs are satisfied:

$$\begin{bmatrix} M_1 & (M_2^-)^\top \\ M_2^- & M_3^- \end{bmatrix} < 0, \quad \begin{bmatrix} M_1 & (M_2^+)^\top \\ M_2^+ & M_3^+ \end{bmatrix} < 0 \quad (20)$$

with

$$\begin{aligned} M_1 &= Q_1 S_1 - X_1 L_1 + S_1^\top Q_1 - L_1^\top X_1^\top, \\ M_2^\pm &= \pm \gamma^* X_2 H_1, \\ M_3^\pm &= Q_2 S_2 - X_2 L_2 \pm \gamma^* X_2 H_2 \\ &\quad + S_2^\top Q_2 - L_2^\top X_2^\top \pm \gamma^* H_2^\top X_2^\top. \end{aligned}$$

Then, the conditions of Theorem 2 are satisfied and the reduced-order Lur'e-type model (8) is globally exponentially convergent according to Definition 1 with model matrices  $\mathcal{G}_i^0 = Q_i^{-1} X_i, i = 1, 2$ . Furthermore, condition (16) is satisfied.

*Proof*: The proof is omitted for the sake of brevity. ■

The LMIs in Theorem 9 can be solved for  $Q_1, Q_2, X_1, X_2$  by publicly available solvers such as SeDuMi [12]. Although studying feasibility of the LMIs is a topic for future work, in our experience, these LMIs are always feasible and a solution can always be found. Once solved, the initial model matrices  $\mathcal{G}_1^0$  and  $\mathcal{G}_2^0$  can be retrieved from  $Q_1, Q_2, X_1, X_2$  and used to launch a gradient-based search to minimize the constrained optimization problem (18).

#### D. Overview of the reduction method

An overview of the reduction method is presented in Algorithm 1 below.

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#### Algorithm 1 Model Order Reduction Algorithm

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**Input**: Full-order model  $\Sigma$  in (1) and the sets of frequencies  $\Omega_0^i, \Omega_{M_i}^i, i = 1, 2$ .

- 1: Define the signal generators  $(S_i, L_i), i = 1, 2$ , in (5) such that  $\sigma(S_i) \simeq \Omega_0^i$ .
- 2: Compute the matrices  $C\Pi_i, i = 1, 2$ , from (15).
- 3: Define the reduced-order model matrices  $F, G_1, G_2, H$ , as in (13).
- 4: Compute initial  $\mathcal{G}_i^0, i = 1, 2$ , using Theorem 9.
- 5: Using  $\Omega_{M_i}^i, i = 1, 2$ , solve the constrained optimization problem (18), starting at  $\mathcal{G}_1^0, \mathcal{G}_2^0$ .

**Output**: Reduced-order model  $\Sigma_r$  in (8).

---

#### IV. ILLUSTRATIVE EXAMPLE

In this example, we consider a convergent Lur'e-type model (1) with state dimension  $n = 100$  for the LTI part and a deadzone nonlinearity defined as follows:

$$\varphi(y) = \text{sign}(y) \max(0, |y| - 0.05). \quad (21)$$

The nonlinear function  $\varphi$  outputs 0 for  $|y| \leq 0.05$  and has a slope of 1 for  $|y| > 0.05$ . The matrices of the LTI block are generated randomly by Matlab's function `rss.m`, and the  $B_1$  matrix is adapted in magnitude such that the Lur'e-type model is exponentially convergent and satisfies Assumption 1 for  $\gamma^* = 1$ . The Bode magnitude diagram of the full-order LTI model is depicted in the solid blue curve in Figure 2.

The signal generators are defined by the triples  $(S_i, L_i, \tau_i^0), i = 1, 2$ , as follows:

$$\begin{aligned} S_i &= \text{blockdiag}(\Xi_1, \dots, \Xi_{\eta_i}), \\ \Xi_k &= \begin{bmatrix} 0 & \omega_k^i \cdot 2\pi \\ -\omega_k^i \cdot 2\pi & 0 \end{bmatrix} \\ L_i &= \tau_i^0 = [1 \ 0 \ \dots \ 1 \ 0]^\top \in \mathbb{R}^{\nu_i}, \end{aligned} \quad (22)$$

where  $\omega_k^i$  is the  $k$ -th element of  $\Omega_0^i \in \mathbb{R}^{\eta_i}$  and  $\tau_i^0$  is the initial condition of the signal generator. If both  $\Omega_0^i$  do not contain repeated frequencies, then Assumption 2 holds for both signal generators  $i = 1, 2$ . In this specific example, we choose the following set of five frequencies:

$$\begin{aligned} \Omega_0^1 &= [0.01 \ 0.1 \ 0.38 \ 1.32 \ 4.1]^\top, \\ \Omega_0^2 &= [0.01 \ 0.1 \ 0.38 \ 1.27 \ 4.1]^\top. \end{aligned} \quad (23)$$

In each  $\Omega_0^i, i = 1, 2$ , the first frequencies are low frequencies and the latter three correspond to the resonance peaks visible in Figure 2. Taking five frequencies in each  $\Omega_0^i, i = 1, 2$ , i.e.,  $\eta_1 = \eta_2 = 5$ , results in a dimension of  $m = 2(\eta_1 + \eta_2) = 20$  for the state  $\xi$  of the reduced-order Lur'e-type model (8).

First, we solve the LMIs in Theorem 9 to obtain  $\mathcal{G}_1^0, \mathcal{G}_2^0$ , that, together with (14), constitute the initial convergent reduced-order Lur'e-type model  $\Sigma_r^0$  in (8). The Bode magnitude diagram of  $\Sigma_r^0$  is shown in Figure 2 in the dashed red curve. It can be observed that the FRFs of the LTI part of  $\Sigma$  and  $\Sigma_r^0$  match at the frequencies  $\Omega_0^1, \Omega_0^2$  in the left and right subfigures, respectively. However, for other frequencies, there is a significant mismatch.

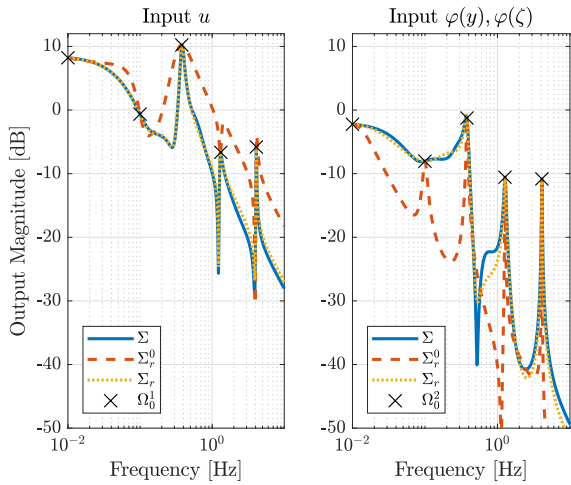


Fig. 2: Bode magnitude diagram of  $\Sigma$ ,  $\Sigma_r^0$  and  $\Sigma_r$ .

Next, we solve the constrained optimization problem (18), starting at the initial  $\mathcal{G}_1^0, \mathcal{G}_2^0$ . Hereto, we define the sets  $\Omega_{M_1}^1 = \Omega_{M_2}^2 = 2\pi \cdot 10^\kappa =: \Omega_M$  with  $\kappa \in \mathbb{R}^M$  linearly spaced with  $M = 100$  elements between -2 and 2, implying that  $\Omega_M$  is logarithmically spaced between 0.01 Hz and 100 Hz. The resulting  $\mathcal{G}_1, \mathcal{G}_2$  define the (final) convergent reduced-order Lur'e-type model  $\Sigma_r$  in (8). The Bode magnitude plot of  $\Sigma_r$  is depicted in the dotted yellow curve in Figure 2. With respect to  $\Sigma_r^0$ , a significant improvement of the fit of  $\Sigma_r$  to  $\Sigma$  can be observed at almost all frequencies.

Finally, we present time-domain simulation results to illustrate the quality of the reduced-order Lur'e-type model in terms of approximating the steady-state response of the full-order Lur'e-type model. For the signal generator in (5) with  $S = S_1, L = L_1, \tau^0 = \tau_1^0$ , the steady-state output  $\bar{y}_t$  of the full-order interconnected model (6) is depicted in Figure 3, where the subscript  $t$  refers to training data. The steady-state response  $\bar{\zeta}_t$  of the reduced-order Lur'e-type model (8) is also depicted in the same figure. It can be observed the steady-state response  $\bar{y}_t$  is approximated accurately by  $\bar{\zeta}_t$ . For validation, Figure 3 also presents a similar result for a validation input signal generated by (22) that now generates  $\Omega_3^0 = [0.02 \ 0.25 \ 2 \ 4]$  Hz. Again, it can be concluded that the steady-state response  $\bar{\zeta}_v$  accurately approximates  $\bar{y}_v$ , where the subscript  $v$  refers to validation data. The mismatch is dominant in the frequencies in  $\Omega_3^0$  due to the mismatch in the FRF at those frequencies, see Figure 2.

## V. CONCLUSIONS

This paper presents a model order reduction technique for exponentially convergent nonlinear models of Lur'e-type form. One of the benefits of our approach is that the Lur'e-type structure of the full-order model is preserved in the reduced-order model. Furthermore, our approach has the interpretation in terms of frequency response functions (FRFs). Namely, the FRF of the LTI block of the reduced-order Lur'e-type model equals the FRF of the LTI block of the full-order Lur'e-type model at user-defined frequencies. Furthermore, the mismatch between the FRF of the LTI block

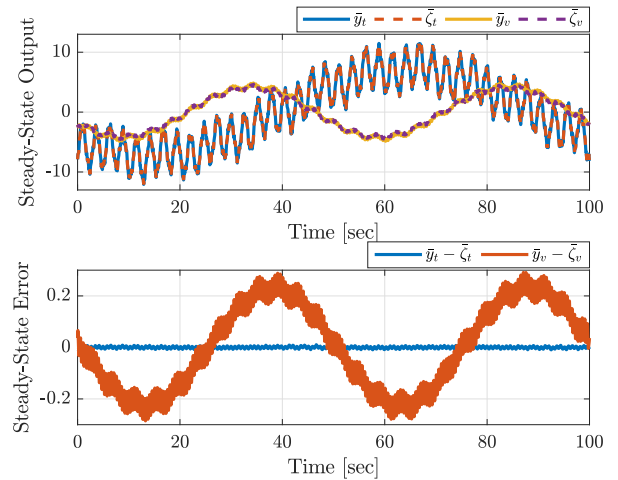


Fig. 3: Top: Steady-state output of  $\Sigma$  and  $\Sigma_r$ . Bottom: Mismatch between the steady-state output of  $\Sigma$  and  $\Sigma_r$ . Subscripts  $t, v$  refer to training and validation data, respectively. The mean of the responses is corrected for visualization.

of the reduced-order model and the full-order model is minimized at another user-defined set of user-defined frequencies. Moreover, our approach guarantees that the reduced-order model also enjoys the exponentially convergent property of the full-order Lur'e-type model. In a numerical example, we illustrated the effectiveness of our approach.

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