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Stochastic Maximum Principle with Control-Dependent Terminal Time and Applications

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Statement of Originality

I certify that this thesis, and the research to which it refers, are the product of my own work, and that any ideas or quotations from the work of other people, published or otherwise, are fully acknowledged in accordance with the standard referencing practices of the discipline.

Riccardo Cesari

Copyright Declaration

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Abstract

In this thesis we study stochastic control problems with control-dependent stopping terminal time. We assess what are the methods and theorems from standard control optimization settings that can be applied to this framework and we introduce new statements where necessary.

In the first part of the thesis we study a general optimal liquidation problem with a control-dependent stopping time which is the first time the stock holding becomes zero or a fixed terminal time, whichever comes first. We prove a stochastic maximum principle (SMP) which is markedly different in its Hamiltonian condition from that of the standard SMP with fixed terminal time. The new version of the SMP involves an innovative definition of the FBSDE associated to the problem and a new type of Hamiltonian. We present several examples in which the optimal solution satisfies the SMP in this thesis but fails the standard SMP in the literature. The generalised version of the SMP Theorem can also be applied to any problem in physics and engineering in which the terminal time of the optimization depends on the control, such as optimal planning problems.

In the second part of thesis, we introduce an optimal liquidation problem with control-dependent stopping time as before. We analyze the case when an agent is trading on a market with two financial assets correlated with each other. The agent's task is to liquidate via market orders an initial position of shares of one of the two financial assets, without having the possibility of trading the other stock. The main results of this part consist in proving a verification theorem and a comparison principle for the viscosity solution to the HJB equation and finding an approximation of the classical solution of the Hamilton-Jacobi-Bellman (HJB) equation associated to this problem.

Keywords: Stochastic maximum principle, control-dependent terminal time, optimal liquidation, variational analysis, backwards stochastic differential equations, price impact, dynamic programming, viscosity solution, FBSDE approximation.

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Chapter 1

Introduction

1.1 Background

A standard service of investment banks is the execution of large trades. Unlike for small trades, the liquidation of a large portfolio is a complex task. It is usually impossible to immediately execute a large liquidation task or it is only possible at a very high cost due to insufficient liquidity. Hence, the ability of exercising an order in a way that minimizes execution costs for the client is of primary importance.

The stock price is usually defined through a specific stochastic process, usually adopting the solution of a Stochastic Differential Equation (SDE). This process may be influenced by many endogenous factors to the market, such as liquidity and volatility of the market and the strategy of the trader itself.

Almgren and Chriss [4] and the subsequent literature analyse the liquidation problem through the use of market orders. The control of the liquidation problem is the amount of sell market orders executed at any time t in the trading period $[0, T]$. Usually the optimal value of the control is calculated by maximizing the terminal wealth of the trader and minimizing some trading cost or some penalty cost, such as urgency to liquidate or risk-aversion to price movements.

The optimal liquidation problem under price impact has been studied extensively in the literature. Bertsimas and Lo [14] use a linear price impact model and solve a discrete optimal control problem to minimize expected trading costs. Almgren and Chriss [4], Huberman and Stanzl [36] introduce the volatility as a trading cost. Almgren [3] employs nonlinear impact functions and discusses the continuous-time limit of the models in Almgren and Chriss [4] in more detail. Almgren [2] considers optimal liquidation in a market with stochastic liquidity and stochastic volatility. Kharroubi and Pham [40] considers real trading that occurs in discrete time. For an overview of continuous-time price impact models, see Cartea et al. [18] and the references therein.

Once a liquidation problem is modelled through the definition of the state variables (such as the stock price, the inventory, liquidity and volatility of the market, ...), the trader is interested in finding the optimal trading strategy in order to maximize her performance criterion. The maximum of the performance criterion over the whole space of admissible trading strategies is defined as the *value function*. Two most commonly used approaches in solving stochastic optimal control problems are Bellman's dynamic programming principle (DPP) and Pontryagin's stochastic maximum principle (SMP).

The first method implies the use of the Hamilton-Jacobi-Bellman (HJB) equation formulation of the problem as in [3, 7, 8, 17, 19, 34, 33, 40, 53]. In general, the value function related to most of the liquidation problems can be proven to be a solution to a PDE, usually known as HJB equation. In Pham [48] it is outlined the form of a general HJB equation deriving from a general performance criterion and general state variables defined through a set of SDEs.

The task of finding the value function as the classical solution of a particular HJB equation is often demanding because of the strict conditions in the Verification Theorem and the difficulty to prove the existence of a solution the HJB equation. Hence, the usual procedure is to prove that the value function is at least the continuous unique viscosity solution to a particular HJB equation. We refer to Pham [48] for definition and properties of viscosity solutions.

The second approach to solving an optimal liquidation problem consists in employing FBSDEs, by applying the Stochastic Maximum Principle, whose statement can be found in Pham [48, Theorem 6.4.6]. One of the most interesting aspect of using the BSDE approach to solve the optimal control problem is that BSDEs can deal with non-Markovian settings. In particular, we can think about some stochastic price impact that depends on the history of the price itself.

Both approaches have been extensively studied in the case when the stochastic optimal control has finite time horizon. In Cartea et al. [18] can be found examples of optimal liquidation problems terminated when the agent finished the inventory to be liquidated. In this case the liquidation problem with random terminal time is solved using the HJB approach by simply adding a boundary condition to the PDE when the inventory variable touches 0. To the best of our knowledge, optimal stochastic problems with stopping random terminal time have never been solved in the literature using the SMP approach.

1.2 Outline of the thesis

In Chapter 2 we study a general optimal liquidation problem with a control-dependent stopping time which is the first time the stock holding becomes zero or a fixed terminal time, whichever comes first. To make the presentation of this chapter easier, we assume the state process to be independent of the control variable. Moreover, since we are considering a liquidation process, we assume the liquidation speed (i.e. the agent's control variable) to be always non-negative and, therefore, not allowing the agent to buy the stock she is liquidating. The two main contributions of this chapter are, firstly, to show several examples of control-dependent terminal time problems that do not satisfy the usual formulation of the SMP and, secondly, to find a new version of the SMP that can be applied to the whole series of these control problems. This new version of the SMP is markedly different in its Hamiltonian condition from that of the standard SMP with fixed terminal time.

In Chapter 3, we consider an extension of the framework studied in the previous chapter, by removing two of the main assumptions considered in previous chapter. Firstly, we introduce the control variable into the SDE defining the state process. Secondly, we allow the control to take negative values. Lastly, we introduce a lighter version of the previously introduced SMP for specific settings. The generalised version of the SMP Theorem makes possible the application of the SMP Theorem to optimal planning problem and other stochastic controls in physics and engineering.

In Chapter 4, we study an example of an optimal liquidation problem with control-dependent terminal time. We analyze the case when an agent is trading on a market with two financial assets, whose difference of log-prices is modelled with a mean-reverting process. The agent's task is to liquidate, using market orders, an initial position of shares of one of the two financial assets, without having the possibility of trading the other stock. The criterion to be optimized consists in maximising the expected final value of the agent, with a running inventory penalty, while the liquidation has multiplicative price impact on stocks. The main result of this Chapter consists in proving a verification theorem and a comparison principle for the viscosity solution to the HJB equation and finding an approximation of the classical solution of the Hamilton-Jacobi-Bellman (HJB) equation associated to this problem, which is proved to not coincide with the value function. However, we find the value function as a solution to the forward-backward stochastic differential equation (FBSDE) associated to the problem. We provide numerical tests showing that the HJB and FBSDE solutions are close to each other and analysing performance of the described model.

1.3 Published papers and preprints

In this section we list all the published as well as working papers related to this thesis.

[20] R. Cesari and H. Zheng. Stochastic maximum principle for optimal liquidation with control-dependent terminal time. *Applied Mathematics and Optimization*, 2021 (accepted)

[21] R. Cesari and H. Zheng. Optimal liquidation in a mean-reverting portfolio. `https://arxiv.org/abs/2010.02624`, preprint, 2020

Chapter 2

Stochastic Maximum Principle for Optimal Liquidation with Control-Dependent Terminal Time

2.1 Introduction

In the last decades stochastic optimal control theory has received an increasing attention mainly driven by applications to financial mathematics. Different approaches have been developed to solve stochastic control problem, broadly divided into two classes, partial differential equations (PDE) methods based on the Hamilton-Jacobi-Bellman (HJB) equation, and methods based on the maximum principle based on backward stochastic differential equations (BSDE) techniques, we refer to Yong and Zhou [59], Fleming and Soner [30] and Pham [48] for a deeper introduction to the topics. In particular, the latter field of BSDE's has proved to be particularly strong in addressing stochastic optimal control problems of various nature. At the same time, motivated by many financial applications, optimal control problems with random terminal time have been deeply studied. In the usual setup, the random events occur by surprise, that is they are totally inaccessible random time for the reference filtration. Recently, this approach has been used in Pham [49], Jiao et al. [38] and Cordoni and Di Persio [23] in order to investigate a general stochastic control problem with multiple default events. In this chapter we investigate a stochastic control problem with multiple random events but, differently from Pham [49] and Jiao et al. [38], we assume the random event to be passively determined by the control strategy.

We denote $(\pi_t)_{t\in[0,T]}$ as the control variable decided by the agent. We define the quantity

Q_t as

$$Q_t^\pi = q_0 - \int_0^t \pi_r \, dr, \tag{2.1.1}$$

where $q_0 > 0$ and we assume that the terminal stopping time is triggered as soon as the quantity Q_t^π gets to 0. In particular, the terminal stopping time is defined as $\tau^\pi = T \wedge \min\{r \geq 0 \mid Q_r^\pi = 0\}$. In the optimal liquidation problem, the control π should be seen as the liquidation rate of the inventory Q_t^π and τ^π is the first time when all stock is liquidated before terminal time T. As we already mentioned, the stopping time τ^π is not completely inaccessible, as it depends on the control π. We also denote the state processes $(\mathbf{X}_t)_{t\in[0,T]}$ to be an $\mathbb{R}^n$ valued stochastic process defined through the following SDE:

$$d\mathbf{X}_t = \boldsymbol{\mu}(t, \mathbf{X}_t)dt + \boldsymbol{\sigma}(t, \mathbf{X}_t)d\mathbf{W}_t. \tag{2.1.2}$$

The optimal liquidation problem is defined by:

$$\sup_{\pi\in\mathcal{A}} \mathbb{E}\left[g(\mathbf{X}_{\tau^\pi}, Q_{\tau^\pi}^\pi) + \int_0^{\tau^\pi} f(r, \pi_r, \mathbf{X}_r, Q_r^\pi) \, dr\right]. \tag{2.1.3}$$

In the following, we indicate π as a generic control and we indicate c as the optimal control associated to a problem of kind (2.1.3). We also avoid having the π in the superscript of Q and τ when it is clear from the context to which control they are referring to.

The optimization problem (2.1.3) is a standard stochastic optimization problem, with the innovative feature of a terminal stopping time that depends on the control. This kind of problem should be thought as a combination of a stochastic optimization problem and a stochastic optimal stopping problem, as the control influences the terminal stopping time but does not directly control it. This kind of problem has been extensively solved using an HJB approach (see for example Cartea et al. [18]). HJB equation approach cannot solve all stochastic optimal control problems, as it requires the proof of a verification theorem based on the close form solution of the equation itself. The HJB equation is a non-linear partial differential equation, whose close form solution can be found only on few simple cases. In more sophisticated examples it becomes impossible to find a solution of the HJB equation, making this approach unproductive. On the other hand, the SMP approach provides a more general necessary condition for optimality. However, to our extent, the SMP approach is only applied to fixed terminal time problems and it has never been applied to any problem with a stopping terminal time.

The two main contributions of this chapter are, firstly, to show that optimal solutions of control problems as in (2.1.3) do not satisfy the usual formulation of the SMP and, secondly, to find a new version of the SMP that can be applied to the whole series of control problems as in (2.1.3). We present two examples of the form as in (2.1.3), whose optimal solution does not satisfy the usual formulation of the SMP. This brings us to the conclusion that the usual SMP is too restrictive for problems with a stopping terminal time and a new formulation is required. In Theorem 2.2.3 we prove a new stopping terminal time version of the SMP that can be applied to problems as in (2.1.3). To strengthen the statement of Theorem 2.2.3, we show that the optimal solutions to the two examples are candidate solution for the new version of the SMP. We also show that our new version of the SMP generalizes the standard version of the SMP Theorem, which can be obtained in the limiting case of our formulation.

Although in this chapter we are restricted to a liquidation problem setting, the SMP approach to a stopped terminal time problem is pioneering and it is groundbraking for other applications to many different problems in stochastic optimization. We can apply our formulation to any stochastic optimization problem in which the agent is controlling a speed and in which the terminal time is determined by the achievement of a predetermined value of the running integral of the speed. Examples of this kind of problems are the optimal path planning problem. Path planning is the task of predicting paths for autonomous vehicles (such as cars, robots, drones, underwater gliders, propelled underwater vehicles and surface crafts) to navigate between any two points while optimizing some or all operational parameters such as time, energy, data collected and safety. Our setting can be applied to different optimal path planning problem in which the control is the speed of the vehicle and the stopping terminal time is achieved when the vehicle reaches an arrival point B in the 2- or 3-dimensional space, starting from a point A. The stopping time may be thought as the first time in which the integral of the component of the velocity parallel to $\overline{AB}$ is equal to the length of $\overline{AB}$. Subramani et al [55], [56] maximize the energy-optimal path for Autonomous Underwater Vehicles travelling in dynamic unsteady currents. Lee et al. [42] and Subramani et al [57] take into account stochastic ocean environmental effects such as current, wind, water depth, and wave effects on a surface vehicle when planning the path of a marine surface vehicle.

To the best of our knowledge, the only paper tackling problems as in (2.1.3) with a SMP approach is Cordoni and Di Persio [23]. However, we recognized a major problem with their proof of both sufficient and necessary SMPs. In standard proofs of SMP (c.f. Bensoussan [11] and Pham [48]), it is shown that the difference between the performance criterion on the optimal control c and the performance criterion on a generic control π must always be non-negative. In particular, that difference can be written as a difference of the functions g evaluated in c and π at time T and the integral of the difference between the functions f evaluated in c and π in the integration interval $[0, T]$. One of the most important aspect related to problem (2.1.3) is that each time a control π is considered, then a corresponding stopping time τ^π must be defined.

This aspect is the most arduous difficulty that need to be overcome in the proof of the SMP associated to problem (2.2.3). Therefore, the standard approach for proving SMP cannot be used in our case, as the terminal time and the endpoint of the integration interval τ^c and τ^π are different. The necessary and sufficient maximum principle in Cordoni and Di Persio [23] (c.f. [23, Theorem 2.8 and Theorem 2.11]) have a wrong proof. In particular, [23, equation (15)] in the proof of necessary SMP and [23, equation (24)] in the proof of sufficient SMP are inaccurate, as the distinction between τ^π and τ^c is not present. In the two proofs, the authors considered as if the stopping time associated with the optimal control $\bar{\alpha}$ and the stopping time associated with any other admissible control α^h were equal to each other. However, as we are going to show in this chapter, one of the main difficulties of the proof is calculating the difference of both integral and terminal part of the value functions related to the two different stopping time associated with the optimal control and the general admissible control.

The rest of the chapter is organized as follows. In Section 2.2 we describe the set up of the studied problem and we state the main Theorem 2.2.3, which is the stopping terminal time version of the Stochastic Maximum Principle. In the three following sections we present three examples that underline the importance of the new version of the SMP for the kind of problems studied in Section 2.2. In the three examples we show that usual statement of Stochastic Maximum Principle is not satisfied. In particular, in Section 2.3 a deterministic example in which function $g = 0$ is presented, in order to keep the description simple. In Section 2.4 a stochastic example is presented. In Section 2.5 a stochastic example where the function g is not trivial is presented. In Section 2.7 we prove main Theorem 2.2.3. Section 5 concludes.

2.2 Model setup

In the following we consider a stochastic optimal control with stochastic terminal time. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ be a filtered probability space, where $(\mathcal{F}_t)_{t\in[0,T]}$ is the natural filtration generated by an m-dimensional standard Brownian motion $\mathbf{W}$, augmented by all $\mathbb{P}$-null sets. Let T be the fixed terminal time. Let $(\pi_t)_{t\in[0,T]}$ denote the rate of selling the stock, which is a decision (control) variable selected by the agent and is said admissible if it is a progressively measurable, non-negative, right-continuous and square integrable process. Right-continuity is necessary as it is going to be pointed out in the proof of main Theorems, in particular in Remark 2.2.1 and proof of Lemma 2.7.4. Denote by $\mathcal{A}$ the set of all admissible control processes. We consider π to be the liquidation rate of the inventory Q_t, which is defined as

$$Q_t^\pi = q_0 - \int_0^t \pi_r \, dr. \tag{2.2.1}$$

We further assume that the control π_t is positive for any $t \in [0,T]$, so that the stocks can only be liquidated. Let $(\mathbf{X}_t)_{t\in[0,T]}$ be an $\mathbb{R}^n$ valued stochastic process defined through the following SDE:

$$d\mathbf{X}_t = \boldsymbol{\mu}(t,\mathbf{X}_t)dt + \boldsymbol{\sigma}(t,\mathbf{X}_t)d\mathbf{W}_t, \tag{2.2.2}$$

with initial condition $\mathbf{X}_0 = \mathbf{x}$ and where $\boldsymbol{\mu} : [0,T]\times\mathbb{R}^n \to \mathbb{R}^n$, $\boldsymbol{\sigma} : [0,T]\times\mathbb{R}^n \to \mathbb{R}^{n\times m}$ are two continuous functions. The optimal liquidation problem is defined by:

$$\sup_{\pi\in\mathcal{A}} \mathbb{E}\left[g(\mathbf{X}_{\tau^\pi}, Q^\pi_{\tau^\pi}) + \int_0^{\tau^\pi} f(r,\pi_r,\mathbf{X}_r,Q^\pi_r)\, dr\right], \tag{2.2.3}$$

where $g : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ and $f : [0,T]\times[0,\infty)\times\mathbb{R}^n\times\mathbb{R} \to \mathbb{R}$ are two continuous functions and τ^π is a stopping time defined by

$$\tau^\pi = T \wedge \min\{r \geq 0 \mid Q^\pi_r = 0\}, \tag{2.2.4}$$

the first time when all stock is liquidated before terminal time T or T otherwise. For the sake of clarity, we intentionally kept processes $\mathbf{X}$ and Q^π separated, because Q^π is defining the moment in which τ^π is triggered. This detachment becomes more evident in the following sections. Denote by $\mathbb{E}^t[\cdot] = \mathbb{E}[\cdot|\mathbf{X}_t = \mathbf{x},\ Q_t = q]$, the conditional expectation operator at time $t \in [0,T]$. We define

$$v(t,\mathbf{x},q) = \sup_{\pi\geq 0} v^\pi(t,\mathbf{x},q), \tag{2.2.5}$$

where

$$v^\pi(t,\mathbf{x},q) = \mathbb{E}^t\left[g(\mathbf{X}_{\tau^\pi}, Q^\pi_{\tau^\pi}) + \int_t^{\tau^\pi} f(r,\pi_r,\mathbf{X}_r,Q^\pi_r)\, dr\right]. \tag{2.2.6}$$

In the following, whenever we choose a $t \in [0,T]$, we assume that $Q_t = q > 0$, meaning that there is still inventory to be liquidated after time t.

To simplify the notation we consider a one-dimensional process X, but all results we show can be obtained in the multi-dimensional case. We denote the state space of the couple (X_r, Q_r) as $\mathcal{O} := \mathbb{R}\times[0,q_0]$. In the following, with a slight abuse of notations, we denote c_r as $c_r\mathbb{1}_{r<\tau}$, which is equal to 0 after τ. Whenever we refer to a time interval $[a,b)$, if $a \geq b$, then we consider it to be an empty set.

In the following, we state a necessary SMP for a problem of kind (2.2.3). To prove such a theorem, we follow the procedure in Bensoussan [11]. We assume that c is the optimal control. Let Q and τ be respectively the inventory and the stopping time associated to the optimal control c, through definitions (2.2.1) and (2.2.4). Let X be the process solving (2.2.2). If $t = T$ or $q = 0$, the problem is already terminated and there is nothing to be analysed. Let $t \in [0,\tau)$,

$q > 0$, $\bar{c} \geq 0$ and $\theta \in \left(0, (T-t) \wedge \frac{q}{\bar{c}}\right)$ be fixed[1]. Then we consider a variation of c as follows

$$c_r^{\theta,\bar{c},t} := \bar{c}\mathbb{1}_{r\in[t,(t+\theta)\wedge\tau)} + c_r\mathbb{1}_{r\in[t+\theta,\tau)} - \frac{\gamma_{t+\theta}^{\theta,\bar{c},t}}{\theta}\mathbb{1}_{r\in[\tau,+\infty)}, \tag{2.2.7}$$

where

$$\gamma_r^{\theta,\bar{c},t} := \int_t^{r\wedge\tau} (\bar{c} - c_s)ds. \tag{2.2.8}$$

The control in (2.2.7) is non-negative and it is an admissible control as it is proved in Lemma 2.7.1. Let $Q_r^{\theta,\bar{c},t}$ be defined as

$$Q_r^{\theta,\bar{c},t} = q - \int_t^r c_s^{\theta,\bar{c},t}\, ds, \tag{2.2.9}$$

where $\tau^{\theta,\bar{c},t}$ is the first hitting time when the inventory $Q_r^{\theta,\bar{c},t}$ gets to 0:

$$\tau^{\theta,\bar{c},t} := T \wedge \min\{r \geq t \mid Q_r^{\theta,\bar{c},t} = 0\}. \tag{2.2.10}$$

In the following, with a slight abuse of notations, we denote $c_r^{\theta,\bar{c},t}$ as $c_r^{\theta,\bar{c},t}\mathbb{1}_{r<\tau^{\theta,\bar{c},t}}$, which is equal to 0 after $\tau^{\theta,\bar{c},t}$. Thanks to this notation, we see that $c_r^{\theta,\bar{c},t}$ is well defined in (2.2.7). Indeed, when $\tau \geq \tau^{\theta,\bar{c},t}$, the last term in (2.2.7) disappears, while if $\tau < \tau^{\theta,\bar{c},t}$, the quantity $\gamma_{t+\theta}^{\theta,\bar{c},t}$ must be negative, making the last term in (2.2.7) a non-negative term.

The idea behind previous definitions is the following. The variation of c is defined so that, when $\bar{c} \geq c_t$, for θ close to 0 in the interval of time $[t, t+\theta]$ the agent is trading more than the optimal strategy c. This means that, if the stopping time τ is hit before terminal time T, then by trading with the faster strategy $c^{\theta,\bar{c},t}$, the new stopping time $\tau^{\theta,\bar{c},t}$ is going to be hit earlier than τ, as it is shown in Figure 2.1. In particular, with the strategy $c^{\theta,\bar{c},t}$ the agent is trading $\gamma_{t+\theta}^{\theta,\bar{c},t}$ more stocks in the period $[t, t+\theta]$, so the new stopping time $\tau^{\theta,\bar{c},t}$ is hit at the time when the inventory Q_t is equal to $\gamma_{t+\theta}^{\theta,\bar{c},t}$, which is the quantity that has already been traded at time $[t, t+\theta]$. On the other hand, when $\bar{c} < c_t$, for θ close to 0 in the interval of time $[t, t+\theta]$ the agent is trading less than the optimal strategy c. This means that, if the stopping time τ is hit before terminal time T, then, by trading with the slower strategy $c^{\theta,\bar{c},t}$, the new stopping time $\tau^{\theta,\bar{c},t}$ is hit later than τ, as it is shown in Figure 2.3. Since the new strategy will be traded even after the stopping time τ, we need to define it after τ as well. We define it to be constantly equal to $-\frac{\gamma_{t+\theta}^{\theta,\bar{c},t}}{\theta}$ after τ. This implies the new strategy to be traded for a time θ after τ in the case when τ is hit before terminal time T, in order to trade the inventory $\gamma_{t+\theta}^{\theta,\bar{c},t}$ that has not been traded earlier in the period $[t, t+\theta]$.

[1]We consider the fraction $\frac{q}{\bar{c}}$ to be equal to $+\infty$ in case when $\bar{c} = 0$

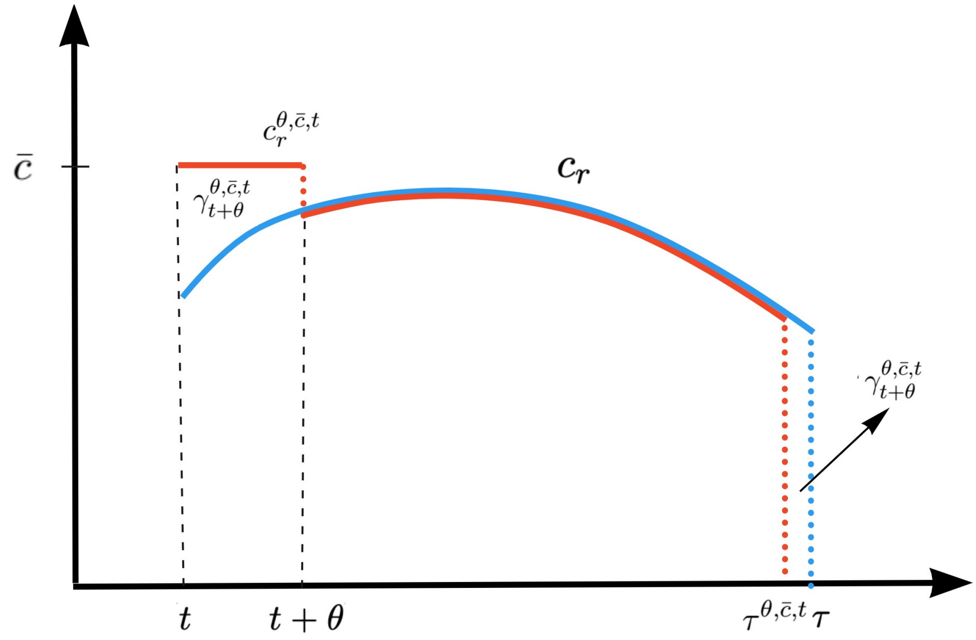


Figure 2.1: Case when $\bar{c} \geq c_t$.

Figure 2.2: Graphical example of c (in blue) and $c^{\theta,\bar{c},t}$ (in red).

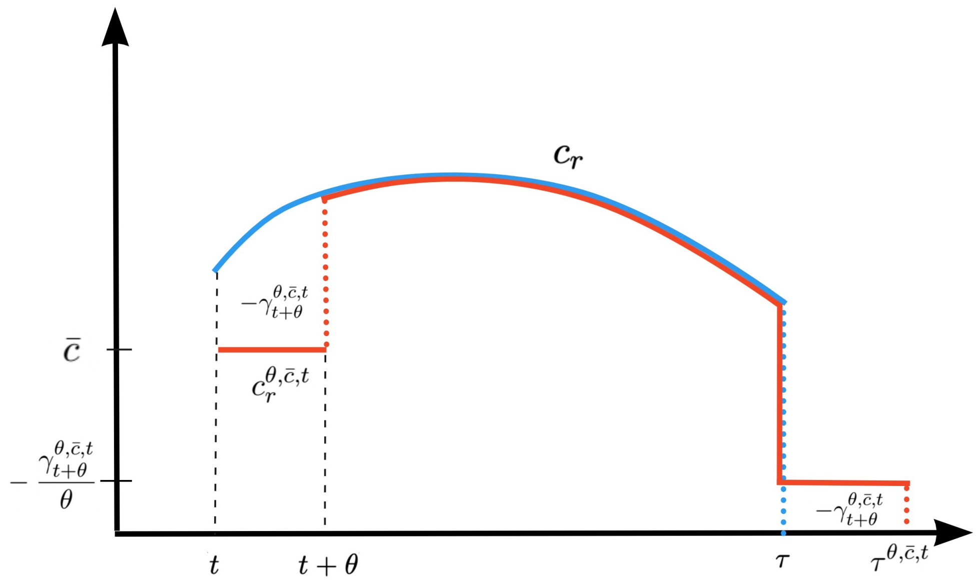


Figure 2.3: Case when $\bar{c} < c_t$.

Figure 2.4: Graphical example of c (in blue) and $c^{\theta,\bar{c},t}$ (in red).

Remark 2.2.1. *Right-continuity of admissible controls is necessary due to the construction of our proofs. The main idea behind the proof is that the two cases* $\bar{c} \geq c_t$ *and* $\bar{c} < c_t$ *correspond respectively to the cases* $\tau^{\theta,\bar{c},t} \leq \tau$ *and* $\tau^{\theta,\bar{c},t} \geq \tau$ *for* θ *small enough. By removing the right-continuity condition of the control* c *it would be easy to have an example with* $\bar{c} \geq c_t$ *and* $\bar{c} < c_r$

for any $r > t$. This would have corresponded to Figure 2.3, making $\tau^{\theta,\bar{c},t} \geq \tau$ for any $\theta > 0$ even if $\bar{c} \geq c_t$. Therefore, to avoid this kind of situation we assume right continuity of controls.

Let $(Y_r, Z_r)_{r\in[0,\tau]}$ be the solution of the following BSDE:

$$\begin{cases} -dY_r = \partial_q f(r, c_r, X_r, Q_r)dr - Z_r dW_r \\ Y_\tau = \partial_q g(X_\tau, Q_\tau). \end{cases} \tag{2.2.11}$$

Previous processes (Y, Z) are conventional in the usual SMP formulation (c.f. Bensoussan [11, Theorem 2.1] and Pham [48, Theorem 6.4.6]). BSDE in (2.2.11) is different from the usual formulation of BSDE with fixed terminal time. In our case we have a random terminal time τ, which is a stopping time with respect to the natural filtration. This kind of BSDEs have been studied in literature, c.f. Darling and Pardoux [26] and in Wu [58].

In the following, we use the following assumptions on functions μ, σ, f and g.

Assumption 2.2.2. *For any $t \in [0, T]$, $\pi, \pi' \geq 0$, $x, x' \in \mathbb{R}$ and for any $q, q' \geq 0$*

$$\begin{aligned} \left|\mu(t,x) - \mu(t,x')\right| + \left|\sigma(t,x) - \sigma(t,x')\right| &\leq K\left|x - x'\right|, \\ \left|\mu(t,x)\right| + \left|\sigma(t,x)\right| &\leq K\left(|x| + 1\right), \\ \left|g(x,q) - g(x,q')\right| &\leq K(1 + |x|)\left|q - q'\right|, \\ \left|f(t,\pi,x,q) - f(t,\pi,x,q')\right| &\leq K\left(\left|q - q'\right| + \left|t - t'\right|\right), \\ \left|f(t,\pi,x,q) - f(t,\pi',x',q)\right| &\leq K\left(\left|x - x'\right| + \left|\pi - \pi'\right|\right)\left(1 + |x| + |x'| + |\pi| + |\pi'|\right). \end{aligned} \tag{2.2.12}$$

We assume that f and g are continuously differentiable functions with respect to the arguments. We also assume that partial derivatives of f and g with respect to q are Lipschitz continuous. In particular,

$$\left|\partial_q f(t,\pi,x,q) - \partial_q f(t,\pi,x,q')\right| + \left|\partial_q g(x,q) - \partial_q g(x,q')\right| \leq K|q - q'| \tag{2.2.13}$$

The above assumptions allow the drift μ and diffusion σ to be linear in x, which allows to consider most of the examples of models in literature, including GBM models and Ornstein-Uhlenbeck models as in Cartea et al. [18, Chapter 6]. g is required to be Lipschitz continuous in q, which would exclude a quadratic penalty g with respect to q. However, in the case of optimal liquidation (as in Example discussed in Chapter 4) the variable q is always limited between the initial inventory q_0 and 0, as the agent can only liquidate the stocks. This allows us to consider a quadratic function g with respect to variable q as in Cartea et al. [18, Section 6.5]. Finally f is allowed to be quadratic in both x and q. This enables us to consider most of the examples in the liquidation literature, in which the impact of the control is quadratic on the price of the stock.

We define the Hamiltonian as it is usually done in the SMP theory:

$$\mathcal{H}(t,\pi,x,q,y) := -\pi y + f(t,\pi,x,q). \tag{2.2.14}$$

We also define the following quantities

$$\begin{aligned} \tau_{\min}^{\theta,\bar{c},t} &= \min\left(\tau, \tau^{\theta,\bar{c},t}\right), & \tau_{\max}^{\theta,\bar{c},t} &= \max\left(\tau, \tau^{\theta,\bar{c},t}\right), \\ \hat{Q}_r^{\theta,\bar{c},t} &= \max\left(Q_r, Q_r^{\theta,\bar{c},t}\right), & \hat{c}_r^{\theta,\bar{c},t} &= \max\left(c_r, c_r^{\theta,\bar{c},t}\right). \end{aligned} \tag{2.2.15}$$

We now state the stochastic maximum principle for the stopped terminal time version.

Theorem 2.2.3. *Let Assumption 2.2.2 be satisfied. Let $(c_r)_{r\in[0,T]}$ be the optimal control for the optimization problem (2.2.5) so that $c \in \mathcal{A}$ and so that*

$$\mathbb{E}\left[\sup_{r\in[0,T]} c_r^2\right] < \infty. \tag{2.2.16}$$

Let $(Q_r)_{r\in[0,T]}$ and $(X_r)_{r\in[0,T]}$ be defined as in (2.2.1) and (2.2.2) with respect to control c. Let $(Y_r)_{r\in[0,\tau]}$ be defined as in (2.2.11) with respect to control c. We assume that there exist $\mathbb{R}$-valued and L^2-integrable functions $\bar{g}(t,\bar{c},x,q)$ and $\bar{f}(t,\bar{c},x,q)$ so that for any $t \in [0,\tau)$, for any $(x,q) \in \mathcal{O}$ and for any $\bar{c} \geq 0$

$$\bar{f}(t,\bar{c},x,q) = \lim_{\theta\to 0} \mathbb{E}^t\left[\frac{sign(\tau - \tau^{\theta,\bar{c},t})}{\theta} \int_{\tau_{\min}^{\theta,\bar{c},t}}^{\tau_{\max}^{\theta,\bar{c},t}} f\left(r, \hat{c}_r^{\theta,\bar{c},t}, X_r, \hat{Q}_r^{\theta,\bar{c},t}\right) dr\right], \tag{2.2.17}$$

$$\bar{g}(t,\bar{c},x,q) = \lim_{\theta\to 0} \mathbb{E}^t\left[\frac{g(X_\tau, Q^{\theta,\bar{c},t}_{\tau^{\theta,\bar{c},t}}) - g(X_{\tau^{\theta,\bar{c},t}}, Q^{\theta,\bar{c},t}_{\tau^{\theta,\bar{c},t}})}{\theta}\right]. \tag{2.2.18}$$

Then, c necessarily satisfies for any $t \in [0,\tau)$, for any $\bar{c} \geq 0$

$$\mathcal{H}(t,\bar{c},X_t,Q_t,Y_t) - \mathcal{H}(t,c_t,X_t,Q_t,Y_t) + \mathcal{G}(t,\bar{c},c_t,X_t,Q_t) \leq 0 \quad a.s., \tag{2.2.19}$$

where $\mathcal{G}(t,\bar{c},c_t,x,q)$ is defined as

$$\mathcal{G}(t,\bar{c},c_t,x,q) := (\bar{c} - c_t)\mathbb{E}^t\left[\partial_q g(X_\tau, Q_\tau)\mathbb{1}_{\Lambda(t,\bar{c})}\right] - \bar{g}(t,\bar{c},x,q) - \bar{f}(t,\bar{c},x,q), \tag{2.2.20}$$

where the event $\Lambda(t,\bar{c})$ is defined as

$$\Lambda(t,\bar{c}) := (\{Q_T = 0\} \cap \{\bar{c} \geq c_t\}) \cup (\{\tau < T\} \cap \{\bar{c} < c_t\}). \tag{2.2.21}$$

Remark 2.2.4. *The definition of $\bar{g}$ in (2.2.18) may look asymmetric in the arguments of functions g. Although we may have defined $\bar{g}$ in a more straightforward way as*

$$\lim_{\theta\to 0} \mathbb{E}^t\left[\frac{g(X_\tau, Q_\tau) - g(X_{\tau^{\theta,\bar{c},t}}, Q^{\theta,\bar{c},t}_{\tau^{\theta,\bar{c},t}})}{\theta}\right],$$

we wanted to state Theorem 2.2.3 so that it recalls the usual statement of the SMP and the definition of the Hamiltonian associated to it. To get such an analogy, we have to define $\bar{g}$ *as above. This argument will get more clear in* (2.8.3) *in the proof of the Theorem.*

Remark 2.2.5. *We would also like to comment on the event* $\Lambda(t, \bar{c})$ *defined in* (2.2.20). *It may look overcomplicate and redundant, but it is necessary as it can be later seen in proof of Lemma 2.7.7. Firstly, the two events in the definition of* $\Lambda(t, \bar{c})$ *are not equal, but we have* $\{\tau < T\} \subsetneq \{Q_T = 0\}$. *Indeed the event in which* Q_r *is touching* 0 *exactly in* T *is included in the second event but not in the first. The reason why we need to split the cases when* $\bar{c} \geq c_t$ *and* $\bar{c} < c_t$ *can be better understood by looking to the proof of Lemma 2.7.7. In particular, if we analyse the following fraction, we see that*

$$\frac{\hat{Q}^{\theta,\bar{c},t}_{\tau^{\theta,\bar{c},t}} - \hat{Q}^{\theta,\bar{c},t}_{\tau}}{\theta} = \begin{cases} \frac{Q_{\tau^{\theta,\bar{c},t}} - Q_\tau}{\theta} & \text{if } \bar{c} \geq c_t \\ \frac{Q^{\theta,\bar{c},t}_{\tau^{\theta,\bar{c},t}} - Q^{\theta,\bar{c},t}_{\tau}}{\theta} & \text{if } \bar{c} < c_t. \end{cases}$$

Therefore, in the event $\{Q_T = 0\} \setminus \{\tau < T\}$, *when* $\bar{c} \geq c_t$, $Q_\tau = 0$, *but* $Q_{\tau^{\theta,\bar{c},t}} > 0$ *making the fraction above not equal to* 0. *On the other hand, in the event* $\{Q_T = 0\} \setminus \{\tau < T\}$, *when* $\bar{c} < c_t$, $\tau = \tau^{\theta,\bar{c},t} = T$ *making the fraction above equal to* 0. *This remark shows an example (among many others that can be found in the proofs) of a random variable that has a different behaviour in* $\{Q_T = 0\}$ *and* $\{\tau < T\}$, *making necessary the introduction of* $\Lambda(t, \bar{c})$.

Remark 2.2.6. *One interesting question raised by one of the reviewers of paper Cesari and Zheng [20] is that if the limit* $\lim_{\theta \to 0} \frac{\tau^{\theta,\bar{c},t} - \tau}{\theta}$ *exists or not. The answer in general is negative. This can be seen by the following simple example. Assume* $t = 0$ *and* $q_0 = T^2/2$. *Assume the optimal control is* $c_t = T - t$ *for* $t \in [0, T]$, *which gives* $Q_r = q_0 - \int_0^r c_s ds = (1/2)(T - r)^2$ *for* $r \in [0, T]$ *and* $Q_r = 0$ *if and only if* $r = \tau := T$. *Now consider a perturbation with* $\bar{c} > T$ *and* $0 < \theta < T^2/(2\bar{c})$, *which gives* $Q^{\theta,\bar{c},t}_r = q_0 - \bar{c}r > 0$ *for* $r \in [0, \theta]$ *and* $Q^{\theta,\bar{c},t}_r = q_0 - \int_0^r c^{\theta,\bar{c},t}_s ds = -\bar{c}\theta + T\theta - \theta^2/2 + (T - r)^2/2$ *for* $r \in [\theta, T]$ *and* $Q^{\theta,\bar{c},t}_r = 0$ *if and only if* $r = \tau^{\theta,\bar{c},t} := T - \sqrt{\theta^2 - 2T\theta + 2\bar{c}\theta}$. *We have* $\tau^{\theta,\bar{c},t} \to \tau$ *as* $\theta \to 0$ *but*

$$\lim_{\theta \to 0} \frac{\tau^{\theta,\bar{c},t} - \tau}{\theta} = -\lim_{\theta \to 0} \frac{\sqrt{\theta^2 - 2T\theta + 2\bar{c}\theta}}{\theta} = -\lim_{\theta \to 0} \sqrt{1 + 2\frac{\bar{c} - T}{\theta}} = -\infty,$$

which shows the limit does not exist.

Remark 2.2.7. *The same result as Theorem 2.2.3 can be obtained in the case when the admissible set is bounded by above as well, i.e. when* π *is required to be in* $\pi \in [0, b]$ *with* $0 < b \leq +\infty$. *Although the proof does not change, the only remark we want to point out is on the admissibility of control* $c^{\theta,\bar{c},t}$. *Since* $\bar{c} \in [0, b]$ *and* $c_r \in [0, b]$ *for every* $r \in [t, T]$, *then* $-\frac{\gamma^{\theta,\bar{c},t}_{t+\theta}}{\theta} = \frac{1}{\theta} \int_t^{(t+\theta)\wedge\tau} (c_r - \bar{c}) dr \leq \frac{(t+\theta)\wedge\tau - t}{\theta} b \leq b$.

We now would like to link Theorem 2.2.3 with the standard version of the SMP (c.f. Pham [48, Theorem 6.4.6]) that we restate here for the sake of exposition.

Theorem 2.2.8 (Standard SMP). *Let $c \in \mathcal{A}$ be the optimal control for the problem*

$$\sup_{\pi \in \mathcal{A}} \mathbb{E}^t \left[g(X_T, Q_T^\pi) + \int_t^T f(r, \pi_r, X_r, Q_r^\pi)\, dr \right],$$

which is the fixed terminal time version of problem (2.2.5). Let processes $(X_r)_{r\in[t,T]}$ be defined as in (2.2.2) and $(Q_r)_{r\in[t,T]}$ be defined as in (2.2.1) with respect to optimal control c. Let $(\bar{Y}_r^1, \bar{Z}_r^1)_{r\in[t,T]}$ and $(\bar{Y}_r^2, \bar{Z}_r^2)_{r\in[t,T]}$ be solution of the following BSDE:

$$\begin{cases} -dY_r^1 = \left(Y_r^1 \partial_x \mu(t, X_t) + Z_r^1 \partial_x \sigma(t, X_t) + \partial_x f(r, c_r, X_r, Q_r)\right) dr - Z_r^1 dW_r \\ -dY_r^2 = \partial_q f(r, c_r, X_r, Q_r) dr - Z_r^2 dW_r \\ Y_T^1 = \partial_x g(X_T, Q_T) \\ Y_T^2 = \partial_q g(X_T, Q_T). \end{cases} \tag{2.2.22}$$

Let $\mathcal{H}$ be the Hamiltonian defined as

$$\mathcal{H}(t, \pi, x, q, y^1, y^2, z^1, z^2) := \mu(t, x) y^1 + \sigma(t, x) z^1 - \pi y^2 + f(t, \pi, x, q).$$

Then, for any $t \in [0, T]$, $(x, q) \in \mathcal{O}$ and $\pi \geq 0$, c necessarily satisfies

$$\mathcal{H}\left(t, \pi, X_t, Q_t, \bar{Y}_t^1, \bar{Y}_t^2, \bar{Z}_t^1, \bar{Z}_t^2\right) \leq \mathcal{H}\left(t, c_t, X_t, Q_t, \bar{Y}_t^1, \bar{Y}_t^2, \bar{Z}_t^1, \bar{Z}_t^2\right) \quad \mathbb{P}\text{-a.s.}. \tag{2.2.23}$$

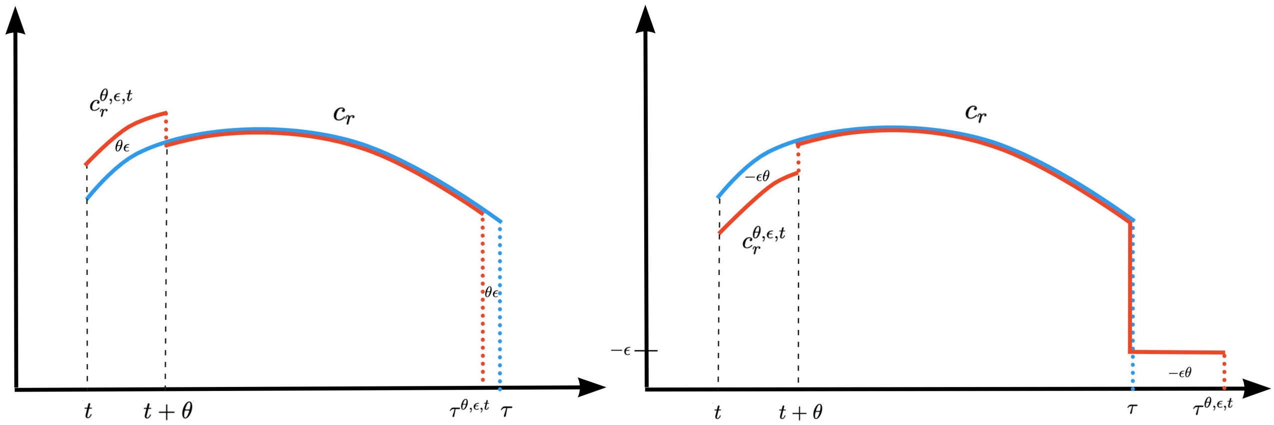
Remark 2.2.9. *When the initial inventory converges asymptotically to infinite $q \to +\infty$, the usual SMP can be recovered from the stopped version in Theorem 2.2.3. In this case, the stopping time τ is never hit and for any $t \in [0, T]$ we have $\tau = T$. Moreover, from (2.2.10), for any $\theta > 0$ we have that $\tau^{\theta, \bar{c}, t} = T$. In this case, $\bar{g} \equiv \bar{f} \equiv 0$ and the indicator function in the first expectation in the definition of $\mathcal{G}$ in (2.2.20) is a.s. equal to 0. Hence, $\mathcal{G} \equiv 0$ and we get the usual statement of the SMP for $q = +\infty$.*

2.2.1 Initial definition of $c^{\theta,\epsilon,t}$

At the beginning of our research we firstly defined $c^{\theta,\epsilon,t}$ as

$$c_r^{\theta,\epsilon,t} := c_r + \epsilon \mathbb{1}_{r\in[t,t+\theta)} - \epsilon \mathbb{1}_{\epsilon<0} \mathbb{1}_{r\in[\tau,+\infty)}. \tag{2.2.24}$$

This would have simplified the proof of the SMP. As it can be inferred from the charts below, the area between the red and the blue charts in the interval $[t, t+\theta]$ would have been linear in ϵ and θ. This would have saved us from defining and introducing the stochastic process $\gamma_t^{\theta,\epsilon,t}$.



(a) Case when $\epsilon \geq 0$. (b) Case when $\epsilon < 0$.

Figure 2.5: Graphical examples of c (in blue) and $c^{\theta,\bar{c},t}$ (in red).

However, we didn't consider that if we had a control c_t touching $c_t = 0$, it would have made $c_t^{\theta,\epsilon,t} < 0$ with a positive probability for any $\epsilon < 0$.

2.3 Deterministic Example

We consider a liquidation problem with linear market impact on trade and no terminal execution. Let $g = 0$ and $f(r, \pi) = \pi(T - r - k\pi)$, where $k > 0$ is the linear impact coefficient. To simplify the example and to be able to find a close form solution, we assume that there is no stochasticity in any process. Let $t \in [0, \tau]$ and $q > 0$ be fixed. The value function associated to this problem is

$$v(t, q) = \sup_{\pi \in \mathcal{A}} \left[\int_t^{\tau^\pi} \pi_r(T - r - k\pi_r)\, dr \right], \tag{2.3.1}$$

where the stopping time τ^π is defined as

$$\tau^\pi := T \wedge \min\{r \geq t | Q_r^\pi = 0\}.$$

We define a control strategy as follows, for any $r \in [t, T]$

$$c_r = \begin{cases} \frac{T-r}{2k} & \text{if } q \geq \frac{(T-r)^2}{4k} \\ \frac{t-r+2\sqrt{kq}}{2k} & \text{if } q < \frac{(T-r)^2}{4k} \end{cases}. \tag{2.3.2}$$

The choice of function f above is purely theoretical and it has been done to create the two different regions in the optimal control c above. In the following proposition we prove that the control c_r, defined in (2.3.2), is non-negative and admissible.

Proposition 2.3.1. *Under trading strategy c_r in* (2.3.2)*, by applying the definition of Q_r in* (2.2.1)*, we have that for any $r \in [t, T]$ the inventory Q_r has the following behaviour, conditionally on $Q_t = q$*

$$Q_r = \begin{cases} q + \frac{(T-r)^2-(T-t)^2}{4k} & \text{if } q \geq \frac{(T-t)^2}{4k} \\ \frac{\left(t-r+2\sqrt{kq}\right)^2}{4k} & \text{if } q < \frac{(T-t)^2}{4k} \end{cases}. \tag{2.3.3}$$

We also have that for any $r \in [t, T]$

$$\begin{aligned} Q_r &\geq \frac{(T-r)^2}{4k} \Leftrightarrow q \geq \frac{(T-t)^2}{4k}, \\ Q_r &< \frac{(T-r)^2}{4k} \Leftrightarrow q < \frac{(T-t)^2}{4k}. \end{aligned} \tag{2.3.4}$$

Using the expression for Q_r that we just found, it can be easily calculated that the first hitting time of $Q_r = 0$ is

$$\tau = \begin{cases} T & \text{if } q \geq \frac{(T-t)^2}{4k} \\ 2\sqrt{kq} + t & \text{if } q < \frac{(T-t)^2}{4k}. \end{cases} \tag{2.3.5}$$

Finally, by applying (2.3.2), (2.3.3) *and* (2.3.5) *into* (2.3.1)*, easy calculus shows that the value function associated with control* (2.3.2) *is equal to*

$$v^c(t, q) = \begin{cases} \frac{(T-t)^3}{12k} & \text{if } q \geq \frac{(T-t)^2}{4k} \\ q\left(T - t - \frac{4\sqrt{kq}}{3}\right) & \text{if } q < \frac{(T-t)^2}{4k}. \end{cases} \tag{2.3.6}$$

The proof of previous Proposition is in the Appendix.

Using following proposition, we prove that control (2.3.2) and its associated value function (2.3.6) are respectively the optimal trading strategy and the value function associated to problem (2.3.1).

Proposition 2.3.2. *Function v^c in* (2.3.6) *coincides with the value function of problem* (2.3.1)*. Therefore, $v(t, q) = v^c(t, q)$ for any $t \in [0, T]$ and $q > 0$.*

Proof. By definition of $v^c(t, q)$, we have that for any $t \in [0, T]$ and $q > 0$

$$v^c(t, q) = \int_t^\tau c_r \left(T - r - kc_r\right) \, dr \leq \sup_{\pi \in \mathcal{A}} \int_t^{\tau^\pi} \pi_r \left(T - r - k\pi_r\right) \, dr = v(t, q). \tag{2.3.7}$$

We now want to show that $v^c \geq v$. Simple calculus on (2.3.6) shows that

$$\partial_t v^c(t, q) = \begin{cases} -\frac{(T-t)^2}{4k} & \text{if } q \geq \frac{(T-t)^2}{4k} \\ -q & \text{if } q < \frac{(T-t)^2}{4k} \end{cases}, \qquad \partial_q v^c(t, q) = \begin{cases} 0 & \text{if } q \geq \frac{(T-t)^2}{4k} \\ T - t - 2\sqrt{kq} & \text{if } q < \frac{(T-t)^2}{4k} \end{cases}. \tag{2.3.8}$$

It can be easily verified that v^c satisfies the HJB equation associated to problem (2.3.1), as it can be derived from Pham [48, Theorem 3.5.2]

$$\partial_t w + \sup_{\pi \geq 0} \left[\pi(T - t - k\pi) - \pi \partial_q w\right] = 0. \tag{2.3.9}$$

Indeed, for $\pi = \frac{1}{2k}(T - t - \partial_q w)$, (2.3.9) is equivalent to $\partial_t w + \frac{1}{4k}\left(T - t - \partial_q w\right)^2 = 0$, which is solved by (2.3.8).

$v^c(t, q)$ satisfies boundary condition $v^c(t, 0) = 0$ for any $t \in [0, T]$ and it satisfies terminal condition $v^c(T, q) = 0$ for any $q > 0$, as it can be easily checked from (2.3.6).

We denote as c_r^* the optimal control for the value function $v(t, q)$ in (2.3.1). We denote Q_r^* and τ^* respectively the inventory and the stopping time associated to control c^*. Then, using properties of derivatives and (2.2.1), we get that for any $r \in [t, T]$

$$\frac{d}{dr} v^c(r, Q_r^*) = \partial_t v^c(r, Q_r^*) + \frac{dQ_r^*}{dr} \partial_q v^c(r, Q_r^*) = \partial_t v^c(r, Q_r^*) - c_r^* \partial_q v^c(r, Q_r^*). \tag{2.3.10}$$

From (2.3.9) we have that for any $r \in [t, T]$ and $q > 0$, $\partial_t v^c(r, q) + c_r^*(T - r - kc_r^*) - c_r^* \partial_q v^c(r, q) \leq 0$. Then, from (2.3.10) we have that

$$\frac{d}{dr} v^c(r, Q_r^*) \leq -c_r^*(T - r - kc_r^*). \tag{2.3.11}$$

Finally, since $Q_t^* = q$, we can write

$$v^c(\tau^*, Q_{\tau^*}^*) = v^c(t, Q_t^*) + \int_t^{\tau^*} \frac{d}{dr} v^c(r, Q_r^*)\, dr \leq v^c(t, q) - \int_t^{\tau^*} c_r^* \left(T - r - kc_r^*\right)\, dr. \tag{2.3.12}$$

However, using boundary and terminal condition $v^c(t, 0) = 0$ for any $t \in [0, T]$ and $v^c(T, q) = 0$ for any $q > 0$, we conclude that $v^c(\tau^*, Q_{\tau^*}^*) = 0$, as either $\tau^* = T$ or $Q_{\tau^*}^* = 0$. Therefore, from (2.3.12) follows that

$$v^c(t, q) \geq \int_t^{\tau^*} c_r^* \left(T - r - kc_r^*\right)\, dr = v(t, q). \tag{2.3.13}$$

Merging (2.3.7) and (2.3.13) we conclude the proof. □

We now want to show that the example presented in this section does not satisfy the standard formulation of the SMP 2.2.8.

Remark 2.3.3. *In the framework of the example presented in this section, the standard SMP is not satisfied. Indeed, if we apply Theorem 2.2.8 to the current example, by recalling that $g = 0$ and $f(t, \pi) = \pi(T - t - k\pi)$, we see that BSDE (2.2.22) becomes*

$$\begin{cases} dY_r = Z_r dW_r \\ Y_T = 0, \end{cases} \tag{2.3.14}$$

whose solution is the processes $(\bar{Y}_r, \bar{Z}_r)_{r\in[t,T]} \equiv (0,0)$ *identically equal to* 0. *Therefore, the necessary condition* (2.2.23) *is equivalent to*

$$\forall t \in [0,T],\ \forall \pi \geq 0, \quad f(t,\pi) \leq f(t,c_t) \tag{2.3.15}$$

However, from (2.3.1) *we get that the maximal point of* $f(t,\cdot)$ *is* $\bar{\pi} = \frac{T-t}{2k}$. *In* (2.3.2) *we derived that, if* $q < \frac{(T-t)^2}{4k}$, *the optimal strategy is* $c_t = \sqrt{\frac{q}{k}} < \frac{T-t}{2k} = \bar{\pi}$. *Therefore, using concavity of* f *(as it can be seen from definition of* f*), we found a positive control* $\bar{\pi}$ *so that* $f(t,\bar{\pi}) > f(t,c_t)$ *for any* $t \in [0,T]$, *which contradicts* (2.3.15) *and contradicts the standard SMP Theorem 2.2.8.*

We want to check that optimal control in (2.3.2) satisfies Theorem 2.2.3 and to do so, we need to show that (2.2.19) holds true. Firstly, we observe that the model setup satisfies Assumption 2.2.2. As it can be derived from (2.2.11), the solution to the BSDE is $Y \equiv 0$ and $Z \equiv 0$. Using the fact that $g = 0$, we show that condition (2.2.19) holds true by proving that for any $\bar{c} \geq 0$ and $t \in [0,\tau)$,

$$f(t,\bar{c}) - f(t,c_t) - \bar{f}(t,\bar{c},Q_t) \leq 0. \tag{2.3.16}$$

To show that the previous inequality holds true, we find the expression for $\bar{f}(t,\bar{c},q)$ in the following proposition.

Proposition 2.3.4. *Let* $t \in [0,\tau)$ *be fixed. Let* $(c_r)_{r\in[0,T]}$ *be the optimal control in* (2.3.2), *then, for any* $\bar{c} \geq 0$ *and* $q > 0$

$$\bar{f}(t,\bar{c},q) = \begin{cases} 0 & \text{if } q \geq \frac{(T-t)^2}{4k} \\ (\bar{c}-c_t)\left(T-t-2\sqrt{kq}+k(\bar{c}-c_t)\mathbb{1}_{\bar{c}<c_t}\right) & \text{if } q < \frac{(T-t)^2}{4k}. \end{cases}$$

The proof of previous Proposition is in the Appendix.

From previous expression of $\bar{f}(t,\bar{c},q)$, (2.3.16) follows. Indeed, if $q \geq \frac{(T-t)^2}{4k}$, then $\bar{f}(t,\bar{c},q) = 0$ and $c_t = \frac{T-t}{2k}$ is the maximal point of $f(t,\cdot)$, making (2.3.16) satisfied for any $\bar{c} \geq 0$. On the other hand, if $q < \frac{(T-t)^2}{4k}$, then left-hand side of (2.3.16) is equal to

$$\begin{aligned} &(\bar{c}-c_t)(T-t) - k(\bar{c}^2 - c_t^2) - (\bar{c}-c_t)(T-2\sqrt{kq}-t+k(\bar{c}-c_t)\mathbb{1}_{\bar{c}<c_t}) \\ &= \frac{\bar{c}-c_t}{k}\left(2\sqrt{\frac{q}{k}} - \bar{c} - c_t + (c_t-\bar{c})\mathbb{1}_{\bar{c}<c_t}\right). \end{aligned} \tag{2.3.17}$$

However, from (2.3.2) we have that $c_t = \frac{t-t+2\sqrt{kq}}{2k} = \sqrt{\frac{q}{k}}$ and so (2.3.16) is equivalent to

$$-\frac{1}{k}\left(\bar{c} - \sqrt{\frac{q}{k}}\right)^2 (\mathbb{1}_{\bar{c}<c_t} + 1) \leq 0,$$

whose left-hand side is always non-positive, implying that (2.3.16) is satisfied.

2.3.1 Heuristic explanation around optimal control

The aim of this section is to explain the reason why the optimal control associated with problem (2.3.1) is the one in (2.3.2) and not the one predicted by the standard formulation of the SMP. Secondly, in this section we want to underline the link between the optimal control in (2.3.2) and the boundary condition of the HJB equation associated to problem (2.3.1), i.e. $v(t,0)=0$. Both condition of continuity of the solution of the previous HJB equation and the boundary condition are necessary in order to coincide with the value function and the solution of the HJB equation associated to problem (2.3.1). By removing the boundary condition, we get the value function associated to the problem

$$v^1(t) = \sup_{c\in\mathcal{A}} \int_t^T c_r(T-r-kc_r)\,dr, \tag{2.3.18}$$

which is the extension of problem (2.1.3) to the case when the terminal time is fixed. Problem (2.3.18) is a completely different problem to the initial (2.3.1) as the trading is allowed to continue even after the inventory becomes negative and so the agent is allowed to short sell the stock, which is not the aim of our problem. Moreover, function v^1 does not even satisfy boundary condition of the HJB equation associated to problem (2.3.1), i.e. $v(t,0)=0$. On the other hand, we can define a new approximation of the value function v by imposing that it does satisfy the boundary condition. We can do that by removing continuity on $q=0$ to the solution of the HJB equation, we get the following value function

$$v^2(t,q) = \begin{cases} \sup_{c\in\mathcal{A}} \int_t^T c_r(T-r-kc_r)\,dr & \text{if } q>0 \\ 0 & \text{if } q=0. \end{cases} \tag{2.3.19}$$

As we are going to show, the change introduced with respect to our initial problem do not only affect its value function by reducing the time in which the integral $\int f(r,c_r)\,dr$ is calculated, but it also indirectly affects the optimal control. v^1 and v^2 above are similar value functions and bring to the same optimal control. We want to compare the value function v in (2.3.1) with v^2 associated to problem (2.3.19).

In (2.3.19) the integration interval does not depend on c, so the maximization with respect to the control c can be done inside the integral sign, for each time r. In this case, the optimal control results to be $c_r^2 = \frac{T-r}{2k}$, which is the optimal control predicted by the standard version of the SMP. Then, the value function v^2 is equal to:

$$v^2(t,q) = \int_t^T \sup_{c\geq 0}\left[c(T-r-kc)\right]\,dr = \int_t^T \frac{(T-r)^2}{4k}\,dr = \frac{(T-t)^3}{12k}. \tag{2.3.20}$$

In region $q \geq \frac{(T-t)^2}{4k}$ the agent has many stocks left to liquidate and shortage of trading time. So, the only goal of the optimal trading strategy is to get rid of the stocks at the highest rate in order to maximize the function $f(t,c)$ at any point in time. In region $q < \frac{(T-t)^2}{4k}$, there is more trading time left. In this way, it is not optimal to trade at the maximal speed, but it is more optimal to balance the speed with the fact that trading too fast impacts on the duration of the trading period by lowering τ. So, it must be found the right balance to have a high value of $f(t,c)$ and a distant stopping time τ in future. The maximum point of $f(t,c)$ is touched in $c = \frac{T-r}{2k} = c^2_r$. Region $q \geq \frac{(T-t)^2}{4k}$ is defined so that, by trading at that speed, the agent holds a positive inventory at terminal time T, while in Region $q < \frac{(T-t)^2}{4k}$ the optimal strategy is to force a lower trading speed in order to get to terminal time precisely when the inventory is 0.

As we already mentioned, in region $q < \frac{(T-t)^2}{4k}$ it is suboptimal to trade at speed c^2, as the trading speed should be decreased to allow for a longer trading period. Indeed, the main difference between problems (2.3.1) and (2.3.19) is that in the latter, we are not considering a stopping time dependent on the trading strategy and we are cutting the strategy as soon as the inventory gets to 0. The main effect in not considering such a stopping time is to get a non-continuous value function at boundary $q = 0$. Indeed, the limit for $q \to 0$ of v^1 converges to a strictly positive value for any $t < T$. The main issue in removing the boundary condition in the HJB equation is that we allow the solution v^1 to be faraway from 0 for $q \to 0$ and so we are not forcing $\partial_q v^1$ to be strongly positive near the boundary. Having a flat value function with respect to variable q allows the strategy to be the fastest possible, as decreasing the variable q does not affect the value function. On the other hand, $\partial_q v$ is evidently positive next to the boundary $q = 0$, and this introduces an obstacle to fast trading, as it would strongly decrease the value function. This leads to a tradeoff between fast trading that maximizes $f(t,c)$ and slow trading that maximizes the trading period $[t,\tau]$.

As a final remark, we would like to compare v with another approximation. We consider a value function v^3 that is equal to the one in (2.3.19) except from the fact that the upper bound of the integration interval is equal to the first time in which the strategy c^2 makes the inventory left equal to 0:

$$v^3(t,q) = \begin{cases} \sup_{c\in\mathcal{A}} \int_t^{\tau^3} c_r(T-r-kc_r)\,dr & \text{if } q > 0 \\ 0 & \text{if } q = 0, \end{cases} \tag{2.3.21}$$

where

$$\tau^3 = \inf\left\{ r \geq t \;\middle|\; q = \int_t^r c^2_s ds \right\} \wedge T = \inf\left\{ r \geq t \;\middle|\; q = \left[\frac{(T-s)^2}{4k} \right]_t^r \right\} \wedge T = T - \sqrt{(T-t)^2 - 4kq}. \tag{2.3.22}$$

Then, the value function v^3 is equal to:

$$\begin{aligned} v^3(t,q) &= \int_t^{T-\sqrt{(T-t)^2-4kq}} \sup_{c\geq 0}\left[c(T-r-kc)\right]\,dr = \frac{(T-t)^3-((T-t)^2-4kq)^{3/2}}{12k} \\ &= \frac{(T-t)^3}{12k}\left(1-\left(1-\frac{4kq}{(T-t)^2}\right)^{3/2}\right). \end{aligned} \tag{2.3.23}$$

We now focus on the region $q < \frac{(T-t)^2}{4k}$, which is the most critical, as we showed earlier. An easy calculation shows that $v(t,q) \geq v^3(t,a,q)$ is equivalent to

$$q\left(T-t-\frac{4\sqrt{kq}}{3}\right) \geq \frac{(T-t)^3}{12k}\left(1-\left(1-\frac{4kq}{(T-t)^2}\right)^{3/2}\right)$$

which is equivalent to

$$\frac{4kq}{(T-t)^2}\left(3-2\sqrt{\frac{4kq}{(T-t)^2}}\right) \geq 1-\left(1-\frac{4kq}{(T-t)^2}\right)^{3/2}$$

which holds true as $x\,(3-2\sqrt{x}) \geq 1-(1-x)^{3/2}$ for $x = \frac{4kq}{(T-t)^2} \in [0,1]$. Hence, we conclude that $v(t,a,q) \geq v^3(t,a,q)$ and so the strategy c is more optimal than strategy c^3, which consists in always trading at the fastest rate to maximize f as long the agent holds stocks.

2.4 Stochastic Example

In this section we describe an extended version of the example in the previous section, including a source of randomness in the stock price. We show that the usual version of the SMP is not satisfied, while Theorem 2.2.3 is satisfied under the new stochastic framework as well. We consider a simple stochastic framework in which we are still able to find a close-form solution. We consider a liquidation problem with no market impact on trade and no terminal execution. In particular, $g = 0$ and $f(\pi,x) = \pi x$. We further assume the control to be non-negative and smaller than a threshold $c^+ > 0$. Let $t \in [0,T]$ and $q > 0$ be fixed. We consider the stock price X to be a Geometric Brownian Motion so that for any $r \in [t,T]$ and for any $x > 0$

$$dX_r = X_r dW_r, \quad X_t = x. \tag{2.4.1}$$

The value function associated to this problem is

$$v(t,x,q) = \sup_{\pi\in\mathcal{A}} \mathbb{E}^t\left[\int_t^{\tau^\pi} \pi_r X_r\,dr\right], \tag{2.4.2}$$

where the stopping time τ^π is defined as in (2.2.4) and $\mathcal{A}$ is the set of all progressively measurable, right-continuous and a.s. bounded processes in the interval $[0,c^+]$. We define a control strategy as follows for any $r \in [t, T]$

$$c_r = \begin{cases} \frac{q}{T-t} & \text{if } q \le c^+(T-t) \\ c^+ & \text{if } q > c^+(T-t) \end{cases}. \tag{2.4.3}$$

Previous control is well defined. Indeed, using the fact that $q > 0$, for $t = T$ the second region is considered. We also observe that the previous control is constant in time.

Proposition 2.4.1. *Under trading strategy c_r in* (2.4.3)*, by applying the definition of Q_r in* (2.2.1)*, we have that for any $r \in [t, T]$ the inventory Q_r has the following behaviour*

$$Q_r = \begin{cases} \frac{q}{T-t}(T-r) & \text{if } q \le c^+(T-t) \\ q-(r-t)c^+ & \text{if } q > c^+(T-t) \end{cases}. \tag{2.4.4}$$

We also have that for any $r \in [t, T]$

$$\begin{aligned} Q_r \le c^+(T-r) &\Leftrightarrow q \le c^+(T-t), \\ Q_r > c^+(T-r) &\Leftrightarrow q > c^+(T-t). \end{aligned} \tag{2.4.5}$$

Using the expression for Q_r that we just found, it can be easily calculated that the first hitting time of $Q_r = 0$ is

$$\tau = T \quad \mathbb{P}\text{-}a.s.. \tag{2.4.6}$$

Finally, by applying (2.4.3)*,* (2.4.4) *and* (2.4.6) *into* (2.4.2)*, easy calculus shows that the value function associated with control* (2.4.3) *is equal to*

$$v^c(t, x, q) = \begin{cases} qx & \text{if } q \le c^+(T-t) \\ xc^+(T-t) & \text{if } q > c^+(T-t) \end{cases}. \tag{2.4.7}$$

The proof of previous Proposition is in the Appendix.

Using following proposition, we prove that control (2.4.3) and its associated value function (2.4.7) are respectively the optimal trading strategy and the value function associated to problem (2.4.2).

Proposition 2.4.2. *Function v^c in* (2.4.7) *coincides with the value function of problem* (2.4.2)*. Therefore, $v(t, x, q) = v^c(t, x, q)$ for any $t \in [0, T]$, $x > 0$ and $q > 0$.*

Proof. By definition of $v^c(t, x, q)$, we have that for any $t \in [0, T]$, $x > 0$ and $q > 0$

$$v^c(t, x, q) = \mathbb{E}^t \left[\int_t^{\tau} c_r X_r dr \right] \le \sup_{\pi \in \mathcal{A}} \mathbb{E}^t \left[\int_t^{\tau^\pi} \pi_r X_r dr \right] = v(t, x, q). \tag{2.4.8}$$

We now want to show that $v^c \geq v$. We firstly define

$$w(t,q) = \begin{cases} q & \text{if } q \leq c^+(T-t) \\ c^+(T-t) & \text{if } q > c^+(T-t) \end{cases}, \tag{2.4.9}$$

so that $v^c(t,x,q) = xw(t,q)$. Simple calculus on w shows that

$$\partial_q w(t,q) = \begin{cases} 1 & \text{if } q \leq c^+(T-t) \\ 0 & \text{if } q > c^+(T-t) \end{cases}, \qquad \partial_t w(t,q) = \begin{cases} 0 & \text{if } q \leq c^+(T-t) \\ -c^+ & \text{if } q > c^+(T-t) \end{cases}$$

It can be easily verified that w satisfies the following HJB equation, for $t \in [0,T]$ and $q > 0$

$$\partial_t w + \sup_{\pi \in [0,c^+]} [\pi - \pi \partial_q w] = 0. \tag{2.4.10}$$

w satisfies boundary condition $w(t,0) = 0$ for any $t \in [0,T]$ and it satisfies terminal condition $w(T,q) = 0$ for any $q > 0$, as it can be easily checked from (2.4.9).

We denote as c^*_r the optimal control for the value function $v(t,x,q)$ in (2.4.2). We denote Q^*_r and τ^* respectively the inventory and the stopping time associated to control c^*. Then, using (2.2.1), for any $r \in [t,T]$

$$dw(r,Q^*_r) = \partial_t w(r,Q^*_r)dr + \partial_q w(r,Q^*_r)dQ^*_r = [\partial_t w(r,Q^*_r) - \partial_q w(r,Q^*_r)c^*_r]\, dr. \tag{2.4.11}$$

From (2.4.10) we have that for any $r \in [t,T]$ and $q > 0$, $\partial_t w(r,q) + c^*_r - c^*_r \partial_q w(r,q) \leq 0$. Then, from (2.4.11) we have that

$$dw(r,Q^*_r) \leq -c^*_r dr. \tag{2.4.12}$$

Finally, recalling that $X_t = x$, $Q^*_t = q$, $v^c(t,x,q) = xw(t,q)$, using stochastic integration by parts and (2.4.12), we have

$$\begin{aligned} v^c(\tau^*, X_{\tau^*}, Q^*_{\tau^*}) &= v^c(t, X_t, Q^*_t) + \int_t^{\tau^*} X_r dw(r,Q^*_r) + \int_t^{\tau^*} w(r,Q^*_r) dX_r \\ &= v^c(t,x,q) + \int_t^{\tau^*} X_r dw(r,Q^*_r) + \int_t^{\tau^*} w(r,Q^*_r) X_r dW_r \\ &\leq v^c(t,x,q) - \int_t^{\tau^*} X_r c^*_r dr + \int_t^{\tau^*} w(r,Q^*_r) X_r dW_r. \end{aligned} \tag{2.4.13}$$

However, using boundary and terminal condition $v^c(t,x,0) = 0$ for any $t \in [0,T]$ and $x > 0$ and $v^c(T,x,q) = 0$ for any $x > 0$ and $q > 0$, we conclude that $v^c(\tau^*, X_{\tau^*}, Q^*_{\tau^*}) = 0$, as either $\tau^* = T$ or $Q^*_{\tau^*} = 0$. Therefore, taking conditional expectations on both sides of (2.4.13) and using the fact that $\int_t^{\tau^*} w(r,Q^*_r) X_r dW_r$ is a martingale, we get

$$v^c(t,x,q) \geq \mathbb{E}^t \left[\int_t^{\tau^*} c^*_r X_r \, dr \right] = v(t,x,q). \tag{2.4.14}$$

Merging (2.4.8) and (2.4.14) we conclude the proof. □

We now want to show that the example presented in this section does not satisfy the standard formulation of the SMP 2.2.8.

Remark 2.4.3. *In the framework of the example presented in this section, the standard SMP is not satisfied. Indeed, if we apply Theorem 2.2.8 to the current example, by recalling that* $g = 0$ *and* $f(\pi, x) = \pi x$*, we see that BSDE* (2.2.22) *becomes*

$$\begin{cases} -dY_r^1 = \left(Z_r^1 + c_r\right) dr - Z_r^1 dW_r \\ dY_r^2 = Z_r^2 dW_r \\ Y_T^1 = 0 \\ Y_T^2 = 0. \end{cases} \tag{2.4.15}$$

We denote as $(\bar{Y}_r^1, \bar{Z}_r^1)_{r\in[t,T]}$ *and* $(\bar{Y}_r^2, \bar{Z}_r^2)_{r\in[t,T]}$ *the solutions to the previous BSDE. We immediately see that the processes* $(\bar{Y}_r^2, \bar{Z}_r^2)_{r\in[t,T]} \equiv (0,0)$ *is identically equal to* 0*. Therefore, recalling that the drift and diffusion coefficient for the equation* (2.4.1) *defining* X *are respectively* $\mu(t,x) = 0$ *and* $\sigma(t,x) = x$*, then the necessary condition* (2.2.23) *is equivalent to*

$$\forall t \in [0,T],\ \forall \pi \in [0, c^+], \quad X_t \bar{Z}_t^1 + f(\pi, X_t) \leq X_t \bar{Z}_t^1 + f(c_t, X_t), \tag{2.4.16}$$

which is equivalent to

$$\forall t \in [0,T],\ \forall \pi \in [0, c^+], \quad f(\pi, X_t) \leq f(c_t, X_t). \tag{2.4.17}$$

However, from (2.4.2) *we get that the maximal point of* $f(\cdot, x)$ *is* $\bar{\pi} = c^+$*. In* (2.4.3) *we derived that if* $q < c^+(T-t)$*, then the optimal strategy is* $c_t = \frac{q}{T-t} < c^+ = \bar{\pi}$*. Therefore, using linearity of* f*, we found a positive control* c^+ *so that* $f(\bar{\pi}, x) > f(c_t, x)$ *for any* $x > 0$ *and* $t \in [0,T]$*, which contradicts* (2.4.17) *and contradicts the standard SMP 2.2.8.*

We want to check that optimal control in (2.4.3) satisfies Theorem 2.2.3. We need to show that (2.2.19) holds true. Firstly, we observe that the model setup satisfies Assumption 2.2.2. Using the fact that $\mu = 0$ and that $g = 0$, we show that (2.2.19) holds true by proving that for any $\bar{c} \in [0, c^+]$ and $t \in [0, \tau)$,

$$f(\bar{c}, X_t) - f(c_t, X_t) - \bar{f}(t, \bar{c}, X_t, Q_t) \leq 0. \tag{2.4.18}$$

To show that the previous inequality holds true, we find the expression for $\bar{f}(t, \bar{c}, x, q)$ in the following proposition.

Proposition 2.4.4. *Let $t \in [0, \tau)$ be fixed. Let $(c_r)_{r\in[0,T]}$ be the optimal control in (2.4.3). Then, for any $\bar{c} \in [0, c^+]$, $x > 0$ and $q > 0$*

$$\bar{f}(t, \bar{c}, x, q) = \begin{cases} (\bar{c} - c_t)x\mathbb{1}_{\bar{c}\geq c_t} & \text{if } q \leq c^+(T-t) \\ 0 & \text{if } q > c^+(T-t). \end{cases}$$

Proof. We consider any $\theta \in (0, (T-t) \wedge \frac{q}{c^+})$, so that $\tau = T > t + \theta$. By combining (2.2.7) and (2.2.9) and using the fact that c_r in (2.4.3) is constant in time, we have for any $r \leq \tau$

$$Q_r^{\theta,\bar{c},t} = q - \bar{c}\theta - \int_{t+\theta}^{r} c_s ds = q - c_t(r-t) + \theta(c_t - \bar{c}). \tag{2.4.19}$$

If $q > c^+(T-t)$, then (2.4.3) implies that $c_t = c^+$ and from (2.4.19),

$$Q_T^{\theta,\bar{c},t} = q - c^+(T-t) + \theta(c^+ - \bar{c}) > \theta(c^+ - \bar{c}) \geq 0, \tag{2.4.20}$$

where we used the fact that $\bar{c}$ is an admissible control and so $\bar{c} \in [0, c^+]$. (2.4.20) implies that $\tau^{\theta,\bar{c},t} = T$. On the other hand, if $q \leq c^+(T-t)$, then $c_t = \frac{q}{T-t}$ and from (2.4.19),

$$Q_T^{\theta,\bar{c},t} = q - \frac{q}{T-t}(T-t) + \theta\left(\frac{q}{T-t} - \bar{c}\right) = \theta(c_t - \bar{c}) = -\theta(\bar{c} - c_t). \tag{2.4.21}$$

Hence, if $\bar{c} \leq c_t, Q_T^{\theta,\bar{c},t} \geq 0$ and so $\tau^{\theta,\bar{c},t} = T$ a.s.. If $\bar{c} > c_t, Q_T^{\theta,\bar{c},t} < 0$ and so $\tau^{\theta,\bar{c},t} < T$ a.s., so by setting (2.4.19) equal to 0, we get that $\tau^{\theta,\bar{c},t}$ must satisfy the following equation

$$0 = Q^{\theta,\bar{c},t}_{\tau^{\theta,\bar{c},t}} = q - \bar{c}\theta - \frac{q}{T-t}(\tau^{\theta,\bar{c},t} - t - \theta).$$

And so,

$$\tau^{\theta,\bar{c},t} = T - t - \bar{c}\theta\frac{T-t}{q} + t + \theta = T - \theta\left(\frac{\bar{c}}{c_t} - 1\right). \tag{2.4.22}$$

Since $\bar{c} > c_t$, then $\theta\left(\frac{\bar{c}}{c_t} - 1\right) > 0$.

In conclusion, if $q > c^+(T-t)$, then $\tau_{\min}^{\theta,\bar{c},t} = \tau_{\max}^{\theta,\bar{c},t} = T$ and so from definition (2.2.17) of $\bar{f}$ we have that $\bar{f} = 0$. On the other hand, if $q \leq c^+(T-t)$ we consider two subcases. If $\bar{c} \leq c_t$, then $\tau_{\min}^{\theta,\bar{c},t} = \tau_{\max}^{\theta,\bar{c},t} = T$, making $\bar{f} = 0$ again. If $\bar{c} > c_t$, then $\tau_{\min}^{\theta,\bar{c},t} = T - \theta\left(\frac{\bar{c}}{c_t} - 1\right)$ and $\tau_{\max}^{\theta,\bar{c},t} = T$ and so

$$\begin{aligned}\bar{f}(t, \bar{c}, x, q) &= \lim_{\theta\to 0} \mathbb{E}^t\left[\frac{1}{\theta}\int_{T-\theta\left(\frac{\bar{c}}{c_t}-1\right)}^{T} c_r X_r \, dr\right] = \lim_{\theta\to 0}\frac{1}{\theta}\int_{T-\theta\left(\frac{\bar{c}}{c_t}-1\right)}^{T} c_t\mathbb{E}^t[X_r] \, dr \\ &= \lim_{\theta\to 0}\frac{\theta\left(\frac{\bar{c}}{c_t}-1\right)}{\theta}c_t x = (\bar{c} - c_t)x.\end{aligned}$$

This concludes the proof of the proposition. □

We show that from previous expression of $\bar{f}(t,\bar{c},x,q)$, (2.4.18) follows. We split the proof of (2.4.18) in two parts. We firstly consider the case when $\bar{c} \geq c_t$. If $q \leq c^+(T-t)$, then the left-hand side of (2.4.18) is equal to $\bar{c}X_t - c_tX_t - (\bar{c}-c_t)X_t\mathbb{1}_{\bar{c}\geq c_t} = X_t \cdot \min(\bar{c}-c_t,\ 0)$, which is always non-positive. If $q > c^+(T-t)$, then the left-hand side of (2.4.18) is equal to $\bar{c}X_t - c_tX_t = (\bar{c}-c^+)X_t$ and so (2.4.18) is satisfied since $\bar{c} \leq c^+$. Hence, (2.4.18) is satisfied for any $c^+ \geq \bar{c} \geq 0$, making Theorem 2.2.3 satisfied.

2.5 Stochastic example with terminal objective g

In previous two examples we always had function $g = 0$. In this section we describe an example with a non trivial terminal objective g. We show that the usual version of the SMP is not satisfied, while Theorem 2.2.3 is satisfied under the new stochastic framework as well. We consider a simple stochastic framework in which we are still able to find a close form solution. We consider a liquidation problem with no market impact. In particular, $f(\pi) = -\pi$. We further assume the control to be non-negative and smaller than a threshold $c^+ > 0$. Let $t \in [0,T]$ and $q > 0$ be fixed. We consider the factor process X to be a solution of the following SDE so that for any $r \in [t,T]$ and for any $x \in \mathbb{R}$

$$dX_r = -dr + dW_r, \quad X_t = x. \tag{2.5.1}$$

The value function associated to this problem is

$$v(t,x,q) = \sup_{\pi\in\mathcal{A}} \mathbb{E}^t \left[X_{\tau^\pi} - \int_t^{\tau^\pi} \pi_r \, dr \right], \tag{2.5.2}$$

where the stopping time τ^π is defined as in (2.2.4) and $\mathcal{A}$ is the set of all progressively measurable, right-continuous and a.s. bounded processes in the interval $[0,c^+]$. The reason why we define the problem as in (2.5.2) is that we have a tradeoff between the ending value X_{τ^π} and the integral $-\int_t^{\tau^\pi} \pi_r \, dr$. Indeed, for control close to 0, the stopping time τ is equal to T, and so the process X_T have a small expected value, due to the negative drift in its definition. On the other hand, for a large control, the stopping time is going to be touched earlier, making the process X_τ greater, even if the integral $-\int_t^{\tau^\pi} \pi_r \, dr$ is going to be negative. We define a control strategy as follows for any $r \in [t,T]$

$$c_r = \begin{cases} c^+ & \text{if } q \leq (T-t)\frac{c^+}{c^++1} \\ 0 & \text{if } q > (T-t)\frac{c^+}{c^++1} \end{cases}. \tag{2.5.3}$$

Proposition 2.5.1. *Under trading strategy c_r in* (2.5.3)*, by applying the definition of Q_r in* (2.2.1)*, we have that for any $r \in [t,T]$ the inventory Q_r has the following behaviour*

$$Q_r = \begin{cases} q-(r-t)c^+ & \text{if } q \le (T-t)\frac{c^+}{c^++1} \\ q & \text{if } q > (T-t)\frac{c^+}{c^++1} \end{cases}. \tag{2.5.4}$$

We also have that for any $r \in [t,T]$

$$q \le (T-t)\frac{c^+}{c^++1} \;\Rightarrow\; Q_r \le (T-r)\frac{c^+}{c^++1}. \tag{2.5.5}$$

Using the expression for Q_r that we just found, it can be easily calculated that the first hitting time of $Q_r = 0$ is

$$\tau = \begin{cases} \frac{q}{c^+}+t & \text{if } q \le (T-t)\frac{c^+}{c^++1} \\ T & \text{if } q > (T-t)\frac{c^+}{c^++1} \end{cases}. \tag{2.5.6}$$

Finally, by applying (2.5.3)*,* (2.5.4) *and* (2.5.6) *into* (2.5.2)*, easy calculus shows that the value function associated with control* (2.5.3) *is equal to*

$$v^c(t,x,q) = \begin{cases} x-(c^++1)\frac{q}{c^+} & \text{if } q \le (T-t)\frac{c^+}{c^++1} \\ x-T+t & \text{if } q > (T-t)\frac{c^+}{c^++1} \end{cases}. \tag{2.5.7}$$

Proof. In the cases when $q > (T-t)\frac{c^+}{c^++1}$, from (2.5.3) we have that $c_r = 0$ and so

$$Q_r = q - \int_t^r c_s \, ds = q.$$

When $q \le (T-t)\frac{c^+}{c^++1}$, from (2.5.3) we get that $c_r = c^+$ and so

$$Q_r = q - \int_t^r c_s \, ds = q - \int_t^r c^+ \, ds = q - c^+(r-t).$$

Hence, we proved (2.5.4). We now prove (2.5.5). When $q \le (T-t)\frac{c^+}{c^++1}$, $Q_r = q - c^+(r-t) \le (T-t)\frac{c^+}{c^++1} - c^+(r-t) = c^+\frac{T-r}{c^++1} - (c^+)^2\frac{r-t}{c^++1} \le c^+\frac{T-r}{c^++1}$. This concludes the proof of (2.5.5).

We now prove (2.5.6). When $q > (T-t)\frac{c^+}{c^++1}$, $Q_r = q$ and so Q_r is strictly positive for any $r \in [t,T]$, making $\tau = T$. On the other hand, if $q \le (T-t)\frac{c^+}{c^++1}$, $Q_r = q - c^+(r-t)$, which is equal to 0 only if $r = \frac{q}{c^+}+t$. Moreover, since $q \le (T-t)\frac{c^+}{c^++1}$, $\frac{q}{c^+}+t \le \frac{T-t}{c^++1}+t \le \frac{T+c^+t}{c^++1} \le T$. This concludes the proof of (2.5.6).

Now we prove (2.5.7). When $q \le (T-t)\frac{c^+}{c^++1}$, by using that $c_r = c^+$ and (2.5.6)

$$v^c(t,x,q) = \mathbb{E}^t\left[X_\tau - \int_t^\tau c_r \, dr\right] = \mathbb{E}^t\left[X_{\frac{q}{c^+}+t}\right] - \int_t^{\frac{q}{c^+}+t} c^+ dr = x - \frac{q}{c^+} - c^+\frac{q}{c^+} = x - \frac{q}{c^+}(c^++1).$$

When $q > (T-t)\frac{c^+}{c^++1}$, by using that $c_r = 0$ and the fact that $\tau = T$, we have

$$v^c(t,x,q) = \mathbb{E}^t\left[X_\tau - \int_t^\tau c_r \, dr\right] = \mathbb{E}^t\left[X_T\right] = x - (T-t).$$

□

Using following proposition, we prove that control (2.5.3) and its associated value function (2.5.7) are respectively the optimal trading strategy and the value function associated to problem (2.5.2).

Proposition 2.5.2. *Function v^c in* (2.5.7) *coincides with the value function of problem* (2.5.2). *Therefore, $v(t,x,q) = v^c(t,x,q)$ for any $t \in [0,T]$, $x > 0$ and $q > 0$.*

Proof. By definition of $v^c(t,x,q)$, we have that for any $t \in [0,T]$, $x > 0$ and $q > 0$

$$v^c(t,x,q) = \mathbb{E}^t\left[X_\tau - \int_t^\tau c_r dr\right] \leq \sup_{\pi\in\mathcal{A}} \mathbb{E}^t\left[\mathbf{X}_{\tau^\pi} - \int_t^{\tau^\pi} \pi_r \, dr\right] = v(t,x,q). \tag{2.5.8}$$

We now want to show that $v^c \geq v$. We denote w

$$w(t,q) = \begin{cases} -(c^+ + 1)\frac{q}{c^+} & \text{if } q \leq (T-t)\frac{c^+}{c^++1} \\ -T + t & \text{if } q > (T-t)\frac{c^+}{c^++1} \end{cases} \tag{2.5.9}$$

so that $v(t,x,q) = x + w(t,q)$. Simple calculus on w shows that

$$\partial_q w(t,q) = \begin{cases} -\frac{c^++1}{c^+} & \text{if } q \leq (T-t)\frac{c^+}{c^++1} \\ 0 & \text{if } q > (T-t)\frac{c^+}{c^++1} \end{cases}, \qquad \partial_t w(t,q) = \begin{cases} 0 & \text{if } q \leq (T-t)\frac{c^+}{c^++1} \\ 1 & \text{if } q > (T-t)\frac{c^+}{c^++1} \end{cases}$$

It can be easily verified that w satisfies the following HJB equation, for $t \in [0,T]$ and $q > 0$

$$\partial_t w + \sup_{\pi\in[0,c^+]} \left[-\pi - \pi\partial_q w\right] = 1. \tag{2.5.10}$$

w satisfies boundary condition $w(t,0) = 0$ for any $t \in [0,T]$ and it satisfies terminal condition $w(T,q) = 0$ for any $q > 0$.

We denote as c_r^* the optimal control for the value function $v(t,x,q)$ in (2.5.2). We denote Q_r^* and τ^* respectively the inventory and the stopping time associated to control c^*. Then, using (2.2.1), for any $r \in [t,T]$

$$dw(r,Q_r^*) = \partial_t w(r,Q_r^*)dr + \partial_q w(r,Q_r^*)dQ_r^* = \left[\partial_t w(r,Q_r^*) - \partial_q w(r,Q_r^*)c_r^*\right] dr. \tag{2.5.11}$$

From (2.5.10) we have that for any $r \in [t,T]$ and $q > 0$, $\partial_t w(r,q) - c_r^* - c_r^*\partial_q w(r,q) \leq 1$. Then, from (2.5.11) we have that

$$dw(r,Q_r^*) \leq (1 + c_r^*)dr. \tag{2.5.12}$$

Finally, recalling that $X_t = x$, $Q^*_t = q$, $v^c(t,x,q) = x + w(t,q)$, using stochastic integration by parts and (2.5.12), we have

$$\begin{aligned}
v^c(\tau^*, X_{\tau^*}, Q^*_{\tau^*}) &= v^c(t, X_t, Q^*_t) + \int_t^{\tau^*} dv^c(r, X_r, Q^*_r) \\
&= v^c(t, X_t, Q^*_t) + \int_t^{\tau^*} dX_r + \int_t^{\tau^*} dw(r, Q^*_r) \\
&= v^c(t, x, q) - \int_t^{\tau^*} dr + \int_t^{\tau^*} dW_r + \int_t^{\tau^*} dw(r, Q^*_r) \\
&\le v^c(t, x, q) - \int_t^{\tau^*} dr + \int_t^{\tau^*} dW_r + \int_t^{\tau^*} (1 + c^*_r) dr.
\end{aligned} \tag{2.5.13}$$

However, using boundary and terminal condition $v^c(t,x,0) = x$ for any $t \in [0,T]$ and $x > 0$ and $v^c(T,x,q) = x$ for any $x > 0$ and $q > 0$, we conclude that $v^c(\tau^*, X_{\tau^*}, Q^*_{\tau^*}) = x$, as either $\tau^* = T$ or $Q^*_{\tau^*} = 0$. Therefore, taking conditional expectations on both sides of (2.5.13) and using the fact that $\int_t^{\tau^*} dW_r$ is a martingale, we get

$$\mathbb{E}^t\left[X_{\tau^*}\right] \le v^c(t,x,q) + \mathbb{E}^t\left[\int_t^{\tau^*} c^*_r \, dr\right],$$

which implies

$$v^c(t,x,q) \ge \mathbb{E}^t\left[X_{\tau^*} - \int_t^{\tau^*} c^*_r \, dr\right] = v(t,x,q). \tag{2.5.14}$$

Merging (2.5.8) and (2.5.14) we conclude the proof. □

We now want to show that the example presented in this section does not satisfy the standard formulation of the SMP 2.2.8.

Remark 2.5.3. *In the framework of the example presented in this section, the standard SMP is not satisfied. Indeed, if we apply Theorem 2.2.8 to the current example, by recalling that $g(x) = x$ and $f(\pi) = \pi$, we see that BSDE* (2.2.22) *becomes*

$$\begin{cases}
dY^1_r = Z^1_r dW_r \\
dY^2_r = Z^2_r dW_r \\
Y^1_T = 1 \\
Y^2_T = 0.
\end{cases} \tag{2.5.15}$$

We denote as $(\bar{Y}^1_r, \bar{Z}^1_r)_{r\in[t,T]}$ and $(\bar{Y}^2_r, \bar{Z}^2_r)_{r\in[t,T]}$ the solutions to the previous BSDE. We immediately see that the processes $(\bar{Y}^2_r, \bar{Z}^2_r)_{r\in[t,T]} \equiv (1,0)$ and $(\bar{Y}^2_r, \bar{Z}^2_r)_{r\in[t,T]} \equiv (0,0)$. Therefore, recalling that the drift and diffusion coefficient for the equation (2.5.1) *defining X are respectively $\mu(t,x) = -1$ and $\sigma(t,x) = 1$, then the necessary condition* (2.2.23) *is equivalent to*

$$\forall t \in [0,T],\ \forall \pi \in [0, c^+], \quad -Y^1_t + \bar{Z}^1_t + f(\pi) \le -Y^1_t + \bar{Z}^1_t + f(c_t), \tag{2.5.16}$$

which is equivalent to

$$\forall t \in [0,T],\ \forall \pi \in [0,c^+], \quad f(\pi) \le f(c_t). \tag{2.5.17}$$

However, from (2.5.2) *we get that the maximal point of* $f(\cdot)$ *is* $\bar{\pi} = 0$. *In* (2.5.3) *we derived that if* $q \le (T-t)\frac{c^+}{c^+ + 1}$, *then the optimal strategy is* $c_t = c^+ > 0 = \bar{\pi}$. *Therefore, we found a positive control* c^+ *so that* $f(\bar{\pi}) > f(c_t)$ *for any* $t \in [0,T]$, *which contradicts* (2.5.17) *and contradicts the standard SMP 2.2.8.*

We want to check that optimal control in (2.5.3) satisfies Theorem 2.2.3. We need to show that (2.2.19) holds true. Firstly, we observe that the model setup satisfies Assumption 2.2.2. Using the fact that $g(x) = x$, we show that (2.2.19) holds true by proving that for any $\bar{c} \in [0, c^+]$ and $t \in [0, \tau)$,

$$f(\bar{c}) - f(c_t) - \bar{f}(t, \bar{c}, X_t, Q_t) - \bar{g}(t, \bar{c}, X_t, Q_t) \le 0. \tag{2.5.18}$$

To show that the previous inequality holds true, we find the expression for $\bar{f}(t, \bar{c}, x, q)$ and $\bar{g}(t, \bar{c}, x, q)$ in the following proposition.

Proposition 2.5.4. *Let* $t \in [0, \tau)$ *be fixed. Let* $(c_r)_{r \in [0,T]}$ *be the optimal control in* (2.5.3). *Then, for any* $\bar{c} \in [0, c^+]$, $x > 0$ *and* $q > 0$

$$\bar{f}(t, \bar{c}, x, q) = \begin{cases} c^+ - \bar{c} & \text{if } q \le (T-t)\frac{c^+}{c^+ + 1} \\ 0 & \text{if } q > (T-t)\frac{c^+}{c^+ + 1}. \end{cases}$$

$$\bar{g}(t, \bar{c}, x, q) = 1$$

The proof of the previous proposition is in the Appendix.

We show that from previous expression of $\bar{f}(t, \bar{c}, x, q)$ and $\bar{g}(t, \bar{c}, x, q)$, (2.5.18) follows. If $q \le (T-t)\frac{c^+}{c^+ + 1}$, then the left-hand side of (2.5.18) is equal to $-\bar{c} + c_t - (c_t - \bar{c}) - 1 = -1$, which is always non-positive. If $q > (T-t)\frac{c^+}{c^+ + 1}$, then the left-hand side of (2.5.18) is equal to $-\bar{c} + c_t - 1$ and so (2.5.18) is satisfied since $c_t = 0$ and $\bar{c} \ge 0$. Hence, (2.5.18) is satisfied for any $\bar{c} \in [0, c^+]$, making Theorem 2.2.3 satisfied.

2.6 Conclusions

In this chapter we have proved a new SMP (Theorem 2.2.3) for an optimal liquidation problem with control-dependent terminal time, which is markedly different in the Hamiltonian condition from that of the standard SMP. We have given several examples to show that the optimal solution satisfies the SMP in Theorem 2.2.3 but not the standard SMP in the literature. The main intuitions gained from the examples are principally two. Firstly, it is highlighted the difference with the standard SMP, in which the optimal control can be found by maximising the Hamiltonian at each step in time, while in the stopping time version it is also important to take into account the terminal time τ embedded in $\bar{f}$ and $\bar{g}$ and it is not enough to maximise the Hamiltonian based on the state process at time t. Secondly, the examples show that the formulation of the problems with a stopping terminal time are close to free-boundary problems, in which the value functions and optimal controls have a representation usually divided in 2 or more regions. In Section 2.3.1 we explained the reasons behind such a division in two regions, that can be summarised in a region in which the terminal stopping time coincides with T and a region in which the optimal control makes the stopping time to happen before T. The function $\bar{f}$ is equal to 0 in the first region, making the usual SMP applicable, while in the second region the function $\bar{f}$ is not equal to 0, making $\mathcal{G}$ non trivial and therefore making the optimal control c and the stopping terminal time τ to have a different formulation that the one in the first region found using the standard SMP.

We also showed that it is difficult to further simplify the expressions in the formulation of Theorem 2.2.3. This is only the first step in the direction of SMP for control-dependent stopping time problems and there remain many open questions to be answered, for example, existence of pointwise limits (2.2.17) and (2.2.18), sufficient SMP for optimality, a jump diffusion control-dependent model for X process, and applications to concrete financial scenarios. We leave these and other questions for future research.

2.7 Proofs

In this section we firstly introduce some results that are needed to prove Theorem 2.2.3, which is proved at the end of the section. In this section we consider all assumptions of Theorem 2.2.3 to be satisfied, In particular, (2.2.16) and right-continuity of process c. As we mentioned in the model setup, any time we fix a $t \in [0, \tau)$, we assume that the initial inventory $q > 0$ at time t. In particular, for any fixed $t \in [0, \tau)$ we consider a partition of the whole event space $\{\tau > t\}$. The partition helps us in stating and proving some lemmas below, which are needed in the proof of Theorem 2.2.3. As general hints for better understanding the discussion in this section, we recall that τ is defined so that $Q_\tau = 0$ if $\tau < T$, while $Q_\tau \geq 0$ if $\tau = T$. Moreover, for any $r \in [t, \tau)$, $Q_r > 0$. On the other hand, $\tau^{\theta,\bar{c},t}$ is defined so that $Q^{\theta,\bar{c},t}_{\tau^{\theta,\bar{c},t}} = 0$ if $\tau^{\theta,\bar{c},t} < T$, while $Q^{\theta,\bar{c},t}_{\tau^{\theta,\bar{c},t}} \geq 0$ if $\tau^{\theta,\bar{c},t} = T$. Moreover, for any $r \in [t, \tau^{\theta,\bar{c},t})$, $Q^{\theta,\bar{c},t}_r > 0$. We firstly observe, combining (2.2.7) and

(2.2.9), that if $\theta \in \left(0, (T-t) \wedge \frac{q}{\bar{c}}\right)$, then for any $r \in [t, (t+\theta) \wedge \tau]$

$$Q_r^{\theta,\bar{c},t} = q - \int_t^r c_s^{\theta,\bar{c},t} ds = q - \bar{c}\,(r-t) \geq q - \bar{c}\theta > 0. \tag{2.7.1}$$

Therefore, if[2] $\theta < (T-t) \wedge \frac{q}{\bar{c}}$, then

$$\tau^{\theta,\bar{c},t} > (t+\theta) \wedge \tau \tag{2.7.2}$$

Let $t \in [0,\tau)$, $0 < \theta < (T-t) \wedge \frac{q}{\bar{c}}$, $\bar{c} \geq 0$ be fixed, we define the following partition of $\{\tau > t\}$, based on the 3 events when $\tau^{\theta,\bar{c},t} < \tau$, $\tau^{\theta,\bar{c},t} > \tau$ and $\tau^{\theta,\bar{c},t} = \tau$:

$$\begin{aligned} E_1^{\theta,\bar{c},t} &:= \left\{\tau^{\theta,\bar{c},t} < \tau\right\}, \\ E_2^{\theta,\bar{c},t} &:= \left\{\tau < \tau^{\theta,\bar{c},t}\right\}, \\ E_3^{\theta,\bar{c},t} &:= \left\{\tau = \tau^{\theta,\bar{c},t}\right\}. \end{aligned}$$

However, using (2.7.2), we have that in $E_1^{\theta,\bar{c},t}$ and $E_3^{\theta,\bar{c},t}$ we must have $\tau^{\theta,\bar{c},t} > t+\theta$, so we can write the previous partition as

$$\begin{aligned} E_1^{\theta,\bar{c},t} &= \left\{t < t+\theta < \tau^{\theta,\bar{c},t} < \tau\right\}, \\ E_2^{\theta,\bar{c},t} &= \left\{t < \tau < \tau^{\theta,\bar{c},t}\right\}, \\ E_3^{\theta,\bar{c},t} &= \left\{t < t+\theta < \tau = \tau^{\theta,\bar{c},t}\right\}. \end{aligned}$$

We now present the properties of the different cases $E_i^{\theta,\bar{c},t}$, for any $i \in \{1,2,3\}$. In particular, for each of the events we show a scheme for the different values of quantities $c^{\theta,\bar{c},t}$ and $Q^{\theta,\bar{c},t}$ in each of the time spans. These schemes help in understanding some steps in the proof of lemmas below.

1) On the event $E_1^{\theta,\bar{c},t}$: combining (2.2.1), (2.2.7) and (2.2.9), for $r \in [t, t+\theta]$

$$\begin{aligned} Q_r^{\theta,\bar{c},t} &= q - \int_t^r c_s^{\theta,\bar{c},t} ds = q - \int_t^r \bar{c}\,dr = q - \int_t^r c_r dr + \int_t^r c_r dr - \int_t^r \bar{c}\,dr \\ &= Q_r - \int_t^r (\bar{c} - c_r) dr = Q_r - \gamma_r^{\theta,\bar{c},t}. \end{aligned} \tag{2.7.3}$$

Using (2.7.3), for $r \in (t+\theta, \tau^{\theta,\bar{c},t}]$ we have

$$Q_r^{\theta,\bar{c},t} = Q_{t+\theta}^{\theta,\bar{c},t} - \int_{t+\theta}^r c_s^{\theta,\bar{c},t} ds = Q_{t+\theta} - \gamma_{t+\theta}^{\theta,\bar{c},t} - \int_{t+\theta}^r c_s ds = Q_r - \gamma_{t+\theta}^{\theta,\bar{c},t}.$$

Therefore, we can outline the following scheme

[2] We consider the fraction $\frac{q}{\bar{c}}$ to be equal to $+\infty$ in case when $\bar{c} = 0$

$r \in$	t – $t+\theta$	$t+\theta$ – $\tau^{\theta,\bar{c},t}$	$\tau^{\theta,\bar{c},t}$ – τ	τ – T
$c_r^{\theta,\bar{c},t} =$	$\bar{c}$	c_r	0	0
$Q_r^{\theta,\bar{c},t} =$	$Q_r - \gamma_r^{\theta,\bar{c},t}$	$Q_r - \gamma_{t+\theta}^{\theta,\bar{c},t}$	0	0
$c_r^{\theta,\bar{c},t} - c_r =$	$\bar{c} - c_r$	0	$-c_r$	0
$Q_r^{\theta,\bar{c},t} - Q_r =$	$-\gamma_r^{\theta,\bar{c},t}$	$-\gamma_{t+\theta}^{\theta,\bar{c},t}$	$-Q_r$	0

We recall that $\tau^{\theta,\bar{c},t} < \tau$ and so $\tau^{\theta,\bar{c},t} < T$, implying that $0 = Q^{\theta,\bar{c},t}_{\tau^{\theta,\bar{c},t}}$.

$$0 = Q^{\theta,\bar{c},t}_{\tau^{\theta,\bar{c},t}} = Q_{\tau^{\theta,\bar{c},t}} - \gamma^{\theta,\bar{c},t}_{t+\theta} \;\Rightarrow\; Q_{\tau^{\theta,\bar{c},t}} = \gamma^{\theta,\bar{c},t}_{t+\theta}, \text{ which also implies that } \gamma^{\theta,\bar{c},t}_{t+\theta} > 0, \tag{2.7.4}$$

since by definition of τ, for any $r \in [t,\tau)$, $Q_r > 0$.

$$\left|Q_r^{\theta,\bar{c},t} - Q_r\right| \le \max\left(\sup_{r\in[t,t+\theta]}\left|\gamma_r^{\theta,\bar{c},t}\right|,\ |Q_{\tau^{\theta,\bar{c},t}}|\right) = \sup_{r\in[t,t+\theta]}\left|\gamma_r^{\theta,\bar{c},t}\right|, \qquad \forall r\in[t,T], \tag{2.7.5}$$

$$Q_{\tau^{\theta,\bar{c},t}} - Q_\tau \le Q_{\tau^{\theta,\bar{c},t}} = \gamma^{\theta,\bar{c},t}_{t+\theta}, \tag{2.7.6}$$

since $Q_\tau \ge 0$.

$$\hat{Q}^{\theta,\bar{c},t}_{\tau^{\theta,\bar{c},t}} = \max\left(Q^{\theta,\bar{c},t}_{\tau^{\theta,\bar{c},t}}, Q_{\tau^{\theta,\bar{c},t}}\right) = Q_{\tau^{\theta,\bar{c},t}}, \qquad \hat{Q}^{\theta,\bar{c},t}_{\tau} = \max\left(Q^{\theta,\bar{c},t}_{\tau}, Q_\tau\right) = Q_\tau. \tag{2.7.7}$$

2) On the event $E_2^{\theta,\bar{c},t}$: We now consider two subcases. Firstly, we assume that $\tau > t+\theta$. Then, combining (2.2.1), (2.2.7) and (2.2.9), for $r \in [t, t+\theta]$

$$\begin{aligned} Q_r^{\theta,\bar{c},t} &= q - \int_t^r c_s^{\theta,\bar{c},t} ds = q - \int_t^r \bar{c}\, dr = q - \int_t^r c_r dr + \int_t^r c_r dr - \int_t^r \bar{c}\, dr \\ &= Q_r - \int_t^r (\bar{c} - c_r) dr = Q_r - \gamma_r^{\theta,\bar{c},t}. \end{aligned} \tag{2.7.8}$$

Using (2.7.8), for $r \in (t+\theta, \tau^{\theta,\bar{c},t}]$ we have

$$Q_r^{\theta,\bar{c},t} = Q_{t+\theta}^{\theta,\bar{c},t} - \int_{t+\theta}^r c_s^{\theta,\bar{c},t} ds = Q_{t+\theta} - \gamma^{\theta,\bar{c},t}_{t+\theta} - \int_{t+\theta}^r c_s ds = Q_r - \gamma^{\theta,\bar{c},t}_{t+\theta}. \tag{2.7.9}$$

We recall that $\tau < \tau^{\theta,\bar{c},t}$ and so $\tau < T$ implying that $Q_\tau = 0$. Then, using (2.7.9), for $r \in (\tau, \tau^{\theta,\bar{c},t}]$ we have

$$Q_r^{\theta,\bar{c},t} = Q_\tau^{\theta,\bar{c},t} - \int_\tau^r c_s^{\theta,\bar{c},t} ds = Q_\tau - \gamma^{\theta,\bar{c},t}_{t+\theta} + \int_\tau^r \frac{\gamma^{\theta,\bar{c},t}_{t+\theta}}{\theta} ds = -\gamma^{\theta,\bar{c},t}_{t+\theta} + \frac{r-\tau}{\theta}\gamma^{\theta,\bar{c},t}_{t+\theta}.$$

Therefore, we can outline the following scheme

$t \qquad t+\theta \qquad \tau \qquad \tau^{\theta,\bar{c},t} \qquad T$

$r \in$				
$c_r^{\theta,\bar{c},t} =$	$\bar{c}$	c_r	$-\frac{\gamma_{t+\theta}^{\theta,\bar{c},t}}{\theta}$	0
$Q_r^{\theta,\bar{c},t} =$	$Q_r - \gamma_r^{\theta,\bar{c},t}$	$Q_r - \gamma_{t+\theta}^{\theta,\bar{c},t}$	$-\gamma_{t+\theta}^{\theta,\bar{c},t}\left(1-\frac{r-\tau}{\theta}\right)$	0
$c_r^{\theta,\bar{c},t} - c_r =$	$\bar{c} - c_r$	0	$-\frac{\gamma_{t+\theta}^{\theta,\bar{c},t}}{\theta}$	0
$Q_r^{\theta,\bar{c},t} - Q_r =$	$-\gamma_r^{\theta,\bar{c},t}$	$-\gamma_{t+\theta}^{\theta,\bar{c},t}$	$-\gamma_{t+\theta}^{\theta,\bar{c},t}\left(1-\frac{r-\tau}{\theta}\right)$	0

Secondly, we assume that $\tau \leq t+\theta$. From (2.2.8), $\gamma_\tau^{\theta,\bar{c},t} = \gamma_{t+\theta}^{\theta,\bar{c},t}$. Then, combining (2.2.1), (2.2.7) and (2.2.9), for $r \in [t,\tau]$

$$\begin{aligned} Q_r^{\theta,\bar{c},t} &= q - \int_t^r c_s^{\theta,\bar{c},t} ds = q - \int_t^r \bar{c}\, dr = q - \int_t^r c_r dr + \int_t^r c_r dr - \int_t^r \bar{c}\, dr \\ &= Q_r - \int_t^r (\bar{c} - c_r) dr = Q_r - \gamma_r^{\theta,\bar{c},t}. \end{aligned} \tag{2.7.10}$$

We recall that $\tau < t+\theta$ and so that $Q_\tau = 0$. Then, using (2.7.10), for $r \in (\tau, \tau^{\theta,\bar{c},t}]$ we have

$$Q_r^{\theta,\bar{c},t} = Q_\tau^{\theta,\bar{c},t} - \int_\tau^r c_s^{\theta,\bar{c},t} ds = Q_\tau - \gamma_{t+\theta}^{\theta,\bar{c},t} + \int_\tau^r \frac{\gamma_{t+\theta}^{\theta,\bar{c},t}}{\theta} ds = -\gamma_{t+\theta}^{\theta,\bar{c},t} + \frac{r-\tau}{\theta}\gamma_{t+\theta}^{\theta,\bar{c},t}.$$

Therefore, we can outline the following scheme

$t \qquad \tau \qquad \tau^{\theta,\bar{c},t} \qquad T$

$r \in$			
$c_r^{\theta,\bar{c},t} =$	$\bar{c}$	$-\frac{\gamma_{t+\theta}^{\theta,\bar{c},t}}{\theta}$	0
$Q_r^{\theta,\bar{c},t} =$	$Q_r - \gamma_r^{\theta,\bar{c},t}$	$-\gamma_{t+\theta}^{\theta,\bar{c},t}\left(1-\frac{r-\tau}{\theta}\right)$	0
$c_r^{\theta,\bar{c},t} - c_r =$	$\bar{c} - c_r$	$-\frac{\gamma_{t+\theta}^{\theta,\bar{c},t}}{\theta}$	0
$Q_r^{\theta,\bar{c},t} - Q_r =$	$-\gamma_r^{\theta,\bar{c},t}$	$-\gamma_{t+\theta}^{\theta,\bar{c},t}\left(1-\frac{r-\tau}{\theta}\right)$	0

Merging the two previous schemes we conclude that on the event $E_2^{\theta,\bar{c},t}$,

$$Q_\tau^{\theta,\bar{c},t} = Q_\tau - \gamma_{t+\theta}^{\theta,\bar{c},t} = -\gamma_{t+\theta}^{\theta,\bar{c},t}, \text{ which implies } \gamma_{t+\theta}^{\theta,\bar{c},t} < 0, \tag{2.7.11}$$

since by definition of $\tau^{\theta,\bar{c},t}$, for any $r \in [t, \tau^{\theta,\bar{c},t})$, $Q_r^{\theta,\bar{c},t} > 0$.

$$\begin{aligned} &\text{if } \tau^{\theta,\bar{c},t} < T,\ 0 = Q_{\tau^{\theta,\bar{c},t}}^{\theta,\bar{c},t} = -\gamma_{t+\theta}^{\theta,\bar{c},t}\left(1 - \frac{\tau^{\theta,\bar{c},t} - \tau}{\theta}\right) \Rightarrow \tau^{\theta,\bar{c},t} = \tau + \theta, \\ &\text{if } \tau^{\theta,\bar{c},t} = T,\ 0 \leq Q_{\tau^{\theta,\bar{c},t}}^{\theta,\bar{c},t} = -\gamma_{t+\theta}^{\theta,\bar{c},t}\left(1 - \frac{\tau^{\theta,\bar{c},t} - \tau}{\theta}\right) \Rightarrow \tau^{\theta,\bar{c},t} \leq \tau + \theta, \end{aligned} \tag{2.7.12}$$

$$\left|Q_r^{\theta,\bar{c},t} - Q_r\right| \leq \sup_{r\in[t,t+\theta]} \left|\gamma_r^{\theta,\bar{c},t}\right|, \qquad \forall r \in [t,T], \tag{2.7.13}$$

$$\text{if } \tau^{\theta,\bar{c},t} < T,\ Q_\tau^{\theta,\bar{c},t} - Q_{\tau^{\theta,\bar{c},t}}^{\theta,\bar{c},t} = -\gamma_{t+\theta}^{\theta,\bar{c},t}, \quad \text{if } \tau^{\theta,\bar{c},t} = T,\ Q_\tau^{\theta,\bar{c},t} - Q_{\tau^{\theta,\bar{c},t}}^{\theta,\bar{c},t} \leq -\gamma_{t+\theta}^{\theta,\bar{c},t}. \tag{2.7.14}$$

$$\hat{Q}_{\tau^{\theta,\bar{c},t}}^{\theta,\bar{c},t} = \max\left(Q_{\tau^{\theta,\bar{c},t}}^{\theta,\bar{c},t}, Q_{\tau^{\theta,\bar{c},t}}\right) = Q_{\tau^{\theta,\bar{c},t}}^{\theta,\bar{c},t}, \qquad \hat{Q}_\tau^{\theta,\bar{c},t} = \max\left(Q_\tau^{\theta,\bar{c},t}, Q_\tau\right) = Q_\tau^{\theta,\bar{c},t}. \tag{2.7.15}$$

3) On the event $E_3^{\theta,\bar{c},t}$: combining (2.2.1), (2.2.7) and (2.2.9), for $r \in [t, t+\theta]$

$$\begin{aligned} Q_r^{\theta,\bar{c},t} &= q - \int_t^r c_s^{\theta,\bar{c},t} ds = q - \int_t^r \bar{c}\, dr = q - \int_t^r c_r dr + \int_t^r c_r dr - \int_t^r \bar{c}\, dr \\ &= Q_r - \int_t^r (\bar{c} - c_r) dr = Q_r - \gamma_r^{\theta,\bar{c},t}. \end{aligned} \tag{2.7.16}$$

Using (2.7.16), for $r \in (t+\theta, \tau^{\theta,\bar{c},t}]$ we have

$$Q_r^{\theta,\bar{c},t} = Q_{t+\theta}^{\theta,\bar{c},t} - \int_{t+\theta}^r c_s^{\theta,\bar{c},t} ds = Q_{t+\theta} - \gamma_{t+\theta}^{\theta,\bar{c},t} - \int_{t+\theta}^r c_s ds = Q_r - \gamma_{t+\theta}^{\theta,\bar{c},t}.$$

Therefore, we can outline the following scheme

$r \in$ t $t+\theta$ $\tau^{\theta,\bar{c},t} = \tau$ T

$c_r^{\theta,\bar{c},t} =$	$\bar{c}$	c_r	0
$Q_r^{\theta,\bar{c},t} =$	$Q_r - \gamma_r^{\theta,\bar{c},t}$	$Q_r - \gamma_{t+\theta}^{\theta,\bar{c},t}$	0
$c_r^{\theta,\bar{c},t} - c_r =$	$\bar{c} - c_r$	0	0
$Q_r^{\theta,\bar{c},t} - Q_r =$	$-\gamma_r^{\theta,\bar{c},t}$	$-\gamma_{t+\theta}^{\theta,\bar{c},t}$	0

From previous scheme we conclude that on the event $E_3^{\theta,\bar{c},t}$

$$\begin{aligned} &\text{if } Q_T = 0,\ 0 \leq Q_{\tau^{\theta,\bar{c},t}}^{\theta,\bar{c},t} = Q_{\tau^{\theta,\bar{c},t}} - \gamma_{t+\theta}^{\theta,\bar{c},t} = Q_\tau - \gamma_{t+\theta}^{\theta,\bar{c},t} = -\gamma_{t+\theta}^{\theta,\bar{c},t},\ \text{and so } \gamma_{t+\theta}^{\theta,\bar{c},t} \leq 0, \\ &\text{if } \tau = \tau^{\theta,\bar{c},t} < T,\ 0 = Q_{\tau^{\theta,\bar{c},t}}^{\theta,\bar{c},t} = Q_{\tau^{\theta,\bar{c},t}} - \gamma_{t+\theta}^{\theta,\bar{c},t} = Q_\tau - \gamma_{t+\theta}^{\theta,\bar{c},t} = -\gamma_{t+\theta}^{\theta,\bar{c},t},\ \text{and so } \gamma_{t+\theta}^{\theta,\bar{c},t} = 0, \end{aligned} \tag{2.7.17}$$

$$\left|Q_r^{\theta,\bar{c},t} - Q_r\right| \leq \sup_{r\in[t,t+\theta]} \left|\gamma_r^{\theta,\bar{c},t}\right|, \qquad \forall r \in [t,T], \tag{2.7.18}$$

$$Q_\tau - Q_{\tau^{\theta,\bar{c},t}} = 0. \tag{2.7.19}$$

From previous schemes we derive the following Lemmas.

Lemma 2.7.1. *Let $t \in [0,\tau)$ be fixed, let $\bar{c} \geq 0$ and let $\theta \in \left(0, (T-t) \wedge \frac{q}{\bar{c}}\right)$. Then the control $c^{\theta,\bar{c},t}$ in (2.2.7) is admissible.*

Proof. Firstly, we observe that control $c_r^{\theta,\bar{c},t}$ is non-negative for any $r \in [t,\tau]$, as both $\bar{c}$ and $(c_r)_{r\in[0,T]}$ are non-negative. If $\tau^{\theta,\bar{c},t} \leq \tau$ there is nothing left to be proved. If $\tau^{\theta,\bar{c},t} > \tau$, i.e. if we are in the event $E_2^{\theta,\bar{c},t}$, then using (2.7.11) we get that $\gamma_{t+\theta}^{\theta,\bar{c},t} < 0$ and so the control $c_r^{\theta,\bar{c},t}$ is non-negative for any $r \geq \tau$ as well. Progressive measurability, right-continuity and square integrability of $c^{\theta,\bar{c},t}$ immediately follow. □

Lemma 2.7.2. *Let $t \in [0,\tau)$ be fixed, let $\bar{c} \geq 0$ and let $\theta \in \left(0,(T-t)\wedge \frac{q}{\bar{c}}\right)$. Then*

$$c_r^{\theta,\bar{c},t} - c_r = 0, \quad \forall r \in \left[t+\theta, \tau_{\min}^{\theta,\bar{c},t} \vee (t+\theta)\right], \tag{2.7.20}$$

$$Q_r^{\theta,\bar{c},t} - Q_r = -\gamma_{t+\theta}^{\theta,\bar{c},t}, \quad \forall r \in \left[t+\theta, \tau_{\min}^{\theta,\bar{c},t} \vee (t+\theta)\right]. \tag{2.7.21}$$

Lemma 2.7.3. *Let $t \in [0,\tau)$ and $\bar{c} \geq 0$ be fixed and let* (2.2.16) *hold true. Then*

$$\lim_{\theta\to 0} \mathbb{E}\left[\sup_{r\in[t,t+\theta]} \left|\gamma_r^{\theta,\bar{c},t}\right|\right] = 0, \tag{2.7.22}$$

$$\lim_{\theta\to 0} \mathbb{E}\left[\sup_{r\in[t,T]} \left|Q_r^{\theta,\bar{c},t} - Q_r\right|\right] = 0. \tag{2.7.23}$$

Proof. Let $\theta \in \left(0,(T-t)\wedge \frac{q}{\bar{c}}\right)$ be fixed. From definition of $\gamma_r^{\theta,\bar{c},t}$ in (2.2.8) we immediately see that

$$\sup_{r\in[t,t+\theta]} \left|\gamma_r^{\theta,\bar{c},t}\right| \leq \int_t^{(t+\theta)\wedge\tau} |\bar{c}-c_s|ds \leq \theta\left(\bar{c} + \sup_{r\in[t,t+\theta]} c_r\right). \tag{2.7.24}$$

Merging (2.7.5), (2.7.13) and (2.7.18), we see that

$$\left|Q_r^{\theta,\bar{c},t} - Q_r\right| \leq \sup_{r\in[t,t+\theta]} \left|\gamma_r^{\theta,\bar{c},t}\right|, \quad \forall r \in [t,T]. \tag{2.7.25}$$

Therefore, merging (2.7.24) and (2.7.25) we get that

$$\mathbb{E}\left[\sup_{r\in[t,T]} \left|Q_r^{\theta,\bar{c},t} - Q_r\right|\right] \leq \theta\left(\bar{c} + \mathbb{E}\left[\sup_{r\in[t,t+\theta]} c_r\right]\right).$$

We conclude the proof by using (2.2.16). □

Lemma 2.7.4. *Let $\bar{c} \geq 0$ and $t \in [0,\tau)$ be fixed. Then*

$$\lim_{\theta\to 0} \mathbb{P}\left(\{\tau \leq t+\theta\}\right) = 0, \tag{2.7.26}$$

$$\lim_{\theta\to 0} \mathbb{P}\left(E_1^{\theta,\bar{c},t} \cap \{Q_T > 0\}\right) = 0, \tag{2.7.27}$$

$$\lim_{\theta\to 0} \mathbb{P}\left(E_1^{\theta,\bar{c},t} \cap \{\bar{c} < c_t\}\right) = 0, \tag{2.7.28}$$

$$\lim_{\theta\to 0} \mathbb{P}\left(E_2^{\theta,\bar{c},t} \cap \{\tau^{\theta,\bar{c},t} = T\}\right) = 0, \tag{2.7.29}$$

$$\lim_{\theta\to 0} \mathbb{P}\left(E_3^{\theta,\bar{c},t} \cap \{\bar{c} > c_t\} \cap \{Q_T = 0\}\right) = 0, \tag{2.7.30}$$

$$\lim_{\theta\to 0} \mathbb{P}\left(E_3^{\theta,\bar{c},t} \cap \{\tau < T\} \cap \{\bar{c} < c_t\}\right) = 0. \tag{2.7.31}$$

Proof. We firstly prove (2.7.26). We have that

$$\lim_{\theta\to 0}\mathbb{P}\left(\{\tau\le t+\theta\}\right)=\lim_{n\to\infty}\mathbb{P}\left(\left\{\tau\le t+\frac{1}{n}\right\}\right)=\mathbb{P}\left(\bigcap_{n\ge\bar n}\left\{\tau\le t+\frac{1}{n}\right\}\right)=\mathbb{P}\left(\{\tau\le t\}\right)=0.$$

In previous calculations we used that for any $n\ge\bar n=\left\lceil\frac{1}{(T-t)\wedge\frac{q}{\bar c}}\right\rceil$, the sequence of events $\left\{\tau\le t+\frac{1}{n}\right\}$ is decreasing. This concludes proof of (2.7.26). We now prove (2.7.27). Using definition of Q, we have that under event $E_1^{\theta,\bar c,t}$, $Q_\tau=Q_{\tau^{\theta,\bar c,t}}-\int_{\tau^{\theta,\bar c,t}}^{\tau}c_r dr\le Q_{\tau^{\theta,\bar c,t}}$. Moreover, if $Q_T>0$, then it necessarily implies that $\tau=T$. Using (2.7.4) we have that

$$\begin{aligned}
\lim_{\theta\to 0}\mathbb{P}\left(E_1^{\theta,\bar c,t}\cap\{Q_T>0\}\right)&\le\lim_{\theta\to 0}\mathbb{P}\left(\{Q_\tau\le Q_{\tau^{\theta,\bar c,t}}=\gamma_{t+\theta}^{\theta,\bar c,t}\}\cap\{Q_\tau>0\}\right)\\
&\le\lim_{\theta\to 0}\mathbb{P}\left(\left\{Q_\tau\le\sup_{r\in[t,t+\theta]}\left|\gamma_r^{\theta,\bar c,t}\right|\right\}\cap\{Q_\tau>0\}\right)\\
&\le\lim_{n\to\infty}\mathbb{P}\left(\left\{Q_\tau\le\sup_{r\in\left[t,t+\frac{1}{n}\right]}\left|\gamma_r^{\frac{1}{n},\bar c,t}\right|\right\}\cap\{Q_\tau>0\}\right)\\
&=\mathbb{P}\left(\bigcap_{n\ge\bar n}\left\{Q_\tau\le\int_t^{t+\frac{1}{n}}|c_r-\bar c|\,dr\right\}\cap\{Q_\tau>0\}\right)\\
&=\mathbb{P}\left(\{Q_\tau=0\}\cap\{Q_\tau>0\}\right)=0.
\end{aligned}$$

In previous calculations we used that the following sequence of events is decreasing for any $n\ge\bar n=\left\lceil\frac{1}{(T-t)\wedge\frac{q}{\bar c}}\right\rceil$

$$\left\{Q_\tau\le\int_t^{t+\frac{1}{n+1}}|c_r-\bar c|\,dr\right\}\subseteq\left\{Q_\tau\le\int_t^{t+\frac{1}{n}}|c_r-\bar c|\,dr\right\}$$

and using right-continuity of c, $\int_t^{t+\frac{1}{n}}|\bar c-c_r|\,dr$ converges to 0 $\mathbb{P}$-a.s., as $n\to\infty$. This concludes proof of (2.7.27). We now prove (2.7.28). Using (2.7.4), we have that under event $E_1^{\theta,\bar c,t}$, $\gamma_{t+\theta}^{\theta,\bar c,t}>0$ and $\tau>t+\theta$ and so

$$\begin{aligned}
\lim_{\theta\to 0}\mathbb{P}\left(E_1^{\theta,\bar c,t}\cap\{\bar c<c_t\}\right)&\le\lim_{\theta\to 0}\mathbb{P}\left(\left\{\gamma_{t+\theta}^{\theta,\bar c,t}>0\right\}\cap\{\bar c<c_t\}\right)\\
&=\lim_{\theta\to 0}\mathbb{P}\left(\left\{\theta\bar c-\int_t^{t+\theta}c_s ds>0\right\}\cap\{\bar c<c_t\}\right)\\
&\le\lim_{\theta\to 0}\mathbb{P}\left(\left\{\bar c>\inf_{r\in[t,t+\theta]}c_r\right\}\cap\{\bar c<c_t\}\right)\\
&=\lim_{n\to\infty}\mathbb{P}\left(\left\{\bar c>\inf_{r\in\left[t,t+\frac{1}{n}\right]}c_r\right\}\cap\{\bar c<c_t\}\right)\\
&=\mathbb{P}\left(\bigcap_{n\ge\bar n}\left\{\bar c>\inf_{r\in\left[t,t+\frac{1}{n}\right]}c_r\right\}\cap\{\bar c<c_t\}\right)\\
&=\mathbb{P}\left(\{\bar c\ge c_t\}\cap\{\bar c<c_t\}\right)=0.
\end{aligned}$$

In previous calculations we used right-continuity of process c and that the following sequence of events is decreasing for any $n \geq \bar{n} = \left\lceil \frac{1}{(T-t)\wedge \frac{q}{\bar{c}}} \right\rceil$

$$\left\{ c_t > \inf_{r \in \left[t, t+\frac{1}{n+1}\right]} c_r \right\} \subseteq \left\{ c_t > \inf_{r \in \left[t, t+\frac{1}{n}\right]} c_r \right\}.$$

This concludes proof of (2.7.28). We now prove (2.7.29). Using (2.7.12), we get

$$\begin{aligned} \lim_{\theta \to 0} \mathbb{P}\left(E_2^{\theta,\bar{c},t} \cap \{\tau^{\theta,\bar{c},t} = T\} \right) &\leq \lim_{\theta \to 0} \mathbb{P}\left(\left\{ \tau + \theta \geq \tau^{\theta,\bar{c},t} \right\} \cap \{\tau < T\} \cap \{\tau^{\theta,\bar{c},t} = T\} \right) \\ &= \lim_{n \to \infty} \mathbb{P}\left(\left\{ \tau + \frac{1}{n} \geq T \right\} \cap \{\tau < T\} \right) \\ &= \mathbb{P}\left(\bigcap_{n \geq \bar{n}} \left\{ \tau + \frac{1}{n} \geq T \right\} \cap \{\tau < T\} \right) = \mathbb{P}\left(\{\tau \geq T\} \cap \{\tau < T\} \right) = 0. \end{aligned}$$

In previous calculations we used that the following sequence of events is decreasing for any $n \geq \bar{n} = \left\lceil \frac{1}{(T-t)\wedge \frac{q}{\bar{c}}} \right\rceil$

$$\left\{ \tau + \frac{1}{n+1} \geq T \right\} \subseteq \left\{ \tau + \frac{1}{n} \geq T \right\}.$$

This concludes proof of (2.7.29). We now prove (2.7.30). Using (2.7.17), we get

$$\begin{aligned} \lim_{\theta \to 0} \mathbb{P}\left(E_3^{\theta,\bar{c},t} \cap \{\bar{c} > c_t\} \cap \{Q_T = 0\} \right) &\leq \lim_{\theta \to 0} \mathbb{P}\left(\{\gamma_{t+\theta}^{\theta,\bar{c},t} \leq 0\} \cap \{\bar{c} > c_t\} \right) \\ &\leq \lim_{\theta \to 0} \mathbb{P}\left(\left\{ \bar{c} \leq \sup_{r \in [t, t+\theta]} c_r \right\} \cap \{\bar{c} > c_t\} \right) \\ &\leq \lim_{n \to \infty} \mathbb{P}\left(\left\{ \bar{c} \leq \sup_{r \in \left[t, t+\frac{1}{n}\right]} c_r \right\} \cap \{\bar{c} > c_t\} \right) \\ &= \mathbb{P}\left(\bigcap_{n \geq \bar{n}} \left\{ \bar{c} \leq \sup_{r \in \left[t, t+\frac{1}{n}\right]} c_r \right\} \cap \{\bar{c} > c_t\} \right) \\ &= \mathbb{P}\left(\{\bar{c} \leq c_t\} \cap \{\bar{c} > c_t\} \right) = 0. \end{aligned}$$

In previous calculations we used right-continuity of process c and that the following sequence of events is decreasing for any $n \geq \bar{n} = \left\lceil \frac{1}{(T-t)\wedge \frac{q}{\bar{c}}} \right\rceil$

$$\left\{ c_t \leq \sup_{r \in \left[t, t+\frac{1}{n+1}\right]} c_r \right\} \subseteq \left\{ c_t \leq \sup_{r \in \left[t, t+\frac{1}{n}\right]} c_r \right\}.$$

This concludes the proof of (2.7.30). We now prove (2.7.31). Recalling that under event $E_3^{\theta,\bar{c},t}$,

$\tau < T$ implies $\tau^{\theta,\bar{c},t} < T$ and using (2.7.17), we get

$$\begin{aligned}\lim_{\theta\to 0}\mathbb{P}\left(E_3^{\theta,\bar{c},t}\cap\{\bar{c}<c_t\}\cap\{\tau<T\}\right) &\le \lim_{\theta\to 0}\mathbb{P}\left(\{\gamma_{t+\theta}^{\theta,\bar{c},t}=0\}\cap\{\bar{c}<c_t\}\right)\\ &\le \lim_{\theta\to 0}\mathbb{P}\left(\left\{\bar{c}\ge\inf_{r\in[t,t+\theta]}c_r\right\}\cap\{\bar{c}<c_t\}\right)\\ &= \lim_{n\to\infty}\mathbb{P}\left(\left\{\bar{c}\ge\inf_{r\in\left[t,t+\frac{1}{n}\right]}c_r\right\}\cap\{\bar{c}<c_t\}\right)\\ &= \mathbb{P}\left(\bigcap_{n\ge\bar{n}}\left\{\bar{c}\ge\inf_{r\in\left[t,t+\frac{1}{n}\right]}c_r\right\}\cap\{\bar{c}<c_t\}\right)\\ &= \mathbb{P}\left(\{\bar{c}\ge c_t\}\cap\{\bar{c}<c_t\}\right)=0.\end{aligned}$$

In previous calculations we used right-continuity of process c and that the following sequence of events is decreasing for any $n\ge\bar{n}=\left\lceil\frac{1}{(T-t)\wedge\frac{q}{\bar{c}}}\right\rceil$

$$\left\{c_t\ge\inf_{r\in\left[t,t+\frac{1}{n+1}\right]}c_r\right\}\subseteq\left\{c_t\ge\inf_{r\in\left[t,t+\frac{1}{n}\right]}c_r\right\}.$$

This concludes the proof of (2.7.31). □

Lemma 2.7.5. *Let $t\in[0,\tau)$ and $\bar{c}\ge 0$ be fixed. Then*

$$\lim_{\theta\to 0}\tau^{\theta,\bar{c},t}=\tau \textit{ pointwise almost everywhere.} \tag{2.7.32}$$

Moreover,

$$\lim_{\theta\to 0}\mathbb{E}\left[\left|\tau^{\theta,\bar{c},t}-\tau\right|\right]=0. \tag{2.7.33}$$

Proof. We firstly prove (2.7.32). We assume on the contrary there exists a non-null event $\mathcal{E}$, so that $\lim_{\theta\to 0}\tau^{\theta,\bar{c},t}\ne\tau$ on $\mathcal{E}$, which means that

$$\exists\gamma>0 \text{ s.t. } \forall\bar{\theta}\in\left(0,(T-t)\wedge\frac{q}{\bar{c}}\wedge\gamma\right),\ \exists\theta\in(0,\bar{\theta}) \text{ s.t. } \left|\tau-\tau^{\theta,\bar{c},t}\right|>\gamma \text{ on } \mathcal{E}. \tag{2.7.34}$$

Using that under event $E_1^{\theta,\bar{c},t}$, $\tau>\tau^{\theta,\bar{c},t}$ and so $\left|\tau-\tau^{\theta,\bar{c},t}\right|>\gamma$ implies that $\tau-\tau^{\theta,\bar{c},t}>\gamma$, which implies $Q_{\tau-\gamma}=Q_{\tau^{\theta,\bar{c},t}}-\int_{\tau^{\theta,\bar{c},t}}^{\tau-\gamma}c_r dr\le Q_{\tau^{\theta,\bar{c},t}}=\gamma_{t+\theta}^{\theta,\bar{c},t}$. Moreover, using that under event $E_2^{\theta,\bar{c},t}$, $\tau^{\theta,\bar{c},t}=(\tau+\theta)\wedge T$, $\left|\tau-\tau^{\theta,\bar{c},t}\right|>\gamma$ implies that $\theta\ge(\tau+\theta)\wedge T-\tau>\gamma$, which is never verified, as $\theta<\bar{\theta}<\gamma$. Moreover, under event $E_3^{\theta,\bar{c},t}$, we have that $\tau^{\theta,\bar{c},t}=\tau$, which never satisfies $\left|\tau-\tau^{\theta,\bar{c},t}\right|>\gamma$. Therefore, we have that (2.7.34) implies that

$$\exists\gamma>0 \text{ s.t. } \forall\bar{\theta}\in\left(0,(T-t)\wedge\frac{q}{\bar{c}}\wedge\gamma\right),\ \exists\theta\in(0,\bar{\theta}) \text{ s.t. } Q_{\tau-\gamma}\le\gamma_{t+\theta}^{\theta,\bar{c},t} \text{ on } \mathcal{E}. \tag{2.7.35}$$

Recalling that $\gamma_{t+\theta}^{\theta,\bar{c},t}=\int_t^{(t+\theta)\wedge\tau}(\bar{c}-c_r)dr\le\int_t^{(t+\theta)\wedge\tau}\bar{c}dr\le\bar{c}\theta$, expression (2.7.35) implies that $Q_{\tau-\gamma}=0$ on $\mathcal{E}$, which contradicts definition of τ, as τ should be the first time in which Q_r hits

0. Therefore, we conclude that $\mathcal{E}$ must be a $\mathbb{P}$-null set and this concludes proof of (2.7.32). To prove (2.7.33) we observe that $\left|\tau^{\theta,\bar{c},t} - \tau\right| \leq T$, independently of θ. Therefore, applying DCT we also get (2.7.33). $\square$

Lemma 2.7.6. *Let $t \in [0,\tau)$, $\bar{c} \geq 0$ and $p \in [1,2]$ be fixed. Then*

$$\lim_{\theta\to 0} \mathbb{E}\left[\left|\bar{c} - c_t - \frac{\gamma_{t+\theta}^{\theta,\bar{c},t}}{\theta}\right|^p\right] = 0, \tag{2.7.36}$$

$$\lim_{\theta\to 0} \mathbb{E}\left[\left|\frac{Q^{\theta,\bar{c},t}_{\tau_{\min}^{\theta,\bar{c},t}} - Q_{\tau_{\min}^{\theta,\bar{c},t}}}{\theta} + \bar{c} - c_t\right|^p\right] = 0. \tag{2.7.37}$$

Proof. Let $t \in [0,\tau)$ and $\theta \in \left(0, (T-t) \wedge \frac{q}{\bar{c}}\right)$ be fixed. We firstly observe that

$$\left|\bar{c} - c_t - \frac{\gamma_{t+\theta}^{\theta,\bar{c},t}}{\theta}\right| \leq \bar{c} + c_t + \frac{1}{\theta}\int_t^{(t+\theta)\wedge\tau} |\bar{c} - c_s| ds \leq 2\bar{c} + 2\sup_{s\in[t,T]} c_s,$$

which is L^p-integrable thanks to assumption (2.2.16). Moreover, we have that

$$\left|\bar{c} - c_t - \frac{\gamma_{t+\theta}^{\theta,\bar{c},t}}{\theta}\right| = \left|\bar{c} - c_t - \frac{1}{\theta}\int_t^{(t+\theta)\wedge\tau} (\bar{c} - c_s) ds\right|.$$

Therefore, by using right-continuity of control c and mean-value theorem, we conclude that the pointwise limit of the expression inside the expectation in (2.7.36) is 0. Finally, by using DCT we conclude the proof of (2.7.36).

We now prove (2.7.37). Looking at schemes above, we can immediately see that a.s.

$$Q^{\theta,\bar{c},t}_{\tau_{\min}^{\theta,\bar{c},t}} - Q_{\tau_{\min}^{\theta,\bar{c},t}} = -\gamma_{t+\theta}^{\theta,\bar{c},t}. \tag{2.7.38}$$

Therefore, by applying (2.7.36) into (2.7.37), we prove the Lemma. $\square$

Lemma 2.7.7. *Let $t \in [0,\tau)$, $\bar{c} \geq 0$ and $p \in [1,2)$ be fixed. Then*

$$\lim_{\theta\to 0} \mathbb{E}\left[\left|\frac{\hat{Q}^{\theta,\bar{c},t}_{\tau^{\theta,\bar{c},t}} - \hat{Q}^{\theta,\bar{c},t}_{\tau}}{\theta} - (\bar{c} - c_t)\mathbb{1}_{(\{Q_T=0\}\cap\{\bar{c}\geq c_t\})\cup(\{\tau<T\}\cap\{\bar{c}<c_t\})}\right|^p\right] = 0. \tag{2.7.39}$$

Proof. Let $t \in [0,\tau)$ and $\theta \in \left(0, (T-t) \wedge \frac{q}{\bar{c}}\right)$ be fixed. Using schemes above, (2.7.4), (2.7.6), (2.7.12), (2.7.14), using Hölder's inequality (with coefficients $\frac{p+2}{2p}$ and $\frac{p+2}{2-p}$) and recalling that $E_2^{\theta,\bar{c},t}$ implies that $\tau < T$, we get

$$\begin{aligned}
&\mathbb{E}\left[\left|\frac{\hat{Q}^{\theta,\bar{c},t}_{\tau^{\theta,\bar{c},t}} - \hat{Q}^{\theta,\bar{c},t}_{\tau}}{\theta} - (\bar{c} - c_t)\mathbb{1}_{(\{Q_T=0\}\cap\{\bar{c}\geq c_t\})\cup(\{\tau<T\}\cap\{\bar{c}<c_t\})}\right|^p\right] \\
&= \mathbb{E}\left[\left|\frac{Q_{\tau^{\theta,\bar{c},t}} - Q_\tau}{\theta} - (\bar{c} - c_t)\mathbb{1}_{(\{Q_T=0\}\cap\{\bar{c}\geq c_t\})\cup(\{\tau<T\}\cap\{\bar{c}<c_t\})}\right|^p \mathbb{1}_{E_1^{\theta,\bar{c},t}}\right] \\
&\quad + \mathbb{E}\left[\left|\frac{Q^{\theta,\bar{c},t}_{\tau^{\theta,\bar{c},t}} - Q^{\theta,\bar{c},t}_{\tau}}{\theta} - (\bar{c} - c_t)\right|^p \mathbb{1}_{E_2^{\theta,\bar{c},t}}\right] + \mathbb{E}\left[\left|-(\bar{c} - c_t)\mathbb{1}_{(\{Q_T=0\}\cap\{\bar{c}\geq c_t\})\cup(\{\tau<T\}\cap\{\bar{c}<c_t\})}\right|^p \mathbb{1}_{E_3^{\theta,\bar{c},t}}\right]
\end{aligned}$$

$$
\begin{aligned}
&\leq \mathbb{E}\left[\left|\frac{Q_{\tau^{\theta,\bar c,t}}-Q_\tau}{\theta}\right|^p \mathbb{1}_{E_1^{\theta,\bar c,t}\cap\{Q_T>0\}}\right] + \mathbb{E}\left[\left|\frac{Q_{\tau^{\theta,\bar c,t}}-Q_\tau}{\theta} - (\bar c - c_t)\mathbb{1}_{\tau<T}\right|^p \mathbb{1}_{E_1^{\theta,\bar c,t}\cap\{Q_T=0\}\cap\{\bar c<c_t\}}\right] \\
&\quad + \mathbb{E}\left[\left|\frac{\gamma_{t+\theta}^{\theta,\bar c,t}}{\theta} - (\bar c - c_t)\right|^p \mathbb{1}_{E_1^{\theta,\bar c,t}\cap\{Q_T=0\}\cap\{\bar c\geq c_t\}}\right] + \mathbb{E}\left[\left|\frac{\gamma_{t+\theta}^{\theta,\bar c,t}}{\theta} - (\bar c - c_t)\right|^p \mathbb{1}_{E_2^{\theta,\bar c,t}\cap\{\tau^{\theta,\bar c,t}<T\}}\right] \\
&\quad + \mathbb{E}\left[\left|\frac{Q^{\theta,\bar c,t}_{\tau^{\theta,\bar c,t}} - Q^{\theta,\bar c,t}_{\tau}}{\theta} - (\bar c - c_t)\right|^p \mathbb{1}_{E_2^{\theta,\bar c,t}\cap\{\tau^{\theta,\bar c,t}=T\}}\right] \\
&\quad + \mathbb{E}\left[\left|-(\bar c - c_t)\mathbb{1}_{\{Q_T=0\}\cap\{\bar c\geq c_t\}}\right|^p \mathbb{1}_{E_3^{\theta,\bar c,t}}\right] + \mathbb{E}\left[\left|-(\bar c - c_t)\mathbb{1}_{\{\tau<T\}\cap\{\bar c< c_t\}}\right|^p \mathbb{1}_{E_3^{\theta,\bar c,t}}\right] \\
&\leq \left(\mathbb{E}\left[\left|\frac{\sup_{r\in[t,t+\theta]}\gamma_r^{\theta,\bar c,t}}{\theta}\right|^{\frac{p+2}{2}}\right]\right)^{\frac{2p}{p+2}} \Bigg(\mathbb{P}\left(E_1^{\theta,\bar c,t}\cap\{Q_T>0\}\right)^{\frac{2-p}{p+2}} \\
&\qquad + 2^{p-1}\mathbb{P}\left(E_1^{\theta,\bar c,t}\cap\{Q_T=0\}\cap\{\bar c<c_t\}\right)^{\frac{2-p}{p+2}} + 2^{p-1}\mathbb{P}\left(E_2^{\theta,\bar c,t}\cap\{\tau^{\theta,\bar c,t}=T\}\right)^{\frac{2-p}{p+2}}\Bigg) \\
&\qquad + \mathbb{E}\left[\left|\frac{\gamma_{t+\theta}^{\theta,\bar c,t}}{\theta} - (\bar c - c_t)\right|^p\right] \\
&\quad + |\bar c - c_t|^p\Bigg(2^{p-1}\mathbb{P}\left(E_1^{\theta,\bar c,t}\cap\{Q_T=0\}\cap\{\bar c<c_t\}\right) + 2^{p-1}\mathbb{P}\left(E_2^{\theta,\bar c,t}\cap\{\tau^{\theta,\bar c,t}=T\}\right) \\
&\qquad\qquad + \mathbb{P}\left(E_3^{\theta,\bar c,t}\cap\{Q_T=0\}\cap\{\bar c>c_t\}\right) + \mathbb{P}\left(E_3^{\theta,\bar c,t}\cap\{\tau<T\}\cap\{\bar c<c_t\}\right)\Bigg).
\end{aligned}
$$

Using (2.7.24), (2.7.27), (2.7.28), (2.7.29), (2.7.30), (2.7.31) and (2.7.36) we conclude the proof of (2.7.39). □

Lemma 2.7.8. *Let $t \in [0,\tau)$ and $\bar c \geq 0$ be fixed. Then,*

$$\lim_{\theta\to 0}\mathbb{E}\left[\left|\hat Q^{\theta,\bar c,t}_{\tau^{\theta,\bar c,t}} - \hat Q^{\theta,\bar c,t}_{\tau}\right|\right] = 0, \tag{2.7.40}$$

$$\lim_{\theta\to 0}\mathbb{E}\left[\left|\hat Q^{\theta,\bar c,t}_{\tau} - Q_\tau\right|\right] = 0. \tag{2.7.41}$$

Proof. Let $\theta \in \left(0, (T-t)\wedge\frac{q}{\bar c}\right)$ be fixed. By using (2.7.6), (2.7.7), (2.7.14), (2.7.15) and recalling that under $E_3^{\theta,\bar c,t}$, $\tau^{\theta,\bar c,t} = \tau$, we have that

$$
\begin{aligned}
\mathbb{E}\left[\left|\hat Q^{\theta,\bar c,t}_{\tau^{\theta,\bar c,t}} - \hat Q^{\theta,\bar c,t}_{\tau}\right|\right] &\leq \mathbb{E}\left[\left|Q_{\tau^{\theta,\bar c,t}} - Q_\tau\right|\mathbb{1}_{E_1^{\theta,\bar c,t}}\right] + \mathbb{E}\left[\left|Q^{\theta,\bar c,t}_{\tau^{\theta,\bar c,t}} - Q^{\theta,\bar c,t}_{\tau}\right|\mathbb{1}_{E_2^{\theta,\bar c,t}}\right] \\
&\leq \mathbb{E}\left[\sup_{r\in[t,t+\theta]}\left|\gamma_r^{\theta,\bar c,t}\right|\right]
\end{aligned}
$$

By using (2.7.22) we conclude the proof of (2.7.40). We now prove (2.7.41). By (2.7.7), (2.7.13), (2.7.15), (2.7.18) and that under $E_3^{\theta,\bar c,t}$ either $\hat Q^{\theta,\bar c,t}_\tau = Q_\tau$ or $\hat Q^{\theta,\bar c,t}_\tau = Q^{\theta,\bar c,t}_\tau$, we have that

$$
\begin{aligned}
\mathbb{E}\left[\left|\hat Q^{\theta,\bar c,t}_{\tau} - Q_\tau\right|\right] &\leq \mathbb{E}\left[\left|Q^{\theta,\bar c,t}_{\tau} - Q_\tau\right|\mathbb{1}_{E_2^{\theta,\bar c,t}}\right] + \mathbb{E}\left[\left|Q^{\theta,\bar c,t}_{\tau} - Q_\tau\right|\mathbb{1}_{E_3^{\theta,\bar c,t}}\right] \\
&\leq \mathbb{E}\left[\sup_{r\in[t,t+\theta]}\left|\gamma_r^{\theta,\bar c,t}\right|\right]
\end{aligned}
$$

By using (2.7.22) we conclude the proof of (2.7.41). □

Lemma 2.7.9. *For any $(x,q),(x,q') \in \mathcal{O}$, with $q \neq q'$, we have that*

$$\left| \frac{g(x,q)-g(x,q')}{q-q'} - \partial_q g\left(x,q'\right) \right| \le K|q-q'|. \tag{2.7.42}$$

Proof. We observe that

$$\frac{g(x,q)-g(x,q')}{q-q'} = \int_0^1 \partial_q g\left(x, q' + \lambda(q-q')\right)\, d\lambda$$

and so using Assumption 2.2.2, we get

$$\begin{aligned}
\left| \frac{g(x,q)-g(x,q')}{q-q'} - \partial_q g\left(x,q'\right) \right| &= \left| \int_0^1 \partial_q g\left(x, q' + \lambda(q-q')\right)\, d\lambda - \partial_q g\left(x,q'\right) \right| \\
&\le \int_0^1 \left| \partial_q g\left(x, q' + \lambda(q-q')\right) - \partial_q g\left(x,q'\right) \right| d\lambda \\
&\le K \int_0^1 \lambda |q-q'|\, d\lambda \\
&\le \frac{K}{2}|q-q'|.
\end{aligned}$$

This proves the lemma. □

We introduce a process used in the proof of Theorem 2.2.3. Let $(\xi_r)_{r\in[t,T]}$ be the solution to the following SDEs

$$\begin{cases} d\xi_r = -(\bar{c}-c_t)\partial_q f(r, c_r, X_r, Q_r)dr \\ \xi_t = f(t,\bar{c},X_t,Q_t) - f(t,c_t,X_t,Q_t). \end{cases} \tag{2.7.43}$$

In the previous definitions process ξ is the same as process ζ in Bensoussan [11]. The corresponding part of z in Bensoussan [11] for process Q would be constantly equal to $\bar{c}-c_t$, as it can be inferred with a simple calculus.

Most of the proof in the following pages relies on arguments in Bensoussan [11]. The main difference with Bensoussan's paper is the presence of the stopping time τ in our setting, which makes necessary the introduction of the stopping time $\tau^{\theta,\bar{c},t}$ as well. This complicates all the proofs and makes necessary many adjustments, especially on those results in Bensoussan [11] that concern terminal time T that must be adapted to τ or $\tau^{\theta,\bar{c},t}$ accordingly.

Lemma 2.7.10. *Let $t \in [0,\tau)$ and $\bar{c} \ge 0$ be fixed. Then,*

$$\lim_{\theta\to 0} \mathbb{E}\left[|\xi_{\tau^{\theta,\bar{c},t}} - \xi_\tau|\right] = 0, \tag{2.7.44}$$

$$\lim_{\theta\to 0} \mathbb{E}\left[|Q_{\tau^{\theta,\bar{c},t}} - Q_\tau|\right] = 0. \tag{2.7.45}$$

Proof. From (2.7.43) and using boundedness of $\partial_q f$, we have that

$$\mathbb{E}\left[|\xi_{\tau^{\theta,\bar{c},t}} - \xi_\tau|\right] = \mathbb{E}\left[\left| -\int_\tau^{\tau^{\theta,\bar{c},t}} (\bar{c}-c_t)\partial_q f(r,c_r,X_r,Q_r)dr \right|\right] \le K|\bar{c}-c_t|\mathbb{E}\left[\left|\tau^{\theta,\bar{c},t} - \tau\right|\right].$$

Therefore, by taking the limit of previous expression and applying (2.7.33), we get (2.7.44). Moreover, from definition of Q_r, we have

$$\mathbb{E}\left[|Q_{\tau^{\theta,\bar{c},t}} - Q_\tau|\right] = \mathbb{E}\left[\left|-\int_\tau^{\tau^{\theta,\bar{c},t}} c_r dr\right|\right] \leq \left(\mathbb{E}\left[\left|\int_t^T c_r^2 dr\right|\right]\right)^{1/2} \left(\mathbb{E}\left[\left|\tau^{\theta,\bar{c},t} - \tau\right|\right]\right)^{1/2}.$$

Therefore, by taking the limit of previous expression, using (2.2.16) and applying (2.7.33), we get (2.7.45). □

Lemma 2.7.11. *Let $t \in [0,\tau)$ and $\bar{c} \geq 0$ be fixed. Then*

$$\lim_{\theta\to 0} \mathbb{E}^t\left[\frac{1}{\theta}\int_t^{\tau_{\min}^{\theta,\bar{c},t}} \left(f\left(s, c_s^{\theta,\bar{c},t}, X_s, Q_s^{\theta,\bar{c},t}\right) - f(s,c_s,X_s,Q_s)\right) ds - \xi_{\tau_{\min}^{\theta,\bar{c},t}}\right] = 0. \tag{2.7.46}$$

Proof. Let $\theta \in \left(0, (T-t)\wedge \frac{q}{\bar{c}}\right)$ be fixed. We denote for any $r \in [t,T]$

$$\tilde{f}_r^{\theta,\bar{c},t} := \frac{1}{\theta}\int_t^r \left(f\left(s, c_s^{\theta,\bar{c},t}, X_s, Q_s^{\theta,\bar{c},t}\right) - f(s,c_s,X_s,Q_s)\right) ds - \xi_r. \tag{2.7.47}$$

The proof of this lemma will be divided in 3 steps. In step 1 we prove that

$$\lim_{\theta\to 0}\mathbb{E}^t\left[\left|\tilde{f}_{t+\theta}^{\theta,\bar{c},t}\right|\right] = 0.$$

In Step 2 we prove that,

$$\lim_{\theta\to 0}\mathbb{E}^t\left[\tilde{f}_{\tau_{\min}^{\theta,\bar{c},t}}^{\theta,\bar{c},t}\mathbb{1}_{\tau_{\min}^{\theta,\bar{c},t}\leq t+\theta}\right] = 0. \tag{2.7.48}$$

In Step 3 we prove that

$$\lim_{\theta\to 0}\mathbb{E}^t\left[\tilde{f}_{\tau_{\min}^{\theta,\bar{c},t}}^{\theta,\bar{c},t}\mathbb{1}_{\tau_{\min}^{\theta,\bar{c},t}> t+\theta}\right] = 0. \tag{2.7.49}$$

Once the proof of the 3 steps is completed, we conclude the proof of the Lemma as follows. By merging (2.7.48) and (2.7.49), we have

$$\lim_{\theta\to 0}\mathbb{E}^t\left[\tilde{f}_{\tau_{\min}^{\theta,\bar{c},t}}^{\theta,\bar{c},t}\right] = \lim_{\theta\to 0}\mathbb{E}^t\left[\tilde{f}_{\tau_{\min}^{\theta,\bar{c},t}}^{\theta,\bar{c},t}\mathbb{1}_{\tau_{\min}^{\theta,\bar{c},t}\leq t+\theta}\right] + \lim_{\theta\to 0}\mathbb{E}^t\left[\tilde{f}_{\tau_{\min}^{\theta,\bar{c},t}}^{\theta,\bar{c},t}\mathbb{1}_{\tau_{\min}^{\theta,\bar{c},t}> t+\theta}\right] = 0.$$

Step 1. From (2.7.43) and (2.7.47) and recalling that $c_t^{\theta,\bar{c},t} = \bar{c}$, we have that for any $r \in [t, t+\theta]$,

$$\begin{aligned}\tilde{f}_r^{\theta,\bar{c},t} = \frac{1}{\theta}\int_t^r \left(f\left(s, c_s^{\theta,\bar{c},t}, X_s, Q_s^{\theta,\bar{c},t}\right) - f(s,c_s,X_s,Q_s)\right) ds + \int_t^r (\bar{c}-c_t)\partial_q f(s,c_s,X_s,Q_s) ds \\ - \left(f\left(t, c_t^{\theta,\bar{c},t}, X_t, Q_t\right) - f(t,c_t,X_t,Q_t)\right).\end{aligned}$$

And so, for any $r \in [t, t+\theta]$

$$
\begin{aligned}
\tilde{f}_r^{\theta,\bar{c},t} &= \frac{1}{\theta}\int_t^r \left(f\left(s, c_s^{\theta,\bar{c},t}, X_s, Q_s^{\theta,\bar{c},t}\right) - f(s, c_s^{\theta,\bar{c},t}, X_s, Q_s)\right) ds \\
&\quad + \frac{1}{\theta}\int_t^r \left(f\left(s, c_s^{\theta,\bar{c},t}, X_s, Q_s\right) - f(t, c_t^{\theta,\bar{c},t}, X_t, Q_t)\right) ds \\
&\quad + \frac{1}{\theta}\int_t^r \left(f(t, c_t^{\theta,\bar{c},t}, X_t, Q_t) - f(t, c_t, X_t, Q_t)\right) ds \\
&\quad + \frac{1}{\theta}\int_t^r \left(f(t, c_t, X_t, Q_t) - f\left(s, c_s, X_s, Q_s\right)\right) ds \\
&\quad + f(t, c_t, X_t, Q_t) - f\left(t, c_t^{\theta,\bar{c},t}, X_t, Q_t\right) \\
&\quad + (\bar{c} - c_t)\int_t^r \partial_q f(s, c_s, X_s, Q_s) ds \qquad (2.7.50)\\
&= \frac{1}{\theta}\int_t^r \left(f\left(s, c_s^{\theta,\bar{c},t}, X_s, Q_s^{\theta,\bar{c},t}\right) - f(s, c_s^{\theta,\bar{c},t}, X_s, Q_s)\right) ds \\
&\quad + \frac{1}{\theta}\int_t^r \left(f\left(s, c_s^{\theta,\bar{c},t}, X_s, Q_s\right) - f(t, c_t^{\theta,\bar{c},t}, X_t, Q_t)\right) ds \\
&\quad + \frac{1}{\theta}\int_t^r \left(f(t, c_t, X_t, Q_t) - f\left(s, c_s, X_s, Q_s\right)\right) ds \\
&\quad + f(t, c_t, X_t, Q_t)\left(1 - \frac{r-t}{\theta}\right) + f(t, c_t^{\theta,\bar{c},t}, X_t, Q_t)\left(\frac{r-t}{\theta} - 1\right) \\
&\quad + (\bar{c} - c_t)\int_t^r \partial_q f(s, c_s, X_s, Q_s) ds.
\end{aligned}
$$

From previous expression we have that for any $r \in [t, t+\theta]$

$$
\begin{aligned}
\left|\tilde{f}_r^{\theta,\bar{c},t}\right| &\leq \frac{1}{\theta}\int_t^{t+\theta} \left|f\left(s, c_s^{\theta,\bar{c},t}, X_s, Q_s^{\theta,\bar{c},t}\right) - f(s, c_s^{\theta,\bar{c},t}, X_s, Q_s)\right| ds \\
&\quad + \frac{1}{\theta}\int_t^{t+\theta} \left|f\left(s, c_s^{\theta,\bar{c},t}, X_s, Q_s\right) - f(t, c_t^{\theta,\bar{c},t}, X_t, Q_t)\right| ds \\
&\quad + \frac{1}{\theta}\int_t^{t+\theta} \left|f(t, c_t, X_t, Q_t) - f\left(s, c_s, X_s, Q_s\right)\right| ds \qquad (2.7.51)\\
&\quad + |f(t, c_t, X_t, Q_t)|\left|1 - \frac{r-t}{\theta}\right| + |f(t, c_t^{\theta,\bar{c},t}, X_t, Q_t)|\left|\frac{r-t}{\theta} - 1\right| \\
&\quad + |\bar{c} - c_t|\int_t^{t+\theta} |\partial_q f(s, c_s, X_s, Q_s)| ds.
\end{aligned}
$$

By taking $r = t+\theta$ in previous expression, so that the second last line disappears and using

Assumption 2.2.2, boundedness of $\partial_q f$ and Hölder's inequality, we get

$$\begin{aligned}
\mathbb{E}^t\left[\left|\tilde{f}^{\theta,\bar{c},t}_{t+\theta}\right|\right] \le K\Bigg(&\mathbb{E}^t\left[\sup_{r\in[t,t+\theta]}\left|Q^{\theta,\bar{c},t}_r - Q_r\right|\right] + 2\mathbb{E}^t\left[\sup_{r\in[t,t+\theta]}|Q_r - Q_t|\right] + \theta|\bar{c}-c_t| \\
&+\left(\left(\mathbb{E}^t\left[\sup_{r\in[t,t+\theta]}|X_r - X_t|^2\right]\right)^{1/2} + \left(\mathbb{E}^t\left[\sup_{r\in[t,t+\theta]}\left|c^{\theta,\bar{c},t}_r - c^{\theta,\bar{c},t}_t\right|^2\right]\right)^{1/2}\right)\cdot \\
&\cdot\left(\mathbb{E}^t\left[\sup_{r\in[t,t+\theta]}\left(1+2|X_r|+2|c^{\theta,\bar{c},t}_r|\right)^2\right]\right)^{1/2} \\
+\frac{2}{\theta}\int_t^{t+\theta}|s-t|ds\quad &+\left(\left(\mathbb{E}^t\left[\sup_{r\in[t,t+\theta]}|X_r - X_t|^2\right]\right)^{1/2} + \left(\mathbb{E}^t\left[\sup_{r\in[t,t+\theta]}|c_r - c_t|^2\right]\right)^{1/2}\right)\cdot \\
&\cdot\left(\mathbb{E}^t\left[\sup_{r\in[t,t+\theta]}\left(1+2|X_r|+2|c_r|\right)^2\right]\right)^{1/2}\Bigg).
\end{aligned} \tag{2.7.52}$$

Using DCT, $\mathbb{E}^t\left[\sup_{r\in[t,t+\theta]}|c_r - c_t|^2\right]$ and $\mathbb{E}^t\left[\sup_{r\in[t,t+\theta]}\left|c^{\theta,\bar{c},t}_r - c^{\theta,\bar{c},t}_t\right|^2\right]$ converge to 0 as $\theta\to 0$. Indeed, c and $c^{\theta,\bar{c},t}$ are right-continuous and thanks to (2.2.16) and (2.7.24), the arguments of the expectations converge to 0 a.s. and they are bounded by $2\sup_{r\in[t,T]}|c_r|^2$ and $2\sup_{r\in[t,T]}|c^{\theta,\bar{c},t}_r|^2$, which are L^1-integrable processes. $\mathbb{E}^t\left[\sup_{r\in[t,t+\theta]}|X_r - X_t|^2\right]$ converges to 0 using standard arguments in SDE theory (c.f. Krylov [41, Corollary 2.5.12]). Moreover, using L^2-integrability of c and $c^{\theta,\bar{c},t}$ and standard arguments in SDE theory (c.f. Krylov [41, Corollary 2.5.12]), we get that $\mathbb{E}^t\left[\sup_{r\in[t,t+\theta]}\left(2|X_r|+2|c^{\theta,\bar{c},t}_r|\right)^2\right]$ and
$\mathbb{E}^t\left[\sup_{r\in[t,t+\theta]}(2|X_r|+2|c_r|)^2\right]$ are bounded independently of θ. Moreover, by definition of Q_r,

$$\mathbb{E}^t\left[\sup_{r\in[t,t+\theta]}|Q_r - Q_t|\right] = \mathbb{E}^t\left[\int_t^{t+\theta}|c_r|\,dr\right] \le \sqrt{\theta}\left(\mathbb{E}^t\left[\int_t^T c_r^2 dr\right]\right)^{1/2},$$

which converges to 0 as $c\in L^2$. Using (2.7.23), we have that $\mathbb{E}^t\left[\left(\sup_{r\in[t,t+\theta]}\left|Q^{\theta,\bar{c},t}_r - Q_r\right|\right)\right]$ converges to 0. Moreover $\frac{2}{\theta}\int_t^{t+\theta}|s-t|ds = \theta$. Therefore, by taking limit of (2.7.52) we conclude the proof of Step 1.

Step 2. From (2.7.50), using Assumption 2.2.2 and boundedness of $\partial_q f$, we get

$$\begin{aligned}
\left|\mathbb{E}^t\left[\tilde{f}^{\theta,\bar{c},t}_{\tau^{\theta,\bar{c},t}_{\min}}\mathbb{1}_{\tau^{\theta,\bar{c},t}_{\min}\le t+\theta}\right]\right| \le K\Bigg(&\mathbb{E}^t\left[\sup_{r\in[t,t+\theta]}\left|Q^{\theta,\bar{c},t}_r - Q_r\right|\right] + 2\mathbb{E}^t\left[\sup_{r\in[t,t+\theta]}|Q_r - Q_t|\right] \\
&+\theta|\bar{c}-c_t| + \frac{2}{\theta}\int_t^{t+\theta}|s-t|ds \\
&+\mathbb{E}^t\left[\sup_{r\in[t,t+\theta]}\left(|X_r - X_t| + \left|c^{\theta,\bar{c},t}_r - c^{\theta,\bar{c},t}_t\right|\right)\left(1+|X_r|+|X_t|+|c^{\theta,\bar{c},t}_r|+|c^{\theta,\bar{c},t}_t|\right)\right] \\
&+\mathbb{E}^t\left[\sup_{r\in[t,t+\theta]}\left(|X_r - X_t| + |c_r - c_t|\right)\left(1+|X_r|+|X_t|+|c_r|+|c_t|\right)\right]
\end{aligned}$$

$$+\left(|f(t,c_t,X_t,Q_t)|+|f(t,c_t^{\theta,\bar{c},t},X_t,Q_t)|\right)\mathbb{E}^t\left[\left|\frac{\tau_{\min}^{\theta,\bar{c},t}-t}{\theta}-1\right|\mathbb{1}_{\tau_{\min}^{\theta,\bar{c},t}\le t+\theta}\right].$$

The first three lines on the right-hand side of the previous expression converge to 0 similarly as we proved that (2.7.52) converges to 0 as $\theta\to 0$ in Step 1. Then, by recalling that $c_t^{\theta,\bar{c},t}=\bar{c}$, that by (2.7.2), $\left\{\tau_{\min}^{\theta,\bar{c},t}\le t+\theta\right\}=\{\tau\le t+\theta\}$, that under event $\tau_{\min}^{\theta,\bar{c},t}\le t+\theta$, then $\left|\frac{\tau_{\min}^{\theta,\bar{c},t}-t}{\theta}-1\right|\le 1$ and by using (2.7.26), we conclude that

$$\begin{aligned}\lim_{\theta\to 0}\mathbb{E}^t\left[\tilde{f}^{\theta,\bar{c},t}_{\tau_{\min}^{\theta,\bar{c},t}}\mathbb{1}_{\tau_{\min}^{\theta,\bar{c},t}\le t+\theta}\right]&\le\left(|f(t,\bar{c},X_t,Q_t)|+|f(t,c_t,X_t,Q_t)|\right)\lim_{\theta\to 0}\mathbb{E}^t\left[\left|\frac{\tau_{\min}^{\theta,\bar{c},t}-t}{\theta}-1\right|\mathbb{1}_{\tau_{\min}^{\theta,\bar{c},t}\le t+\theta}\right]\\&\le\left(|f(t,\bar{c},X_t,Q_t)|+|f(t,c_t,X_t,Q_t)|\right)\lim_{\theta\to 0}\mathbb{P}\left(\{\tau\le t+\theta\}\right)=0.\end{aligned}$$

This concludes the proof of Step 2.

Step 3. From (2.7.43) and (2.7.47) we have that for any $r\in[t+\theta,T]$,

$$\begin{aligned}\tilde{f}_r^{\theta,\bar{c},t}&=\tilde{f}_{t+\theta}^{\theta,\bar{c},t}+\frac{1}{\theta}\int_{t+\theta}^r\left(f\left(s,c_s^{\theta,\bar{c},t},X_s,Q_s^{\theta,\bar{c},t}\right)-f(s,c_s,X_s,Q_s)\right)ds+\int_{t+\theta}^r(\bar{c}-c_t)\partial_q f(s,c_s,X_s,Q_s)ds\\&=\tilde{f}_{t+\theta}^{\theta,\bar{c},t}+\frac{1}{\theta}\int_{t+\theta}^r\left(f\left(s,c_s^{\theta,\bar{c},t},X_s,Q_s^{\theta,\bar{c},t}\right)-f(s,c_s,X_s,Q_s^{\theta,\bar{c},t})\right)ds\\&\quad+\frac{1}{\theta}\int_{t+\theta}^r\left(f\left(s,c_s,X_s,Q_s^{\theta,\bar{c},t}\right)-f(s,c_s,X_s,Q_s)\right)ds+\int_{t+\theta}^r(\bar{c}-c_t)\partial_q f(s,c_s,X_s,Q_s)\\&=\tilde{f}_{t+\theta}^{\theta,\bar{c},t}+\frac{1}{\theta}\int_{t+\theta}^r\left(f\left(s,c_s^{\theta,\bar{c},t},X_s,Q_s^{\theta,\bar{c},t}\right)-f(s,c_s,X_s,Q_s^{\theta,\bar{c},t})\right)ds\\&\quad+\int_{t+\theta}^r\int_0^1\left(\partial_q f\left(s,c_s,X_s,Q_s+\lambda\left(Q_s^{\theta,\bar{c},t}-Q_s\right)\right)\frac{Q_s^{\theta,\bar{c},t}-Q_s}{\theta}+(\bar{c}-c_t)\partial_q f(s,c_s,X_s,Q_s)\right)d\lambda ds\\&=\tilde{f}_{t+\theta}^{\theta,\bar{c},t}+\frac{1}{\theta}\int_{t+\theta}^r\left(f\left(s,c_s^{\theta,\bar{c},t},X_s,Q_s^{\theta,\bar{c},t}\right)-f(s,c_s,X_s,Q_s^{\theta,\bar{c},t})\right)ds\\&\quad+\int_{t+\theta}^r\int_0^1\left(\bar{c}-c_t-\frac{Q_s-Q_s^{\theta,\bar{c},t}}{\theta}\right)\partial_q f\left(s,c_s,X_s,Q_s+\lambda\left(Q_s^{\theta,\bar{c},t}-Q_s\right)\right)d\lambda ds\\&\quad+\int_{t+\theta}^r\int_0^1(\bar{c}-c_t)\left(\partial_q f(s,c_s,X_s,Q_s)-\partial_q f\left(s,c_s,X_s,Q_s+\lambda\left(Q_s^{\theta,\bar{c},t}-Q_s\right)\right)\right)d\lambda ds.\end{aligned}$$

Therefore, by applying Assumption 2.2.2, then boundedness and Lipschitz continuity of $\partial_q f$ follows, we have that

$$\begin{aligned}\left|\tilde{f}_r^{\theta,\bar{c},t}\right|&\le\left|\tilde{f}_{t+\theta}^{\theta,\bar{c},t}\right|+\frac{1}{\theta}\int_{t+\theta}^r\left|f\left(s,c_s^{\theta,\bar{c},t},X_s,Q_s^{\theta,\bar{c},t}\right)-f(s,c_s,X_s,Q_s^{\theta,\bar{c},t})\right|ds\\&\quad+\int_{t+\theta}^r\int_0^1\left|\bar{c}-c_t-\frac{Q_s-Q_s^{\theta,\bar{c},t}}{\theta}\right|\left|\partial_q f\left(s,c_s,X_s,Q_s+\lambda\left(Q_s^{\theta,\bar{c},t}-Q_s\right)\right)\right|d\lambda ds\\&\quad+|\bar{c}-c_t|\int_{t+\theta}^r\int_0^1\left|\partial_q f(s,c_s,X_s,Q_s)-\partial_q f\left(s,c_s,X_s,Q_s+\lambda\left(Q_s^{\theta,\bar{c},t}-Q_s\right)\right)\right|d\lambda ds\\&\le\left|\tilde{f}_{t+\theta}^{\theta,\bar{c},t}\right|+\frac{K}{\theta}\int_{t+\theta}^r\left|c_s^{\theta,\bar{c},t}-c_s\right|ds+K\int_{t+\theta}^r\int_0^1\left|\bar{c}-c_t-\frac{Q_s-Q_s^{\theta,\bar{c},t}}{\theta}\right|d\lambda ds\end{aligned}$$

$$
\begin{aligned}
&\quad + K|\bar{c} - c_t| \int_0^1 \lambda d\lambda \int_{t+\theta}^r \left| Q_s^{\theta,\bar{c},t} - Q_s \right| ds \\
&= \left| \tilde{f}_{t+\theta}^{\theta,\bar{c},t} \right| + \frac{K}{\theta} \int_{t+\theta}^r \left| c_s^{\theta,\bar{c},t} - c_s \right| ds \\
&\quad + K \int_{t+\theta}^r \left| \frac{Q_s - Q_s^{\theta,\bar{c},t}}{\theta} - (\bar{c} - c_t) \right| ds + \frac{K|\bar{c} - c_t|}{2} \int_{t+\theta}^r \left| Q_s^{\theta,\bar{c},t} - Q_s \right| ds.
\end{aligned}
$$

Therefore, from previous expression and using (2.7.20) and (2.7.21), we get that

$$
\begin{aligned}
\left| \mathbb{E}^t \left[\tilde{f}_{\tau_{\min}^{\theta,\bar{c},t}}^{\theta,\bar{c},t} \mathbb{1}_{\tau_{\min}^{\theta,\bar{c},t} > t+\theta} \right] \right| &\leq \mathbb{E}^t \left[\left| \tilde{f}_{t+\theta}^{\theta,\bar{c},t} \right| \right] + K \mathbb{E}^t \left[\frac{\mathbb{1}_{\tau_{\min}^{\theta,\bar{c},t} > t+\theta}}{\theta} \int_{t+\theta}^{\tau_{\min}^{\theta,\bar{c},t}} \left| c_s^{\theta,\bar{c},t} - c_s \right| ds \right] \\
&\quad + K \mathbb{E}^t \left[\mathbb{1}_{\tau_{\min}^{\theta,\bar{c},t} > t+\theta} \int_{t+\theta}^{\tau_{\min}^{\theta,\bar{c},t}} \left| \frac{Q_s - Q_s^{\theta,\bar{c},t}}{\theta} - (\bar{c} - c_t) \right| ds \right] \\
&\quad + K \frac{|\bar{c} - c_t|}{2} \mathbb{E}^t \left[\mathbb{1}_{\tau_{\min}^{\theta,\bar{c},t} > t+\theta} \int_{t+\theta}^{\tau_{\min}^{\theta,\bar{c},t}} \left| Q_s^{\theta,\bar{c},t} - Q_s \right| ds \right] \\
&\leq \mathbb{E}^t \left[\left| \tilde{f}_{t+\theta}^{\theta,\bar{c},t} \right| \right] + K \mathbb{E}^t \left[\mathbb{1}_{\tau_{\min}^{\theta,\bar{c},t} > t+\theta} \int_{t+\theta}^{\tau_{\min}^{\theta,\bar{c},t}} \left| \frac{\gamma_{t+\theta}^{\theta,\bar{c},t}}{\theta} - (\bar{c} - c_t) \right| ds \right] \\
&\quad + K \frac{|\bar{c} - c_t|}{2} \mathbb{E}^t \left[\mathbb{1}_{\tau_{\min}^{\theta,\bar{c},t} > t+\theta} \int_{t+\theta}^{\tau_{\min}^{\theta,\bar{c},t}} \left| \gamma_{t+\theta}^{\theta,\bar{c},t} \right| ds \right] \\
&\leq \mathbb{E}^t \left[\left| \tilde{f}_{t+\theta}^{\theta,\bar{c},t} \right| \right] + KT \mathbb{E}^t \left[\left| \frac{\gamma_{t+\theta}^{\theta,\bar{c},t}}{\theta} - (\bar{c} - c_t) \right| \right] + KT \frac{|\bar{c} - c_t|}{2} \mathbb{E}^t \left[\left| \gamma_{t+\theta}^{\theta,\bar{c},t} \right| \right].
\end{aligned}
$$

By taking limit of previous expression for $\theta \to 0$, by using (2.7.22) and (2.7.36) together with Step 1, we conclude the proof of (2.7.49). This concludes the proof of Step 3 and the proof of the Lemma as well. □

2.8 Proof of Theorem 2.2.3

Let $t \in [0, \tau)$ be fixed. Since control c is optimal, it necessarily follows that for any $\bar{c} \geq 0$ and for any $\theta > 0$

$$
v^{c^{\theta,\bar{c},t}}(t, x, q) \leq v^c(t, x, q).
$$

Therefore, if the limit of previous expression exists, then we need to necessarily have that for any $\bar{c} \geq 0$

$$
\lim_{\theta \to 0} \frac{v^{c^{\theta,\bar{c},t}}(t, x, q) - v^c(t, x, q)}{\theta} \leq 0. \tag{2.8.1}
$$

By definition of v^π in (2.2.6), recalling that when $\tau_{\min}^{\theta,\bar{c},t} = \tau^{\theta,\bar{c},t}$, then for $r \geq \tau^{\theta,\bar{c},t}$, $\hat{Q}_r^{\theta,\bar{c},t} = Q_r$ and $\hat{c}_r^{\theta,\bar{c},t} = c_r$ and when $\tau_{\min}^{\theta,\bar{c},t} = \tau$, then for $r \geq \tau$, $\hat{Q}_r^{\theta,\bar{c},t} = Q_r^{\theta,\bar{c},t}$ and $\hat{c}_r^{\theta,\bar{c},t} = c_r^{\theta,\bar{c},t}$

$$\begin{aligned}
\lim_{\theta\to 0} \frac{v^{c^{\theta,\bar{c},t}}(t,x,q) - v^c(t,x,q)}{\theta} &= \lim_{\theta\to 0} \mathbb{E}^t \left[\frac{g(X_{\tau^{\theta,\bar{c},t}}, Q^{\theta,\bar{c},t}_{\tau^{\theta,\bar{c},t}})}{\theta} + \frac{1}{\theta}\int_t^{\tau^{\theta,\bar{c},t}} f(r, c_r^{\theta,\bar{c},t}, X_r, Q_r^{\theta,\bar{c},t}) dr \right] \\
&\quad - \lim_{\theta\to 0} \mathbb{E}^t \left[\frac{g(X_\tau, Q_\tau)}{\theta} + \frac{1}{\theta}\int_t^{\tau} f(r, c_r, X_r, Q_r) dr \right] \\
&= \lim_{\theta\to 0} \mathbb{E}^t \left[\frac{g(X_{\tau^{\theta,\bar{c},t}}, Q^{\theta,\bar{c},t}_{\tau^{\theta,\bar{c},t}}) - g(X_\tau, Q_\tau)}{\theta} \right] \\
&\quad + \lim_{\theta\to 0} \mathbb{E}^t \left[\frac{1}{\theta}\int_t^{\tau_{\min}^{\theta,\bar{c},t}} \Big(f(r, c_r^{\theta,\bar{c},t}, X_r, Q_r^{\theta,\bar{c},t}) - f(r, c_r, X_r, Q_r) \Big) dr \right] \\
&\quad + \lim_{\theta\to 0} \mathbb{E}^t \left[-\frac{\operatorname{sign}(\tau - \tau^{\theta,\bar{c},t})}{\theta} \int_{\tau_{\min}^{\theta,\bar{c},t}}^{\tau_{\max}^{\theta,\bar{c},t}} f(r, \hat{c}_r^{\theta,\bar{c},t}, X_r, \hat{Q}_r^{\theta,\bar{c},t}) dr \right],
\end{aligned} \tag{2.8.2}$$

where in last line we used the fact that if $\tau_{\min}^{\theta,\bar{c},t} = \tau$ and $\tau_{\max}^{\theta,\bar{c},t} = \tau^{\theta,\bar{c},t}$ then it means we are under case $E_2^{\theta,\bar{c},t}$ and so

$$-\frac{\operatorname{sign}(\tau - \tau^{\theta,\bar{c},t})}{\theta} \int_{\tau_{\min}^{\theta,\bar{c},t}}^{\tau_{\max}^{\theta,\bar{c},t}} f(r, \hat{c}_r^{\theta,\bar{c},t}, X_r, \hat{Q}_r^{\theta,\bar{c},t}) dr = \frac{1}{\theta}\int_\tau^{\tau^{\theta,\bar{c},t}} f(r, c_r^{\theta,\bar{c},t}, X_r, Q_r^{\theta,\bar{c},t}) dr.$$

On the other hand, if $\tau_{\max}^{\theta,\bar{c},t} = \tau$ and $\tau_{\min}^{\theta,\bar{c},t} = \tau^{\theta,\bar{c},t}$ then it means we are under case $E_1^{\theta,\bar{c},t}$ and so

$$-\frac{\operatorname{sign}(\tau - \tau^{\theta,\bar{c},t})}{\theta} \int_{\tau_{\min}^{\theta,\bar{c},t}}^{\tau_{\max}^{\theta,\bar{c},t}} f(r, \hat{c}_r^{\theta,\bar{c},t}, X_r, \hat{Q}_r^{\theta,\bar{c},t}) dr = -\frac{1}{\theta}\int_{\tau^{\theta,\bar{c},t}}^{\tau} f(r, c_r, X_r, Q_r) dr.$$

The first line on the right-hand side of (2.8.2) can be written as

$$\begin{aligned}
\lim_{\theta\to 0} \mathbb{E}^t \left[\frac{g(X_{\tau^{\theta,\bar{c},t}}, Q^{\theta,\bar{c},t}_{\tau^{\theta,\bar{c},t}}) - g(X_\tau, Q_\tau)}{\theta} \right] &= \lim_{\theta\to 0} \mathbb{E}^t \left[\frac{g(X_{\tau^{\theta,\bar{c},t}}, Q^{\theta,\bar{c},t}_{\tau^{\theta,\bar{c},t}}) - g(X_\tau, Q^{\theta,\bar{c},t}_{\tau^{\theta,\bar{c},t}})}{\theta} \right] \\
+ \lim_{\theta\to 0} \mathbb{E}^t \left[\frac{g(X_\tau, Q^{\theta,\bar{c},t}_{\tau_{\min}^{\theta,\bar{c},t}}) - g(X_\tau, Q_{\tau_{\min}^{\theta,\bar{c},t}})}{\theta} \right] &+ \lim_{\theta\to 0} \mathbb{E}^t \left[\frac{g(X_\tau, \hat{Q}^{\theta,\bar{c},t}_{\tau^{\theta,\bar{c},t}}) - g(X_\tau, \hat{Q}^{\theta,\bar{c},t}_{\tau})}{\theta} \right] \\
+ \lim_{\theta\to 0} \mathbb{E}^t \Bigg[\frac{g(X_\tau, Q^{\theta,\bar{c},t}_{\tau^{\theta,\bar{c},t}}) - g(X_\tau, Q^{\theta,\bar{c},t}_{\tau_{\min}^{\theta,\bar{c},t}}) + g(X_\tau, Q_{\tau_{\min}^{\theta,\bar{c},t}}) - g(X_\tau, \hat{Q}^{\theta,\bar{c},t}_{\tau^{\theta,\bar{c},t}}) + g(X_\tau, \hat{Q}^{\theta,\bar{c},t}_{\tau}) - g(X_\tau, Q_\tau)}{\theta} \Bigg]&.
\end{aligned} \tag{2.8.3}$$

Recalling that when $\tau_{\min}^{\theta,\bar{c},t} = \tau^{\theta,\bar{c},t}$, then $\hat{Q}^{\theta,\bar{c},t}_{\tau^{\theta,\bar{c},t}} = Q_{\tau^{\theta,\bar{c},t}}$ and $\hat{Q}^{\theta,\bar{c},t}_{\tau} = Q_\tau$ and when $\tau_{\min}^{\theta,\bar{c},t} = \tau$, then $\hat{Q}^{\theta,\bar{c},t}_{\tau^{\theta,\bar{c},t}} = Q^{\theta,\bar{c},t}_{\tau^{\theta,\bar{c},t}}$ and $\hat{Q}^{\theta,\bar{c},t}_{\tau} = Q^{\theta,\bar{c},t}_{\tau}$, then we have that the last element on the right-hand side of (2.8.3) is equal to 0. First line on the right-hand side of (2.8.3) is equal to $-\bar{g}(t,\bar{c},x,q)$ by its definition (2.2.18). We define $\tilde{g}$ for any $(x,q) \in \mathcal{O}, (x,q') \in \mathcal{O}$ as

$$\tilde{g}(x,q,q') := \begin{cases} \frac{g(x,q)-g(x,q')}{q-q'} & \text{if } q \neq q' \\ \partial_q g(x,q') & \text{if } q = q'. \end{cases} \tag{2.8.4}$$

From Assumption 2.2.12 we have that $\tilde{g}$ is bounded by $K(1+|x|)$. The second element on the right-hand side of (2.8.3) is equal to

$$\begin{aligned}
&\lim_{\theta\to 0}\mathbb{E}^t\left[\tilde{g}\left(X_\tau, Q^{\theta,\bar{c},t}_{\tau^{\theta,\bar{c},t}_{\min}}, Q_{\tau^{\theta,\bar{c},t}_{\min}}\right)\frac{Q^{\theta,\bar{c},t}_{\tau^{\theta,\bar{c},t}_{\min}} - Q_{\tau^{\theta,\bar{c},t}_{\min}}}{\theta}\right] \\
&\quad= \lim_{\theta\to 0}\mathbb{E}^t\left[\tilde{g}\left(X_\tau, Q^{\theta,\bar{c},t}_{\tau^{\theta,\bar{c},t}_{\min}}, Q_{\tau^{\theta,\bar{c},t}_{\min}}\right)\left(\frac{Q^{\theta,\bar{c},t}_{\tau^{\theta,\bar{c},t}_{\min}} - Q_{\tau^{\theta,\bar{c},t}_{\min}}}{\theta} + \bar{c} - c_t\right)\right] \\
&\qquad- \lim_{\theta\to 0}\mathbb{E}^t\left[(\bar{c}-c_t)\left(\tilde{g}\left(X_\tau, Q^{\theta,\bar{c},t}_{\tau^{\theta,\bar{c},t}_{\min}}, Q_{\tau^{\theta,\bar{c},t}_{\min}}\right) - \partial_q g\left(X_\tau, Q_{\tau^{\theta,\bar{c},t}_{\min}}\right)\right)\right] \\
&\qquad- (\bar{c}-c_t)\lim_{\theta\to 0}\mathbb{E}^t\left[\partial_q g\left(X_\tau, Q_{\tau^{\theta,\bar{c},t}_{\min}}\right) - \partial_q g\left(X_\tau, Q_\tau\right)\right] \\
&\qquad- (\bar{c}-c_t)\mathbb{E}^t\left[\partial_q g\left(X_\tau, Q_\tau\right)\right].
\end{aligned} \tag{2.8.5}$$

Using Hölder's inequality, boundedness of $\tilde{g}$ and (2.7.37), we get

$$\begin{aligned}
&\lim_{\theta\to 0}\mathbb{E}\left[\left|\tilde{g}\left(X_\tau, Q^{\theta,\bar{c},t}_{\tau^{\theta,\bar{c},t}_{\min}}, Q_{\tau^{\theta,\bar{c},t}_{\min}}\right)\left(\frac{Q^{\theta,\bar{c},t}_{\tau^{\theta,\bar{c},t}_{\min}} - Q_{\tau^{\theta,\bar{c},t}_{\min}}}{\theta} + \bar{c} - c_t\right)\right|\right] \\
&\quad\le K\left(\mathbb{E}\left[(1+|X_\tau|)^4\right]\right)^{\frac{1}{4}}\lim_{\theta\to 0}\left(\mathbb{E}\left[\left|\frac{Q^{\theta,\bar{c},t}_{\tau^{\theta,\bar{c},t}_{\min}} - Q_{\tau^{\theta,\bar{c},t}_{\min}}}{\theta} + \bar{c} - c_t\right|^{\frac{4}{3}}\right]\right)^{\frac{3}{4}} = 0.
\end{aligned} \tag{2.8.6}$$

Here we used standard arguments of SDE theory, i.e. $\mathbb{E}\left[\sup_{r\in[0,T]}|X_r|^4\right] < \infty$. Moreover, using (2.7.42) in Lemma 2.7.9 together with definition of $\tilde{g}$ in (2.8.4), we get that

$$\begin{aligned}
&\lim_{\theta\to 0}\mathbb{E}\left[|\bar{c}-c_t|\left|\tilde{g}\left(X_\tau, Q^{\theta,\bar{c},t}_{\tau^{\theta,\bar{c},t}_{\min}}, Q_{\tau^{\theta,\bar{c},t}_{\min}}\right) - \partial_q g\left(X_\tau, Q_{\tau^{\theta,\bar{c},t}_{\min}}\right)\right|\right] \\
&\quad\le \frac{K}{2}|\bar{c}-c_t|\lim_{\theta\to 0}\mathbb{E}\left[\left|Q^{\theta,\bar{c},t}_{\tau^{\theta,\bar{c},t}_{\min}} - Q_{\tau^{\theta,\bar{c},t}_{\min}}\right|\mathbb{1}_{Q^{\theta,\bar{c},t}_{\tau^{\theta,\bar{c},t}_{\min}} \neq Q_{\tau^{\theta,\bar{c},t}_{\min}}}\right] = 0,
\end{aligned} \tag{2.8.7}$$

where in the last line we used (2.7.23) in Lemma 2.7.3. Moreover, using Lipschitz continuity of $\partial_q g$ and (2.7.45), we get that

$$\begin{aligned}
\lim_{\theta\to 0}\mathbb{E}\left[\left|\partial_q g\left(X_\tau, Q_{\tau^{\theta,\bar{c},t}_{\min}}\right) - \partial_q g\left(X_\tau, Q_\tau\right)\right|\right] &\le K\lim_{\theta\to 0}\mathbb{E}\left[\left|Q_{\tau^{\theta,\bar{c},t}_{\min}} - Q_\tau\right|\right] \\
&\le K\lim_{\theta\to 0}\mathbb{E}\left[|Q_{\tau^{\theta,\bar{c},t}} - Q_\tau|\,\mathbb{1}_{\tau^{\theta,\bar{c},t}_{\min} = \tau^{\theta,\bar{c},t}}\right] = 0.
\end{aligned} \tag{2.8.8}$$

Hence, merging (2.8.6), (2.8.7) and (2.8.8) into (2.8.5), we get

$$\lim_{\theta\to 0}\mathbb{E}^t\left[\frac{g(X_\tau, Q^{\theta,\bar{c},t}_{\tau^{\theta,\bar{c},t}_{\min}}) - g(X_\tau, Q_{\tau^{\theta,\bar{c},t}_{\min}})}{\theta}\right] = -(\bar{c}-c_t)\mathbb{E}^t\left[\partial_q g\left(X_\tau, Q_\tau\right)\right]. \tag{2.8.9}$$

The third element on the right-hand side of (2.8.3) is equal to

$$
\begin{aligned}
&\lim_{\theta\to 0}\mathbb{E}^t\left[\tilde{g}\left(X_\tau,\hat{Q}^{\theta,\bar{c},t}_{\tau^{\theta,\bar{c},t}},\hat{Q}^{\theta,\bar{c},t}_{\tau}\right)\frac{\hat{Q}^{\theta,\bar{c},t}_{\tau^{\theta,\bar{c},t}}-\hat{Q}^{\theta,\bar{c},t}_{\tau}}{\theta}\right]\\
&=\lim_{\theta\to 0}\mathbb{E}^t\left[\tilde{g}\left(X_\tau,\hat{Q}^{\theta,\bar{c},t}_{\tau^{\theta,\bar{c},t}},\hat{Q}^{\theta,\bar{c},t}_{\tau}\right)\left(\frac{\hat{Q}^{\theta,\bar{c},t}_{\tau^{\theta,\bar{c},t}}-\hat{Q}^{\theta,\bar{c},t}_{\tau}}{\theta}-(\bar{c}-c_t)\mathbb{1}_{\Lambda(t,\bar{c})}\right)\right]\\
&\quad+\lim_{\theta\to 0}\mathbb{E}^t\left[\left(\tilde{g}\left(X_\tau,\hat{Q}^{\theta,\bar{c},t}_{\tau^{\theta,\bar{c},t}},\hat{Q}^{\theta,\bar{c},t}_{\tau}\right)-\partial_q g\left(X_\tau,\hat{Q}^{\theta,\bar{c},t}_{\tau}\right)\right)(\bar{c}-c_t)\mathbb{1}_{\Lambda(t,\bar{c})}\right]\\
&\quad+\lim_{\theta\to 0}\mathbb{E}^t\left[\left(\partial_q g\left(X_\tau,\hat{Q}^{\theta,\bar{c},t}_{\tau}\right)-\partial_q g\left(X_\tau,Q_\tau\right)\right)(\bar{c}-c_t)\mathbb{1}_{\Lambda(t,\bar{c})}\right]\\
&\quad+(\bar{c}-c_t)\mathbb{E}^t\left[\partial_q g\left(X_\tau,Q_\tau\right)\mathbb{1}_{\Lambda(t,\bar{c})}\right].
\end{aligned}
\tag{2.8.10}
$$

Using Hölder's inequality, boundedness of $\tilde{g}$ and Lemma 2.7.7, we get

$$
\begin{aligned}
&\lim_{\theta\to 0}\mathbb{E}\left[\left|\tilde{g}\left(X_\tau,\hat{Q}^{\theta,\bar{c},t}_{\tau^{\theta,\bar{c},t}},\hat{Q}^{\theta,\bar{c},t}_{\tau}\right)\left(\frac{\hat{Q}^{\theta,\bar{c},t}_{\tau^{\theta,\bar{c},t}}-\hat{Q}^{\theta,\bar{c},t}_{\tau}}{\theta}-(\bar{c}-c_t)\mathbb{1}_{\Lambda(t,\bar{c})}\right)\right|\right]\\
&\le K\left(\mathbb{E}\left[(1+|X_\tau|)^4\right]\right)^{\frac{1}{4}}\lim_{\theta\to 0}\left(\mathbb{E}\left[\left|\frac{\hat{Q}^{\theta,\bar{c},t}_{\tau^{\theta,\bar{c},t}}-\hat{Q}^{\theta,\bar{c},t}_{\tau}}{\theta}-(\bar{c}-c_t)\mathbb{1}_{\Lambda(t,\bar{c})}\right|^{\frac{4}{3}}\right]\right)^{\frac{3}{4}}\\
&=0.
\end{aligned}
\tag{2.8.11}
$$

Here we used standard arguments of SDE theory, i.e. $\mathbb{E}\left[\sup_{r\in[0,T]}|X_r|^4\right]<\infty$. Moreover, using (2.7.42) in Lemma 2.7.9 together with definition of $\tilde{g}$ in (2.8.4), we get that

$$
\begin{aligned}
&\lim_{\theta\to 0}\mathbb{E}\left[|\bar{c}-c_t|\left|\tilde{g}\left(X_\tau,\hat{Q}^{\theta,\bar{c},t}_{\tau^{\theta,\bar{c},t}},\hat{Q}^{\theta,\bar{c},t}_{\tau}\right)-\partial_q g\left(X_\tau,\hat{Q}^{\theta,\bar{c},t}_{\tau}\right)\right|\mathbb{1}_{\Lambda(t,\bar{c})}\right]\\
&\quad\le\lim_{\theta\to 0}\mathbb{E}\left[|\bar{c}-c_t|\left|\tilde{g}\left(X_\tau,\hat{Q}^{\theta,\bar{c},t}_{\tau^{\theta,\bar{c},t}},\hat{Q}^{\theta,\bar{c},t}_{\tau}\right)-\partial_q g\left(X_\tau,\hat{Q}^{\theta,\bar{c},t}_{\tau}\right)\right|\right]\\
&\quad\le\frac{K}{2}|\bar{c}-c_t|\lim_{\theta\to 0}\mathbb{E}\left[\left|\hat{Q}^{\theta,\bar{c},t}_{\tau^{\theta,\bar{c},t}}-\hat{Q}^{\theta,\bar{c},t}_{\tau}\right|\mathbb{1}_{\hat{Q}^{\theta,\bar{c},t}_{\tau^{\theta,\bar{c},t}}\neq\hat{Q}^{\theta,\bar{c},t}_{\tau}}\right]=0,
\end{aligned}
\tag{2.8.12}
$$

where in the last line we used (2.7.40) in Lemma 2.7.8. Moreover, using Lipschitz continuity of $\partial_q g$ in Assumption 2.2.2, we have that

$$
\begin{aligned}
&\lim_{\theta\to 0}\mathbb{E}\left[\left|(\bar{c}-c_t)\mathbb{1}_{\Lambda(t,\bar{c})}\left(\partial_q g\left(X_\tau,\hat{Q}^{\theta,\bar{c},t}_{\tau}\right)-\partial_q g\left(X_\tau,Q_\tau\right)\right)\right|\right]\\
&\quad\le|\bar{c}-c_t|\lim_{\theta\to 0}\mathbb{E}\left[\left|\partial_q g\left(X_\tau,\hat{Q}^{\theta,\bar{c},t}_{\tau}\right)-\partial_q g\left(X_\tau,Q_\tau\right)\right|\right]\\
&\quad\le K|\bar{c}-c_t|\lim_{\theta\to 0}\mathbb{E}\left[\left|\hat{Q}^{\theta,\bar{c},t}_{\tau}-Q_\tau\right|\right]=0,
\end{aligned}
\tag{2.8.13}
$$

where in the last equality we used (2.7.41) in Lemma 2.7.8. Hence, merging (2.8.11), (2.8.12) and (2.8.13) into (2.8.10), we get

$$
\lim_{\theta\to 0}\mathbb{E}^t\left[\frac{g(X_\tau,\hat{Q}^{\theta,\bar{c},t}_{\tau^{\theta,\bar{c},t}})-g(X_\tau,\hat{Q}^{\theta,\bar{c},t}_{\tau})}{\theta}\right]=(\bar{c}-c_t)\mathbb{E}^t\left[\partial_q g\left(X_\tau,Q_\tau\right)\mathbb{1}_{\Lambda(t,\bar{c})}\right].
\tag{2.8.14}
$$

Merging (2.2.18), (2.8.9) and (2.8.14) into (2.8.3), we conclude that the first line of the right-hand side of (2.8.2) is equal to

$$\begin{aligned}\lim_{\theta\to 0}\mathbb{E}^t\left[\frac{g(X_{\tau^{\theta,\bar{c},t}},Q^{\theta,\bar{c},t}_{\tau^{\theta,\bar{c},t}})-g(X_\tau,Q_\tau)}{\theta}\right]&=-\bar{g}(t,\bar{c},x,q)-\mathbb{E}^t\left[(\bar{c}-c_t)\partial_q g(X_\tau,Q_\tau)\right]\\&\quad+(\bar{c}-c_t)\mathbb{E}^t\left[\partial_q g(X_\tau,Q_\tau)\mathbb{1}_{\Lambda(t,\bar{c})}\right].\end{aligned}\tag{2.8.15}$$

The second and third lines of right-hand side of (2.8.2) can be written as

$$\begin{aligned}&\lim_{\theta\to 0}\mathbb{E}^t\left[\frac{1}{\theta}\int_t^{\tau^{\theta,\bar{c},t}_{\min}}\Big(f(r,c^{\theta,\bar{c},t}_r,X_r,Q^{\theta,\bar{c},t}_r)-f(r,c_r,X_r,Q_r)\Big)\,dr\right]\\&\quad-\lim_{\theta\to 0}\mathbb{E}^t\left[\frac{\operatorname{sign}(\tau-\tau^{\theta,\bar{c},t})}{\theta}\int_{\tau^{\theta,\bar{c},t}_{\min}}^{\tau^{\theta,\bar{c},t}_{\max}}f(r,\hat{c}^{\theta,\bar{c},t}_r,X_r,\hat{Q}^{\theta,\bar{c},t}_r)dr\right]\\&\quad=\lim_{\theta\to 0}\mathbb{E}^t\left[\frac{1}{\theta}\int_t^{\tau^{\theta,\bar{c},t}_{\min}}\Big(f(r,c^{\theta,\bar{c},t}_r,X_r,Q^{\theta,\bar{c},t}_r)-f(r,c_r,X_r,Q_r)\Big)\,dr-\xi_{\tau^{\theta,\bar{c},t}_{\min}}\right]+\lim_{\theta\to 0}\mathbb{E}^t\left[\xi_{\tau^{\theta,\bar{c},t}_{\min}}\right]\\&\qquad-\lim_{\theta\to 0}\mathbb{E}^t\left[\frac{\operatorname{sign}(\tau-\tau^{\theta,\bar{c},t})}{\theta}\int_{\tau^{\theta,\bar{c},t}_{\min}}^{\tau^{\theta,\bar{c},t}_{\max}}f(r,\hat{c}^{\theta,\bar{c},t}_r,X_r,\hat{Q}^{\theta,\bar{c},t}_r)dr\right].\end{aligned}\tag{2.8.16}$$

Using Lemma 2.7.11, we have that

$$\lim_{\theta\to 0}\mathbb{E}^t\left[\frac{1}{\theta}\int_t^{\tau^{\theta,\bar{c},t}_{\min}}\Big(f(r,c^{\theta,\bar{c},t}_r,X_r,Q^{\theta,\bar{c},t}_r)-f(r,c_r,X_r,Q_r)\Big)\,dr-\xi_{\tau^{\theta,\bar{c},t}_{\min}}\right]=0.\tag{2.8.17}$$

Using (2.7.44) in Lemma 2.7.10, we have that

$$\begin{aligned}\lim_{\theta\to 0}\mathbb{E}^t\left[\xi_{\tau^{\theta,\bar{c},t}_{\min}}\right]&=\lim_{\theta\to 0}\mathbb{E}^t\left[\xi_{\tau^{\theta,\bar{c},t}}\mathbb{1}_{\tau^{\theta,\bar{c},t}_{\min}=\tau^{\theta,\bar{c},t}}\right]+\lim_{\theta\to 0}\mathbb{E}^t\left[\xi_\tau\mathbb{1}_{\tau^{\theta,\bar{c},t}_{\min}=\tau}\right]\\&=\lim_{\theta\to 0}\mathbb{E}^t\left[\left(\xi_{\tau^{\theta,\bar{c},t}}-\xi_\tau\right)\mathbb{1}_{\tau^{\theta,\bar{c},t}_{\min}=\tau^{\theta,\bar{c},t}}\right]+\mathbb{E}^t\left[\xi_\tau\right]=\mathbb{E}^t\left[\xi_\tau\right].\end{aligned}\tag{2.8.18}$$

Using (2.2.17), the third limit on the right-hand side of (2.8.16) converges to $\bar{f}(t,\bar{c},x,q)$. Merging (2.8.17), (2.8.18) and (2.2.17) into (2.8.16), we get that

$$\begin{aligned}&\lim_{\theta\to 0}\mathbb{E}^t\left[\frac{1}{\theta}\int_t^{\tau^{\theta,\bar{c},t}_{\min}}\Big(f(r,c^{\theta,\bar{c},t}_r,X_r,Q^{\theta,\bar{c},t}_r)-f(r,c_r,X_r,Q_r)\Big)\,dr\right]\\&\quad-\lim_{\theta\to 0}\mathbb{E}^t\left[\frac{\operatorname{sign}(\tau-\tau^{\theta,\bar{c},t})}{\theta}\int_{\tau^{\theta,\bar{c},t}_{\min}}^{\tau^{\theta,\bar{c},t}_{\max}}f(r,\hat{c}^{\theta,\bar{c},t}_r,X_r,\hat{Q}^{\theta,\bar{c},t}_r)dr\right]=\mathbb{E}^t\left[\xi_\tau\right]-\bar{f}(t,\bar{c},x,q).\end{aligned}\tag{2.8.19}$$

Then, merging (2.8.2) together with (2.8.15) and (2.8.19), we get

$$\begin{aligned}\lim_{\theta\to 0}\frac{v^{c^{\theta,\bar{c},t}}(t,x,q)-v^c(t,x,q)}{\theta}&=\mathbb{E}^t\left[-(\bar{c}-c_t)\partial_q g(X_\tau,Q_\tau)+\xi_\tau\right]\\&\quad+(\bar{c}-c_t)\mathbb{E}^t\left[\partial_q g(X_\tau,Q_\tau)\mathbb{1}_{\Lambda(t,\bar{c})}\right]\\&\quad-\left(\bar{g}(t,\bar{c},X_t,Q_t)+\bar{f}(t,\bar{c},X_t,Q_t)\right).\end{aligned}\tag{2.8.20}$$

However, from (2.2.11) and (2.7.43), also noting $c_t^{\theta,\bar{c},t} = \bar{c}$, we have that

$$
\begin{aligned}
&\mathbb{E}^t\left[-(\bar{c}-c_t)\partial_q g(X_\tau,Q_\tau)+\xi_\tau\right] = \mathbb{E}^t\left[-(\bar{c}-c_t)Y_\tau+\xi_\tau\right]\\
&\quad= \mathbb{E}^t\left[-(\bar{c}-c_t)Y_t-(\bar{c}-c_t)\int_t^\tau dY_r+\xi_t+\int_t^\tau d\xi_r\right]\\
&\quad= \mathbb{E}^t\left[-(\bar{c}-c_t)Y_t+(\bar{c}-c_t)\int_t^\tau \partial_q f(r,c_r,X_r,Q_r)\,dr+\xi_t-\int_t^\tau(\bar{c}-c_t)\partial_q f(r,c_r,X_r,Q_r)dr\right]\\
&\quad= \mathbb{E}^t\left[-(\bar{c}-c_t)Y_t+\xi_t\right] = \mathbb{E}^t\left[-(\bar{c}-c_t)Y_t+f(t,c_t^{\theta,\bar{c},t},X_t,Q_t)-f(t,c_t,X_t,Q_t)\right]\\
&\quad= \mathbb{E}^t\left[-(\bar{c}-c_t)Y_t+f(t,\bar{c},X_t,Q_t)-f(t,c_t,X_t,Q_t)\right].
\end{aligned}
\tag{2.8.21}
$$

Moreover, under Assumption 3.2.1, the BSDE (3.2.8) admits an unique solution, as it has been proved in Royer-Carenzi [52, Theorem 2.1]. So, merging (2.8.20) and (2.8.21), we get that

$$
\begin{aligned}
\lim_{\theta\to 0}\frac{v^{c^{\theta,\bar{c},t}}(t,x,q)-v^c(t,x,q)}{\theta} &= \mathbb{E}^t\left[-(\bar{c}-c_t)Y_t+f(t,\bar{c},X_t,Q_t)-f(t,c_t,X_t,Q_t)\right]\\
&+(\bar{c}-c_t)\mathbb{E}^t\left[\partial_q g(X_\tau,Q_\tau)\mathbb{1}_{\Lambda(t,\bar{c})}\right]-\bar{g}(t,\bar{c},X_t,Q_t)-\bar{f}(t,\bar{c},X_t,Q_t).
\end{aligned}
\tag{2.8.22}
$$

Therefore, merging (2.8.22) together with (2.8.1), we get that for any $\bar{c}\geq 0$ and for any $t\in[0,\tau)$

$$
\begin{aligned}
&\mathbb{E}^t\left[-(\bar{c}-c_t)Y_t+f(t,\bar{c},X_t,Q_t)-f(t,c_t,X_t,Q_t)\right]\\
&\quad+(\bar{c}-c_t)\mathbb{E}^t\left[\partial_q g(X_\tau,Q_\tau)\mathbb{1}_{\Lambda(t,\bar{c})}\right]-\bar{g}(t,\bar{c},X_t,Q_t)-\bar{f}(t,\bar{c},X_t,Q_t)\leq 0.
\end{aligned}
$$

Since the argument of the first conditional expectation is $\mathcal{F}^t$-measurable, we have that for any $\bar{c}\geq 0$ and for any $t\in[0,\tau)$ a.s.

$$
\begin{aligned}
0&\geq -(\bar{c}-c_t)Y_t+f(t,\bar{c},X_t,Q_t)-f(t,c_t,X_t,Q_t)\\
&\quad+(\bar{c}-c_t)\mathbb{E}^t\left[\partial_q g(X_\tau,Q_\tau)\mathbb{1}_{\Lambda(t,\bar{c})}\right]-\bar{g}(t,\bar{c},X_t,Q_t)-\bar{f}(t,\bar{c},X_t,Q_t)\\
&= -\bar{c}Y_t+f(t,\bar{c},X_t,Q_t)-(-c_tY_t+f(t,c_t,X_t,Q_t))\\
&\quad+(\bar{c}-c_t)\mathbb{E}^t\left[\partial_q g(X_\tau,Q_\tau)\mathbb{1}_{\Lambda(t,\bar{c})}\right]-\bar{g}(t,\bar{c},X_t,Q_t)-\bar{f}(t,\bar{c},X_t,Q_t)\\
&= \mathcal{H}(t,\bar{c},X_t,Q_t,Y_t)-\mathcal{H}(t,c_t,X_t,Q_t,Y_t)+\mathcal{G}(t,\bar{c},c_t,X_t,Q_t),
\end{aligned}
$$

which concludes the proof of Theorem 2.2.3.

Chapter 3

Stochastic Maximum Principle for Optimal Liquidation with Control-Dependent Terminal Time and controlled State Process and Generalizations

3.1 Introduction

The main goal of this chapter is to introduce several generalizations to Theorem 2.2.3. In the previous chapter, we introduced an easy setting that could allow the reader to familiarise with the control-dependent terminal time setting. We now want to answer some questions that are left open from the previous chapter. In particular, could we introduce the control variable into the definition of the state variable X in (2.2.2)? Does the new version of the SMP Theorem 2.2.3 hold true in the case when we allow the control variable c to be negative as well? Are we able to find a sufficient statement of SMP Theorem 2.2.3?

In this chapter we try to answer previous questions. In particular, the rest of this chapter is organised as follows. In Section 3.2 we introduce the control variable into the in the drift term in the definition of the state variable X in 2.2.2. This introduces an additional difficulty in the proof of the SMP, as most of the limit in the proof of Theorem 2.2.3 must be proved again, to consider the variational process of the state process. In Subsection 3.2.1 we explain why we extended the presence of the control c only into the drift term and not into the diffusion term of the SDE defining the state process X. In Section 3.3 we allow the control variable c to take also negative values. The main reason why we only considered non-negative control in previous chapter is chronological. We initially wanted to consider a liquidation problem in which the agent is only allowed to liquidate the stock and never to buy it. This setting was limiting the application of the new version of the SMP. In particular, if we consider engineering and physical problems, the agents are usually allowed to proceed in the opposite direction to their final goal as well. In Section 3.5, our main aim is to simplify expressions of $\bar{f}$ and $\bar{g}$ in (2.2.17) and (2.2.18). The reason why we want to achieve such a simplification would be to separate the expression of $\mathcal{G}$ in (2.2.20) into two addends one referred to c and one referred to $\bar{c}$. This would allow us to separate all expressions with c on one side of inequality (2.2.19) and all expression with $\bar{c}$ on the other side of inequality (2.2.19). This would be the first step to get a formulation for a sufficient condition of the SMP with stopping terminal time. However, as we show in Section 3.5, the separation of c and $\bar{c}$ in expression of $\bar{f}$ is not possible, even if we consider simple examples of f.

3.2 Generalization of Theorem 2.2.3 with control-dependent state process

We would like to prove a generalization of Theorem 2.2.3 to include the control variable inside the SDE defining process X_r (2.2.2). We begin to include the control c_r in the drift coefficient as follows

$$dX_r = \mu(r, \pi_r, X_r)dr + \sigma(r, X_r)dW_r, \qquad X_t = x. \tag{3.2.1}$$

We define

$$v(t, \mathbf{x}, q) = \sup_{\pi \geq 0} v^{\pi}(t, \mathbf{x}, q), \tag{3.2.2}$$

where

$$v^{\pi}(t, \mathbf{x}, q) = \mathbb{E}^t \left[g(\mathbf{X}_{\tau^{\pi}}, Q^{\pi}_{\tau^{\pi}}) + \int_t^{\tau^{\pi}} f(r, \pi_r, \mathbf{X}_r, Q^{\pi}_r)\, dr \right]. \tag{3.2.3}$$

We also define

$$dX_r^{\theta,\bar{c},t} = \mu(r, \pi_r^{\theta,\bar{c},t}, X_r^{\theta,\bar{c},t})dr + \sigma(r, X_r^{\theta,\bar{c},t})dW_r, \qquad X_t = x. \tag{3.2.4}$$

In the following, we use the following assumptions on functions μ, σ, f and g.

Assumption 3.2.1. *For any $t \in [0,T]$, $\pi, \pi' \geq 0$, $x, x' \in \mathbb{R}$ and for any $q, q' \geq 0$*

$$\begin{aligned}
\left|\mu(t,\pi,x) - \mu(t,\pi',x')\right| + \left|\sigma(t,x) - \sigma(t,x')\right| &\leq K\left|x - x'\right| + K\left|\pi - \pi'\right|, \\
|\mu(t,x)| + |\sigma(t,x)| &\leq K\left(|x| + 1\right), \\
\left|g(x,q) - g(x',q)\right| &\leq K(1 + |q|)\left|x - x'\right|, \\
\left|g(x,q) - g(x,q')\right| &\leq K(1 + |x|)\left|q - q'\right|, \\
\left|f(t,\pi,x,q) - f(t,\pi,x,q')\right| &\leq K\left|q - q'\right|, \\
\left|f(t,\pi,x,q) - f(t,\pi',x',q)\right| &\leq K\left(\left|x - x'\right| + \left|\pi - \pi'\right|\right)\left(1 + |x| + |x'| + |\pi| + |\pi'|\right).
\end{aligned} \tag{3.2.5}$$

We assume that f and g are continuously differentiable functions with respect to the arguments. We also assume that partial derivatives of f and g with respect to q are Lipschitz continuous. In particular

$$\begin{aligned}
\left|\partial_x\mu(t,\pi,x) - \partial_x\mu(t,\pi,x')\right| + \left|\partial_x\sigma(t,x) - \partial_x\sigma(t,x')\right| &\leq K|x - x'| \\
\left|\partial_x f(t,\pi,x,q) - \partial_x f(t,\pi,x',q')\right| + \left|\partial_x g(x,q) - \partial_x g(x',q')\right| &\leq K\left(|x - x'| + |q - q'|\right) \\
\left|\partial_q f(t,\pi,x,q) - \partial_q f(t,\pi,x,q')\right| + \left|\partial_q g(x,q) - \partial_q g(x,q')\right| &\leq K|q - q'|
\end{aligned}$$

We define the Hamiltonian as it is usually done in the SMP theory:

$$\mathcal{H}(t,\pi,x,q,\mathbf{y}) := y^1\mu(t,\pi,x) - \pi y^2 + f(t,\pi,x,q). \tag{3.2.6}$$

Let Q_r, τ, $Q_r^{\theta,\bar{c},t}$, $\tau^{\theta,\bar{c},t}$ be defined as in the previous chapter respectively as in (2.2.1), (2.2.4), (2.2.9) and (2.2.10). We also define the following quantities

$$\begin{aligned}
&\tau_{\min}^{\theta,\bar{c},t} = \min\left(\tau, \tau^{\theta,\bar{c},t}\right), \qquad \tau_{\max}^{\theta,\bar{c},t} = \max\left(\tau, \tau^{\theta,\bar{c},t}\right), \\
&\hat{Q}_r^{\theta,\bar{c},t} = \max\left(Q_r, Q_r^{\theta,\bar{c},t}\right), \qquad \hat{c}_r^{\theta,\bar{c},t} = \max\left(c_r, c_r^{\theta,\bar{c},t}\right), \\
&\hat{X}_r^{\theta,\bar{c},t} = \begin{cases} X_r^{\theta,\bar{c},t} & \text{if } \tau \leq \tau^{\theta,\bar{c},t} \\ X_r & \text{if } \tau^{\theta,\bar{c},t} < \tau \end{cases}.
\end{aligned} \tag{3.2.7}$$

Let $(\mathbf{Y}_r, \mathbf{Z}_r)_{r\in[0,\tau]} = ((Y_r^1, Y_r^2), (Z_r^1, Z_r^2))_{r\in[0,\tau]}$ be solution of the following BSDE:

$$\begin{cases}
-dY_r^1 = \left(Y_r^1\partial_x\mu(r,c_r,X_r) + Z_r^1\partial_x\sigma(r,X_r) + \partial_x f(r,c_r,X_r,Q_r)\right)dr - Z_r^1 dW_r \\
-dY_r^2 = \partial_q f(r,c_r,X_r,Q_r)dr - Z_r^2 dW_r \\
Y_\tau^1 = \partial_x g(X_\tau, Q_\tau) \\
Y_\tau^2 = \partial_q g(X_\tau, Q_\tau).
\end{cases} \tag{3.2.8}$$

We now state the stochastic maximum principle for the stopped terminal time version with drift coefficient of state process SDE dependent on control.

Theorem 3.2.2. *Let Assumption 3.2.1 be satisfied. Let $(c_r)_{r\in[0,T]}$ be the optimal control for the optimization problem* (3.2.2) *so that $c \in \mathcal{A}$ and so that*

$$\mathbb{E}\left[\sup_{r\in[0,T]} c_r^2\right] < \infty. \tag{3.2.9}$$

Let $(Q_r)_{r\in[0,T]}$ and $(X_r)_{r\in[0,T]}$ be defined as in (2.2.1) *and* (3.2.1) *with respect to control c. Let $(Y_r)_{r\in[0,\tau]}$ be defined as in* (3.2.8) *with respect to control c. We assume that there exist $\mathbb{R}$-valued and L^2-integrable functions $\bar{g}(t,\bar{c},x,q)$ and $\bar{f}(t,\bar{c},x,q)$ so that for any $t \in [0,\tau)$, for any $(x,q) \in \mathcal{O}$ and for any $\bar{c} \geq 0$*

$$\bar{g}(t,\bar{c},x,q) = \lim_{\theta\to 0} \mathbb{E}^t\left[\frac{g(\hat{X}_\tau^{\theta,\bar{c},t}, Q_{\tau^{\theta,\bar{c},t}}^{\theta,\bar{c},t}) - g(\hat{X}_{\tau^{\theta,\bar{c},t}}^{\theta,\bar{c},t}, Q_{\tau^{\theta,\bar{c},t}}^{\theta,\bar{c},t})}{\theta}\right], \tag{3.2.10}$$

$$\bar{f}(t,\bar{c},x,q) = \lim_{\theta\to 0} \mathbb{E}^t\left[\frac{sign(\tau-\tau^{\theta,\bar{c},t})}{\theta}\int_{\tau_{\min}^{\theta,\bar{c},t}}^{\tau_{\max}^{\theta,\bar{c},t}} f\left(r, \hat{c}_r^{\theta,\bar{c},t}, \hat{X}_r^{\theta,\bar{c},t}, \hat{Q}_r^{\theta,\bar{c},t}\right) dr\right]. \tag{3.2.11}$$

Then, c necessarily satisfies for any $t \in [0,\tau)$, for any $\bar{c} \geq 0$

$$\mathcal{H}(t,\bar{c},X_t,Q_t,\mathbf{Y}_t) - \mathcal{H}(t,c_t,X_t,Q_t,\mathbf{Y}_t) + \mathcal{G}(t,\bar{c},c_t,X_t,Q_t) \leq 0 \quad a.s., \tag{3.2.12}$$

where $\mathcal{G}(t,\bar{c},c_t,x,q)$ is defined as

$$\mathcal{G}(t,\bar{c},c_t,x,q) := (\bar{c}-c_t)\mathbb{E}^t\left[\partial_q g(X_\tau,Q_\tau)\mathbb{1}_{\Lambda(t,\bar{c})}\right] - \bar{g}(t,\bar{c},x,q) - \bar{f}(t,\bar{c},x,q), \tag{3.2.13}$$

where the event $\Lambda(t,\bar{c})$ is defined as

$$\Lambda(t,\bar{c}) := (\{Q_T = 0\} \cap \{\bar{c} \geq c_t\}) \cup (\{\tau < T\} \cap \{\bar{c} < c_t\}). \tag{3.2.14}$$

Remark 3.2.3. *The main difference of the proof of the generalized version, with respect to the lighter version in previous chapter relies on the decomposition in* (2.8.2)*. Indeed, in the new case, we are going to have the variational process $X^{\theta,\bar{c},t}$. Therefore, in the proof of the generalized version Theorem 3.2.2, we are going to have also decompositions regarding the difference between the two processes $X^{\theta,\bar{c},t}$ and X and the limit of that difference. The main differences with respect to the proof presented in the previous chapter is an extension of Lemma 2.7.11, that includes the presence of $X^{\theta,\bar{c},t}$ and the introduction of the process η, which is going to be proved as the generalized derivative of the process $X^{\theta,\bar{c},t}$ with respect to θ in Lemma 3.7.5.*

Remark 3.2.4. *Theorem 2.2.3 can be recovered from Theorem 3.2.2 by observing that if μ is independent of π, in* (3.2.12)*, the difference between the two Hamiltonians cancels out the two terms μy^1 from the equation. This removes the presence of the process Y^1 from* (3.2.12)*. This allows the process $\mathbf{Y}$ to be reduced to its only second component Y^2, as in the definition* (2.2.11) *in previous chapter.*

3.2.1 Introducing the control in the diffusion term in definition of X_r

In the SDE defining process X_r (3.2.1), the diffusion coefficient is independent of control process c. This is due to the necessity of proving that

$$\mathbb{E}^t\left[\left|\frac{X_{t+\theta}^{\theta,\bar{c},t}-X_{t+\theta}}{\theta}\right|^2\right] \tag{3.2.15}$$

is bounded as $\theta \to 0$ in proof of Theorem 3.2.2. To prove the previous property, we use that Lipschitz continuity implies that

$$\mathbb{E}^t\left[\left|\frac{1}{\theta}\int_t^{t+\theta}\mu\left(r,c_r^{\theta,\bar{c},t},X_r\right)-\mu\left(r,c_r,X_r\right)\,dr\right|^2\right]\le\mathbb{E}^t\left[\left(\frac{K}{\theta}\int_t^{t+\theta}\left|c_r^{\theta,\bar{c},t}-c_r\right|\,dr\right)^2\right]\le K(\bar{c}-c_t)^2 \tag{3.2.16}$$

which is bounded independently of θ. On the other hand, if the diffusion coefficient was dependent on control process c, to prove boundedness of (3.2.15) we would have needed to show that

$$\mathbb{E}^t\left[\left|\frac{1}{\theta}\int_t^{t+\theta}\sigma\left(r,c_r^{\theta,\bar{c},t},X_r\right)-\sigma\left(r,c_r,X_r\right)\,dW_r\right|^2\right] \tag{3.2.17}$$

is bounded as $\theta \to 0$. However, by simply considering the diffusion coefficient to be $\sigma(t,c,x)=c$, by applying Ito's isometry to (3.2.17), it would be equal to

$$\mathbb{E}^t\left[\frac{1}{\theta^2}\int_t^{t+\theta}\left|c_r^{\theta,\bar{c},t}-c_r\right|^2\,dr\right]=\frac{\theta(\bar{c}-c_t)}{\theta^2},$$

which explodes to $+\infty$ as $\theta \to 0$. The main difference with respect to drift term is that the time-integral in (3.2.16) can be upper bounded by bringing the absolute value operator inside the integration and leaving the power 2 outside the integration. On the other hand, by applying Ito's isometry to the stochastic integral in (3.2.17), both absolute value operator and power 2 are brought inside the integration, not allowing the compensation of the $1/\theta$ term with the length of the integration interval anymore.

Therefore, we have not been able to introduce dependency on the control also in the diffusion term. This weakness is due to the fact we decided to approach the proof of Theorem 3.2.2 using Bensoussan [11] approach. As we previously mentioned Bensoussan's technique accommodates the need of having a variational approach that allows us to guess in the proof what is the variation in the stopping time τ given the variation of the control π. The drawback of using this approach is that it makes impossible to deal with more general diffusion coefficients in the state variable SDE definition.

3.3 Admitting negative control π

In this section we want to show that settings of Theorem 3.2.2 can be extended also to the case of negative control variable π. In order to admit control π to have the whole real line as admissible set, we need to make few changes in the proofs in Chapter 2. There are 3 main observations we need to point out in order to allow for negative controls in Theorem 3.2.2. Firstly, we need to point out that anywhere we mention the expression $\theta \in \left(0, (T-t) \wedge \frac{q}{\bar{c}}\right)$, we interpret $\frac{q}{\bar{c}}$ as equal to $+\infty$ when $\bar{c} \leq 0$. This observation is especially important to underline that expression (2.7.2) is still valid, as for $\bar{c} \leq 0$ expression (2.7.1) is clearly non-negative. Secondly, Lemma 2.7.1 is not needed anymore, as it is not necessary to show that c is not negative. Progressive measurability, right-continuity and square integrability of $c^{\theta,\bar{c},t}$ immediately follow, without the necessity of having the Lemma. Finally, we need to slightly change proofs of Lemma 2.7.4 and Lemma 2.7.5. In particular, we use continuity of process Q_r, thanks to its definition as an integral. Therefore, we define the process $\check{Q}_r$ as

$$\check{Q}_r = \min_{s \leq r} Q_s. \tag{3.3.1}$$

The minimum is well defined as Q is continuous and $[0, r]$ is a compact interval. We state and prove the new versions of Lemma 2.7.4 and Lemma 2.7.5. In Lemma 2.7.4, the only limit that needs to be proved again is (2.7.27). All the other limits can be proved as in Lemma 2.7.4.

Lemma 3.3.1. *Let $\bar{c} \in \mathbb{R}$ and $t \in [0, \tau)$ be fixed. Then*

$$\lim_{\theta \to 0} \mathbb{P}\left(E_1^{\theta,\bar{c},t} \cap \{Q_T > 0\}\right) = 0. \tag{3.3.2}$$

Proof. Using definition of Q, we have that under event $E_1^{\theta,\bar{c},t}$, $\check{Q}_\tau \leq \check{Q}_{\tau^{\theta,\bar{c},t}} \leq Q_{\tau^{\theta,\bar{c},t}} = \gamma_{t+\theta}^{\theta,\bar{c},t}$. Moreover, if $Q_T > 0$, then it necessarily implies that $\tau = T$. Using (2.7.4) we have that

$$\begin{aligned}
\lim_{\theta \to 0} \mathbb{P}\left(E_1^{\theta,\bar{c},t} \cap \{Q_T > 0\}\right) &\leq \lim_{\theta \to 0} \mathbb{P}\left(\{\check{Q}_{\tau^{\theta,\bar{c},t}} \leq \gamma_{t+\theta}^{\theta,\bar{c},t}\} \cap \{Q_\tau > 0\}\right) \\
&\leq \lim_{\theta \to 0} \mathbb{P}\left(\left\{\check{Q}_\tau \leq \sup_{r \in [t, t+\theta]} \left|\gamma_r^{\theta,\bar{c},t}\right|\right\} \cap \{Q_\tau > 0\}\right) \\
&\leq \lim_{n \to \infty} \mathbb{P}\left(\left\{\check{Q}_\tau \leq \sup_{r \in \left[t, t+\frac{1}{n}\right]} \left|\gamma_r^{\frac{1}{n},\bar{c},t}\right|\right\} \cap \{Q_\tau > 0\}\right) \\
&= \mathbb{P}\left(\bigcap_{n \geq \bar{n}} \left\{\check{Q}_\tau \leq \int_t^{t+\frac{1}{n}} |c_r - \bar{c}| \, dr\right\} \cap \{Q_\tau > 0\}\right) \\
&= \mathbb{P}\left(\{\check{Q}_\tau = 0\} \cap \{Q_\tau > 0\}\right) = 0,
\end{aligned}$$

as event $\left\{\check{Q}_\tau = 0\right\}$ implies that $\{Q_\tau = 0\}$. In previous calculations we used that the following sequence of events is decreasing for any $n \geq \bar{n} = \left\lceil \frac{1}{(T-t)\wedge\frac{q}{\bar{c}}} \right\rceil$

$$\left\{ \check{Q}_\tau \leq \int_t^{t+\frac{1}{n+1}} |c_r - \bar{c}|\, dr \right\} \subseteq \left\{ \check{Q}_\tau \leq \int_t^{t+\frac{1}{n}} |c_r - \bar{c}|\, dr \right\}$$

and using right-continuity of c, $\int_t^{t+\frac{1}{n}} |\bar{c} - c_r|\, dr$ converges to 0 $\mathbb{P}$-a.s., as $n \to \infty$. This concludes proof of (3.3.2). $\square$

Lemma 3.3.2. *Let $t \in [0, \tau)$ and $\bar{c} \in \mathbb{R}$ be fixed. Then*

$$\lim_{\theta\to 0} \tau^{\theta,\bar{c},t} = \tau \textit{ pointwise almost everywhere.} \tag{3.3.3}$$

Moreover,

$$\lim_{\theta\to 0} \mathbb{E}\left[\left|\tau^{\theta,\bar{c},t} - \tau\right|\right] = 0. \tag{3.3.4}$$

Proof. We firstly prove (3.3.3). We assume on the contrary there exists a non-null event $\mathcal{E}$, so that $\lim_{\theta\to 0}\tau^{\theta,\bar{c},t} \neq \tau$ on $\mathcal{E}$, which means that

$$\exists\gamma > 0 \;\text{ s.t. } \forall\bar{\theta} \in \left(0, (T-t)\wedge\frac{q}{\bar{c}}\wedge\gamma\right),\; \exists\theta \in (0,\bar{\theta}) \text{ s.t. } \left|\tau - \tau^{\theta,\bar{c},t}\right| > \gamma \;\text{ on } \mathcal{E}. \tag{3.3.5}$$

Using that under event $E_1^{\theta,\bar{c},t}$, $\tau > \tau^{\theta,\bar{c},t}$ and so $\left|\tau - \tau^{\theta,\bar{c},t}\right| > \gamma$ implies that $\tau - \tau^{\theta,\bar{c},t} > \gamma$, which implies $\check{Q}_{\tau-\gamma} \leq \check{Q}_{\tau^{\theta,\bar{c},t}} \leq Q_{\tau^{\theta,\bar{c},t}} = \gamma_{t+\theta}^{\theta,\bar{c},t}$. Moreover, using that under event $E_2^{\theta,\bar{c},t}$, $\tau^{\theta,\bar{c},t} = (\tau+\theta)\wedge T$, $\left|\tau - \tau^{\theta,\bar{c},t}\right| > \gamma$ implies that $\theta \geq (\tau+\theta)\wedge T - \tau > \gamma$, which is never verified, as $\theta < \bar{\theta} < \gamma$. Moreover, under event $E_3^{\theta,\bar{c},t}$, we have that $\tau^{\theta,\bar{c},t} = \tau$, which never satisfies $\left|\tau - \tau^{\theta,\bar{c},t}\right| > \gamma$. Therefore, we have that (3.3.5) implies that

$$\exists\gamma > 0 \;\text{ s.t. } \forall\bar{\theta} \in \left(0, (T-t)\wedge\frac{q}{\bar{c}}\wedge\gamma\right),\; \exists\theta \in (0,\bar{\theta}) \text{ s.t. } \check{Q}_{\tau-\gamma} \leq \gamma_{t+\theta}^{\theta,\bar{c},t} \;\text{ on } \mathcal{E}. \tag{3.3.6}$$

Recalling that $\left|\gamma_{t+\theta}^{\theta,\bar{c},t}\right| = \int_t^{(t+\theta)\wedge\tau} |\bar{c} - c_r|\, dr \leq \int_t^{(t+\theta)\wedge\tau} |c_r| dr + |\bar{c}|\theta$ and using right-continuity of c_r, we have that expression (3.3.6) implies that $\check{Q}_{\tau-\gamma} = 0$ on $\mathcal{E}$, which contradicts definition of τ, as τ should be the first time in which Q_r hits 0. Therefore, we conclude that $\mathcal{E}$ must be a $\mathbb{P}$-null set and this conclude proof of (3.3.3). To prove (3.3.4) we observe that $\left|\tau^{\theta,\bar{c},t} - \tau\right| \leq T$, independently of θ. Therefore, applying DCT we also get (3.3.4). $\square$

3.4 Introducing general linear drift in (2.2.1)

In the definition of Q_r in (2.2.1) we had a simple drift function. In this section we introduce a time-dependant linear drift with respect to the control c and the inventory Q itself. Let $\alpha, \beta, \delta : [0,T] :\to \mathbb{R}$ be bounded functions. Let Q_r be defined as

$$Q_t = q_0 - \int_0^t (\alpha(r)c_r + \beta(r) + \delta(r)Q_r)\, dr. \tag{3.4.1}$$

However, we see that Theorem 3.2.2 still holds with the new definition (3.4.1). Indeed, introducing the following substitutions

$$\tilde{c}_t = e^{\int_0^t \delta(s)ds} \left(\alpha(t)c_t + \beta(t)\right), \tag{3.4.2}$$

$$\tilde{Q}_t = q_0 - \int_0^t \tilde{c}_r dr, \tag{3.4.3}$$

we have that Theorem 3.2.2 is satisfied using processes $\tilde{Q}_t$ and $\tilde{c}_t$. Moreover $\tilde{Q}_t = e^{\int_0^t \delta(s)ds} Q_t$, indeed simple calculus shows that

$$\begin{aligned} d\tilde{Q}_r &= -\tilde{c}_r dr = -e^{\int_0^r \delta(s)ds} \left(\alpha(r)c_r + \beta(r)\right) \\ &= -e^{\int_0^r \delta(s)ds} \left(\alpha(r)c_r + \beta(r) + \delta(r)Q_r\right) + e^{\int_0^r \delta(s)ds}\delta(r)Q_r \\ &= e^{\int_0^r \delta(s)ds} dQ_r + Q_r d\left(e^{\int_0^r \delta(s)ds}\right) \\ &= d\left(e^{\int_0^r \delta(s)ds} Q_r\right) \end{aligned}.$$

Therefore, using the substitutions above, the definitions in (3.4.1) holds true. Finally, to confirm that Theorem 3.2.2 can be used with new control $\tilde{c}$ and inventory $\tilde{Q}$, we observe that Lipschitz conditions in Assumption 3.2.1 are satisfied as δ is a bounded function and so for any $r \in [0, T]$, $e^{\int_0^r \delta(s)ds}$ is bounded.

We now state the stochastic maximum principle for the stopped terminal time version including the extensions introduced in the current section and in the previous one. We define

$$v(t, \mathbf{x}, q) = \sup_{\pi \in \mathbb{R}} v^\pi(t, \mathbf{x}, q), \tag{3.4.4}$$

where

$$v^\pi(t, \mathbf{x}, q) = \mathbb{E}^t \left[g(\mathbf{X}_{\tau^\pi}, Q^\pi_{\tau^\pi}) + \int_t^{\tau^\pi} f(r, \pi_r, \mathbf{X}_r, Q^\pi_r)\, dr \right]. \tag{3.4.5}$$

We also define the following quantities

$$\hat{Q}_r^{\theta,\bar{c},t} = \max\left(\tilde{Q}_r, \tilde{Q}_r^{\theta,\bar{c},t}\right), \qquad \hat{c}_r^{\theta,\bar{c},t} = \max\left(\tilde{c}_r, \tilde{c}_r^{\theta,\bar{c},t}\right), \tag{3.4.6}$$

Theorem 3.4.1. *Let Assumption 3.2.1 be satisfied. Let $(c_r)_{r\in[0,T]}$ be the optimal control for the optimization problem* (3.4.4) *so that $c \in \mathcal{A}$ and so that*

$$\mathbb{E}\left[\sup_{r\in[0,T]} c_r^2\right] < \infty. \tag{3.4.7}$$

Let $(Q_r)_{r\in[0,T]}$ and $(X_r)_{r\in[0,T]}$ be defined as in (3.4.1) *and* (3.2.1) *with respect to control c. Let $(Y_r)_{r\in[0,\tau]}$ be defined as in* (3.2.8) *with respect to control c. We assume that there exist $\mathbb{R}$-valued and L^2-integrable functions $\bar{g}(t, \bar{c}, x, q)$ and $\bar{f}(t, \bar{c}, x, q)$ so that for any $t \in [0, \tau)$, for any*

$(x, q) \in \mathcal{O}$ and for any $\bar{c} \in \mathbb{R}$

$$\bar{g}(t, \bar{c}, x, q) = \lim_{\theta \to 0} \mathbb{E}^t \left[\frac{g(\hat{X}_{\tau}^{\theta,\bar{c},t}, e^{\int_t^{\tau^{\theta,\bar{c},t}} \delta(s)ds} Q_{\tau^{\theta,\bar{c},t}}^{\theta,\bar{c},t}) - g(\hat{X}_{\tau^{\theta,\bar{c},t}}^{\theta,\bar{c},t}, e^{\int_t^{\tau^{\theta,\bar{c},t}} \delta(s)ds} Q_{\tau^{\theta,\bar{c},t}}^{\theta,\bar{c},t})}{\theta} \right], \tag{3.4.8}$$

$$\bar{f}(t, \bar{c}, x, q) = \lim_{\theta \to 0} \mathbb{E}^t \left[\frac{sign(\tau - \tau^{\theta,\bar{c},t})}{\theta} \int_{\tau_{\min}^{\theta,\bar{c},t}}^{\tau_{\max}^{\theta,\bar{c},t}} f\left(r, \hat{c}_r^{\theta,\bar{c},t}, \hat{X}_r^{\theta,\bar{c},t}, \hat{Q}_r^{\theta,\bar{c},t}\right) dr \right]. \tag{3.4.9}$$

Then, c necessarily satisfies for any $t \in [0, \tau)$, for any $\bar{c} \geq 0$

$$\mathcal{H}(t, \bar{c}, X_t, Q_t, \mathbf{Y}_t) - \mathcal{H}(t, c_t, X_t, Q_t, \mathbf{Y}_t) + \mathcal{G}(t, \bar{c}, c_t, X_t, Q_t) \leq 0 \quad a.s., \tag{3.4.10}$$

where $\mathcal{G}(t, \bar{c}, c_t, x, q)$ is defined as

$$\mathcal{G}(t, \bar{c}, c_t, x, q) := (\bar{c} - c_t)\mathbb{E}^t \left[\partial_q g(X_\tau, e^{\int_t^\tau \delta(s)ds} Q_\tau) \mathbb{1}_{\Lambda(t,\bar{c})}\right] - \bar{g}(t, \bar{c}, x, q) - \bar{f}(t, \bar{c}, x, q), \tag{3.4.11}$$

where the event $\Lambda(t, \bar{c})$ is defined as

$$\Lambda(t, \bar{c}) := (\{Q_T = 0\} \cap \{\bar{c} \geq c_t\}) \cup (\{\tau < T\} \cap \{\bar{c} < c_t\}). \tag{3.4.12}$$

3.5 Sufficient condition for the new version of SMP

The standard version of the necessary SMP is usually linked with its sufficient version. By simply requiring that the Hamiltonian $\mathcal{H}$ in the standard SMP has some global convexity properties, it can be easily verified that the optimal control is the control that maximizes the Hamiltonian itself (c.f. Pham [48, Theorem 6.4.6]). We are interested in extending the necessary condition in Theorem 3.2.2 to a sufficient condition as well, by trying to find a formulation that enables us to guess the optimal control c and the optimal stopping time associated directly from a maximisation of the Hamiltonian $\mathcal{H}$ and the new term $\mathcal{G}$ introduced in this thesis. In order to achieve a sufficient formulation in the case of the SMP with stopping terminal time as well we need to simplify expressions (3.2.10) and (3.2.11). The main aim of this simplification would be to separate c and $\bar{c}$ inside the expression of $\mathcal{G}$ in (3.2.12). In particular, we would like to be able to split the functional $\mathcal{G}(t, \bar{c}, c_t, x, q)$ into the difference of two functionals $\mathcal{G}_1(t, \bar{c}, x, q) - \mathcal{G}_1(t, c_t, x, q)$, where the first one only depends on c and the second one only on c_t. This would allow us to separate expression regarding c on one side of the inequality and expression regarding $\bar{c}$ on the other side of the inequality. Once we had obtained this separation would be enough to assume some sort of convexity in the Hamiltonian $\mathcal{H}$ and in the functions forming $\mathcal{G}$ to get a sufficient formulation of the SMP.

In this Section we are going to consider an example where f is a polynomial function and we try to find a sufficient formulation of Theorem 3.2.2. However, even in a simple case such as the one presented, the task to state a sufficient condition of the SMP is complicated. We also tried other examples, without being able to build a case in which a sufficient formulation of Theorem 3.2.2 could be found.

The first step to simplify expressions (3.2.10) and (3.2.11) consists in trying to find a pointwise limit for the function $\bar{f}$ defined in (2.2.17). As we show in the following, we are able to find an Heuristic pointwise limit for functions f that are polynomial in the control π. In particular, we consider f to be of the following form

$$f(t,\pi,x,q) = f_0(t,x) + \pi f_1(t,\pi,x,q). \tag{3.5.1}$$

In this case we are able to exploit some properties of process Q that simplify definition of $\bar{f}$. We also define the process $(F_r)_{r\in[t,T]}$ to satisfy the following SDE

$$dF_r = f_0(r,X_r)dr, \quad F_t = 0 \tag{3.5.2}$$

Corollary 3.5.1 (Heuristic). *Let f be of the form* (3.5.1). *Let assumptions of Theorem 2.2.3 be satisfied. Let F be defined as in* (3.5.2). *Let $\bar{g}$ be defined as*

$$\bar{g}(t,\bar{c},x,q) = \lim_{\theta\to 0}\mathbb{E}^t\left[\frac{g(X_\tau, Q^{\theta,\bar{c},t}_{\tau^{\theta,\bar{c},t}}) - g(X_{\tau^{\theta,\bar{c},t}}, Q^{\theta,\bar{c},t}_{\tau^{\theta,\bar{c},t}}) + F_\tau - F_{\tau^{\theta,\bar{c},t}}}{\theta}\right]. \tag{3.5.3}$$

Then, $\bar{f}$ in (2.2.17) *is equal to*

$$\bar{f}(t,\bar{c},x,q) = (\bar{c}-c_t)\mathbb{E}^t\left[f_1\left(\tau, c_\tau \mathbb{1}_{\bar{c}\geq c_t} - (\bar{c}-c_t)\mathbb{1}_{\bar{c}<c_t}, X_\tau, 0\right)\mathbb{1}_{(\{Q_T=0\}\cap\{\bar{c}\geq c_t\})\cup(\{\tau<T\}\cap\{\bar{c}<c_t\})}\right], \tag{3.5.4}$$

which is equivalent to

$$\bar{f}(t,\bar{c},x,q) = \begin{cases} (\bar{c}-c_t)\mathbb{E}^t\left[f_1\left(\tau,c_\tau,X_\tau,0\right)\mathbb{1}_{Q_T=0}\right] & \text{if } \bar{c}\geq c_t \\ (\bar{c}-c_t)\mathbb{E}^t\left[f_1\left(\tau,-(\bar{c}-c_t),X_\tau,0\right)\mathbb{1}_{\tau<T}\right] & \text{if } \bar{c}< c_t \end{cases}. \tag{3.5.5}$$

As it can be seen in (3.5.5), we managed to simplify the expression of $\bar{f}$. However, inside the expression of $\bar{f}$ in (3.5.5) it is not possible to separate the two control variables c and $\bar{c}$. This makes impossible to separate the functional $\mathcal{G}$ into two pieces, as it can be done for $\mathcal{F}$ in the standard formulation of the SMP. Therefore, we conclude that, even in a simplified case of a polynomial function f, the task to state a sufficient condition of the SMP is complicated.

In the following we present a sketch of the proof for the corollary above.

Sketch of the proof. Firstly, we analyze the limit inside the definition of $\bar{f}$ in (2.2.17)

$$\begin{aligned}
&\mathbb{E}^t\left[\frac{\operatorname{sign}(\tau-\tau^{\theta,\bar{c},t})}{\theta}\int_{\tau_{\min}^{\theta,\bar{c},t}}^{\tau_{\max}^{\theta,\bar{c},t}} f\left(r,\hat{c}_r^{\theta,\bar{c},t},X_r,\hat{Q}_r^{\theta,\bar{c},t}\right)\,dr\right]\\
&\quad=\mathbb{E}^t\left[\frac{\operatorname{sign}(\tau-\tau^{\theta,\bar{c},t})}{\theta}\int_{\tau_{\min}^{\theta,\bar{c},t}}^{\tau_{\max}^{\theta,\bar{c},t}} f_0\left(r,X_r\right)\,dr\right]\\
&\qquad+\mathbb{E}^t\left[\frac{\operatorname{sign}(\tau-\tau^{\theta,\bar{c},t})}{\theta}\int_{\tau_{\min}^{\theta,\bar{c},t}}^{\tau_{\max}^{\theta,\bar{c},t}} \hat{c}_r^{\theta,\bar{c},t}f_1\left(r,\hat{c}_r^{\theta,\bar{c},t},X_r,\hat{Q}_r^{\theta,\bar{c},t}\right)\,dr\right]
\end{aligned}\tag{3.5.6}$$

However, the first item on the right-hand side of (3.5.6) can be written as

$$\begin{aligned}
&\mathbb{E}^t\left[\frac{\operatorname{sign}(\tau-\tau^{\theta,\bar{c},t})}{\theta}\int_{\tau_{\min}^{\theta,\bar{c},t}}^{\tau_{\max}^{\theta,\bar{c},t}} f_0\left(r,X_r\right)\,dr\right]\\
&\quad=\mathbb{E}^t\left[\frac{1}{\theta}\int_{\tau^{\theta,\bar{c},t}}^{\tau} f_0\left(r,X_r\right)\,dr\right]=\mathbb{E}^t\left[\frac{F_\tau-F_{\tau^{\theta,\bar{c},t}}}{\theta}\right].
\end{aligned}\tag{3.5.7}$$

Moreover, the second item on the right-hand side of (3.5.6) is equal to

$$\begin{aligned}
&\mathbb{E}^t\left[\frac{\operatorname{sign}(\tau-\tau^{\theta,\bar{c},t})}{\theta}\int_{\tau_{\min}^{\theta,\bar{c},t}}^{\tau_{\max}^{\theta,\bar{c},t}} \hat{c}_r^{\theta,\bar{c},t}f_1\left(r,\hat{c}_r^{\theta,\bar{c},t},X_r,\hat{Q}_r^{\theta,\bar{c},t}\right)\,dr\right]\\
&=\mathbb{E}^t\Bigg[\frac{\operatorname{sign}(\tau-\tau^{\theta,\bar{c},t})}{\theta}\mathbb{1}_{E_1^{\theta,\bar{c},t}}\int_{\tau_{\min}^{\theta,\bar{c},t}}^{\tau_{\max}^{\theta,\bar{c},t}} \hat{c}_r^{\theta,\bar{c},t}f_1\left(r,\hat{c}_r^{\theta,\bar{c},t},X_r,\hat{Q}_r^{\theta,\bar{c},t}\right)\,dr\\
&\qquad+\frac{\operatorname{sign}(\tau-\tau^{\theta,\bar{c},t})}{\theta}\mathbb{1}_{E_2^{\theta,\bar{c},t}}\int_{\tau_{\min}^{\theta,\bar{c},t}}^{\tau_{\max}^{\theta,\bar{c},t}} \hat{c}_r^{\theta,\bar{c},t}f_1\left(r,\hat{c}_r^{\theta,\bar{c},t},X_r,\hat{Q}_r^{\theta,\bar{c},t}\right)\,dr\\
&\qquad+\frac{\operatorname{sign}(\tau-\tau^{\theta,\bar{c},t})}{\theta}\mathbb{1}_{E_3^{\theta,\bar{c},t}}\int_{\tau_{\min}^{\theta,\bar{c},t}}^{\tau_{\max}^{\theta,\bar{c},t}} \hat{c}_r^{\theta,\bar{c},t}f_1\left(r,\hat{c}_r^{\theta,\bar{c},t},X_r,\hat{Q}_r^{\theta,\bar{c},t}\right)\,dr\Bigg]\\
&=\mathbb{E}^t\Bigg[\frac{\mathbb{1}_{E_1^{\theta,\bar{c},t}}}{\theta}\int_{\tau^{\theta,\bar{c},t}}^{\tau} \hat{c}_r^{\theta,\bar{c},t}f_1\left(r,\hat{c}_r^{\theta,\bar{c},t},X_r,\hat{Q}_r^{\theta,\bar{c},t}\right)\,dr\\
&\qquad-\frac{\mathbb{1}_{E_2^{\theta,\bar{c},t}}}{\theta}\int_{\tau}^{\tau^{\theta,\bar{c},t}} \hat{c}_r^{\theta,\bar{c},t}f_1\left(r,\hat{c}_r^{\theta,\bar{c},t},X_r,\hat{Q}_r^{\theta,\bar{c},t}\right)\,dr\\
&\qquad+\frac{\mathbb{1}_{E_3^{\theta,\bar{c},t}\cap((\{\bar{c}>c_t\}\cap\{Q_T=0\})\cup(\{\bar{c}<c_t\}\cap\{\tau<T\}))}}{\theta}\int_{\tau_{\min}^{\theta,\bar{c},t}}^{\tau_{\max}^{\theta,\bar{c},t}} \hat{c}_r^{\theta,\bar{c},t}f_1\left(r,\hat{c}_r^{\theta,\bar{c},t},X_r,\hat{Q}_r^{\theta,\bar{c},t}\right)\,dr\\
&\qquad+\frac{\mathbb{1}_{E_3^{\theta,\bar{c},t}\cap(\{\bar{c}=c_t\}\cup(\{\bar{c}>c_t\}\cap\{Q_T>0\})\cup(\{\bar{c}<c_t\}\cap\{\tau=T\}))}}{\theta}\int_{\tau_{\min}^{\theta,\bar{c},t}}^{\tau_{\max}^{\theta,\bar{c},t}} \hat{c}_r^{\theta,\bar{c},t}f_1\left(r,\hat{c}_r^{\theta,\bar{c},t},X_r,\hat{Q}_r^{\theta,\bar{c},t}\right)\,dr\Bigg]\\
&=\mathbb{E}^t\Bigg[\frac{\mathbb{1}_{E_1^{\theta,\bar{c},t}}}{\theta}\int_{\tau^{\theta,\bar{c},t}}^{\tau} c_rf_1\left(r,c_r,X_r,Q_r\right)\,dr\\
&\qquad+\frac{\mathbb{1}_{E_2^{\theta,\bar{c},t}}}{\theta}\int_{\tau}^{(\tau+\theta)\wedge T} \frac{\gamma_{t+\theta}^{\theta,\bar{c},t}}{\theta}f_1\left(r,-\frac{\gamma_{t+\theta}^{\theta,\bar{c},t}}{\theta},X_r,Q_r^{\theta,\bar{c},t}\right)\,dr\\
&\qquad+\frac{\mathbb{1}_{E_3^{\theta,\bar{c},t}\cap((\{\bar{c}>c_t\}\cap\{Q_T=0\})\cup(\{\bar{c}<c_t\}\cap\{\tau<T\}))}}{\theta}\cdot 0\\
&\qquad+\frac{\mathbb{1}_{E_3^{\theta,\bar{c},t}\cap(\{\bar{c}=c_t\}\cup(\{\bar{c}>c_t\}\cap\{Q_T>0\})\cup(\{\bar{c}<c_t\}\cap\{\tau=T\}))}}{\theta}\cdot 0\Bigg]
\end{aligned}\tag{3.5.8}$$

where in the last equality we used that under $E_3^{\theta,\bar{c},t}$, $\tau_{\max}^{\theta,\bar{c},t} = \tau_{\min}^{\theta,\bar{c},t}$.

From now on, the proof is an Heuristic proof, whose main focus is to give an idea on how the actual proof could work in this case. Applying the change of variable $\varphi = Q_r$ (recalling that $d\varphi = -c_r dr$) to the first integral in the right-hand side of previous expression, and recalling that under event $E_1^{\theta,\bar{c},t}$, $Q_{\tau^{\theta,\bar{c},t}} = \gamma_{t+\theta}^{\theta,\bar{c},t}$, we get

$$\frac{\mathbb{1}_{E_1^{\theta,\bar{c},t}}}{\theta} \int_{\tau^{\theta,\bar{c},t}}^{\tau} c_r f_1\left(r, c_r, X_r, Q_r\right)\, dr = -\frac{\mathbb{1}_{E_1^{\theta,\bar{c},t}}}{\theta} \int_{\gamma_{t+\theta}^{\theta,\bar{c},t}}^{Q_\tau} f_1\left(Q^{-1}(\varphi), c_{Q^{-1}(\varphi)}, X_{Q^{-1}(\varphi)}, \varphi\right)\, d\varphi. \tag{3.5.9}$$

By applying the mean value theorem, we get that there exists $\bar{\varphi} \in [\gamma_{t+\theta}^{\theta,\bar{c},t}, Q_\tau]$ and $\bar{t} \in [\tau, (\tau+\theta)\wedge T]$ so that

$$\begin{aligned}
&\mathbb{E}^t\left[\frac{\text{sign}(\tau - \tau^{\theta,\bar{c},t})}{\theta} \int_{\tau_{\min}^{\theta,\bar{c},t}}^{\tau_{\max}^{\theta,\bar{c},t}} \hat{c}_r^{\theta,\bar{c},t} f_1\left(r, \hat{c}_r^{\theta,\bar{c},t}, X_r, \hat{Q}_r^{\theta,\bar{c},t}\right)\, dr\right] \\
&= \mathbb{E}^t\Bigg[-\frac{\mathbb{1}_{E_1^{\theta,\bar{c},t}}}{\theta}\left(Q_\tau - \gamma_{t+\theta}^{\theta,\bar{c},t}\right) f_1\left(Q^{-1}(\bar{\varphi}), c_{Q^{-1}(\bar{\varphi})}, X_{Q^{-1}(\bar{\varphi})}, \bar{\varphi}\right) \\
&\qquad + \frac{\mathbb{1}_{E_2^{\theta,\bar{c},t}}}{\theta}\left((\tau+\theta)\wedge T - \tau\right) \frac{\gamma_{t+\theta}^{\theta,\bar{c},t}}{\theta} f_1\left(\bar{t}, -\frac{\gamma_{t+\theta}^{\theta,\bar{c},t}}{\theta}, X_{\bar{t}}, Q_{\bar{t}}^{\theta,\bar{c},t}\right)\Bigg] \\
&= \mathbb{E}^t\Bigg[\mathbb{1}_{E_1^{\theta,\bar{c},t}\cap\{Q_T=0\}} \frac{\gamma_{t+\theta}^{\theta,\bar{c},t}}{\theta} f_1\left(Q^{-1}(\bar{\varphi}), c_{Q^{-1}(\bar{\varphi})}, X_{Q^{-1}(\bar{\varphi})}, \bar{\varphi}\right) \\
&\qquad - \frac{\mathbb{1}_{E_1^{\theta,\bar{c},t}\cap\{Q_T>0\}}}{\theta}\left(Q_\tau - \gamma_{t+\theta}^{\theta,\bar{c},t}\right) f_1\left(Q^{-1}(\bar{\varphi}), c_{Q^{-1}(\bar{\varphi})}, X_{Q^{-1}(\bar{\varphi})}, \bar{\varphi}\right) \\
&\qquad + \frac{\mathbb{1}_{E_2^{\theta,\bar{c},t}\cap\{\tau<T\}}}{\theta}\left((\tau+\theta)\wedge T - \tau\right) \frac{\gamma_{t+\theta}^{\theta,\bar{c},t}}{\theta} f_1\left(\bar{t}, -\frac{\gamma_{t+\theta}^{\theta,\bar{c},t}}{\theta}, X_{\bar{t}}, Q_{\bar{t}}^{\theta,\bar{c},t}\right) \\
&\qquad + \frac{\mathbb{1}_{E_2^{\theta,\bar{c},t}\cap\{\tau=T\}}}{\theta}\cdot 0\Bigg]
\end{aligned} \tag{3.5.10}$$

and the limit converges to the desired result, by using Lemmas, especially the one for limit of probabilities. □

3.6 Conclusions

In this chapter we introduce several generalizations to Theorem 2.2.3. In particular, we generalized the definition of the state process X, by introducing the control into the drift term. Although proving a generalization of the SMP 2.2.3 with the control into the drift term has been possible, by using a similar approach to the proof of Theorem 2.2.3, in Subsection 3.2.1 we show that this is not possible for the diffusion term as well. Moreover, we proved that the new version of SMP can be also stated in the case when the control is negative. These new extensions enable Theorem 3.2.2 to be applied to more examples not only related to mathematical finance.

Finally, we showed that it is a complicated task to simplify expressions of $\bar{f}$ and $\bar{g}$ in (3.2.10) and (3.2.11). This introduces a further difficulty to prove a sufficient condition to the SMP. We leave the study of a sufficient condition for future research.

3.7 Proofs

Differently from (2.7.43), we define $(\eta_r)_{r\in[t,T]}, (\xi_r)_{r\in[t,T]}$ to be the solutions to the following SDEs:

$$\begin{cases} d\eta_r = \eta_r \partial_x \mu(r, c_r, X_r)dr + \eta_r \partial_x \sigma(r, X_r)dW_r \\ \eta_t = \mu(t, \bar{c}, X_t) - \mu(t, c_t, X_t), \end{cases} \tag{3.7.1}$$

$$\begin{cases} d\xi_r = [\eta_r \partial_x f(r, c_r, X_r, Q_r) - (\bar{c} - c_t)\partial_q f(r, c_r, X_r, Q_r)]\, dr \\ \xi_t = f(t, \bar{c}, X_t, Q_t) - f(t, c_t, X_t, Q_t). \end{cases} \tag{3.7.2}$$

Lemma 3.7.1. *Let $t \in [0, \tau)$ and $\bar{c} \geq 0$ be fixed and let* (3.2.9) *hold true. Then*

$$\lim_{\theta\to 0} \mathbb{E}\left[\theta \cdot \sup_{r\in[t,t+\theta]} \left|\frac{\gamma_r^{\theta,\bar{c},t}}{\theta}\right|^2\right] = 0. \tag{3.7.3}$$

Proof. Let $\theta \in \left(0, (T-t) \wedge \frac{q}{\bar{c}}\right)$ be fixed. From definition of $\gamma_r^{\theta,\bar{c},t}$ in (2.2.8) we immediately see that

$$\sup_{r\in[t,t+\theta]} \left|\gamma_r^{\theta,\bar{c},t}\right| \leq \int_t^{(t+\theta)\wedge\tau} |\bar{c} - c_s|ds \leq \theta \left(\bar{c} + \sup_{r\in[t,T]} c_r\right) \quad \text{a.s..} \tag{3.7.4}$$

Therefore,

$$\mathbb{E}\left[\sup_{r\in[t,t+\theta]} \left|\frac{\gamma_r^{\theta,\bar{c},t}}{\theta}\right|^2\right] \leq \mathbb{E}\left[\bar{c} + \sup_{r\in[t,T]} c_r\right]$$

We conclude the proof by taking the limit of previous expression and using (3.2.9). □

Lemma 3.7.2. *Let $t \in [0, \tau)$ and $\bar{c} \geq 0$ be fixed. Then,*

$$\lim_{\theta\to 0} \mathbb{E}\left[\int_t^{t+\theta} \left|c_r^{\theta,\bar{c},t} - c_r\right|^2 dr\right] = 0. \tag{3.7.5}$$

Proof. Let $\theta \in \left(0, (T-t) \wedge \frac{q}{\bar{c}}\right)$ be fixed. By using schemes above,

$$\begin{aligned} \int_t^{t+\theta} \left|c_r^{\theta,\bar{c},t} - c_r\right|^2 dr &= \int_t^{(t+\theta)\wedge\tau_{\min}^{\theta,\bar{c},t}} \left|c_r^{\theta,\bar{c},t} - c_r\right|^2 dr + \int_{(t+\theta)\wedge\tau_{\min}^{\theta,\bar{c},t}}^{t+\theta} \left|c_r^{\theta,\bar{c},t} - c_r\right|^2 dr \\ &= \int_t^{(t+\theta)\wedge\tau_{\min}^{\theta,\bar{c},t}} |\bar{c} - c_r|^2 dr + \int_{(t+\theta)\wedge\tau_{\min}^{\theta,\bar{c},t}}^{t+\theta} \left|\frac{\gamma_{t+\theta}^{\theta,\bar{c},t}}{\theta}\right|^2 dr \\ &\leq 2\theta\bar{c}^2 + 2\theta \sup_{r\in[t,T]} c_r^2 + \theta \left|\frac{\gamma_{t+\theta}^{\theta,\bar{c},t}}{\theta}\right|^2 \end{aligned}$$

Therefore,

$$\mathbb{E}\left[\int_t^{t+\theta}\left|c_r^{\theta,\bar{c},t}-c_r\right|^2 dr\right] \leq \theta\left(2\bar{c}^2+2\mathbb{E}\left[\sup_{r\in[t,T]} c_r^2 dr\right]\right)+\mathbb{E}\left[\theta\left|\frac{\gamma_{t+\theta}^{\theta,\bar{c},t}}{\theta}\right|^2\right].$$

Taking the limit of previous expression and using (3.2.9), (2.7.22) and (3.7.3) we conclude the proof. □

Lemma 3.7.3. *Let $t \in [0,\tau)$ and $\bar{c} \geq 0$ be fixed. Then,*

$$\lim_{\theta\to 0}\mathbb{E}\left[\left|\xi_{\tau^{\theta,\bar{c},t}}-\xi_\tau\right|\right]=0, \tag{3.7.6}$$

$$\lim_{\theta\to 0}\mathbb{E}\left[\left|\eta_{\tau^{\theta,\bar{c},t}}-\eta_\tau\right|^2\right]=0, \tag{3.7.7}$$

$$\lim_{\theta\to 0}\mathbb{E}\left[\left|X_{\tau^{\theta,\bar{c},t}}-X_\tau\right|^2\right]=0. \tag{3.7.8}$$

Proof. From (3.7.1), (3.7.2) and using [41, Corollary 2.5.12] we have that $\mathbb{E}\left[\sup_{r\in[t,T]}|\xi_r|\right]<\infty$, $\mathbb{E}\left[\sup_{r\in[t,T]}|\eta_r|^2\right]<\infty$ and $\mathbb{E}\left[\sup_{r\in[t,T]}|X_r|^2\right]<\infty$. Therefore, $|\xi_{\tau^{\theta,\bar{c},t}}-\xi_\tau|\leq 2\sup_{r\in[t,T]}|\xi_r|$, $|\eta_{\tau^{\theta,\bar{c},t}}-\eta_\tau|^2\leq 2\sup_{r\in[t,T]}|\eta_r|^2$ and $|X_{\tau^{\theta,\bar{c},t}}-X_\tau|^2\leq 2\sup_{r\in[t,T]}|X_r|^2$, which are all L^1-integrable functions. Finally, using (2.7.32), a.s. continuity of the three processes and DCT we conclude the Lemma. □

Lemma 3.7.4. *Let $t \in [0,\tau)$ and $\bar{c} \geq 0$ be fixed. Then,*

$$\lim_{\theta\to 0}\mathbb{E}\left[\sup_{r\in[t,t+\theta]}\left|X_r^{\theta,\bar{c},t}-X_r\right|^2\right]=0, \tag{3.7.9}$$

$$\lim_{\theta\to 0}\mathbb{E}\left[\left|X^{\theta,\bar{c},t}_{\tau_{\min}^{\theta,\bar{c},t}}-X_{\tau_{\min}^{\theta,\bar{c},t}}\right|^2\right]=0, \tag{3.7.10}$$

$$\lim_{\theta\to 0}\mathbb{E}\left[\mathbb{1}_{\tau_{\min}^{\theta,\bar{c},t}>t+\theta}\int_{t+\theta}^{\tau_{\min}^{\theta,\bar{c},t}}\left|X_s^{\theta,\bar{c},t}-X_s\right|^2 ds\right]=0. \tag{3.7.11}$$

Proof. Let $\theta \in \left(0,(T-t)\wedge\frac{q}{\bar{c}}\right)$ be fixed. From definition of X_r in (2.2.2), we have that for any $r \in [t,t+\theta]$,

$$X_r^{\theta,\bar{c},t}-X_r=\int_t^r\left(\mu\left(s,c_s^{\theta,\bar{c},t},X_s^{\theta,\bar{c},t}\right)-\mu(s,c_s,X_s)\right)ds+\int_t^r\left(\sigma\left(s,X_s^{\theta,\bar{c},t}\right)-\sigma(s,X_s)\right)dW_s.$$

Using [41, Theorem 2.5.9], we conclude that there exists $K_1>0$ such that

$$\mathbb{E}\left[\sup_{r\in[t,t+\theta]}\left|X_r^{\theta,\bar{c},t}-X_r\right|^2\right]\leq K_1\mathbb{E}\left[\int_t^{t+\theta}\left|c_s^{\theta,\bar{c},t}-c_s\right|^2 ds\right].$$

From Lemma 3.7.2 we have that

$$\lim_{\theta\to 0}\mathbb{E}\left[\sup_{r\in[t,t+\theta]}\left|X_r^{\theta,\bar{c},t}-X_r\right|^2\right]\leq\lim_{\theta\to 0}\mathbb{E}\left[\int_t^{t+\theta}\left|c_s^{\theta,\bar{c},t}-c_s\right|^2 ds\right]=0, \tag{3.7.12}$$

which concludes the proof of (3.7.9). To prove (3.7.11), under the event $\tau_{\min}^{\theta,\bar{c},t} > t+\theta$, from definition of X_r and $X_r^{\theta,\bar{c},t}$ in (3.2.1) and (3.2.1), and from (2.7.20) for any $r \in [t+\theta, \tau_{\min}^{\theta,\bar{c},t}]$, $c_r^{\theta,\bar{c},t} = c_r$, we have that for any $r \in [t+\theta, \tau_{\min}^{\theta,\bar{c},t}]$,

$$\begin{aligned}
X_r^{\theta,\bar{c},t} - X_r &= X_{t+\theta}^{\theta,\bar{c},t} - X_{t+\theta} + \int_{t+\theta}^{r} \left(\mu\left(s, c_s^{\theta,\bar{c},t}, X_s^{\theta,\bar{c},t}\right) - \mu(s, c_s, X_s)\right) ds \\
&\quad + \int_{t+\theta}^{r} \left(\sigma\left(s, X_s^{\theta,\bar{c},t}\right) - \sigma(s, X_s)\right) dW_s \\
&= X_{t+\theta}^{\theta,\bar{c},t} - X_{t+\theta} + \int_{t+\theta}^{r} \left(\mu\left(s, c_s, X_s^{\theta,\bar{c},t}\right) - \mu(s, c_s, X_s)\right) ds \\
&\quad + \int_{t+\theta}^{r} \left(\sigma\left(s, X_s^{\theta,\bar{c},t}\right) - \sigma(s, X_s)\right) dW_s.
\end{aligned}$$

Using [41, Theorem 2.5.9], we conclude that there exists $K_1 > 0$ such that

$$\begin{aligned}
\mathbb{E}\left[\left|X_{\tau_{\min}^{\theta,\bar{c},t}}^{\theta,\bar{c},t} - X_{\tau_{\min}^{\theta,\bar{c},t}}\right|^2\right] &\le \mathbb{E}\left[\mathbb{1}_{\tau_{\min}^{\theta,\bar{c},t} > t+\theta} \sup_{r\in[t+\theta,T]} \left|X_r^{\theta,\bar{c},t} - X_r\right|^2 \mathbb{1}_{r \le \tau_{\min}^{\theta,\bar{c},t}}\right] \\
&\quad + \mathbb{E}\left[\mathbb{1}_{\tau_{\min}^{\theta,\bar{c},t} \le t+\theta} \sup_{r\in[t,t+\theta]} \left|X_r^{\theta,\bar{c},t} - X_r\right|^2\right] \\
&\le \mathbb{E}\left[\left|X_{t+\theta}^{\theta,\bar{c},t} - X_{t+\theta}\right|^2\right] + \mathbb{E}\left[\sup_{r\in[t,t+\theta]} \left|X_r^{\theta,\bar{c},t} - X_r\right|^2\right] \\
&\le (1+K_1)\mathbb{E}\left[\sup_{r\in[t,t+\theta]} \left|X_r^{\theta,\bar{c},t} - X_r\right|^2\right].
\end{aligned}$$

If we take the limit of previous expression and we use (3.7.9), then we conclude the proof of (3.7.10). Finally,

$$\begin{aligned}
\mathbb{E}\left[\mathbb{1}_{\tau_{\min}^{\theta,\bar{c},t} > t+\theta} \int_{t+\theta}^{\tau_{\min}^{\theta,\bar{c},t}} \left|X_s^{\theta,\bar{c},t} - X_s\right|^2 ds\right] &\le T\mathbb{E}\left[\mathbb{1}_{\tau_{\min}^{\theta,\bar{c},t} > t+\theta} \sup_{r\in[t+\theta,T]} \left|X_r^{\theta,\bar{c},t} - X_r\right|^2 \mathbb{1}_{r \le \tau_{\min}^{\theta,\bar{c},t}}\right] \\
&\le K_1 T\mathbb{E}\left[\left|X_{t+\theta}^{\theta,\bar{c},t} - X_{t+\theta}\right|^2\right].
\end{aligned}$$

If we take the limit of previous expression and we use (3.7.9), then we conclude the proof of (3.7.11). □

Lemma 3.7.5. *Let $t \in [0,\tau)$ and $\bar{c} \ge 0$ be fixed. Then,*

$$\lim_{\theta\to 0} \mathbb{E}^t\left[\left|\frac{X_{\tau_{\min}^{\theta,\bar{c},t}}^{\theta,\bar{c},t} - X_{\tau_{\min}^{\theta,\bar{c},t}}}{\theta} - \eta_{\tau_{\min}^{\theta,\bar{c},t}}\right|^2 ds\right] = 0, \tag{3.7.13}$$

$$\lim_{\theta\to 0} \mathbb{E}^t\left[\mathbb{1}_{\tau_{\min}^{\theta,\bar{c},t} > t+\theta} \int_{t+\theta}^{\tau_{\min}^{\theta,\bar{c},t}} \left|\frac{X_s^{\theta,\bar{c},t} - X_s}{\theta} - \eta_s\right|^2 ds\right] = 0. \tag{3.7.14}$$

Proof. Let $\theta \in \left(0, (T-t) \wedge \frac{q}{\bar{c}}\right)$ be fixed. We denote

$$\tilde{X}_r^{\theta,\bar{c},t} := \frac{X_r^{\theta,\bar{c},t} - X_r}{\theta} - \eta_r.$$

The proof of this lemma will be divided in 2 steps. In Step 1 we prove that

$$\lim_{\theta\to 0}\mathbb{E}^t\left[\sup_{r\in[t,t+\theta]}\left|\tilde{X}_r^{\theta,\bar{c},t}\right|^2\right]=0.$$

In Step 2 we prove that,

$$\lim_{\theta\to 0}\mathbb{E}^t\left[\mathbb{1}_{\tau_{\min}^{\theta,\bar{c},t}>t+\theta}\sup_{r\in[t+\theta,T]}\left|\tilde{X}^{\theta,\bar{c},t}_{r\wedge\tau_{\min}^{\theta,\bar{c},t}}\right|^2\right]=0. \tag{3.7.15}$$

and

$$\lim_{\theta\to 0}\mathbb{E}^t\left[\mathbb{1}_{\tau_{\min}^{\theta,\bar{c},t}>t+\theta}\int_{t+\theta}^{\tau_{\min}^{\theta,\bar{c},t}}\left|\tilde{X}_s^{\theta,\bar{c},t}\right|^2ds\right]=0. \tag{3.7.16}$$

Step 1. From (3.7.1) we have that for any $r\in[t,t+\theta]$,

$$\begin{aligned}
\tilde{X}_r^{\theta,\bar{c},t}=&-\left(\mu(t,c_t^{\theta,\bar{c},t},X_t)-\mu(t,c_t,X_t)\right)+\frac{1}{\theta}\int_t^r\left(\mu\left(s,c_s^{\theta,\bar{c},t},X_s^{\theta,\bar{c},t}\right)-\mu(s,c_s,X_s^{\theta,\bar{c},t})\right)ds\\
&+\frac{1}{\theta}\int_t^r\left(\mu\left(s,c_s,X_s^{\theta,\bar{c},t}\right)-\mu(s,c_s,X_s)-\theta\eta_s\partial_x\mu(s,c_s,X_s)\right)ds\\
&+\frac{1}{\theta}\int_t^r\left(\sigma\left(s,X_s^{\theta,\bar{c},t}\right)-\sigma(s,X_s)-\theta\eta_s\partial_x\sigma(s,X_s)\right)dW_s\\
=&-\left(\mu(t,c_t^{\theta,\bar{c},t},X_t)-\mu(t,c_t,X_t)\right)+\frac{1}{\theta}\int_t^r\left(\mu\left(s,c_s^{\theta,\bar{c},t},X_s^{\theta,\bar{c},t}\right)-\mu(s,c_s,X_s^{\theta,\bar{c},t})\right)ds\\
&+\frac{1}{\theta}\int_t^r\left(\mu\left(s,c_s,X_s+\theta\left(\tilde{X}_s^{\theta,\bar{c},t}+\eta_s\right)\right)-\mu(s,c_s,X_s)-\theta\eta_s\partial_x\mu(s,c_s,X_s)\right)ds\\
&+\frac{1}{\theta}\int_t^r\left(\sigma\left(s,X_s+\theta\left(\tilde{X}_s^{\theta,\bar{c},t}+\eta_s\right)\right)-\sigma(s,X_s)-\theta\eta_s\partial_x\sigma(s,X_s)\right)dW_s\\
=&-\left(\mu(t,c_t^{\theta,\bar{c},t},X_t)-\mu(t,c_t,X_t)\right)+\frac{1}{\theta}\int_t^r\left(\mu\left(s,c_s^{\theta,\bar{c},t},X_s^{\theta,\bar{c},t}\right)-\mu(s,c_s,X_s^{\theta,\bar{c},t})\right)ds\\
&+\int_t^r\int_0^1\left(\partial_x\mu\left(s,c_s,X_s+\lambda\theta\left(\tilde{X}_s^{\theta,\bar{c},t}+\eta_s\right)\right)-\partial_x\mu\left(s,c_s,X_s\right)\right)\eta_sd\lambda ds\\
&+\int_t^r\int_0^1\partial_x\mu\left(s,c_s,X_s+\lambda\theta\left(\tilde{X}_s^{\theta,\bar{c},t}+\eta_s\right)\right)\tilde{X}_s^{\theta,\bar{c},t}d\lambda ds\\
&+\int_t^r\int_0^1\left(\partial_x\sigma\left(s,X_s+\lambda\theta\left(\tilde{X}_s^{\theta,\bar{c},t}+\eta_s\right)\right)-\partial_x\sigma(s,X_s)\right)\eta_sd\lambda dW_s\\
&+\int_t^r\int_0^1\partial_x\sigma\left(s,X_s+\lambda\theta\left(\tilde{X}_s^{\theta,\bar{c},t}+\eta_s\right)\right)\tilde{X}_s^{\theta,\bar{c},t}d\lambda dW_s.
\end{aligned}$$

Using the notation of Krylov [41, Section 2.5], the drift coefficient $\mu^{\theta,\bar{c},t}(s,\tilde{x})$, diffusion coefficient $\sigma^{\theta,\bar{c},t}(s,\tilde{x})$ and initial value $\xi^{\theta,\bar{c},t}(s)$ referred to previous SDE are respectively

$$\begin{aligned}
\mu^{\theta,\bar{c},t}(s,\tilde{x})=&\int_0^1\left(\partial_x\mu\left(s,c_s,X_s+\lambda\theta\left(\tilde{x}+\eta_s\right)\right)-\partial_x\mu(s,c_s,X_s)\right)\eta_sd\lambda\\
&+\int_0^1\partial_x\mu\left(s,c_s,X_s+\lambda\theta\left(\tilde{x}+\eta_s\right)\right)\tilde{x}d\lambda,\\
\sigma^{\theta,\bar{c},t}(s,\tilde{x})=&\int_0^1\left(\partial_x\sigma\left(s,X_s+\lambda\theta\left(\tilde{x}+\eta_s\right)\right)-\partial_x\sigma(s,X_s)\right)\eta_sd\lambda\\
&+\int_0^1\partial_x\sigma\left(s,X_s+\lambda\theta\left(\tilde{x}+\eta_s\right)\right)\tilde{x}d\lambda,
\end{aligned}$$

$$
\begin{aligned}
\xi_r^{\theta,\bar{c},t} &= -\left(\mu(t, c_t^{\theta,\bar{c},t}, X_t) - \mu(t, c_t, X_t)\right) + \frac{1}{\theta}\int_t^r \left(\mu\left(s, c_s^{\theta,\bar{c},t}, X_s^{\theta,\bar{c},t}\right) - \mu(s, c_s, X_s^{\theta,\bar{c},t})\right) ds \\
&= \frac{1}{\theta}\int_t^r \left(\mu\left(s, c_s^{\theta,\bar{c},t}, X_s^{\theta,\bar{c},t}\right) - \mu(t, c_t^{\theta,\bar{c},t}, X_t)\right) ds + \mu(t, c_t^{\theta,\bar{c},t}, X_t)\left(\frac{r-t}{\theta} - 1\right) \\
&\quad - \frac{1}{\theta}\int_t^r \left(\mu\left(s, c_s, X_s^{\theta,\bar{c},t}\right) - \mu(t, c_t, X_t)\right) ds - \mu(t, c_t, X_t)\left(\frac{r-t}{\theta} - 1\right).
\end{aligned}
$$

Using Lipschitz continuity and boundedness of $\partial_x\mu$ and $\partial_x\sigma$, we get that a.s.

$$
\begin{aligned}
\left|\mu^{\theta,\bar{c},t}(s,\tilde{x})\right| &\le \frac{K}{2}\theta\,|\tilde{x}+\eta_s|\,|\eta_s| + K|\tilde{x}|, \\
\left|\sigma^{\theta,\bar{c},t}(s,\tilde{x})\right| &\le \frac{K}{2}\theta\,|\tilde{x}+\eta_s|\,|\eta_s| + K|\tilde{x}|.
\end{aligned}
\tag{3.7.17}
$$

Using Lipschitz continuity of μ and σ and recalling that $X_t^{\theta,\bar{c},t} = X_t$, we get that a.s.

$$
\begin{aligned}
\left|\xi_{t+\theta}^{\theta,\bar{c},t}\right| &\le \frac{1}{\theta}\int_t^{t+\theta} \left|\mu\left(s, c_s^{\theta,\bar{c},t}, X_s^{\theta,\bar{c},t}\right) - \mu(t, c_t^{\theta,\bar{c},t}, X_t)\right| ds \\
&\quad + \frac{1}{\theta}\int_t^{t+\theta} \left|\mu\left(s, c_s, X_s^{\theta,\bar{c},t}\right) - \mu(t, c_t, X_t)\right| ds \\
&\le \frac{K}{\theta}\int_t^{t+\theta} \left(\left|c_s^{\theta,\bar{c},t} - c_t^{\theta,\bar{c},t}\right| + \left|X_s^{\theta,\bar{c},t} - X_t\right|\right) ds \\
&\quad + \frac{K}{\theta}\int_t^{t+\theta} \left(|c_s - c_t| + \left|X_s^{\theta,\bar{c},t} - X_t\right|\right) ds \\
&\le K \sup_{r\in[t,t+\theta]} \left|c_r^{\theta,\bar{c},t} - c_t^{\theta,\bar{c},t}\right| + 2K \sup_{r\in[t,t+\theta]} \left|X_r^{\theta,\bar{c},t} - X_t^{\theta,\bar{c},t}\right| + K \sup_{r\in[t,t+\theta]} |c_r - c_t|.
\end{aligned}
\tag{3.7.18}
$$

Moreover,

$$
\begin{aligned}
\int_t^{t+\theta} \left|\xi_r^{\theta,\bar{c},t}\right|^2 dr &= \int_t^{t+\theta} \left| -\left(\mu(t, c_t^{\theta,\bar{c},t}, X_t) - \mu(t, c_t, X_t)\right) + \frac{1}{\theta}\int_t^r \left(\mu\left(s, c_s^{\theta,\bar{c},t}, X_s^{\theta,\bar{c},t}\right) - \mu(s, c_s, X_s^{\theta,\bar{c},t})\right) ds\right|^2 dr \\
&= \frac{1}{\theta^2}\int_t^{t+\theta}\int_t^r \left|\mu\left(s, c_s^{\theta,\bar{c},t}, X_s^{\theta,\bar{c},t}\right) - \mu(t, c_t^{\theta,\bar{c},t}, X_t)\right|^2 ds dr \\
&\quad + \mu(t, c_t^{\theta,\bar{c},t}, X_t)^2 \int_t^{t+\theta} \left(\frac{r-t}{\theta} - 1\right)^2 dr \\
&\quad + \frac{1}{\theta^2}\int_t^{t+\theta}\int_t^r \left|\mu\left(s, c_s, X_s^{\theta,\bar{c},t}\right) - \mu(t, c_t, X_t)\right|^2 ds dr \\
&\quad + \mu(t, c_t, X_t)^2 \int_t^{t+\theta} \left(\frac{r-t}{\theta} - 1\right)^2 dr \\
&= \sup_{r\in[t,t+\theta]} \left|\mu\left(r, c_r^{\theta,\bar{c},t}, X_r^{\theta,\bar{c},t}\right) - \mu(t, c_t^{\theta,\bar{c},t}, X_t)\right|^2 \\
&\quad + \mu(t, c_t^{\theta,\bar{c},t}, X_t)^2 \frac{\theta}{3}\left[\left(\frac{r-t}{\theta} - 1\right)^3\right]_t^{t+\theta} \\
&\quad + \sup_{r\in[t,t+\theta]} \left|\mu\left(r, c_r, X_r^{\theta,\bar{c},t}\right) - \mu(t, c_t, X_t)\right|^2 \\
&\quad + \mu(t, c_t, X_t)^2 \frac{\theta}{3}\left[\left(\frac{r-t}{\theta} - 1\right)^3\right]_t^{t+\theta} \\
&\le K \sup_{r\in[t,t+\theta]} \left|c_r^{\theta,\bar{c},t} - c_t^{\theta,\bar{c},t}\right|^2 + 2K \sup_{r\in[t,t+\theta]} \left|X_r^{\theta,\bar{c},t} - X_t^{\theta,\bar{c},t}\right|^2 + K \sup_{r\in[t,t+\theta]} |c_r - c_t|^2 \\
&\quad + \frac{\theta}{3}\left(\mu(t, c_t^{\theta,\bar{c},t}, X_t)^2 + \mu(t, c_t, X_t)^2\right)
\end{aligned} \tag{3.7.19}
$$

We now show that the limit of the expectations of the expressions in (3.7.18) and (3.7.19) converge to 0. Using DCT, $\mathbb{E}^t\left[\sup_{r\in[t,t+\theta]} |c_r - c_t|^2\right]$ and $\mathbb{E}^t\left[\sup_{r\in[t,t+\theta]} \left|c_r^{\theta,\bar{c},t} - c_t^{\theta,\bar{c},t}\right|^2\right]$ converge to 0 as $\theta \to 0$. Indeed, c and $c^{\theta,\bar{c},t}$ are right-continuous and thanks to (3.2.9) and (2.7.24), the arguments of the expectations converge to 0 a.s. and they are bounded by $2\sup_{r\in[t,T]} |c_r|^2$ and $2\sup_{r\in[t,T]} |c_r^{\theta,\bar{c},t}|^2$, which are L^1-integrable processes. $\mathbb{E}^t\left[\sup_{r\in[t,t+\theta]} \left|X_r^{\theta,\bar{c},t} - X_t^{\theta,\bar{c},t}\right|^2\right]$ converges to 0 using standard arguments in SDE theory (c.f. Krylov [41, Corollary 2.5.12]). Moreover, using standard arguments in SDE theory (c.f. Krylov [41, Corollary 2.5.12]) and Assumption 3.2.5, we get that $\mu(t, c_t^{\theta,\bar{c},t}, X_t)^2$ and $\mu(t, c_t, X_t)^2$ are bounded independently of θ. Therefore we conclude that

$$
\lim_{\theta\to 0} \mathbb{E}^t\left[\left|\xi_{t+\theta}^{\theta,\bar{c},t}\right|^2\right] = 0, \tag{3.7.20}
$$

$$
\lim_{\theta\to 0} \mathbb{E}^t\left[\int_t^{t+\theta} \left|\xi_r^{\theta,\bar{c},t}\right|^2 dr\right] = 0. \tag{3.7.21}
$$

Using Krylov [41, Corollary 2.5.10] and (3.7.17), we have that there exists $K_1 > 0$ such that

$$\begin{aligned}
\mathbb{E}^t\left[\sup_{r\in[t,t+\theta]}\left|\tilde{X}^{\theta,\bar{c},t}_r\right|^2\right] &\le K_1\mathbb{E}^t\left[\sup_{r\in[t,t+\theta]}\left|\xi^{\theta,\bar{c},t}_r\right|^2\right] \\
&\quad + K_1\mathbb{E}^t\left[\int_t^{t+\theta}\left(\left|\xi^{\theta,\bar{c},t}_s\right|^2 + \left|\mu^{\theta,\bar{c},t}(s,0)\right|^2 + \left|\sigma^{\theta,\bar{c},t}(s,0)\right|^2\right)ds\right] \\
&\le K_1\left(\mathbb{E}^t\left[\sup_{r\in[t,t+\theta]}\left|\xi^{\theta,\bar{c},t}_r\right|^2\right] + \mathbb{E}^t\left[\int_t^{t+\theta}\left|\xi^{\theta,\bar{c},t}_s\right|^2\right] + \frac{K^2\theta^2}{2}\mathbb{E}^t\left[\int_t^{t+\theta}|\eta_s|^4 ds\right]\right),
\end{aligned} \tag{3.7.22}$$

which converges to 0 when $\theta \to 0$, by using (3.7.20), (3.7.21) and standard arguments in SDE theory for boundedness of $\mathbb{E}^t\left[\int_t^T |\eta_s|^4 ds\right]$.

Step 2. Under the event $\tau^{\theta,\bar{c},t}_{\min} > t+\theta$, using (3.7.1) and recalling that by (2.7.20), $c^{\theta,\bar{c},t}_r = c_r$ for any $r \in [t+\theta, \tau^{\theta,\bar{c},t}_{\min}]$, we have that for any $r \in [t+\theta, \tau^{\theta,\bar{c},t}_{\min}]$

$$\begin{aligned}
\tilde{X}^{\theta,\bar{c},t}_r &= \tilde{X}^{\theta,\bar{c},t}_{t+\theta} + \frac{1}{\theta}\int_t^r \left(\mu\left(s, c^{\theta,\bar{c},t}_s, X^{\theta,\bar{c},t}_s\right) - \mu(s, c_s, X^{\theta,\bar{c},t}_s)\right)ds \\
&\quad + \frac{1}{\theta}\int_t^r\left(\mu\left(s, c_s, X^{\theta,\bar{c},t}_s\right) - \mu(s,c_s,X_s) - \theta\eta_s\partial_x\mu(s,c_s,X_s)\right)ds \\
&\quad + \frac{1}{\theta}\int_t^r\left(\sigma\left(s, X^{\theta,\bar{c},t}_s\right) - \sigma(s,X_s) - \theta\eta_s\partial_x\sigma(s,X_s)\right)dW_s \\
&= \tilde{X}^{\theta,\bar{c},t}_{t+\theta} + 0 + \frac{1}{\theta}\int_t^r\left(\mu\left(s,c_s,X_s+\theta\left(\tilde{X}^{\theta,\bar{c},t}_s+\eta_s\right)\right) - \mu(s,c_s,X_s) - \theta\eta_s\partial_x\mu(s,c_s,X_s)\right)ds \\
&\quad + \frac{1}{\theta}\int_t^r\left(\sigma\left(s,X_s+\theta\left(\tilde{X}^{\theta,\bar{c},t}_s+\eta_s\right)\right) - \sigma(s,X_s) - \theta\eta_s\partial_x\sigma(s,X_s)\right)dW_s \\
&= \tilde{X}^{\theta,\bar{c},t}_{t+\theta} + \int_t^r\int_0^1\left(\partial_x\mu\left(s,c_s,X_s+\lambda\theta\left(\tilde{X}^{\theta,\bar{c},t}_s+\eta_s\right)\right) - \partial_x\mu\left(s,c_s,X_s\right)\right)\eta_s d\lambda ds \\
&\quad + \int_t^r\int_0^1\partial_x\mu\left(s,c_s,X_s+\lambda\theta\left(\tilde{X}^{\theta,\bar{c},t}_s+\eta_s\right)\right)\tilde{X}^{\theta,\bar{c},t}_s d\lambda ds \\
&\quad + \int_t^r\int_0^1\left(\partial_x\sigma\left(s,X_s+\lambda\theta\left(\tilde{X}^{\theta,\bar{c},t}_s+\eta_s\right)\right) - \partial_x\sigma(s,X_s)\right)\eta_s d\lambda dW_s \\
&\quad + \int_t^r\int_0^1\partial_x\sigma\left(s,X_s+\lambda\theta\left(\tilde{X}^{\theta,\bar{c},t}_s+\eta_s\right)\right)\tilde{X}^{\theta,\bar{c},t}_s d\lambda dW_s.
\end{aligned}$$

Therefore, under event $\tau^{\theta,\bar{c},t}_{\min} > t+\theta$, for any $r \in [t+\theta, T]$

$$\begin{aligned}
\tilde{X}^{\theta,\bar{c},t}_{r\wedge\tau^{\theta,\bar{c},t}_{\min}} &= \tilde{X}^{\theta,\bar{c},t}_{t+\theta} + \int_{t+\theta}^r \mathbb{1}_{s\le\tau^{\theta,\bar{c},t}_{\min}}\int_0^1\left(\partial_x\mu\left(s,c_s,X_s+\lambda\theta\left(\tilde{X}^{\theta,\bar{c},t}_s+\eta_s\right)\right) - \partial_x\mu(s,c_s,X_s)\right)\eta_s d\lambda ds \\
&\quad + \int_{t+\theta}^r \mathbb{1}_{s\le\tau^{\theta,\bar{c},t}_{\min}}\int_0^1\partial_x\mu\left(s,c_s,X_s+\lambda\theta\left(\tilde{X}^{\theta,\bar{c},t}_s+\eta_s\right)\right)\tilde{X}^{\theta,\bar{c},t}_s d\lambda ds \\
&\quad + \int_{t+\theta}^r \mathbb{1}_{s\le\tau^{\theta,\bar{c},t}_{\min}}\int_0^1\left(\partial_x\sigma\left(s,X_s+\lambda\theta\left(\tilde{X}^{\theta,\bar{c},t}_s+\eta_s\right)\right) - \partial_x\sigma(s,X_s)\right)\eta_s d\lambda dW_s \\
&\quad + \int_{t+\theta}^r \mathbb{1}_{s\le\tau^{\theta,\bar{c},t}_{\min}}\int_0^1\partial_x\sigma\left(s,X_s+\lambda\theta\left(\tilde{X}^{\theta,\bar{c},t}_s+\eta_s\right)\right)\tilde{X}^{\theta,\bar{c},t}_s d\lambda dW_s.
\end{aligned} \tag{3.7.23}$$

Similarly to Step 1, we have that, using the notation of Krylov [41, Section 2.5], the drift coefficient $\mu^{\theta,\bar{c},t}(s,\tilde{x})$, diffusion coefficient $\sigma^{\theta,\bar{c},t}(s,\tilde{x})$ and initial value $\xi^{\theta,\bar{c},t}(s)$ referred to SDE (3.7.23) are respectively

$$\begin{aligned}
\mu^{\theta,\bar{c},t}(s,\tilde{x}) &= \mathbb{1}_{s\le\tau_{\min}^{\theta,\bar{c},t}}\int_0^1\left(\partial_x\mu\left(s,c_s^{\theta,\bar{c},t},X_s+\lambda\theta\left(\tilde{x}+\eta_s\right)\right)-\partial_x\mu(s,c_s^{\theta,\bar{c},t},X_s)\right)\eta_s d\lambda\\
&\quad+\mathbb{1}_{s\le\tau_{\min}^{\theta,\bar{c},t}}\int_0^1\partial_x\mu\left(s,c_s^{\theta,\bar{c},t},X_s+\lambda\theta\left(\tilde{x}+\eta_s\right)\right)\tilde{x}d\lambda,\\
\sigma^{\theta,\bar{c},t}(s,\tilde{x}) &= \mathbb{1}_{s\le\tau_{\min}^{\theta,\bar{c},t}}\int_0^1\left(\partial_x\sigma\left(s,X_s+\lambda\theta\left(\tilde{x}+\eta_s\right)\right)-\partial_x\sigma(s,X_s)\right)\eta_s d\lambda\\
&\quad+\mathbb{1}_{s\le\tau_{\min}^{\theta,\bar{c},t}}\int_0^1\partial_x\sigma\left(s,X_s+\lambda\theta\left(\tilde{x}+\eta_s\right)\right)\tilde{x}d\lambda,\\
\xi_r^{\theta,\bar{c},t} &= \tilde{X}_{t+\theta}^{\theta,\bar{c},t}.
\end{aligned}$$

Using Lipschitz continuity of μ and σ and boundedness of $\partial_x\mu$ and $\partial_x\sigma$, we get that a.s.

$$\begin{aligned}
\left|\mu^{\theta,\bar{c},t}(s,\tilde{x})\right| &\le \frac{K}{2}\theta\left|\tilde{x}+\eta_s\right|\left|\eta_s\right|+K|\tilde{x}|,\\
\left|\sigma^{\theta,\bar{c},t}(s,\tilde{x})\right| &\le \frac{K}{2}\theta\left|\tilde{x}+\eta_s\right|\left|\eta_s\right|+K|\tilde{x}|.
\end{aligned}$$

Using Krylov [41, Corollary 2.5.10], we have that there exists $K_1>0$ such that

$$\begin{aligned}
\mathbb{E}^t\left[\mathbb{1}_{\tau_{\min}^{\theta,\bar{c},t}>t+\theta}\sup_{r\in[t+\theta,T]}\left|\tilde{X}_{r\wedge\tau_{\min}^{\theta,\bar{c},t}}^{\theta,\bar{c},t}\right|^2\right] &\le K_1\mathbb{E}^t\left[\sup_{r\in[t+\theta,T]}\left|\xi_r^{\theta,\bar{c},t}\right|^2\right]\\
&\quad+K_1\mathbb{E}^t\left[\int_{t+\theta}^T\left(\left|\xi_s^{\theta,\bar{c},t}\right|^2+\left|\mu^{\theta,\bar{c},t}(s,0)\right|^2+\left|\sigma^{\theta,\bar{c},t}(s,0)\right|^2\right)ds\right]\\
&\le K_1\left((1+T)\mathbb{E}^t\left[\left|\tilde{X}_{t+\theta}^{\theta,\bar{c},t}\right|^2\right]+\frac{K^2\theta^2}{2}\mathbb{E}^t\left[\int_{t+\theta}^T|\eta_s|^4ds\right]\right),
\end{aligned}\tag{3.7.24}$$

which converges to 0 when $\theta\to0$, as $\lim_{\theta\to0}\mathbb{E}^t\left[\left|\tilde{X}_{t+\theta}^{\theta,\bar{c},t}\right|^2\right]=0$ as proved in Step 1 and using standard arguments in SDE theory for boundedness of $\mathbb{E}^t\left[\int_t^T|\eta_s|^4ds\right]$. Hence,(3.7.15) is proved, which concludes the proof of (3.7.13). To conclude the proof of (3.7.14), we just observe that

$$\mathbb{E}^t\left[\mathbb{1}_{\tau_{\min}^{\theta,\bar{c},t}>t+\theta}\int_{t+\theta}^{\tau_{\min}^{\theta,\bar{c},t}}\left|\tilde{X}_r^{\theta,\bar{c},t}\right|^2dr\right]\le T\mathbb{E}^t\left[\sup_{r\in[t+\theta,T]}\left|\tilde{X}_{r\wedge\tau_{\min}^{\theta,\bar{c},t}}^{\theta,\bar{c},t}\right|^2\mathbb{1}_{\tau_{\min}^{\theta,\bar{c},t}>t+\theta}\right]$$

which converges to 0, as shown above. □

Lemma 3.7.6. *Let $t\in[0,\tau)$ and $\bar{c}\ge0$ be fixed. Then,*

$$\lim_{\theta\to0}\mathbb{E}^t\left[\frac{1}{\theta}\int_t^{\tau_{\min}^{\theta,\bar{c},t}}\left(f\left(s,c_s^{\theta,\bar{c},t},X_s^{\theta,\bar{c},t},Q_s^{\theta,\bar{c},t}\right)-f(s,c_s,X_s,Q_s)\right)ds-\xi_{\tau_{\min}^{\theta,\bar{c},t}}\right]=0.\tag{3.7.25}$$

Proof. Let $\theta \in \big(0, (T-t) \wedge \frac{q}{\bar{c}}\big)$ be fixed. We denote for any $r \in [t, T]$

$$\tilde{f}_r^{\theta,\bar{c},t} := \frac{1}{\theta} \int_t^r \left(f\left(s, c_s^{\theta,\bar{c},t}, X_s^{\theta,\bar{c},t}, Q_s^{\theta,\bar{c},t}\right) - f(s, c_s, X_s, Q_s) \right) ds - \xi_r. \tag{3.7.26}$$

The proof of this lemma will be divided in 3 steps. In step 1 we prove that

$$\lim_{\theta \to 0} \mathbb{E}^t \left[\left| \tilde{f}_{t+\theta}^{\theta,\bar{c},t} \right| \right] = 0.$$

In Step 2 we prove that,

$$\lim_{\theta \to 0} \mathbb{E}^t \left[\tilde{f}^{\theta,\bar{c},t}_{\tau_{\min}^{\theta,\bar{c},t}} \mathbb{1}_{\tau_{\min}^{\theta,\bar{c},t} \le t+\theta} \right] = 0. \tag{3.7.27}$$

In Step 3 we prove that

$$\lim_{\theta \to 0} \mathbb{E}^t \left[\tilde{f}^{\theta,\bar{c},t}_{\tau_{\min}^{\theta,\bar{c},t}} \mathbb{1}_{\tau_{\min}^{\theta,\bar{c},t} > t+\theta} \right] = 0. \tag{3.7.28}$$

Once the proof of the 3 steps is completed, we conclude the proof of the Lemma as follows. By merging (3.7.27) and (3.7.28), we have

$$\lim_{\theta \to 0} \mathbb{E}^t \left[\tilde{f}^{\theta,\bar{c},t}_{\tau_{\min}^{\theta,\bar{c},t}} \right] = \lim_{\theta \to 0} \mathbb{E}^t \left[\tilde{f}^{\theta,\bar{c},t}_{\tau_{\min}^{\theta,\bar{c},t}} \mathbb{1}_{\tau_{\min}^{\theta,\bar{c},t} \le t+\theta} \right] + \lim_{\theta \to 0} \mathbb{E}^t \left[\tilde{f}^{\theta,\bar{c},t}_{\tau_{\min}^{\theta,\bar{c},t}} \mathbb{1}_{\tau_{\min}^{\theta,\bar{c},t} > t+\theta} \right] = 0.$$

Step 1. From (3.7.2), recalling that $c_t^{\theta,\bar{c},t} = \bar{c}$ and (3.7.26) we have that for any $r \in [t, t+\theta]$, recalling that $X_t^{\theta,\bar{c},t} = X_t$ and $Q_t^{\theta,\bar{c},t} = Q_t$,

$$\begin{aligned}
\tilde{f}_r^{\theta,\bar{c},t} = \frac{1}{\theta} \int_t^r \left(f\left(s, c_s^{\theta,\bar{c},t}, X_s^{\theta,\bar{c},t}, Q_s^{\theta,\bar{c},t}\right) - f(s, c_s, X_s, Q_s) \right) ds + \int_t^r (\bar{c} - c_t) \partial_q f(s, c_s, X_s, Q_s) ds \\
- \int_t^r \eta_s \partial_x f(s, c_s, X_s, Q_s) ds - \left(f\left(t, c_t^{\theta,\bar{c},t}, X_t, Q_t\right) - f(t, c_t, X_t, Q_t) \right).
\end{aligned}$$

And so, for any $r \in [t, t+\theta]$

$$
\begin{aligned}
\tilde{f}_r^{\theta,\bar{c},t} &= \frac{1}{\theta}\int_t^r \left(f\left(s, c_s^{\theta,\bar{c},t}, X_s^{\theta,\bar{c},t}, Q_s^{\theta,\bar{c},t}\right) - f(s, c_s^{\theta,\bar{c},t}, X_s^{\theta,\bar{c},t}, Q_s)\right) ds \\
&\quad + \frac{1}{\theta}\int_t^r \left(f\left(s, c_s^{\theta,\bar{c},t}, X_s^{\theta,\bar{c},t}, Q_s\right) - f(s, c_s^{\theta,\bar{c},t}, X_s, Q_s)\right) ds \\
&\quad + \frac{1}{\theta}\int_t^r \left(f\left(s, c_s^{\theta,\bar{c},t}, X_s, Q_s\right) - f(t, c_t^{\theta,\bar{c},t}, X_t, Q_t)\right) ds \\
&\quad + \frac{1}{\theta}\int_t^r \left(f(t, c_t^{\theta,\bar{c},t}, X_t, Q_t) - f(t, c_t, X_t, Q_t)\right) ds \\
&\quad + \frac{1}{\theta}\int_t^r \left(f(t, c_t, X_t, Q_t) - f\left(s, c_s, X_s, Q_s\right)\right) ds \\
&\quad + f(t, c_t, X_t, Q_t) - f\left(t, c_t^{\theta,\bar{c},t}, X_t, Q_t\right) \\
&\quad + (\bar{c} - c_t)\int_t^r \partial_q f(s, c_s, X_s, Q_s) ds \\
&\quad - \int_t^r \eta_s \partial_x f(s, c_s, X_s, Q_s) ds \qquad (3.7.29)\\
&= \frac{1}{\theta}\int_t^r \left(f\left(s, c_s^{\theta,\bar{c},t}, X_s, Q_s^{\theta,\bar{c},t}\right) - f(s, c_s^{\theta,\bar{c},t}, X_s, Q_s)\right) ds \\
&\quad + \frac{1}{\theta}\int_t^r \left(f\left(s, c_s^{\theta,\bar{c},t}, X_s^{\theta,\bar{c},t}, Q_s\right) - f(s, c_s^{\theta,\bar{c},t}, X_s, Q_s)\right) ds \\
&\quad + \frac{1}{\theta}\int_t^r \left(f\left(s, c_s^{\theta,\bar{c},t}, X_s, Q_s\right) - f(t, c_t^{\theta,\bar{c},t}, X_t, Q_t)\right) ds \\
&\quad + \frac{1}{\theta}\int_t^r \left(f(t, c_t, X_t, Q_t) - f\left(s, c_s, X_s, Q_s\right)\right) ds \\
&\quad + f(t, c_t, X_t, Q_t)\left(1 - \frac{r-t}{\theta}\right) + f(t, c_t^{\theta,\bar{c},t}, X_t, Q_t)\left(\frac{r-t}{\theta} - 1\right) \\
&\quad + (\bar{c} - c_t)\int_t^r \partial_q f(s, c_s, X_s, Q_s) ds \\
&\quad - \int_t^r \eta_s \partial_x f(s, c_s, X_s, Q_s) ds.
\end{aligned}
$$

From previous expression we have that for any $r \in [t, t+\theta]$

$$
\begin{aligned}
\left|\tilde{f}_r^{\theta,\bar{c},t}\right| &\le \frac{1}{\theta}\int_t^{t+\theta} \left|f\left(s, c_s^{\theta,\bar{c},t}, X_s^{\theta,\bar{c},t}, Q_s^{\theta,\bar{c},t}\right) - f(s, c_s^{\theta,\bar{c},t}, X_s^{\theta,\bar{c},t}, Q_s)\right| ds \\
&\quad + \frac{1}{\theta}\int_t^r \left|f\left(s, c_s^{\theta,\bar{c},t}, X_s^{\theta,\bar{c},t}, Q_s\right) - f(s, c_s^{\theta,\bar{c},t}, X_s, Q_s)\right| ds \\
&\quad + \frac{1}{\theta}\int_t^{t+\theta} \left|f\left(s, c_s^{\theta,\bar{c},t}, X_s, Q_s\right) - f(t, c_t^{\theta,\bar{c},t}, X_t, Q_t)\right| ds \\
&\quad + \frac{1}{\theta}\int_t^{t+\theta} \left|f(t, c_t, X_t, Q_t) - f\left(s, c_s, X_s, Q_s\right)\right| ds \qquad (3.7.30)\\
&\quad + |f(t, c_t, X_t, Q_t)|\left|1 - \frac{r-t}{\theta}\right| + |f(t, c_t^{\theta,\bar{c},t}, X_t, Q_t)|\left|\frac{r-t}{\theta} - 1\right| \\
&\quad + |\bar{c} - c_t|\int_t^{t+\theta} \left|\partial_q f(s, c_s, X_s, Q_s)\right| ds \\
&\quad + \int_t^r |\eta_s||\partial_x f(s, c_s, X_s, Q_s)| ds.
\end{aligned}
$$

By taking $r = t + \theta$ in previous expression, so that the second last line disappears and using Assumption 3.2.1, boundedness of $\partial_q f$ and $\partial_x f$ and Hölder's inequality, we get

$$\begin{aligned}
\mathbb{E}^t\left[\left|\tilde{f}^{\theta,\bar{c},t}_{t+\theta}\right|\right] \leq K\Bigg(&\mathbb{E}^t\left[\sup_{r\in[t,t+\theta]}\left|Q^{\theta,\bar{c},t}_r - Q_r\right|\right] + \mathbb{E}^t\left[\sup_{r\in[t,t+\theta]}\left|X^{\theta,\bar{c},t}_r - X_r\right|\right] \\
&+ 2\mathbb{E}^t\left[\sup_{r\in[t,t+\theta]}|Q_r - Q_t|\right] + \theta|\bar{c} - c_t| + \theta\mathbb{E}^t\left[\sup_{r\in[t,t+\theta]}|\eta_r|\right] \\
&+ \left(\left(\mathbb{E}^t\left[\sup_{r\in[t,t+\theta]}|X_r - X_t|^2\right]\right)^{1/2} + \left(\mathbb{E}^t\left[\sup_{r\in[t,t+\theta]}\left|c^{\theta,\bar{c},t}_r - c^{\theta,\bar{c},t}_t\right|^2\right]\right)^{1/2}\right)\cdot \\
&\cdot\left(\mathbb{E}^t\left[\sup_{r\in[t,t+\theta]}\left(1 + 2|X_r| + 2|c^{\theta,\bar{c},t}_r|\right)^2\right]\right)^{1/2} + \frac{2}{\theta}\int_t^{t+\theta}|s-t|ds \\
&+ \left(\left(\mathbb{E}^t\left[\sup_{r\in[t,t+\theta]}|X_r - X_t|^2\right]\right)^{1/2} + \left(\mathbb{E}^t\left[\sup_{r\in[t,t+\theta]}|c_r - c_t|^2\right]\right)^{1/2}\right)\cdot \\
&\cdot\left(\mathbb{E}^t\left[\sup_{r\in[t,t+\theta]}(1 + 2|X_r| + 2|c_r|)^2\right]\right)^{1/2}\Bigg).
\end{aligned} \tag{3.7.31}$$

Using DCT, $\mathbb{E}^t\left[\sup_{r\in[t,t+\theta]}|c_r - c_t|^2\right]$ and $\mathbb{E}^t\left[\sup_{r\in[t,t+\theta]}\left|c^{\theta,\bar{c},t}_r - c^{\theta,\bar{c},t}_t\right|^2\right]$ converge to 0 as $\theta \to 0$. Indeed, c and $c^{\theta,\bar{c},t}$ are right-continuous and thanks to (3.2.9) and (2.7.24), the arguments of the expectations converge to 0 a.s. and they are bounded by $2\sup_{r\in[t,T]}|c_r|^2$ and $2\sup_{r\in[t,T]}|c^{\theta,\bar{c},t}_r|^2$, which are L^1-integrable processes. $\mathbb{E}^t\left[\sup_{r\in[t,t+\theta]}|X_r - X_t|^2\right]$ converges to 0 using standard arguments in SDE theory (c.f. Krylov [41, Corollary 2.5.12]). Moreover, using L^2-integrability of c and $c^{\theta,\bar{c},t}$ and standard arguments in SDE theory (c.f. Krylov [41, Corollary 2.5.12]), we get that $\mathbb{E}^t\left[\sup_{r\in[t,t+\theta]}\left(2|X_r| + 2|c^{\theta,\bar{c},t}_r|\right)^2\right]$,
$\mathbb{E}^t\left[\sup_{r\in[t,t+\theta]}(2|X_r| + 2|c_r|)^2\right]$ and $\mathbb{E}^t\left[\sup_{r\in[t,t+\theta]}|\eta_r|\right]$ are bounded independently of θ. Moreover, by definition of Q_r,

$$\mathbb{E}^t\left[\sup_{r\in[t,t+\theta]}|Q_r - Q_t|\right] = \mathbb{E}^t\left[\int_t^{t+\theta}|c_r|\,dr\right] \leq \sqrt{\theta}\left(\mathbb{E}^t\left[\int_t^T c_r^2 dr\right]\right)^{1/2},$$

which converges to 0 as $c \in L^2$. Using (2.7.23), we have that $\mathbb{E}^t\left[\left(\sup_{r\in[t,t+\theta]}\left|Q^{\theta,\bar{c},t}_r - Q_r\right|\right)\right]$ converges to 0. Using Lemma 3.7.4, we have that $\mathbb{E}^t\left[\left(\sup_{r\in[t,t+\theta]}\left|X^{\theta,\bar{c},t}_r - X_r\right|\right)\right]$ converges to 0. Moreover, $\frac{2}{\theta}\int_t^{t+\theta}|s-t|ds = \theta$. Therefore, by taking limit of (2.7.52) we conclude the proof of Step 1.

Step 2. From (3.7.29), using Assumption 3.2.1 and boundedness of $\partial_q f$, we get

$$\left|\mathbb{E}^t\left[\tilde{f}^{\theta,\bar{c},t}_{\tau^{\theta,\bar{c},t}_{\min}}\mathbb{1}_{\tau^{\theta,\bar{c},t}_{\min}\leq t+\theta}\right]\right| \leq K\Bigg(\mathbb{E}^t\left[\sup_{r\in[t,t+\theta]}\left|Q^{\theta,\bar{c},t}_r - Q_r\right|\right] + \mathbb{E}^t\left[\sup_{r\in[t,t+\theta]}\left|X^{\theta,\bar{c},t}_r - X_r\right|\right]$$

$$
\begin{aligned}
&+\frac{2}{\theta}\int_t^{t+\theta}|s-t|ds+2\mathbb{E}^t\left[\sup_{r\in[t,t+\theta]}|Q_r-Q_t|\right]+\theta|\bar{c}-c_t|+\theta\mathbb{E}^t\left[\sup_{r\in[t,t+\theta]}|\eta_r|\right]\\
&+\mathbb{E}^t\left[\sup_{r\in[t,t+\theta]}\left(|X_r-X_t|+\left|c_r^{\theta,\bar{c},t}-c_t^{\theta,\bar{c},t}\right|\right)\left(1+|X_r|+|X_t|+|c_r^{\theta,\bar{c},t}|+|c_t^{\theta,\bar{c},t}|\right)\right]\\
&+\mathbb{E}^t\left[\sup_{r\in[t,t+\theta]}\left(|X_r-X_t|+|c_r-c_t|\right)\left(1+|X_r|+|X_t|+|c_r|+|c_t|\right)\right]\\
&+\left(|f(t,c_t,X_t,Q_t)|+|f(t,c_t^{\theta,\bar{c},t},X_t,Q_t)|\right)\mathbb{E}^t\left[\left|\frac{\tau_{\min}^{\theta,\bar{c},t}-t}{\theta}-1\right|\mathbb{1}_{\tau_{\min}^{\theta,\bar{c},t}\le t+\theta}\right].
\end{aligned}
$$

The first three lines on the right-hand side of the previous expression converge to 0 similarly as we proved that (3.7.31) converges to 0 as $\theta\to 0$ in Step 1. Then, by recalling that $c_t^{\theta,\bar{c},t}=\bar{c}$, that by (2.7.2), $\left\{\tau_{\min}^{\theta,\bar{c},t}\le t+\theta\right\}=\{\tau\le t+\theta\}$, that under event $\tau_{\min}^{\theta,\bar{c},t}\le t+\theta$, then $\left|\frac{\tau_{\min}^{\theta,\bar{c},t}-t}{\theta}-1\right|\le 1$ and by using (2.7.26), we conclude that

$$
\begin{aligned}
\lim_{\theta\to 0}\mathbb{E}^t\left[\tilde{f}^{\theta,\bar{c},t}_{\tau_{\min}^{\theta,\bar{c},t}}\mathbb{1}_{\tau_{\min}^{\theta,\bar{c},t}\le t+\theta}\right]&\le\left(|f(t,\bar{c},X_t,Q_t)|+|f(t,c_t,X_t,Q_t)|\right)\lim_{\theta\to 0}\mathbb{E}^t\left[\left|\frac{\tau_{\min}^{\theta,\bar{c},t}-t}{\theta}-1\right|\mathbb{1}_{\tau_{\min}^{\theta,\bar{c},t}\le t+\theta}\right]\\
&\le\left(|f(t,\bar{c},X_t,Q_t)|+|f(t,c_t,X_t,Q_t)|\right)\lim_{\theta\to 0}\mathbb{P}\left(\{\tau\le t+\theta\}\right)=0.
\end{aligned}
$$

This concludes the proof of Step 2.

Step 3. From (3.7.2) and (3.7.26) we have that for any $r\in[t+\theta,T]$,

$$
\begin{aligned}
\tilde{f}_r^{\theta,\bar{c},t}&=\tilde{f}_{t+\theta}^{\theta,\bar{c},t}+\frac{1}{\theta}\int_{t+\theta}^r\left(f\left(s,c_s^{\theta,\bar{c},t},X_s^{\theta,\bar{c},t},Q_s^{\theta,\bar{c},t}\right)-f(s,c_s,X_s,Q_s)\right)ds\\
&\quad+\int_{t+\theta}^r(\bar{c}-c_t)\partial_q f(s,c_s,X_s,Q_s)ds-\int_{t+\theta}^r\eta_s\partial_x f(s,c_s,X_s,Q_s)ds\\
&=\tilde{f}_{t+\theta}^{\theta,\bar{c},t}+\frac{1}{\theta}\int_{t+\theta}^r\left(f\left(s,c_s^{\theta,\bar{c},t},X_s^{\theta,\bar{c},t},Q_s^{\theta,\bar{c},t}\right)-f(s,c_s,X_s^{\theta,\bar{c},t},Q_s^{\theta,\bar{c},t})\right)ds\\
&\quad+\frac{1}{\theta}\int_{t+\theta}^r\left(f\left(s,c_s,X_s^{\theta,\bar{c},t},Q_s^{\theta,\bar{c},t}\right)-f(s,c_s,X_s,Q_s^{\theta,\bar{c},t})\right)ds\\
&\quad+\frac{1}{\theta}\int_{t+\theta}^r\left(f\left(s,c_s,X_s,Q_s^{\theta,\bar{c},t}\right)-f(s,c_s,X_s,Q_s)\right)ds\\
&\quad+\int_{t+\theta}^r(\bar{c}-c_t)\partial_q f(s,c_s,X_s,Q_s)ds-\int_{t+\theta}^r\eta_s\partial_x f(s,c_s,X_s,Q_s^{\theta,\bar{c},t})ds\\
&\quad+\int_{t+\theta}^r\eta_s\left(\partial_x f(s,c_s,X_s,Q_s^{\theta,\bar{c},t})-\partial_x f(s,c_s,X_s,Q_s)\right)ds\\
&=\tilde{f}_{t+\theta}^{\theta,\bar{c},t}+\frac{1}{\theta}\int_{t+\theta}^r\left(f\left(s,c_s^{\theta,\bar{c},t},X_s^{\theta,\bar{c},t},Q_s^{\theta,\bar{c},t}\right)-f(s,c_s,X_s^{\theta,\bar{c},t},Q_s^{\theta,\bar{c},t})\right)ds\\
&\quad+\int_{t+\theta}^r\int_0^1\Bigg(\partial_x f\left(s,c_s,X_s+\lambda\left(X_s^{\theta,\bar{c},t}-X_s\right),Q_s^{\theta,\bar{c},t}\right)\frac{X_s^{\theta,\bar{c},t}-X_s}{\theta}\\
&\qquad\qquad-\eta_s\partial_x f(s,c_s,X_s,Q_s^{\theta,\bar{c},t})\Bigg)d\lambda ds\\
&\quad+\int_{t+\theta}^r\eta_s\left(\partial_x f(s,c_s,X_s,Q_s^{\theta,\bar{c},t})-\partial_x f(s,c_s,X_s,Q_s)\right)ds
\end{aligned}
$$

$$
\begin{aligned}
&+ \int_{t+\theta}^{r}\int_0^1 \Bigg(\partial_q f\left(s, c_s, X_s, Q_s + \lambda\left(Q_s^{\theta,\bar{c},t} - Q_s\right)\right)\frac{Q_s^{\theta,\bar{c},t} - Q_s}{\theta} \\
&\qquad\qquad + (\bar{c} - c_t)\partial_q f(s, c_s, X_s, Q_s)\Bigg) d\lambda ds \\
= \tilde{f}_{t+\theta}^{\theta,\bar{c},t} &+ \frac{1}{\theta}\int_{t+\theta}^{r}\left(f\left(s, c_s^{\theta,\bar{c},t}, X_s^{\theta,\bar{c},t}, Q_s^{\theta,\bar{c},t}\right) - f(s, c_s, X_s^{\theta,\bar{c},t}, Q_s^{\theta,\bar{c},t})\right) ds \\
&+ \int_{t+\theta}^{r}\int_0^1 \left(\frac{X_s^{\theta,\bar{c},t} - X_s}{\theta} - \eta_s\right)\partial_x f\left(s, c_s, X_s + \lambda\left(X_s^{\theta,\bar{c},t} - X_s\right), Q_s^{\theta,\bar{c},t}\right) d\lambda ds \\
&+ \int_{t+\theta}^{r}\int_0^1 \eta_s\left(\partial_x f\left(s, c_s, X_s + \lambda\left(X_s^{\theta,\bar{c},t} - X_s\right), Q_s^{\theta,\bar{c},t}\right) - \partial_x f(s, c_s, X_s, Q_s^{\theta,\bar{c},t})\right) d\lambda ds \\
&+ \int_{t+\theta}^{r} \eta_s\left(\partial_x f(s, c_s, X_s, Q_s^{\theta,\bar{c},t}) - \partial_x f(s, c_s, X_s, Q_s)\right) ds \\
&+ \int_{t+\theta}^{r}\int_0^1 \left(\bar{c} - c_t - \frac{Q_s - Q_s^{\theta,\bar{c},t}}{\theta}\right)\partial_q f\left(s, c_s, X_s, Q_s + \lambda\left(Q_s^{\theta,\bar{c},t} - Q_s\right)\right) d\lambda ds \\
&+ \int_{t+\theta}^{r}\int_0^1 (\bar{c} - c_t)\left(\partial_q f(s, c_s, X_s, Q_s) - \partial_q f\left(s, c_s, X_s, Q_s + \lambda\left(Q_s^{\theta,\bar{c},t} - Q_s\right)\right)\right) d\lambda ds.
\end{aligned}
$$

Therefore, by applying Assumption 3.2.1, then boundedness and Lipschitz continuity of $\partial_x f$ and $\partial_q f$ follows, we have that

$$
\begin{aligned}
\left|\tilde{f}_r^{\theta,\bar{c},t}\right| \le{}& \left|\tilde{f}_{t+\theta}^{\theta,\bar{c},t}\right| + \frac{1}{\theta}\int_{t+\theta}^{r}\left|f\left(s, c_s^{\theta,\bar{c},t}, X_s^{\theta,\bar{c},t}, Q_s^{\theta,\bar{c},t}\right) - f(s, c_s, X_s^{\theta,\bar{c},t}, Q_s^{\theta,\bar{c},t})\right| ds \\
&+ \int_{t+\theta}^{r}\int_0^1 \left|\frac{X_s^{\theta,\bar{c},t} - X_s}{\theta} - \eta_s\right|\left|\partial_x f\left(s, c_s, X_s + \lambda\left(X_s^{\theta,\bar{c},t} - X_s\right), Q_s^{\theta,\bar{c},t}\right)\right| d\lambda ds \\
&+ \int_{t+\theta}^{r}\int_0^1 |\eta_s|\left|\partial_x f\left(s, c_s, X_s + \lambda\left(X_s^{\theta,\bar{c},t} - X_s\right), Q_s^{\theta,\bar{c},t}\right) - \partial_x f(s, c_s, X_s, Q_s^{\theta,\bar{c},t})\right| d\lambda ds \\
&+ \int_{t+\theta}^{r} |\eta_s|\left|\partial_x f(s, c_s, X_s, Q_s^{\theta,\bar{c},t}) - \partial_x f(s, c_s, X_s, Q_s)\right| ds \\
&+ \int_{t+\theta}^{r}\int_0^1 \left|\bar{c} - c_t - \frac{Q_s - Q_s^{\theta,\bar{c},t}}{\theta}\right|\left|\partial_q f\left(s, c_s, X_s, Q_s + \lambda\left(Q_s^{\theta,\bar{c},t} - Q_s\right)\right)\right| d\lambda ds \\
&+ |\bar{c} - c_t|\int_{t+\theta}^{r}\int_0^1 \left|\partial_q f(s, c_s, X_s, Q_s) - \partial_q f\left(s, c_s, X_s, Q_s + \lambda\left(Q_s^{\theta,\bar{c},t} - Q_s\right)\right)\right| d\lambda ds \\
\le{}& \left|\tilde{f}_{t+\theta}^{\theta,\bar{c},t}\right| + \frac{K}{\theta}\int_{t+\theta}^{r}\left|c_s^{\theta,\bar{c},t} - c_s\right| ds + K\int_{t+\theta}^{r}\int_0^1\left|\frac{X_s^{\theta,\bar{c},t} - X_s}{\theta} - \eta_s\right| d\lambda ds \\
&+ K\int_{t+\theta}^{r} |\eta_s|\left|X_s^{\theta,\bar{c},t} - X_s\right| ds \int_0^1 \lambda d\lambda + K\int_{t+\theta}^{r}|\eta_s|\left|Q_s^{\theta,\bar{c},t} - Q_s\right| ds \\
&+ K\int_{t+\theta}^{r}\int_0^1\left|\bar{c} - c_t - \frac{Q_s - Q_s^{\theta,\bar{c},t}}{\theta}\right| d\lambda ds + K|\bar{c} - c_t|\int_0^1 \lambda d\lambda\int_{t+\theta}^{r}\left|Q_s^{\theta,\bar{c},t} - Q_s\right| ds \\
={}& \left|\tilde{f}_{t+\theta}^{\theta,\bar{c},t}\right| + \frac{K}{\theta}\int_{t+\theta}^{r}\left|c_s^{\theta,\bar{c},t} - c_s\right| ds + K\int_{t+\theta}^{r}\left|\frac{X_s^{\theta,\bar{c},t} - X_s}{\theta} - \eta_s\right| ds \\
&+ \frac{K}{2}\int_{t+\theta}^{r}|\eta_s|\left|X_s^{\theta,\bar{c},t} - X_s\right| ds + K\int_{t+\theta}^{r}|\eta_s|\left|Q_s^{\theta,\bar{c},t} - Q_s\right| ds
\end{aligned}
$$

$$+ K \int_{t+\theta}^{r} \left| \frac{Q_s - Q_s^{\theta,\bar{c},t}}{\theta} - (\bar{c} - c_t) \right| ds + \frac{K|\bar{c} - c_t|}{2} \int_{t+\theta}^{r} \left| Q_s^{\theta,\bar{c},t} - Q_s \right| ds.$$

Therefore, from previous expression and using (2.7.20) and (2.7.21), we get that

$$\begin{aligned}
\left| \mathbb{E}^t \left[\tilde{f}^{\theta,\bar{c},t}_{\tau^{\theta,\bar{c},t}_{\min}} \mathbb{1}_{\tau^{\theta,\bar{c},t}_{\min} > t+\theta} \right] \right| &\leq \mathbb{E}^t \left[\left| \tilde{f}^{\theta,\bar{c},t}_{t+\theta} \right| \right] + K \mathbb{E}^t \left[\frac{\mathbb{1}_{\tau^{\theta,\bar{c},t}_{\min} > t+\theta}}{\theta} \int_{t+\theta}^{\tau^{\theta,\bar{c},t}_{\min}} \left| c_s^{\theta,\bar{c},t} - c_s \right| ds \right] \\
&+ K \mathbb{E}^t \left[\mathbb{1}_{\tau^{\theta,\bar{c},t}_{\min} > t+\theta} \int_{t+\theta}^{\tau^{\theta,\bar{c},t}_{\min}} \left| \frac{X_s^{\theta,\bar{c},t} - X_s}{\theta} - \eta_s \right| ds \right] \\
&+ \frac{K}{2} \mathbb{E}^t \left[\mathbb{1}_{\tau^{\theta,\bar{c},t}_{\min} > t+\theta} \int_{t+\theta}^{\tau^{\theta,\bar{c},t}_{\min}} |\eta_s| \left| X_s^{\theta,\bar{c},t} - X_s \right| ds \right] \\
&+ K \mathbb{E}^t \left[\mathbb{1}_{\tau^{\theta,\bar{c},t}_{\min} > t+\theta} \int_{t+\theta}^{\tau^{\theta,\bar{c},t}_{\min}} |\eta_s| \left| Q_s^{\theta,\bar{c},t} - Q_s \right| ds \right] \\
&+ K \mathbb{E}^t \left[\mathbb{1}_{\tau^{\theta,\bar{c},t}_{\min} > t+\theta} \int_{t+\theta}^{\tau^{\theta,\bar{c},t}_{\min}} \left| \frac{Q_s - Q_s^{\theta,\bar{c},t}}{\theta} - (\bar{c} - c_t) \right| ds \right] \\
&+ K \frac{|\bar{c} - c_t|}{2} \mathbb{E}^t \left[\mathbb{1}_{\tau^{\theta,\bar{c},t}_{\min} > t+\theta} \int_{t+\theta}^{\tau^{\theta,\bar{c},t}_{\min}} \left| Q_s^{\theta,\bar{c},t} - Q_s \right| ds \right] \\
&\leq \mathbb{E}^t \left[\left| \tilde{f}^{\theta,\bar{c},t}_{t+\theta} \right| \right] + K \mathbb{E}^t \left[\mathbb{1}_{\tau^{\theta,\bar{c},t}_{\min} > t+\theta} \int_{t+\theta}^{\tau^{\theta,\bar{c},t}_{\min}} \left| \frac{X_s^{\theta,\bar{c},t} - X_s}{\theta} - \eta_s \right| ds \right] \\
&+ \frac{K}{2} \left(\mathbb{E}^t \left[\mathbb{1}_{\tau^{\theta,\bar{c},t}_{\min} > t+\theta} \int_{t+\theta}^{\tau^{\theta,\bar{c},t}_{\min}} \eta_s^2 ds \right] \mathbb{E}^t \left[\mathbb{1}_{\tau^{\theta,\bar{c},t}_{\min} > t+\theta} \int_{t+\theta}^{\tau^{\theta,\bar{c},t}_{\min}} \left| X_s^{\theta,\bar{c},t} - X_s \right|^2 ds \right] \right)^{1/2} \\
&+ K \left(\mathbb{E}^t \left[\mathbb{1}_{\tau^{\theta,\bar{c},t}_{\min} > t+\theta} \int_{t+\theta}^{\tau^{\theta,\bar{c},t}_{\min}} \eta_s^2 ds \right] \mathbb{E}^t \left[\mathbb{1}_{\tau^{\theta,\bar{c},t}_{\min} > t+\theta} \int_{t+\theta}^{\tau^{\theta,\bar{c},t}_{\min}} \left| \gamma_{t+\theta}^{\theta,\bar{c},t} \right|^2 ds \right] \right)^{1/2} \\
&+ K \mathbb{E}^t \left[\mathbb{1}_{\tau^{\theta,\bar{c},t}_{\min} > t+\theta} \int_{t+\theta}^{\tau^{\theta,\bar{c},t}_{\min}} \left| \frac{\gamma_{t+\theta}^{\theta,\bar{c},t}}{\theta} - (\bar{c} - c_t) \right| ds \right] \\
&+ K \frac{|\bar{c} - c_t|}{2} \mathbb{E}^t \left[\mathbb{1}_{\tau^{\theta,\bar{c},t}_{\min} > t+\theta} \int_{t+\theta}^{\tau^{\theta,\bar{c},t}_{\min}} \left| \gamma_{t+\theta}^{\theta,\bar{c},t} \right| ds \right] \Bigg) \\
&\leq \mathbb{E}^t \left[\left| \tilde{f}^{\theta,\bar{c},t}_{t+\theta} \right| \right] + K \mathbb{E}^t \left[\mathbb{1}_{\tau^{\theta,\bar{c},t}_{\min} > t+\theta} \int_{t+\theta}^{\tau^{\theta,\bar{c},t}_{\min}} \left| \frac{X_s^{\theta,\bar{c},t} - X_s}{\theta} - \eta_s \right| ds \right] \\
&+ \frac{KT}{2} \left(\mathbb{E}^t \left[\sup_{s \in [t,T]} \eta_s^2 \right] \mathbb{E}^t \left[\mathbb{1}_{\tau^{\theta,\bar{c},t}_{\min} > t+\theta} \int_{t+\theta}^{\tau^{\theta,\bar{c},t}_{\min}} \left| X_s^{\theta,\bar{c},t} - X_s \right|^2 ds \right] \right)^{1/2} \\
&+ KT \left(\mathbb{E}^t \left[\sup_{s \in [t,T]} \eta_s^2 \right] \mathbb{E}^t \left[\left| \gamma_{t+\theta}^{\theta,\bar{c},t} \right|^2 \right] \right)^{1/2} \\
&+ KT \mathbb{E}^t \left[\left| \frac{\gamma_{t+\theta}^{\theta,\bar{c},t}}{\theta} - (\bar{c} - c_t) \right| \right] + \frac{KT}{2} |\bar{c} - c_t| \mathbb{E}^t \left[\left| \gamma_{t+\theta}^{\theta,\bar{c},t} \right| \right].
\end{aligned}$$

By taking limit of previous expression for $\theta \to 0$, by using (2.7.22) and (2.7.36) together with Step 1, we conclude the proof of (3.7.28). This concludes the proof of Step 3 and the proof of the Lemma as well. $\square$

Lemma 3.7.7. *For any $(x,q),(x',q')\in\mathcal{O}$, if $x\neq x'$, we have that*

$$\left|\frac{g(x,q)-g(x',q)}{x-x'}-\partial_x g\left(x',q\right)\right|\leq K|x-x'|. \tag{3.7.32}$$

if $q\neq q'$, we have that

$$\left|\frac{g(x,q)-g(x,q')}{q-q'}-\partial_q g\left(x,q'\right)\right|\leq K|q-q'|. \tag{3.7.33}$$

Proof. Let $x\neq x'$, then we observe that

$$\frac{g(x,q)-g(x',q)}{x-x'}=\int_0^1 \partial_x g\left(x'+\lambda(x-x'),q\right)\,d\lambda$$

and so using Assumption 3.2.1, we get

$$\begin{aligned}\left|\frac{g(x,q)-g(x',q)}{x-x'}-\partial_x g\left(x',q\right)\right| &= \left|\int_0^1 \partial_x g\left(x'+\lambda(x-x'),q\right)\,d\lambda-\partial_x g\left(x',q\right)\right|\\ &\leq \int_0^1 \left|\partial_x g\left(x'+\lambda(x-x'),q\right)-\partial_x g\left(x',q\right)\right|d\lambda\\ &\leq K\int_0^1 \lambda|x-x'|\,d\lambda\\ &\leq \frac{K}{2}|x-x'|.\end{aligned}$$

This proves (3.7.32). On the other hand, if $q\neq q'$, we observe that

$$\frac{g(x,q)-g(x,q')}{q-q'}=\int_0^1 \partial_q g\left(x,q'+\lambda(q-q')\right)\,d\lambda$$

and so using Assumption 3.2.1, we get

$$\begin{aligned}\left|\frac{g(x,q)-g(x,q')}{q-q'}-\partial_q g\left(x,q'\right)\right| &= \left|\int_0^1 \partial_q g\left(x,q'+\lambda(q-q')\right)\,d\lambda-\partial_q g\left(x,q'\right)\right|\\ &\leq \int_0^1 \left|\partial_q g\left(x,q'+\lambda(q-q')\right)-\partial_q g\left(x,q'\right)\right|d\lambda\\ &\leq K\int_0^1 \lambda|q-q'|\,d\lambda\\ &\leq \frac{K}{2}|q-q'|.\end{aligned}$$

This proves the lemma. □

Proof of Theorem 3.2.2. Let $t\in[0,\tau)$ be fixed. Since control c is optimal, it necessarily follows that for any $\bar{c}\geq 0$ and for any $\theta>0$

$$v^{c^{\theta,\bar{c},t}}(t,x,q)\leq v^c(t,x,q).$$

Therefore, if the limit of previous expression exists, then we need to necessarily have that for any $\bar{c}\geq 0$

$$\lim_{\theta\to 0}\frac{v^{c^{\theta,\bar{c},t}}(t,x,q)-v^c(t,x,q)}{\theta}\leq 0. \tag{3.7.34}$$

By definition of v^π in (2.2.6), recalling that when $\tau_{\min}^{\theta,\bar{c},t} = \tau^{\theta,\bar{c},t}$, then for $r \geq \tau^{\theta,\bar{c},t}$, $\hat{Q}_r^{\theta,\bar{c},t} = Q_r$ and $\hat{c}_r^{\theta,\bar{c},t} = c_r$ and when $\tau_{\min}^{\theta,\bar{c},t} = \tau$, then for $r \geq \tau$, $\hat{Q}_r^{\theta,\bar{c},t} = Q_r^{\theta,\bar{c},t}$ and $\hat{c}_r^{\theta,\bar{c},t} = c_r^{\theta,\bar{c},t}$

$$\begin{aligned}
\lim_{\theta\to 0} \frac{v^{c^{\theta,\bar{c},t}}(t,x,q) - v^c(t,x,q)}{\theta} &= \lim_{\theta\to 0} \mathbb{E}^t \left[\frac{g(X_{\tau^{\theta,\bar{c},t}}^{\theta,\bar{c},t}, Q_{\tau^{\theta,\bar{c},t}}^{\theta,\bar{c},t})}{\theta} + \frac{1}{\theta} \int_t^{\tau^{\theta,\bar{c},t}} f(r, c_r^{\theta,\bar{c},t}, X_r^{\theta,\bar{c},t}, Q_t^{\theta,\bar{c},t}) dr \right] \\
&\quad - \lim_{\theta\to 0} \mathbb{E}^t \left[\frac{g(X_\tau, Q_\tau)}{\theta} + \frac{1}{\theta} \int_t^{\tau} f(r, c_r, X_r, Q_r) dr \right] \\
&= \lim_{\theta\to 0} \mathbb{E}^t \left[\frac{g(X_{\tau^{\theta,\bar{c},t}}^{\theta,\bar{c},t}, Q_{\tau^{\theta,\bar{c},t}}^{\theta,\bar{c},t}) - g(X_\tau, Q_\tau)}{\theta} \right] \\
&\quad + \lim_{\theta\to 0} \mathbb{E}^t \left[\frac{1}{\theta} \int_t^{\tau_{\min}^{\theta,\bar{c},t}} \Big(f(r, c_r^{\theta,\bar{c},t}, X_r^{\theta,\bar{c},t}, Q_t^{\theta,\bar{c},t}) - f(r, c_r, X_r, Q_r) \Big) dr \right] \\
&\quad + \lim_{\theta\to 0} \mathbb{E}^t \left[- \frac{\operatorname{sign}(\tau - \tau^{\theta,\bar{c},t})}{\theta} \int_{\tau_{\min}^{\theta,\bar{c},t}}^{\tau_{\max}^{\theta,\bar{c},t}} f(r, \hat{c}_r^{\theta,\bar{c},t}, \hat{X}_r^{\theta,\bar{c},t}, \hat{Q}_r^{\theta,\bar{c},t}) dr \right],
\end{aligned} \tag{3.7.35}$$

where in last line we used the fact that if $\tau_{\min}^{\theta,\bar{c},t} = \tau$ and $\tau_{\max}^{\theta,\bar{c},t} = \tau^{\theta,\bar{c},t}$ then it means we are under case $E_2^{\theta,\bar{c},t}$ and so

$$- \frac{\operatorname{sign}(\tau - \tau^{\theta,\bar{c},t})}{\theta} \int_{\tau_{\min}^{\theta,\bar{c},t}}^{\tau_{\max}^{\theta,\bar{c},t}} f(r, \hat{c}_r^{\theta,\bar{c},t}, \hat{X}_r^{\theta,\bar{c},t}, \hat{Q}_r^{\theta,\bar{c},t}) dr = \frac{1}{\theta} \int_{\tau}^{\tau^{\theta,\bar{c},t}} f(r, c_r^{\theta,\bar{c},t}, X_r^{\theta,\bar{c},t}, Q_r^{\theta,\bar{c},t}) dr.$$

On the other hand, if $\tau_{\max}^{\theta,\bar{c},t} = \tau$ and $\tau_{\min}^{\theta,\bar{c},t} = \tau^{\theta,\bar{c},t}$ then it means we are under case $E_1^{\theta,\bar{c},t}$ and so

$$- \frac{\operatorname{sign}(\tau - \tau^{\theta,\bar{c},t})}{\theta} \int_{\tau_{\min}^{\theta,\bar{c},t}}^{\tau_{\max}^{\theta,\bar{c},t}} f(r, \hat{c}_r^{\theta,\bar{c},t}, \hat{X}_r^{\theta,\bar{c},t}, \hat{Q}_r^{\theta,\bar{c},t}) dr = - \frac{1}{\theta} \int_{\tau^{\theta,\bar{c},t}}^{\tau} f(r, c_r, X_r, Q_r) dr.$$

The first line on the right-hand side of (2.8.2) can be written as

$$
\begin{aligned}
&\lim_{\theta\to 0}\mathbb{E}^t\left[\frac{g(X^{\theta,\bar{c},t}_{\tau^{\theta,\bar{c},t}}, Q^{\theta,\bar{c},t}_{\tau^{\theta,\bar{c},t}}) - g(X_\tau, Q_\tau)}{\theta}\right]\\
&= \lim_{\theta\to 0}\mathbb{E}^t\left[\frac{g(X^{\theta,\bar{c},t}_{\tau^{\theta,\bar{c},t}_{\min}}, Q^{\theta,\bar{c},t}_{\tau^{\theta,\bar{c},t}}) - g(X_{\tau^{\theta,\bar{c},t}_{\min}}, Q^{\theta,\bar{c},t}_{\tau^{\theta,\bar{c},t}})}{\theta}\right]\\
&+ \lim_{\theta\to 0}\mathbb{E}^t\left[\frac{g(\hat{X}^{\theta,\bar{c},t}_{\tau^{\theta,\bar{c},t}}, Q^{\theta,\bar{c},t}_{\tau^{\theta,\bar{c},t}}) - g(\hat{X}^{\theta,\bar{c},t}_{\tau}, Q^{\theta,\bar{c},t}_{\tau^{\theta,\bar{c},t}})}{\theta}\right]\\
&+ \lim_{\theta\to 0}\frac{1}{\theta}\mathbb{E}^t\Big[g(X^{\theta,\bar{c},t}_{\tau^{\theta,\bar{c},t}}, Q^{\theta,\bar{c},t}_{\tau^{\theta,\bar{c},t}}) - g(X^{\theta,\bar{c},t}_{\tau^{\theta,\bar{c},t}_{\min}}, Q^{\theta,\bar{c},t}_{\tau^{\theta,\bar{c},t}}) + g(X_{\tau^{\theta,\bar{c},t}_{\min}}, Q^{\theta,\bar{c},t}_{\tau^{\theta,\bar{c},t}}) - g(\hat{X}^{\theta,\bar{c},t}_{\tau^{\theta,\bar{c},t}}, Q^{\theta,\bar{c},t}_{\tau^{\theta,\bar{c},t}})\\
&\qquad + g(\hat{X}^{\theta,\bar{c},t}_{\tau}, Q^{\theta,\bar{c},t}_{\tau^{\theta,\bar{c},t}}) - g(X_\tau, Q^{\theta,\bar{c},t}_{\tau^{\theta,\bar{c},t}})\Big]\\
&+ \lim_{\theta\to 0}\mathbb{E}^t\left[\frac{g(X_\tau, Q^{\theta,\bar{c},t}_{\tau^{\theta,\bar{c},t}_{\min}}) - g(X_\tau, Q_{\tau^{\theta,\bar{c},t}_{\min}})}{\theta}\right] + \lim_{\theta\to 0}\mathbb{E}^t\left[\frac{g(X_\tau, \hat{Q}^{\theta,\bar{c},t}_{\tau^{\theta,\bar{c},t}}) - g(X_\tau, \hat{Q}^{\theta,\bar{c},t}_{\tau})}{\theta}\right]\\
&+ \lim_{\theta\to 0}\frac{1}{\theta}\mathbb{E}^t\Big[g(X_\tau, Q^{\theta,\bar{c},t}_{\tau^{\theta,\bar{c},t}}) - g(X_\tau, Q^{\theta,\bar{c},t}_{\tau^{\theta,\bar{c},t}_{\min}}) + g(X_\tau, Q_{\tau^{\theta,\bar{c},t}_{\min}})\\
&\qquad - g(X_\tau, \hat{Q}^{\theta,\bar{c},t}_{\tau^{\theta,\bar{c},t}}) + g(X_\tau, \hat{Q}^{\theta,\bar{c},t}_{\tau}) - g(X_\tau, Q_\tau)\Big].
\end{aligned}
\tag{3.7.36}
$$

Recalling that when $\tau^{\theta,\bar{c},t}_{\min} = \tau^{\theta,\bar{c},t}$, then $\hat{X}^{\theta,\bar{c},t}_{\tau^{\theta,\bar{c},t}} = X_{\tau^{\theta,\bar{c},t}}$, $\hat{X}^{\theta,\bar{c},t}_{\tau} = X_\tau$, $\hat{Q}^{\theta,\bar{c},t}_{\tau^{\theta,\bar{c},t}} = Q_{\tau^{\theta,\bar{c},t}}$ and $\hat{Q}^{\theta,\bar{c},t}_{\tau} = Q_\tau$ and when $\tau^{\theta,\bar{c},t}_{\min} = \tau$, then $\hat{X}^{\theta,\bar{c},t}_{\tau^{\theta,\bar{c},t}} = X^{\theta,\bar{c},t}_{\tau^{\theta,\bar{c},t}}$, $\hat{X}^{\theta,\bar{c},t}_{\tau} = X^{\theta,\bar{c},t}_{\tau}$, $\hat{Q}^{\theta,\bar{c},t}_{\tau^{\theta,\bar{c},t}} = Q^{\theta,\bar{c},t}_{\tau^{\theta,\bar{c},t}}$ and $\hat{Q}^{\theta,\bar{c},t}_{\tau} = Q^{\theta,\bar{c},t}_{\tau}$, then we have that the third element and the last element on the right-hand side of (3.7.36) are equal to 0. We define $\tilde{g}^x$ and $\tilde{g}^q$ for any $(x,q)\in\mathcal{O}, (x',q')\in\mathcal{O}$ as

$$
\begin{aligned}
\tilde{g}^x(x,x',q) &:= \begin{cases} \frac{g(x,q)-g(x',q)}{x-x'} & \text{if } x \neq x' \\ \partial_x g(x',q) & \text{if } x = x', \end{cases}\\
\tilde{g}^q(x,q,q') &:= \begin{cases} \frac{g(x,q)-g(x,q')}{q-q'} & \text{if } q \neq q' \\ \partial_q g(x,q') & \text{if } q = q'. \end{cases}
\end{aligned}
\tag{3.7.37}
$$

From Assumption 3.2.1 we have that $\tilde{g}^x$ is bounded by $K(1+|q|)$ and $\tilde{g}^q$ is bounded by $K(1+|x|)$.

First element on the right-hand side of (3.7.36) is equal to

$$
\begin{aligned}
&\lim_{\theta\to 0}\mathbb{E}^t\left[\tilde g^x\left(X^{\theta,\bar c,t}_{\tau^{\theta,\bar c,t}_{\min}},X_{\tau^{\theta,\bar c,t}_{\min}},Q^{\theta,\bar c,t}_{\tau^{\theta,\bar c,t}}\right)\frac{X^{\theta,\bar c,t}_{\tau^{\theta,\bar c,t}_{\min}}-X_{\tau^{\theta,\bar c,t}_{\min}}}{\theta}\right]\\
&\quad=\lim_{\theta\to 0}\mathbb{E}^t\left[\tilde g^x\left(X^{\theta,\bar c,t}_{\tau^{\theta,\bar c,t}_{\min}},X_{\tau^{\theta,\bar c,t}_{\min}},Q^{\theta,\bar c,t}_{\tau^{\theta,\bar c,t}}\right)\left(\frac{X^{\theta,\bar c,t}_{\tau^{\theta,\bar c,t}_{\min}}-X_{\tau^{\theta,\bar c,t}_{\min}}}{\theta}-\eta_{\tau^{\theta,\bar c,t}_{\min}}\right)\right]\\
&\qquad+\lim_{\theta\to 0}\mathbb{E}^t\left[\eta_{\tau^{\theta,\bar c,t}_{\min}}\left(\tilde g^x\left(X^{\theta,\bar c,t}_{\tau^{\theta,\bar c,t}_{\min}},X_{\tau^{\theta,\bar c,t}_{\min}},Q^{\theta,\bar c,t}_{\tau^{\theta,\bar c,t}}\right)-\partial_x g\left(X_{\tau^{\theta,\bar c,t}_{\min}},Q^{\theta,\bar c,t}_{\tau^{\theta,\bar c,t}}\right)\right)\right]\\
&\qquad+\lim_{\theta\to 0}\mathbb{E}^t\left[\eta_{\tau^{\theta,\bar c,t}_{\min}}\left(\partial_x g\left(X_{\tau^{\theta,\bar c,t}_{\min}},Q^{\theta,\bar c,t}_{\tau^{\theta,\bar c,t}}\right)-\partial_x g\left(X_\tau,Q^{\theta,\bar c,t}_{\tau^{\theta,\bar c,t}}\right)\right)\right]\\
&\qquad+\lim_{\theta\to 0}\mathbb{E}^t\left[\left(\eta_{\tau^{\theta,\bar c,t}_{\min}}-\eta_\tau\right)\partial_x g\left(X_\tau,Q^{\theta,\bar c,t}_{\tau^{\theta,\bar c,t}}\right)\right]\\
&\qquad+\lim_{\theta\to 0}\mathbb{E}^t\Bigg[\eta_\tau\Bigg(\partial_x g(X_\tau,Q^{\theta,\bar c,t}_{\tau^{\theta,\bar c,t}})-\partial_x g(X_\tau,Q^{\theta,\bar c,t}_{\tau^{\theta,\bar c,t}_{\min}})+\partial_x g(X_\tau,Q_{\tau^{\theta,\bar c,t}_{\min}})-\partial_x g(X_\tau,\hat Q^{\theta,\bar c,t}_{\tau^{\theta,\bar c,t}})\\
&\qquad\qquad+\partial_x g(X_\tau,\hat Q^{\theta,\bar c,t}_{\tau})-\partial_x g(X_\tau,Q_\tau)\Bigg)\Bigg]\\
&\qquad+\mathbb{E}^t\left[\eta_\tau\partial_x g(X_\tau,Q_\tau)\right].
\end{aligned}
\tag{3.7.38}
$$

Recalling that when $\tau^{\theta,\bar c,t}_{\min}=\tau^{\theta,\bar c,t}$, then $\hat X^{\theta,\bar c,t}_{\tau^{\theta,\bar c,t}}=X_{\tau^{\theta,\bar c,t}}$ and $\hat X^{\theta,\bar c,t}_{\tau}=X_\tau$ and when $\tau^{\theta,\bar c,t}_{\min}=\tau$, then $\hat X^{\theta,\bar c,t}_{\tau^{\theta,\bar c,t}}=X^{\theta,\bar c,t}_{\tau^{\theta,\bar c,t}}$ and $\hat X^{\theta,\bar c,t}_{\tau}=X^{\theta,\bar c,t}_{\tau}$, then we have that the fifth element on the right-hand side of (3.7.38) is equal to 0. Using Hölder's inequality, boundedness of $\tilde g^x$, (3.7.13) and the fact that $Q^{\theta,\bar c,t}_r$ is always bounded by q, we get

$$
\begin{aligned}
&\lim_{\theta\to 0}\mathbb{E}\left[\left|\tilde g^x\left(X^{\theta,\bar c,t}_{\tau^{\theta,\bar c,t}_{\min}},X_{\tau^{\theta,\bar c,t}_{\min}},Q^{\theta,\bar c,t}_{\tau^{\theta,\bar c,t}}\right)\left(\frac{X^{\theta,\bar c,t}_{\tau^{\theta,\bar c,t}_{\min}}-X_{\tau^{\theta,\bar c,t}_{\min}}}{\theta}-\eta_{\tau^{\theta,\bar c,t}_{\min}}\right)\right|\right]\\
&\quad\le K\lim_{\theta\to 0}\left(\mathbb{E}\left[\left(1+\left|Q^{\theta,\bar c,t}_{\tau^{\theta,\bar c,t}}\right|\right)^2\right]\right)^{\frac12}\left(\mathbb{E}\left[\left|\frac{X^{\theta,\bar c,t}_{\tau^{\theta,\bar c,t}_{\min}}-X_{\tau^{\theta,\bar c,t}_{\min}}}{\theta}-\eta_{\tau^{\theta,\bar c,t}_{\min}}\right|^2\right]\right)^{\frac12}=0,
\end{aligned}
\tag{3.7.39}
$$

where we used the fact that $Q^{\theta,\bar c,t}_r$ is globally bounded from above by q. Moreover, using (3.7.32) in Lemma 3.7.7 together with definition of $\tilde g^x$ in (3.7.37), we get that

$$
\begin{aligned}
&\lim_{\theta\to 0}\mathbb{E}^t\left[\left|\eta_{\tau^{\theta,\bar c,t}_{\min}}\right|\left|\tilde g^x\left(X^{\theta,\bar c,t}_{\tau^{\theta,\bar c,t}_{\min}},X_{\tau^{\theta,\bar c,t}_{\min}},Q^{\theta,\bar c,t}_{\tau^{\theta,\bar c,t}}\right)-\partial_x g\left(X_{\tau^{\theta,\bar c,t}_{\min}},Q^{\theta,\bar c,t}_{\tau^{\theta,\bar c,t}}\right)\right|\right]\\
&\quad\le\frac{K}{2}\lim_{\theta\to 0}\left(\mathbb{E}^t\left[\left|\eta_{\tau^{\theta,\bar c,t}_{\min}}\right|^2\right]\right)^{\frac12}\left(\mathbb{E}^t\left[\left|X^{\theta,\bar c,t}_{\tau^{\theta,\bar c,t}_{\min}}-X_{\tau^{\theta,\bar c,t}_{\min}}\right|^2\mathbb{1}_{X^{\theta,\bar c,t}_{\tau^{\theta,\bar c,t}_{\min}}\neq X_{\tau^{\theta,\bar c,t}_{\min}}}\right]\right)^{\frac12}=0,
\end{aligned}
\tag{3.7.40}
$$

where in the last line we used standard arguments of SDE theory, i.e. $\mathbb{E}^t\left[\sup_{r\in[t,T]}|\eta_r|^2\right]<\infty$ and we used (3.7.10) in Lemma 3.7.4. Moreover, using Lipschitz continuity of $\partial_x g$ and (3.7.8), we get that

$$\begin{aligned}
&\lim_{\theta\to 0}\mathbb{E}^t\left[\left|\eta_{\tau_{\min}^{\theta,\bar{c},t}}\right|\left|\partial_x g\left(X_{\tau_{\min}^{\theta,\bar{c},t}},Q^{\theta,\bar{c},t}_{\tau^{\theta,\bar{c},t}}\right)-\partial_x g\left(X_\tau,Q^{\theta,\bar{c},t}_{\tau^{\theta,\bar{c},t}}\right)\right|\right]\\
&\quad\le\lim_{\theta\to 0}\left(\mathbb{E}^t\left[\left|\eta_{\tau_{\min}^{\theta,\bar{c},t}}\right|^2\right]\right)^{\frac{1}{2}}\left(\mathbb{E}^t\left[\left|\partial_x g\left(X_{\tau_{\min}^{\theta,\bar{c},t}},Q^{\theta,\bar{c},t}_{\tau^{\theta,\bar{c},t}}\right)-\partial_x g\left(X_\tau,Q^{\theta,\bar{c},t}_{\tau^{\theta,\bar{c},t}}\right)\right|^2\right]\right)^{\frac{1}{2}}\\
&\quad\le K\lim_{\theta\to 0}\left(\mathbb{E}^t\left[\left|\eta_{\tau_{\min}^{\theta,\bar{c},t}}\right|^2\right]\right)^{\frac{1}{2}}\left(\mathbb{E}^t\left[\left|X_{\tau_{\min}^{\theta,\bar{c},t}}-X_\tau\right|^2\right]\right)^{\frac{1}{2}}\\
&\quad\le K\lim_{\theta\to 0}\left(\mathbb{E}^t\left[\left|\eta_{\tau_{\min}^{\theta,\bar{c},t}}\right|^2\right]\right)^{\frac{1}{2}}\left(\mathbb{E}^t\left[\left|X_{\tau^{\theta,\bar{c},t}}-X_\tau\right|^2\mathbb{1}_{\tau_{\min}^{\theta,\bar{c},t}=\tau^{\theta,\bar{c},t}}\right]\right)^{\frac{1}{2}}=0.
\end{aligned}\tag{3.7.41}$$

Finally, using boundedness of $\partial_x g$ and (3.7.7), we get that

$$\begin{aligned}
&\lim_{\theta\to 0}\mathbb{E}^t\left[\left|\eta_{\tau_{\min}^{\theta,\bar{c},t}}-\eta_\tau\right|\left|\partial_x g\left(X_\tau,Q^{\theta,\bar{c},t}_{\tau^{\theta,\bar{c},t}}\right)\right|\right]\\
&\quad\le\lim_{\theta\to 0}\left(\mathbb{E}^t\left[\left|\eta_{\tau_{\min}^{\theta,\bar{c},t}}-\eta_\tau\right|^2\right]\right)^{\frac{1}{2}}\left(\mathbb{E}^t\left[\left|\partial_x g\left(X_\tau,Q^{\theta,\bar{c},t}_{\tau^{\theta,\bar{c},t}}\right)\right|^2\right]\right)^{\frac{1}{2}}\\
&\quad\le K\lim_{\theta\to 0}\left(\mathbb{E}^t\left[\left|\eta_{\tau_{\min}^{\theta,\bar{c},t}}-\eta_\tau\right|^2\right]\right)^{\frac{1}{2}}\\
&\quad\le K\lim_{\theta\to 0}\left(\mathbb{E}^t\left[\left|\eta_{\tau^{\theta,\bar{c},t}}-\eta_\tau\right|^2\mathbb{1}_{\tau_{\min}^{\theta,\bar{c},t}=\tau^{\theta,\bar{c},t}}\right]\right)^{\frac{1}{2}}=0.
\end{aligned}\tag{3.7.42}$$

Hence, merging (3.7.39), (3.7.40), (3.7.41) and (3.7.42) into (3.7.38), we get

$$\begin{aligned}
&\lim_{\theta\to 0}\mathbb{E}^t\left[\frac{g(X^{\theta,\bar{c},t}_{\tau_{\min}^{\theta,\bar{c},t}},Q^{\theta,\bar{c},t}_{\tau^{\theta,\bar{c},t}})-g(X_{\tau_{\min}^{\theta,\bar{c},t}},Q^{\theta,\bar{c},t}_{\tau^{\theta,\bar{c},t}})}{\theta}\right]\\
&\quad=\lim_{\theta\to 0}\mathbb{E}^t\left[\tilde{g}^x\left(X^{\theta,\bar{c},t}_{\tau_{\min}^{\theta,\bar{c},t}},X_{\tau_{\min}^{\theta,\bar{c},t}},Q^{\theta,\bar{c},t}_{\tau^{\theta,\bar{c},t}}\right)\frac{X^{\theta,\bar{c},t}_{\tau_{\min}^{\theta,\bar{c},t}}-X_{\tau_{\min}^{\theta,\bar{c},t}}}{\theta}\right]\\
&\quad=\mathbb{E}^t\left[\eta_\tau\partial_x g\left(X_\tau,Q_\tau\right)\right].
\end{aligned}\tag{3.7.43}$$

Second element on the right-hand side of (3.7.36) is equal to $-\bar{g}(t,\bar{c},x,q)$ by its definition (3.2.10). Fourth element on the right-hand side of (3.7.36) is equal to

$$\begin{aligned}
&\lim_{\theta\to 0}\mathbb{E}^t\left[\tilde{g}^q\left(X_\tau,Q^{\theta,\bar{c},t}_{\tau_{\min}^{\theta,\bar{c},t}},Q_{\tau_{\min}^{\theta,\bar{c},t}}\right)\frac{Q^{\theta,\bar{c},t}_{\tau_{\min}^{\theta,\bar{c},t}}-Q_{\tau_{\min}^{\theta,\bar{c},t}}}{\theta}\right]\\
&\quad=\lim_{\theta\to 0}\mathbb{E}^t\left[\tilde{g}^q\left(X_\tau,Q^{\theta,\bar{c},t}_{\tau_{\min}^{\theta,\bar{c},t}},Q_{\tau_{\min}^{\theta,\bar{c},t}}\right)\left(\frac{Q^{\theta,\bar{c},t}_{\tau_{\min}^{\theta,\bar{c},t}}-Q_{\tau_{\min}^{\theta,\bar{c},t}}}{\theta}+\bar{c}-c_t\right)\right]\\
&\qquad-\lim_{\theta\to 0}\mathbb{E}^t\left[(\bar{c}-c_t)\left(\tilde{g}^q\left(X_\tau,Q^{\theta,\bar{c},t}_{\tau_{\min}^{\theta,\bar{c},t}},Q_{\tau_{\min}^{\theta,\bar{c},t}}\right)-\partial_q g\left(X_\tau,Q_{\tau_{\min}^{\theta,\bar{c},t}}\right)\right)\right]\\
&\qquad-(\bar{c}-c_t)\lim_{\theta\to 0}\mathbb{E}^t\left[\partial_q g\left(X_\tau,Q_{\tau_{\min}^{\theta,\bar{c},t}}\right)-\partial_q g\left(X_\tau,Q_\tau\right)\right]\\
&\qquad-(\bar{c}-c_t)\mathbb{E}^t\left[\partial_q g\left(X_\tau,Q_\tau\right)\right].
\end{aligned}\tag{3.7.44}$$

Using Hölder's inequality, boundedness of $\tilde{g}^q$ and (2.7.37), we get

$$\begin{aligned}
&\lim_{\theta\to 0}\mathbb{E}\left[\left|\tilde{g}^q\left(X_\tau, Q^{\theta,\bar{c},t}_{\tau^{\theta,\bar{c},t}_{\min}}, Q_{\tau^{\theta,\bar{c},t}_{\min}}\right)\left(\frac{Q^{\theta,\bar{c},t}_{\tau^{\theta,\bar{c},t}_{\min}} - Q_{\tau^{\theta,\bar{c},t}_{\min}}}{\theta} + \bar{c} - c_t\right)\right|\right] \\
&\qquad \leq K\left(\mathbb{E}\left[(1+|X_\tau|)^4\right]\right)^{\frac{1}{4}}\lim_{\theta\to 0}\left(\mathbb{E}\left[\left|\frac{Q^{\theta,\bar{c},t}_{\tau^{\theta,\bar{c},t}_{\min}} - Q_{\tau^{\theta,\bar{c},t}_{\min}}}{\theta} + \bar{c} - c_t\right|^{\frac{4}{3}}\right]\right)^{\frac{3}{4}} = 0.
\end{aligned} \tag{3.7.45}$$

Here we used standard arguments of SDE theory, i.e. $\mathbb{E}\left[\sup_{r\in[0,T]}|X_r|^4\right] < \infty$. Moreover, using (3.7.33) in Lemma 3.7.7 together with definition of $\tilde{g}^q$ in (3.7.37), we get that

$$\begin{aligned}
&\lim_{\theta\to 0}\mathbb{E}\left[|\bar{c}-c_t|\left|\tilde{g}^q\left(X_\tau, Q^{\theta,\bar{c},t}_{\tau^{\theta,\bar{c},t}_{\min}}, Q_{\tau^{\theta,\bar{c},t}_{\min}}\right) - \partial_q g\left(X_\tau, Q_{\tau^{\theta,\bar{c},t}_{\min}}\right)\right|\right] \\
&\qquad \leq \frac{K}{2}|\bar{c}-c_t|\lim_{\theta\to 0}\mathbb{E}\left[\left|Q^{\theta,\bar{c},t}_{\tau^{\theta,\bar{c},t}_{\min}} - Q_{\tau^{\theta,\bar{c},t}_{\min}}\right|\mathbb{1}_{Q^{\theta,\bar{c},t}_{\tau^{\theta,\bar{c},t}_{\min}} \neq Q_{\tau^{\theta,\bar{c},t}_{\min}}}\right] = 0,
\end{aligned} \tag{3.7.46}$$

where in the last line we used (2.7.23) in Lemma 2.7.3. Moreover, using Lipschitz continuity of $\partial_q g$ and (2.7.45), we get that

$$\begin{aligned}
\lim_{\theta\to 0}\mathbb{E}\left[\left|\partial_q g\left(X_\tau, Q_{\tau^{\theta,\bar{c},t}_{\min}}\right) - \partial_q g\left(X_\tau, Q_\tau\right)\right|\right] &\leq K\lim_{\theta\to 0}\mathbb{E}\left[\left|Q_{\tau^{\theta,\bar{c},t}_{\min}} - Q_\tau\right|\right] \\
&\leq K\lim_{\theta\to 0}\mathbb{E}\left[\left|Q_{\tau^{\theta,\bar{c},t}} - Q_\tau\right|\mathbb{1}_{\tau^{\theta,\bar{c},t}_{\min} = \tau^{\theta,\bar{c},t}}\right] = 0.
\end{aligned} \tag{3.7.47}$$

Hence, merging (3.7.45), (3.7.46) and (3.7.47) into (3.7.44), we get

$$\lim_{\theta\to 0}\mathbb{E}^t\left[\frac{g(X_\tau, Q^{\theta,\bar{c},t}_{\tau^{\theta,\bar{c},t}_{\min}}) - g(X_\tau, Q_{\tau^{\theta,\bar{c},t}_{\min}})}{\theta}\right] = -(\bar{c}-c_t)\mathbb{E}^t\left[\partial_q g\left(X_\tau, Q_\tau\right)\right]. \tag{3.7.48}$$

Fifth element on the right-hand side of (3.7.36) is equal to

$$\begin{aligned}
&\lim_{\theta\to 0}\mathbb{E}^t\left[\tilde{g}^q\left(X_\tau, \hat{Q}^{\theta,\bar{c},t}_{\tau^{\theta,\bar{c},t}}, \hat{Q}^{\theta,\bar{c},t}_\tau\right)\frac{\hat{Q}^{\theta,\bar{c},t}_{\tau^{\theta,\bar{c},t}} - \hat{Q}^{\theta,\bar{c},t}_\tau}{\theta}\right] \\
&= \lim_{\theta\to 0}\mathbb{E}^t\left[\tilde{g}^q\left(X_\tau, \hat{Q}^{\theta,\bar{c},t}_{\tau^{\theta,\bar{c},t}}, \hat{Q}^{\theta,\bar{c},t}_\tau\right)\left(\frac{\hat{Q}^{\theta,\bar{c},t}_{\tau^{\theta,\bar{c},t}} - \hat{Q}^{\theta,\bar{c},t}_\tau}{\theta} - (\bar{c}-c_t)\mathbb{1}_{\Lambda(t,\bar{c})}\right)\right] \\
&\quad + \lim_{\theta\to 0}\mathbb{E}^t\left[\left(\tilde{g}^q\left(X_\tau, \hat{Q}^{\theta,\bar{c},t}_{\tau^{\theta,\bar{c},t}}, \hat{Q}^{\theta,\bar{c},t}_\tau\right) - \partial_q g\left(X_\tau, \hat{Q}^{\theta,\bar{c},t}_\tau\right)\right)(\bar{c}-c_t)\mathbb{1}_{\Lambda(t,\bar{c})}\right] \\
&\quad + \lim_{\theta\to 0}\mathbb{E}^t\left[\left(\partial_q g\left(X_\tau, \hat{Q}^{\theta,\bar{c},t}_\tau\right) - \partial_q g\left(X_\tau, Q_\tau\right)\right)(\bar{c}-c_t)\mathbb{1}_{\Lambda(t,\bar{c})}\right] \\
&\quad + (\bar{c}-c_t)\mathbb{E}^t\left[\partial_q g\left(X_\tau, Q_\tau\right)\mathbb{1}_{\Lambda(t,\bar{c})}\right].
\end{aligned} \tag{3.7.49}$$

Using Hölder's inequality, boundedness of $\tilde{g}^q$ and Lemma 2.7.7, we get

$$\begin{aligned}
&\lim_{\theta\to 0}\mathbb{E}\left[\left|\tilde{g}^q\left(X_\tau, \hat{Q}^{\theta,\bar{c},t}_{\tau^{\theta,\bar{c},t}}, \hat{Q}^{\theta,\bar{c},t}_\tau\right)\left(\frac{\hat{Q}^{\theta,\bar{c},t}_{\tau^{\theta,\bar{c},t}} - \hat{Q}^{\theta,\bar{c},t}_\tau}{\theta} - (\bar{c}-c_t)\mathbb{1}_{\Lambda(t,\bar{c})}\right)\right|\right] \\
&\leq K\left(\mathbb{E}\left[(1+|X_\tau|)^4\right]\right)^{\frac{1}{4}}\lim_{\theta\to 0}\left(\mathbb{E}\left[\left|\frac{\hat{Q}^{\theta,\bar{c},t}_{\tau^{\theta,\bar{c},t}} - \hat{Q}^{\theta,\bar{c},t}_\tau}{\theta} - (\bar{c}-c_t)\mathbb{1}_{\Lambda(t,\bar{c})}\right|^{\frac{4}{3}}\right]\right)^{\frac{3}{4}} = 0.
\end{aligned} \tag{3.7.50}$$

Here we used standard arguments of SDE theory, i.e. $\mathbb{E}\left[\sup_{r\in[0,T]}|X_r|^4\right]<\infty$. Moreover, using (3.7.33) in Lemma 3.7.7 together with definition of $\tilde{g}^q$ in (3.7.37), we get that

$$\begin{aligned}&\lim_{\theta\to 0}\mathbb{E}\left[|\bar{c}-c_t|\left|\tilde{g}^q\left(X_\tau,\hat{Q}^{\theta,\bar{c},t}_{\tau^{\theta,\bar{c},t}},\hat{Q}^{\theta,\bar{c},t}_\tau\right)-\partial_q g\left(X_\tau,\hat{Q}^{\theta,\bar{c},t}_\tau\right)\right|\mathbb{1}_{\Lambda(t,\bar{c})}\right]\\&\quad\le\lim_{\theta\to 0}\mathbb{E}\left[|\bar{c}-c_t|\left|\tilde{g}^q\left(X_\tau,\hat{Q}^{\theta,\bar{c},t}_{\tau^{\theta,\bar{c},t}},\hat{Q}^{\theta,\bar{c},t}_\tau\right)-\partial_q g\left(X_\tau,\hat{Q}^{\theta,\bar{c},t}_\tau\right)\right|\right]\\&\quad\le\frac{K}{2}|\bar{c}-c_t|\lim_{\theta\to 0}\mathbb{E}\left[\left|\hat{Q}^{\theta,\bar{c},t}_{\tau^{\theta,\bar{c},t}}-\hat{Q}^{\theta,\bar{c},t}_\tau\right|\mathbb{1}_{\hat{Q}^{\theta,\bar{c},t}_{\tau^{\theta,\bar{c},t}}\neq\hat{Q}^{\theta,\bar{c},t}_\tau}\right]=0,\end{aligned}\tag{3.7.51}$$

where in the last line we used (2.7.40) in Lemma 2.7.8. Moreover, using Lipschitz continuity of $\partial_q g$ in Assumption 3.2.1, we have that

$$\begin{aligned}&\lim_{\theta\to 0}\mathbb{E}\left[\left|(\bar{c}-c_t)\mathbb{1}_{\Lambda(t,\bar{c})}\left(\partial_q g\left(X_\tau,\hat{Q}^{\theta,\bar{c},t}_\tau\right)-\partial_q g\left(X_\tau,Q_\tau\right)\right)\right|\right]\\&\quad\le|\bar{c}-c_t|\lim_{\theta\to 0}\mathbb{E}\left[\left|\partial_q g\left(X_\tau,\hat{Q}^{\theta,\bar{c},t}_\tau\right)-\partial_q g\left(X_\tau,Q_\tau\right)\right|\right]\\&\quad\le K|\bar{c}-c_t|\lim_{\theta\to 0}\mathbb{E}\left[\left|\hat{Q}^{\theta,\bar{c},t}_\tau-Q_\tau\right|\right]=0,\end{aligned}\tag{3.7.52}$$

where in the last equality we used (2.7.41) in Lemma 2.7.8. Hence, merging (3.7.50), (3.7.51) and (3.7.52) into (3.7.49), we get

$$\lim_{\theta\to 0}\mathbb{E}^t\left[\frac{g(X_\tau,\hat{Q}^{\theta,\bar{c},t}_{\tau^{\theta,\bar{c},t}})-g(X_\tau,\hat{Q}^{\theta,\bar{c},t}_\tau)}{\theta}\right]=(\bar{c}-c_t)\mathbb{E}^t\left[\partial_q g\left(X_\tau,Q_\tau\right)\mathbb{1}_{\Lambda(t,\bar{c})}\right].\tag{3.7.53}$$

Merging (3.2.10), (3.7.43), (3.7.48) and (3.7.53) into (3.7.36), we conclude that the first line of the right-hand side of (3.7.35) is equal to

$$\begin{aligned}\lim_{\theta\to 0}\mathbb{E}^t\left[\frac{g(X_{\tau^{\theta,\bar{c},t}},Q^{\theta,\bar{c},t}_{\tau^{\theta,\bar{c},t}})-g(X_\tau,Q_\tau)}{\theta}\right]&=\mathbb{E}^t\left[\eta_\tau\partial_q g(X_\tau,Q_\tau)\right]-\bar{g}(t,\bar{c},x,q)\\&\quad-\mathbb{E}^t\left[(\bar{c}-c_t)\partial_q g(X_\tau,Q_\tau)\right]+(\bar{c}-c_t)\mathbb{E}^t\left[\partial_q g(X_\tau,Q_\tau)\mathbb{1}_{\Lambda(t,\bar{c})}\right].\end{aligned}\tag{3.7.54}$$

The second and third lines of right-hand side of (3.7.35) can be written as

$$\begin{aligned}&\lim_{\theta\to 0}\mathbb{E}^t\left[\frac{1}{\theta}\int_t^{\tau^{\theta,\bar{c},t}_{\min}}\left(f(r,c^{\theta,\bar{c},t}_r,X^{\theta,\bar{c},t}_r,Q^{\theta,\bar{c},t}_r)-f(r,c_r,X_r,Q_r)\right)dr\right]\\&\quad-\lim_{\theta\to 0}\mathbb{E}^t\left[\frac{\operatorname{sign}(\tau-\tau^{\theta,\bar{c},t})}{\theta}\int_{\tau^{\theta,\bar{c},t}_{\min}}^{\tau^{\theta,\bar{c},t}_{\max}}f(r,\hat{c}^{\theta,\bar{c},t}_r,\hat{X}^{\theta,\bar{c},t}_r,\hat{Q}^{\theta,\bar{c},t}_r)dr\right]\\&\quad=\lim_{\theta\to 0}\mathbb{E}^t\left[\frac{1}{\theta}\int_t^{\tau^{\theta,\bar{c},t}_{\min}}\left(f(r,c^{\theta,\bar{c},t}_r,X^{\theta,\bar{c},t}_r,Q^{\theta,\bar{c},t}_r)-f(r,c_r,X_r,Q_r)\right)dr-\xi_{\tau^{\theta,\bar{c},t}_{\min}}\right]+\lim_{\theta\to 0}\mathbb{E}^t\left[\xi_{\tau^{\theta,\bar{c},t}_{\min}}\right]\\&\qquad-\lim_{\theta\to 0}\mathbb{E}^t\left[\frac{\operatorname{sign}(\tau-\tau^{\theta,\bar{c},t})}{\theta}\int_{\tau^{\theta,\bar{c},t}_{\min}}^{\tau^{\theta,\bar{c},t}_{\max}}f(r,\hat{c}^{\theta,\bar{c},t}_r,\hat{X}^{\theta,\bar{c},t}_r,\hat{Q}^{\theta,\bar{c},t}_r)dr\right].\end{aligned}\tag{3.7.55}$$

Using Lemma 3.7.6, we have that

$$\lim_{\theta\to 0}\mathbb{E}^t\left[\frac{1}{\theta}\int_t^{\tau^{\theta,\bar{c},t}_{\min}}\left(f(r,c^{\theta,\bar{c},t}_r,X^{\theta,\bar{c},t}_r,Q^{\theta,\bar{c},t}_r)-f(r,c_r,X_r,Q_r)\right)dr-\xi_{\tau^{\theta,\bar{c},t}_{\min}}\right]=0.\tag{3.7.56}$$

Using (3.7.6) in Lemma 3.7.3, we have that

$$\begin{aligned}\lim_{\theta\to 0}\mathbb{E}^t\left[\xi_{\tau^{\theta,\bar c,t}_{\min}}\right] &= \lim_{\theta\to 0}\mathbb{E}^t\left[\xi_{\tau^{\theta,\bar c,t}}\mathbb{1}_{\tau^{\theta,\bar c,t}_{\min}=\tau^{\theta,\bar c,t}}\right] + \lim_{\theta\to 0}\mathbb{E}^t\left[\xi_\tau\mathbb{1}_{\tau^{\theta,\bar c,t}_{\min}=\tau}\right]\\ &= \lim_{\theta\to 0}\mathbb{E}^t\left[\left(\xi_{\tau^{\theta,\bar c,t}}-\xi_\tau\right)\mathbb{1}_{\tau^{\theta,\bar c,t}_{\min}=\tau^{\theta,\bar c,t}}\right] + \mathbb{E}^t\left[\xi_\tau\right] = \mathbb{E}^t\left[\xi_\tau\right].\end{aligned} \tag{3.7.57}$$

Using (3.2.11), the third limit on the right-hand side of (3.7.55) converges to $\bar f(t,\bar c,x,q)$. Merging (2.8.17), (3.7.57) and (3.2.11) into (3.7.55), we get that

$$\begin{aligned}&\lim_{\theta\to 0}\mathbb{E}^t\left[\frac{1}{\theta}\int_t^{\tau^{\theta,\bar c,t}_{\min}}\left(f(r,c^{\theta,\bar c,t}_r,X^{\theta,\bar c,t}_r,Q^{\theta,\bar c,t}_r)-f(r,c_r,X_r,Q_r)\right)dr\right]\\ &\quad-\lim_{\theta\to 0}\mathbb{E}^t\left[\frac{\operatorname{sign}(\tau-\tau^{\theta,\bar c,t})}{\theta}\int_{\tau^{\theta,\bar c,t}_{\min}}^{\tau^{\theta,\bar c,t}_{\max}} f(r,\hat c^{\theta,\bar c,t}_r,\hat X^{\theta,\bar c,t}_r,\hat Q^{\theta,\bar c,t}_r)dr\right] = \mathbb{E}^t\left[\xi_\tau\right]-\bar f(t,\bar c,x,q).\end{aligned} \tag{3.7.58}$$

Then, merging (3.7.35) together with (3.7.54) and (3.7.58), we get

$$\begin{aligned}\lim_{\theta\to 0}\frac{v^{c^{\theta,\bar c,t}}(t,x,q)-v^c(t,x,q)}{\theta} &= \mathbb{E}^t\left[\eta_\tau\partial_q g(X_\tau,Q_\tau)-(\bar c-c_t)\partial_q g(X_\tau,Q_\tau)+\xi_\tau\right]\\ &\quad+(\bar c-c_t)\mathbb{E}^t\left[\partial_q g(X_\tau,Q_\tau)\mathbb{1}_{\Lambda(t,\bar c)}\right]\\ &\quad-\left(\bar g(t,\bar c,X_t,Q_t)+\bar f(t,\bar c,X_t,Q_t)\right).\end{aligned} \tag{3.7.59}$$

However, from (3.2.8) and (3.7.2), noting that $c^{\theta,\bar c,t}_t=\bar c$, we have that

$$\begin{aligned}&\mathbb{E}^t\left[\eta_\tau\partial_q g\left(X_\tau,Q_\tau\right)-(\bar c-c_t)\partial_q g(X_\tau,Q_\tau)+\xi_\tau\right] = \mathbb{E}^t\left[\eta_\tau Y^1_\tau-(\bar c-c_t)Y^2_\tau+\xi_\tau\right]\\ &= \mathbb{E}^t\left[Y^1_t\eta_t+\int_t^\tau d\left(Y^1_r\eta_r\right)-(\bar c-c_t)Y^2_t-(\bar c-c_t)\int_t^\tau dY^2_r+\xi_t+\int_t^\tau d\xi_r\right]\\ &= \mathbb{E}^t\Bigg[Y^1_t\eta_t-\int_t^\tau\left(Y^1_r\partial_x\mu(r,c_r,X_r)+Z^1_r\partial_x\sigma(r,X_r)+\partial_x f(r,c_r,X_r,Q_r)\right)\eta_r dr\\ &\qquad+\int_t^\tau Y^1_r\eta_r\partial_x\mu(r,c_r,X_r)\,dr+\int_t^\tau Z^1_r\eta_r\partial_x\sigma(r,X_r)\,dr-(\bar c-c_t)Y^2_t\\ &\qquad+(\bar c-c_t)\int_t^\tau\partial_q f(r,c_r,X_r,Q_r)\,dr+\xi_t-\int_t^\tau(\bar c-c_t)\partial_q f(r,c_r,X_r,Q_r)dr\\ &\qquad+\int_t^\tau\eta_r\partial_x f(r,c_r,X_r,Q_r)dr\Bigg]\\ &= \mathbb{E}^t\left[Y^1_t\eta_t-(\bar c-c_t)Y^2_t+\xi_t\right]\\ &= \mathbb{E}^t\left[Y^1_t\left(\mu(t,c^{\theta,\bar c,t}_t,X_t)-\mu(t,c_t,X_t)\right)-(\bar c-c_t)Y^2_t+f(t,c^{\theta,\bar c,t}_t,X_t,Q_t)-f(t,c_t,X_t,Q_t)\right]\\ &= \mathbb{E}^t\left[Y^1_t\left(\mu(t,\bar c,X_t)-\mu(t,c_t,X_t)\right)-(\bar c-c_t)Y^2_t+f(t,\bar c,X_t,Q_t)-f(t,c_t,X_t,Q_t)\right].\end{aligned} \tag{3.7.60}$$

Moreover, under Assumption 3.2.1, the BSDE (3.2.8) admits an unique solution, as it has been proved in Royer-Carenzi [52, Theorem 2.1]. So, merging (3.7.59) and (3.7.60), we get that

$$\begin{aligned}
&\lim_{\theta\to 0}\frac{v^{c^{\theta,\bar c,t}}(t,x,q)-v^{c}(t,x,q)}{\theta}\\
&\quad=\mathbb{E}^t\left[Y_t^1\left(\mu(t,\bar c,X_t)-\mu(t,c_t,X_t)\right)-(\bar c-c_t)Y_t^2+f(t,\bar c,X_t,Q_t)-f(t,c_t,X_t,Q_t)\right]\\
&\qquad+(\bar c-c_t)\mathbb{E}^t\left[\partial_q g(X_\tau,Q_\tau)\mathbb{1}_{\Lambda(t,\bar c)}\right]-\bar g(t,\bar c,X_t,Q_t)-\bar f(t,\bar c,X_t,Q_t).
\end{aligned}\tag{3.7.61}$$

Therefore, merging (3.7.61) together with (3.7.34), we get that for any $\bar c\geq 0$ and for any $t\in[0,\tau)$

$$\begin{aligned}
&\mathbb{E}^t\left[Y_t^1\left(\mu(t,\bar c,X_t)-\mu(t,c_t,X_t)\right)-(\bar c-c_t)Y_t^2+f(t,\bar c,X_t,Q_t)-f(t,c_t,X_t,Q_t)\right]\\
&\qquad+(\bar c-c_t)\mathbb{E}^t\left[\partial_q g(X_\tau,Q_\tau)\mathbb{1}_{\Lambda(t,\bar c)}\right]-\bar g(t,\bar c,X_t,Q_t)-\bar f(t,\bar c,X_t,Q_t)\leq 0.
\end{aligned}$$

Since the argument of the first conditional expectation is $\mathcal{F}^t$-measurable, we have that for any $\bar c\geq 0$ and for any $t\in[0,\tau)$ a.s.

$$\begin{aligned}
0&\geq Y_t^1\left(\mu(t,\bar c,X_t)-\mu(t,c_t,X_t)\right)-(\bar c-c_t)Y_t^2+f(t,\bar c,X_t,Q_t)-f(t,c_t,X_t,Q_t)\\
&\quad+(\bar c-c_t)\mathbb{E}^t\left[\partial_q g(X_\tau,Q_\tau)\mathbb{1}_{\Lambda(t,\bar c)}\right]-\bar g(t,\bar c,X_t,Q_t)-\bar f(t,\bar c,X_t,Q_t)\\
&=Y_t^1\mu(t,\bar c,X_t)-\bar cY_t^2+f(t,\bar c,X_t,Q_t)-\left(Y_t^1\mu(t,c_t,X_t)-c_tY_t^2+f(t,c_t,X_t,Q_t)\right)\\
&\quad+(\bar c-c_t)\mathbb{E}^t\left[\partial_q g(X_\tau,Q_\tau)\mathbb{1}_{\Lambda(t,\bar c)}\right]-\bar g(t,\bar c,X_t,Q_t)-\bar f(t,\bar c,X_t,Q_t)\\
&=\mathcal{H}(t,\bar c,X_t,Q_t,\mathbf{Y}_t)-\mathcal{H}(t,c_t,X_t,Q_t,\mathbf{Y}_t)+\mathcal{G}(t,\bar c,c_t,X_t,Q_t),
\end{aligned}$$

which concludes the proof of Theorem 3.2.2.

Chapter 4

Optimal Liquidation in a Mean-reverting Portfolio

4.1 Introduction

As we mentioned in the Preface of this thesis, the optimal liquidation problem has been extensively studied in the literature, mainly driven from investment banks offering execution of large trades as a standard service. All the literature we have inspected on the optimal liquidation strategy is based only on the stock that the agent needs to liquidate. However, there may be additional information available in the market, such as the price of a correlated stock, which could be helpful to better predict the stock price movements. A model based on both asset prices may generate a more reliable adaptive liquidation strategy, which not only relies on the price of the liquidating stock, but also on that of the correlated stock.

In this chapter we analyze the case when an agent trades on a market with two financial assets whose difference of log-prices has a mean-reverting behavior. The agent's task is to liquidate the initial position of shares of one stock, without the possibility of trading the other stock. This technique is often employed when modeling a pair of stocks in pair trading in which the agent tries to make money out of a couple of correlated stocks by selling one stock and buying the other, to take advantage of the mean-reverting behavior of the co-integration factor between the two stocks. In our setting the agent can only sell stock, but cannot trade the other stock. Moreover, we define the difference of the log-prices to be an Ornstein-Uhlenbeck process which is the continuous-time analogue of the discrete-time AR(1) process and makes its parametrization an easy task, see Cartea et al. [18, Section 3.7] and Brockwell and Davis [16, Chapter 3] for further details on parametrization of such processes.

We would like to confide that the results presented in this chapter are the chronologically first topics we researched on during our PhD. Initially, our first goal was to find a solution to the optimal liquidation problem that we are presenting in this chapter. In Section 4.3.2 we try to apply the standard version of the SMP to the example presented in this chapter, which is a stopping terminal time optimization problem. Our initial thought was that the usual formulation of the SMP could have been applied also to the stopping terminal time case, by simply introducing the stopping time into the FBSDE associated to the problem, without changing the Hamiltonian structure. This led us to understand if this was really the case and gave birth to the research topic we addressed in previous two chapters.

The main contributions of this chapter are that we prove the value function is the unique continuous viscosity solution to the HJB equation which is complicated with three state variables, that we find an approximation of the classical solution under some mild conditions, which opens the way of finding the optimal value and strategy with the Monte-Carlo simulation, and that we show the value function and the optimal liquidation rate depend only on observable data which allows a straightforward calculation at each moment in time. Although the approximation of the classical solution to the HJB equation is proved to be not coincident with the value function, numerical tests show that it is close to the value function, by proving that it is close to the approximated solution of the FBSDE associated to the optimization problem.

The rest of the chapter is structured as follows. Section 4.2 describes the settings of the problem, defines the value function, writes the HJB equation and states the main theorem (Theorem 4.2.2) that the value function is the unique continuous viscosity solution to the HJB equation. Section 4.3 introduces the approximations that are going to be discussed in the following subsections. Subsection 4.3.1 finds an approximation of the solution of the HJB equation under some mild conditions on the model parameters and derives the objective function as a sum of classical solutions to three different parabolic PDEs which can be solved one by one. Subsection 4.3.2 finds value function and optimal trading speed as solution to an FBSDE obtained by applying stochastic maximum principle to our problem. Section 4.4 is the numerical section and it is divided in two parts. Subsection 4.4.1 compares the closed form solution obtained in section 4.3.1 with the solution of the FBSDE in section 4.3.2, which is approximated using a deep learning algorithm. Subsection 4.4.3 provides some numerical tests to assess our model and compares its performance with that of two other strategies based on two geometric Brownian motion approximations of the liquidating stock price. Section 4.5 concludes. Appendix 4.6 contains the proofs of Theorem 4.2.2, Propositions 4.3.1 and 4.3.6.

4.2 Model

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ be a filtered probability space, where $(\mathcal{F}_t)_{t\in[0,T]}$ is the natural filtration generated by two independent standard Brownian motions W^1 and W^2, augmented by all $\mathbb{P}$-null sets. Let T be the fixed terminal time, $(A_r)_{r\in[0,T]}$ the price of a stock in the market, satisfying the following geometric Brownian motion (GBM):

$$dA_r = \mu_1 A_r dr + \sigma_1 A_r dW_r^1, \quad A_0 = a, \tag{4.2.1}$$

where μ_1, σ_1 are constants, μ_1 is the growth rate, $\sigma_1 \geq 0$ the volatility rate, $(S_r)_{r\in[0,T]}$ the price of the stock that the agent aims to liquidate, $(\varepsilon_r)_{r\in[0,T]}$ the co-integration factor between stocks S_r and A_r, defined by $\varepsilon_r = \ln\left(\frac{S_r}{A_r}\right)$, and follows an Ornstein-Uhlenbeck (OU) process

$$d\varepsilon_r = -k\varepsilon_r dr + \sigma_2\left(\rho dW_r^1 + \sqrt{1-\rho^2}dW_r^2\right), \quad \varepsilon_0 = \epsilon, \tag{4.2.2}$$

where k, σ_2 are constants, $\rho \in [-1, 1]$ the correlation coefficient, k the mean reversion speed, σ_2 the volatility. The co-integration factor ε_r behaves as a mean-reverting process, which implies a period of time in which the process S_r outperforms (or underperform) A_r is followed by a moment in which the two stocks have similar prices.

Let $(\pi_r)_{r\in[0,T]}$ denote the rate of selling the stock, which is a decision (control) variable decided by the agent and is said admissible if it is a progressively measurable, right continuous, non-negative and square integrable process. Denote by $\mathcal{A}$ the set of all admissible control processes.

Let $(Q_r)_{r\in[0,T]}$ denote the inventory left at time r and $q_0 > 0$ the initial amount of stock S owned by the agent. The process Q_r depends on the trading strategy π and follows the equation:

$$dQ_r = -\pi_r dr, \quad Q_0 = q_0. \tag{4.2.3}$$

Let $(M_r)_{r\in[0,T]}$ denote the wealth process, satisfying the following equation:

$$dM_r = \pi_r(S_r - \eta\pi_r)dr, \quad M_0 = 0,$$

where $\eta \geq 0$ is the temporary price impact factor, which is the same as that in Cartea et al. [18, Section 6.1]. The term $\pi_r S_r$ is the dollar amount of the stock liquidated at time r, while the term $\eta\pi_r^2$ is a penalty for selling the stock too quickly. Indeed, as analysed in Cartea et al. [18, Section 6.1], if many market orders are executed simultaneously, usually the limit order book is climbed and the orders are executed at a lower price than the spot price S_r. The speed of trading with market orders has been modelled in different ways in the literature and we chose to use the quadratic impact on the price of the stock as in Cartea et al. [18, Section 6.1].

Denote by $\mathbf{x}$ the vector of two state variables (a, ϵ) and $\mathcal{O}$ the state space, given by $\mathcal{O} := (0, \infty) \times \mathbb{R} \times [0, \bar{Q}_0)$ with $q_0 < \bar{Q}_0 < \infty$. Moreover, denote the initial price of the stock S by $s := ae^{\epsilon}$. We group the two state processes (A, ε) into a vector $\mathbf{X}$. Let $t \in [0, T]$, we define the 2-dimensional stochastic process $(\mathbf{X}_r)_{r\in[t,T]} := (A_r, \varepsilon_r)_{r\in[t,T]}$ as the solution to the following SDE

$$d\mathbf{X}_r = \mu(\mathbf{X}_r)dt + \sigma(\mathbf{X}_r)d\mathbf{W}_r, \tag{4.2.4}$$

where

$$\mu(c, \mathbf{x}) = \begin{pmatrix} \mu_1 \\ -k\epsilon \end{pmatrix}, \quad \sigma(\mathbf{x}) = \begin{pmatrix} \sigma_1 a & 0 \\ \sigma_2 \rho & \sigma_2\sqrt{1-\rho^2} \end{pmatrix}. \tag{4.2.5}$$

The optimal liquidation problem is defined by:

$$\sup_{\pi\in\mathcal{A}} \mathbb{E}_t\left[\int_0^\tau \pi_r(S_r - \eta\pi_r)dr + Q_\tau\left(S_\tau - \chi Q_\tau\right) - \phi_1\int_0^\tau Q_r^2\, dr - \phi_2\int_0^\tau S_r Q_r\, dr - \phi_3\int_0^\tau A_r Q_r\, dr\right], \tag{4.2.6}$$

where τ is a stopping time defined by $\tau = T \wedge \min\{r \geq 0 \mid Q_r = 0\}$, the first time when all stock is liquidated before terminal time T or T otherwise. The first term inside expectation is the wealth value at τ, the second the terminal liquidation value and the last three the running inventory penalties. The terminal liquidation value is the cash from liquidating all the inventory left at terminal time T at a price S_T penalized by a quantity proportional to the amount of remaining stocks. Inventory penalties are not financial costs, but incorporate the agent's urgency for executing the trade. Denote by $\mathbb{E}_t[\cdot] = \mathbb{E}[\cdot|A_t = a,\ \varepsilon_t = \epsilon,\ Q_t = q]$, the conditional expectation operator at time $t \in [0, T]$.

The value function of problem (4.2.6) is defined by

$$v(t, a, \epsilon, q) = \sup_{\pi\in\mathcal{A}} v^\pi(t, a, \epsilon, q), \tag{4.2.7}$$

where

$$v^\pi(t, a, \epsilon, q) = \mathbb{E}_t\left[\int_t^\tau \pi_r(S_r - \eta\pi_r) + Q_\tau\left(S_\tau - \chi Q_\tau\right) - \phi_1\int_t^\tau Q_r^2\, dr - \phi_2\int_t^\tau S_r Q_r\, dr - \phi_3\int_t^\tau A_r Q_r\, dr\right], \tag{4.2.8}$$

where τ is defined by $\tau = T \wedge \min\{r \geq t \mid Q_r = 0\}$.

To solve the control problem (4.2.7), we adopt the dynamic programming principle and derive the following HJB equation for the value function:

$$\frac{\partial w}{\partial t} + \mathcal{L}w + \sup_{\pi\geq 0}\left[-\pi\frac{\partial w}{\partial q} + ae^{\epsilon}\pi - \eta\pi^2\right] - \phi_1 q^2 - \phi_2 qae^{\epsilon} - \phi_3 qa = 0 \tag{4.2.9}$$

on $[0,T) \times \mathcal{O}$, with terminal condition $w(T,a,\epsilon,q) = q\,(ae^{\epsilon} - \chi q)$ and boundary condition $w(t,a,\epsilon,0) = 0$, where $\mathcal{L}$ is the operator defined by

$$\mathcal{L}w = \frac{\sigma_1^2}{2}a^2\frac{\partial^2 w}{\partial a^2} + \rho\sigma_1\sigma_2 a\frac{\partial^2 w}{\partial a \partial \epsilon} + \frac{\sigma_2^2}{2}\frac{\partial^2 w}{\partial \epsilon^2} + \mu_1 a\frac{\partial w}{\partial a} - k\epsilon\frac{\partial w}{\partial \epsilon}.$$

Theorem 4.2.1 (Verification Theorem)**.** *Let w be a function in $C^{1,2}([0,T)\times\mathcal{O})\cap C^0([0,T]\times\bar{\mathcal{O}})$ and satisfy the following growth condition*

$$|w(t,\mathbf{x},q)| \le C(1+q^2)(1+a^{p_1})(1+e^{p_2\epsilon}) \qquad \forall (t,\mathbf{x},q) \in [0,T]\times\mathcal{O}$$

for fixed $p_1, p_2, C > 0$. Assume there exists a measurable function $c(t,\mathbf{x},q)$ such that

$$\begin{aligned}
&\frac{\partial w}{\partial t} + \mathcal{L}w + \sup_{\pi\ge 0}\left[-\pi\frac{\partial w}{\partial q} + ae^{\epsilon}\pi - \eta\pi^2\right] - \phi_1 q^2 - \phi_2 qae^{\epsilon} - \phi_3 qa \\
&= \frac{\partial w}{\partial t} + \mathcal{L}w - c\frac{\partial w}{\partial q} + ae^{\epsilon}c - \eta c^2 - \phi_1 q^2 - \phi_2 qae^{\epsilon} - \phi_3 qa = 0
\end{aligned}$$

with terminal condition $w(T,a,\epsilon,q) = q\,(ae^{\epsilon} - \chi q)$ and boundary condition $w(t,a,\epsilon,0) = 0$. Let the SDE

$$d\mathbf{X}_r = \mu(c_r, \mathbf{X}_r)dt + \sigma(\mathbf{X}_r)d\mathbf{W}_r \tag{4.2.10}$$

admit a unique solution, given an initial condition $\mathbf{X}_t = \mathbf{x}$, where μ and σ are defined in (4.2.5)*. Let $(c_r)_{r\in[t,T]} \in \mathcal{A}$. Then w coincides with the value function v.*

Equation (4.2.9) is a nonlinear PDE with three state variables a, ϵ and q. We show the value function is a viscosity solution of (4.2.9), see Pham [48] for its definition and properties.

Theorem 4.2.2. *The value function v defined in* (4.2.7) *is the unique viscosity solution of the HJB equation* (4.2.9)*.*

If we strengthen the condition on the control set, we have continuity of the value function. Let $(t,\mathbf{x},q) \in [0,T]\times\mathcal{O}$ be fixed and let $\gamma, N > 0$. Then, we define the set $\tilde{\mathcal{A}}_{\gamma,N}(t,\mathbf{x},q)$ as

$$\tilde{\mathcal{A}}_{\gamma,N}(t,\mathbf{x},q) = \left\{\pi \in \mathcal{A}(t,\mathbf{x},q) \,\middle|\, \left(\mathbb{E}_t\left[\int_t^T \pi_r^{2+\gamma}\,dr\right]\right)^{\frac{1}{2+\gamma}} \le N(1+a)\left(1+e^{N\epsilon}\right)\right\}. \tag{4.2.11}$$

Proposition 4.2.3. *Let the set of admissible controls be reduced to $\tilde{\mathcal{A}}_{\gamma,N}$ for fixed $\gamma, N > 0$. Then the value function v, defined in* (4.2.7)*, is continuous on $[0,T]\times\mathcal{O}$.*

4.3 Approximation to find value function

It is in general difficult to find a classical solution of equation (4.2.9). In this chapter we are going to present 3 different approaches to find an approximation of the value function v and the optimal control c. In Section 4.4 we are going to compare the 3 approximated solutions among each other. The first method consists in finding an approximation to HJB equation (4.2.9). The second approach consists in solving the FBSDE associated to the maximisation problem with a Neural Network approximation. In the third approach we approximate the process S_r to a GBM.

4.3.1 Approximated solution to HJB equation (4.2.9)

In this section we approximate the terminal condition of the HJB equation (4.2.9) and we show that, under some mild conditions, we find a classical solution to the approximated HJB equation. In particular, we are going to remove the terminal condition $w(t, a, \epsilon, 0) = 0$ and we are going to look for a solution to the following HJB equation

$$\begin{cases} \frac{\partial w}{\partial t} + \mathcal{L}w + \sup_{\pi \geq 0} \left[-\pi \frac{\partial w}{\partial q} + ae^{\epsilon}\pi - \eta\pi^2 \right] - \phi_1 q^2 - \phi_2 qae^{\epsilon} - \phi_3 qa = 0 \quad \text{on } [0,T) \times \mathcal{O} \\ w(T, a, \epsilon, q) = q\left(ae^{\epsilon} - \chi q\right). \end{cases} \tag{4.3.1}$$

From equation (4.3.1) we get the optimal rate of trading as

$$c(t, a, \epsilon, q) = \frac{1}{2\eta} \max\left\{ ae^{\epsilon} - \frac{\partial w}{\partial q},\ 0 \right\}. \tag{4.3.2}$$

Substituting c in equation (4.3.1), we have

$$\frac{\partial w}{\partial t} + \mathcal{L}w - \phi_1 q^2 - \phi_2 qae^{\epsilon} - \phi_3 qa + \frac{1}{4\eta} \left(\max\left\{ ae^{\epsilon} - \frac{\partial w}{\partial q},\ 0 \right\} \right)^2 = 0. \tag{4.3.3}$$

The PDE (4.3.3) is nonlinear and difficult to solve. To simplify it we try to eliminate the last term containing the max operator. In the following, we assume that the function w satisfies $ae^{\epsilon} - \frac{\partial w}{\partial q} \geq 0$ for any $(t, \mathbf{x}, q) \in [0,T] \times \mathcal{O}$. Under this assumption, (4.3.3) reduces to

$$\frac{\partial w}{\partial t} + \mathcal{L}w - \phi_1 q^2 - \phi_2 qae^{\epsilon} - \phi_3 qa + \frac{1}{4\eta} \left(ae^{\epsilon} - \frac{\partial w}{\partial q} \right)^2 = 0. \tag{4.3.4}$$

By inspecting the terminal conditions, we postulate a solution of the following form:

$$w(t, a, \epsilon, q) = g_1(t, \epsilon, a) + qae^{\epsilon} g_2(t, \epsilon) + q^2 g_3(t). \tag{4.3.5}$$

Substituting (4.3.5) to equation (4.3.4), collecting terms with coefficients 1, $ae^{\epsilon}q$ and q^2, and setting each term equal to 0, we derive the following system of PDEs on $[0,T] \times (\mathcal{O} \cap \{q > 0\})$:

$$\begin{cases} 0 &= \frac{\partial g_1}{\partial t} + \mathcal{L}g_1 + \frac{1}{4\eta} a^2 e^{2\epsilon} \left(1 - g_2\right)^2, \\ 0 &= \frac{\partial g_2}{\partial t} + \frac{\sigma_2^2}{2} \frac{\partial^2 g_2}{\partial \epsilon^2} + \left(\sigma_2^2 + \rho\sigma_1\sigma_2 - k\epsilon\right) \frac{\partial g_2}{\partial \epsilon} + \left(\frac{\sigma_2^2}{2} + \rho\sigma_1\sigma_2 + \mu_1 - k\epsilon + \frac{g_3}{\eta} \right) g_2 - \phi_2 - \phi_3 e^{-\epsilon} - \frac{g_3}{\eta}, \\ 0 &= g_3' - \phi_1 + \frac{g_3^2}{\eta}, \end{cases} \tag{4.3.6}$$

with terminal conditions $g_1(T,a,\epsilon)=0,\ g_2(T,\epsilon)=1,\ g_3(T)=-\chi$.

The last equation in (4.3.6) is a Riccati type equation and has a closed form solution given by

$$g_3(t)=\sqrt{\phi_1\eta}\frac{e^{2t\sqrt{\frac{\phi_1}{\eta}}}(\sqrt{\phi_1\eta}-\chi)-e^{2T\sqrt{\frac{\phi_1}{\eta}}}(\sqrt{\phi_1\eta}+\chi)}{e^{2t\sqrt{\frac{\phi_1}{\eta}}}(\sqrt{\phi_1\eta}-\chi)+e^{2T\sqrt{\frac{\phi_1}{\eta}}}(\sqrt{\phi_1\eta}+\chi)},\qquad \forall t\in[0,T]. \tag{4.3.7}$$

It is easy to verify that g_3 is a negative and increasing function.

Recall that function w must satisfy $ae^{\epsilon}-\frac{\partial w}{\partial q}\geq 0$ for any $(t,\mathbf{x},q)\in[0,T]\times\mathcal{O}$, which is equivalent to the following:

$$ae^{\epsilon}\left(1-g_2(t,\epsilon)\right)-2qg_3(t)\geq 0,\quad \forall(t,a,\epsilon,q)\in[0,T]\times\mathcal{O}. \tag{4.3.8}$$

Since a is positive and g_3 is negative, condition (4.3.8) holds if

$$g_2(t,\epsilon)\leq 1,\quad \forall(t,\epsilon)\in[0,T]\times\mathbb{R}. \tag{4.3.9}$$

Proposition 4.3.1. *Assume the model parameters satisfy the following condition:*

$$\phi_3 e^{1+\frac{\phi_2}{k}}\geq ke^{\frac{\sigma_2^2}{2k}+\frac{\mu_1}{k}+\frac{\rho}{k}\sigma_1\sigma_2}. \tag{4.3.10}$$

Then solution g_2 in (4.3.6) *satisfies condition* (4.3.9) *and is given by*

$$\begin{aligned}g_2(t,\epsilon)=1-\phi_3e^{-\epsilon}\int_t^T\hat{g}(r;t)\,dr-e^{-\epsilon}\int_t^T\hat{g}(r;t)e^{\bar{\mu}(r-t)\epsilon+\frac{\bar{\sigma}(r-t)^2}{2}+\frac{\rho}{k}\sigma_1\sigma_2(1-\bar{\mu}(r-t))}\cdot\\ \cdot\left(k\bar{\mu}(r-t)\epsilon+k\bar{\sigma}(r-t)^2-\frac{\sigma_2^2}{2}-\rho\sigma_1\sigma_2\bar{\mu}(r-t)-\mu_1+\phi_2\right)\,dr\end{aligned} \tag{4.3.11}$$

for $(t,\epsilon)\in[0,T]\times\mathbb{R}$, where $\bar{\mu}$, $\bar{\sigma}$ and $\hat{g}$ are functions defined by

$$\bar{\mu}(s)=e^{-ks},\quad \bar{\sigma}(s)^2=\frac{\sigma_2^2}{2k}\left(1-e^{-2ks}\right),\quad \hat{g}(r;t)=\exp\left(\frac{1}{\eta}\int_t^r g_3(s)\,ds+\mu_1(r-t)\right). \tag{4.3.12}$$

Moreover, the optimal control is given by

$$c(t,a,\epsilon,q)=\frac{1}{2\eta}\left[ae^{\epsilon}\left(1-g_2(t,\epsilon)\right)-2qg_3(t)\right]. \tag{4.3.13}$$

Note that for any fixed parameters $k,\sigma_1,\sigma_2,\mu_1,\rho$, one can always choose ϕ_2 and ϕ_3 such that (4.3.10) is satisfied, and that g_1 in (4.3.6) can be written, with the help of the Feynman-Kac formula, as

$$g_1(t,a,\epsilon)=\frac{1}{4\eta}\mathbb{E}_t\left[\int_t^T S_r^2\left(1-g_2(r,\varepsilon_r)\right)^2\,dr\right]. \tag{4.3.14}$$

Combining (4.3.14) with Proposition 4.3.1, we have the following result:

Theorem 4.3.2. *Assume condition* (4.3.10) *is satisfied. Then equations in* (4.3.6) *admit classical solutions g_1, g_2 and g_3 given by* (4.3.14), (4.3.11) *and* (4.3.7) *respectively.*

Proposition 4.3.3. *Let c be defined as in* (4.3.13)*, let γ be any positive real number, let $N > 0$ be big enough and let $(t, \mathbf{x}, q) \in [0, T] \times \mathcal{O}$. Then, $c_r \in \tilde{\mathcal{A}}_{\gamma,N}(t, \mathbf{x}, q)$, as defined in* (4.2.11).

Remark 4.3.4. *Let c be defined as in* (4.3.13)*. By the fact that g_2 is locally Lipschitz, we get that c, drift and diffusion coefficients of SDE* (4.2.10) *are locally Lipschitz. Existence and uniqueness of solution to equation* (4.2.10) *follow by applying Karatzas and Shreve [39, Theorem 2.5].*

Remark 4.3.5. *Let condition* (4.3.10) *be satisfied and g_1, g_2 and g_3 be the classical solutions to the equations in* (4.3.6) *as in Theorem 4.3.2. As proved in Remark 4.3.4 and Proposition 4.3.3, all conditions of verification Thorem 4.2.1 are satisfied except for continuity of w. Indeed, function w is not continuous for $q \to 0$, unless $g_1 \equiv 0$ on $[0, T] \times \mathcal{O}$. Hence, w does not necessarily coincide with the value function v in* (4.2.7)*, unless it is proved to be continuous on $q = 0$.*

As proved in Proposition 4.3.3, the optimal control c lies in the more restrictive control set $\tilde{\mathcal{A}}_{\gamma,N}$ defined in (4.2.11). Proposition 4.2.3 ensures that if the control set is reduced to $\tilde{\mathcal{A}}_{\gamma,N}$, then the value function v is continuous. We conclude that if g_1 is not identically equal to 0, then the solution w to the HJB equation does not coincide with the value function.

If we reduce our model to a one stock model as that in Cartea et al. [18] with the same parameters, i.e., $\epsilon_0 = 0, \sigma_2 = 0,\ \mu_1 = 0,\ \rho = 0,\ k = 0,\ \phi_2 = \phi_3 = 0$, then condition (4.3.10) is satisfied. Using (4.3.11) and (4.3.14), we get $g_2(t, \epsilon) = 1$ and $g_1(t, a, \epsilon) = 0$ for any $(t, a, \epsilon) \in [0, T] \times (0, \infty) \times \mathbb{R}$, which makes w in (4.3.5) continuous in the whole domain. We can apply Theorem 4.2.1 to verify that w coincides with the value function v.

Proposition 4.3.6. *Assume condition* (4.3.10) *is satisfied and g_1, g_2 and g_3 are the classical solutions to the equations* (4.3.6)*. Then, function*

$$w(t, a, \epsilon, q) = qae^{\epsilon} g_2(t, \epsilon) + q^2 g_3(t).$$

coincides with the value function v in (4.2.7) *on $[0, T] \times \mathcal{O}$.*

If condition (4.3.10) is not satisfied, then it is not clear if HJB equation (4.3.1) admits a classical solution, however, Theorem 4.2.2 states that the value function v is the unique viscosity solution to the HJB equation (4.3.1).

Remark 4.3.7. *The model can be extended to cover limit orders as well by introducing a premium for executing limit orders instead of market orders (see Cartea et al. [18]) and a new state variable D_t as a measure of uncertainty in filling limit orders. We can prove all theorems in* *this more general setup and, under similar conditions to that in* (4.3.10)*, prove the existence of a classical solution to the HJB equation that coincides with the value function. The model can also be extended to the multi-dimensional case, in which an agent aims to liquidate m different stocks $S^1, \ldots, S^m$ on a basket of n correlated stocks $A^1, \ldots, A^n$.*

4.3.2 FBSDE approach

In this section we approach the control problem (4.2.6), by using the stochastic maximum principle (c.f. Pham [48, Theorem 6.4.6] and Li and Zheng [43]). It is a standard approach to write the value function v defined in (4.2.7) as a solution to an FBSDE and to find the optimal control from the maximization of the Hamiltonian associated to the optimization problem. As control problem (4.2.6) is a stopping terminal time problem, we should use the Stochastic Maximum Principle as in Chapter 2. However, the terms $\bar{f}$ and $\bar{g}$ in Chapter 2 are too complicated to be evaluated in the case of the model introduced in the current Chapter. The main blocker is that, to evaluate $\bar{f}$ and $\bar{g}$, it is required to know the optimal stopping time τ of the optimal strategy c. Since we are not able to calculate precisely the value of τ, we are not able to apply the theory introduced in Chapter 2. Therefore, to apply stochastic maximum principle, we approximate problem (4.2.6), by replacing the stochastic terminal time with a fixed terminal time T. Replacing the stopping time τ with a fixed terminal time T makes possible for the inventory Q to assume also negative values. However, the inventory penalty $\phi_1 \int_t^T Q_r^2 \, dr$ pushes the terminal inventory to be close to 0 and not too negative. We rewrite the value function as

$$
\begin{aligned}
v(t, a, \epsilon, q) = \sup_{\pi \in \mathcal{A}} \mathbb{E}_t \bigg[& Q_T \left(S_T - \chi Q_T \right) + \int_t^T \pi_r (S_r - \eta \pi_r) \, dr \\
& - \phi_1 \int_t^T Q_r^2 \, dr - \phi_2 \int_t^T S_r Q_r \, dr - \phi_3 \int_t^T A_r Q_r \, dr \bigg].
\end{aligned}
\tag{4.3.15}
$$

From SDE (4.2.4), we get that the process $\mathbf{X}_t$ is independent of the control c. We define the Hamiltonian $\mathcal{H} : [0, \infty) \times \mathcal{O} \times \mathbb{R} \to \mathbb{R}$ as

$$
\mathcal{H}(\pi, \mathbf{x}, q, y) = -\pi y + \pi (a e^{\epsilon} - \eta \pi) - \phi_1 q^2 - \phi_2 a e^{\epsilon} q - \phi_3 a q.
$$

The BSDE associated to our problem is

$$
dY_r = -\mathcal{G}_2(\mathbf{X}_r, Q_r) dr + \mathbf{Z}_r d\boldsymbol{W}_r,
$$

with terminal condition $Y_T = \mathcal{S}(\mathbf{X}_T, Q_T)$, where

$$
\mathcal{G}_2(\mathbf{x}, q) = \frac{\partial}{\partial q} \mathcal{H}(\mathbf{x}, q) = -2\phi_1 q - \phi_2 a e^{\epsilon} - \phi_3 a, \qquad \mathcal{S}(\mathbf{x}, q) = a e^{\epsilon} - 2\chi q.
$$

Using the stochastic maximum principle approach, it follows that the optimal trading strategy c coincides with the control function c that maximizes the Hamiltonian $\mathcal{H}$, which is

$$c(\mathbf{x}, y) = \frac{1}{2\eta}\left(ae^{\epsilon} - y\right)^{+}. \tag{4.3.16}$$

In the following we prove that the solution to the above BSDE can be used to find the optimal strategy of the optimization problem (4.3.15). The proof of the theorem below immediately follows from Pham [48, Theorem 6.4.6] using concavity of Hamiltonian $\mathcal{H}$ with respect to variables $(\mathbf{x}, \pi)$ and maximality of c in (4.3.16) for the Hamiltonian $\mathcal{H}$.

Theorem 4.3.8 (Stochastic maximum principle)**.** *Suppose that the FBSDE*

$$\begin{cases} dA_r = \mu_1 A_r dr + \sigma_1 A_r dW_r^1 \\ d\varepsilon_r = -k\varepsilon_r dr + \sigma_2\left(\rho dW_r^1 + \sqrt{1-\rho^2}dW_r^2\right) \\ dQ_r = -\frac{1}{2\eta}(A_r e^{\varepsilon_r} - Y_r)^{+} dr \\ dY_r = -\mathcal{G}_2(A_r, \varepsilon_r, Q_r)dr + \mathbf{Z}_r \cdot d\mathbf{W}_r \\ A_t = a \\ \varepsilon_t = \epsilon \\ Q_t = q \\ Y_T = \mathcal{S}(A_T, \varepsilon_T, Q_T) \end{cases}. \tag{4.3.17}$$

admits a solution $(Q_t, Y_t, \mathbf{Z}_t)_{t\in[0,T]}$ *and that* $(Y_t)_{t\in[0,T]}$ *is a progressively measurable, non-negative and square integrable process. Then* c*, defined in* (4.3.16)*, is the optimal control of problem* (4.3.15)*.*

In the following section we focus on finding a solution to FBSDE (4.3.17), which is a coupled non-linear Forward-Backward SDE and, to our knowledge, cannot be explicitly solved. In the numerical section below, we use a deep learning-based method, following the one presented in E et al. [29], to find an approximated solution to FBSDE (4.3.17) and we show that the closed form control in (4.3.13) is close to the approximated version of the optimal control in (4.3.16).

4.3.3 Single stock models

In this section we compare our model based on both processes A_t and ε_t with two simplified models based only on one stock price: one is to approximate the stock price S with a GBM $\tilde{S}$, whose first two moments are equal to those of S, and the other is to set the co-integration factor ε to 0, whose effect is to approximate the stock price S with that of A. Although in both cases the stock price S is approximated with a GBM, the first one is more accurate as it uses the information of the co-integration factor ε. To get the optimal strategy related to approximation $\tilde{S}$, we compare the stock price S_r with a GBM $\tilde{S}_r$ satisfying the following stochastic differential equation (SDE):

$$d\tilde{S}_r = \tilde{\mu}(r)\tilde{S}_r dr + \tilde{\sigma}(r)\tilde{S}_r dW_r, \quad \tilde{S}_0 = s,$$

where $\tilde{\mu}$ and $\tilde{\sigma}$ are deterministic functions that ensure the first two moments of S_r and $\tilde{S}_r$ are the same for $0 \le r \le T$, seen at time 0, which leads to (see (4.6.2) and (4.6.3) with $t = 0$)

$$\begin{aligned} ae^{\epsilon e^{-kr} + \frac{\sigma_2^2}{4k}\left(1-e^{-2kr}\right) + \frac{\rho\sigma_1\sigma_2}{k}\left(1-e^{-kr}\right)} &= se^{\int_0^r \tilde{\mu}(s)\, ds}, \\ e^{\sigma_1^2 r + \frac{\sigma_2^2}{2k}\left(1-e^{-2kr}\right) + \frac{2\rho\sigma_1\sigma_2}{k}\left(1-e^{-kr}\right)} &= e^{\int_0^r \tilde{\sigma}(s)^2\, ds}. \end{aligned}$$

Since $S_r = A_r e^{\varepsilon_r}$ and $\tilde{S}_r$ are log-normal variables, simple calculus gives

$$\begin{cases} \tilde{\mu}(r) = -k\epsilon_0 e^{-kr} + \mu_1 + \frac{\sigma_2^2}{2} e^{-2kr} + \rho\sigma_1\sigma_2 e^{-kr} \\ \tilde{\sigma}(r)^2 = \sigma_1^2 + \sigma_2^2 e^{-2kr} + 2\rho\sigma_1\sigma_2 e^{-kr} \end{cases}.$$

Remark 4.3.9. *Note that the initial value of the co-integration factor ϵ_0 appears in $\tilde{\mu}$, which ensures the two processes S and $\tilde{S}$, seen at time 0, are the same in distribution. They are different, seen at later time $t > 0$, as S is determined by two Brownian motions but $\tilde{S}$ by one only. In our numerical test, we approximate the price S_r with the GBM $\tilde{S}_r$ by fixing the co-integration factor to its initial values ϵ_0 throughout the whole trading period $[0,T]$.*

We solve the stochastic control problem with the same objective function as the one in (4.2.6) without the last term and with $\tilde{S}_r$ instead of S_r. The HJB equation is given by

$$\frac{\partial w}{\partial t} + \sup_{\tilde{\pi}\ge 0}\left[\frac{\tilde{\sigma}(t)^2}{2}\tilde{s}^2\frac{\partial^2 w}{\partial \tilde{s}^2} + \tilde{\mu}(t)\tilde{s}\frac{\partial w}{\partial \tilde{s}} - \tilde{\pi}\frac{\partial w}{\partial q} + (\tilde{s} - \eta\tilde{\pi})\tilde{\pi}\right] - \phi_1 q^2 - 2\phi_2 q\tilde{s} = 0 \qquad (4.3.18)$$

on $[0,T)\times(0,\infty)\times[0,q_0]$, with terminal condition $w(T,\tilde{s},q) = q\left(\tilde{s} - \chi q\right)$ and boundary condition $w(t,\tilde{s},0) = 0$. The optimal trading strategy $\tilde{c}$ has the following form

$$\tilde{c} = \frac{1}{2\eta}\max\left\{\tilde{s} - \frac{\partial w}{\partial q}, 0\right\}.$$

Moreover, equation (4.3.18) can be solved using a method similar to the one used in Section 3. Since the solution $w(t,\tilde{s})$ does not depend on ϵ, the equation is easier to be solved.

The second approximation is to use only the price A, the optimal trading strategy c^A has the same formula as that in (4.3.2) with ϵ equal to 0.

4.4 Numerical Tests

This section is divided in three parts. The first subsection shows that the closed form control (4.3.13) deriving from the HJB equation and the neural network (NN) approximated control (4.3.16) deriving from the FBSDE approach are close to each other. The second subsection shows that the function g_1 in (4.3.14) is close to 0 for q close to 0, making the solution w in Proposition 4.3.6 close to the value function in (4.2.7). In the last subsection we compare the performance of the closed form control (4.3.13) based on A_t and ε_t with respect to the optimal strategy based on two simplified models based on geometric Brownian motion approximations of the liquidating stock price.

4.4.1 Neural network approximation vs. closed form control

In this subsection we compare the control obtained through the NN approximated solution of the FBSDE with the closed form control in (4.3.13). To numerically find the solution of the FBSDE (4.3.17) we apply a similar method to the one in Weinan et al. [29]. We adapt [29, Framework 3.2] to our case by generalizing the implementation to a coupled FBSDE setting with a multi-dimensional backward equation. The method consists into a neural network approximation of the two solutions Y and $\mathbf{Z}$ of the FBSDE (4.3.17), where the backward equation is transformed into a forward equation and initial condition Y_0 and process $\mathbf{Z}_t$ are chosen in order to minimize the loss

$$\text{loss} := \mathbb{E}[|Y_T - \mathcal{S}(\mathbf{X}_T, Q_T)|], \tag{4.4.1}$$

in order to guarantee the terminal condition $Y_T = \mathcal{S}(\mathbf{X}_T, Q_T)$. In Algorithm 1 can be found the pseudo-code for the used algorithm and at the following GitHub link can be found the complete source of the Python code used to run the tests: `https://github.com/RiccardoCesari/PhD_neural_network`

We run several neural network approximations for different model parameters choices and we compare the results of the FBSDE method with the closed form control from Section 4.3.1. To compare the two methods, we divide the time interval $[0, T]$ in 40 time steps. To calculate the approximated solution of the FBSDE (4.3.17), we use a 4 layers neural network as in Weinan et al. [29] with a batch set made of 64 realizations of $\mathbf{W}$ and a validation set made of 256 realizations. In all numerical examples, we stop training the neural network after 40.000 steps. To calculate the integrals in (4.3.11) used in the representation of the control c (4.3.13), we apply a quadrature approximation formula. We denote optimal control calculated using NN approximation of the FBSDE solution as c_t^{NN}, the inventory process Q_t^{NN} and the wealth

Algorithm 1 NN algorithm

$\mathbf{dW} \leftarrow$ Initialised as an array of $\mathcal{N}(0,h)$ r.v.

$A \leftarrow a$

$M \leftarrow 0$

$\varepsilon \leftarrow \epsilon$

$Q \leftarrow q_0$

$Y_0, \mathbf{Z}_0 \leftarrow$ Initialised as NN variables

$t \leftarrow 0$

loss $\leftarrow +\infty$

$h = T/n$

while loss $>$ desired loss and $i <$ max steps **do**

    while $t \leq n$ **do**

        $\mathbf{W} \leftarrow \mathbf{W} + \mathbf{dW}$

        $Y \leftarrow Y - \mathcal{G}_2(A, \varepsilon, Q) \cdot h + Z \cdot \mathbf{dW}$

        $(A, \varepsilon) \leftarrow (A, \varepsilon) + \mu(A, \varepsilon, Y, \mathbf{W}) \cdot h + \sigma(A, W_t)$

        $S \leftarrow Ae^{\varepsilon}$

        $c \leftarrow (S - Y)^+/(2\eta)$

        $Q \leftarrow Q - ch$

        $M \leftarrow M + c(S - \eta c)h$

        Z is passed through neurons layer as function of X and then normalised

        $t \leftarrow t + 1$

    end while

    $M \leftarrow M + Q(S - \chi Q)$ ▷ for terminal time

    loss $\leftarrow \sqrt{|Y - \mathcal{S}(A, \varepsilon, Q)|}$

end while

process M_t^{NN}. For each parameter choice we compare the closed form control c_t with c_t^{NN}, the inventory process Q_t with Q_t^{NN} and the wealth process M_t with M_t^{NN}.

In the following we show numerical results for 2 different sets of parameters, both satisfying condition (4.3.10). The only differences between the two following settings are volatilities σ_1, σ_2 and the terminal time T.

Setting 4.4.1. $A_0 = 1,\ \epsilon_0 = 0,\ M_0 = 1,\ Q_0 = 20,\ T = 0.5,\ \chi = 0.5,\ \phi_1 = 0.003,\ \phi_2 = 0.06,\ \phi_3 = 0.06,\ \sigma_1 = 0.1,\ \sigma_2 = 0.1,\ k = 0.2,\ \eta = 0.003,\ \rho = -0.4.$

Setting 4.4.2. $A_0 = 1,\ \epsilon_0 = 0,\ M_0 = 1,\ Q_0 = 20,\ T = 1,\ \chi = 0.5,\ \phi_1 = 0.003,\ \phi_2 = 0.06,\ \phi_3 = 0.06,\ \sigma_1 = 0.4,\ \sigma_2 = 0.4,\ k = 0.2,\ \eta = 0.003,\ \rho = -0.4.$

In Figure 4.1 is displayed the convergence of loss function (4.4.1) under Settings 4.4.1 and 4.4.2, which reaches a value lower than 10^{-6} after 40.000 training steps in all cases.

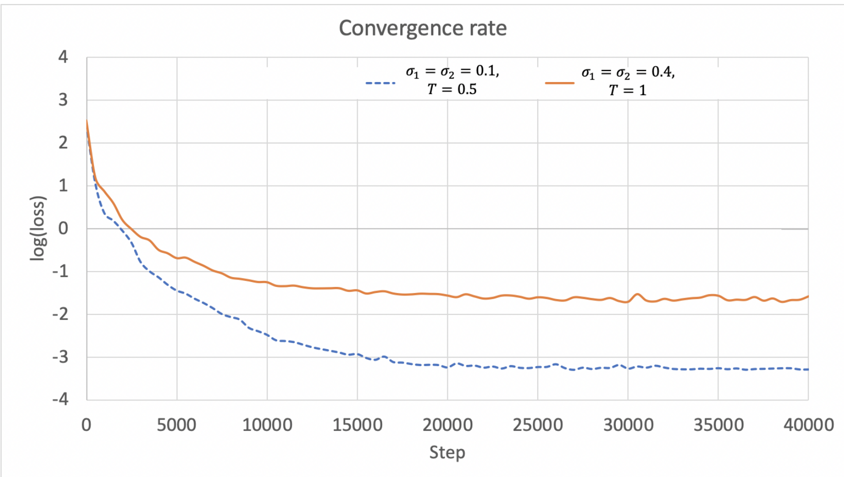


Figure 4.1: Convergences of logarithm of losses of NNs for the different Settings 4.4.1 and 4.4.2.

As a second step, we calculate the average relative discrepancy between M_t^{NN} and M_t over many different realizations of the Brownian motion $(\mathbf{W}_t)_{t\in[0,T]}$ under the two different model parameters sets defined above. In Figure 4.2 is drawn the average and standard deviation of the quantity $\frac{|M_t^{NN} - M_t|}{M_t^{NN}}$ along 400 different realizations of $\mathbf{W}$, for each time step t. We notice that in the low volatility case the relative errors $\frac{|M_t^{NN} - M_t|}{M_t^{NN}}$ is low and never exceeding 0.3%, while in the high volatility case the discrepancy increases its magnitude to a value of 1%.

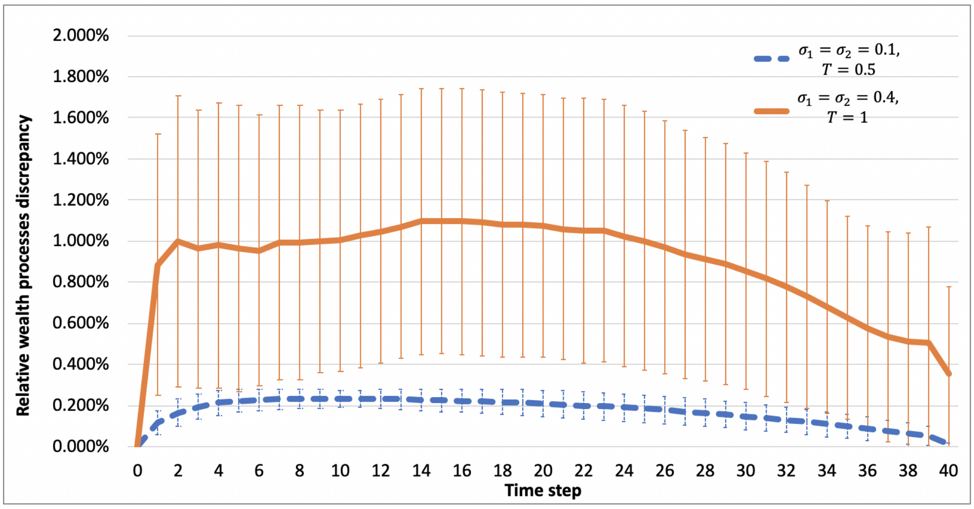


Figure 4.2: Average and standard deviation of $\frac{|M_t^{NN} - M_t|}{M_t^{NN}}$ along 400 different realizations of $\mathbf{W}$, for each time step t and for different Settings 4.4.1 and 4.4.2.

In Table 4.1 we group the relative discrepancies in Figure 4.2 and we also consider relative discrepancies of inventories over many different realizations of Brownian motion $(\mathbf{W}_t)_{t\in[0,T]}$ and for different model parameters choices. In Table 4.1 is shown the average and standard deviation of the quantities $\frac{1}{40}\sum_{t=1}^{40} \frac{|M_t^{NN} - M_t|}{M_t^{NN}}$ and $\frac{1}{40}\sum_{t=1}^{40} \frac{|Q_t^{NN} - Q_t|}{Q_0}$ along 400 different realizations of $\mathbf{W}$. We calculate these figures for both Settings 4.4.1 and 4.4.2.

Table 4.1 Average and standard deviation of $\frac{1}{40}\sum_{t=1}^{40} \frac{|M_t^{NN} - M_t|}{M_t^{NN}}$ and $\frac{1}{40}\sum_{t=1}^{40} \frac{|Q_t^{NN} - Q_t|}{Q_0}$ along 400 different realizations of $\mathbf{W}$ for Settings 4.4.1 and 4.4.2.

Settings	Av. Rel. Discr. M		Av. Rel. Discr. Q		Run-time closed form	Run-time NN
	Mean	St. Dev.	Mean	St. Dev.		
$\sigma_1 = \sigma_2 = 0.1$, $T = 0.5$	0.170%	0.00057	0.105%	0.00044	210 sec.	6850 sec.
$\sigma_1 = \sigma_2 = 0.4$, $T = 1$	0.886%	0.00601	0.665%	0.00463	213 sec.	7120 sec.

In Table 4.2 we show that the approximation made by removing the stopping time τ from optimization problem (4.2.6) and fixing it to a terminal time T as in in (4.3.15) scarcely affects the value function. Indeed, in Table 4.2 is shown that the average and standard deviation of the relative discrepancy $|Q_T|/Q_0$ along 400 different realizations of $\mathbf{W}$ are close to 0. We calculate these figures for both Settings 4.4.1 and 4.4.2.

Table 4.2 Average and standard deviation of $|Q_T|/Q_0$ along 400 different realizations of $\mathbf{W}$ for Settings 4.4.1 and 4.4.2.

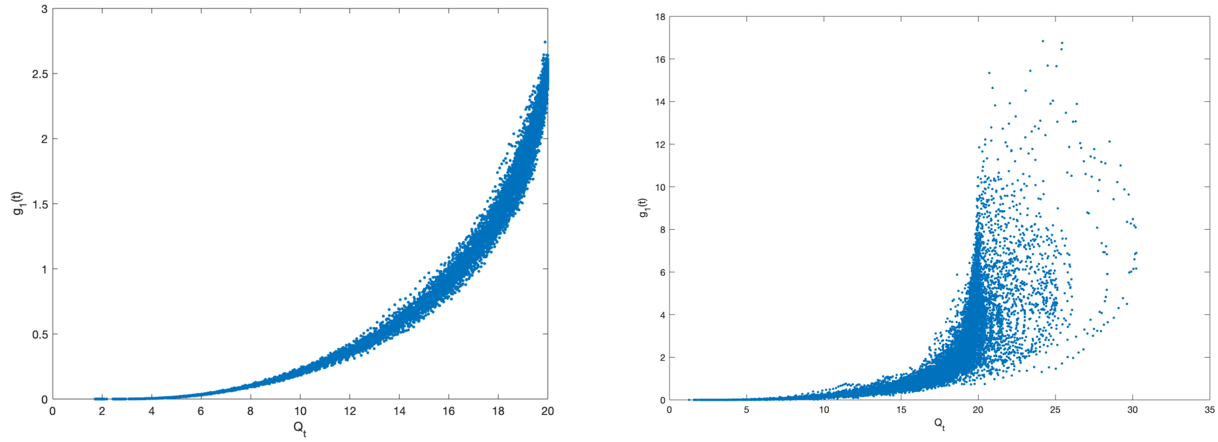
Settings	Mean($|Q_T|/Q_0$)	St. Dev.($|Q_T|/Q_0$)
$\sigma_1 = \sigma_2 = 0.1$, $T = 0.5$	0.0079	0.0041
$\sigma_1 = \sigma_2 = 0.4$, $T = 1$	0.0271	0.0181

In all examples we have shown, the results of the two different methods are close to each other. This increases our confidence in considering the solution of the HJB equation and the trading speed found in Section 4.3.1 respectively equal to the value function and the optimal trading speed of the problem. The computing time necessary to approximate integrals inside the closed form control representation is around 0.5 seconds for each realization of $\mathbf{W}$. To get an acceptable convergence of the neural network we waited 40.000 steps, taking around 110 minutes for each setting. Once the NN is trained, the computational time for the optimal strategy is around 0.6 seconds for each realization. In conclusion, the NN solution requires a time-consuming initial training that may cause delays any time the model needs to be recalibrated. Once the NN has been trained, the run-times of the two methods are almost equivalent.

4.4.2 Function w as close approximation of v

In this subsection we show that the solution w in Proposition 4.3.6 is close to the value function in (4.2.7). As it is mentioned in Proposition 4.3.6, w and v would coincide if $g_1 \equiv 0$, but as it can be seen from (4.3.14) this is not the case. From (4.3.5) it can be inferred that in order to get w as a continuous function, it would be necessary to have $g_1(t, \epsilon, a)\mathbb{1}_{\{q>0\}}$ continuous on $q = 0$. g_1 is independent of q, however, in the following, we want to show that for the examples presented above, whenever Q_t gets close to 0 it is most likely that t is close to T, making g_1 close to 0. In the following we show the numerical results obtained from the same numerical examples we presented in the previous subsection. Figures 4.3a and 4.3b show that the inventory Q_t usually gets to 0 only next to terminal time T. Thanks to the definition of g_1 in (4.3.14), we get that for t getting closer to T, the integration interval in (4.3.14) gets smaller and so the function g_1 gets closer to 0. This justifies that for any realizations of Q_t close to 0, it is likely that g_1 is close to 0 as well. This is verified from the two charts below, in which we show that the function g_1 is clearly increasing with respect to the realization of Q_t and is close to 0 whenever Q_t is close to

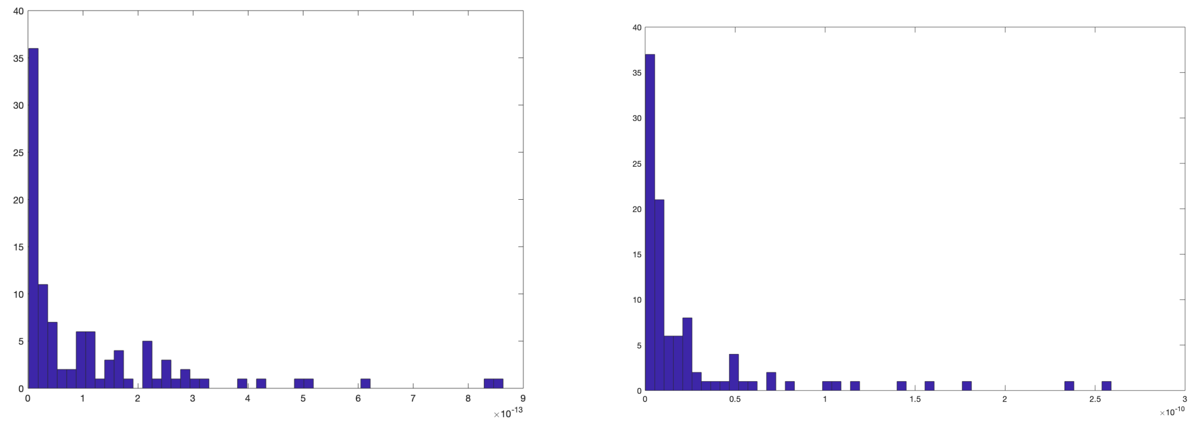
0.



(a) Setting 4.4.1. (b) Setting 4.4.2.

Figure 4.3: Plot of all realizations of Q_t versus all realizations of $g_1(t, A_t, \epsilon_t)$ for all $t \in [0, \tau]$.

Moreover, from the chart below, we can see that the value of g_1 at the specific time τ is close to 0 as well. In particular, we can see that 90% of realizations of $g_1(\tau, A_\tau, \epsilon_\tau)$ are smaller than 10^{-12} for Setting 4.4.1 and 10^{-9} for Setting 4.4.2. This result underlines that whenever the inventory q gets closer to 0, g_1 tends to 0 as well.



(a) Setting 4.4.1. (b) Setting 4.4.2.

Figure 4.4: Histogram of all realizations of $g_1(\tau, A_\tau, \epsilon_\tau)$.

The arguments above support the hypothesis that g_1 is close to 0 for $q \to 0$, making w a continuous function around $q = 0$.

4.4.3 Closed form model vs. single stock models

In this section we compare our model based on both processes A_t and ε_t with the two simplified models based only on one stock price as presented in Subsection 4.3.3.

We compare the performance of our strategy with those of the approximations in different settings. By simulating S_τ, we can evaluate the performances of the strategies c, $\tilde{c}$ and c^A respectively based on the price S, the GBM price $\tilde{S}$ and the price A. To compare the distributions of the cash value $M_\tau + Q_\tau(S_\tau - \chi Q_\tau)$, we run 100 different realizations of process S_t and, by calculating the trading rate for each realization, get the agent's final wealth. We assume that the trader executes orders at equally spaced moment in the interval $[0, T]$. In particular, we consider 100 trades, occurring every $T/100$. The data used for numerical tests are the following: $a_0 = 6,\ \epsilon_0 = 0,\ q_0 = 120,\ T = 1,\ \sigma_1 = 0.3,\ \sigma_2 = 0.05,\ \mu_1 = 0,\ \rho = 0.5,\ k = 0.1,\ \eta = 0.01,\ \chi = 0.007,\ \phi_1 = \phi_2 = \phi_3 = 0.07$. Similar numbers are used in Cartea et al. [18]. These parameters satisfy condition (4.3.10).

Table 4.3 summarizes the key statistics of agent's final wealth using the three different strategies.

Table 4.3 Key statistics of agent's final wealth based on simulations with different optimal strategies. Percentages in brackets represent the discrepancies with respect to the strategy c based on stock price S.

Strategy based on	Exp. Val.	St. Dev.	5^{th} Perc.	95^{th} Perc.
Price S	723.3	79.8	598.2	861.8
GBM approx. $\tilde{S}$	718.8 (-0.6%)	95.2 (19.3%)	567.3 (-3.5%)	877.3 (1.8%)
Stock A	718.3 (-0.7%)	95.8 (20.1%)	561.6(-6.1%)	877.8 (1.9%)

Table 4.3 shows that the strategy c has the best performance in producing the highest expected value and the lowest standard deviation for agent's final wealth, which indicates using the information of both stocks is highly useful in increasing the final wealth and reducing the risk. The strategy c is also the one that guarantees the highest final wealth with 95% confidence.

Table 4.4 summarizes the key statistics of agent's final wealth with change of one parameter while all other parameters are kept the same. In particular, we compare the performance for different correlation coefficient ρ, penalty coefficients ϕ_i, and volatility σ_2. Table 4.4 shows again that strategy using the information of two stocks outperforms those using only one stock.

We also observe that for increasing values of penalty parameters ϕ_i, we get decreasing ex-

Table 4.4 Sensitivity analysis to model parameters, by slightly modifying parameters.

Param. Choice	Strategy based on	Exp. Val.	St. Dev.	5^{th} Perc.	95^{th} Perc.
$\rho = 0$	Price S	713.3	68.4	619.0	824.0
	GBM approx. $\tilde{S}$	711.2	91.6	583.8	866.5
	Stock A	710.7	95.6	579.7	883.9
$\rho = -0.5$	Price S	718.2	73.6	610.8	850.4
	GBM approx. $\tilde{S}$	712.7	93.2	579.3	877.1
	Stock A	713.0	102.8	568.5	880.4
$\phi_1 = \phi_2 = \phi_3 = 0.05$	Price S	724.9	105.4	586.4	931.9
	GBM approx. $\tilde{S}$	720.9	110.3	541.2	982.9
	Stock A	714.9	115.1	539.4	986.9
$\phi_1 = \phi_2 = \phi_3 = 0.09$	Price S	715.7	90.8	586.0	857.7
	GBM approx. $\tilde{S}$	715.1	109.3	548.9	887.0
	Stock A	714.9	113.9	541.7	897.8
$\sigma_2 = 0.04$	Price S	709.0	85.0	579.6	854.9
	GBM approx. $\tilde{S}$	708.0	115.2	549.8	921.7
	Stock A	707.5	116.2	542.7	923.7
$\sigma_2 = 0.06$	Price S	729.5	87.7	599.5	881.8
	GBM approx. $\tilde{S}$	726.9	112.9	564.5	928.2
	Stock A	725.7	113.7	560.5	924.6

pected value and standard deviation of terminal wealth. This is due to the urgency of liquidation introduced by these penalizations, which implies that when trader's risk aversion is higher, the optimal strategy concentrates the liquidation on the initial part of period $[0, T]$, leading to a less volatile but lower expected final wealth.

The opposite behavior can be inferred from different choices of volatility σ_2. The higher the volatility of co-integration process, the lower the expected final wealth (and the higher the standard deviation).

We also perform a robustness test on three strategies by randomly choosing volatilities σ_1 and σ_2 from uniform distributions in which $\sigma_1 \in [0.25, 0.35]$ and $\sigma_2 \in [0.04, 0.06]$. We run 300 different simulations of stock price S.

Table 4.5 Robustness test for uniformly randomly chosen values of $\sigma_1 \in [0.25, 0.35]$ and $\sigma_2 \in [0.04, 0.06]$

Strategy based on	Exp. Val.	St. Dev.	5^{th} Perc.	95^{th} Perc.
Price S	712.7	90.6	566.2	892.2
GBM approx. $\tilde{S}$	711.9	117.5	558.2	982.1
Stock A	709.4	125.2	545.6	991.8

Table 4.5 shows the conclusions are largely the same as those in Tables 4.3 and 4.4.

4.5 Conclusion

We have proved that the value function is the unique viscosity solution of the HJB equation associated to our model, that, under some mild conditions, an approximation of the HJB equation admits a semi-closed integral representation which makes the calculation for agent's optimal liquidation rate easy and fast. Moreover, we showed that the solution to the approximated HJB equation is close to the value function. We attacked the problem from another perspective, using stochastic maximum principle to solve it. Numerical tests show that the approximate solution of the FBSDE is close to the solution of the HJB equation. This fact increases our confidence in considering the solution of the HJB equation and the trading speed found in Section 4.3.1 respectively equal to the value function and the optimal trading speed of the problem. Numerical tests show that, independent of market conditions, our strategy based on two stock prices outperforms other single stock strategies and approximations with the highest expected final wealth and the lowest standard deviation, is as robust as other strategies known in the literature, based on a single stock.

4.6 Proofs

We first introduce some notations and relations that are used in the proofs. Denote by $\mathbf{x} := (a, \epsilon, q)$ and

$$\begin{aligned} \mathcal{G}_1(\pi, \mathbf{x}) &:= a e^{\epsilon}\pi - \eta\pi^2, \\ \mathcal{G}_2(\mathbf{x}, q) &:= -\phi_1 q^2 - \phi_2 q a e^{\epsilon} - \phi_3 q a, \end{aligned} \tag{4.6.1}$$

$$\mathcal{S}(\mathbf{x}, q) := q\,(ae^{\epsilon} - \chi q)\,.$$

The objective function (4.2.8) can be written as

$$v^{\pi}(t, \mathbf{x}) := \mathbb{E}_t\Bigg[\int_t^{\tau} \mathcal{G}_1(\pi_r, \mathbf{X}_r^{t,\mathbf{x}})dr + \int_t^{\tau} \mathcal{G}_2(\mathbf{X}_r^{t,\mathbf{x}}, Q_r^{t,q})dr + \mathcal{S}(\mathbf{X}_{\tau}^{t,\mathbf{x}}, Q_r^{t,q})\Bigg].$$

Denote by $\mathbf{X}_r^{t,\mathbf{x}} := (A_r^{t,\mathbf{x}}, \varepsilon_r^{t,\mathbf{x}})$, the solution of (4.2.1), (4.2.2), and (4.2.3) with the initial condition $(\mathbf{X}_t, Q_t) = (\mathbf{x}, q) \in \mathcal{O}$ and square integrable feasible control $c \in \mathcal{A}$ and $t \in [0, T]$. We omit the superscript in $X^{t,\mathbf{x}}$ and we denote it as X when the initial conditions are clear from the context.

Lemma 4.6.1. *The first two moments of the stock price S are*

$$\mathbb{E}\left[S_r\right] = \mathbb{E}\left[A_r e^{\varepsilon_r}\right] = a e^{\epsilon e^{-k(r-t)} + \mu_1(r-t) + \frac{\sigma_2^2}{4k}\left(1-e^{-2k(r-t)}\right) + \frac{\rho\sigma_1\sigma_2}{k}\left(1-e^{-k(r-t)}\right)} \tag{4.6.2}$$

and

$$\mathrm{Var}(S_r) = \mathrm{Var}\left(A_r e^{\varepsilon_r}\right) = \mathbb{E}\left[A_r e^{\varepsilon_r}\right]^2 \left(e^{\sigma_1^2(r-t) + \frac{\sigma_2^2}{2k}\left(1-e^{-2k(r-t)}\right) + \frac{2\rho\sigma_1\sigma_2}{k}\left(1-e^{-k(r-t)}\right)} - 1\right). \tag{4.6.3}$$

Proof. Define the following processes:

$$\begin{aligned} F_r &:= \exp\left(\sqrt{1-\rho^2}\sigma_2 \int_t^r e^{-k(r-s)} dW_s^2\right), \\ G_r &:= \exp\left(\int_t^r \left(\rho\sigma_2 e^{-k(r-s)} + \sigma_1\right) dW_s^1\right). \end{aligned}$$

From definition (4.2.1) and (4.2.2) of A_r and ε_r we get

$$\mathbb{E}\left[A_r e^{\varepsilon_r}\right] = a e^{\left(\mu_1 - \frac{\sigma_1^2}{2}\right)(r-t)} e^{\epsilon e^{-k(r-t)}} \mathbb{E}\left[F_r\right] \mathbb{E}\left[G_r\right]. \tag{4.6.4}$$

The exponents of F_r and G_r are sums of normal distributed random variables whose means are 0 and variances are respectively $\frac{(1-\rho^2)\sigma_2^2}{2k}\left(1 - e^{-2k(r-t)}\right)$ and $\sigma_1^2(r-t) + \frac{\rho^2\sigma_2^2}{2k}\left(1 - e^{-2k(r-t)}\right) + \frac{2\rho\sigma_1\sigma_2}{k}\left(1 - e^{-k(r-t)}\right)$. Hence, F_r and G_r are log-normal random variables and simple calculus on their expected value proves result in (4.6.2).

Similarly to (4.6.4) we have

$$\begin{aligned} \mathrm{Var}\left[A_r e^{\varepsilon_r}\right] &= a^2 e^{\left(2\mu_1 - \sigma_1^2\right)(r-t)} e^{2\epsilon e^{-k(r-t)}} \mathrm{Var}(F_r G_r) \\ &= a^2 e^{\left(2\mu_1 - \sigma_1^2\right)(r-t)} e^{2\epsilon e^{-k(r-t)}} \left(\mathrm{Var}(F_r)\mathrm{Var}(G_r) + \mathrm{Var}(F_r)\mathbb{E}[G_r]^2 + \mathbb{E}[F_r]^2\mathrm{Var}(G_r)\right). \end{aligned}$$

We recall that if a random variable $K \sim \mathrm{Lognormal}(0, b^2)$, then $\mathbb{E}[K] = e^{b^2/2}$ and $\mathrm{Var}(K) = \left(e^{b^2} - 1\right) e^{b^2} = \left(\mathbb{E}[K]^2 - 1\right)\mathbb{E}[K]^2$. Then, for any $r \in [t, T]$, $\mathrm{Var}(G_r) = \left(\mathbb{E}[G_r]^2 - 1\right)\mathbb{E}[G_r]^2$ and $\mathrm{Var}(F_r) = \left(\mathbb{E}[F_r]^2 - 1\right)\mathbb{E}[F_r]^2$. We conclude that

$$\mathrm{Var}\left[A_r e^{\varepsilon_r}\right] = a^2 e^{\left(2\mu_1 - \sigma_1^2\right)(r-t)} e^{2\epsilon e^{-k(r-t)}} \mathbb{E}[G_r]^2\mathbb{E}[F_r]^2 \left(\mathbb{E}[F_r]^2\mathbb{E}[G_r]^2 - 1\right),$$

which is the desired result in (4.6.3). □

To prove Theorem 4.2.1 and Theorem 4.2.2, we first give some technical lemmas.

Lemma 4.6.2. *Let $c \in \mathcal{A}$ and $(\mathbf{x}, q) \in \mathcal{O}$, then for any $t \in [0,T]$,*

$$\lim_{h \searrow 0^+} \mathbb{E}\left[\sup_{r\in[t,t+h]} |\mathbf{X}_r - \mathbf{x}|^2 + |Q_r - q|^2\right] = 0. \tag{4.6.5}$$

Proof. Expression (4.6.5) has the following equivalent formulation:

$$\lim_{h \searrow 0^+} \mathbb{E}\left[\sup_{r\in[t,t+h]} |A_r - a|^2\right] + \lim_{h \searrow 0^+} \mathbb{E}\left[\sup_{r\in[t,t+h]} |\varepsilon_r - \epsilon|^2\right] + \lim_{h \searrow 0^+} \mathbb{E}\left[\sup_{r\in[t,t+h]} |Q_r - q|^2\right]. \tag{4.6.6}$$

The first two limits in previous expression tend to 0 as it is proved in Krylov [41, Corollary 2.6.12]. Indeed, both SDEs defining A_r and ε_r have linearly growing drift and diffusion terms independent of the control process c.

Let $h > 0$ be fixed,

$$\mathbb{E}\left[\sup_{r\in[t,t+h]} |Q_r - q|^2\right] \le \mathbb{E}\left[\sup_{r\in[t,t+h]} \left|\int_t^r \pi_s \, ds\right|^2\right] \le \mathbb{E}\left[\left(\int_t^{t+h} \pi_s \, ds\right)^2\right].$$

Using Cauchy-Schwarz inequality $\|\pi_s\|_{L^1(t,t+h)} \le \|\pi_s\|_{L^2(0,T)}\sqrt{h}$, also noting $(\pi_s)_{s\in[0,T]} \in L^2(\mathbb{P};[0,T])$, we conclude that there exists $K > 0$ such that

$$\mathbb{E}\left[\sup_{r\in[t,t+h]} |Q_r - q|^2\right] \le h\mathbb{E}\left[\int_0^T \pi_s^2 ds\right] \le Kh \xrightarrow[h\to 0]{} 0. \tag{4.6.7}$$

□

Lemma 4.6.3. *Let $p > 0$ be a constant and $t \in [0,T]$, then there exists a constant $C > 0$ such that for any $(\mathbf{x}, q) \in \mathcal{O}$*

$$\mathbb{E}_t\left[\sup_{r\in[t,T]} e^{p\varepsilon_r^{t,\mathbf{x}}}\right] \le Ce^{p|\epsilon|} \tag{4.6.8}$$

and for any $r \in [t,T]$

$$\mathbb{E}_t\left[e^{p\left(\varepsilon_r^{t,\mathbf{x}} - \epsilon\right)}\right] \le Ce^{p|\epsilon|\left(1-e^{-k(r-t)}\right)}. \tag{4.6.9}$$

Proof. Process ε_r, defined in (4.2.2), has explicit formulation

$$\varepsilon_r^{t,\mathbf{x}} = \epsilon e^{-k(r-t)} + \sigma_2 \int_t^r e^{-k(r-s)}\left(\rho dW_s^1 + \sqrt{1-\rho^2} dW_s^2\right) =: \epsilon e^{-k(r-t)} + \sigma_2 e^{-kr} M_r^t.$$

Using Ito's formula, we have the process $N_r^t := e^{p\sigma_2 M_r^t}$ satisfies the following SDE

$$dN_r^t = \frac{p^2\sigma_2^2}{2} e^{2kr} N_r^t dr + p\sigma_2 e^{kr} N_r^t \left(\rho dW_r^1 + \sqrt{1-\rho^2} dW_r^2\right), \quad N_t^t = 1.$$

The SDE above satisfies conditions of Krylov [41, Corollary 2.6.12], then there exists a constant $C > 0$ such that $\mathbb{E}\left[\sup_{r\in[t,T]} e^{p\sigma_2 M_r^t}\right] \le \mathbb{E}\left[\sup_{r\in[t,T]} |N_r^t|\right] \le \mathbb{E}\left[\sup_{r\in[0,T]} |N_r^0|\right] \le C$. Using that for any $\kappa \in [0,1]$, $e^{\kappa x} \le e^x + 1$ on $\mathbb{R}$, we get (4.6.8):

$$\mathbb{E}_t\left[\sup_{r\in[t,T]} e^{p\varepsilon_r^{t,\mathbf{x}}}\right] \le \mathbb{E}_t\left[\sup_{r\in[t,T]} e^{p\epsilon e^{-k(r-t)}}\left(e^{p\sigma_2 M_r^t} + 1\right)\right] \le e^{p|\epsilon|}\left(\mathbb{E}_t\left[\sup_{r\in[t,T]} e^{p\sigma_2 M_r^t}\right] + 1\right)$$

$$\le (C+1)e^{p|\epsilon|}.$$

Finally, we apply similar arguments to get (4.6.9):

$$\mathbb{E}_t\left[e^{p\left(\varepsilon_r^{t,\mathbf{x}}-\epsilon\right)}\right] \le \mathbb{E}_t\left[e^{p\epsilon\left(e^{-k(r-t)}-1\right)}\left(e^{p\sigma_2 M_r^t}+1\right)\right] \le (C+1)e^{p|\epsilon|\left(1-e^{-k(r-t)}\right)}.$$

□

Lemma 4.6.4. *Let $t \in [0,T]$ and $(\mathbf{x}, q), (\mathbf{x}', q) \in \mathcal{O}$. There exists $C_p > 0$, independent of t, such that*

$$\mathbb{E}_t\left[\sup_{r\in[t,T]}\left|S_r^{t,\mathbf{x}} - S_r^{t,\mathbf{x}'}\right|^p\right] \le C_p e^{p|\epsilon'|}\left(|a-a'|^p + a^p\left|e^{\epsilon-\epsilon'}-1\right|^p\right). \tag{4.6.10}$$

Proof. Using results on GBM (cf. [48, Theorem 1.3.15] and [41, Corollary 2.6.12]) and using (4.6.8), we have that, for any $p \ge 1$, there exists a constant $C_p > 0$ independent of t and $\mathbf{x}$ such that

$$\mathbb{E}_t\left[\sup_{r\in[t,T]}\left(A_r^{t,\mathbf{x}}\right)^p\right] < C_p a^p, \qquad \mathbb{E}_t\left[\sup_{r\in[t,T]} e^{p\varepsilon_r^{t,\mathbf{x}}}\right] < C_p e^{p|\epsilon|}. \tag{4.6.11}$$

Hence, for any $p \ge 1$ there exists $C_p > 0$, independent of t, so that

$$\begin{aligned}
\mathbb{E}_t\left[\sup_{r\in[t,T]}\left|S_r^{t,\mathbf{x}} - S_r^{t,\mathbf{x}'}\right|^p\right] &\le 2^p\left(\mathbb{E}_t\left[\sup_{r\in[t,T]}\left(A_r^{t,\mathbf{x}}\right)^{2p}\right]\right)^{\frac{1}{2}}\left(\mathbb{E}_t\left[\sup_{r\in[t,T]}\left|e^{\varepsilon_r^{t,\mathbf{x}}} - e^{\varepsilon_r^{t,\mathbf{x}'}}\right|^{2p}\right]\right)^{\frac{1}{2}} \\
&\quad + 2^p\left(\mathbb{E}_t\left[\sup_{r\in[t,T]} e^{2p\varepsilon_r^{t,\mathbf{x}'}}\right]\right)^{\frac{1}{2}}\left(\mathbb{E}_t\left[\sup_{r\in[t,T]}\left|A_r^{t,\mathbf{x}} - A_r^{t,\mathbf{x}'}\right|^{2p}\right]\right)^{\frac{1}{2}} \\
&\le C_p\Bigg(a^p\left(\mathbb{E}_t\left[\sup_{r\in[t,T]}\left|e^{\varepsilon_r^{t,\mathbf{x}}} - e^{\varepsilon_r^{t,\mathbf{x}'}}\right|^{2p}\right]\right)^{\frac{1}{2}} \\
&\quad + e^{p|\epsilon'|}\left(\mathbb{E}_t\left[\sup_{r\in[t,T]}\left|A_r^{t,\mathbf{x}} - A_r^{t,\mathbf{x}'}\right|^{2p}\right]\right)^{\frac{1}{2}}\Bigg).
\end{aligned}$$

The explicit formulations for processes A_r and ε_r give

$$\begin{aligned}
\left|A_r^{t,\mathbf{x}} - A_r^{t,\mathbf{x}'}\right| &= |a-a'|e^{\left(\mu_1-\frac{\sigma_1^2}{2}\right)(r-t)+\sigma_1 W_{r-t}^1}, \\
\left|e^{\varepsilon_r^{t,\mathbf{x}}} - e^{\varepsilon_r^{t,\mathbf{x}'}}\right| &= e^{\varepsilon_r^{t,\mathbf{x}'}}\left|e^{(\epsilon-\epsilon')e^{-k(r-t)}}-1\right| \le e^{\varepsilon_r^{t,\mathbf{x}'}}\left|e^{\epsilon-\epsilon'}-1\right|.
\end{aligned} \tag{4.6.12}$$

Here we have used the fact that for $\kappa \in [0,1]$, $|e^{\kappa x}-1| \le |e^x - 1|$ on $\mathbb{R}$. Using similar argument as in (4.6.11), we get that there exists $C_p > 0$ such that

$$\begin{aligned}
\left(\mathbb{E}_t\left[\sup_{r\in[t,T]}\left|A_r^{t,\mathbf{x}} - A_r^{t,\mathbf{x}'}\right|^{2p}\right]\right)^{\frac{1}{2}} &\le C_p|a-a'|^p, \\
\left(\mathbb{E}_t\left[\sup_{r\in[t,T]}\left|e^{\varepsilon_r^{t,\mathbf{x}}} - e^{\varepsilon_r^{t,\mathbf{x}'}}\right|^{2p}\right]\right)^{\frac{1}{2}} &\le C_p e^{p|\epsilon'|}\left|e^{\epsilon-\epsilon'}-1\right|^p.
\end{aligned}$$

□

4.6.1 Proof of Theorem 4.2.1

We follow the proof of Verification Theorem Pham [48, Theorem 3.5.2] to show that v and w coincide on $[0,T]\times\mathcal{O}$. The two main differences between our setting and Pham's setting are that solution w does not satisfy a quadratic growth condition and the presence of a stopping time τ in the definition of value function in our case. We define sequence of stopping time τ_n similarly as in [48, proof of Theorem 3.5.2], by capping it with the stopping time τ:

$$\tau_n := \tau \wedge \inf_{s\geq t}\left\{\int_t^s |\nabla_{\mathbf{x}} w(r,\mathbf{X}_r,Q_r)'\sigma(\mathbf{X}_r)|^2\,dr \geq n\right\}.$$

We notice that $\tau_n \nearrow \tau$ and the stopped process $(\int_t^{s\wedge\tau_n}\nabla_{\mathbf{x}} w(r,\mathbf{X}_r,Q_r)'\sigma(\mathbf{X}_r)\ dr)_{s\in[t,T]}$ is a martingale. Let $\pi\in\mathcal{A}$ be fixed. By taking the expectation of the Ito's representation of $w(s,\mathbf{X}_s,Q_s)$, we get

$$\mathbb{E}_t\left[w(s\wedge\tau_n,\mathbf{X}_{s\wedge\tau_n},Q_{s\wedge\tau_n})\right] = w(t,\mathbf{x},q) + \mathbb{E}_t\left[\int_t^{s\wedge\tau_n}\left(\frac{\partial w}{\partial t}(r,\mathbf{X}_r,Q_r) + \mathcal{L}w(r,\mathbf{X}_r,Q_r) - \pi\frac{\partial w}{\partial q}(r,\mathbf{X}_r,Q_r)\right)dr\right]. \tag{4.6.13}$$

Since w is a solution to the HJB equation (4.2.9), for a general $c\in\mathcal{A}$

$$\frac{\partial w}{\partial t}(r,\mathbf{X}_r,Q_r) + \mathcal{L}w(r,\mathbf{X}_r,Q_r) + \mathcal{G}_1(\pi_r,\mathbf{X}_r) + \mathcal{G}_2(\mathbf{X}_r,Q_r) - \pi_r\frac{\partial w}{\partial q}(r,\mathbf{X}_r,Q_r) \leq 0$$

and applying it to (4.6.13), we get

$$\mathbb{E}_t\left[w(s\wedge\tau_n,\mathbf{X}_{s\wedge\tau_n},Q_{s\wedge\tau_n})\right] \leq w(t,\mathbf{x},q) - \mathbb{E}_t\left[\int_t^{s\wedge\tau_n}\left(\mathcal{G}_1(\pi_r,\mathbf{X}_r) + \mathcal{G}_2(\mathbf{X}_r,Q_r)\right)dr\right]. \tag{4.6.14}$$

We apply the dominated convergence theorem to previous inequality. Both sides are bounded by an integrable process independent of n. By using boundedness of process $(Q_r)_{r\in[0,T]}$ and Hölder's inequality, we have that

$$\begin{aligned}\mathbb{E}_t\left[\left|\int_t^{s\wedge\tau_n}\left(\mathcal{G}_1(\pi_r,\mathbf{X}_r) + \mathcal{G}_2(\mathbf{X}_r,Q_r)\right)dr\right|\right] &\leq \left(\mathbb{E}_t\left[\int_t^T A_r^2 e^{2\varepsilon_r}\ dr\right]\right)^{\frac{1}{2}}\left(\mathbb{E}_t\left[\int_t^T \pi_r^2\ dr\right]\right)^{\frac{1}{2}}\\ &+ \eta\mathbb{E}_t\left[\int_t^T \pi_r^2\ dr\right] + \phi_1 q^2 + \phi_2 q\mathbb{E}_t\left[\int_t^T A_r e^{\varepsilon_r}\ dr\right] + \phi_3 q\mathbb{E}_t\left[\int_t^T A_r\ dr\right],\end{aligned}$$

which is bounded independently of n, using (4.6.11) and square integrability of control process $(\pi_r)_{r\in[t,T]}$. By (4.6.11), Hölder's inequality, growth condition on w and recalling that Q_r is bounded, we conclude that for any $(t,\mathbf{x},q)\in[0,T]\times\mathcal{O}$, there exists $C>0$ independent of n such that

$$\mathbb{E}_t\left[w(s\wedge\tau_n,\mathbf{X}_{s\wedge\tau_n},Q_{s\wedge\tau_n})\right] \leq C(1+q^2)(1+a^{p_1})(1+e^{p_2\epsilon}),$$

We apply the dominated convergence theorem to (4.6.14) by sending $n\to\infty$:

$$\mathbb{E}_t\left[w(s\wedge\tau,\mathbf{X}_{s\wedge\tau},Q_{s\wedge\tau_n})\right] \leq w(t,\mathbf{x},q) - \mathbb{E}_t\left[\int_t^{s\wedge\tau}\left(\mathcal{G}_1(\pi_r,\mathbf{X}_r) + \mathcal{G}_2(\mathbf{X}_r,Q_r)\right)dr\right].$$

Since w is continuous on $[0,T]\times\mathcal{O}$, by sending s to T, we obtain by the dominated convergence theorem:

$$\mathbb{E}_t\left[w(\tau,\mathbf{X}_\tau,Q_\tau)\right] \le w(t,\mathbf{x},q) - \mathbb{E}_t\left[\int_t^\tau \left(\mathcal{G}_1(\pi_r,\mathbf{X}_r)+\mathcal{G}_2(\mathbf{X}_r,Q_r)\right)dr\right].$$

By terminal and boundary condition of HJB equation (4.2.9), we know that $w(\tau,\mathbf{x},q)=\mathcal{S}(\mathbf{x},q)$, so we have

$$\mathbb{E}_t\left[\mathcal{S}(\mathbf{X}_\tau,Q_\tau)\right] \le w(t,\mathbf{x},q) - \mathbb{E}_t\left[\int_t^\tau \left(\mathcal{G}_1(\pi_r,\mathbf{X}_r)+\mathcal{G}_2(\mathbf{X}_r,Q_r)\right)dr\right],$$

which implies that

$$w(t,\mathbf{x},q) \ge \mathbb{E}_t\left[\mathcal{S}(\mathbf{X}_\tau,Q_\tau)+\int_t^\tau \left(\mathcal{G}_1(\pi_r,\mathbf{X}_r)+\mathcal{G}_2(\mathbf{X}_r,Q_r)\right)dr\right] = v^c(t,\mathbf{x},q).$$

From arbitrariness of $c\in\mathcal{A}$, it follows that $v\le w$ on $[0,T]\times\mathcal{O}$.

To prove that $v\ge w$ on $[0,T]\times\mathcal{O}$, we proceed as before, by getting a similar version of (4.6.13) in which the control process c_r is substituted by the optimal control $c(r,\mathbf{X}_r,Q_r)$:

$$\begin{aligned}\mathbb{E}_t&\left[w(s\wedge\tau_n,\mathbf{X}_{s\wedge\tau_n},Q_{s\wedge\tau_n})\right] = w(t,\mathbf{x},q)\\ &+\mathbb{E}_t\left[\int_t^{s\wedge\tau_n}\left(\frac{\partial w}{\partial t}(r,\mathbf{X}_r,Q_r)+\mathcal{L}w(r,\mathbf{X}_r,Q_r)-c(r,\mathbf{X}_r,Q_r)\frac{\partial w}{\partial q}(r,\mathbf{X}_r,Q_r)\right)dr\right].\end{aligned}$$

By applying optimality of c, we get

$$\mathbb{E}_t\left[w(s\wedge\tau_n,\mathbf{X}_{s\wedge\tau_n},Q_{s\wedge\tau_n})\right] = w(t,\mathbf{x},q)+\mathbb{E}_t\left[\int_t^{s\wedge\tau_n}\left(\mathcal{G}_1(c(r,\mathbf{X}_r,Q_r),\mathbf{X}_r)+\mathcal{G}_2(\mathbf{X}_r,Q_r)\right)dr\right].$$

Proceeding as before, we apply dominated convergence theorem to both sides of previous expression. By sending $n\to\infty$ and then sending s to T, we get

$$\mathbb{E}_t\left[w(\tau,\mathbf{X}_\tau,Q_\tau)\right] = w(t,\mathbf{x},q) - \mathbb{E}_t\left[\int_t^\tau \left(\mathcal{G}_1(c(r,\mathbf{X}_r,Q_r),\mathbf{X}_r)+\mathcal{G}_2(\mathbf{X}_r,Q_r)\right)dr\right].$$

By terminal condition of the HJB equation, $w(\tau,\mathbf{x},q)=\mathcal{S}(\mathbf{x},q)$, so we have

$$w(t,\mathbf{x},q) = \mathbb{E}_t\left[\mathcal{S}(\mathbf{X}_\tau,Q_\tau)+\int_t^\tau \left(\mathcal{G}_1(c(r,\mathbf{X}_r,Q_r),\mathbf{X}_r)+\mathcal{G}_2(\mathbf{X}_r,Q_r)\right)dr\right] = v^c(t,\mathbf{x},q).$$

This shows that $w(t,\mathbf{x},q)=v^c(t,\mathbf{x},q)\le v(t,\mathbf{x},q)$ on $[0,T]\times\mathcal{O}$.

4.6.2 Proof of Theorem 4.2.2

To prove the result, we first give a technical lemma.

Lemma 4.6.5 (Comparison Principle). *Let U (respectively V) be an upper semicontinuous viscosity subsolution (resp. lower semicontinuous viscosity supersolution) to the following HJB equation*

$$-\frac{\partial v}{\partial t}(t,\mathbf{x},q)-\mathcal{L}v(t,\mathbf{x},q)-\sup_{\pi\geq 0}\left[-\pi\frac{\partial v}{\partial q}(t,\mathbf{x},q)+\mathcal{G}_1(\pi,\mathbf{x})\right]-\mathcal{G}_2(\mathbf{x},q)=0 \qquad (4.6.15)$$

for any $(t,\mathbf{x},q)\in[0,T)\times(0,\infty)\times\mathbb{R}\times(0,\bar{Q}_0)$. Assume there exist $C,\kappa>0$ and $m\in\mathbb{N}$ such that

$$|U(t,\mathbf{x},q)|+|V(t,\mathbf{x},q)|\leq C\left(1+a^m\right)\left(1+e^{\kappa|\epsilon|}\right) \qquad (4.6.16)$$

for any $(t,\mathbf{x},q)\in[0,T]\times\mathcal{O}$. If

$$U(T,\cdot,\cdot)\leq V(T,\cdot,\cdot) \text{ on } \mathcal{O} \text{ and } U(t,a,\epsilon,0)\leq V(t,a,\epsilon,0) \text{ for any } (t,a,\epsilon)\in[0,T]\times(0,\infty)\times\mathbb{R}, \qquad (4.6.17)$$

then $U\leq V$ on $[0,T]\times\mathcal{O}$.

Proof. The Lemma is proved following Pham [48, proof of Theorem 4.4.5]. The main difference between our statement and Pham [48, Theorem 4.4.5] is our functions U and V are not polynomially growing and are defined in a subset of $\mathbb{R}^n$ space. We apply the first step in [48, proof of Theorem 4.4.3], which provides an equivalent formulation for the HJB equation (4.6.15). Let $\beta>0$ be specified later, $\bar{U}(t,\mathbf{x},q)=e^{\beta t}U(t,\mathbf{x},q)$ and $\bar{V}(t,\mathbf{x},q)=e^{\beta t}V(t,\mathbf{x},q)$, then $\bar{U}$ and $\bar{V}$ are respectively subsolution and supersolution to

$$-\frac{\partial w}{\partial t}(t,\mathbf{x},q)+\beta w(t,\mathbf{x},q)-\mathcal{L}w(t,\mathbf{x},q)-\sup_{\pi\geq 0}\left[-\pi\frac{\partial w}{\partial q}(t,\mathbf{x},q)+e^{\beta t}\mathcal{G}_1(\pi,\mathbf{x})\right]-e^{\beta t}\mathcal{G}_2(\mathbf{x},q)=0 \qquad (4.6.18)$$

for any $(t,\mathbf{x},q)\in[0,T)\times\mathcal{O}$. With a slight abuse of notation, in the remaining of the proof, we denote $\bar{U}$, $\bar{V}$ respectively U, V and we replace equation (4.6.15) with (4.6.18).

We adapt second step of [48, proof of Theorem 4.4.3] to show that there exists a function $\varphi(t,\mathbf{x})$ such that for any $\delta>0$, $V+\delta\varphi$ is a supersolution to (4.6.18). Define $p(a)=C\left(1+a^d\right)$, where $d>\max(m,2)$ and C,m are as in (4.6.16). Define for any $(\mathbf{x},q)\in\mathcal{O}$

$$\varphi(\mathbf{x},q)=\frac{1}{a^2}-\ln\left(\frac{\bar{Q}_0-q}{\bar{Q}_0+1}\right)+\left(1+p(a)^2\right)\left(1+e^{b\epsilon}+e^{-b\epsilon}\right)$$

where $b>\max(\kappa,2)$ and κ is defined in (4.6.16). We observe that $\varphi(\mathbf{x},q)$ is non-negative and infinitely many times differentiable on $\mathcal{O}$. An explicit calculation shows that

$$\begin{aligned}&-\frac{\partial\varphi}{\partial t}(\mathbf{x},q)+\beta\varphi(\mathbf{x},q)-\mathcal{L}\varphi(\mathbf{x},q)-\sup_{\pi\geq 0}\left[-\pi\frac{\partial\varphi}{\partial q}(\mathbf{x},\mathbf{q})\right]\\&\quad=\beta\varphi+\frac{2\mu_1-3\sigma_1^2}{a^2}-b\left(e^{b\epsilon}-e^{-b\epsilon}\right)\left(-k\epsilon\left(1+p^2\right)+2pp'\rho\sigma_1\sigma_2 a\right)-\frac{\sigma_2^2}{2}b^2\left(1+p^2\right)\left(e^{b\epsilon}+e^{-b\epsilon}\right)\end{aligned}$$

$$- \left(\sigma_1^2 a^2 \left(\left(p'\right)^2 + pp''\right) + 2\mu_1 app'\right)\left(1 + e^{b\epsilon} + e^{-b\epsilon}\right) - \sup_{c \geq 0}\left[-\frac{c}{\bar{Q}_0 - q}\right].$$

We observe that $\sup_{\pi \geq 0}\left[-\frac{\pi}{\bar{Q}_0 - q}\right]$ and there exists a constant $C_1 > 0$ such that $a^2\left(\left(p'(a)\right)^2 + p(a)p''(a)\right) \leq C_1 p(a)^2$ and $2ap(a)p'(a) \leq C_1 p(a)^2$ for any $a \geq 0$. Simple calculus shows that $2\left(e^{b\epsilon} - e^{-b\epsilon}\right) pp'a \leq 2\left(e^{b\epsilon} + e^{-b\epsilon}\right) pp'a \leq C_1\left(e^{b\epsilon} + e^{-b\epsilon}\right) p^2 \leq C_1 \varphi$. Then, we get

$$\begin{aligned}
&- \frac{\partial \varphi}{\partial t}(\mathbf{x}, q) + \beta\varphi(\mathbf{x}, q) - \mathcal{L}\varphi(\mathbf{x}, q) - \sup_{\pi \geq 0}\left[-\pi \frac{\partial \varphi}{\partial q}(\mathbf{x}, q)\right] \\
&\geq \left(\beta - C_1 \rho\sigma_1\sigma_2 b\right)\varphi + \underbrace{k\epsilon b\left(e^{b\epsilon} - e^{-b\epsilon}\right)\left(1 + p^2\right)}_{\geq 0} \\
&\quad - \left[\frac{\sigma_2^2}{2} b^2\left(1 + p^2\right)\left(e^{b\epsilon} + e^{-b\epsilon}\right) + \frac{3\sigma_1^2}{a^2} + C_1\left(\sigma_1^2 + \mu_1\right) p^2\left(1 + e^{b\epsilon} + e^{-b\epsilon}\right)\right] \\
&\geq \left(\beta - C_1\left(\sigma_1^2 + \mu_1 + \rho\sigma_1\sigma_2 b\right) - \frac{\sigma_2^2}{2} b^2 - 3\sigma_1^2\right)\varphi. \qquad (4.6.19)
\end{aligned}$$

Choosing $\beta > 0$ so that $\beta > C_1\left(\sigma_1^2 + \mu_1 + \rho\sigma_1\sigma_2 b\right) + \frac{\sigma_2^2}{2} b^2 + 3\sigma_1^2$, we get that for any $\delta > 0$, the function $V_\delta = V + \delta\varphi$ is, as V, a supersolution to (4.6.18). Moreover, from definition of φ, and from growth conditions on U, V we have that for $\epsilon \to \pm\infty$ and $a \to +\infty$, φ grows more rapidly than U and V. For $a \to 0$ and $q \to \bar{Q}_0$, U and V are finite, while $\varphi \to +\infty$. This implies that for any $\delta > 0$, there exists an open and bounded set $\mathcal{O}_\delta$ so that $\bar{\mathcal{O}}_\delta \subset \mathcal{O}$ and

$$\sup_{(t,\mathbf{x},q) \in [0,T] \times \mathcal{O}} (U - V_\delta)(t, \mathbf{x}, q) = \max_{(t,\mathbf{x},q) \in [0,T] \times \{\mathbf{x} \in \mathbb{R}^3 | \mathbf{x} \in \mathcal{O}_\delta \text{ or } q = 0\}} (U - V_\delta)(t, \mathbf{x}, q). \qquad (4.6.20)$$

To conclude the proof of the Lemma, we need to show that

$$\forall \delta > 0, \quad \sup_{(t,\mathbf{x},q) \in [0,T] \times \mathcal{O}} (U - V_\delta)(t, \mathbf{x}, q) \leq 0. \qquad (4.6.21)$$

However, using (4.6.17), upper semicontinuity of U, lower semicontinuity of V and that $\varphi(\cdot, \cdot) \geq 1$, we get that

$$\forall \delta > 0 \text{ exists } \gamma > 0 \text{ s.t. } (U - V_\delta)(t, \mathbf{x}, q) < 0 \text{ when } t \in (T - \gamma, T] \text{ and} \qquad (4.6.22)$$

$$\forall \delta > 0 \text{ exists } \gamma > 0 \text{ s.t. } (U - V_\delta)(t, \mathbf{x}, q) < 0 \text{ when } q < \gamma. \qquad (4.6.23)$$

By applying (4.6.20), (4.6.22) and (4.6.23) we reduce our objective from (4.6.21) to the proof of

$$\forall \delta > 0, \; M_\delta := \max_{(t,\mathbf{x},q) \in [0,T) \times \mathcal{O}_\delta} (U - V_\delta)(t, \mathbf{x}, q) \leq 0. \qquad (4.6.24)$$

To prove the above statement, we assume by contradiction that $M_\delta > 0$. On the bounded set $\mathcal{O}_\delta$, functions μ and σ are uniformly Lipschitz and $\mathcal{G}_2$ is uniformly continuous. Then, by following Pham [48, proof of Theorem 4.4.5], we get that for any $\delta > 0$, $\beta M_\delta \leq 0$, which is a contradiction.

We conclude that for any $\delta > 0$, $M_\delta \le 0$ and so both (4.6.24) and (4.6.21) hold true. By taking limit of δ going to 0 in (4.6.21), we get that $(U - V)(\cdot,\cdot,\cdot) \le 0$ in $[0,T] \times \mathcal{O}$, which concludes the proof. $\square$

We now prove Theorem 4.2.2. By analysing value function v, we get the following upper and lower bounds. Using boundedness of process $(Q_r)_{r\in[t,T]}$ and (4.6.11), there exists $C > 0$ such that for any $(t,\mathbf{x},q) \in [0,T]\times\mathcal{O}$,

$$\begin{aligned}
v(t,\mathbf{x},q) &\le \sup_{\pi\in\mathcal{A}} \mathbb{E}_t[M_\tau] + \sup_{\pi\in\mathcal{A}} \mathbb{E}_t[Q_\tau S_\tau] \\
&\quad - \inf_{\pi\in\mathcal{A}} \mathbb{E}_t\left[\chi Q_\tau^2 + \phi_1 \int_t^\tau Q_r^2\, dr + \phi_2 \int_t^\tau S_r Q_r\, dr + \phi_3 \int_t^\tau A_r Q_r\, dr\right] \\
&\le \sup_{\pi\in\mathcal{A}} \mathbb{E}_t\left[\int_t^\tau \pi_r(S_r - \eta\pi_r)\, dr\right] + q\mathbb{E}_t\left[\sup_{r\in[t,T]} S_r\right] \le \frac{T}{4\eta}\mathbb{E}_t\left[\sup_{r\in[t,T]} S_r^2\right] + q\mathbb{E}_t\left[\sup_{r\in[t,T]} S_r\right] \\
&\le C(a + a^2)e^{C|\epsilon|}.
\end{aligned}$$

On the other hand, by choosing $c \equiv 0$, there exists $C > 0$ such that for any $(t,\mathbf{x},q) \in [0,T]\times\mathcal{O}$

$$\begin{aligned}
v(t,\mathbf{x},q) &\ge v^0(t,\mathbf{x},q) = \mathbb{E}_t[qS_T] - \mathbb{E}_t\left[\chi q^2 + \phi_1 \int_t^T q^2\, dr + \phi_2 \int_t^T S_r q\, dr + \phi_3 \int_t^T A_r q\, dr\right] \\
&\ge -\mathbb{E}_t\left[\chi q^2 + \phi_1 T q^2 + \phi_2 T q \sup_{r\in[t,T]} S_r + \phi_3 T q \sup_{r\in[t,T]} A_r\right] \\
&\ge -C(1+a)\left(1 + e^{C|\epsilon|}\right).
\end{aligned}$$

Here in the last inequality we have used (4.6.11). All conditions in [48, Propositions 4.3.1 and 4.3.2] are satisfied. In particular, [48, Condition (3.5)] holds true in (4.6.5) and v is locally bounded as proved in upper and lower bounds above. Then, by applying Pham [48, Propositions 4.3.1 and 4.3.2], we prove that the value function v is a viscosity solution to the HJB equation (4.2.9).

Using the above upper and lower bounds we get that v satisfies the growth condition (4.6.16). Then, using Comparison Principle Lemma 4.6.5, we conclude that value function v is the unique viscosity solution of HJB equation (4.2.9). $\square$

4.6.3 Proof of Proposition 4.2.3

To prove the result, we first give one technical lemma.

Lemma 4.6.6. *Let $\gamma > 0$ be fixed and let $\tilde{\mathcal{A}}_\gamma$, defined in (4.2.11), be the set of admissible controls. Then, the value function v, defined in (4.2.7), has the following property:*

$$\begin{aligned}
|v(t,\mathbf{x},q) - v(t,\mathbf{x}',q')| \le C\left(|a-a'| + \left|e^{\epsilon-\epsilon'} - 1\right| + |q-q'|^{\frac{\gamma}{\gamma+1}}\right)\left(1+a+a'\right)^C \cdot \\
\cdot \left(1 + e^{C|\epsilon|} + e^{C|\epsilon'|}\right)\left(1+q+q'\right)^C
\end{aligned}$$

for any $t \in [0,T]$ and $(\mathbf{x},q),(\mathbf{x}',q') \in \mathcal{O}$, where $C>0$ is a constant independent of t.

Proof. Let $t \in [0,T]$ be fixed and $(\mathbf{x},q),(\mathbf{x}',q') \in \mathcal{O}$. We assume w.l.o.g. that $q \geq q'$. Denote $\mathbf{X}_r^{t,\mathbf{x}}$ and $\mathbf{X}_r^{t,\mathbf{x}'}$ the two solutions to (4.2.4) with initial conditions $(t,\mathbf{x},q)$ and $(t,\mathbf{x}',q')$ respectively and the stopping times $\tau = T \wedge \min\{r \geq t \mid Q_r^{t,q} = 0\}$ and $\tau' = T \wedge \min\{r \geq t \mid Q_r^{t,q'} = 0\}$. We observe that

$$|v(t,\mathbf{x},q) - v(t,\mathbf{x}',q')| \leq \sup_{\pi \in \tilde{\mathcal{A}}_\gamma(t,\mathbf{x},q)\cap\tilde{\mathcal{A}}_\gamma(t,\mathbf{x}',q')} \mathbb{E}_t\Bigg[\left|M_\tau^{t,\mathbf{x}} - M_{\tau'}^{t,\mathbf{x}'}\right| + \left|Q_\tau^{t,q}\left(S_\tau^{t,\mathbf{x}} - \chi Q_\tau^{t,q}\right) - Q_{\tau'}^{t,q'}\left(S_{\tau'}^{t,\mathbf{x}'} - \chi Q_{\tau'}^{t,q'}\right)\right| + \left|\int_t^\tau \mathcal{G}_2(\mathbf{X}_r^{t,\mathbf{x}}, Q_r^{t,q})\, dr - \int_t^{\tau'} \mathcal{G}_2(\mathbf{X}_r^{t,\mathbf{x}'}, Q_r^{t,q'})\, dr\right|\Bigg]. \tag{4.6.25}$$

We fix a control $\pi \in \tilde{\mathcal{A}}_\gamma(t,\mathbf{x},q)\cap\tilde{\mathcal{A}}_\gamma(t,\mathbf{x}',q')$. We observe that $\tau \geq \tau'$ $\mathbb{P}$-a.s., since $Q_r^{t,\mathbf{x}} \geq Q_r^{t,q'}$ $\mathbb{P}$-a.s. for any $r \geq t$, by the assumption $q \geq q'$. Recall that

$$\forall \omega \in \{\tau' < T\},\ \forall r \geq \tau'(\omega), \quad Q_r^{t,q}(\omega) \leq q - q', \tag{4.6.26}$$

we get that

$$\begin{aligned}\mathbb{E}_t\left[|Q_\tau^{t,q} - Q_{\tau'}^{t,q'}|\right] &\leq \mathbb{E}_t\left[\mathbb{1}_{\tau<T,\tau'<T}\cdot 0 + \mathbb{1}_{\tau'<\tau=T} Q_T^{t,q} + \mathbb{1}_{\tau=\tau'=T}\left|Q_T^{t,q} - Q_T^{t,q'}\right|\right] \\ &= \mathbb{E}_t\left[\mathbb{1}_{\tau'<\tau=T}\right]|q-q'| + \mathbb{E}_t\left[\mathbb{1}_{\tau=\tau'=T}\right]|q-q'| \leq 2|q-q'|.\end{aligned} \tag{4.6.27}$$

Using uniformly boundedness of $\mathbb{E}_t\left[S_T^{t,\mathbf{x}}\right]$ with respect to t, obtained by (4.6.11), we get that there exists $C>0$ independent of t and of control c such that

$$\begin{aligned}\mathbb{E}_t\left[\left|S_\tau^{t,\mathbf{x}} Q_\tau^{t,q} - S_{\tau'}^{t,\mathbf{x}'} Q_{\tau'}^{t,q'}\right|\right] &\leq \mathbb{E}_t\Bigg[\mathbb{1}_{\{\tau'=\tau=T\}}\Bigg(S_T^{t,\mathbf{x}} \underbrace{\left|Q_T^{t,q} - Q_T^{t,q'}\right|}_{=|q-q'|} + \underbrace{Q_T^{t,q'}}_{\leq q'}\left|S_T^{t,\mathbf{x}} - S_T^{t,q'}\right|\Bigg)\Bigg] \\ &\quad + \mathbb{E}_t\Bigg[\mathbb{1}_{\{\tau'<T,\tau<T\}}\Bigg|S_\tau^{t,\mathbf{x}} \underbrace{Q_\tau^{t,q}}_{=0} - S_{\tau'}^{t,\mathbf{x}'} \underbrace{Q_{\tau'}^{t,\mathbf{x}'}}_{=0}\Bigg|\Bigg] + \mathbb{E}_t\Bigg[\mathbb{1}_{\{\tau'<T=\tau\}}\Bigg|S_T^{t,\mathbf{x}} \underbrace{Q_T^{t,q}}_{\leq q-q'} - S_{\tau'}^{t,\mathbf{x}'} \underbrace{Q_{\tau'}^{t,q'}}_{=0}\Bigg|\Bigg] \\ &\leq 2|q-q'|\mathbb{E}_t\left[S_T^{t,\mathbf{x}}\right] + \mathbb{E}_t\left[\left|S_T^{t,\mathbf{x}} - S_T^{t,\mathbf{x}'}\right|\right] q' \leq C|q-q'|ae^{|\epsilon|} + \mathbb{E}_t\left[\left|S_T^{t,\mathbf{x}} - S_T^{t,\mathbf{x}'}\right|\right] q'.\end{aligned} \tag{4.6.28}$$

Using (4.6.10) and merging (4.6.27) and (4.6.28) we get that there exists $C>0$ independent of t and of control c such that

$$\begin{aligned}&\mathbb{E}_t\left[\left|Q_\tau^{t,q}\left(S_\tau^{t,\mathbf{x}} - \chi Q_\tau^{t,q}\right) - Q_{\tau'}^{t,q'}\left(S_{\tau'}^{t,\mathbf{x}'} - \chi Q_{\tau'}^{t,q'}\right)\right|\right] \\ &\quad \leq \chi\mathbb{E}_t\left[\left|Q_\tau^{t,q} - Q_{\tau'}^{t',q}\right| \cdot \left|Q_\tau^{t,q} + Q_{\tau'}^{t',q}\right|\right] + \mathbb{E}_t\left[\left|S_\tau^{t,\mathbf{x}} Q_\tau^{t,q} - S_{\tau'}^{t,\mathbf{x}'} Q_{\tau'}^{t,q'}\right|\right]\end{aligned}$$

$$\leq 2\chi q|q-q'| + C|q-q'|ae^{|\epsilon|} + Cq'e^{|\epsilon'|}\left(|a-a'| + a\left|e^{\epsilon-\epsilon'}-1\right|\right). \tag{4.6.29}$$

Similarly to (4.6.28), we get that there exists $C > 0$ independent of t and of control c such that

$$\begin{aligned}
&\mathbb{E}_t\left[\left|\int_t^{\tau} S_r^{t,\mathbf{x}} Q_r^{t,q}\, dr - \int_t^{\tau'} S_r^{t,\mathbf{x}'} Q_r^{t,q'}\, dr\right|\right] \\
&\quad \leq \mathbb{E}_t\left[\int_t^{\tau'} \left|S_r^{t,\mathbf{x}} Q_r^{t,q} - S_r^{t,\mathbf{x}'} Q_r^{t,q'}\right|\, dr\right] + \mathbb{E}_t\left[\mathbb{1}_{\{\tau'=T\}}\cdot 0\right] + \mathbb{E}_t\left[\mathbb{1}_{\{\tau'<T\}}\int_{\tau'}^{\tau} S_r^{t,\mathbf{x}} \underbrace{Q_r^{t,q}}_{\leq q-q'}\, dr\right] \\
&\quad \leq C\left(\mathbb{E}_t\left[\sup_{r\in[t,T]}\left|S_r^{t,\mathbf{x}} - S_r^{t,\mathbf{x}'}\right|\right] q' + |q-q'|ae^{|\epsilon|}\right) \qquad (4.6.30)
\end{aligned}$$

and that

$$\mathbb{E}_t\left[\left|\int_t^{\tau} A_r^{t,\mathbf{x}} Q_r^{t,q}\, dr - \int_t^{\tau'} A_r^{t,\mathbf{x}'} Q_r^{t,q'}\, dr\right|\right] \leq C\left(\mathbb{E}_t\left[\sup_{r\in[t,T]}\left|A_r^{t,\mathbf{x}} - A_r^{t,\mathbf{x}'}\right|\right] q' + |q-q'|a\right). \tag{4.6.31}$$

Using boundedness of Q_s and (4.6.26), we get

$$\begin{aligned}
&\mathbb{E}_t\left[\left|\int_t^{\tau} \left(Q_r^{t,q}\right)^2\, dr - \int_t^{\tau'} \left(Q_r^{t,q'}\right)^2\, dr\right|\right] \leq 2q\mathbb{E}_t\left[\left|\int_t^{\tau'} \underbrace{\left(Q_r^{t,q} - Q_r^{t,q'}\right)}_{=q-q'}\, dr\right|\right] + \mathbb{E}_t\left[\int_{\tau'}^{\tau} \left(Q_r^{t,q}\right)^2\, dr\right] \\
&\quad \leq 2qT|q-q'| + \mathbb{E}_t\left[\mathbb{1}_{\{\tau'=T\}}\cdot 0\right] + T\mathbb{E}_t\left[\mathbb{1}_{\{\tau'<T\}}|q-q'|^2\right] \leq 2T\left(q|q-q'| + |q-q'|^2\right). \qquad (4.6.32)
\end{aligned}$$

Merging (4.6.30), (4.6.31) and (4.6.32) and applying (4.6.10) and (4.6.12), we conclude that there exists $C > 0$ independent of t and of control π such that

$$\begin{aligned}
&\mathbb{E}_t\left[\left|\int_t^{\tau} \mathcal{G}_2(\mathbf{X}_r^{t,\mathbf{x}}, Q_r^{t,q})\, dr - \int_t^{\tau'} \mathcal{G}_2(\mathbf{X}_r^{t,\mathbf{x}'}, Q_r^{t,q})\, dr\right|\right] \qquad (4.6.33) \\
&\quad \leq C\left(q|q-q'| + |q-q'|^2 + q'\left(e^{|\epsilon'|}+1\right)\left(|a-a'| + a\left|e^{\epsilon-\epsilon'}-1\right|\right) + |q-q'|a\left(e^{|\epsilon|}+1\right)\right).
\end{aligned}$$

Finally,

$$\begin{aligned}
\mathbb{E}_t\left[\left|M_{\tau}^{t,\mathbf{x}} - M_{\tau'}^{t,\mathbf{x}'}\right|\right] &\leq \mathbb{E}_t\left[\int_t^{\tau'} \left|\pi_r\left(S_r^{t,\mathbf{x}} - S_r^{t,\mathbf{x}'}\right)\right|\, dr\right] + \mathbb{E}_t\left[\int_{\tau'}^{\tau} \left|\pi_r\left(S_r^{t,\mathbf{x}} - \eta\pi_r\right)\right|\, dr\right] \\
&\leq \left(\mathbb{E}_t\left[\left(\int_t^{\tau'} \pi_r\, dr\right)^2\right]\right)^{\frac{1}{2}} \left(\mathbb{E}_t\left[\sup_{r\in[t,T]}\left|S_r^{t,\mathbf{x}} - S_r^{t,\mathbf{x}'}\right|^2\right]\right)^{\frac{1}{2}} \\
&\quad + \underbrace{\left(\mathbb{E}_t\left[\left(\int_{\tau'}^{\tau} \pi_r\, dr\right)^2\right]\right)^{\frac{1}{2}}}_{\leq|q-q'|} \left(\mathbb{E}_t\left[\sup_{r\in[t,T]}\left|S_r^{t,\mathbf{x}}\right|^2\right]\right)^{\frac{1}{2}} + \eta\mathbb{E}_t\left[\int_{\tau'}^{\tau} \pi_r^2\, dr\right].
\end{aligned}$$

By using Hölder's inequality with parameters $\frac{\gamma+1}{\gamma}$ and $1+\gamma$, we get

$$\mathbb{E}_t\left[\int_{\tau'}^{\tau}\pi_r^2\,dr\right] \leq \left(\mathbb{E}_t\left[\int_{\tau'}^{\tau}\left(\pi_r^{\frac{\gamma}{\gamma+1}}\right)^{\frac{\gamma+1}{\gamma}}dr\right]\right)^{\frac{\gamma}{\gamma+1}}\left(\mathbb{E}_t\left[\int_{\tau'}^{\tau}\left(\pi_r^{\frac{\gamma+2}{\gamma+1}}\right)^{1+\gamma}dr\right]\right)^{\frac{1}{1+\gamma}}$$
$$\leq \left(\underbrace{\mathbb{E}_t\left[\int_{\tau'}^{\tau}\pi_r\,dr\right]}_{\leq |q-q'|}\right)^{\frac{\gamma}{\gamma+1}}\left(\underbrace{\mathbb{E}_t\left[\int_{t}^{T}\pi_r^{\gamma+2}\,dr\right]}_{\leq N(1+a)^{2+\gamma}\left(1+e^{N(2+\gamma)\epsilon}\right)}\right)^{\frac{1}{1+\gamma}}.$$

Hence, using $L^{2+\gamma}$ boundedness of process $(\pi_r)_{r\in[t,T]}$, for any $(\pi_r)_{r\in[t,T]}\in\tilde{\mathcal{A}}_\gamma(t,\mathbf{x},q)\cap\tilde{\mathcal{A}}_\gamma(t,\mathbf{x}',q')$,

$$\mathbb{E}_t\left[\left|M_\tau^{t,\mathbf{x}}-M_{\tau'}^{t,\mathbf{x}'}\right|\right] \leq q'\left(\mathbb{E}_t\left[\sup_{r\in[t,T]}\left|S_r^{t,\mathbf{x}}-S_r^{t,\mathbf{x}'}\right|^2\right]\right)^{\frac{1}{2}} + |q-q'|\left(\mathbb{E}_t\left[\sup_{r\in[t,T]}\left(S_r^{t,\mathbf{x}}\right)^2\right]\right)^{\frac{1}{2}}$$
$$+\eta N^{\frac{1}{1+\gamma}}|q-q'|^{\frac{\gamma}{\gamma+1}}(1+a)^{\frac{2+\gamma}{1+\gamma}}\left(1+e^{N(2+\gamma)\epsilon}\right)^{\frac{1}{1+\gamma}}. \tag{4.6.34}$$

All previous inequality can also be obtained when $q\leq q'$. By merging inequalities (4.6.29), (4.6.33) and (4.6.34) into (4.6.25) and using arbitrariness of control c and (4.6.10), we have proved (4.6.25). □

Continuity of value function v is proved using Lemma 4.6.6. Let $(t',\mathbf{x}',q')\in[0,T]\times\mathcal{O}$ be fixed. We assume w.l.o.g. that $t\leq t'$. We observe that

$$|v(t,\mathbf{x},q)-v(t',\mathbf{x}',q')| \leq |v(t,\mathbf{x},q)-v(t,\mathbf{x}',q')| + |v(t,\mathbf{x}',q')-v(t',\mathbf{x}',q')|. \tag{4.6.35}$$

However, $|v(t,\mathbf{x},q)-v(t,\mathbf{x}',q')|\to 0$ uniformly on t for $\mathbf{x}\to\mathbf{x}'$ as stated in Lemma 4.6.6. If we apply Dynamic Programming Principle [48, Remark 3.3.3], we get that for any $\delta>0$ there exists $\pi\in\tilde{\mathcal{A}}_\gamma(t,\mathbf{x}',q')$ such that

$$|v(t,\mathbf{x}',q')-v(t',\mathbf{x}',q')|-\delta \leq \mathbb{E}_t\left[\left|\int_t^{t'}\mathcal{G}_2(\mathbf{X}_r^{t,\mathbf{x}'},Q_r^{t,q'})\,dr\right| + \left|v(t',\mathbf{X}_{t'}^{t,\mathbf{x}'},Q_r^{t,q})-v(t',\mathbf{x}',q')\right|\right].$$

Using boundedness of Q_r and (4.6.11), it is easy to show that there exists C such that

$$\mathbb{E}_t\left[\left|\int_t^{t'}\mathcal{G}_2(\mathbf{X}_r^{t,\mathbf{x}'},Q_r^{t,q'})\,dr\right|\right] \leq C|t-t'|^{\frac{1}{2}}.$$

By using Lemma 4.6.6, boundedness of Q_r, L^p-integrability of A_r and e^{ε_r} for any $p\geq 1$ and Hölder's inequality, we get that there exists $C>0$ independent of t and δ such that

$$|v(t,\mathbf{x}',q')-v(t',\mathbf{x}',q')| \leq \delta + C\Bigg(|t'-t|^{\frac{1}{2}} + \left(\mathbb{E}_t\left[|A_{t'}^{t,\mathbf{x}'}-a'|^2\right]\right)^{\frac{1}{2}} \tag{4.6.36}$$

$$+ \left(\mathbb{E}_t \left[\left| e^{\varepsilon^{t,\mathbf{x}'}_{t'} - \epsilon'} - 1 \right|^2 \right] \right)^{\frac{1}{2}} + \left(\mathbb{E}_t \left[|Q^{t,q'}_{t'} - q'|^{2\frac{\gamma}{\gamma+1}} \right] \right)^{\frac{1}{2}} \Bigg).$$

We observe that using Hölder's inequality with coefficients $2+\gamma$ and $\frac{2+\gamma}{1+\gamma}$

$$\begin{aligned}
\mathbb{E}_t \left[|Q^{t,q'}_{t'} - q'|^{\frac{2\gamma}{\gamma+1}} \right] &\le \mathbb{E}_t \left[\left(\int_t^{t'} \pi_r \, dr \right)^{\frac{2\gamma}{\gamma+1}} \right] \le |t-t'|^{\frac{2\gamma}{2+\gamma}} \mathbb{E}_t \left[\left(\int_0^T \pi_r^{2+\gamma} \, dr \right)^{\frac{2\gamma}{(2+\gamma)(1+\gamma)}} \right] \\
&\le |t-t'|^{\frac{2\gamma}{2+\gamma}} \left(\mathbb{E}_t \left[\int_0^T \pi_r^{2+\gamma} \, dr \right] \right)^{\frac{2\gamma}{(2+\gamma)(1+\gamma)}} . \qquad (4.6.37)
\end{aligned}$$

Here in the last inequality we used Jensen's inequality for $\frac{2\gamma}{(2+\gamma)(1+\gamma)} \le 1$. Recalling that $\pi \in \tilde{\mathcal{A}}_\gamma(t, \mathbf{x}', q')$ and applying (4.6.5), (4.6.9), (4.6.11) and (4.6.37) to (4.6.36), we get that, uniformly on $\pi \in \tilde{\mathcal{A}}_\gamma(t, \mathbf{x}', q')$

$$\lim_{t \to (t')^-} |v(t, \mathbf{x}', q') - v(t', \mathbf{x}', q')| \le \delta.$$

From arbitrariness of δ we conclude that previous limit converges to 0. Continuity of v follows from (4.6.35), by sending $(t, \mathbf{x}, q) \to (t', \mathbf{x}', q')$. The same results can be obtained when $t \ge t'$.

4.6.4 Proof of Proposition 4.3.1

Define for any $(t,\epsilon) \in [0,T] \times \mathbb{R}$, $g^*(t,\epsilon) := e^\epsilon (1 - g_2(t,\epsilon))$. Condition (4.3.9) is equivalent to proving that $g^*(t,\epsilon) \ge 0$ for any $(t,\epsilon) \in [0,T] \times \mathbb{R}$. A simple calculus on second PDE in (4.3.6) shows that function g^* satisfies the following PDE:

$$\frac{\partial g^*}{\partial t} + \frac{\sigma_2^2}{2} \frac{\partial^2 g^*}{\partial \epsilon^2} + (\rho\sigma_1\sigma_2 - k\epsilon) \frac{\partial g^*}{\partial \epsilon} + \left(\frac{g_3(t)}{\eta} + \mu_1 \right) g^* + e^\epsilon \left(k\epsilon - \frac{\sigma_2^2}{2} - \rho\sigma_1\sigma_2 - \mu_1 + \phi_2 \right) + \phi_3 = 0$$

on $[0,T) \times \mathbb{R}$, with terminal condition $g^*(T,\epsilon) = e^\epsilon(1 - g_2(T,\epsilon)) = 0$.

We check that conditions of Feynman-Kac Theorem are fulfilled for function g^*. As we have proved in (4.3.7), g_3 is twice differentiable and bounded from above and function $e^\epsilon \left(k\epsilon - \frac{\sigma_2^2}{2} - \rho\sigma_1\sigma_2 - \mu_1 + \phi_2 \right)$ is linearly exponential on variable ϵ. Hence, we get the following Feynmann-Kac representation for g^*:

$$g^*(t,\epsilon) = \mathbb{E}_t \left[\int_t^T \exp\left(\int_t^r \left(\frac{g_3(s)}{\eta} + \mu_1 \right) \, ds \right) \left(e^{\tilde{\varepsilon}_r} \left(k\tilde{\varepsilon}_r - \frac{\sigma_2^2}{2} - \rho\sigma_1\sigma_2 - \mu_1 + \phi_2 \right) + \phi_3 \right) \, dr \right], \tag{4.6.38}$$

where $\tilde{\varepsilon}_r$ is the solution to the following SDE:

$$d\tilde{\varepsilon}_r = (\rho\sigma_1\sigma_2 - k\tilde{\varepsilon}_r)dr + \sigma_2 dW_r, \quad \tilde{\varepsilon}_t = \epsilon.$$

$(\tilde{\varepsilon}_r)_{0\le r\le T}$ is an OU process and for any fixed $r \in [0,T]$, $\tilde{\varepsilon}_r$ is a normal distributed random variable with first two moments equal to

$$\mathbb{E}_t\left[\tilde{\varepsilon}_r\right] = \epsilon e^{-k(r-t)} + \frac{\rho}{k}\sigma_1\sigma_2(1-e^{-k(r-t)}) = \epsilon\bar{\mu}(r-t) + \frac{\rho}{k}\sigma_1\sigma_2\left(1-\bar{\mu}(r-t)\right),$$
$$\mathrm{Var}_t\left(\tilde{\varepsilon}_r\right) = \frac{\sigma_2^2}{2k}\left(1-e^{-2k(r-t)}\right) = \bar{\sigma}(r-t)^2.$$

Calculus on normally and log-normally distributed random variables gives

$$\mathbb{E}_t\left[e^{\tilde{\varepsilon}_r}\left(k\tilde{\varepsilon}_r - \frac{\sigma_2^2}{2} - \rho\sigma_1\sigma_2 - \mu_1 + \phi_2\right)\right] \tag{4.6.39}$$
$$= \left(k\bar{\mu}(r-t)\epsilon + k\bar{\sigma}(r-t)^2 - \frac{\sigma_2^2}{2} - \rho\sigma_1\sigma_2\bar{\mu}(r-t) - \mu_1 + \phi_2\right)e^{\bar{\mu}(r-t)\epsilon + \frac{\bar{\sigma}(r-t)^2}{2} + \frac{\rho}{k}\sigma_1\sigma_2(1-\bar{\mu}(r-t))}.$$

By applying (4.6.39) to (4.6.38) we get result in (4.3.11).

We now prove that integral in (4.3.11) is non-negative. From definition of $\bar{\mu}(r)$ and $\bar{\sigma}(r)^2$ in (4.3.12) we have

$$\frac{\partial\bar{\mu}(r)}{\partial r} = -k\bar{\mu}(r), \quad \frac{\partial\bar{\sigma}(r)^2}{\partial r} = -2k\bar{\sigma}(r)^2 + \sigma_2^2.$$

g^* can be written as

$$g^*(t,\epsilon) = \int_t^T \hat{g}(r;t) f(r,\epsilon;t)\, dr, \tag{4.6.40}$$

where f is defined as

$$f(r,\epsilon;t) := e^{\bar{\mu}(r-t)\epsilon + \frac{\bar{\sigma}(r-t)^2}{2} + \frac{\rho}{k}\sigma_1\sigma_2(1-\bar{\mu}(r-t))}\cdot$$
$$\cdot\left(k\bar{\mu}(r-t)\epsilon + k\bar{\sigma}(r-t)^2 - \frac{\sigma_2^2}{2} - \rho\sigma_1\sigma_2\bar{\mu}(r-t) - \mu_1 + \phi_2\right) + \phi_3.$$

To prove that $g^*(t,\epsilon) \ge 0$ for any $(t,\epsilon) \in [0,T]\times\mathbb{R}$, we show that, under condition (4.3.10), f is non-negative. Observing that

$$\partial_\epsilon f(r,\epsilon;t) = e^{\bar{\mu}(r-t)\epsilon + \frac{\bar{\sigma}(r-t)^2}{2} + \frac{\rho}{k}\sigma_1\sigma_2(1-\bar{\mu}(r-t))}\bar{\mu}(r-t)\cdot$$
$$\cdot\left(k\bar{\mu}(r-t)\epsilon + k\bar{\sigma}(r-t)^2 - \frac{\sigma_2^2}{2} - \rho\sigma_1\sigma_2\bar{\mu}(r-t) - \mu_1 + \phi_2 + k\right),$$

we get that for any $0 \le t \le r \le T$, the minimum point $\epsilon^*(r;t)$ of $f(r,\epsilon;t)$ satisfies the following equation

$$\bar{\mu}(r-t)\epsilon^*(r;t) + \bar{\sigma}(r-t)^2 - \frac{\sigma_2^2}{2k} - \frac{\rho}{k}\sigma_1\sigma_2\bar{\mu}(r-t) - \frac{\mu_1}{k} + \frac{\phi_2}{k} + 1 = 0.$$

By evaluating $f(r,\epsilon;t)$ in $\epsilon^*(r;t)$,

$$f(r,\epsilon^*(r;t);t) = -ke^{-\frac{\bar{\sigma}(r-t)^2}{2} + \frac{\sigma_2^2}{2k} - 1 + \frac{\mu_1}{k} + \frac{\rho}{k}\sigma_1\sigma_2 - \frac{\phi_2}{k}} + \phi_3.$$

If condition (4.3.10) is satisfied, then $f(r,\epsilon^*(r;t);t) \ge 0$ and from (4.6.40) we conclude that g^* is non-negative. □

4.6.5 Proof of Proposition 4.3.3

Using boundedness of Q_r and g_3, linearity of g_2 in a and linear exponential growth of g_2 with respect to ϵ, we conclude there exist $C > 0$ such that

$$|c(r, \mathbf{X}_r, Q_r)| \leq C(1+q)(1+A_r)(1+e^{C\varepsilon_r}). \tag{4.6.41}$$

Applying Hölder's inequality, we get there exists $C_1 > 0$ such that

$$\begin{aligned}\mathbb{E}_t\left[\int_0^T |c(r, \mathbf{X}_r, Q_r)|^{2+\gamma} dr\right] \leq C_1 (1+q)^{2+\gamma} &\left(1 + \mathbb{E}_t\left[\sup_{r\in[0,T]} A_r^{2(2+\gamma)}\right]\right)^{\frac{1}{2}} \cdot \\ &\cdot \left(1 + \mathbb{E}_t\left[\sup_{r\in[0,T]} e^{2C(2+\gamma)\varepsilon_r}\right]\right)^{\frac{1}{2}}.\end{aligned}$$

Using (4.6.11), we conclude that there exists $C_2 > 0$, independent of t and $\mathbf{x}$, such that

$$\left(\mathbb{E}_t\left[\int_t^T |c(r, \mathbf{X}_r, Q_r)|^{2+\gamma} dr\right]\right)^{\frac{1}{2+\gamma}} \leq C_2 (1+a)\left(1 + e^{C_2|\epsilon|}\right),$$

which implies $(c(r, \mathbf{X}_r, Q_r))_{r\in[t,T]} \in \tilde{\mathcal{A}}_{\gamma,C_2}(t, \mathbf{x}, q)$. □

Chapter 5

Conclusions and Further research

The main contribution of this PhD thesis is to prove that the standard Stochastic Maximum Principle cannot be applied to the control-dependent terminal time setting. In Section 2.3, 2.4 and 2.5 we found three counterexamples for the standard formulation of the Stochastic Maximum Principle. In Theorem 2.2.3 we stated a new version of the SMP that generalizes the standard formulation and can be applied to any optimal liquidation problem with a terminal stopping time and in particular to the examples in Sections 2.3 and 2.4.

As regards future research, there are many different implementations that can be done in order to improve our piece of research. We list them in the following.

1. Extend the state process $\mathbf{X}$ to more general definitions, e.g. considering a jump diffusion control-dependent model.

2. Theorem 2.2.3 gives a necessary condition for the optimality. We would like to find a sufficient condition as well. This is not straightforward, as the functional $\mathcal{G}$ is not convex with respect to the control π, as in the case of $\mathcal{F}$ and as we showed in Section 3.5.

3. We can extend the stopping time with a more general definition such as

$$\tau = T \wedge \inf \left\{ r \geq t \mid \mathcal{L}\left(r, (\pi_s)_{s\in[0,T]}, (\mathbf{X}_s)_{s\in[0,T]}, (Q_s)_{s\in[0,T]}\right) = 0 \right\}, \tag{5.0.1}$$

   where $\mathcal{L}$ is a general functional. In our case $\mathcal{L}\left(r, (\pi_s)_{s\in[0,T]}, (X_s)_{s\in[0,T]}, (Q_s)_{s\in[0,T]}\right) = Q_t - \int_t^r \pi_s \, ds$.

4. Find other applications for the control-dependent terminal time setting we introduced in this thesis. Many path optimizations in engineering and robotic belong to this framework. Indeed, whenever an agent is required to maximize a process that depends on a path between point A and point B, it is necessary the introduction of a stopping time τ corresponding to the instant in which the point B is reached.

Appendix A

Proofs of statements in Example Sections in Chapter 2

In the appendix we insert some proofs of theorems that are particularly tedious and don't bring much insights in the main discussion of Examples Sections in Chapter 2.

A.1 Proof of Proposition 2.3.1

To prove (2.3.3), we use (2.3.2) together with definition of process Q_r in (2.1.1). When $q \geq \frac{(T-t)^2}{4k}$, we get that for any $r \in [t,T]$

$$Q_r = q - \int_t^r c_s\, ds = q - \int_t^r \frac{T-s}{2k}\, ds = q + \left[\frac{(T-s)^2}{4k}\right]_t^r = q + \frac{(T-r)^2 - (T-t)^2}{4k}.$$

When $q < \frac{(T-t)^2}{4k}$, we get that for any $r \in [t,T]$

$$Q_r = q - \int_t^r c_s\, ds = q - \int_t^r \frac{t-s+2\sqrt{kq}}{2k}\, ds = q + \left[\frac{(t-s+2\sqrt{kq})^2}{4k}\right]_t^r = \frac{(t-r+2\sqrt{kq})^2}{4k}.$$

This concludes the proof of (2.3.3). We now prove (2.3.4) and (2.3.5). From (2.3.3), when $q \geq \frac{(T-t)^2}{4k}$, we get that

$$Q_r = q + \frac{(T-r)^2 - (T-t)^2}{4k} \geq \frac{(T-r)^2}{4k}.$$

Therefore, if $q \geq \frac{(T-t)^2}{4k}$, then it follows that $Q_r \geq \frac{(T-r)^2}{4k}$ for any $r \in [t,T]$. We also conclude that for $q > \frac{(T-t)^2}{4k}$, Q_r is strictly positive for any $r \in [t,T]$, meaning that $\tau = T$. On the other hand, when $q = \frac{(T-t)^2}{4k}$, then $Q_r = \frac{(T-r)^2}{4k}$ implying that $\tau = T$. If $q < \frac{(T-t)^2}{4k}$, from (2.3.3) we get that $Q_r = 0$ whenever $t - r + 2\sqrt{kq} = 0$, implying that $\tau = t + 2\sqrt{kq}$. We also observe that if $q < \frac{(T-t)^2}{4k}$, then for any $r \in [t, T]$

$$Q_r = \frac{(t-r+2\sqrt{kq})^2}{4k} < \frac{\left(t-r+2\sqrt{k\frac{(T-t)^2}{4k}}\right)^2}{4k} = \frac{(T-r)^2}{4k}.$$

Merging last result together with the fact that when $q \geq \frac{(T-t)^2}{4k}$, for any $r \in [t, T]$, $Q_r \geq \frac{(T-r)^2}{4k}$, we conclude that (2.3.4) holds true.

We now prove (2.3.6). If $q \geq \frac{(T-t)^2}{4k}$, then, using the fact that $\tau = T$, we have that the value function associated is

$$\begin{aligned} v^c(t,q) &= \int_t^\tau c_r\left(T-r-kc_r\right)\,dr = \int_t^\tau \frac{T-r}{2k}\left(T-r-\frac{T-r}{2}\right)\,dr = \int_t^T \frac{(T-r)^2}{4k}\,dr \\ &= -\left[\frac{(T-r)^3}{12k}\right]_t^T = \frac{(T-t)^3}{12k}. \end{aligned}$$

If $q < \frac{(T-t)^2}{4k}$, then, using expression (2.3.3) and the fact that $\tau = t + 2\sqrt{kq}$, we have

$$\begin{aligned} v^c(t,q) &= \int_t^\tau c_r\left(T-r-kc_r\right)\,dr \\ &= \int_t^\tau \frac{t-r+2\sqrt{kq}}{2k}\left(T-r-\frac{t-r+2\sqrt{kq}}{2}\right)\,dr \\ &= \frac{1}{4k}\int_t^\tau \left(t-r+2\sqrt{kq}\right)\left(2T-r-t-2\sqrt{kq}\right)\,dr \\ &= \frac{1}{4k}\left[\int_t^\tau 2T\left(t-r+2\sqrt{kq}\right)\,dr + \int_t^\tau \left(r^2-(t+2\sqrt{kq})^2\right)\,dr\right] \\ &= \frac{1}{4k}\left[-T\left(t-r+2\sqrt{kq}\right)^2+\frac{r^3}{3}-r\left(t+2\sqrt{kq}\right)^2\right]_t^{t+2\sqrt{kq}} \\ &= \frac{1}{4k}\left[\frac{\left(t+2\sqrt{kq}\right)^3}{3}-\left(t+2\sqrt{kq}\right)^3+4Tkq-\frac{t^3}{3}+t\left(t+2\sqrt{kq}\right)^2\right] \\ &= \frac{1}{4k}\left[\left(t+2\sqrt{kq}\right)^2\left(-\frac{2}{3}\left(t+2\sqrt{kq}\right)+t\right)+4Tkq-\frac{t^3}{3}\right] \\ &= \frac{1}{4k}\left[\left(t^2+4t\sqrt{kq}+4kq\right)\left(\frac{t}{3}-\frac{4}{3}\sqrt{kq}\right)+4Tkq-\frac{t^3}{3}\right] \\ &= q\left(T-t-\frac{4}{3}\sqrt{kq}\right). \end{aligned}$$

A.2 Proof of Proposition 2.3.4

This is a deterministic example, then the definition of function $\bar{f}$ in (2.2.17) becomes

$$\bar{f}(t,\bar{c},q) = \operatorname{sign}(\tau-\tau^{\theta,\bar{c},t})\lim_{\theta\to 0}\frac{1}{\theta}\int_{\tau_{\min}^{\theta,\bar{c},t}}^{\tau_{\max}^{\theta,\bar{c},t}} f\left(r,\hat{c}_r^{\theta,\bar{c},t}\right)\,dr. \tag{A.2.1}$$

We firstly consider the case $q \geq \frac{(T-t)^2}{4k}$, which implies $\tau = T$. We consider any $\theta \in (0, T-t)$, so that $\tau = T > t + \theta$. By combining (2.2.7) and (2.2.9) and using the fact that c_r is as in

(2.3.2), we have for any $r \le \tau$

$$\begin{aligned} Q_r^{\theta,\bar{c},t} &= q - \bar{c}\theta - \int_{t+\theta}^{r} c_s ds = q - \bar{c}\theta - \int_{t+\theta}^{r} \frac{T-s}{2k} ds \\ &= q - \bar{c}\theta + \left[\frac{(T-s)^2}{4k}\right]_{t+\theta}^{r} = q - \bar{c}\theta - \frac{(T-t+\theta)^2}{4k} + \frac{(T-r)^2}{4k}. \end{aligned} \tag{A.2.2}$$

Hence,

$$\begin{aligned} Q_\tau^{\theta,\bar{c},t} &= q - \bar{c}\theta - \frac{(T-t+\theta)^2}{4k} \\ &= q + \frac{(T-t)^2}{4k} - \frac{(T-t)^2}{4k} - \bar{c}\theta - \frac{(T-t)^2 - 2\theta(T-t) + \theta^2}{4k} \\ &= q - \frac{(T-t)^2}{4k} + \theta\left(\frac{T-t}{2k} - \bar{c}\right) - \frac{\theta^2}{4k}. \end{aligned} \tag{A.2.3}$$

If $q > \frac{(T-t)^2}{4k}$, then $\exists \bar{\theta}_1 \in (0, T-t)$ such that $Q_\tau^{\theta,\bar{c},t} > 0$, $\forall \theta \in (0, \bar{\theta}_1)$ and so $\tau^{\theta,\bar{c},t} = T = \tau$. If $q = \frac{(T-t)^2}{4k}$ and $\bar{c} < \frac{T-t}{2k}$, then $\exists \bar{\theta}_2 \in (0, T-t)$ such that $Q_\tau^{\theta,\bar{c},t} > 0$, $\forall \theta \in (0, \bar{\theta}_2)$ and so $\tau^{\theta,\bar{c},t} = T = \tau$. If $q = \frac{(T-t)^2}{4k}$ and $\bar{c} \ge \frac{T-t}{2k}$, then $Q_\tau^{\theta,\bar{c},t} < 0$ and so $\tau^{\theta,\bar{c},t} < \tau$. Then, in this last case, from (A.2.2) we have

$$\begin{aligned} Q_r^{\theta,\bar{c},t} &= \frac{(T-t)^2}{4k} - \bar{c}\theta - \frac{(T-t)^2 - 2\theta(T-t) + \theta^2}{4k} + \frac{(T-r)^2}{4k} \\ &= -\frac{1}{4k}\left(\theta(4k\bar{c} - 2(T-t) + \theta) - (T-r)^2\right), \end{aligned}$$

so $\tau^{\theta,\bar{c},t} = T - \sqrt{\theta(4k\bar{c} - 2(T-t) + \theta)}$. By recalling that for $q \ge \frac{(T-t)^2}{4k}$ then $c_t = \frac{T-t}{2k}$, we conclude that if $q > \frac{(T-t)^2}{4k}$ or if $q = \frac{(T-t)^2}{4k}$ and $\bar{c} < c_t$ then $\tau_{\max}^{\theta,\bar{c},t} = \tau_{\min}^{\theta,\bar{c},t} = T$, $\forall \theta \in (0, \bar{\theta}_1 \wedge \bar{\theta}_2)$ and so from (A.2.1), $\bar{f} = 0$. On the other hand if $q = \frac{(T-t)^2}{4k}$ and $\bar{c} \ge c_t$ then $\tau_{\max}^{\theta,\bar{c},t} = T$ and $\tau_{\min}^{\theta,\bar{c},t} = T - \sqrt{\theta(4k\bar{c} - 2(T-t) + \theta)}$ and so from (A.2.1), recalling that $\hat{c}_r^{\theta,\bar{c},t} = c_r^{\theta,\bar{c},t} = c_r$ for $r \in [\tau_{\min}^{\theta,\bar{c},t}, \tau_{\max}^{\theta,\bar{c},t}]$

$$\begin{aligned} \bar{f}(t,\bar{c},q) &= \lim_{\theta\to 0} \frac{1}{\theta} \int_{T-\sqrt{\theta(4k\bar{c}-2(T-t)+\theta)}}^{T} c_r(T - r - kc_r)\, dr \\ &= \lim_{\theta\to 0} \frac{1}{\theta} \int_{T-\sqrt{\theta(4k\bar{c}-2(T-t)+\theta)}}^{T} \frac{(T-r)^2}{4k}\, dr \\ &= \lim_{\theta\to 0} -\frac{1}{\theta}\left[\frac{(T-r)^3}{12k}\right]_{T-\sqrt{\theta(4k\bar{c}-2(T-t)+\theta)}}^{T} \\ &= \lim_{\theta\to 0} \frac{\left(\theta(4k\bar{c} - 2(T-t) + \theta)\right)^{3/2}}{12k\theta} \\ &= \lim_{\theta\to 0} \sqrt{\theta}\frac{(4k\bar{c} - 2(T-t) + \theta)^{3/2}}{12k} = 0. \end{aligned}$$

Therefore when $q \ge \frac{(T-t)^2}{4k}$, $\bar{f} = 0$.

We now analyse the case when $q < \frac{(T-t)^2}{4k}$. For $\theta \in (0, 2\sqrt{kq})$, we have that $\tau = t + 2\sqrt{kq} > t + \theta$. Then by combining (2.2.7) and (2.2.9) and using the fact that c_r is as in (2.3.2), we have for any $r \in [t+\theta, \tau]$

$$
\begin{aligned}
Q_r^{\theta,\bar{c},t} &= q - \bar{c}\theta - \int_{t+\theta}^{r} c_s ds = q - \bar{c}\theta - \int_{t+\theta}^{r} \frac{t-s+2\sqrt{kq}}{2k} ds \\
&= q - \bar{c}\theta + \left[\frac{(t-s+2\sqrt{kq})^2}{4k}\right]_{t+\theta}^{r} = q - \bar{c}\theta - \frac{(2\sqrt{kq}-\theta)^2}{4k} + \frac{(t-r+2\sqrt{kq})^2}{4k} \\
&= q - \bar{c}\theta - \frac{4kq}{4k} + \frac{4\sqrt{kq}\theta}{4k} - \frac{\theta^2}{4k} + \frac{(t-r+2\sqrt{kq})^2}{4k} = -\theta\left(\bar{c} - \sqrt{\frac{q}{k}} + \frac{\theta}{4k}\right) + \frac{(\tau-r)^2}{4k}.
\end{aligned} \tag{A.2.4}
$$

If $\bar{c} \geq \sqrt{\frac{q}{k}}$, then by setting (A.2.4) equal to 0, we have that $\tau^{\theta,\bar{c},t} = \tau - \sqrt{\theta\left(4k\bar{c} - 4\sqrt{kq} + \theta\right)}$. By recalling that for $q < \frac{(T-t)^2}{4k}$ then $c_t = \sqrt{\frac{q}{k}}$, we conclude that if $q < \frac{(T-t)^2}{4k}$ and $\bar{c} \geq c_t$ then $\tau_{\max}^{\theta,\bar{c},t} = \tau$ and $\tau_{\min}^{\theta,\bar{c},t} = \tau - \sqrt{\theta\left(4k\bar{c} - 4\sqrt{kq} + \theta\right)}$, $\forall \theta \in (0, 2\sqrt{kq})$ and so from (A.2.1), recalling that $\hat{c}_r^{\theta,\bar{c},t} = c_r^{\theta,\bar{c},t} = c_r$ for $r \in [\tau_{\min}^{\theta,\bar{c},t}, \tau_{\max}^{\theta,\bar{c},t}]$ and using the change of variable $s = r - \tau$

$$
\begin{aligned}
\bar{f}(t,\bar{c},q) &= \lim_{\theta\to 0} \frac{1}{\theta} \int_{\tau-\sqrt{\theta\left(4k\bar{c}-4\sqrt{kq}+\theta\right)}}^{\tau} \hat{c}_r^{\theta,\bar{c},t}(T - r - k\hat{c}_r^{\theta,\bar{c},t})\, dr \\
&= \lim_{\theta\to 0} \frac{1}{\theta} \int_{\tau-\sqrt{\theta\left(4k\bar{c}-4\sqrt{kq}+\theta\right)}}^{\tau} c_r(T - r - kc_r)\, dr \\
&= \lim_{\theta\to 0} \frac{1}{\theta} \int_{\tau-\sqrt{\theta\left(4k\bar{c}-4\sqrt{kq}+\theta\right)}}^{\tau} \frac{\tau-r}{2k}\left(T - r - \frac{\tau-r}{2}\right) dr \\
&= \lim_{\theta\to 0} \frac{1}{\theta} \int_{-\sqrt{\theta\left(4k\bar{c}-4\sqrt{kq}+\theta\right)}}^{0} -\frac{s}{2k}\left(T - \frac{s}{2} - \tau\right) ds \\
&= \lim_{\theta\to 0} \frac{1}{\theta} \left(-\left[\frac{s^2}{4k}(T-\tau)\right]_{-\sqrt{\theta\left(4k\bar{c}-4\sqrt{kq}+\theta\right)}}^{0} + \left[\frac{s^3}{12k}\right]_{-\sqrt{\theta\left(4k\bar{c}-4\sqrt{kq}+\theta\right)}}^{0}\right) \\
&= \lim_{\theta\to 0} \frac{\theta\left(4k\bar{c} - 4\sqrt{kq} + \theta\right)}{4k\theta}(T-\tau) + \lim_{\theta\to 0} \frac{\left(\theta\left(4k\bar{c} - 4\sqrt{kq} + \theta\right)\right)^{3/2}}{12k\theta} \\
&= (T-\tau)\left(\bar{c} - \sqrt{\frac{q}{k}}\right) = (T-\tau)(\bar{c} - c_t).
\end{aligned}
$$

If $\bar{c} < \sqrt{\frac{q}{k}}$, from (A.2.4) we have that $Q_\tau^{\theta,\bar{c},t} = -\theta\left(\bar{c} - \sqrt{\frac{q}{k}} + \frac{\theta}{4k}\right)$ and so $\exists \bar{\theta}_3 \in (0, 2\sqrt{kq})$ such that $Q_\tau^{\theta,\bar{c},t} > 0$, $\forall \theta \in (0, \bar{\theta}_3)$, that implies $\tau^{\theta,\bar{c},t} > \tau$ for any $\theta \in (0, \bar{\theta}_3)$. Therefore, by combining

(2.2.7) and (2.2.9) and using the fact that c_r is as in (2.3.2), we have for any $r \in (\tau, T]$

$$
\begin{aligned}
Q_r^{\theta,\bar{c},t} &= q - \bar{c}\theta - \int_{t+\theta}^{\tau} c_s ds + \frac{r-\tau}{\theta} \int_t^{t+\theta} (\bar{c} - c_s) ds \\
&= q - \int_t^{t+\theta} c_s ds - \int_{t+\theta}^{\tau} c_s ds - \bar{c}\theta + (r-\tau)\bar{c} - \frac{r-\tau-\theta}{\theta} \int_t^{t+\theta} c_s ds \\
&= Q_\tau - \bar{c}\,(\theta + \tau - r) + \frac{\tau+\theta-r}{\theta} \int_t^{t+\theta} \frac{t-s+2\sqrt{kq}}{2k} ds \\
&= 0 - \bar{c}\,(\theta+\tau-r) - \frac{\tau+\theta-r}{\theta} \left[\frac{(t-s+2\sqrt{kq})^2}{4k} \right]_t^{t+\theta} \\
&= -\bar{c}\,(\theta+\tau-r) + \frac{\tau+\theta-r}{\theta} \left(q - \frac{(2\sqrt{kq}-\theta)^2}{4k} \right) \\
&= (\theta+\tau-r) \left(\frac{1}{\theta} \left(q - \frac{4kq - 4\theta\sqrt{kq} + \theta^2}{4k} \right) - \bar{c} \right) \\
&= (\theta+\tau-r) \left(\sqrt{\frac{q}{k}} - \frac{\theta}{4k} - \bar{c} \right).
\end{aligned}
$$

Since $\bar{c} < \sqrt{\frac{q}{k}}$, then $\exists \bar{\theta}_4 \in (0, \bar{\theta}_3)$ such that $\sqrt{\frac{q}{k}} - \frac{\theta}{4k} - \bar{c} > 0$, $\forall \theta \in (0, \bar{\theta}_4)$ and so $\tau^{\theta,\bar{c},t} = (\tau+\theta) \wedge T$. Using the fact that for $q < \frac{(T-t)^2}{4k}$ and $\tau = t + 2\sqrt{kq}$, we conclude that $\exists \bar{\theta}_5 \in (0, \bar{\theta}_4)$ such that $\tau + \theta < T$ and so $\tau^{\theta,\bar{c},t} = \tau + \theta$, $\forall \theta \in (0, \bar{\theta}_5)$. By recalling that for $q < \frac{(T-t)^2}{4k}$ then $c_t = \sqrt{\frac{q}{k}}$, we conclude that if $q < \frac{(T-t)^2}{4k}$ and $\bar{c} < c_t$ then $\tau_{\min}^{\theta,\bar{c},t} = \tau$ and $\tau_{\max}^{\theta,\bar{c},t} = \tau + \theta$, $\forall \theta \in (0, \bar{\theta}_5)$. Moreover, for any $r \in (\tau, T]$

$$
\hat{c}_r^{\theta,\bar{c},t} = -\frac{\gamma_{t+\theta}^{\theta,\bar{c},t}}{\theta} = -\left(\bar{c} - \frac{1}{\theta} \int_t^{t+\theta} c_r dr \right) = -\bar{c} + \frac{q}{\theta} - \frac{(2\sqrt{kq}-\theta)^2}{4k\theta} = -\bar{c} + \sqrt{\frac{q}{k}} - \frac{\theta}{4k} = -\left(\bar{c} - c_t + \frac{\theta}{4k} \right).
$$

Hence, from (A.2.1),

$$
\begin{aligned}
\bar{f}(t, \bar{c}, q) &= -\lim_{\theta \to 0} \frac{1}{\theta} \int_\tau^{\tau+\theta} \hat{c}_r^{\theta,\bar{c},t} (T - r - k\hat{c}_r^{\theta,\bar{c},t})\, dr \\
&= -\lim_{\theta \to 0} \frac{1}{\theta} \int_\tau^{\tau+\theta} -\left(\bar{c} - c_t + \frac{\theta}{4k} \right) \left(T - r + k\left(\bar{c} - c_t + \frac{\theta}{4k} \right) \right) dr \\
&= \lim_{\theta \to 0} \frac{\bar{c} - c_t + \frac{\theta}{4k}}{\theta} \left(\left(T + k\left(\bar{c} - c_t + \frac{\theta}{4k} \right) \right) \theta - \left[\frac{r^2}{2} \right]_\tau^{\tau+\theta} \right) \\
&= \lim_{\theta \to 0} \left(\bar{c} - c_t + \frac{\theta}{4k} \right) \left(T + k\left(\bar{c} - c_t + \frac{\theta}{4k} \right) + \frac{\tau^2 - (\tau+\theta)^2}{2\theta} \right) \\
&= \lim_{\theta \to 0} \left(\bar{c} - c_t + \frac{\theta}{4k} \right) \left(T + k\left(\bar{c} - c_t + \frac{\theta}{4k} \right) - \tau - \frac{\theta}{2} \right) \\
&= (T - \tau + k(\bar{c} - c_t))(\bar{c} - c_t).
\end{aligned}
$$

Therefore, the result follows.

A.3 Proof of Proposition 2.4.1

In the cases when $q > c^+(T-t)$, from (2.4.3) we have that $c_r = c^+$ and so

$$Q_r = q - \int_t^r c_s \, ds = q - \int_t^r c^+ \, ds = q - (r-t)c^+.$$

When $q \le c^+(T-t)$, from (2.4.3) we get that $c_r = \frac{q}{T-t}$ and so

$$Q_r = q - \int_t^r c_s \, ds = q - \int_t^r \frac{q}{T-t} \, ds = q - \frac{q(r-t)}{T-t} = \frac{q(T-r)}{T-t}.$$

Hence, we proved (2.4.4). We now prove (2.4.5). When $q > c^+(T-t)$, $Q_r = q-(r-t)c^+ > c^+(T-t) - (r-t)c^+ = (T-r)c^+$. On the other hand, if $q \le c^+(T-t)$, $Q_r = \frac{q(T-r)}{T-t} \le \frac{c^+(T-t)(T-r)}{T-t} = (T-r)c^+$. This concludes the proof of (2.4.5).

We now prove (2.4.6). When $q > c^+(T-t)$, $Q_r = q-(r-t)c^+ > c^+(T-t)-(r-t)c^+ = (T-r)c^+ \ge 0$ and so Q_r is strictly positive for any $r \in [t,T]$, making $\tau = T$. On the other hand, if $q \le c^+(T-t)$, $Q_r = \frac{q(T-r)}{T-t}$, which is equal to 0 only if $r = T$. This concludes the proof of (2.4.6).

Now we prove (2.4.7). When $q > c^+(T-t)$, by using that $c_r = c^+$ and the fact that $\tau = T$

$$v^c(t,x,q) = \mathbb{E}^t\left[\int_t^\tau c_r X_r \, dr\right] = \int_t^T c^+ \mathbb{E}^t\left[X_r\right] \, dr = (T-t)c^+ x.$$

When $q \le c^+(T-t)$, by using that $c_r = \frac{q}{T-t}$ and the fact that $\tau = T$, we have

$$v^c(t,x,q) = \mathbb{E}^t\left[\int_t^\tau c_r X_r \, dr\right] = \int_t^T \frac{q}{T-t}\mathbb{E}^t\left[X_r\right] \, dr = \int_t^T \frac{q}{T-t} x \, dr = qx.$$

A.4 Proof of Proposition 2.5.4

We consider any $\theta \in \left(0, c^+\frac{T-t}{c^++1} \wedge \frac{q}{c^+}\right)$. By combining (2.2.7) and (2.2.9) and using the fact that c_r in (2.5.3) is constant in time, we have for any $r \le \tau$

$$Q_r^{\theta,\bar{c},t} = q - \bar{c}\theta - \int_{t+\theta}^r c_s ds = q - c_t(r-t) + \theta(c_t - \bar{c}). \tag{A.4.1}$$

If $q \le (T-t)\frac{c^+}{c^++1}$, then (2.5.3) implies that $c_t = c^+$ and from (A.4.1) we have that $Q_\tau^{\theta,\bar{c},t} = \theta(c^+ - \bar{c})$ and using the fact that $\bar{c}$ is an admissible control and so $\bar{c} \in [0, c^+]$, $Q_\tau^{\theta,\bar{c},t} \le Q_\tau = 0$. Therefore, for $r \ge \tau$

$$\begin{aligned} Q_r &= q - \int_t^{t+\theta} \bar{c} - \int_{t+\theta}^\tau c_r + \int_\tau^r \frac{\gamma_{t+\theta}}{\theta} = q - \theta\bar{c} - (\tau - t - \theta)c^+ + (\bar{c} - c^+)(r-\tau) \\ &= q - \bar{c}\left(\frac{q}{c^+} + t + \theta\right) + c^+(t+\theta) + r(\bar{c} - c^+) = \left(c^+ - \bar{c}\right)\left(t + \theta + \frac{q}{c^+} - r\right). \end{aligned} \tag{A.4.2}$$

Therefore, by setting previous equation equal to 0, we get that

$$\tau^{\theta,\bar{c},t} = \begin{cases} t+\theta+\frac{q}{c^+} & \text{if } \bar{c}\in[0,c^+) \\ t+\frac{q}{c^+} & \text{if } \bar{c}=c^+. \end{cases} \tag{A.4.3}$$

Since, $q \le (T-t)\frac{c^+}{c^++1}$ then $\tau^{\theta,\bar{c},t} \le t+\theta+(T-t)\frac{1}{c^++1} = \theta + \frac{T+c^+t}{c^++1}$ and using that $\theta < c^+\frac{T-t}{c^++1}$ we have that $\tau^{\theta,\bar{c},t} < T$. On the other hand, if $q > (T-t)\frac{c^+}{c^++1}$, then $c_t = 0$ and from (A.4.1),

$$Q_T^{\theta,\bar{c},t} = q - \bar{c}\theta \ge q - c^+\theta. \tag{A.4.4}$$

And so, using that $\theta < \frac{q}{c^+}$, we have that

$$\tau^{\theta,\bar{c},t} = T. \tag{A.4.5}$$

In conclusion, if $q > (T-t)\frac{c^+}{c^++1}$, then $\tau_{\min}^{\theta,\bar{c},t} = \tau_{\max}^{\theta,\bar{c},t} = T$ and so from definition (2.2.17) of $\bar{f}$ we have that $\bar{f} = 0$. On the other hand, if $q \le (T-t)\frac{c^+}{c^++1}$, then $\tau_{\min}^{\theta,\bar{c},t} = t + \frac{q}{c^+}$ and $\tau_{\max}^{\theta,\bar{c},t} = t+\theta+\frac{q}{c^+}$ and so

$$\begin{aligned} \bar{f}(t,\bar{c},x,q) &= \lim_{\theta\to 0}\mathbb{E}^t\left[\frac{\operatorname{sign}(\tau-\tau^{\theta,\bar{c},t})}{\theta}\int_{\tau_{\min}^{\theta,\bar{c},t}}^{\tau_{\max}^{\theta,\bar{c},t}} f\left(\hat{c}_r^{\theta,\bar{c},t}\right)\,dr\right] = \lim_{\theta\to 0}\mathbb{E}^t\left[-\frac{1}{\theta}\int_{\tau_{\min}^{\theta,\bar{c},t}}^{\tau_{\max}^{\theta,\bar{c},t}} -\left(-\frac{\gamma_{t+\theta}}{\theta}\right)\,dr\right] \\ &= -\lim_{\theta\to 0}\mathbb{E}^t\left[\frac{1}{\theta}\int_{t+\frac{q}{c^+}}^{t+\theta+\frac{q}{c^+}}\frac{\gamma_{t+\theta}}{\theta}\,dr\right] = -\lim_{\theta\to 0}\frac{1}{\theta}\int_{t+\frac{q}{c^+}}^{t+\theta+\frac{q}{c^+}}\left(\bar{c}-c^+\right)\,dr = c^+ - \bar{c}. \end{aligned}$$

We now apply formula (2.2.18) to find $\bar{g}$

$$\begin{aligned} \bar{g}(t,\bar{c},x,q) &= \lim_{\theta\to 0}\mathbb{E}^t\left[\frac{g(X_\tau, Q^{\theta,\bar{c},t}_{\tau^{\theta,\bar{c},t}}) - g(X_{\tau^{\theta,\bar{c},t}}, Q^{\theta,\bar{c},t}_{\tau^{\theta,\bar{c},t}})}{\theta}\right] \\ &= \lim_{\theta\to 0}\mathbb{E}^t\left[\frac{X_\tau - X_{\tau^{\theta,\bar{c},t}}}{\theta}\right] = \lim_{\theta\to 0}\mathbb{E}^t\left[\frac{X_{t+\frac{q}{c^+}} - X_{t+\theta+\frac{q}{c^+}}}{\theta}\right] \\ &= \lim_{\theta\to 0}\mathbb{E}^t\left[\frac{\int_{t+\frac{q}{c^+}}^{t+\theta+\frac{q}{c^+}}dr - \int_{t+\frac{q}{c^+}}^{t+\theta+\frac{q}{c^+}}dW_r}{\theta}\right] = \lim_{\theta\to 0}\mathbb{E}^t\left[\frac{\theta - \int_{t+\frac{q}{c^+}}^{t+\theta+\frac{q}{c^+}}dW_r}{\theta}\right] \\ &= 1 - \lim_{\theta\to 0}\frac{\mathbb{E}^t\left[\int_{t+\frac{q}{c^+}}^{t+\theta+\frac{q}{c^+}}dW_r\right]}{\theta} = 1. \end{aligned} \tag{A.4.6}$$

This concludes the proof of the proposition.

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