Constructing Imprecise Probabilities Using Arguments as Evidence

Paul-Amaury MATT and Francesca TONI
Imperial College London, Department of Computing

Abstract. This paper addresses the problem of constructing subjective imprecise probabilities using qualitative and conflicting pieces of information (arguments) as evidence. We propose formulae for the calculus of imprecise probabilities and show that the probabilities obtained reflect the indeterminacy of the subject, faithfully quantify the support offered by the arguments and constitute previsions that are mathematically coherent in the sense of [Walley, 1991].

Keywords. Arguments, evidence, indeterminacy, imprecise probability, coherence.

1. INTRODUCTION

Coherent previsions (imprecise probabilities) have recently been proposed as a general way of representing subjective uncertainty [Walley, 1991; 1996; 2000]. Subjective probabilities serve to measure the confidence of a *subject* in the truth of a particular proposition [Savage, 1972]. They allow to model the complete spectrum of epistemic states about a given situation or domain, from total ignorance to absolute certainty and therefore measure not only the uncertainty of a subject, but also his indeterminacy. Imprecise probabilities have a fundamental role to play in Artificial Intelligence as they may be used for measuring the strength of beliefs [Shafer, 1976], assess risk [Pelessoni and Vicig, 2001], for statistical inference and robust decision making or optimisation [Walley, 1991; 1996; Jaffray, 1999; Bertsimas and Brown, 2005; Natarajan *et al.*, 2005; Aughenbaugh and Paredis, 2006]. To the best of our knowledge, this paper introduces the first method for constructing imprecise probabilities using qualitative pieces of information. We will refer to such pieces of information as *arguments*.

Arguments may be seen as first class objects that have the form of proofs but whose conclusions are not certain [Krause *et al.*, 1995; Fox, 2003]. In practical domains of human knowledge such as e.g. Law, Medicine, Biology or Psychology, evidence takes most often the form of arguments. In general, arguments may be borrowed from empirical theories, elicited from experts or trusted people, extracted from databases and built from information provided by physical devices such as instruments, captors, sensors, etc. To reduce indeterminacy, one usually seeks to collect arguments from more than one source. Unfortunately, the people or systems consulted may provide various arguments with contradictory conclusions [Finkelstein *et al.*, 1994; Amgoud and Cayrol, 2002] and one is then confronted with the difficulty of combining inconsistent pieces of evidence.

Let us for instance consider the domain of auctions. There is no established theory that allows auctioneers to predetermine their chances at an auction of making a sale nor of selling their item at a high price. However, auction specialists have experience and valuable knowledge that you can exploit to make a rough assessment of your chance of success in an auction. According to them, the most important things for being successful in an auction are the following three: the auctioned item should be popular, its market availability should be low and it should also be in good material condition. The item popularity is essential for making a sale. In order to sell the item at a high price, there must be some competition between the bidders. Either of the two other criteria (a low market availability and a good material condition) are usually sufficient to foster competition. Your chances of success thus boil down to the extent to which each one of the three criteria is met. To assess this, suppose you decide to consult two friends experienced in on-line auctions. After discussion, suppose you have elicited the four following arguments

- Argument A_1: "This type of item is not available on the market because it is not manufactured anymore."
- Argument A_2: "I know many people who collect items of this kind, and I therefore believe that the item is popular."
- Argument A_3: "This item hasn't lost the functionality that makes it so popular; it can be said to be in good material condition and can be expected to be still popular nowadays."
- Argument A_4: "The problem with the item is that it has several scratches and anyway, it is not really considered a rare item insofar as counterfeits can be found easily."

According to A_1, the item should have a low availability on the market. A_2 strongly suggests that the item is popular. A_3 corroborates this and besides shows that the item can be considered as being in good condition. All this should make us quite confident in the success of the auction. However, A_4 casts major doubt on the auction's success. It basically contradicts both claims that the item is in good condition and that its market availability is low. Given these conflicting arguments as evidence, how would you now assess your chances of making a sale and of selling the item at a good price ?

This paper proposes formulae for the calculus of imprecise probabilities in situations that are especially of this kind, viz. situations in which no or little probabilistic information is available, but in which a few qualitative and unfortunately conflicting arguments can be obtained from common sense or expert knowledge. Basically, the paper offers mathematical guidance on how to come up with probability values in the light of qualitative information. We will show that the probabilities obtained with the proposed formulae adequately reflect the subject's overall indeterminacy, faithfully quantify the support offered by the arguments and also constitute previsions which are mathematically coherent in the sense of [Walley, 1991].

The remainder of the paper is organised as follows. Section 2 introduces all the necessary background concerning the theory of imprecise probabilities and coherent previsions. In section 3, we propose two new formulae for calculating lower and upper probabilities from a set of weighted arguments. For convenience, we will refer to such probabilities as *argumentative probabilities* throughout the paper. Section 4 presents a detailed mathematical study of the properties satisfied by argumentative probabilities. In section 5, we provide a short survey of related work and then conclude in section 6.

2. IMPRECISE PROBABILITIES

Probability and possibility are the two classical theories of uncertainty [Klir, 2006]. It has been established in [Walley, 2000] that all possibility measures are coherent upper probabilities, so that in fact, the theory of imprecise probabilities constitutes a possible generalisation of the two classical theories of uncertainty. In order to define imprecise probabilities, one first needs to introduce mathematically several notions, which are those of universe, event, gamble, lower and upper previsions and coherence. Coherence is the most important and difficult one, so it will be explained separately in the next subsection. In the last subsection, we will finally show how to use imprecise probabilities for statistical inference and decision making, although these concerns fall outside the scope of this paper.

Uncertainty is always relative to a system or object of interest for a subject. The term uncertain means that the subject does not know in exactly which state this system or object will be. For simplicity of presentation, let us assume that the there exists a finite number of possible states of the system or object and let us denote these $w_1, ..., w_n$. These possible states are commonly referred to as *scenarios* and are assumed to be mutually exclusive and exhaustive. The set of scenarios $\Omega = \{w_1, ..., w_n\}$ is called the *universe*. Only one of the scenarios of Ω will occur, but the subject is not certain which one it will be. We call this scenario the *true* scenario. Subsets $E \subseteq \Omega$ are referred to as *events* and we say that E occurs when the true scenario belongs to E. The powerset 2^Ω of Ω is the set of all events. We are basically interested in the probability of events, i.e. in the likelihood of their occurrence. In the auction problem for instance, the events of interest correspond informally to making a sale and selling the item for a good price.

A *gamble* is any function $X : \Omega \to \mathbb{R}$. It mathematically represents a ticket for a lottery where the true scenario w is revealed and a price of $X(w)$ (in units of money or utility) is won. The true scenario is not known in advance so the outcome of the lottery is uncertain, hence the name gamble. Each event E can be seen as gamble with a price equal to 1 if E occurs and 0 if E does not occur. Formally, it is easy to understand that the gamble corresponding to an event E is the *indicator function* $I_E : \Omega \to \{0, 1\}$ of the set E, defined $\forall w \in \Omega$ as $I_E(w) = 1$ if $w \in E$ and $I_E(w) = 0$ otherwise. By convenience, we use de Finetti's notation and systematically write E instead of I_E. For any gambles X and Y, we write $X \leq Y$ if and only if $\forall w \in \Omega$ we have $X(w) \leq Y(w)$, define $\sup X = \max\{X(w) \mid w \in \Omega\}$ and the operations of addition $+$, substraction $-$ and multiplication $*$ of gambles in exactly the same way as done for real-valued functions in general, e.g. $(X + Y)(w) = X(w) + Y(w)$.

Given a nonempty set $\mathcal{X}$ of gambles of interest[1], a *lower prevision* is a function $\underline{P} : \mathcal{X} \to \mathbb{R}$ and an *upper prevision* a function $\overline{P} : \mathcal{X} \to \mathbb{R}$. For any gamble $X \in \mathcal{X}$, the value $\underline{P}(X)$ is interpreted as the subject's maximum buying price for the lottery ticket X and $\overline{P}(X)$ as its minimum selling price. If the only gambles of interest $\mathcal{X}$ are events, then lower and upper previsions are called *lower* and *upper probabilities*. The lower probability $\underline{P}(E)$ is interpreted as a maximum betting rate for betting on event E and the upper probability $\overline{P}(E)$ as a minimum rate for betting against it. Intuitively, the more arguments we have that support an event, the higher the price a subject (which trusts these arguments) would be willing to bet in favour of the event. In the auction problem

[1] In this paper, $\mathcal{X}$ is the set of all events 2^Ω.

for instance, the first three arguments A_1, A_2, A_3 indicate that you are likely to make a sale (this event is denoted E_{sale}) and thus should all contribute to increasing your betting rate $\underline{P}(E_{\text{sale}})$. But the last argument A_4 is in conflict with the first three one and shall thus have the opposite effect on this betting rate.

Intuitively, $\underline{P}(E)$ and $\overline{P}(E)$ represent lower and upper bounds for our fair price for E. Thus, the less we know about the likelihood of E, the more indeterminate we are regarding its fair price and the larger the gap between $\underline{P}(E)$ and $\overline{P}(E)$. In other terms, the imprecision of probabilities reveals how indeterminate the subject is and should intuitively be directly related to the amount of information or arguments available. In case of complete ignorance, one should normally have $\underline{P}(E) = 0$ and $\overline{P}(E) = 1$. At the other extreme, when a large sample of statistical trials have been undertaken on E or more generally, when the subject has a huge amount of information concerning the occurrence of event E, his indeterminacy shall be very low and the betting rates / probabilities approximately equal $\underline{P}(E) \approx \overline{P}(E)$. The common value of these two probabilities may then be in practise taken as equal to the past frequency of occurrence of E (if the subject is a Bayesian which has chosen to adopt the frequentist approach). Thus, Bayesian probabilities simply correspond to maximally precise probabilities and frequency based probabilities can in turn be seen as a special type of precise probabilities.

2.1. Rationality principles

Walley proposes two fundamental principles of rationality to characterise mathematically the "coherence" of previsions. The first principle states that the subject should avoid sure losses, i.e. never be sure to lose money by engaging in any number of lotteries. We can formalise this principle using the notion of marginal gamble. For any gamble X, we define the *marginal gamble* $G(X) = X - \underline{P}(X)$. $G(X)$ corresponds to a transaction in which we pay $\underline{P}(X)$ to obtain X. Since $\underline{P}(X)$ is our maximal price for X, this transaction is acceptable. Formally, we may say that $\underline{P}$ *avoids sure loss* if and only if $\sup\sum_{j=1}^{n} G(X_j) \geq 0$ whenever the gambles $X_1, ..., X_n$ are in $\mathcal{X}$ and $n \geq 1$. The most basic previsions that avoid sure loss are vacuous previsions, defined for all $X \in \mathcal{X}$ as $\underline{P}(X) = \inf X$ and $\overline{P}(X) = \sup X$ and serve to model the state of complete ignorance about the gambles in $\mathcal{X}$.

The second principle states in essence that the subject's prices should be consistent as a whole and not contradict one another. If for some $\alpha \in \mathbb{R}$, the gamble $X - \alpha$ is greater or equal to a positive combination of acceptable/marginal gambles, then $X - \alpha$ should also be acceptable, that is to say $\underline{P}(X) \geq \alpha$. This should hold for any such α, which means that, to make sense, $\underline{P}(X)$ should be greater or equal to the quantity

$$\underline{E}(X) = \sup\{\alpha \in \mathbb{R} \mid \exists n \geq 1, X_j \in \mathcal{X}, \lambda_j \geq 0 : X - \alpha \geq \sum_{j=1}^{n} \lambda_j G(X_j)\}$$

The function $\underline{E}$ is called the *natural extension*[2] of $\underline{P}$. The natural extension $\underline{E}(X)$ of the gamble X depends on the previsions of the other gambles in $\mathcal{X}$. Since $\underline{E}$ remarkably dominates $\underline{P}$ on $\mathcal{X}$ (take $n = 1$, $X_1 = X$ and $\alpha = \underline{P}(X)$ in the formula given above), we may conclude that the lower prevision of a fully rational gambler should not only avoid

[2]The actual computation of natural extension generally requires linear programming [Dantzig *et al.*, 1955].

sure loss, but also always coincide with its natural extension. We therefore say that a lower prevision is *coherent* if and only if it avoids sure loss and coincides with its natural extension on $\mathcal{X}$. When $\underline{P}$ is a coherent lower prevision, the upper prevision $\overline{P}$ defined for all gambles $X \in \mathcal{X}$ as $\overline{P}(X) = -\underline{E}(-X)$ is called its *conjugate* and an upper previsions is said to be coherent when it is the conjugate of some coherent lower prevision.

Imprecise probabilities are the name given to coherent lower and upper probabilities. Vacuous probabilities which are defined as $\underline{P}(E) = 0$ ($\forall E \subset \Omega$), $\underline{P}(\Omega) = 1$, $\overline{P}(E) = 1$ ($\forall E \supset \emptyset$) and $\overline{P}(\emptyset) = 0$ are coherent. Thus a subject that is totally ignorant and whose previsions or probabilities are vacuous is still deemed rational in the Walleysian sense. Other important examples of probabilities that are known to be coherent [Walley, 2000] are naturally all the Bayesian probabilities, but also possibility and necessity measures [Dubois and Prade, 1988; 1995], the Dempster-Shafer belief and plausibility functions [Shafer, 1976] and all Choquet capacities of order 2 [Klir, 2006].

2.2. *Statistical inference and decision making*

This short subsection discusses the role of natural extension in the theory of coherent previsions. This discussion is not essential to the presentation of the results of this paper and may be skipped by the reader. It is however presented for the sake of completeness and to give an idea of how to make statistical inference and decisions when using imprecise probabilities or coherent previsions.

Natural extension allows to extend coherent lower and upper previsions from their original domain of definition $\mathcal{X}$ to the entire set of gambles. When $\underline{P}$ is a coherent lower prevision on $\mathcal{X}$ and $\overline{P}$ is the conjugate upper prevision, the natural extension $\underline{E}(X)$ exists for any gamble X and $\underline{E}$ defines a coherent lower prevision on the set of all real-valued gambles. The conjugate of $\underline{E}$ is defined for every gamble X as $-\underline{E}(-X)$. Natural extension can also be used for comparing the likelihood of events, for computing conditional probabilities and comparing decisions. For two events A and B, we say that A is more likely than B if $\underline{E}(A - B) \geq 0$, A is at least twice as likely as B if $\underline{E}(A - 2B) \geq 0$, etc. The conditional lower probability $\underline{P}(A|B)$ of A given B is the solution $z \in \mathbb{R}$ of the equation $\underline{E}(B * A - z) = 0$ called generalised Bayes rule. When all probabilities are precise, the generalised Bayes rule collapses to the standard Bayes rules of conditioning.

Finally, let us assume that D is a set of possible decisions and $U : D \times \Omega \to \mathbb{R}$ a utility function (i.e. $U(d, w)$ measures the utility of the consequence of d under scenario $w \in \Omega$ for the subject). The gamble X_d defined $\forall w \in \Omega$ as $X_d(w) = U(d, w)$ models the decision d. We say that d is preferred to d' whenever $\underline{E}(X_d - X_{d'}) \geq 0$. This means that the transaction where the subject gets X_d in exchange of $X_{d'}$ is judged acceptable. One may also say that d is twice as much preferred as d' whenever $\underline{E}(X_d - 2X_{d'})$, etc. Intuitively, $\underline{E}(X_d - X_{d'})$ and $-\overline{E}(X_d - X_{d'})$ provide a robust and imprecise estimation of the expected value of d in comparison to d'. When $P = \underline{P} = \overline{P}$ is a Bayesian probability, the expression of $\underline{E}(X_d)$ collapses to the classical criterion of expected utility, i.e. $\underline{E}(X_d) = \sum_{i=1}^{n} P(w_i)U(d, w_i)$. Thus, the function of natural extension generalises the one of expected utility to the case of imprecise probabilities.

3. FORMULAE FOR ARGUMENTATIVE PROBABILITIES

Let us consider a subject (typically a real person or an artificially intelligent agent) and assume given a universe Ω, a set of arguments Arg and a mapping $f : Arg \to 2^{\Omega}$ such that for all $\forall A \in Arg$, $f(A) = X_A$ represents the event (set of scenarios) supported by argument A. We may assume that for all $A \in Arg$ it holds that $\emptyset \subset X_A \subset \Omega$. Indeed, an argument A for which $X_A = \emptyset$ directly contradicts the fact that one of the scenarios in Ω is the true one, and an argument such that $X_A = \Omega$ brings no new information (we already know that the true scenario belongs to Ω by exhaustiveness of the universe).

The auction universe describes your uncertainty concerning the item to sell. Each one of the three factors used to describe its status (popularity, low availability and good physical condition) can be either true of false and may be modelled by a Boolean variable. Let us then represent by the notation s_{ijk} the scenario in which the truth value of the first (respectively second and third) factor is i (respectively j and k) where $i = 0$ (respectively $j = 0$ and $k = 0$) means false and $i = 1$ (respectively $j = 1$ and $k = 1$) means true. For example s_{100} is the scenario in which the item is popular but does not have a low availability and is not in good condition. Remark that s_{000} is the worst case scenario for an auctioneer whilst s_{111} is the best case scenario. The universe used to model your uncertainty in the auction problem is thus $\Omega = \{s_{000}, s_{001}, s_{010}, ..., s_{111}\}$ and contains eight scenarios in total.

We are interested in two events, namely, making a sale and selling the item for a good price. In order to make a sale, the item should be popular and in order to sell the item for a good price, the item should on top of that be either in good condition or have a low availability on the market. Using the previously introduced notations for the scenarios, the events of interest can be formalised as $E_{\text{sale}} = \{s_{100}, s_{101}, s_{110}, s_{111}\}$ and $E_{\text{good price}} = \{s_{101}, s_{110}, s_{111}\}$. Let us now examine the events supported by each argument. A_1 supports the event containing all the scenarios of the form s_{i1k} where i and k take the value 0 or 1, which implies that $X_{A_1} = \{s_{010}, s_{011}, s_{110}, s_{111}\}$. This event contains 4 scenarios. Similarly, A_2 supports the events containing all the scenarios of the form s_{1jk} and thus corresponds to $X_{A_2} = \{s_{100}, s_{101}, s_{110}, s_{111}\}$. By proceeding in the same fashion for the remaining two arguments, we would obtain $X_{A_3} = \{s_{101}, s_{111}\}$ and $X_{A_4} = \{s_{000}, s_{010}\}$.

Arguments provide information and this information must be quantified by the subject. Indeed, the total amount of information collected should be used to determine his overall indeterminacy. Thus, for every $A \in Arg$, let $w(A) \geq 0$ denote the amount of information of A from his/her viewpoint. Intuitively, $w(A)$ may be thought of as the *strength* of argument A. If all arguments are of equal importance to the subject, then he may choose to assign a strength of 1 to all arguments. If the subject however wishes to ignore or minimise the importance of certain arguments, then he may assign them a strength of zero or a small value close to zero. In general, the subject is free to specify and assess the strength of arguments in whichever way that suits him/her, as long as $w(A)$ can be interpreted as the amount of information of argument A from his/her viewpoint. Then, the value of the sum $W = \sum_{A \in Arg} w(A)$ represents the total amount of information available to the subject. It is this very quantity which will essentially determine the subject's overall indeterminacy and the precision with which he/she will eventually assesses the probability of events. For the auction, you may for instance choose $w(A_i) = 1$ $\forall i \in \{1, 2, 3\}$ and assign to the last argument the strength $w(A_4) = 2$ if for example the argument has been repeated by both of your friends.

Intuitively, when W is big the indeterminacy of the subject should be small. But the term big does not mean anything without a comparison value. Let us then introduce a constant $K > 0$ called *fair amount of information* to make comparisons on the subjective information scale. We shall see later that the adjective fair means that K is the amount of information required to reduce the indeterminacy of an initially totally ignorant subject by one half, that is to say, to reduce the imprecision of the probability of events from 1 to $1/2$. To make good probability estimates and decisions, one would need to experiment on the (ideally long-term) performance of the decisions made by the subject depending on his value of K. Sceptical and risk-averse subjects shall use high values of K and credulous and risk-neutral subjects shall on the opposite use small value for the constant K. The optimal value of K obviously completely depends on the type of situation confronted by the subject or domain of application. In the auction problem however we choose (completely arbitrarily) to set K as equal to 1.

We may now explain how we intend to construct imprecise probabilities from a set of scenarios Ω, a set of arguments Arg, a measure of their strength $w : Arg \to \mathbb{R}^+$, a mapping from arguments A to the events $f(A) = X_A$ they support and a value for the fair amount of information K. The lower and upper argumentative probabilities are given by the following expressions.

Definition 1 (lower argumentative probability) $\underline{P}(\Omega) = 1$ *and* $\forall E \subset \Omega$ *:*

$$\underline{P}(E) = \frac{1}{W + K} \sum_{A \in Arg} w(A). \frac{|X_A \cap E|}{|X_A|}$$

Definition 2 (upper argumentative probability) $\overline{P}(\emptyset) = 0$ *and* $\forall E \neq \emptyset$ *:*

$$\overline{P}(E) = \frac{1}{W + K} (K + \sum_{A \in Arg} w(A). \frac{|X_A \cap E|}{|X_A|})$$

where $|S|$ denotes the cardinality of $S \subseteq \Omega$. Figure 1 below shows how each one of the four arguments given by the experts successively impacts on the subject's estimation of the probabilities. As we can observe, argumentative probabilities are initially vacuous be-

Arguments considered	E_{sale}	$E_{\text{good price}}$
{} (total ignorance)	[0.000, 1.000]	[0.000, 1.000]
$\{A_1\}$	$[0.250, 0.750]$	$[0.250, 0.750]$
$\{A_1, A_2\}$	$[0.500, 0.833]$	$[0.417, 0.750]$
$\{A_1, A_2, A_3\}$	$[0.625, 0.875]$	$[0.562, 0.812]$
$\{A_1, A_2, A_3, A_4\}$	$[0.417, 0.583]$	$[0.375, 0.542]$

Figure 1. Lower and upper probabilities of E_{sale} and $E_{\text{good price}}$ depending on the evidence at hand.

cause the subject is completely ignorant and then progressively become more precise as new arguments are obtained from the sources. The more arguments are given in support of (respectively against) an event, the more its lower probability increases (respectively decreases).

4. PROPERTIES OF ARGUMENTATIVE PROBABILITIES

In this section, we study the substantive and normative goodness of argumentative probabilities [Winkler and Murphy, 1968]. The expression substantive goodness is used to express the fact that the probabilities estimated really reflect the substance of the problem and the expression normative goodness instead relates to the mathematical correctness of these estimations. Both types of goodness are obviously quite crucial in the elicitation/assessment and application of probabilities.

4.1. Substantive goodness

We shall first make sure that the imprecision of argumentative probabilities reflect the subject's indeterminacy. As we will show, the difference between the upper and lower probability of an event varies as K/W. This is a desirable substantive property, as more information should reduce indeterminacy, but also, the speed at which indeterminacy is reduced should decrease as the fair amount of information K for the subject gets larger. Formally, we prove that

Property 1 (indeterminacy varies as K/W**)** *For the extreme cases in which* $E = \emptyset$ *or* $E = \Omega$*, the subject's indeterminacy is null:* $\overline{P}(E) - \underline{P}(E) = 0$. *Otherwise, for any event* $\emptyset \subset E \subset \Omega$*, the indeterminacy* $\overline{P}(E) - \underline{P}(E)$ *is strictly positive, inferior or equal to* K/W *and equivalent to* K/W *as* W *diverges towards infinity.*

Proof 1 $\underline{P}(\emptyset) = \overline{P}(\emptyset) = 0$ *and* $\underline{P}(\Omega) = \overline{P}(\Omega) = 1$ *(by definition). For all* E *such that* $\emptyset \subset E \subset \Omega$*, we have* $\overline{P}(E) - \underline{P}(E) = \frac{K}{W+K} > 0$ *and since* $K \geq 0$, $\frac{K}{W+K} \leq \frac{K}{W}$. *Finally, we have* $\frac{K}{W+K} \sim_{W\to\infty} \frac{K}{W}$.

Remarkably, the imprecision is reduced to $1/n$ when $W = (n-1)K$. Thus, if the subject accumulates information and W increases according to the sequence $W = 0, K, 2K, 3K, ..., (n-1)K$ his/her indeterminacy decreases as the sequence 1, 1/2, 1/3, 1/4,..., $1/n$. As mentioned earlier, K is fair in that it represents the amount of information required to reduce the subject's initial indeterminacy from 1 to 1/2. Also note that the subject can force argumentative probabilities to become precise (or Bayesian) by making K converge to zero. It can be proved that when doing so, the Bayesian probability obtained is the same [3] as the one that one would obtain by application of Smets' pignistic transform [Smets, 1991].

The second substantive property of argumentative probabilities is that these are proportionate to the (even partial) support offered by arguments. The exact meaning of the expression "support" is provided in body of the proof of the next property. Essentially, the support of an argument A in favour of an event E corresponds to the degree of relevance of the argument multiplied by the strength of the argument.

Property 2 (evidential support) $\underline{P}(E)$ *and* $\overline{P}(E)$ *vary as the fraction of the evidence that supports* E.

[3] $\forall w \in \Omega : P(\{w\}) = \sum_{A \in Arg, w \in X_A} \frac{1}{|X_A|} * m(X_A)$, where $m(X_A) = \frac{w(A)}{W}$.

Proof 2 *Let us introduce the relevance function* $r : 2^\Omega \times 2^\Omega \to [0,1]$ *defined for all pairs of events* (X_A, E) *as*

$$r(X_A, E) = \frac{|X_A \cap E|}{|X_A|}$$

The value $r(X_A, E)$ *is always positive and takes a maximal value of one when the supported set of scenarios* X_A *is equal to the event* E*. The ratio* $r(X_A, E)$ *is interpreted as the degree of relevance of the argument* A *with respect to the event* E*. A possible Bayesian justification of this interpretation is the following. By using Johann Bernoulli's insufficient-reason principle, in the only light of argument* A*, all scenarios* $w \in X_A$ *are equally likely to occur and this probability* $p_A(w)$ *shall thus be equal to* $1/|X_A|$*. Thus, the probability for* E *to occur according to argument* A *only is equal to* $p_A(E) = |X_A \cap E|/|X_A| = r(X_A, E)$*.*

Let us now also formalise the notion of support. Let the function $s : 2^\Omega \to \mathbb{R}$ *be defined for every event* E *as the weighted sum*

$$s(E) = \sum_{A \in Arg} w(A).r(X_A, E)$$

$s(E)$ *shall be interpreted as the total support that the corpus of arguments* Arg *provides in favour of the occurrence of event* E*. The substantive property of evidential support becomes now clear. Since* $s(E)$ *is the weighted sum of the relevance of all* X_A *with respect to* E*,* $s(E)$ *is a measure of the subjective amount of evidence supporting* E*. Remarkably* $s(\Omega) = W$ *and* $s(\emptyset) = 0$*. When* $E = \Omega$*, we have* $\underline{P}(E) = \overline{P}(E) = 1 = \frac{s(E)}{W}$*. When* $E \subset \Omega$ *we have* $\underline{P}(E) = \frac{s(E)}{W+K} \leq \frac{s(E)}{W}$ *and* $\frac{s(E)}{W+K} \sim_{W\to\infty} \frac{s(E)}{W}$*. Similarly,* $\overline{P}(E) = \frac{s(E)+K}{s(\Omega)+K} \geq \frac{s(E)}{s(\Omega)}$ *since* $s(\Omega) \geq s(E)$ *and* $\frac{s(E)+K}{s(\Omega)+K} \sim_{W\to\infty} \frac{s(E)}{W}$*.*

The two properties exposed in this subsection justify why, or at least clarify the extent in which argumentative probabilities can be said to have substantive goodness.

4.2. Normative goodness

It is without surprise the notion of coherence that we choose to use in order to guarantee the normative goodness of argumentative probabilities, as this rich notion encapsulates by construction the principles of rationality that are essential for a subject betting on beliefs or decisions. Coherence is not a simple notion and to establish this property, we will need to proceed in three steps. We shall first prove that lower argumentative probabilities avoid sure loss, then show that they coincide with their natural extension (as explained in section 2.1) and finally prove that upper argumentative probabilities are conjugate of the lower ones.

Theorem 1 ($\underline{P}$ **avoids sure loss)** *Lower argumentative probabilities avoid sure loss.*

Proof 3 *Let* $n \geq 1$ *and* $X_1, ..., X_n \in 2^\Omega$*. Since* $G(\Omega) = \Omega - \underline{P}(\Omega) = 1 - 1 = 0$*, we may assume without loss of generality that* $\forall i \in \{1, ..., n\}$*:* $X_i \subset \Omega$*. By definition of a lower argumentative probability*

$$\sum_{j=1}^{n} G(X_j) = \sum_{j=1}^{n}(X_j - \frac{1}{W+K}\sum_{A\in Arg} w(A).\frac{|X_A \cap X_j|}{|X_A|})$$

Let us then consider the functions $g : \Omega \to \mathbb{R}$ and $f : \Omega \to \mathbb{R}$ defined $\forall w \in \Omega$ as

$$g(w) = \sum_{j=1}^{n}(X_j(w) - \frac{1}{W+K}\sum_{A\in Arg} w(A).\frac{|X_A \cap X_j|}{|X_A|})$$

$$f(w) = \sum_{j=1}^{n}(X_j(w) - \frac{1}{W}\sum_{A\in Arg} w(A).\frac{|X_A \cap X_j|}{|X_A|})$$

To prove that $\underline{P}$ avoids sure loss, we must prove that $\sup_{w\in\Omega} g(w) \geq 0$. Remark that $g \geq f$, so it is sufficient to prove that $\sup_{w\in\Omega} f(w) \geq 0$. This obviously holds if $w(A) = 0$ for all $A \in Arg$. Otherwise there exists some $A \in Arg$ such that $w(A) > 0$. Then, let us denote $x_{A,j} = \frac{|X_A \cap X_j|}{|X_A|}$, $\lambda_A = \frac{w(A)}{W}$ and compute the value of

$$\sum_{w\in X_A} \frac{1}{|X_A|}.f(w) = \sum_{w\in X_A} \frac{1}{|X_A|}\sum_{j=1}^{n} X_j(w) - \sum_{w\in X_A}\frac{1}{|X_A|}\sum_{j=1}^{n}\frac{1}{W}\sum_{A'\in Arg} w(A')x_{A',j} =$$

$$\sum_{j=1}^{n}\frac{1}{|X_A|}.|X_A \cap X_j| - \sum_{j=1}^{n}\frac{1}{W}\sum_{A'\in Arg} w(A')x_{A',j} = \sum_{j=1}^{n}(x_{A,j} - \sum_{A\in Arg}\lambda_A x_{A,j})$$

By summing over $A \in Arg$ with weights λ_A we obtain

$$\sum_{A\in Arg}\lambda_A \sum_{w\in X_A}\frac{1}{|X_A|}.f(w) = \sum_{A\in Arg}\sum_{j=1}^{n}\lambda_A x_{A,j} - \sum_{A\in Arg}\lambda_A \sum_{j=1}^{n}\sum_{A'\in Arg}\lambda_{A'}x_{A',j} =$$

$$\sum_{A\in Arg}\sum_{j=1}^{n}\lambda_A x_{A,j} - \sum_{A'\in Arg}\sum_{j=1}^{n}\lambda_{A'}x_{A',j} = 0$$

We thus have found a non-trivial positive combination of some values of f that is null. This implies that all the values of f cannot be strictly negative, or in other words, that $\sup f$ *is indeed positive.*

Theorem 2 (coherence of $\underline{P}$) *Every lower argumentative probability is coherent.*

Proof 4 *Since lower argumentative probabilities avoid sure loss, they admit a natural extension $\underline{E}$. We shall now prove to establish the result of the theorem, that every lower argumentative probability coincides with its natural extension. $\underline{E}(X)$ is defined as the solution of the linear optimisation problem of maximising the variable α subject to the positivity constraints $\lambda_j \geq 0$ and $\forall w \in \Omega$:*

$$X(w) - \alpha \geq \sum_{X_j\subseteq\Omega}\lambda_j.(X_j(w) - \frac{s(X_j)}{W+K})$$

where s is the support function introduced in proof 2. Gambles X are now seen as vectors of dimension $n = |\Omega|$ with components $X_i = X(w_i)$, $\langle X, Y\rangle$ denotes the scalar product $X^T.Y$ and $\|X\|^2 = \langle X, X\rangle$ the squared Euclidean norm of X. With these new notations, the support $s(X_j)$ can be rewritten under the form

$$s(X_j) = \sum_{A\in Arg} w(A).\frac{\langle X_A, X_j\rangle}{\|X_A\|^2}$$

Let us then introduce the linear form defined on the vector space $\mathbb{R}^n$ as

$$\sigma(\bullet) = \frac{1}{W+K}\sum_{A\in Arg} w(A).\frac{\langle X_A, \bullet\rangle}{\|X_A\|^2}$$

By linearity, the above inequality constraint is equivalent to

$$X - \alpha \geq \sum_{X_j\in\Omega}\lambda_j X_j - \sigma(\sum_{X_j\subseteq\Omega}\lambda_j X_j)$$

The problem is to maximise α such that for some $Y \in \mathbb{R}^n$ with positive components $X - \alpha \geq Y - \sigma(Y)$. So, we need to compute

$$\alpha^* = \max_{y_i\geq 0}\min_{i\in\{1,\ldots,n\}} x_i - (y_i - \sum_{j=1}^{n}\sigma_j y_j) = \max_{y_i\geq 0}\sum_{j=1}^{n}\sigma_j y_j + \min_{i\in\{1,\ldots,n\}}(x_i - y_i)$$

Clearly, the value of α^* is unchanged if we add the further contraint that the values of $x_i - y_i$ should all be equal to the same value m. The value of m would then be such that $m = x_i - y_i$ for all i and since y_i are positive, m must satisfy $m \leq \min_{i\in\{1,\ldots,n\}} x_i$. Now, we also have $y_i = x_i - m$, so

$$\begin{aligned}\alpha^* &= \max_{m\leq\min_{i\in\{1,\ldots,n\}} x_i}\sum_{j=1}^{n}\sigma_j(x_j - m) + m = \sum_{j=1}^{n}\sigma_j x_j + \max_{m\leq\min_{i\in\{1,\ldots,n\}} x_i} m(1 - \sum_{j=1}^{n}\sigma_j)\\ &= \sum_{j=1}^{n}\sigma_j x_j + (\min_{i\in\{1,\ldots,n\}} x_i).(1 - \sum_{j=1}^{n}\sigma_j)\end{aligned}$$

$\forall X \subseteq \Omega$, $\min_{i\in\{1,\ldots,n\}} x_i = 1$ if $X = \Omega$ and 0 otherwise. Therefore, $\alpha^* = 1 = \underline{P}(\Omega)$ if $X = \Omega$, and otherwise

$$\begin{aligned}\alpha^* &= \sum_{j=1}^{n}\sigma_j x_j = \sum_{w\in X}\sigma(\{w\}) = \sum_{w\in X}\frac{1}{W+K}\sum_{A\in Arg} w(A).\frac{|X_A\cap\{w\}|}{|X_A|}\\ &= \frac{1}{W+K}\sum_{A\in Arg} w(A).\sum_{w\in X}\frac{|X_A\cap\{w\}|}{|X_A|} = \frac{1}{W+K}\sum_{A\in Arg} w(A).\frac{|X_A\cap X|}{|X_A|}\\ &= \underline{P}(X)\end{aligned}$$

Since by definition of natural extension $\alpha^* = \underline{E}(X)$, it holds that $\underline{P}(X) = \underline{E}(X)$ for any arbitrarily chosen X.

Theorem 3 (conjugacy of $\overline{P}$) *Upper argumentative probabilities are the conjugate of the lower ones.*

Proof 5 *We must prove that $\forall X \subseteq \Omega$, $\overline{P}(X) = -\underline{E}(-X)$. For any event X, we can rewrite $-\underline{E}(-X)$ as $-\underline{E}((\Omega - X) - \Omega)$. By coherence of $\underline{E}$ and the property that for any coherent lower prevision and constant μ: $\underline{P}(X + \mu) = \underline{P}(X) + \mu$, we also have that $-\underline{E}(-X) = 1 - \underline{E}(\Omega - X)$. This means that the conjugate of the lower prevision $\underline{P}$ is given for every event X as $1 - \underline{P}(\Omega - X)$. Note here that $\Omega - X$ is the complement or "contrary" of the event X. It can be checked easily from the definition formula of $\overline{P}$ that for any event X we indeed have $\overline{P}(X) = 1 - \underline{P}(\Omega - X)$.*

Argumentative probabilities are thus coherent no matter what the arguments and their strength are. All the techniques discussed in subsection 2.2 for statistical inference and decision making can be employed when working with argumentative probabilities. Upper probabilities and previsions are also useful for assessing financial risk and constructing coherent risk measures. The reader interested in this topic may refer to the references [Pelessoni and Vicig, 2001; Artzner *et al.*, 1999]. The role and use of coherent risk measures for robust optimisation is discussed in [Bertsimas and Brown, 2005; Natarajan *et al.*, 2005].

5. DISCUSSION OF RELATED WORK

This section discusses the relationship existing between our approach and statistical methods based either on objective or subjective probability as well as some of the work that has been done in the areas of Logic, Artificial Intelligence and Decision Theory.

Frequency probability [Neyman, 1950; Fishburn, 1964] is the interpretation of probability that defines an event's probability as the limit of its relative frequency in a large number ofobservationss. Argumentative and frequency probability match in the following sense. Assume that each observation i in a sample $1, ..., N$ is modelled as an argument A_i, and the outcome $\{\omega_i\}$ of the i-thobservationn corresponds to the support of A_i, that all arguments in A_i are assigned the same strength and that $K > 0$. Then, for any event $E \subseteq \Omega$, the lower and upper argumentative probabilities $\underline{P}_N(E)$ and $\overline{P}_N(E)$ bound below and above the frequency estimator $\hat{F}_N(E) = \frac{1}{N} \sum_{i=1}^{N} I_E(\omega_i)$ (I_E is the indicator function of the set E), and if $\lim_{N\to\infty} \hat{F}_N(E)$ exists and represents the true probability $p(E)$ of E, then the argumentative probabilities also converge to the same limit, i.e. $\lim_{N\to\infty} \underline{P}_N(E) = \lim_{N\to\infty} \overline{P}_N(E) = p(E)$.

Frequentists talk about probabilities only when dealing with well-defined random experiments. Different approaches are used though when no random experiment can be defined or when no or few trials can be undertaken as was the case in the auction problem considered throughout the paper. These approaches rely on the more general concept of *subjective probability*. Subjective probabilities measure the confidence of a subject in the truth of a particular proposition [Savage, 1972]. The intuitive comparative school (Koopman and Good) exploits the concept of ordering relation – not more probable than – to derive an empirical probability measure. The second school is based on Ramsey and

Savage utility decision making approach, whereby the choices of a decision maker (modelled as an expected utility maximiser) are used to reveal a set of probabilities. Subjective probabilities, when elicited by such means, are referred to as psychological probabilities to recognise the lack of total rationality and coherence expected from human responses [Chesley, 1975]. Psychological probabilities are descriptive of human beliefs and may not conform to the basic axioms of probability theory. Instead, this paper has provided an approach that always guarantees the normative goodness of probabilities.

A general procedure for eliciting imprecise probabilities from a subject is proposed in [Walley, 1991]. The idea is to obtain a number of qualitative or quantitative probability judgements and model them as marginally acceptable gambles. If this set of marginal gambles avoids sure loss, then one can use natural extension to assess the probability of any event. This method is not easily applicable. The first problem is that the method fails whenever conflicting judgements are employed. The second problem is that checking that the property of avoiding sure loss holds and computing natural extension is computationally expensive. Argumentative probabilities avoid sure loss and are coherent even when arguments conflict. Moreover, we dispose of simple formulae to compute them directly so we notably avoid the use of optimisation. Finally, the design of artificial agents using arguments seems conceptually simpler than the one of agents making probabilistic judgements autonomously.

In Artificial Intelligence, the theory of evidence, originally exposed in [Shafer, 1976] and later developed by other authors – refer to [Sentz and Scott, 2002] for a survey on this topic – provides a number of rules for combining numerical representations of beliefs called belief functions. Belief functions are computed by summing evidence mass functions over 2^{Ω} (whose size is exponential unlike Arg). This theory essentially focuses on the problem of combining the belief functions of several sources and offers a general way of constructing imprecise probabilities from numerical pieces of evidence. Belief functions are known to be coherent [Walley, 2000] and have been used for the design of many expert systems [Biswas *et al.*, 1988; Hsia and Shenoy, 1989; Kak *et al.*, 1990; Krause and Clark, 1993; Saffiotti and Umkehrer, 1991]. The construction of belief functions requires the user to input quantitative information whilst in this paper we have focused on qualitative pieces of information.

In Logic, argumentation theory (surveyed in [Chesñevar *et al.*, 2000; Bench-Capon and Dunne, 2007] is fundamentally concerned with the characterisation and computation of rationally acceptable sets of arguments [Dung, 1995] or assumptions [Bondarenko *et al.*, 1997; Dung *et al.*, 2006]. The primal goal of this theory is not to assess uncertainty but rather to study what a rational subject may believe. Argumentation can be seen essentially as a qualitative and logical approach to uncertainty. Probability is therefore not considered as essential to argumentation theory. However, a few researchers in Logic and Artificial Intelligence have advanced the idea of estimating both objective [Poole, 1993] and subjective probabilities [Krause *et al.*, 1995; Ambler, 1996] by means of argument aggregation. The basic idea behind Poole's probabilistic calculus is to estimate the probability of a claim being true by summing the probability of statistically independent, logically derived arguments that support that same claim. In other words, Poole uses argumentation aggregation for statistical inference, i.e. the inference of probabilistic information from an initial set of probabilistic judgements. On the opposite, we focus on problems in which no probabilistic information is available to the subject. The authors of [Krause *et al.*, 1995;

Ambler, 1996] embrace a more general and abstract view on argument aggregation than Poole and interestingly propose the use of algebraic calculus to assess (symbolically or numerically) the strength of arguments depending on their internal structure. The method used in this paper also involves a notion of argument strength, which may be assessed from the arguments internal structure, but which may as well be derived from an analysis of the dialectical relationships existing between the arguments.

Finally, textbooks on Decision Theory say little about the actual construction of probabilities [French, 1987]. In fact, standard Decision Theory does not try to answer how probabilities should be constructed but considers them as given *a priori*. The authors of [Tan and Pearl, 1994], amongst others, indicate that the specification of complete sets of probabilities and utilities makes decision theory impractical in complex tasks involving common sense knowledge. This important issue has favoured the emergence of qualitative approaches to decision making [Tan and Pearl, 1994; Boutilier, 1994; Dubois and Prade, 1995; Bonet and Geffner, 1996; Amgoud and Prade, 2004]. These works rely on mathematical models of uncertainty that not only differ from probability but as for Decision Theory, do not seek to explain how to measure uncertainty. Instead, we have provided a concrete recipe for constructing robust probability estimates, given empirical and practical arguments.

6. SUMMARY AND DISCUSSION OF FUTURE WORK

We have proposed formulae showing that a set of arguments and a measure of their strength allow to estimate the lower and upper probability of any event. The constructed probabilities are "good" in both a substantive and normative sense because they truly reflect the indeterminacy of the subject, relate clearly to the support offered by arguments and constitute coherent previsions. The property of coherence alone implies that these probabilities may be used for assessing risks [Pelessoni and Vicig, 2001], for statistical inference and rational decision making [Walley, 1991; 1996; 2000] in decision-theoretic settings.

In future work, we will prove that argumentative probabilities fall within the class of belief functions [Shafer, 1976] and show that the corresponding combination operator satisfies the algebraic properties of commutativity, continuity and associativity. We will provide a general closed form formula for the natural extension $\underline{E}(X)$ of any real-valued gamble X and show how to use it for comparing decisions or constructing coherent risk measures [Artzner *et al.*, 1999; Pelessoni and Vicig, 2001]. We will provide an additional formula for estimating lower and upper conditional argumentative probabilities or previsions and variances. We intend to demonstrate the applicability and usefulness of these results in the domain of quantitative Finance [Lhabitant, 2001].

We have used arguments as evidence to construct probabilities but have so far assumed given a measure of their strength. In the restricted context of this paper, all arguments have been envisaged to be put forward and justified by some sources playing the role of their proponents, but in real life, nothing would prevent other sources of evidence to act as opponents and undermine such arguments. These attacks should intuitively reduce the strength of arguments and in our framework automatically augment the subject's indeterminacy. We believe that by looking at the dialectical structure of arguments in controversial debates – as done in abstract argumentation theory [Dung, 1995] – it is possible to analyse the status of arguments, notably by looking at the acceptability [Dung, 1995; Bondarenko *et al.*, 1997; Dung *et al.*, 2006] of the opinions embracing them, but also by studying the game-theoretical equilibrium resulting from the interactions with the other opinions that do not embrace them and eventually define a purely dialectical measure argument strength. This topic is the object of another working paper.

References

[Ambler, 1996] Simon Ambler. A categorial approach to the semantics of argumentation. *Mathematical Structures in Computer Science*, 6(2):167–188, 1996.

[Amgoud and Cayrol, 2002] L. Amgoud and C. Cayrol. Inferring from inconsistency in preference-based argumentation frameworks. *Journal of Automated Reasoning*, 29:125–169, 2002.

[Amgoud and Prade, 2004] L. Amgoud and H. Prade. Using arguments for making decisions: A possibilistic logic approach. In *Proceedings 20th Conference of Uncertainty in AI*, pages 10–17, 2004.

[Artzner *et al.*, 1999] P. Artzner, F. Delbaen, J.M. Eber, and D. Heath. Coherent measures of risk. *Mathematical Finance*, 1999.

[Aughenbaugh and Paredis, 2006] J.M. Aughenbaugh and C.J.J. Paredis. The value of using imprecise probabilities in engineering design. *Journal of Mechanical Design*, 128, 2006.

[Bench-Capon and Dunne, 2007] T. J. M. Bench-Capon and Paul E. Dunne. Argumentation in artificial intelligence. *Artificial Intelligence*, 171(10–15):619–641, 2007.

[Bertsimas and Brown, 2005] D. Bertsimas and D.B. Brown. Robust linear optimization and coherent risk measures. Submitted for publication, Department of Electrical Engineering and Computer Science, Massachusetts Institute of Technology, 2005.

[Biswas *et al.*, 1988] G. Biswas, M. Oliff, and R. Abramczyk. OASES (operations analysis expert system): an application in fiberglass manufacturing. *International Journal of Expert Systems: Research and Applications*, 1:193–216, 1988.

[Bondarenko *et al.*, 1997] A. Bondarenko, P. M. Dung, R. A. Kowalski, and F. Toni. An abstract, argumentation-theoretic approach to default reasoning. *Artificial Intelligence*, 93(1–2):63–101, 1997.

[Bonet and Geffner, 1996] Blai Bonet and H. Geffner. Arguing for decisions: A qualitative model of decision making. In Eric Horvitz and Finn Jensen, editors, *Proceedings of the 12th Conference on Uncertainty in Artificial Intelligence (UAI-96)*, pages 98–105, San Francisco, August 1–4 1996. Morgan Kaufmann Publishers.

[Boutilier, 1994] Craig Boutilier. Toward a logic for qualitative decision theory. In *KR*, pages 75–86, 1994.

[Chesley, 1975] G.R. Chesley. Elicitation of subjective probabilities: a review. *The Accounting Review*, 1975.

[Chesñevar *et al.*, 2000] C. I. Chesñevar, A. Gabriela Maguitman, and R. Prescott Loui. Logical models of argument. *ACM Computing Surveys*, 32(4):337–383, 2000.

[Dantzig *et al.*, 1955] G. B. Dantzig, A. Orden, and P. Wolfe. The generalized simplex method for minimizing a linear form under linear inequality constraints. *Pacific J. Math.*, 5(2):183–195, 1955.

[Dubois and Prade, 1988] Didier Dubois and H. Prade. An introduction to possibilistic and fuzzy logics. In et al Smets, editor, *Non-Standard Logics for Automated Reasoning*. Academic Press, 1988. Reprinted in Readings in Uncertain Reasoning.

[Dubois and Prade, 1995] Didier Dubois and H. Prade. Possibility theory as a basis for qualitative decision theory. In *IJCAI*, pages 1924–1932, 1995.

[Dung *et al.*, 2006] P. M. Dung, R. A. Kowalski, and F. Toni. Dialectic proof procedures for assumption-based, admissible argumentation. *Artificial Intelligence*, 170(2):114–159, 2006.

[Dung, 1995] P. M. Dung. On the acceptability of arguments and its fundamental role in nonmonotonic reasoning, logic programming, and n-person games. *Artificial Intelligence*, 77(2):321–257, 1995.

[Finkelstein *et al.*, 1994] A. Finkelstein, D. Gabbay, A. Hunter, J. Kramer, and B. Nuseibeh. Inconsistency handling in multiperspective specifications. *IEEE Transactions on Software Engineering*, 20(8):569–578, August 1994.

[Fishburn, 1964] Peter C. Fishburn. *Decision and Value Theory*. John Wiley and Sons, Inc., New York, 1964.

[Fox, 2003] J. Fox. Probability, logic and the cognitive foundations of rational belief. *Journal of Applied Logic*, 1:197–224, 2003.

[French, 1987] S. French. *Decision theory: an introduction to the mathematics of rationality*. Ellis Horwood, 1987.

[Hsia and Shenoy, 1989] Y.-T. Hsia and P.P. Shenoy. An evidential language for expert systems. *Methodologies for Intelligent Systems*, 4:9–16, 1989.

[Jaffray, 1999] J.-Y. Jaffray. Rational decision making with imprecise probabilities. In *ISIPTA*, pages 183–188, 1999.

[Kak *et al.*, 1990] A. C. Kak, K. M. Andress, C. Lopez-Abadia, M. S. Carroll, and J. R. Lewis. Hierarchical evidence accumulation in the Pseiki system and experiments in model-driven mobile robot navigation. In M. Henrion, R. D. Shachter, L. N. Kanal, and J. F. Lemmer, editors, *Uncertainty in Artificial Intelligence 5*, volume 10 of *Machine Intelligence and Pattern Recognition*, pages 353–370. North-Holland, Amsterdam, 1990.

[Klir, 2006] G.J. Klir. *Uncertainty and Information*. John Wiley and Sons, Inc., 2006.

[Krause and Clark, 1993] P. Krause and D. Clark. *Representing uncertain knowledge*. Kluwer Academic Press, Dordrecht, NL, 1993.

[Krause *et al.*, 1995] P. Krause, S. Ambler, M. Elvang-Gøransson, and J. Fox. A logic of argumentation for reasoning under uncertainty. *Computational Intelligence*, 11:113–131, 1995.

[Lhabitant, 2001] F.-S. Lhabitant. *Hedge Funds: Quantitative Insights*. Wiley Finance Series, 2001.

[Natarajan *et al.*, 2005] K. Natarajan, D. Pachamanova, and M. Sim. Constructing risk measures from uncertainty sets. Working Paper, Department of Mathematics, National University of Singapore, 2005.

[Neyman, 1950] J. Neyman. *First Course in Probability and Statistics*. Henry Holt, 1950.

[Pelessoni and Vicig, 2001] R. Pelessoni and P. Vicig. Coherent risk measures and upper previsions. In *ISIPTA*, pages 377–386, 2001.

[Poole, 1993] D. Poole. Probabilistic horn abduction and bayesian networks. *Artificial Intelligence*, 64(1):81–129, 1993.

[Saffiotti and Umkehrer, 1991] A. Saffiotti and E. Umkehrer. Pulcinella: A general tool for propagating uncertainty in valuation networks. In D'Ambrosio, Bruce D., Smets, Philippe, and Piero P. Bonissone, editors, *Proceedings of the 7th Conference on Uncertainty in Artificial Intelligence*, pages 323–331, San Mateo, CA, USA, July 1991. Morgan Kaufmann Publishers.

[Savage, 1972] L. J. Savage. *The foundations of statistics*. Dover, New York, 1972.

[Sentz and Scott, 2002] K. Sentz and F. Scott. Combination of evidence in Dempster-Shafer Theory – Sandia National Laboratories, 2002.

[Shafer, 1976] Glenn Shafer. *A Mathematical Theory of Evidence*. Princeton University Press, 1976.

[Smets, 1991] P. Smets. Constructing the pignistic probability function in a context of uncertainty. In M. Henrion, R. Shachter, L. N. Kanal, and J. Lemmer, editors, *Uncertainty in Artificial Intelligence 5*, pages 29–39, Amsterdam, 1991. Elsevier Science Publishers.

[Tan and Pearl, 1994] S. Tan and Judea Pearl. Qualitative decision theory. In *Proceedings of the Twelfth National Conference on Artificial Intelligence*, pages 928–932, 1994.

[Walley, 1991] Peter Walley. *Statistical reasoning with Imprecise Probabilities*. Chapman and Hall, 1991.

[Walley, 1996] Peter Walley. Measures of uncertainty in expert systems. *Artificial Intelligence*, 83(1):1–58, 1996.

[Walley, 2000] P. Walley. Towards a unified theory of imprecise probability. *International Journal of Approximate Reasoning*, 24:125–148, 2000.

[Winkler and Murphy, 1968] R.L. Winkler and A.H. Murphy. Good probability assessors. *Journal of Applied Meteorology*, pages 751–758, 1968.