Differentiation in Logical Form

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***Abstract*—We introduce a logical theory of differentiation for a real-valued function on a finite dimensional real Euclidean space. A real-valued continuous function is represented by a localic approximable mapping between two semi-strong proximity lattices, representing the two stably locally compact Euclidean spaces for the domain and the range of the function. Similarly, the Clarke subgradient, equivalently the L-derivative, of a locally Lipschitz map, which is non-empty, compact and convex valued, is represented by an approximable mapping. Approximable mappings of the latter type form a bounded complete domain isomorphic with the function space of Scott continuous functions of a real variable into the domain of non-empty compact and convex subsets of the finite dimensional Euclidean space partially ordered with reverse inclusion. Corresponding to the notion of a single-tie of a locally Lipschitz function, used to derive the domain-theoretic L-derivative of the function, we introduce the dual notion of a single-knot of approximable mappings which gives rise to Lipschitzian approximable mappings. We then develop the notion of a strong single-tie and that of a strong knot leading to a Stone duality result for locally Lipschitz maps and Lipschitzian approximable mappings. The strong single-knots, in which a Lipschitzian approximable mapping belongs, are employed to define the Lipschitzian derivative of the approximable mapping. The latter is dual to the Clarke subgradient of the corresponding locally Lipschitz map defined domain-theoretically using strong single-ties. A stricter notion of strong single-knots is subsequently developed which captures approximable mappings of continuously differentiable maps providing a gradient Stone duality for these maps. Finally, we derive a calculus for Lipschitzian derivative of approximable mapping for some basic constructors and show that it is dual to the calculus satisfied by the Clarke subgradient.**

I. Introduction

Differentiation is one of the two pillars of Calculus and thereby a fundamental basis of modern mathematics, science and engineering. The derivative of a real-valued function of a real variable and the gradient of a multivariate function are used as basic tools in many areas of computer science including Optimisation, Neural Networks, Machine Learning and Robotics. The gradient descent algorithm, for example, is at the very heart of these and other areas of computer science which deal with optimisation and learning. There are now tools for automatic differentiation [18], which is used for example in the fundamental Backpropagation algorithm in neural networks. A logical representation of differentiation in the sense of programme logics is thus long overdue. We address this task here.

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There are two completely different threads in the background to the present work. The first thread is that of programme logics. The study of properties of programmes was pioneered by the work of Flyod [16], Hoare [19] and later that of Dijkstra in his work on predicate transformers and the weakest precondition for obtaining a given post-condition for a programme [8]. In this same period, Scott introduced Domain theory as a mathematical foundation of computation [24]. The seminal work of Johnstone on point free topology [20] illustrated mathematically how, instead of working with points and point maps as in classical mathematics, one can work with the lattice of open sets of sober topological spaces, i.e., spaces for which we can recover the points from the lattice of open sets by taking completely prime filters of open sets. Moreover, instead of working with point maps on spaces, one can work with frame homomorphisms between the complete lattices of open sets of these spaces, which preserve finite intersections and arbitrary unions of open sets. In Locale theory one considers the frame homomorphisms as morphisms in the opposite direction, i.e., the same direction as the point maps. This has extended the work of Stone to far more general topological spaces than those generated by Boolean algebras [28], providing a duality between the category of sober topological spaces and spatial locales. Many properties of topological spaces have their counterparts in Locale theory [20].

In [25], Scott showed how, instead of the infinitary operations required in the complete lattice of open sets, one can develop finitary structures, called *information systems* and *approximable mappings* respectively, to represent domains of computation and maps between them respectively. Subsequently, Smyth [27] provided a topological view of predicate transformers using topology and Domain Theory: open sets represent properties of points of topological spaces, points of the space represent logical theories and frame homomorphisms correspond to predicate transformers. The underlying logic has been called *geometric logic*; see Subsection I-A. The stage was by now set for the application of Stone duality in Computer Science, which has in particular been promoted in Vickers' book [29].

In his elaborate work [1] in the framework of Locale theory, Abramsky provided a comprehensive account of Domain Theory in logical form for stably locally compact algebraic domains, which have a topological basis of compact-open sets. Continuous domains, which can be used to represent classical Hausdorff spaces, do not have such a basis. Smyth [26], Abramsky and Jung [2] and Vickers [31] have extended the notion of Scott's information systems to continuous domains. Jung and his collaborators have worked in the past two decades to extend Abramsky's Stone duality for stably locally compact algebraic domains to the case of continuous domains in order to obtain Stone duality for stably locally compact spaces, defined as spaces which are locally compact and sober and in which the intersection of two compact saturated sets (i.e., sets that are the intersection of their open neighbourhoods) is compact. See [21] for a historical account of this endeavour [22]. For developing a logical form of differentiation to represent finite dimensional Euclidean spaces, we start with a weaker notion of what was introduced by Jung and Sünderhauf [22] and called strong proximity lattices. These lattices allow a finitary representation of points by prime filters of open sets instead of completely prime filters.

There is a completely different thread of research in the background of this work however. Locally Lipschitz real-valued functions on $\mathbb{R}^n$ form a useful class of functions that have several desirable closure and extension properties. They are closed under composition and contain the important class of piecewise polynomial functions, which are widely used in geometric modelling, approximation and interpolation and are supported in MatLab [6]. They are uniformly continuous and have much better invariant properties than differentiable maps, in particular they are closed under the fundamental $\min$ and $\max$ operations that are frequently used in optimisation and control theory.

In 1970's, Frank Clarke developed a set-valued derivative for real-valued locally Lipschitz maps on $\mathbb{R}^n$, which is now called the Clarke subgradient and is a main tool in optimisation theory and convex analysis. Clarke was motivated by applications in non-smooth analysis, where one deals with functions that are locally Lipschitz but not necessarily differentiable. For real-valued functions on finite dimensional Euclidean spaces, the Clarke subgradient at any point is a non-empty compact and convex subset of the Euclidean space. For example, the absolute value function, which is not classically differentiable at zero, is a Lipschitz map that has Clarke gradient $[-1, 1]$ at zero. A crucial property of the Clarke subgradient is that it extends the classical gradient of continuously differentiable functions to locally Lipschitz maps.

The key link with Domain Theory lies in the fact that the Clarke subgradient is continuous with respect to the Scott topology on the set of non-empty convex and compact subsets of $\mathbb{R}^n$ partially ordered with reverse inclusion. In [11], a domain-theoretic derivative was introduced for functions with real interval input and output. This was later extended to real-valued functions on $\mathbb{R}^n$ and shown to be mathematically equivalent to the Clarke subgradient in this case [9], [12]. Whereas the Clarke subgradient of a Lipschitz map is defined by using the generalized directional derivative based on taking a double limsup of the rate of change of the function along a given direction, the L-derivative is constructed by collecting together some finitary generalized Lipschitz properties of the map that allow a natural way of approximating the L-derivative using domain theory. The L-derivative has been used to develop an extension of Real PCF with a differentiation operator so that both Lipschitz maps and their derivatives can be numerically computed [7]. The L-derivative has also a wider scope than real maps and has been extended to complex Lipschitz maps [10].

In this paper, we aim to synthesise the above two different threads and formulate a Stone duality for the Clarke subgradient of locally Lipschitz maps and a Stone duality for the gradient of a continuously differentiable function. This amounts to a localic representations of the Clarke subgradient and the classical gradient. The problem we solve in this paper can be described non-technically as follows. Consider a real-valued function f on $\mathbb{R}^n$ as a programme with input in $\mathbb{R}^n$ and output in $\mathbb{R}$. Then f can be represented via its predicate transformer as a relation between properties (open sets) of inputs in $\mathbb{R}^n$ which under f guarantee to satisfy a given property of outputs in $\mathbb{R}$. Given this relation which provides a logical presentation of f, we show how we can construct the corresponding relation between the properties of inputs and those of the outputs for the Clarke subgradient or the classical gradient of f. We envisage that this approach can be used in automatic differentiation.

In [14], Ehrhard and Regnier have introduced the differential lambda calculus which syntactically models the derivative operation on power series in a typed lambda calculus or a full linear logic. Differential lambda calculus proposes to model smooth (infinitely differentiable) maps on non-normed vector spaces [3], which is orthogonal to the work present in this paper that addresses the classical derivative on Euclidean spaces.

The rest of the paper is organised as follows. In the rest of this section, we provide the basic elements, as we here require, of geometric logic, domain theory and topological notions, operations on convex open sets and the support function of convex subsets in finite dimensional Euclidean spaces. In Section II, the definition of the Clarke gradient with its basic properties and those of the L-derivative and the ties of functions are presented. In Section III, the category of semi-strong proximity lattices and localic approximable mappings is defined and is shown to be equivalent to the category of stably locally compact spaces and continuous maps. In Section IV, we construct a domain for localic approximable mappings. In Section V, several constructors of approximable mappings are derived. In Section VI, the notion of knots of Lipschitzian approximable mappings is introduced which corresponds to the ties of functions. In Section VII, the notions of strong tie and strong knots are formulated and are shown to provide a duality between Lipschitz maps and Lipschitzian approximable mappings. In addition, the Lipschitzian derivative of a Lipschitzian approximable mapping is defined and is shown to be the approximable mapping of the Clarke gradient of the Lipschitz map corresponding to the Lipschitzian approximable mapping. In Section VIII, the strong ties and knots are further refined to provide a duality between continuously differentiable functions and their localic approximable mappings. In Section IX, a calculus for the Lipschitzian derivative of the constructors of approximable mappings is developed which is shown to be dual to the calculus of the Clarke gradient. Finally, in Section X, we list a number of topics for further research. All proofs in the article are given in the full version of the paper [13].

A. Geometric logic

We present a brief account of geometric logic, also referred to as the logic of finite observations, or equivalently as the logic of semi-decidable properties. The idea is to use the open sets of a topological space as propositions or semi-decidable properties [26]. We will here use the notations in [30].

Let X be a topological space and $\Omega(X)$ be its lattice of open sets. We define a propositional geometric theory as follows: For every open set $a \in \Omega(X)$, we define a proposition P_a, i.e., every open set of X provides a property or predicate. For open sets a and b with $a \subseteq b$, we have an axiom: (i) $P_a \vdash P_b$. For a family of open sets S, we have an axiom: (ii) $P_{\cup S} \vdash \bigvee_{a\in S} P_a$. For a finite family of open sets S, we have an axiom: (iii) $\bigwedge_{a\in S} P_a \vdash P_{\cap S}$. The converses of (ii) and (iii) follow from (i). The nullary disjunction in (ii) is interpreted as **false** and the nullary conjunction in the converse of (iii) is interpreted as **truth**, i.e, $P_\emptyset \vdash \mathbf{false}$ and $P_X \vdash \mathbf{truth}$.

Each point $x \in X$ gives a model of the theory in which P_a is interpreted as **true** iff $x \in a$, i.e, $x \models a$ iff $x \in a$, or, a point is a model of a proposition if it is in the open set representing the proposition. In general, there is no negation in propositional geometric logic as the complement of an open set is not in general open. In addition, in general, it is possible that different points give rise to the same model, i.e., satisfy the same open sets, and it is also possible that a model does not arise by points in X in this way. For so-called sober spaces, as we will define below, we do have a one-to-one correspondence between points and models.

Recall that for a set A with a transitive order $\prec$, a non-empty subset $F \subset A$ is a *filter* if (i) it is upwards closed, i.e., $x \in F$ and $x \prec y$ implies $y \in F$, and (ii) $x, y \in F$ implies there exists $z \in F$ with $z \prec x, y$. In a lattice L, a filter F is said to be *completely prime* if $F \cap \bigvee M \neq \emptyset$ implies $F \cap M \neq \emptyset$ for every subset $M \subset L$. For every point $x \in X$, the set of its open neighbourhoods, i.e., the set of open sets containing x, is a completely prime filter.

A *frame* is a complete lattice with the infinite distributive law:

$$a \wedge \bigvee S = \bigvee \{a \wedge s | s \in S\}$$

In particular, $\Omega(X)$ is a frame for any topological space X. A frame homomorphism is map between two frames that preserves finite meets and arbitrary joins. **Frm** denotes the category whose objects are frames and its morphisms are frame homomorphisms. If $f : X_1 \to X_2$ is a continuous map of two topological spaces then the map $\Omega f = f^{-1} : \Omega X_2 \to \Omega X_1$ is a frame homomorphism. It is convenient to have a morphism in the opposite direction i.e. $\Omega X_1 \to \Omega X_2$. Thus, the category **Loc** of locales is defined to be the opposite of the category of **Frm** and $\Omega : \mathbf{Top} \to \mathbf{Loc}$ defines a functor, where **Top** is the category of topological spaces and continuous maps.

There is also a functor from **Loc** to **Top**. Given a frame L, let $\text{pt}(L)$ be the set of all completely prime filters of L. Consider the map $\eta_X : X \to \text{pt}(\Omega(X))$ defined by $\eta_X(x) = F_x$ where F_x is the completely prime filter of open neighborhoods of x.

Given a locale L, let $O_a = \{F \in \text{pt}(L) | a \in F\}$ for every $a \in L$. Then the collection $\{O_a : a \in L\}$ forms a base for a topology on $\text{pt}(L)$. If $h : L_1 \to L_2$ is a frame-homomorphism then $\text{pt}(h) : \text{pt}(L_2) \to \text{pt}(L_1)$ with $\text{pt}(h)(F) = h^{-1}(F)$ is well-defined, i.e., $h^{-1}(F) \in \text{pt}(L_1)$. It can also be easily checked that $\text{pt}(h)$ is a continuous map. Thus $\text{pt} : \mathbf{Loc} \to \mathbf{Top}$ is a functor.

A topological space is said to be *sober* if η_X is bijective. If X is a sober space, then X is homeomorphic with $\text{pt}(\Omega(X))$.

We can also define a frame homomorphism $\varepsilon_L : L \to \Omega(\text{pt}(L))$, by $\varepsilon_L(a) = O_a$. A locale L is called *spatial* if ε_L is bijective. Thus, a spatial locale L is a locale of open sets of the topological space $\text{pt}(L)$.

The functors pt and Ω establish an equivalence between the category of sober spaces and the category of spatial locales.

B. Domain theory and topology notions

We assume the reader is essentially familiar with the very basic elements of domain theory as introduced by Dana Scott and developed in [2], [17]. We adopt the notion of a domain and the notation for a single-step function as used in [17]. Thus, a domain is a continuous directed complete partial order. Given an open set $O \subset X$ of a topological space X and an element $b \in D$ in a domain D with bottom element $\perp$, we denote by $b\chi_O : X \to D$ the single-step function with values $b\chi_O(x) = b$ if $x \in O$ and $\perp$ otherwise.

Let $U \subset \mathbb{R}^n$ be a non-empty open subset with respect to the Euclidean topology. The lattice of open sets of U is denoted by $\Omega(U)$. We denote by $\mathbf{C}(\mathbb{R}^n)$ the set of non-empty compact and convex subsets of $\mathbb{R}^n$ ordered with reverse inclusion and augmented with $\mathbb{R}^n$ as the bottom element. Then, $\mathbf{C}(\mathbb{R}^n)$ is a bounded complete domain with $C_1 \ll C_2$ iff $C_1^\circ \supset C_2$, where C° is the interior of a subset C. If $(C_i)_{i\in I}$ is a directed family in $\mathbf{C}(\mathbb{R}^n)$, then $\sup_{i\in I} C_i = \bigcap_{i\in I} C_i$. When $n = 1$, we have $\mathbf{C}(\mathbb{R}) = \mathbf{I}\mathbb{R}$ the domain of non-empty compact intervals of $\mathbb{R}$. The domain $\mathbf{C}(\mathbb{R}^n)$ has a basis given by non-empty, convex and compact sets with non-empty interior. The Scott topology of a domain D is denoted by σ_D. A basic open set of the Scott topology, equivalently the upper topology, for $\mathbf{C}(\mathbb{R}^n)$ is given by

$$\Box O = \{C \in \mathbf{C}(\mathbb{R}^n) : C \subset O\},$$

where $O \subset \mathbb{R}^n$ is an open set. The function space $(U \to \mathbf{C}(\mathbb{R}^n))$, the collection of Scott continuous maps of type $U \to \mathbf{C}(\mathbb{R}^n)$ partially ordered by pointwise ordering induced from $\mathbf{C}(\mathbb{R}^n)$ is itself a bounded complete domain with a basis consisting of step functions. For $b\chi_a, f \in (U \to \mathbf{C}(\mathbb{R}^n))$, we have $b\chi_a \ll f$ iff $a \ll_{\Omega(U)} f^{-1}(\twoheaduparrow b)$ [15, Proposition 5], where for any element $x \in D$ of a domain D we write $\twoheaduparrow x = \{y \in D : x \ll y\}$.

The closure of a subset $S \subset \mathbb{R}^n$ is denoted by $\overline{S}$. When S is bounded, its diameter is denoted by $\text{diam}(S)$. The convex hull of S is denoted by $\text{conv}(S)$. Given a map $f : X \to Y$

and a subset $A \subset X$, the forward image of A is written as $f[A]$. For $x \in \mathbb{R}$, we identify x with the singleton $\{x\}$. The Euclidean norm of $v \in \mathbb{R}^n$ is denoted by $\|v\|$. We use pointwise extension of operations on real numbers and real vectors to operations on sets of real numbers and vectors. For example, the inner product of two open sets $O_1, O_2 \subset \mathbb{R}^n$ is given by

$$O_1 \cdot O_2 = \{v \cdot w : v \in O_1, w \in O_2\}.$$

C. Basic operations on convex open sets

In this section, we look at the extension of the basic operations of vector addition and inner product to convex open sets in $\mathbb{R}^n$ and show that these extended operations are Scott continuous on $\Omega(\mathbb{R}^n)$.

Definition I.1. *Suppose we have non-empty subsets $A, B \subset \mathbb{R}^n$. The* Minkowski *sum of A and B is given by*

$$A + B = \{x + y : x \in A, y \in B\} \subset \mathbb{R}^n.$$

The inner product *of A and B is given by*

$$A \cdot B = \{x \cdot y : x \in A, y \in B\} \subset \mathbb{R}.$$

For $r \in \mathbb{R}$, scalar multiplication *of A with r is given by*

$$rA = \{rx : x \in A\} \subset \mathbb{R}^n.$$

Note that subtraction $A - B = A + (-1)B$ can be defined by composing addition with scalar multiplication by -1 on the second argument. We show that these operations preserve convex and open sets and give rise to Scott continuous maps on the lattice $\Omega(\mathbb{R}^n)$ of open sets of $\mathbb{R}^n$.

Proposition I.2. *The three maps*

(i) $(-) + (-) : \Omega(\mathbb{R}^n) \times \Omega(\mathbb{R}^n) \to \Omega(\mathbb{R}^n)$,
(ii) $(-) \cdot (-) : \Omega(\mathbb{R}^n) \times \Omega(\mathbb{R}^n) \to \Omega(\mathbb{R})$ *,and,*
(iii) $r \cdot (-) : \Omega(\mathbb{R}^n) \to \Omega(\mathbb{R}^n)$.

are well-defined, Scott continuous and preserve convex sets.

Note that for convex polytopes $O_1, O_2 \subset \mathbb{R}^n$ (each considered as an open set) with sets of vertices V_1 and V_2 respectively, we have $O_1 + O_2 = \text{conv}\{x+y : x \in V_1, y \in V_2\}$ and $O_1 \cdot O_2 = (m, M)$ with $m = \min\{x \cdot y : x \in O_1, y \in O_2\}$ and $M = \max\{x \cdot y : x \in O_1, y \in O_2\}$, whereas $\text{diam}(O_1) = \max\{\|x - y\| : x, y \in V_1\}$. These simple expressions allow us to develop an effective structure for a countable basis of the lattice $\Omega(\mathbb{R}^n)$ of open sets of $\mathbb{R}^n$ as explained in Subsection III-D.

D. Support function of convex sets

Given a non-empty and bounded convex set $A \subset \mathbb{R}^n$, its support function is defined as [1]

$$\begin{array}{rccc} S_A : & \mathbb{R}^n & \to & \mathbb{R} \\ & v & \mapsto & \sup\{v \cdot x : x \in A\} \end{array}$$

The support function has a number of key properties which makes it an important tool in convex analysis.

[1] In optimisation theory the support function of A is usually denoted by σ_A. This notation, however, clashes in domain theory with the use of σ_X for the Scott topology of a partial order X.

Proposition I.3. *[23, page 37]*

(i) *If A is a non-empty and compact convex set, then S_A is a real-valued continuous and convex function.*
(ii) *For A and B non-empty, compact and convex sets, $A \subset B$ iff $S_A \le S_B$.*

Note that if $O \subset \mathbb{R}^n$ is a non-empty bounded and convex open set, then from the definition we immediately obtain: $S_O = S_{\overline{O}}$. We will use the support function in some of the proofs in the paper.

II. Clarke's subgradient and the L-derivative

In this section, we recall the definition and basic properties of the Clarke subgradient and that of the L-derivative.

Let $U \subset \mathbb{R}^n$ be an open subset. First, recall that the *Fréchet* derivative, also called the *gradient*, of a map $f : U \to \mathbb{R}$ at $x \in U$, when it exists, is a linear map $T : \mathbb{R}^n \to \mathbb{R}$ with

$$\lim_{\|x-y\| \to 0} \frac{|f(x) - f(y) - T(x - y)|}{\|x - y\|} = 0.$$

As usual, given a basis of the vector space $\mathbb{R}^n$, we identify $T = \nabla f(x)$ as a vector in $\mathbb{R}^n$, which is simply the n-tuple of partial derivatives of f at x with respect to the n coordinates.

A. Clarke's subgradient

Let $f : U \to \mathbb{R}$ be locally Lipschitz, i.e., for each $v \in U$, there exists an open neighbourhood O of v and $k \ge 0$ such that $|f(x) - f(y)| \le k\|x - y\|$ for all $x, y \in O$. The *generalized directional derivative* [4, Chapter 2] of f at x in the direction of v is

$$f^\circ(x; v) = \limsup_{y \to x \;\; t \downarrow 0} \frac{f(y + tv) - f(y)}{t}. \qquad (1)$$

The *(Clarke) subgradient* of f at x, denoted by $\partial f(x)$ is the subset of $\mathbb{R}^n$ given by

$$\partial f(x) = \{w \in \mathbb{R}^n : f^\circ(x; v) \ge w \cdot v \text{ for all } v \in \mathbb{R}^n\},$$

where $w \cdot v = \sum_{i=1}^n w_i v_i$ is the inner product of w and v.

Example II.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be the absolute value function $f(x) = |x|$. Then*

$$\partial f(x) = \begin{cases} [-1, 1] & x = 0 \\ \{-1\} & x < 0 \\ \{1\} & x > 0 \end{cases}$$

Let $C^1(U)$ be the set of continuously differentiable real-valued functions.

Proposition II.2. *[4, 2.1-2.3] Let $f, g : U \subset \mathbb{R}^n \to \mathbb{R}$, where U is an open set, be locally Lipschitz.*

(i) *$\partial f(x)$ is a non-empty, convex, compact subset of $\mathbb{R}^n$.*
(ii) *For $v \in X$, we have:*

$$f^\circ(x; v) = \sup\{w \cdot v : w \in \partial f(x)\}. \qquad (2)$$

(iii) *$\partial f : U \to \mathbf{C}(\mathbb{R}^n)$ is upper (or Scott) continuous.*
(iv) *If $f \in C^1(U)$ then $\partial f(x) = \{\nabla f(x)\}$.*
(v) *If $\partial f(x)$ is a singleton for each $x \in U$, then $f \in C^1(U)$ and $\partial f(x) = \{\nabla f(x)\}$ for each $x \in U$.*

(vi) *(**Sum**)* $\partial f(x) + \partial g(x) \supset \partial (f+g)(x)$.
(vii) *(**Product**)* $\partial f(x) \cdot g(x) + f(x) \cdot \partial g(x) \supset \partial (f \cdot g)(x)$.
(viii) *(**Chain rule**) If $h : U_0 \subset \mathbb{R} \to \mathbb{R}$ is locally Lipschitz with $Im(f) \subset U_0$, then $\partial h(f(x)) \cdot \partial f(x) \supset \partial (h \circ f)(x)$.*

Moreover, in (vi) *and* (vii)*, if f or g is differentiable at x, then we obtain equality. In* (viii)*, if f is differentiable at x or if h is differentiable at $f(x)$, then we obtain equality.*

There is an alternative characterization of the Clarke subgradient. By Rademacher's Theorem [5, page 148], if $f : U \to \mathbb{R}$ is Lipschitz, then it is differentiable almost everywhere with respect to the Lebesgue measure. Suppose Ω_f is the null set where f fails to be differentiable. Then:

$$\partial f(x) = \text{conv}\{\lim \nabla f(x_i) : x_i \to x,\ x_i \notin \Omega_f\}, \quad (3)$$

where conv(S) is the convex hull of a subset $S \subset \mathbb{R}^n$ [4, page 63]. The above expression is interpreted as follows. Consider all sequences $(x_i)_{i \geq 0}$, with $x_i \notin \Omega_f$, for $i \geq 0$, which converge to x such that $\lim_{i \to \infty} \nabla f(x_i)$ exists. Then the generalized gradient is the convex hull of all such limits.

B. L-derivative

The L-derivative of a locally Lipschitz map $f : U \subset \mathbb{R}^n \to \mathbb{R}$ provides a finitary representation of the Clarke subgradient.

Definition II.3. *[9] The continuous function $f : U \to \mathbb{R}$ has* a non-empty, convex and compact set-valued Lipschitz constant *$b \in \mathbf{C}(\mathbb{R}^n) \setminus \{\bot\}$ in a non-empty convex open subset $a \subset U$ if for all $x, y \in a$ we have: $f(x) - f(y) \in b \cdot (x - y)$. The* single-step tie *$\delta(a, b)$ of a with b is the collection of all partial functions f on U with $a \subset dom(f)$ which have b as a non-empty convex compact set-valued Lipschitz constant in a.*

Note that we have used the extension of the inner product to subsets of $\mathbb{R}^n$.

To understand the definition, first note that if $f \in \delta(a, b)$, then f is Lipschitz in a with Lipschitz constant $\|b\| = \sup\{\|v\| : v \in b\}$. Next let $n = 1$. If $f : U \subset \mathbb{R} \to \mathbb{R}$ has a Lipschitz constant $k \geq 0$ in $a \subset U$, then we have $|f(x) - f(y)| \leq k|x - y|$ or equivalently $g \in \delta(a, b)$ with $b = [-k, k]$. By allowing b in Definition II.3 to be any non-empty compact interval (and not just of the form $[-k, k]$), we can collect all the locally differential properties of f expressed in $\delta(a, b)$ and then construct its Clarke subgradient as we will see below. First, we consider a simple example. Given an open interval $a \subset \mathbb{R}$, we write $a = (a^-, a^+)$.

Example II.4. *Let $f : \mathbb{R} \to \mathbb{R}$ be the absolute value function $f(x) = |x|$. If $0 \in a$ then $f \in \delta(a, b)$ iff $[-1, 1] \subseteq b$. If $0 \leq a^-$ then $f \in \delta(a, b)$ iff $1 \in b$. Finally, if $a^+ \leq 0$ then $f \in \delta(a, b)$ iff $-1 \in b$.*

Let $f : U \subset \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz map. The set of single-step functions $\{b\chi_a : U \to \mathbf{C}(\mathbb{R}^n) : f \in \delta(a, b)\}$ are bounded above [9, Proposition 3.9]. Thus

$$\mathcal{L}f = \sup\{b\chi_a : f \in \delta(a, b)\}$$

is a Scott continuous function of type $U \to \mathbf{C}(\mathbb{R}^n)$. Moreover:

Theorem II.5. *[9, Corollary 8.2] If $f : U \subset \mathbb{R}^n \to \mathbb{R}$ is locally Lipschitz, then:*

$$\mathcal{L}f = \partial f$$

III. Stably locally compact spaces

In this section, we explain the the first step for formulating a theory of differentiation in logical form. Consider a continuous function f of type $U \to \mathbb{R}$. Then its L-derivative $\mathcal{L}f$, developed domain-theoretically as described in Section II-B, has type $U \to \mathbf{C}(\mathbb{R}^n)$ and is constructed as the supremum of single-step functions $b\chi_a$ of type $U \to \mathbf{C}(\mathbb{R}^n)$. We will likewise develop a logical characterisation of $\mathcal{L}f$ as a localic approximable mapping of type $B_U \to B_{\mathbf{C}(\mathbb{R}^n)}$ of semi-strong proximity lattices.

We will deal with functions of two different types as follows. We consider (Lipschitz) continuous functions of type $U \subset \mathbb{R}^n \to \mathbb{R}$ and their Clarke subgradient, equivalently L-derivative, of type $U \to \mathbf{C}(\mathbb{R}^n)$, and develop a logical characterisation of their L-derivatives using approximable mappings.

The spaces we deal with namely $U \subset \mathbb{R}^n$ and $\mathbf{C}(\mathbb{R}^n)$ are basic examples of so-called *stably locally compact* spaces [17]. Recall that a topological space is stably locally compact if it is sober, locally compact, and if the intersection of any two compact saturated sets is compact. (A set is saturated if it is the intersection of its open neighbourhoods.)

There is a simple characterisation of stably locally compact spaces in terms of their lattice of open sets: A topological space is stably locally compact if its lattice of open sets is a distributive continuous lattice which is also arithmetic, i.e., its way-below relation satisfies

$$O \ll O_1, O_2 \Rightarrow O \ll O_1 \wedge O_2.$$

The way-below relation for the lattice $\Omega(U)$ of Euclidean open sets of U is characterised by $O_1 \ll O_2$ iff $\overline{O_1}$ is compact and $\overline{O_1} \subset O_2$, whereas that of the basic Scott open sets of $\Omega(\mathbf{C}(\mathbb{R}^n))$ is given by $\Box O_1 \ll \Box O_2$ iff $O_1 \ll O_2$.

Proposition III.1. *The domain $\mathbf{C}(\mathbb{R}^n)$ is a stably locally compact space as is any open set $U \subset \mathbb{R}^n$ with its subspace topology.*

A. Semi-strong proximity lattices

We will represent the lattice of open sets $\Omega(U)$ and $\Omega(\mathbf{C}(U))$ by *proximity lattices* [22]. Assume we have a basis B_U and $B_{\mathbf{C}(\mathbb{R}^n)}$ of the topologies of $\Omega(U)$ and $\Omega(\mathbf{C}(\mathbb{R}^n))$ which are both closed under finite intersections. We will discuss our particular choices for these bases later on in this section. By considering these bases for $\Omega(U)$ and $\Omega(\mathbf{C}(\mathbb{R}^n))$ and the restriction $\prec$ of the way-below relation to them, we obtain a proximity lattice.

Definition III.2. *A* semi-strong proximity lattice *denoted by $B = (B; \vee, \wedge, 0, 1; \prec)$ is given by a distributive lattice $(B; \vee, \wedge, 0, 1)$ such that $\prec$ is a binary relation on B with $\prec = \prec \circ \prec$ and satisfying:*

O-1 $\forall a \in B\ M \subset_f B. M \prec a \iff \bigvee M \prec a.$
O-2 $\forall a \in B. a \neq 1 \Rightarrow a \prec 1.$

O-3 $\forall a, a_1, a_2 \in B. a \prec a_1, a_2 \iff a \prec a_1 \wedge a_2$.
O-4 $\forall a, x, y \in B. a \prec x \vee y \Rightarrow$
$\exists x', y' \in B. x' \prec x \,\&\, y' \prec y \,\&\, a \prec x' \vee y'$.

Here, $M \subset_f B$ means that M is a finite (possibly empty) subset of B, and $M \prec a$ means that $\forall m \in M. m \prec a$.

This definition is the same as in [22, Definition 18] for strong proximity lattice except for two differences.

The first difference, which is minor, is that we do not in general require $1 \prec 1$ since we would like to represent stably locally compact spaces such as $\mathbb{R}$ that are not compact. When our stably locally compact space is actually compact, which is the case with $\mathbf{C}(\mathbb{R}^n)$, we will simply add the axiom $1 \prec 1$ to the axioms of a semi-strong proximity lattice.

The second difference, which is major, is that we do not require an additional axiom dual to O-4 for $\wedge$, which was adopted in [22], and in later work by Jung and his collaborators [21], in order to obtain a complete symmetry between compact and open sets for representation of stably locally compact spaces. The inclusion of this dual axiom leads to an interesting framework in which both open and compact sets are required for constructing points and thus for representation of stably locally compact spaces. However, the addition of this dual axiom in particular implies that standard spaces like $[0, 1]$ can no longer be represented in the usual way by their open sets; see [22, Example 22]. In the current work, in which for the first time a localic representation of the gradient is aimed at, we adhere to the more traditional approach to point free topology and although compact sets do play a vital role in tandem with open sets in our setting, representation of stably locally compact spaces is obtained by working with open sets only. For this reason, we have not included the dual of O-4 for $\wedge$ and thus the notion of semi-strong proximity lattice defined above is weaker than that of strong proximity lattice.

If O-4 itself is removed, we have what is called a *proximity lattice*, which satisfies the following properties, shared by semi-strong proximity lattices as well.

Proposition III.3. *[22, Lemma 7] Let $(B; \prec, \vee, \wedge, 0, 1)$ be a proximity lattice. We have:*

1) $\forall a \in B. 0 \prec a$,
2) $\forall a \in B. a \neq 1 \Rightarrow a \prec 1$,
3) $\forall a, a_1, a_2 \in B. a_1 \prec a_2 \Rightarrow a_1 \prec a_2 \vee a$,
4) $\forall a, a_1, a_2 \in B. a_1 \prec a_2 \Rightarrow a_1 \wedge a \prec a_2$,
5) $\forall a_1, a_2, a_1', a_2' \in B. a_1 \prec a_2 \,\&\, a_1' \prec a_2' \Rightarrow a_1 \vee a_1' \prec a_2 \vee a_2'$,
6) $\forall a_1, a_2, a_1', a_2' \in B. a_1 \prec a_2 \,\&\, a_1' \prec a_2' \Rightarrow a_1 \wedge a_1' \prec a_2 \wedge a_2'$,.

B. Localic approximable mapping

Suppose two stably locally compact spaces X_1 and X_2 are represented by the two semi-strong proximity lattices B_1 and B_2. We represent a continuous map $f : X_1 \to X_2$ by a *localic approximable mapping* $R : B_1 \to B_2$ as a relation $R \subset B_1 \times B_2$ as in [22, Definition 25]. We, however, write the composition of relations in the same order as for maps, i.e., if $R : B_1 \to B_2$ and $S : B_2 \to B_3$ are relations then their composition is written as $S \circ R : B_1 \to B_3$.

Definition III.4. *The relation $R \subset B_1 \times B_2$ is a* localic approximable mapping *if it satisfies:*

M-1 $R \circ \prec_1 = R$
M-2 $\prec_2 \circ R = R$.
M-3 $\forall M \subset_f B_1 \forall a' \in B_2. M \, R \, a' \iff \bigvee M \, R \, a'$.
M-4 $\forall a \in B_1. a \neq 1 \Rightarrow a \, R \, 1$.
M-5 $\forall a \in B_1 \forall a_1, a_2 \in B_2$.
$a \, R \, a_1 \,\&\, a R a_2 \iff a \, R \, a_1 \wedge a_2$.
M-6 $\forall a \in B_1 \forall M \subset_f B_2. a \, R \bigvee M \Rightarrow$
$\exists N \subset_f B_1. a \prec_1 \bigvee N \,\&\, \forall n \in N \exists m \in M. n \, R \, m$.

The identity approximable mapping $\mathrm{Id}_B : B \to B$ on B is the relation $\prec$, and composition of approximable mappings corresponds to composition of relations as usual. We now have two categories: the category of stably locally compact spaces and continuous functions and the category of semi-strong proximity lattices and approximable mappings.

C. Equivalence of two categories

Let $\mathbf{SL-Compact}$ denote the category of stably locally compact spaces with continuous functions and let **Semi-Strong PL** denote the category of semi-strong proximity lattices and localic approximable mappings. We will follow [22] to define two functors between these two categories which induce an equivalence between them:

$$A : \textbf{SL-Compact} \to \textbf{Semi-Strong PL}$$

$$G : \textbf{Semi-Strong PL} \to \textbf{SL-Compact}$$

with their actions on morphisms given by $A(f) = A_f$ and $G(R) = G_R$. (This notation reduces clutter in many expressions in the rest of the paper.) The reader is referred to the above paper for full details.

We first define the functor A. Given a stably locally compact space X, we fix a basis B of its topology which is closed under finite intersections and let $A(X)$ be the semi-strong proximity lattice represented by B. Next, suppose $f : X_1 \to X_2$ is a continuous map of stably locally compact spaces X_1 and X_2 represented by semi-strong proximity lattices B_1 and B_2, induced by two topological bases for X_1 and X_2 that are closed under finite intersections. Then we have a localic approximable mapping $A_f : B_1 \to B_2$ given by $a \, A_f \, a'$ iff $a \ll f^{-1}(a')$. It is simple to check that A_f satisfies the axioms in Definition III.4 and is thus a localic approximable mapping.

We need to do more work to define the functor G. Given a semi-strong proximity lattice $(B; \vee, \wedge, 0, 1; \prec)$, one obtains a frame $(\mathsf{Idl}(B), \subset)$ where $\mathsf{Idl}(B)$ is the set of ideals of B with respect to $\prec$. Recall that an ideal is the dual notion to a filter, i.e., a non-empty subset $S \subset B$ is an ideal if it is downwards closed under $\prec$ and for any set of elements $s_i \in S$, where $i \in I$ with I finite, there exists $s \in S$ with $s_i \prec s$ for $i \in I$. Moreover, the map $\downarrow(\cdot) : B \to (\mathsf{Idl}(B), \subset)$, which takes $a \in B$ to the principal ideal $\downarrow a = \{a' : a' \prec a\}$, preserves binary suprema as well as finite infima and is hence a lattice homomorphism. This essentially follows from [22, Proposition 17].

However, since, in contrast to [22], we do not in general assume that $1 \prec 1$, we need to check that $\downarrow(\cdot)$ indeed preserves 0-ary infima, i.e., that $\downarrow 1$ is the largest ideal. If indeed $1 \prec 1$, then clearly $\downarrow 1 = B$ is the largest ideal. Otherwise, if $1 \prec 1$ does not hold, then $\downarrow 1 = B \setminus \{1\}$ by O-2. Thus, to verify that $\downarrow 1$ is the largest ideal, we need to show that B itself is not an ideal. Suppose B is an ideal. Then, since $1 \in B$, by the definition of an ideal, there is $t \in B$ with $1 \prec t$. There are now two possible cases: (i) either $t = 1$ in which case we get $1 \prec 1$, a contradiction, or (ii) $t \neq 1$, in which case $t \prec 1$ and $1 \prec t$ from which, by part (3) of Proposition III.3, we obtain $1 \prec 1 \vee t = 1$, a contradiction. Thus $\downarrow(\cdot)$ indeed preserves all infima.

The frame $(\mathsf{Idl}(B), \subset)$ is a distributive continuous lattice which is arithmetic and thus represents the lattice of open sets of a stably locally compact space. We can however obtain this space by using a finitary construction as follows. Recall that a filter in B is an ideal with respect to $\succ$. Let $\mathsf{filt}(B)$ be the set of all filters of B and let $\mathsf{spec}(B)$ be the spectrum of B consisting of all *prime* filters of B, i.e.,

$$\mathsf{spec}(B) =$$
$$\{F \in \mathsf{filt}(B) : M \subset_f B \& \bigvee M \in F \Rightarrow M \cap F \neq \emptyset\}$$

For $x \in B$ let $\mathcal{O}_x = \{F \in \mathsf{spec}(B) : x \in F\}$ and consider the topology on $\mathsf{spec}(B)$ given by the open sets $\mathcal{O}_x$ for $x \in B$. We have: $\mathcal{O}_a \cap \mathcal{O}_b = \mathcal{O}_{a \wedge b}$ and $\mathcal{O}_a \cup \mathcal{O}_b = \mathcal{O}_{a \vee b}$.

Consider now the Scott topology $\sigma_{(\mathsf{Idl}(B), \subset)}$ of the continuous lattice $(\mathsf{Idl}(B), \subset)$. Let the map $\psi : \mathsf{spec}(B) \to \sigma_{\mathsf{Idl}(B)}$ be given by

$$\psi(F) = \{I \in \mathsf{Idl}(B) : I \cap F \neq \emptyset\}.$$

Then, considering $\mathsf{Idl}(B)$ with subset inclusion, $\psi(F)$ is a completely prime filter for any prime filter F and ψ establishes a homeomorphism between $\mathsf{spec}(B)$ and $\mathsf{pt}(\mathsf{Idl}(B))$.

We can now define the functor

$$G : \textbf{Semi-Strong PL} \to \textbf{SL-Compact}$$

For a semi-strong proximity lattice B, we put $G(B) = \mathsf{spec}(B)$ which, as we have seen, is a stably locally compact space. Given a localic approximable mapping $R : B_1 \to B_2$, we obtain a map $G_R : \mathsf{spec}(B_1) \to \mathsf{spec}(B_2)$ by putting

$$G_R(F) = \{b_2 \in B_2 : \exists b_1 \in F.\, b_1 \, R \, b_2\},$$

for $F \in \mathsf{spec}(B_1)$. It follows from M-2 that $G_R(F) \in \mathsf{spec}(B_2)$ is a filter. To see that it is a prime filter, let $\bigvee M \in G_R(F)$. Thus, there exists $b_1 \in F$ with $b_1 \, R \bigvee M$. By O-4, there exists a finite set $N \subset B_1$ such that $b_1 \prec \bigvee N$ and for all $n \in N$ there exists $m \in M$ with $n \, R \, m$. Since $\bigvee N \in F$ and F is a prime filter, there exists $n \in N$ with $n \in F$. We conclude that there exists $m \in M$ be such that $n \, R \, m$, i.e., $m \in G_R(F)$, or, $G_R(F) \in \mathsf{spec}(B_2)$. It remains to check that G_R is a continuous function. Given any neighbourhood open set $\mathcal{O}_y$ we have:

$$(G_R)^{-1}(\mathcal{O}_y) = \{F : G_R(F) \in \mathcal{O}_y\} = \{F : y \in G_R(F)\}$$
$$= \{F : \exists b \in F.\, b \, R \, y\} = \bigcup_{b \, R \, y} \mathcal{O}_b$$

establishing the continuity of G_R.

The Stone duality theorem proved in [22, Theorem 21] also applies to semi-strong proximity lattices.

Theorem III.5. *[22, Theorem 26] We have the two identities $G_{A_f} = f$ and $A_{G_R} = R$, i.e., $\mathbf{SL-Compact}$ is equivalent to* **Semi-Strong PL** *via the functors A and G.*

D. Semi-strong proximity lattices for $U \subset \mathbb{R}^n$, $\mathbb{R}$ and $\mathbf{C}(\mathbb{R}^n)$

Let's now turn back to the type of spaces we deal with, namely an open subset $U \subset \mathbb{R}^n$, $\mathbb{R}$ and $\mathbf{C}(\mathbb{R}^n)$. In view of the linear structure of $\mathbb{R}^n$, we can choose our basic open sets to have an additional very useful property, namely that they be convex. Let $\mathcal{D}$ be the basis consisting of all open convex sets in $\mathbb{R}^n$ with compact closure, including the empty set. And let $\mathcal{D}^r$ be the basis consisting of all open convex polytopes, including the empty polytope, whose vertices have rational coordinates. Since convex sets are closed under finite intersections, both $\mathcal{D}$ and $\mathcal{D}^r$ are closed under finite intersections. We denote by $B^0_{\mathbb{R}}$, respectively B^0_U, any basis of $\mathbb{R}$, respectively U, such as $\mathcal{D}$ or $\mathcal{D}^r$, that consists of convex open sets and is closed under binary intersections. We then let $B_{\mathbb{R}}$, respectively B_U, denote the semi-strong proximity lattice generated by $B^0_{\mathbb{R}}$, respectively B^0_U. This means that every element of $B_{\mathbb{R}}$, respectively B_U, is a finite join of elements of $B^0_{\mathbb{R}}$, respectively B^0_U.

Recall that the way-below relation in $\Omega(\mathbb{R}^n)$ is given by $O_1 \ll O_2$ iff $\overline{O_1}$ is compact and $\overline{O_1} \subset O_2$. Since the elements of $\mathcal{D}$ and $\mathcal{D}^r$ have compact closure, the restriction of the way-below relation to them will simply be $O_1 \ll O_2$ iff $\overline{O_1} \subset O_2$. Moreover, for elements of $\mathcal{D}^r$, the way-below relation is decidable, which allows the semi-strong proximity lattice generated by it to be given an effective structure.

Each of the two sets $\mathcal{D}$ and $\mathcal{D}^r$ induces a basis for the bounded complete domain $\mathbf{C}(\mathbb{R}^n)$ as follows. The set $\{\overline{a} : a \in \mathcal{D}, a \neq \emptyset\} \subset \mathbf{C}(\mathbb{R}^n)$ and the countable set $\{\overline{a} : a \in \mathcal{D}^r, a \neq \emptyset\} \subset \mathbf{C}(\mathbb{R}^n)$, both augmented with $\{\bot\}$, each provide a basis with compact convex sets with non-empty interior for the domain $\mathbf{C}(\mathbb{R}^n)$. Similarly, the set $\mathcal{D}$, respectively $\mathcal{D}^r$, induces a basis, respectively a countable basis, for the Scott topology of $\mathbf{C}(\mathbb{R}^n)$ as follows. The set $\mathcal{D}_{\mathbf{C}(\mathbb{R}^n)} := \{\Box a : a \in \mathcal{D}\}$ and the countable set $\mathcal{D}^r_{\mathbf{C}(\mathbb{R}^n)} := \{\Box a : a \in \mathcal{D}^r\}$ are two bases of the Scott topology $\sigma_{\mathbf{C}(\mathbb{R}^n)}$ of $\mathbf{C}(\mathbb{R}^n)$. We let $B^0_{\mathbf{C}(\mathbb{R}^n)}$ be the basis of the Scott topology $\sigma_{\mathbf{C}(\mathbb{R}^n)}$ whose elements are of the type $\Box a$ where $a \in B^0_{\mathbb{R}^n}$, and we let $B_{\mathbf{C}(\mathbb{R}^n)}$ be the semi-strong proximity lattice generated by $B^0_{\mathbf{C}(\mathbb{R}^n)}$. Thus, each element of the semi-strong proximity lattice $B_{\mathbf{C}(\mathbb{R}^n)}$ is the finite join of elements of $B^0_{\mathbf{C}(\mathbb{R}^n)}$.

The bases $B_{\mathbb{R}}$, B_U and $B_{\mathbf{C}(\mathbb{R}^n)}$, when generated by $\mathcal{D}$, provide a general mathematical and logical theory of localic differentiation. When these bases are instead generated by $\mathcal{D}^r$, they can be employed to develop a computability theory for localic differentiation.

We also stress here that nearly all proofs in the paper use both open sets and compact sets. For this reason, when proving results related to one of the semi-strong proximity lattices, we often implicitly use Stone duality to work in the stably locally compact space dual to the semi-strong proximity lattice to derive the proof. For example, this means that for elements

$a, a' \in B$ with $a \prec a'$ in a proximity lattice B, we work with the open sets in the stably locally compact space $X := G(B)$ that a and a' represent which we simply denote by a and a' and will satisfy the relation $a \ll a'$ in the lattice of open sets of X.

IV. Domain of approximable mappings

In this section, we will develop a logical characterisation of $\mathcal{L}f$ as a localic approximable mapping of type $B_U \to B_{\mathbf{C}(\mathbb{R}^n)}$ of semi-strong proximity lattices, and show that these localic approximable mappings form a domain. We will see that similar to the construction of the L-derivative $\mathcal{L}f$ as the supremum of single-step functions $b\chi_a$ of type $U \to \mathbf{C}(\mathbb{R}^n)$, the corresponding localic approximable mapping can be constructed as the supremum of single-step approximable mappings.

Our task then is now to consider the domain of approximable mappings of type $B_U \to B_{\mathbf{C}(\mathbb{R}^n)}$ and derive the corresponding properties. Consider the collection of Scott continuous functions of type $U \subset \mathbb{R}^n \to \mathbf{C}(\mathbb{R}^n)$ ordered with pointwise ordering induced by $\mathbf{C}(\mathbb{R}^n)$, i.e., superset pointwise ordering.

Proposition IV.1. *(i) For $f_1, f_2 : U \subset \mathbb{R}^n \to \mathbf{C}(\mathbb{R}^n)$ we have:*

$$f_1 \sqsubseteq f_2 \Longleftrightarrow A_{f_1} \subseteq A_{f_2}$$

(ii) For $R_1, R_2 : B_U \to B_{\mathbf{C}(\mathbb{R}^n)}$ for $U \subset \mathbb{R}^n$ we have:

$$R_1 \subseteq R_2 \Longleftrightarrow G_{R_1} \sqsubseteq G_{R_2}$$

Definition IV.2. *The partially ordered hom-set of localic approximable mappings of type $U \to \mathbf{C}(\mathbb{R}^n)$ is denoted by* $\mathsf{App}(B_U, B_{\mathbf{C}(\mathbb{R}^n)})$.

It is straightforward to check that $\mathsf{App}(B_U, B_{\mathbf{C}(\mathbb{R}^n)})$ is directed complete. The supremum of a directed set of approximable mappings with respect to subset inclusion is simply the union of the approximable mappings in the directed set: $\sup_{i\in I} R_i = \bigcup_{i\in I} R_i$.

We now consider

$$A : (U \to \mathbf{C}(\mathbb{R}^n)) \to \mathsf{App}(B_U, B_{\mathbf{C}(\mathbb{R}^n)})$$
$$G : \mathsf{App}(B_U, B_{\mathbf{C}(\mathbb{R}^n)}) \to (U \to \mathbf{C}(\mathbb{R}^n))$$

given as before by $A(f) = A_f$ and $G(R) = G_R$, as maps between the partial orders.

Corollary IV.3. *The function space $(U \to \mathbf{C}(\mathbb{R}^n))$ is isomorphic to $\mathsf{App}(B_U, B_{\mathbf{C}(\mathbb{R}^n)})$ via the maps A and G which are inverses of each other.*

Proposition IV.4. (i) *If f_i is a directed set in $(U \to \mathbf{C}(\mathbb{R}^n))$, with supremum $f = \sup_{i\in I} f_i$, then $\bigcup_{i\in I} A_{f_i} = A_f$ in $\mathsf{App}(B_U, B_{\mathbf{C}(\mathbb{R}^n)})$.*
(ii) *If R_i for $i \in I$, is a directed set in $\mathsf{App}(B_U, B_{\mathbf{C}(\mathbb{R}^n)})$, then $\sup_{i\in I} G_{R_i} = G_R$ in $(U \to \mathbf{C}(\mathbb{R}^n))$, where $R = \sup_{i\in I} R_i$.*

The function space $(U \to \mathbf{C}(\mathbb{R}^n))$ is a bounded complete domain. Consider the space $\mathsf{App}(B_U, B_{\mathbf{C}(\mathbb{R}^n)})$ of approximable mappings of type $U \to \mathbf{C}(\mathbb{R}^n)$ ordered by subset inclusion:

Corollary IV.5. *The space of approximable mappings $\mathsf{App}(B_U, B_{\mathbf{C}(\mathbb{R}^n)})$ is a bounded complete domain with*

$$R_1 \ll R_2 \text{ in } \mathsf{App}(B_U, B_{\mathbf{C}(\mathbb{R}^n)}) \text{ iff}$$
$$G_{R_1} \ll G_{R_2} \text{ in } (U \to \mathbf{C}(\mathbb{R}^n)).$$

The function space $(U \to \mathbf{C}(\mathbb{R}^n))$ has a basis consisting of step functions, each generated by a bounded finite set of single-step functions of the form $b\chi_a$ where $a \subset U$ is a basic convex open set and $b \in \mathbf{C}(\mathbb{R}^n)$ has non-empty interior. Consider a basic open set $c \subset U$ and a basic open set $\Box V \subset \mathbf{C}(\mathbb{R}^n)$ where $V \subset \mathbb{R}^n$ is itself a basic open set.

Proposition IV.6. *We have $c\, A_{b\chi_a}\, \Box V$ iff $c \ll a \,\&\, b \in \Box V$.*

Definition IV.7. *Given a pair (a, O), where $a \subset U$ is a basic open set and $O \subset \mathbb{R}^n$ is a basic open set, the* single-step approximable mapping $\eta_{(a,O)}$ *is defined as $\eta_{(a,O)} = A_{\overline{O}\chi_a}$.*

V. Constructors of approximable mappings

In this section we first formulate the product type in the category of semi-strong proximity lattices and approximable mappings and then present the constructors for addition and multiplication of approximable mappings, which are specific for $\mathsf{App}(B_U, B_{\mathbb{R}})$.

A. Product

Since **SL-Compact**, as a category of topological spaces, has products, we already know by the equivalence result that **Semi-Strong PL** has also product. Given semi-strong proximity lattices B_1 and B_2 the product $B_1 \times B_2$ is given by the Cartesian product of B_1 and B_2 generated by $a_1 \times a_2$ for $a_1 \in B_1$ and $a_2 \in B_2$ so that an element of $B_1 \times B_2$ is of the form $\bigvee_{i\in I}(a_{i1} \times a_{i2})$ for a finite set I. The semi-strong proximity lattice properties are generated componentwise, e.g., $a_1 \times a_2 \prec a_1' \times a_2'$ if $a_1 \prec a_1'$ and $a_2 \prec a_2'$. It is clear that this gives the product of two objects in **Semi-Strong PL** with $B_1 \times B_2 = A(X_1 \times X_2)$.

Definition V.1. *Given approximable mappings $R_1 : B \to B_1$ and $R_2 : B \to B_2$ their product is an approximable mapping of type $\langle R_1, R_2\rangle : B \to B_1 \times B_2$, where*

$$a\,(\langle R_1, R_2\rangle) \bigvee_{i\in I}(a_{i1} \times a_{i2})$$

if there exist $a_i \in B$ for $i \in I$, such that $a \prec \bigvee_{i\in I} a_i$ and $a_i\, R_1\, a_{i1}$ and $a_i\, R_2\, a_{i2}$.

We will prove that $\langle R_1, R_2\rangle$ actually gives the categorical product of R_1 and R_2. This can be done either directly by showing that $\langle R_1, R_2\rangle$ has the universal property of the product or by using Stone duality. We opt for the latter method. Let $f_1 : X \to X_1$, $f_2 : X \to X_2$ be continuous functions of stably locally compact spaces with $A(X) = B$, $A(X_1) = B_1$, $A(X_2) = B_2$, $A_{f_1} = R_1$ and $A_{f_2} = R_2$.

Proposition V.2.

$$A_{\langle f_1, f_2\rangle} = \langle A_{f_1}, A_{f_2}\rangle$$

It remains to define the projection approximable mappings. Let $P_i : B_1 \times B_2 \to B_i$ for $i = 1, 2$ be defined by $(a_1 \times a_2) P_i a$ if $a_i \prec a$ for $i = 1, 2$. It is easy to check that P_i is an approximable mapping for $i = 1, 2$. Let $\pi_i : X_1 \times X_2 \to X_i$, for $i = 1, 2$ be the projection maps, i.e, $\pi_i(x_1, x_2) = x_i$.

Proposition V.3. *We have $A_{\pi_i} = P_i$ for $i = 1, 2$.*

B. Addition

We now define addition as an approximable mapping $R^+ : B_{\mathbb{R}} \times B_{\mathbb{R}} \to B_{\mathbb{R}}$. The sum of two open intervals a_1 and a_2 is defined as $a_1 + a_2 := (a_1^- + a_2^-, a_1^+ + a_2^+)$. For $a \in B_{\mathbb{R}} \times B_{\mathbb{R}}$ and $O \in B_{\mathbb{R}}$, define the binary relation R^+ as follows: $a\, R^+\, O$ if there exist $a_{i1}, a_{i2} \in B^0_{\mathbb{R}}$, $i \in I$, with I a finite indexing set, such that $a \prec \bigvee_{i \in I} a_{i1} \times a_{i2}$ and $a_{i1} + a_{i2} \prec O$. Let $\mathsf{sum} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be the addition of two real numbers given by $\mathsf{sum}(x, y) = x + y$.

Proposition V.4. $A_{\mathsf{sum}} = R^+$.

We can now define sum of two approximable mappings. Let $R_1, R_2 : B_U \to B_{\mathbb{R}}$ be approximable mappings.

Definition V.5. *The* sum $R_1 + R_2 : B_U \to B_{\mathbb{R}}$ *is defined as* $R_1 + R_2 := R^+ \circ (\langle R_1, R_2 \rangle)$.

By Proposition V.4, Proposition V.2 and Theorem III.5, we obtain:

Corollary V.6. $R_1 + R_2 = A_{G_{R_1} + G_{R_2}}$.

C. Multiplication, constant and subtraction

The approximable mapping for multiplication $R^\times : B_{\mathbb{R}} \times B_{\mathbb{R}} \to B_{\mathbb{R}}$ is defined as follows. For $a \in B_{\mathbb{R}} \times B_{\mathbb{R}}$ and $O \in B_{\mathbb{R}}$, define the binary relation $R^\times$, we put: $a\, R^\times\, O$ if there exist $a_{i1}, a_{i2} \in B^0_{\mathbb{R}}$, $i \in I$, a finite indexing set, such that $a \prec \bigvee_{i \in I} a_{i1} \cdot a_{i2}$ and $a_{i1} \cdot a_{i2} \prec O$, where $a_1 \cdot a_2 = (m, M)$ is the product of the open intervals a_1 and a_2 with

$$m = \min\{a_1^- a_2^-, a_1^- a_2^+, a_1^+ a_2^-, a_1^+ a_2^+\}$$
$$M = \max\{a_1^- a_2^-, a_1^- a_2^+, a_1^+ a_2^-, a_1^+ a_2^+\}.$$

If $\mathsf{multip} : \mathbb{R}^2 \to \mathbb{R}$ is the multiplication map $\mathsf{multip}(x, y) = x \cdot y$, then similar to Proposition V.4, we obtain:

Proposition V.7. $A_{\mathsf{multip}} = R^\times$.

Definition V.8. *Given $R_1, R_2 : B_U \to B_{\mathbb{R}}$, their* multiplication $R_1 \cdot R_2 : B_U \to B_{\mathbb{R}}$ *is defined as* $R_1 \cdot R_2 := R^\times \circ (\langle R_1, R_2 \rangle)$.

By Proposition V.7, Proposition V.2 and Theorem III.5, we obtain:

Corollary V.9. $R_1 \cdot R_2 = A_{G_{R_1} \cdot G_{R_2}}$.

The constant approximable mapping $R^c : B_{\mathbb{R}} \to B_{\mathbb{R}}$ with constant value $c \in \mathbb{R}$ is defined as $a\, R^c\, O$ if $c \in O$. Let the constant function with value $c \in \mathbb{R}$ be given by $\mathsf{const_c} : \mathbb{R} \to \mathbb{R}$ with $x \mapsto c$ for all $x \in \mathbb{R}$. Then we have $A_{\mathsf{const_c}} = R^c$.

We will also need to define the constant approximable mapping of type $R^c_{\mathbf{I}\mathbb{R}} : B_U \to B_{\mathbf{I}\mathbb{R}}$, where $U \subset \mathbb{R}$ and $c \in \mathbb{R}$, by $a\, R^c_{\mathbf{I}\mathbb{R}}\, \square O$ iff $c \in O$.

Finally, multiplication with -1 is obtained as $R^{-1\times} = R^\times \circ (R^{-1} \times \mathrm{Id})$, from which we can construct subtraction by using addition.

VI. Lipschitzian approximable mappings

In this section, we introduce the notion of a Lipschitzian approximable mapping of type $B_U \to B_{\mathbb{R}}$, which corresponds to locally Lipschitz maps. First, we define a predicate to imply that two basic open sets are separated in the sense that their closures are disjoint.

Definition VI.1. *We say a pair of elements $a_1, a_2 \in B^0_U$, are* separated, *denoted as $(a_1, a_2) \in \mathsf{Sep}$, if there exist $a_1', a_2' \in B^0_U$ such that $a_1 \prec a_1'$, $a_2 \prec a_2'$ and $a_1' \wedge a_2' = 0$.*

For two points $x_1, x_2 \in U$, if $x_1 \neq x_2$, then, by the Hausdorff separation property, there are open sets O_1 and O_2 such that $x_1 \in O_1$, $x_2 \in O_2$ and $O_1 \cap O_2 = \emptyset$. Therefore, the notion of separated open sets is precisely the dual concept of Hausdorff separation for points.

We now define a notion of Lipschitzian constant for two approximable mappings, which in this localic form is quite similar to the definition of a set-valued Lipschitz constant for locally Lipschitz maps.

Definition VI.2. *We say an approximable mapping $R : B_U \to B_{\mathbb{R}}$ has* Lipschitzian constant *$O \in B^0_{\mathbb{R}^n}$ in $a \in B^0_U$, $a, O \neq \mathsf{0}$ and $O \neq \mathsf{1}$, denoted by $R \in \Delta(a, O)$, if*

$$\forall a_1 \forall a_2 \in B^0_U.\ a_1, a_2 \prec a\, \&\, (a_1, a_2) \in \mathsf{Sep} \ \Rightarrow$$
$$\exists a_1' \exists a_2' \in B^0_{\mathbb{R}}. a_1\, R\, a_1' \ \&\ a_2\, R\, a_2' \ \&\ a_1' - a_2' \prec O \cdot (a_1 - a_2).$$

We call $\Delta(a, O)$, as a family of approximable mappings, the knot *of (a, O).*

Observe that $\Delta(a, O)$ is only defined locally for basic open sets a and O, which is analogous to the definition of $\delta(a, b)$ referred to in Section II-B. Note also that, given the predicate Sep, the formula in the Definition VI.2 has quantifier rank four, i.e., it uses two instances of $\forall$ and two instances of $\exists$. In contrast, the formula in the Definition II.3 of a tie has quantifier two, i.e., it uses two instances of $\forall$. From the definition, we immediately obtain the following inclusion property.

Proposition VI.3. *If $a \subset a'$ and $O' \subset O$, then $\Delta(a', O') \subset \Delta(a, O)$.*

Our next task is to show that the point map of any Lipschitzian approximable mapping is Lipschitz.

Proposition VI.4. *If $R : B_U \to B_{\mathbb{R}}$ is an approximable mapping such that $R \in \Delta(a, O)$ then:*

$$\forall x, y \in a, G_R(x) - G_R(y) \in \overline{O} \cdot (x - y)$$

Corollary VI.5. *If $R \in \Delta(a, O)$ then $G_R \in \delta(a, \overline{O})$ and G_R is Lipschitz.*

We note that the compact and convex set $\overline{O}$ in Corollary VI.5 is tight in the following sense. Assuming that $R \in \Delta(a, O)$ there may be no compact and convex $b \subset O$ with $G_R \in \delta(a, b)$. The following example illustrates this point.

Example VI.6. *Let $f : (0,1) \to \mathbb{R}$ be the square function $x \mapsto x^2$. Then, $A_f \in \Delta((0,1),(0,2))$. But there is no compact interval $b \subset (0,2)$ for which $f \in \delta((0,1),b)$. In fact, $f \in \delta((0,1),[0,2])$ where $[0,2]$ is minimal, i.e., if $f \in \delta((0,1),b)$ then $[0,2] \subseteq b$.*

We now need several lemmas to deduce a type of converse for Corollary VI.5 with respect to single-knots of approximable mappings and single-ties of continuous function.

Lemma VI.7. *If O and a are bounded convex open sets in $\mathbb{R}^n$ with $0 \notin \overline{a}$, then $O \cdot a = O \cdot \overline{a}$.*

Lemma VI.8. *Suppose O, O' and a are open and convex subsets of $\mathbb{R}^n$ such that $\overline{O} \subset O'$ and suppose $0 \notin \overline{a}$ then $\overline{O \cdot a} \subset O' \cdot a$.*

Lemma VI.9. *If a_1, a_2 are bounded open subsets of $\mathbb{R}^n$ and $\overline{a_1} \cap \overline{a_2} = \emptyset$ then $0 \notin \overline{a_1 - a_2}$.*

We can finally deduce a type of converse for Corollary VI.5.

Theorem VI.10. *If $f \in \delta(a,b)$ then for all open convex subsets $a_0 \in B^0_{\mathbb{R}^n}$ with $\overline{a}_0 \subset a$ and for all open convex sets $O \in B^0_{\mathbb{R}^n}$ with $b \subset O$ we have $A_f \in \Delta(a_0, O)$.*

The following example shows that Theorem VI.10 fails if we replace the separation condition $(a_1, a_2) \in \mathsf{Sep}$ with the weaker condition $a_1 \cap a_2 = \emptyset$.

Example VI.11. *Let $f : (-1,4) \to \mathbb{R}$ be the identity map. Then $f \in \delta((-1,3),\{1\})$. Take any open set $O = (1-r, 1+s)$, with $r, s > 0$ and $r < 1$, that contains $b = \{1\}$. If $a_1 = (1,3)$ and $a_2 = (0,1)$, then $a_1 \, A_f \, a_1'$ and $a_2 \, A_f \, a_2'$ imply that $a_1' - a_2' = (-t, 3+u)$ for some $t, u > 0$. On the other hand, $a_1 - a_2 = \underline{(0,3)}$ and $O \cdot (a_1 - a_2) = (0, 3(1+s))$. Thus, we cannot have $\overline{a_1' - a_2'} \subset O \cdot (a_1 - a_2)$.*

Example VI.6 also shows that the condition $\overline{a}_0 \subset a$ in the statement of Theorem VI.10 cannot be replaced with $a_0 \subset a$.

Example VI.12. *Let $f : \mathbb{R} \to \mathbb{R}$ be the absolute value function $f(x) = |x|$. If $0 \in a$, then $A_f \in \Delta(a,O)$ iff $[-1,1] \subset O$. If $a^+ \leq 0$, then $A_f \in \Delta(a,O)$ iff $-1 \in O$. And if $0 \leq a^-$, then $A_f \in \Delta(a,O)$ iff $1 \in O$.*

We have established a close connection between $\delta(a,b)$ and $\Delta(a,O)$ in Corollary VI.5 and Theorem VI.10. However, as we can see from the statements of these two results the connection is not completely symmetric: The statement of Theorem VI.10 requires $a_0 \ll a$ whereas Corollary VI.5 has no such condition. Moreover, the conclusions of the two statements are not really symmetric. We strive to redress this anomaly in the next section in order to obtain a duality between the single-ties and the single-knots.

VII. Gradient Stone Duality

In this section, we will sharpen the close connection between the differential properties of Lipschitz maps and Lipschitzian approximable mappings and establish a Stone duality for differentiation by introducing a stronger notion of single-ties and a stronger notion of single-knots.

Let us digress to re-examine the property of single-ties as presented in Section II-B. Recall that the notion of a single-tie $\delta(a,b)$ was used to define the L-derivative of a Lipschitz function, which can be written as

$$b\chi_a \sqsubseteq \mathcal{L}f \iff f \in \delta(a,b). \quad (4)$$

We see that the single-tie $\delta(a,b)$ induces a single-step function $b\chi_a \sqsubseteq \mathcal{L}f$, where $\sqsubseteq$ is the partial order provided by pointwise ordering in the function space of type: $U \to \mathbf{C}(\mathbb{R}^n)$. While the use of these single-ties and the induced single-step functions with the partial order on the function space $(U \to \mathbf{C}(\mathbb{R}^n))$ has enabled us to develop a domain-theoretic account of differential calculus, it falls short of allowing us to develop Stone duality in a symmetric fashion.

The solution lies in using what we call a strong single-tie so that the partial order $\sqsubseteq$ in Relation (4) is replaced with the way-below relation $\ll$ of the function space $(U \to \mathbf{C}(\mathbb{R}^n))$. Let a non-empty convex open set $a \subset U$ and a compact convex set $b \in \mathbf{C}(\mathbb{R}^n)$ with non-empty interior be given.

Definition VII.1. *We say $f : U \subset \mathbb{R}^n \to \mathbb{R}$ has a strong set-valued Lipschitz constant b in a, denoted by $f \in \delta_s(a,b)$, if there exist a' with $a \ll_{\Omega(U)} a'$ and $b' \in \mathbf{C}(\mathbb{R}^n)$ with $b \ll_{\mathbf{C}(\mathbb{R}^n)} b'$ such that $f \in \delta(a',b')$. We call $\delta_s(a,b)$ the* strong *single-tie of a with b.*

Observe that the formula used in the definition of a strong single-tie has now quantifier rank four in total, since there are two instances of $\exists$ in addition to the two instances of $\forall$ in Definition II.3 of a single-tie. The following properties are immediate from the definition of a strong single-tie and the interpolation property of $\ll$ in ΩU and $\mathbf{C}(\mathbb{R}^n)$:

Proposition VII.2. (i) *If $a \subset a'$ and $b \sqsubseteq b'$ then $\delta_s(a',b') \subset \delta_s(a,b)$.*
(ii) *$\delta_s(a,b) \subset \delta(a,b)$.*
(iii) *If $f \in \delta_s(a,b)$ then there exist a' with $a \ll_{\Omega U} a'$ and $b' \in \mathbf{C}(\mathbb{R}^n)$ with $b \ll b'$ such that $f \in \delta_s(a',b')$.*

Proposition VII.2(ii) justifies the word strong in the definition of a strong tie. In addition, we have:

Proposition VII.3. *If $f : U \subset \mathbb{R}^n \to \mathbb{R}$, then:*

$$f \in \delta_s(a,b) \iff b\chi_a \ll \mathcal{L}f$$

Example VII.4. *Let $f : \mathbb{R} \to \mathbb{R}$ be the absolute value function $f(x) = |x|$. If $0 \in \overline{a}$ then $f \in \delta_s(a,b)$ iff $[-1,1] \subseteq b^\circ$. If $0 < a^-$ then $f \in \delta_s(a,b)$ iff $1 \in b^\circ$. Finally, if $a^+ < 0$ then $f \in \delta_s(a,b)$ iff $-1 \in b^\circ$.*

Corollary VII.5. *If $f : U \subset \mathbb{R}^n \to \mathbb{R}$ be locally Lipschitz, then we have:*

$$\mathcal{L}f = \sup\{b\chi_a | b\chi_a \ll \mathcal{L}f\} = \sup\{b\chi_a | f \in \delta_s(a,b)\}$$

Next, we define a stronger notion of single-knots to match with that of strong single-ties.

Definition VII.6. *We say an approximable mapping $R : B_U \to B_{\mathbb{R}}$ has* strong Lipschitzian constant *$O \in B^0_{\mathbb{R}^n}$ in*

$a \in B_U^0$, denoted by $R \in \Delta_s(a, O)$, if there exist $a' \in B_U^0$ with $a \prec a'$ and $O' \in B_{\mathbb{R}^n}^0$ with $O' \prec O$ such that $R \in \Delta(a', O')$.

The formula in Definition VII.6 of a strong single-knot has quantifier rank six in total since there are two instances of $\exists$ in addition to the four quantifiers in Definition VI.2 of a single-knot.

Proposition VII.7. (i) *If we have $a \subset a'$ and $O' \subset O$, then $\Delta_s(a', O') \subset \Delta_s(a, O)$.*
(ii) *$\Delta_s(a, O) \subseteq \Delta(a, O)$.*
(iii) *If $f \in \Delta_s(a, O)$, then there exist a' with $a \prec a'$ and O' with $O' \prec O$ such that $R \in \Delta_s(a', O')$.*

The inclusion in the Proposition VII.7(iii) is strict. This can be seen from Example VI.6 again. Let $f(x) = x^2$ then $A_f \in \Delta(a, O)$ for $a = (0, 1)$ and $O = (0, 2)$. But if $a' > a$ and $O' < O$ then there exists $x \in a'$ such that $x > 1$. Therefore:

$$\frac{f(x) - f(1)}{x - 1} \notin O$$

and thus $\frac{f(x)-f(1)}{x-1} \notin O' \subset O$ which implies $A_f \notin \Delta(a', O')$. We conclude that $A_f \notin \Delta_s(a, O)$.

We can now improve the results in the previous section.

Theorem VII.8. (i) *If $f \in \delta_s(a, b)$ then for all basic convex open sets O with $b \subset O$ we have $A_f \in \Delta_s(a, O)$.*
(ii) *If $A_f \in \Delta_s(a, O)$, then there exists a convex and compact subset $b \subset O$ such that $f \in \delta_s(a, b)$.*

We can finally make the correspondence in Theorem VII.8 sharper and obtain the duality result we have aimed at. Let $O \subset U$ be a basic convex open set.

Corollary VII.9. *We have $R \in \Delta_s(a, O)$ iff $G_R \in \delta_s(a, \overline{O})$. Dually, we have $f \in \delta_s(a, b)$ iff $A_f \in \Delta_s(a, b^\circ)$.*

Example VII.10. *Let $f : \mathbb{R} \to \mathbb{R}$ be the absolute value function $f(x) = |x|$. If $0 \in \overline{a}$, then $A_f \in \Delta_s(a, O)$ iff $[-1, 1] \subset O$. If $a^+ < 0$, then $A_f \in \Delta(a, O)$ iff $-1 \in O$. And if $0 < a^-$, then $A_f \in \Delta(a, O)$ iff $1 \in O$.*

Having found the proper duality between strong single-ties and strong single-knots, we can define the dual notion of the L-derivative for Lipschitzian approximable mappings by dualising the definition of the L-derivative for Lipschitz maps as in Corollary VII.5. Recall the definition of $\eta_{(a,O)}$ from Definition IV.7.

Definition VII.11. *Let $R \in \mathsf{App}(B_U, B_{\mathbb{R}})$ be a Lipschitzian approximable mapping. The* Lipschitzian derivative *of R is defined as*

$$\mathsf{L}(R) = \sup\{\eta_{(a,O)} : R \in \Delta_s(a, O)\}$$

We need to ensure that $\mathsf{L}(R)$ is well-defined by checking that the supremum in Definition VII.11 is bounded so that $\mathsf{L}(R)$ is indeed an approximable mapping in $\mathsf{App}(B_U, B_{\mathbf{C}(\mathbb{R}^n)})$. By now, we have developed all the tools required to show, using Stone duality, that in fact $\mathsf{L}(R)$ is the approximable mapping for the point map $\mathcal{L}G_R$.

Theorem VII.12. *The Lipschitzian derivative of a Lipschitzian approximable mapping is an approximable mapping and we have: $\mathsf{L}(R) = A_{\mathcal{L}G_R}$.*

The Lipschitzian derivative of the constant approximable map is the constant approximable map with value zero as expected, i.e., $\mathsf{L}(R^c) = R^0_{\mathbf{I}\mathbb{R}}$.

VIII. Continuously differentiable functions

In this section, we formulate a duality theory for the class $C^1(U)$ of continuously differentiable functions $f : U \subset \mathbb{R}^n \to \mathbb{R}$. Any function in $C^1(U)$ is locally Lipschitz and is in fact Lipschitz on any compact subset of U. We therefore need a stricter notion of a strong single-knot in order to represent $C^1(U)$ functions by approximable mappings. We will use a basic property of the Clarke subgradient, equivalently the L-derivative, as in Proposition II.2(iv). Thus, our aim will be to formulate an axiom such that the L-derivative of $f(x)$ everywhere in U is a singleton, equivalently, that it has diameter zero.

Definition VIII.1. *Given a map $f : U \to \mathbb{R}$, we say $f \in \delta^1(U)$ if for all $a_0 \ll U$ and all $\epsilon > 0$ there exists $\delta > 0$ such that whenever $a \ll a_0$ and diam$(a) < \delta$, there exists a basic convex compact set b with diam$(b) < \epsilon$ and $f \in \delta_s(a, b)$.*

For a computable theory, we can as usual restrict ϵ and δ to be rational numbers.

Proposition VIII.2. *$f \in \delta^1(U)$ iff $f \in C^1(U)$.*

It is now straightforward to formulate the dual notion corresponding to $\delta^1(U)$ for approximable mappings.

Definition VIII.3. *We say an approximable mapping $R \in \mathsf{App}(B_U, B_{\mathbf{C}(\mathbb{R}^n)})$ is* differentiable *in U, denoted $R \in \Delta^1(U)$, if for all $a_0 \prec 1$ and all $\epsilon > 0$ there exists $\delta > 0$ such that whenever $a \prec a_0$ and diam$(a) < \delta$, there exists a basic convex open set $O \in B^0_{\mathbb{R}^n}$ with diam$(O) < \epsilon$ and $R \in \Delta_s(a, O)$.*

We can now show that $\Delta^1(U)$ is dual to $\delta^1(U)$:

Proposition VIII.4. *We have $f \in \delta^1(U)$ iff $A_f \in \Delta^1(U)$.*

Corollary VIII.5. *$R \in \Delta^1(U)$ iff $G_R \in C^1(U)$.*

IX. Lipschitzian derivative of constructors

In this section, we derive a calculus for the Lipschitzian derivative of the basic constructors we have defined in Section V and show that the calculus is dual to the calculus of the Clarke subgradient as in Proposition II.2(vi-viii).

Proposition IX.1. *(**Sum**) Let $R_1, R_2 : B_U \to B_{\mathbb{R}}$, where $U \subseteq \mathbb{R}^n$ are approximable mappings. Then:*

$$\mathsf{L}(R_1) + \mathsf{L}(R_2) \subseteq \mathsf{L}(R_1 + R_2)$$

and equality holds if at least one of R_1 or R_2 is in $\Delta^1(U)$.

A similar result holds for the product as follows.

Proposition IX.2. *(**Product**) Let $R_1, R_2 : B_U \to B_{\mathbb{R}}$ be approximable mappings. Then:*

$$R_1 \cdot \mathsf{L}(R_2) + R_2 \cdot \mathsf{L}(R_1) \subseteq \mathsf{L}(R_1 \cdot R_2)$$

and equality holds if one of R_1 or R_2 is in $\Delta^1(U)$.

Proposition IX.3. *(**Chain rule**) Let $R_1 : B_{U_1} \to B_{\mathbb{R}}$ and $R_2 : B_{U_2} \to B_{\mathbb{R}}$ be approximable mappings, where $U_1 \subseteq \mathbb{R}^n$, $U_2 \subseteq \mathbb{R}$ and Im$(R_1) \subseteq B_{U_2}$. Then:*

$$(\mathsf{L}(R_2) \circ R_1) \cdot \mathsf{L}(R_1) \subseteq \mathsf{L}(R_2 \circ R_1)$$

and the equality holds if at least one of the following predicates holds: $R_1 \in \Delta(U_1)$ or $R_2 \in \Delta(U_2)$. Here by Im(R_1) we mean the image of R_1.

X. Conclusion and further work

We have developed a localic representation of the Clarke subgradient of locally Lipschitz maps and a localic representation of the classical gradient of continuously differentiable maps. There are clear similarities between the way strong single-ties and strong single-knots construct the subgradient in the spatial theory and in the localic theory respectively. If the quantifier rank of formulas is considered as a measure of complexity of definitions, then for the subgradient this rank has increased from four, with respect to strong single-ties for the spatial theory, to six, with respect to strong single-knots for the localic theory. The advantage of using the locale theory is clearly marked out when we use $\mathcal{D}^r$ which generates countable semi-strong proximity lattices. We envisage that the localic representation would then allow an implementation of the Clarke subgradient in theorem provers.

We have worked out all our proofs using topological spaces and continuous maps as we often had to deal with open sets and compact sets at the same time in a proof. What will happen if, instead of using semi-strong proximity lattices and approximable mappings, we actually use strong proximity lattices and sequent calculi as in the subsequent work of Jung and his collaborators [21] in which both compact sets as well as open sets are integrated in the representation of the stably locally compact space by the strong proximity lattice? Would we be able to carry out most of our proofs in strong proximity lattices without going to the stably locally compact spaces represented by them?

There are a number of themes for further work:

1) extending the set of constructors to include division, $\min$, $\max$ and basic logical predicates,
2) extension to maps with imprecise input represented as hyper-rectangles in $\mathbb{R}^n$ and imprecise output, i.e., elements in $\mathbf{I}\mathbb{R}$.
3) extension to locally Lipschitz vector maps of type $\mathbb{R}^n \to \mathbb{R}^m$ to develop a localic representation of the generalised Jacobian [4],
4) extension to locally Lipschitz complex maps to develop a localic representation of analytic maps using the L-derivative for complex maps [10], and,
5) a localic representation of the initial value problem for solving ODE's.

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Appendix

Proposition I.2
The three maps

(i) $(-)+(-):\Omega(\mathbb{R}^n)\times\Omega(\mathbb{R}^n)\to\Omega(\mathbb{R}^n)$,
(ii) $(-)\cdot(-):\Omega(\mathbb{R}^n)\times\Omega(\mathbb{R}^n)\to\Omega(\mathbb{R})$ *,and,*
(iii) $r\cdot(-):\Omega(\mathbb{R}^n)\to\Omega(\mathbb{R}^n)$.

are well-defined, Scott continuous and preserve convex sets.

Proof: (i) Note that for any vector $x\in\mathbb{R}^n$ the set $x+B$, the translation of B by x is open for any non-empty open set B. Therefore, for any non-empty set A and any non-empty open set $B\subset\mathbb{R}^n$, the set $A+B=\bigcup_{x\in A}(x+B)$ is the union of open sets and is thus open, and in particular it is open for A open. Suppose A and B are both convex as well as non-empty and open. If $x=x_1+x_2\in A+B$ and $y=y_1+y_2\in A+B$ with $x_1,y_1\in A$ and $x_2,y_2\in B$, then $cx+(1-c)y=c(x_1+x_2)+(1-c)(y_1+y_2)=(cx_1+(1-c)y_1)+(cx_2+(1-c)y_2)\in A+B$ since $cx_1+(1-c)y_1\in A$ and $cx_2+(1-c)y_2\in B$ as a consequence of convexity of A and B. To show Scott continuity, it is sufficient to prove Scott continuity separately for the two input open sets. Let $u\in A+\bigcup_{i\in I}B_i$ where B_i is a directed family of open sets in $\sigma_{\Omega\mathbb{R}^n}$. Then $u=x+y$ with $x\in A$ and $y\in\bigcup_{i\in I}B_i$. Then $y\in B_i$ for some $i\in I$ and thus $u\in\bigcup_{i\in I}(A+B_i)$. Hence, $A+\bigcup_{i\in I}B_i\subset\bigcup_{i\in I}(A+B_i)$. Since $(-)+(-)$ is clearly monotone, it follows that $A+\bigcup_{i\in I}B_i\supset\bigcup_{i\in I}(A+B_i)$. Thus, $A+\bigcup_{i\in I}B_i=\bigcup_{i\in I}(A+B_i)$. By symmetry $(-)+(-)$ is Scott continuous with respect to the first argument as well when the second argument is fixed.

(ii) The inner product map $(-)\cdot(-):\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}$ is continuous as it can be written as $(x,y)\mapsto\sum_{i=1}^n\pi_i(x)\pi_i(y)$ where the projections map $\pi_i:\mathbb{R}^n\to\mathbb{R}$ with $(x_1,\cdots,x_n)\mapsto x_i$ is continuous for $1\leq i\leq n$. Since a continuous map sends connected sets to connected sets, it follows that for nonempty and convex $A,B\in\Omega(\mathbb{R}^n)$, the image $A\cdot B\subset\mathbb{R}$ is connected and hence is an interval. To see that it is in fact an open interval when A and B are convex, consider $x\in A$ and $y\in B$. Let $r>0$ be small such that $O_r(x)\subset A$ and $O_r(y)\subset B$, where $O_r(u)$ is the open ball with radius r centred at $u\in\mathbb{R}^n$. Then, $x\cdot y\in O_r(x)\cdot O_r(y)\subset A\cdot B$ and $O_r(x)\cdot O_r(y)$ is an open interval. We conclude that $A\cdot B$ is open.

(iii) Straightforward.

Proposition III.1 *The domain $\mathbf{C}(\mathbb{R}^n)$ is a stably locally compact space as is any open set $U\subset\mathbb{R}^n$ with its subspace topology.*

Proof: We check that the lattice of open sets of U and $\mathbf{C}(\mathbb{R}^n)$ are both arithmetic. If $O\ll O_1,O_2$ for $O,O_1,O_2\subset U$ then $\overline{O}\subset O_1,O_2$ and hence $\overline{O}\subset O_1\cap O_2$, which implies $O\ll O_1\cap O_2$. For $\mathbf{C}(\mathbb{R}^n)$ we use basic open sets of the form $\Box O$ to verify the arithmetic property. Suppose $\Box O\ll\Box O_1,\Box O_2$. Then $O\ll_{\Omega\mathbb{R}^n}O_1,O_2$ and hence $O\ll_{\Omega\mathbb{R}^n}O_1\cap O_2$, by the first part. Thus, $\Box O\ll\Box O_1\cap O_2=\Box O_1\cap\Box O_2$.

■

Proposition IV.1

(i) For $f_1,f_2:U\subset\mathbb{R}^n\to\mathbf{C}(\mathbb{R}^n)$ we have:

$$f_1\sqsubseteq f_2\Longleftrightarrow A_{f_1}\subseteq A_{f_2}$$

(ii) For $R_1,R_2:B_U\to B_{\mathbf{C}(\mathbb{R}^n)}$ for $U\subset\mathbb{R}^n$ we have:

$$R_1\subseteq R_2\Longleftrightarrow G_{R_1}\sqsubseteq G_{R_2}$$

Proof:

(i) Suppose $f_1\sqsubseteq f_2$ and let $a\,A_{f_1}\,O$. Then $a\ll f_1^{-1}(O)$. Since any Scott open set $O\subset\mathbf{C}(\mathbb{R}^n)$ is upwards closed, it follows that $f_1^{-1}(O)\subset f_2^{-1}(O)$ and thus $a\ll f_1^{-1}(O)\subset f_2^{-1}(O)$, i.e., $a\,A_{f_2}\,O$.
Next, let $A_{f_1}\subseteq A_{f_2}$. Since $a\,A_f\,O$ iff $a\ll f_1^{-1}(O)$ it follows that

$$G_{A_{f_1}}^{-1}(O)=\bigcup_{a\,A_{f_1}\,O}a\subset\bigcup_{a\,A_{f_2}\,O}a=G_{A_{f_2}}^{-1}(O).$$

Hence, for any Scott open set $O\subset\mathbf{C}(\mathbb{R}^n)$, we have $G_{A_{f_1}}^{-1}(O)\subset G_{A_{f_2}}^{-1}(O)$ and therefore $f_1(x)\sqsubseteq f_2(x)$.

(ii) This follows from the previous part as a result of the equivalence of the two categories.

■

Corollary IV.3

(i) *If f_i is a directed set in $(U\to\mathbf{C}(\mathbb{R}^n))$, with supremum $f=\sup_{i\in I}f_i$, then $\bigcup_{i\in I}A_{f_i}=A_f$ in* $\mathsf{App}(B_U,B_{\mathbf{C}(\mathbb{R}^n)})$.

(ii) *If R_i for $i\in I$, is a directed set in $\mathsf{App}(B_U,B_{\mathbf{C}(\mathbb{R}^n)})$, then $\sup_{i\in I}G_{R_i}=G_R$ in $(U\to\mathbf{C}(\mathbb{R}^n))$, where $R=\sup_{i\in I}R_i$.*

Proof: (i) If $a\bigcup_{i\in I}A_{f_i}\,O$, then there exists $i\in I$ such that $a\,A_{f_i}\,O$ which implies $a\ll f_i^{-1}(O)\subset f^{-1}(O)$ since $f_i\sqsubseteq f$. Thus, $a\,A_f\,O$. Conversely, if $a\,A_f\,O$, then there exists a' with $a\ll a'\ll f^{-1}(O)$. Since $f=\sup_{i\in I}f_i$ we have for their frame maps $f^{-1}=\sup_{i\in I}f_i^{-1}$. In particular, $f^{-1}(O)=\bigcup_{i\in I}f_i^{-1}(O)$ and thus there exists $i\in I$ such that $a\ll a'\subset f_i^{-1}(O)$, which implies $a\,A_{f_i}\,O$.

(ii) We already know by monotonicity that $\sup_{i\in I}G_{R_i}\sqsubseteq G_R$. For the reverse, note that for basic open sets a and O we have: $a\ll G_R^{-1}(O)$ iff $a\bigcup_{i\in I}R_i\,O$ iff $\exists i\in I.a\,R_i\,O$ iff $a\ll G_{R_i}^{-1}(O)$. Thus, $G_R^{-1}(O)\subset\sup_{i\in I}G_{R_i}^{-1}(O)$ and, hence, $G_R\sqsubseteq\sup_{i\in I}G_{R_i}$.

■

Proposition IV.6
We have $c\,A_{b\chi_a}\,\Box V$ iff $c\ll a\,\&\,b\in\Box V$.

Proof: By the definition of an approximable mapping and the definition of $b\chi_a$, we obtain: $c\,A_{b\chi_a}\,\Box V$ iff $c\ll(b\chi_a)^{-1}(\Box V)$ iff $c\ll a\,\&\,b\in\Box V$.

■

Proposition V.2

$$A_{\langle f_1,f_2\rangle}=\langle A_{f_1},A_{f_2}\rangle$$

Proof: Suppose $aA_{\langle f_1,f_2\rangle}(\bigvee_{i\in I}a_{i1}\times a_{i2})$. By M-1, let a' be a basic open set with $a\prec a'A_{\langle f_1,f_2\rangle}(\bigvee_{i\in I}a_{i1}\times a_{i2})$. i.e.,

$$a'\ll\bigcup_{i\in I}((\langle f_1,f_2\rangle)^{-1}(a_{i1}\times a_{i2})=\bigcup_{i\in I}f_1^{-1}(a_{i1})\cap f_2^{-1}(a_{i2}).$$

Let $x\in\overline{a}$. Then, there exists $i\in I$ with $x\in(f_1^{-1}(a_{i1})\cap f_2^{-1}(a_{i2}))$. Let $O_x\in B$ be an open set containing x such that

$O_x \ll a', O_x \ll (f_1^{-1}(a_{i1}) \cap f_2^{-1}(a_{i2}))$. By compactness of $\overline{a}$, there exists a finite set of points $\{x_j : j \in J\}$ with $x_j \in \overline{a}$ such that $\overline{a} \subset \bigcup_{j\in J} O_{x_j} \ll a'$. Let

$$a_i = \bigcup \{O_{x_j} : j \in J, O_{x_j} \ll f_1^{-1}(a_{i1}) \cap f_2^{-1}(a_{i2})\}.$$

Note that the union above can be empty which means that a_i is the empty set. Then $a_i \in B$ for each $i \in I$ with $a \ll \bigcup_{i\in I} a_i$ and $a_i \ll f_1^{-1}(a_{i1}) \cap f_2^{-1}(a_{i2})$, i.e., $a_i A_{f_1} a_{i1}$ and $a_i A_{f_2} a_{i2}$. Thus, $a(\langle A_{f_1}, A_{f_2}\rangle) \bigvee_{i\in I} a_{i1} \times a_{i2}$.

Conversely, suppose $a(\langle A_{f_1}, A_{f_2}\rangle)(\bigvee_{i\in I} a_{i1} \times a_{i2})$. Then, by definition, there exists a_i for each $i \in I$ with $a \prec \bigvee_{i\in I} a_i$ such that $a_i A_{f_1} a_{i1}$ and $a_i A_{f_2} a_{i2}$ for each $i \in I$. Thus, $a_i \ll f_1^{-1}(a_{i1})$ and $a_i \ll f_2^{-1}(a_{i2})$. Hence $a_i \ll f_1^{-1}(a_{i1}) \cap f_2^{-1}(a_{i2})$, i.e., $a_i A_{\langle f_1, f_2\rangle}(a_{i1} \times a_{i2})$ for each $i \in I$. It follows that $aA_{\langle f_1, f_2\rangle}(\bigvee_{i\in I} a_{i1} \times a_{i2})$.

■

Proposition V.3

We have $A_{\pi_i} = P_i$ for $i = 1, 2$.

Proof: Let $(a_1 \times a_2)A_{\pi_1} a$ then $a_1 \times a_2 \ll (\pi_1)^{-1}(a)$. But $(\pi_1)^{-1}(a) = a \times 1$. Thus, $a_1 \times a_2 \ll a \times 1$ which implies $a_1 \ll a$. Hence, $(a_1 \times a_2)P_1 a$. Next suppose $a_1 \times a_2 P_1 a$. Then $a_1 \ll a$ and hence $a_1 \times a_2 \ll a \times 1$. This means $(a_1 \times a_2)A_{\pi_1} a$. The same holds for A_{π_2}. ■

Proposition V.4

$A_{\mathsf{sum}} = R^+$.

Proof: Let $aA_{\mathsf{sum}}O$. Then, we have:

$$a \ll (\mathsf{sum})^{-1}(O) = \bigcup \{a_{i1} \times a_{i2} : a_{i1} + a_{i2} \ll O, a_{i1}, a_{i2} \in B^0_{\mathbb{R}}\}$$

By compactness of $\overline{a}$, there is a finite indexing set I such that:

$$a \ll \bigcup \{a_{i_1} \times a_{i2} : a_{i_1} + a_{i2} \ll O, a_{i1}, a_{i2} \in B^0_{\mathbb{R}}, i \in I\}$$

i.e, aR^+O. Next, suppose aR^+O. Thus, there exist $a_{i1}, a_{i2} \in B^0_{\mathbb{R}}$, for $i \in I$, with I a finite indexing set, such that $a \prec \bigvee_{i\in I} a_{i1} \times a_{i2}$ and $a_{i1} + a_{i2} \prec O$. Therefore:

$$a \ll \bigvee_{i\in I} a_{i1} \times a_{i2} \ll (\mathsf{sum})^{-1}(O)$$

Hence, $aA_{\mathsf{sum}}O$. ■

■

Lemma VI.7

If O and a are bounded convex open sets in $\mathbb{R}^n$ with $0 \notin \overline{a}$, then $O \cdot a = O \cdot \overline{a}$.

Proof: Note that for any non-zero vector $v \in \mathbb{R}^n$, the set $O \cdot v \subset \mathbb{R}$ is an open interval (which is in fact the projection of the open set O in the direction of v scaled by $\|v\|$). It follows that the convex set $O \cdot \overline{a} = \bigcup_{v\in\overline{a}} O \cdot v$ is an open interval. We have $\sup(O \cdot \overline{a}) = S_O(x)$ for some $x \in \overline{a}$. Let $x_n \in O$ $(n \geq 0)$ be a sequence with $x = \lim_{n\to\infty} x_n$. Then $S_O(x_n) \leq \sup(O.a)$ and $S_O(x) = \lim_{n\to\infty} S_O(x_n)$ since the support function S_O is continuous. It follows that $S_O(x) \leq \sup(O.a)$ and hence $\sup(O \cdot \overline{a}) \leq \sup(O \cdot a)$, which means $\sup(O \cdot \overline{a}) = \sup(O \cdot a)$, as $O \cdot a \subset O \cdot \overline{a}$. Similarly, using $\inf O \cdot v = -\sup O \cdot (-v)$, we obtain $\inf(O \cdot \overline{a}) = \inf(O \cdot a)$ and therefore $O \cdot a$ and $O \cdot \overline{a}$ are the same open intervals.

■

Lemma VI.8

Suppose O, O' and a are open and convex subsets of $\mathbb{R}^n$ such that $\overline{O} \subset O'$ and suppose $0 \notin \overline{a}$ then $\overline{O \cdot a} \subset O' \cdot a$.

Proof: We know from lemma VI.7 that $O \cdot a = O \cdot \overline{a}$ and thus there exists $x \in \overline{a}$ such that $\sup(O \cdot a) = \sup(O) \cdot x = S_O(x)$. But $S_O(x) < S_{O'}(x) \leq \sup(O' \cdot a)$, because $\overline{O} \subset O'$. Thus, $\sup(O \cdot a) < \sup(O' \cdot a)$. Similarly, $\inf(O \cdot a) > \inf(O' \cdot a)$, and we conclude that $O \cdot a < O' \cdot a$.

■

Lemma VI.9

If a_1, a_2 are bounded open subsets of $\mathbb{R}^n$ and $\overline{a_1} \cap \overline{a_2} = \emptyset$ then $0 \notin \overline{a_1 - a_2}$.

Proof: Assume for the sake of a contradiction that $0 \in \overline{a_1 - a_2}$. Then there exists a sequence $x_n \in a_1 - a_2$ such that $x_n \to 0$. This implies that there exist sequences $s_n \in a_1$ and $t_n \in a_2$ such that $x_n = s_n - t_n$. Because $\overline{a_1}$ and $\overline{a_2}$ are compact sets, there exist convergent subsequences s_{n_k} and t_{n_k} with limits $s \in \overline{a_1}$ and $t \in \overline{a_2}$ respectively. But then $s - t = \lim_{k\to\infty} s_{n_k} - t_{n_k} = \lim_{k\to\infty} x_{n_k} = 0$, which is a contradiction since $\overline{a_1} \cap \overline{a_2} = \emptyset$. ■

Theorem VI.10

If $f \in \delta(a, b)$ then for all open convex subsets $a_0 \in B^0_{\mathbb{R}^n}$ with $\overline{a}_0 \subset a$ and for all open convex sets $O \in B^0_{\mathbb{R}^n}$ with $b \subset O$ we have $A_f \in \Delta(a_0, O)$.

Proof: Suppose $\overline{a}_1, \overline{a}_2 \in B^0_{\mathbb{R}^n}$ with $\overline{a}_1, \overline{a}_2 \subset a_0$ and $(a_1, a_2) \in \mathsf{Sep}$. From $f \in \delta(a, b)$, we obtain:

$$f[\overline{a_1}] - f[\overline{a_2}] \subset b \cdot (\overline{a_1} - \overline{a_2}).$$

Since $\overline{a_1} - \overline{a_2} = \overline{a_1 - a_2}$ and $b \subset O$, we deduce:

$$f[\overline{a_1}] - f[\overline{a_2}] \subset O \cdot (\overline{a_1 - a_2}).$$

By Lemma VI.9, $0 \notin \overline{a_1 - a_2}$, and, hence, by Lemma VI.7, we have: $O \cdot (\overline{a_1 - a_2}) = O \cdot (a_1 - a_2)$. It follows that

$$f[\overline{a_1}] - f[\overline{a_2}] \subset O \cdot (a_1 - a_2).$$

Since f is continuous, the set $f[\overline{a_1}] - f[\overline{a_2}]$ is compact and convex. Thus, we can find open convex sets a'_1, a'_2 such that $f[\overline{a_i}] \subset a'_i$ for $i = 1, 2$, and $\overline{a'_1 - a'_2} \subset O \cdot (a_1 - a_2)$. Hence $A_f \in \Delta(a_0, O)$. ■

Proposition VII.3

If $f : U \subset \mathbb{R}^n \to \mathbb{R}$, then:

$$f \in \delta_s(a, b) \iff b\chi_a \ll \mathcal{L}f$$

Proof: Suppose $b\chi_a \ll \mathcal{L}f$. Then $a \ll (\mathcal{L}f)^{-1}(\Uparrow b)$ which in particular implies that $\Uparrow b \neq \emptyset$, i.e., $b^\circ \neq \emptyset$. Take an open convex set $a' \subset U$ with $a \ll a' \ll (\mathcal{L}f)^{-1}(\Uparrow b)$. Let $D_\delta(x)$ denote the closed disk of radius $\delta > 0$ centred at $x \in \mathbb{R}^n$. We have

$$\Uparrow b = \bigcup_{n\geq 1} \Uparrow b_{-1/n},$$

where, for $r > 0$, $b_{-r} := \{x \in b : D_r(x) \subset b\}$, which is a compact, convex set, and is non-empty for sufficiently small r. Since the frame map $(\mathcal{L}f)^{-1} : \sigma_{\mathbf{C}(\mathbb{R}^n)} \to \Omega(U)$ is Scott continuous, there exists $n \geq 1$ such that $a' \subset (\mathcal{L}f)^{-1}(b_{-1/n})$. i.e., $f \in \delta(a', b')$ with $b' = b_{-1/n}$ and hence $f \in \delta_s(a, b)$.

For the converse, suppose that $f \in \delta_s(a,b)$. Therefore, there exist a' with $a \ll a'$ and b' with $b \ll b'$ such that $b'\chi_{a'} \sqsubseteq \mathcal{L}f$. Then, $b\chi_a \ll b'\chi_{a'} \sqsubseteq \mathcal{L}f$ and hence $b\chi_a \ll \mathcal{L}f$. ■

Proposition VII.6

(i) *If we have $a \subset a'$ and $O' \subset O$, then $\Delta_s(a',O') \subset \Delta_s(a,O)$.*
(ii) *$\Delta_s(a,O) \subseteq \Delta(a,O)$.*
(iii) *If $f \in \Delta_s(a,O)$, then there exist a' with $a \prec a'$ and O' with $O' \prec O$ such that $R \in \Delta_s(a',O')$.*

Proof: (i) This follows immediately from the definition.

(ii) Let $R \in \Delta_s(a,O)$ so that there exist a_0 with $a \ll a_0$ and O_0 with $O_0 \ll O$ such that $R \in \Delta(a_0,O_0)$. Assume $a_1,a_2 \ll a$ with $(a_1,a_2) \in \mathsf{Sep}$. From $a \ll a_0$, we have $a_1,a_2 \ll a_0$. Since $R \in \Delta(a_0,O_0)$ there exist a_1',a_2' such that a_iRa_i' for $i=1,2$ and $a_1'-a_2' \ll O_0\cdot(a_1-a_2)$. By Lemma VI.8 and Lemma VI.9 $a_1'-a_2' \ll O_0\cdot(a_1-a_2) \subset O\cdot(a_1-a_2)$. Hence $R \in \Delta(a,O)$.

(iii) This is straightforward using the interpolation property. ■

Theorem VII.7

(i) *If $f \in \delta_s(a,b)$ then for all basic convex open sets O with $b \subset O$ we have $A_f \in \Delta_s(a,O)$.*
(ii) *If $A_f \in \Delta_s(a,O)$, then there exists a convex and compact subset $b \subset O$ such that $f \in \delta_s(a,b)$.*

Proof: (i) Let $f \in \delta_s(a,b)$ and $O \supset b$. Then, by the definition of a strong tie, there exist a_0 with $a \ll a_0$ and b_0 with $b_0 \ll b$ such that $f \in \delta(a_0,b_0)$. By the interpolation property, there exist a' with $a \ll a' \ll a_0$, and O' with $O' \ll O$ and $b \subset O'$. By Theorem VI.10, $A_f \in \Delta(a',O')$ and hence $A_f \in \Delta_s(a,O)$.

(ii) Suppose $A_f \in \Delta_s(a,O)$. Then, by the definition of a strong single-knot, there exist a' with $a \ll a'$ and O_0 with $O_0 \ll O$ such that $A_f \in \Delta(a',O_0)$. From Corollary VI.5, $f \in \delta(a',\overline{O'})$. By the interpolation property, there exists O' such that $O_0 \ll O' \ll O$. Put $b' := \overline{O_0}$ and $b := \overline{O'}$. Then $b \ll b'$ and $f \in \delta(a',b')$ and it follows that $f \in \delta_s(a,b)$. ■

Corollary VII.8

We have $R \in \Delta_s(a,O)$ iff $G_R \in \delta_s(a,\overline{O})$. Dually, we have $f \in \delta_s(a,b)$ iff $A_f \in \Delta_s(a,b^\circ)$.

Proof: Suppose $R \in \Delta_s(a,O)$. Then, by Theorem VII.8, there exists $b \subset O$ such that $G_R \in \delta_s(a,b)$, which implies $G_R \in \delta_s(a,\overline{O})$. Next assume $G_R \in \delta_s(a,\overline{O})$. By Proposition VII.2(iii), there exists b with $\overline{O} \ll b$ such that $G_R \in \delta_s(a,b)$. Hence, by Theorem VII.8, $R \in \Delta_s(a,O)$ since $b \subset O$. The dual statement follows immediately from the property of a basic convex open set O and a basic convex compact set b, namely that $\overline{(b^\circ)} = b$ and $(\overline{O})^\circ = O$. ■

Theorem VII.10

The Lipschitzian derivative of a Lipschitzian approximable mapping is an approximable mapping and we have: $\mathsf{L}(R) = A_{\mathcal{L}G_R}$.

Proof: We have, by Corollary VII.9, Corollary VII.5 and Definition IV.7 of $\eta_{(a,O)}$, the following three equivalent statements: $R \in \Delta_s(a,O)$ iff $G_R \in \delta_s(a,\overline{O})$ iff $\overline{O}\chi_a \ll \mathcal{L}G_R$. Thus, since $A : (U \to \mathbf{C}(\mathbb{R}^n)) \to \mathsf{App}(B_U, B_{\mathbf{C}(\mathbb{R}^n)})$ is an isomorphism, we obtain:

$$\mathsf{L}(R) = \sup\{A_{\overline{O}\chi_a} : \overline{O}\chi_a \ll \mathcal{L}G_R\} = A_{\mathcal{L}G_R}$$

■

Proposition VIII.2

$f \in \delta^1(U)$ iff $f \in C^1(U)$.

Proof: Let $f \in \delta^1(U)$ and $x \in U$. Take a basic convex open set $a_0 \ll U$ with $x \in a_0$ and let $\epsilon > 0$ be given. Assume $\delta > 0$ is the witness for $f \in \delta^1(U)$ with $a \ll a_0$ satisfying $x \in a$ and $\mathrm{diam}(a) < \delta$. Then there exists a basic convex compact set b with $\mathrm{diam}(b) < \epsilon$ and $f \in \delta_s(a,b)$. Hence $b\chi_a \ll \mathcal{L}f$ and in particular $b \ll \mathcal{L}f(x)$. It follows that $\mathrm{diam}(\mathcal{L}f(x)) < \epsilon$. Since $\epsilon > 0$ is arbitrary, we conclude that $\mathcal{L}f(x)$ is a singleton. This is true for all $x \in U$. Thus, by Proposition II.2(iv), $f \in C^1(U)$.

For the other direction, assume $f \in C^1(U)$ and let $a_0 \ll U$ and $\epsilon > 0$. Then $\overline{a_0} \subset U$ and hence ∇f is continuous, and thus uniformly continuous, on the compact set $\overline{a_0}$. Therefore, there exists $\delta > 0$ such that $\|\nabla f(x) - \nabla f(y)\| < \epsilon$ for $x,y \in \overline{a_0}$ with $\|x-y\| < \delta$. Let $a \ll a_0$ with $\mathrm{diam}(a) < \delta$. Take a_1 with $a \ll a_1 \ll a_0$ and $\mathrm{diam}(a_1) < \delta$. The convex set $b_1 := \mathrm{conv}\{\nabla f(x) : x \in \overline{a_1}\}$ is compact with $\mathrm{diam}(b_1) < \epsilon$. Thus, there exists a convex compact basic set b with $b \subset b_1^\circ$ and $\mathrm{diam}(b) < \epsilon$. Since $\nabla f = \mathcal{L}f$ on U, it follows that $b\chi_a \ll \mathcal{L}f$ and thus $f \in \delta_s(a,b)$, yielding $f \in \delta^1(U)$. ■

Proposition VIII.4

We have $f \in \delta^1(U)$ iff $A_f \in \Delta^1(U)$.

Proof: Suppose $f \in \delta^1(U)$, $a_0 \ll U$ and $\epsilon > 0$. Then, there exists $\delta > 0$ such that whenever $a \ll a_0$ and $\mathrm{diam}(a) < \delta$, there exists a basic convex compact set b with $\mathrm{diam}(b) < \epsilon$ and $f \in \delta_s(a,b)$. From Corollary VII.9, we have $A_f \in \Delta_s(a,b^\circ)$. Since $\mathrm{diam}(b^\circ) = \mathrm{diam}(b) < \epsilon$, we deduce that $A_f \in \Delta^1(U)$. Next, assume $R \in \Delta^1(U)$, $a_0 \ll U$ and $\epsilon > 0$. There exists $\delta > 0$ such that whenever $a \ll a_0$ and $\mathrm{diam}(a) < \delta$, there exists a basic convex open set O with $\mathrm{diam}(O) < \epsilon$ and $R \in \Delta_s(a,O)$. By Corollary VII.9 again, we have $G_R \in \delta_s(a,\overline{O})$. Since $\mathrm{diam}(\overline{O}) = \mathrm{dima}(O) < \epsilon$, we deduce that $f \in \delta^1(U)$. ■

Corollary VIII.5

$R \in \Delta^1(U)$ iff $G_R \in C^1(U)$.

Proof: Combine Proposition VIII.2 and Corollary VIII.4. ■

Proposition IX.1

Let $R_1, R_2 : B_U \to B_{\mathbb{R}}$, where $U \subseteq \mathbb{R}^n$ are approximable mappings. Then:

$$\mathsf{L}(R_1) + \mathsf{L}(R_2) \subseteq \mathsf{L}(R_1 + R_2)$$

and equality holds if at least one of R_1 or R_2 is in $\Delta^1(U)$.

Proof: We have:

$\mathsf{L}(R_1) + \mathsf{L}(R_2)$	
$= A_{\mathcal{L}G_{R_1}} + A_{\mathcal{L}G_{R_2}}$	Theorem VII.12
$= A_{\mathcal{L}G_{R_1} + \mathcal{L}G_{R_2}}$	Corollary V.6
$\subseteq A_{\mathcal{L}(G_{R_1} + G_{R_2})}$	Propositions II.2(vi), IV.1(i)
$= \mathsf{L}(A_{G_{R_1} + G_{R_2}})$	Theorem VII.12
$= \mathsf{L}(R_1 + R_2)$	Theorem III.5

If, for example, $R_1 \in \Delta(U)$ then $G_{R_1} \in C^1(U)$ and by Proposition II.2(vi), we have $\mathcal{L}(G_{R_1}) + \mathcal{L}(G_{R_2}) = \mathcal{L}(G_{R_1} + G_{R_2})$ and we obtain equality in the above derivation:

$$\mathsf{L}(R_1) + \mathsf{L}(R_2) = \mathsf{L}(R_1 + R_2)$$

■

Proposition IX.2

Let $R_1, R_2 : B_U \to B_{\mathbb{R}}$ be approximable mappings. Then:

$$R_1 \cdot \mathsf{L}(R_2) + R_2 \cdot \mathsf{L}(R_1) \subseteq \mathsf{L}(R_1 \cdot R_2)$$

and equality holds if one of R_1 or R_2 is in $\Delta^1(U)$.

Proof: We have:

$R_1 \cdot \mathsf{L}(R_2) + R_2 \cdot \mathsf{L}(R_1)$	
$= A_{G_{R_1}} \cdot A_{\mathcal{L}G_{R_2}} + A_{G_{R_2}} \cdot A_{\mathcal{L}G_{R_1}}$	Theorems III.5, VII.12
$= A_{(G_{R_1} \cdot \mathcal{L}G_{R_2} + G_{R_2} \cdot \mathcal{L}G_{R_1})}$	Corollaries V.6, V.9
$\subseteq A_{\mathcal{L}(G_{R_1} \cdot G_{R_2})}$	Prop. II.2(vii), IV.1(i)
$= \mathsf{L}(A_{G_{R_1} \cdot G_{R_2}})$	Theorem VII.12
$= \mathsf{L}(R_1 \cdot R_2)$	Theorem III.5

If $R_1 \in \Delta^1(U)$ then $G_{R_1} \in C^1(U)$ and Proposition II.2(vii) yields: $G_{R_1} \cdot \mathcal{L}G_{R_2} + G_{R_2} \cdot \mathcal{L}G_{R_1} = \mathcal{L}(G_{R_1} \cdot G_{R_2})$. Thus:

$$R_1 \cdot \mathsf{L}(R_2) + R_2 \cdot \mathsf{L}(R_1) = \mathsf{L}(R_1 \cdot R_2)$$

■

Proposition IX.3

Let $R_1 : B_{U_1} \to B_{\mathbb{R}}$ and $R_2 : B_{U_2} \to B_{\mathbb{R}}$ be approximable mappings, where $U_1 \subseteq \mathbb{R}^n$, $U_2 \subseteq \mathbb{R}$ and Im$(R_1) \subseteq B_{U_2}$. Then:

$$(\mathsf{L}(R_2) \circ R_1) \cdot \mathsf{L}(R_1) \subseteq \mathsf{L}(R_2 \circ R_1)$$

and the equality holds if at least one of the following predicates holds: $R_1 \in \Delta(U_1)$ or $R_2 \in \Delta(U_2)$. Here by Im(R_1) we mean the image of R_1.

Proof:

$(\mathsf{L}(R_2) \circ R_1) \cdot \mathsf{L}(R_1)$	
$= (A_{\mathcal{L}G_{R_2} \circ G_{R_1}}) \cdot A_{\mathcal{L}G_{R_1}}$	Theorems III.5, VII.12
$= A_{(\mathcal{L}G_{R_2} \circ G_{R_1}) \cdot \mathcal{L}G_{R_1}}$	Corollary V.9
$\subseteq A_{\mathcal{L}(G_{R_2} \circ G_{R_1})}$	Prop. II.2(viii), IV.1(i)
$= \mathsf{L}(A_{G_{R_2} \circ G_{R_1}})$	Theorem VII.12
$= \mathsf{L}(A_{G_{R_2}} \circ A_{G_{R_1}})$	Theorem III.5
$= \mathsf{L}(R_2 \circ R_1)$	Theorem III.5

If at least one of two predicates $R_1 \in \Delta(U_1)$ or $R_2 \in \Delta(U_2)$ holds, then, by Proposition II.2(viii), we have the equality $\mathcal{L}(G_{R_2} \circ G_{R_1}) = (\mathcal{L}G_{R_2} \circ G_{R_1}) \cdot \mathcal{L}G_{R_1}$ and we obtain: $\mathsf{L}(R_1) \cdot (\mathsf{L}(R_2) \circ R_1) = \mathsf{L}(R_1 \circ R_2)$. ■