Parameter Estimation for the McKean-Vlasov Stochastic Differential Equation *

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Abstract. We consider the problem of parameter estimation for a stochastic McKean-Vlasov equation, and the associated system of weakly interacting particles. We first establish consistency and asymptotic normality of the offline maximum likelihood estimator for the interacting particle system in the limit as the number of particles $N \to \infty$. We then propose an online estimator for the parameters of the McKean-Vlasov SDE, which evolves according to a continuous-time stochastic gradient descent algorithm on the asymptotic log-likelihood of the interacting particle system. We prove that this estimator converges in $\mathbb{L}^1$ to the stationary points of the asymptotic log-likelihood of the McKean-Vlasov SDE in the joint limit as $N \to \infty$ and $t \to \infty$, under suitable conditions which guarantee ergodicity and uniform-in-time propagation of chaos. We then demonstrate, under the additional condition of global strong concavity, that our estimator converges in $\mathbb{L}^2$ to the unique maximiser of this asymptotic log-likelihood function, and establish an $\mathbb{L}^2$ convergence rate. We also obtain analogous results under the condition that, rather than observing multiple trajectories of the interacting particle system, we instead observe multiple independent replicates of the McKean-Vlasov SDE itself or, less realistically, a single sample path of the McKean-Vlasov SDE and its law. Our theoretical results are demonstrated via two numerical examples, a linear mean field model and a stochastic opinion dynamics model.

Key words. McKean-Vlasov equation, nonlinear diffusion, maximum likelihood, parameter estimation, consistency, asymptotic normality, stochastic gradient descent

AMS subject classifications. 60F05, 60F25, 60H10, 62F12

1. Introduction.

In this paper, we consider a family of McKean-Vlasov stochastic differential equations (SDEs) on $\mathbb{R}^d$, parametrised by $\theta \in \mathbb{R}^p$, of the form

$$\mathrm{d}x_t^\theta = B(\theta, x_t^\theta, \mu_t^\theta)\mathrm{d}t + \sigma(x_t^\theta)\mathrm{d}w_t, \quad t \geq 0 \tag{1.1}$$

$$\mu_t^\theta = \mathcal{L}(x_t^\theta), \tag{1.2}$$

where $B : \mathbb{R}^p \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}^d$, $\sigma : \mathbb{R}^d \to \mathbb{R}^{d \times d}$ are Borel measurable functions, $(w_t)_{t\geq 0}$ is a $\mathbb{R}^d$-valued standard Brownian motion, and $\mathcal{L}(x_t^\theta)$ denotes the law of of x_t^θ. We assume that $x_0 \in \mathbb{R}^d$, or that x_0 is a $\mathbb{R}^d$-valued random variable with law μ_0, independent of $(w_t)_{t\geq 0}$. This equation is non-linear in the sense of McKean [60, 61, 79]; in particular, the coefficients depend on the law of the solution, in addition to the solution itself. We will restrict our attention to the case in which the dependence on the law only enters linearly in the drift, namely, that

$$B(\theta, x, \mu) = b(\theta, x) + \int_{\mathbb{R}^d} \phi(\theta, x, y)\mu(\mathrm{d}y), \tag{1.3}$$

***Funding:** The first author was funded by the EPRSC CDT in the Mathematics of Planet Earth (grant number EP/L016613/1) and the National Physical Laboratory. The second, third, and fourth authors were partially funded under a J.P. Morgan A.I. Research Award (2022). The fourth author was partially supported by the EPSRC (grant number EP/P031587/1).

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for some Borel measurable functions $b : \mathbb{R}^p \times \mathbb{R}^d \to \mathbb{R}^d$ and $\phi : \mathbb{R}^p \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$. This choice of dynamics, while not the most general possible, is sufficiently broad for many applications of interest. Moreover, it includes the popular case in which b and ϕ both have gradient forms, that is, $b(\theta, x) = \nabla V_\theta(x)$ and $\phi(\theta, x, y) = \nabla W_\theta(x - y)$, in which case V_θ and W_θ are referred to as the confinement potential and the interaction potential, respectively (e.g., [28, 58]).

The McKean-Vlasov SDE arises naturally as the hydrodynamical limit ($N \to \infty$) of the mean-field interacting particle system (IPS)

$$\mathrm{d}x_t^{\theta,i,N} = B(\theta, x_t^{\theta,i,N}, \mu_t^{\theta,N})\mathrm{d}t + \sigma(x_t^{\theta,i,N})\mathrm{d}w_t^i \ , \quad i = 1, \dots, N \tag{1.4}$$

where $(w_t^i)_{t\geq 0}$ are N independent $\mathbb{R}^d$-valued independent standard Brownian motions, x_0^i are a family of i.i.d. $\mathbb{R}^d$-valued random variables with common law μ_0, independent of $(w_t^i)_{t\geq 0}$, and $\mu_t^{\theta,N} = \frac{1}{N}\sum_{i=1}^N \delta_{x_t^{\theta,i,N}}$ is the empirical law of the interacting particles. In particular, under relatively weak assumptions, it is well known that the empirical law $\mu_t^{\theta,N} \to \mu_t^\theta$ weakly as $N \to \infty$ (e.g., [66]). This phenomenon is commonly known as the propagation of chaos [79].

The McKean-Vlasov SDE also has a natural connection to a non-linear, non-local partial differential equation on the space of probability measures (e.g., [21]). In particular, under some regularity conditions on b and ϕ, one can show that $\mathcal{L}(x_t^\theta)$ is absolutely continuous with respect to the Lebesgue measure for all $t \geq 0$ [61, 80] and its density, which we will denote by u_t^θ, satisfies a non-linear partial differential equation of the form

$$\frac{\partial u_t^\theta(x)}{\partial t} = \nabla \left[\frac{1}{2}\sigma(x)\sigma^T(x)\nabla u_t^\theta(x) + u_t^\theta(x)\left[b(\theta, x) + \int_{\mathbb{R}^d} \phi(\theta, x, y)u_t^\theta(y)\mathrm{d}y\right]\right]. \tag{1.5}$$

In the particular case that $b(x) = \nabla V(x)$ and $\phi(x, y) = \nabla W(x-y)$, this is commonly referred to as the granular media equation or the kinetic Fokker-Planck equation (e.g., [4, 21]).

1.1. Literature Review. The systematic study of McKean-Vlasov SDEs was first initiated by McKean [60] in the 1960s, inspired by Kac's programme in Kinetic Theory [43]. We refer to [31, 63, 79, 84] for some other classical references. In the last two decades, the study of non-linear diffusions has continued to receive considerable attention, with extensive results on well-posedness (e.g., [22, 40]), existence and uniqueness (e.g., [3, 42, 64]), ergodicity (e.g., [10, 20, 21, 29, 37, 58, 81]), and propagation of chaos (e.g., [4, 16, 28, 58, 59]). This has no doubt been motivated, at least in part, by the increasing number of applications for McKean-Vlasov SDEs, including in statistical physics [5], multi-agent systems [4], mean-field games [18], stochastic control [14], filtering [26], mathematical biology (including neuroscience [1] and structured models of population dynamics [15]), epidemic dynamics [2], social sciences (including opinion dynamics [23] and cooperative behaviours [17]), financial mathematics [35], and, perhaps most recently, high dimensional sampling [51] and neural networks [76].

Despite the recent renewed interest in the study of McKean-Vlasov SDEs, however, the problem of parameter estimation for this class of equations has received relatively little attention. This is contrast to the wealth of literature on parameter inference in linear (i.e., not measure dependent) diffusion processes (e.g., [8, 12, 46, 49]). Recently, Wen et al. [86] established the asymptotic consistency and asymptotic normality of the (offline) maximum likelihood estimator (MLE) for a broad class of McKean-Vlasov SDEs, based on continuous observation of $(x_t)_{t\in[0,T]}$. These results have since been extended by Liu et al. to the path-dependent case [50]. We also mention the work of Catalot and Laredo [33, 34], who have studied parametric inference for a particular class of one-dimensional nonlinear self-stabilising SDEs using an approximate log-likelihood function, again based on continuous observation of the non-linear diffusion process, and established the asymptotic properties (consistency, normality, convergence rates) of the resulting estimators in several asymptotic regimes (e.g., small noise and long time limit). More recently, Gomes et al. [36] have considered parameter estimation for a McKean-Vlasov PDE, based on independent realisation of the associated non-linear SDE, in the context of models for pedestrian dynamics.

In a slightly different framework, Maestra and Hoffmann [57] consider non-parametric estimation of the drift-term in a McKean-Vlasov SDE, and the solution of the corresponding non-linear Fokker-Planck equation, based on continuous observation of the associated IPS over a fixed time horizon, namely $(x_t^{i,N})_{t\in[0,T]}^{i=1,\dots,N}$, in the limit as $N\to\infty$. The authors obtain adaptive estimators based on the solution map of the Fokker-Planck equation, and prove their optimality in a minimax sense. Moreover, in the case of the so-called Vlasov model, which in our notation corresponds to the case in which $b(x)=-\nabla V(x)$ and $\phi(x,y)=-\nabla W(x-y)$, the authors derive an estimator of the interaction potential, and establish its consistency. We also refer to [56, 54, 55] for some other recent contributions on non-parametric inference for IPSs. While these approaches are interesting and potentially very useful, we should emphasise that they are tangential and very different to this contribution.

Despite these recent contributions, however, to our knowledge there are no existing works which tackle the problem of online parameter estimation for McKean-Vlasov SDEs. The main purpose of this paper is to address this gap. There is significant motivation for this approach. Indeed, in comparison to classical (offline) methods, which process observations in a batch fashion, online methods perform inference in real time, can track changes in parameters over time, are more computationally efficient, and have significantly smaller storage requirements. Even for standard diffusion processes, literature on online parameter estimation is somewhat sparse, with some notable recent exceptions [75, 77, 78]. The problem of recursive estimation in continuous-time stochastic processes was first rigorously analysed by Levanony et al. [47], who proposed an online MLE which, irrespective of initial conditions, was shown to be consistent and asymptotically efficient. This estimator, however, involves computing gradients of a Girsanov log-likelihood, $\mathcal{L}_t(\theta)$, every time a new observation arrives; as a result, it is computationally expensive, and cannot be implemented in a truly online fashion, since $\nabla_\theta\mathcal{L}_t(\theta)$ depends on the entire trajectory of the process x_t. This problem has recently been revisited by Sirignano and Spiliopoulos [75, 77], who propose an online statistical learning algorithm - 'stochastic gradient descent in continuous time' - for the estimation of the parameters in a fully observed ergodic diffusion process. These authors establish the almost sure convergence of this estimator in the sense that $||\nabla_\theta g(\theta_t)||\to 0$ as $t\to\infty$ a.s., for some suitably defined objective function $g(\theta)$ [75], and, under additional assumptions, also obtain an $\mathbb{L}^p$ convergence rate and a central limit theorem [77]. These results have since also been extended to partially observed diffusion processes [78] and jump-diffusion processes [7].

There also exists relatively little previous literature on statistical inference for IPSs, in the limit as the number of particles $N\to\infty$. In the context of parameter estimation, the mean field regime was first analysed by Kasonga [44], who considered a system of interacting diffusion processes, depending linearly on some unknown parameter, and established that the MLE based on continuous observations over a fixed time interval $[0, T]$ is consistent and asymptotically normal in the limit as $N \to \infty$. Bishwal [9] later extended these results to the case in which the parameter to be estimated is a function of time, proving consistency and asymptotic normality of the sieve estimator (in the case of continuous observations) and an approximate MLE (in the case of discrete observations). In this paper, we extend the results in [44] in another direction, establishing consistency and asymptotic normality of the offline MLE when the parametrisation is not linear.

More recently, Giesecke et al. [35] have established the asymptotic properties (consistency, asymptotic normality, and asymptotic efficiency) of an approximate MLE for a much broader class of dynamic interacting stochastic systems, widely applicable in financial mathematics, which additionally allow for discontinuous (i.e., jump) dynamics. In addition, Chen [24] has established the optimal convergence rate for the MLE in an interacting parameter system with linear interaction for ϕ, simultaneously in the large N (mean-field limit) and large T (long-time dynamics) regimes. None of the these works, however, considers parameter estimation for the IPS in the online setting.

1.2. Contributions. The main contributions of this paper relate to both the methodology and the theory of parameter estimation for the McKean-Vlasov SDE (1.1) - (1.2). Regarding methodology:

- We discuss how one can formulate an appropriate approximation to the true likelihood function in this problem, under various modelling assumptions.
- We distinguish between cases in which the data consists of multiple paths of the IPS (Case I), multiple independent samples of the McKean-Vlasov SDE (Case II), or, less realistically, a single sample path of the McKean-Vlasov SDE and its law (Case III).

In each of these cases, we perform a rigorous asymptotic analysis of the MLE, with a focus on online parameter estimation. Our main theoretical contributions can be summarised as follows:

- In Case I, we establish asymptotic consistency and asymptotic normality of the offline MLE, in the limit as the number of particles $N \to \infty$. Our results generalise those in [44] to the case in which b and ϕ depend non-linearly on the parameter.
- In all three cases, we propose online estimators for the parameters of the McKean-Vlasov SDE, which evolve according to continuous-time stochastic gradient descent algorithms with respect to appropriate asymptotic log-likelihood functions.
- We prove that each of these estimators converges in $\mathbb{L}^1$ to the stationary points of the asymptotic log-likelihood of the McKean-Vlasov SDE, under suitable conditions which guarantee ergodicity and uniform-in-time propagation of chaos. In Cases I - II, this convergence holds in the joint limit as $N \to \infty$ and $t \to \infty$. In Case III, it holds solely in the limit as $t \to \infty$.
- We prove, under the additional condition that the asymptotic log-likelihood of the McKean Vlasov SDE is strongly concave, that these estimators converge in $\mathbb{L}^2$ to its unique global maximiser, in the same limits outlined above. In each case, we also obtain explicit convergence rates.

Finally, we provide numerical examples to illustrate the application of these results to two cases of interest, namely, a linear mean-field model, and a stochastic opinion dynamics model. It is worth emphasising that, given the connection between the McKean-Vlasov SDE (1.1) - (1.2) and the non-linear, non-local PDE (1.5), the results of this paper are also applicable when one is primarily interested in parameter estimation for the non-linear PDE (1.5).

1.3. Paper Organisation. The remainder of this paper is organised as follows. In Section 2, we formulate the estimation problem, and propose a recursive estimator for the McKean-Vlasov SDE. In Section 3, we state our conditions and our main results regarding the asymptotic properties of the offline and online MLEs. In Section 4, we provide the proofs of these results. In Section 5, we provide several numerical examples illustrating the performance of the proposed algorithm. Finally, in Section 6, we provide some concluding remarks.

1.4. Additional Notation. We will assume throughout this paper that all stochastic processes are defined canonically on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, equipped with filtration $(\mathcal{F}_t)_{t\geq 0}$. We will use $\langle \cdot, \cdot \rangle$ and $||\cdot||$ to denote, respectively, the Euclidean inner product and the corresponding norm on $\mathbb{R}^d$. We write $\mathcal{P}(\mathbb{R}^d)$ and $\mathcal{P}_p(\mathbb{R}^d)$, $p > 0$, for the collection of all probability measures on $\mathbb{R}^d$, and the collection of all probability measures on $\mathbb{R}^d$ with finite p^{th} moment. In a slight abuse of notation, we will frequently write $\mu(||\cdot||^p)$ for the p^{th} moment of μ; that is, $\mu(||\cdot||^p) = \int_{\mathbb{R}^d} ||x||^p \mu(\mathrm{d}x)$. For $\mu, \nu \in \mathcal{P}_p(\mathbb{R}^d)$, we write $\mathbb{W}_p(\mu,\nu)$ to denote the Wasserstein distance between μ and ν, viz

$$\mathbb{W}_p(\mu,\nu) = \inf_{\pi \in \Pi(\mu,\nu)} \left[\int_{\mathbb{R}^d \times \mathbb{R}^d} ||x-y||^p \pi(\mathrm{d}x, \mathrm{d}y) \right]^{\frac{1}{\max\{1,p\}}}.$$

where $\Pi(\mu,\nu)$ for the set of all couplings of μ, ν. That is, if $\pi \in \Pi(\mu,\nu)$, then $\pi(A \times \mathbb{R}^d) = \mu(A)$ and $\pi(\mathbb{R}^d \times A) = \nu(A)$ for all $A \in \mathcal{B}(\mathbb{R}^d)$. Finally, if $(x_t)_{t\geq 0}$ is a solution of the McKean-Vlasov SDE with $x_0 = x \in \mathbb{R}^d$, we will occasionally make explicit the dependence on the initial condition by writing $\mu_t^x = \mathcal{L}(x_t)$ for the law of x_t. We can also then write $\mathbb{E}_x[f(x_t)] = \int_{\mathbb{R}^d} f(y) \mu_t^x(\mathrm{d}y)$.

2. Parameter Estimation for the McKean-Vlasov SDE. We will assume, throughout this paper, that there exists a true (static) parameter $\theta_0 \in \mathbb{R}^p$ which generates observations $(x_t)_{t\geq 0} := (x_t^{\theta_0})_{t \geq 0}$ of the McKean-Vlasov SDE (1.1). Thus, we operate under the standard well specified regime, and in our notation will suppress the dependence of the observed path on the true parameter θ_0. We will make the same condition when instead we observe trajectories the IPS (1.4), in which case the observations are given by $(x_t^{i,N})_{t\geq 0}^{i=1,\dots,N} = (x_t^{\theta_0,i,N})_{t\geq 0}^{i=1,\dots,N}$.

2.1. The Likelihood Function. Let $\mathbb{P}_t^\theta$ denote the probability measure induced by a path $(x_s^\theta)_{s\in[0,t]}$ of the McKean-Vlasov SDE (1.1). Then, under certain regularity conditions, to be specified below, one can use the Girsanov formula to obtain a likelihood function as (e.g., [86])

$$\begin{aligned}\mathcal{L}_t(\theta) = \log \frac{\mathrm{d}\mathbb{P}_t^\theta}{\mathrm{d}\mathbb{P}_t^{\theta_0}} &= \int_0^t \left\langle \left[B(\theta, x_s, \mu_s) - B(\theta_0, x_s, \mu_s) \right], (\sigma(x_s)\sigma^T(x_s))^{-1} \mathrm{d}x_s \right\rangle \\ &\quad - \frac{1}{2} \int_0^t \left[\left|\left| \sigma^{-1}(x_s) B(\theta, x_s, \mu_s) \right|\right|^2 - \left|\left| \sigma^{-1}(x_s) B(\theta_0, x_s, \mu_s) \right|\right|^2 \right] \mathrm{d}s.\end{aligned}$$

Suppose, for a moment, that the diffusion coefficient σ also depended on the parameter θ. In this case, the measures $\{\mathbb{P}_\theta\}$ would, in general, be mutually singular, and the likelihood function would not be well defined. We thus adopt the standard condition of parameter independence for the diffusion coefficient, and for convenience set $\sigma = 1$ (e.g., [12, 47, 86]). In the case that σ is an unknown constant, it can be estimated separately using standard methods (e.g., [32]). In fact, there are various different approaches in this case, including those based on a quasi log-likelihood function [39], or on a least squares type function for the diffusion coefficient [75]. The methods outlined in this paper can be extended to either of these cases, as well as to parameter estimation under other criteria.

In order to proceed, it will be convenient to define the functions $G : \mathbb{R}^p \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}^d$ and $L : \mathbb{R}^p \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}$ according to

$$G(\theta, x, \mu) := B(\theta, x, \mu) - B(\theta_0, x, \mu) \tag{2.1}$$

$$L(\theta, x, \mu) := -\frac{1}{2}||G(\theta, x, \mu)||^2. \tag{2.2}$$

We are now ready to state our first basic assumption. This is Novikov-type condition which ensures that $\frac{\mathrm{d}\mathbb{P}_t^\theta}{\mathrm{d}\mathbb{P}_t^{\theta_0}}$ exists and is a martingale. We note that several slightly weaker versions of this condition are also possible (e.g., [50, 86]).

Assumption A.1. *For all $\theta \in \mathbb{R}^p$, $t \geq 0$, the function $G : \mathbb{R}^p \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}^d$ satisfies*

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^t ||G(\theta, x_s, \mu_s)||^2 \mathrm{d}s\right)\right] < \infty.$$

Under this assumption, it follows immediately from Girsanov's Theorem that $\mathbb{P}_t^\theta$ is absolutely continuous with respect to $\mathbb{P}_t^{\theta_0}$ for all $\theta \in \mathbb{R}^p$, $t > 0$ (e.g. [49, Theorem 7.19], [50, 86]), and that the log-likelihood for an observed path of the McKean-Vlasov SDE (1.1) - (1.2) is given by

$$\mathcal{L}_t(\theta) = \int_0^t L(\theta, x_s, \mu_s)\mathrm{d}s + \int_0^t \langle G(\theta, x_s, \mu_s), \mathrm{d}w_s\rangle. \tag{2.3}$$

While, in general, it is possible to observe a sample path $(x_t)_{t\geq 0}$ of a (McKean-Vlasov) SDE, in general one does not have direct access to its law $(\mu_t)_{t\geq 0}$. As such, it is generally not to possible to compute the likelihood function $\mathcal{L}_t(\theta)$ in (2.3) directly. On this basis, even if one is interested in fitting data to the McKean-Vlasov SDE, it will typically be necessary to approximate the corresponding likelihood function.

In order to make such an approximation, we will henceforth assume that we can simultaneously observe multiple continuous sample paths, which is much more typical of the data that we observe in practice. There are now two possibilities. The first is to assume that the observed paths correspond to the trajectories of N particles $(x_t^{i,N})_{t\geq 0}^{i=1,\dots,N}$ from the IPS (1.4). In this case, we can approximate $\mathcal{L}_t(\theta)$ by the Girsanov log-likelihood for the IPS, which is given by (e.g., [9, 24, 44])

$$\mathcal{L}_t^N(\theta) := \frac{1}{N}\sum_{i=1}^N \mathcal{L}_t^{i,N}(\theta) = \frac{1}{N}\sum_{i=1}^N \left[\int_0^t L(\theta, x_s^{i,N}, \mu_s^N)\mathrm{d}s + \int_0^t \left\langle G(\theta, x_s^{i,N}, \mu_s^N), \mathrm{d}w_s^i\right\rangle\right], \tag{2.4}$$

Case	Data-Generating Model	Observation(s)	Likelihood Function *Approximate*	*Ideal*
Case I	IPS (1.4)	$(x_t^{i,N})_{t\geq 0}^{i=1,\dots,N}$	$\mathcal{L}_t^N(\theta)$ in (2.4)	$\mathcal{L}_t(\theta)$ in (2.3)
Case II	MVSDE (1.1) - (1.2)	$(x_t^i)_{t\geq 0}^{i=1,\dots,N}$	$\mathcal{L}_t^{[N]}(\theta)$ in (2.5)	$\mathcal{L}_t(\theta)$ in (2.3)
Case III	MVSDE (1.1) - (1.2)	$(x_t,\mu_t)_{t\geq 0}$	-	$\mathcal{L}_t(\theta)$ in (2.3)

Table 1
Parameter Estimation: Summary of Different Cases

where $\mu_t^N = \frac{1}{N}\sum_{j=1}^N \delta_{x_t^{j,N}}$ denotes the empirical measure of the IPS, and we have included $\frac{1}{N}$ as a normalisation factor. We will refer to this as **Case I**. The second possibility is to instead assume the observed paths are N independent instances $(x_t^i)_{t\geq 0}^{i=1,\dots,N}$ of the McKean-Vlasov SDE (1.1). In this case, we can approximate $\mathcal{L}_t(\theta)$ by

$$\mathcal{L}_t^{[N]}(\theta) := \frac{1}{N}\sum_{i=1}^N \mathcal{L}_t^{[i,N]}(\theta) = \frac{1}{N}\sum_{i=1}^N \left[\int_0^t L(\theta,x_s^i,\mu_s^{[N]})\mathrm{d}s + \int_0^t \left\langle G(\theta,x_s^i,\mu_s^{[N]}),\mathrm{d}w_s^i\right\rangle\right], \tag{2.5}$$

where $\mu_t^{[N]} = \frac{1}{N}\sum_{i=1}^N \delta_{x_t^i}$ denotes the empirical measure of the sample paths. In this approximation, the the functions $\mathcal{L}_t^{[i,N]}(\theta)$, $i=1,\dots,N$, correspond to N Monte Carlo approximations of $\mathcal{L}_t(\theta)$, obtained by substituting $\mu_t^{[N]}$ for μ_t. The approximation $\mathcal{L}_t^{[N]}(\theta)$ then follows by independence. We will refer to this case as **Case II**. Finally, we will refer to the rather unrealistic case in which we directly observe a single path $(x_t)_{t\geq 0}$ of the McKean-Vlasov SDE (1.1), as well as its law $(\mu_t)_{t\geq 0}$, as **Case III**. These cases are summarised in Table 1.

In what follows, our exposition will primarily focus on Case I, which provides the most interesting and challenging case in which to perform asymptotic analysis in both N and t. One can consider Case I and Case II as approximations to Case III that are amenable to implementation. In the limit as $N\to\infty$, standard propagation-of-chaos results (e.g., [58]) show that the dynamics of the observations in Cases I and II will coincide. In our results, we will establish rigorously that this also holds for the different implied likelihood functions, $\mathcal{L}_t^N(\theta)$ and $\mathcal{L}_t^{[N]}(\theta)$. This should not be a surprise given the similarities between these two functions: in particular, they are identical as functions of x and μ^N. We will also demonstrate that, as $N\to\infty$, these two 'approximations' also coincide with $\mathcal{L}_t(\theta)$, the 'ideal' likelihood function implied by the less realistic Case III. Moreover, we show that the same is true of the resulting parameter estimates.

2.2. Offline Parameter Estimation. In the offline setting, the objective is to estimate the true parameter θ_0 after receiving a batch of data over a fixed time interval $[0,t]$. Let us first consider the 'idealised' framework of Case III, in which one directly observes both $(x_s)_{s\in[0,t]}$ and $(\mu_s)_{s\in[0,t]}$ from the McKean-Vlasov SDE (1.1) - (1.2). In this case, one can achieve this objective directly by seeking to maximise the value of $\mathcal{L}_t(\theta)$ in order to obtain the MLE

$$\hat{\theta}_t = \arg\sup_{\theta\in\mathbb{R}^p} \mathcal{L}_t(\theta).$$

The asymptotic properties (i.e., consistency, asymptotic normality) of this estimator in the limit as $t \to \infty$, under similar conditions to our own (see Section 3), have recently been established [50, 86]. In this paper, we are more interested in Case I, in which we assume that we observe N sample paths $(x_t^{i,N})_{s\in[0,t]}^{i=1,\dots,N}$ following the dynamics of the IPS (1.4). In this case, we aim instead to maximise the value of $\mathcal{L}_t^N(\theta)$, and are thus interested in the asymptotic properties of the following MLE

$$\hat{\theta}_t^N = \arg\sup_{\theta\in\mathbb{R}^p} \mathcal{L}_t^N(\theta).$$

The asymptotic properties of this estimator as $t \to \infty$, for fixed N, are covered by well established results for parameter estimation in standard SDEs (e.g., [8, 47, 49]). Conversely, there are very few results on the properties of this MLE in the limit as $N \to \infty$, aside from in the case of a linear parametrisation [9, 44]. We thus find it instructive to revisit this problem. In Theorems 3.1 - 3.2, we extend previous results to the more general and possible non-linear setting (in the sense of parametrisation), establishing consistency and asymptotic normality of this estimator as $N \to \infty$, for fixed t.

2.3. Online Parameter Estimation. In the online setting, our objective is to estimate the true parameter θ_0 in real time, using the continuous stream of observations. Once more, let us begin in the 'idealised' framework of Case III. In this case, a standard approach to this task would be to seek to recursively maximise the asymptotic log-likelihood function $\tilde{\mathcal{L}}(\theta)$ of the McKean-Vlasov SDE, which, provided the limit exists, could be defined according to

$$\tilde{\mathcal{L}}(\theta) = \lim_{t\to\infty} \frac{1}{t}\mathcal{L}_t(\theta).$$

In the spirit of [7, 75, 78], this could be achieved using stochastic approximation by defining an estimator $\theta = (\theta_t)_{t\geq 0}$ which follows the gradient of the integrand of the log-likelihood in (2.3), evaluated with the current parameter estimate. Thus, initialised at $\theta_{\text{init}} \in \mathbb{R}^p$, θ_t evolves according to a McKean-Vlasov SDE of the form

$$\mathrm{d}\theta_t = \gamma_t\Big(\underbrace{\nabla_\theta L(\theta_t, x_t, \mu_t)\mathrm{d}t}_{\text{(noisy) ascent term}} + \underbrace{\nabla_\theta B(\theta_t, x_t, \mu_t)\mathrm{d}w_t}_{\text{noise term}} \Big) \tag{2.6}$$

where $\gamma_t : \mathbb{R}_+ \to \mathbb{R}_+^p$ is a positive, non-increasing function known as the *learning rate*. One also arrives at this estimator by considering a 'least-squares' type objective, i.e., minimisation of the function $||G(\theta, x, \mu)||^2$ (see also [75]). This evolution equation represents a continuous-time stochastic gradient ascent scheme on the asymptotic log-likelihood function. To see this, let us rewrite the parameter update equation (2.6) in the form

$$\mathrm{d}\theta_t = \gamma_t\Big(\underbrace{\nabla_\theta \tilde{\mathcal{L}}(\theta_t)\mathrm{d}t}_{\text{(true) ascent term}} + \underbrace{(\nabla_\theta L(\theta_t, x_t, \mu_t) - \nabla_\theta \tilde{\mathcal{L}}(\theta_t))\mathrm{d}t}_{\text{fluctuations term}} + \underbrace{\nabla_\theta B(\theta_t, x_t, \mu_t)\mathrm{d}w_t}_{\text{noise term}} \Big) \tag{2.7}$$

The first term in this decomposition represents the true ascent direction $\nabla_\theta\tilde{\mathcal{L}}(\theta_t)$, the second term the deviation between the stochastic gradient ascent direction $\nabla_\theta L(\theta_t, x_t, \mu_t)$ and the true (deterministic) gradient ascent direction $\nabla_\theta\tilde{\mathcal{L}}(\theta_t)$, while the third term is a zero-mean noise term. Heuristically, we might expect that, provided the learning rate γ_t decreases (sufficiently quickly) with time, the ascent term will dominate the fluctuations term and the noise term when t is sufficiently large. If this is the case, we could then reasonably expect that θ_t will converge to a local maximum of $\tilde{\mathcal{L}}(\theta)$.

Similarly to the offline case, the 'ideal' online estimator (2.6) cannot typically be implemented in practice, since we do not have access to the law $(\mu_t)_{t\geq 0}$. Instead, as remarked previously, we will typically observe multiple continuous sample paths. Once again, let us first consider the case in which the N sample paths $(x_t)_{s\in[0,t]}^{i=1,\dots,N}$ are assumed to correspond to the trajectories of the IPS (1.4) (Case I). In this case, it is natural to consider the 'approximate' update equation

$$\mathrm{d}\theta_t^{i,N} = \gamma_t\left[\nabla_\theta L(\theta_t, x_t^{i,N}, \mu_t^N)\mathrm{d}t + \nabla_\theta B(\theta_t, x_t^{i,N}, \mu_t^N)\mathrm{d}w_t^i\right], \tag{2.8}$$

for some $i = 1,\dots,N$, or, averaging over all of the interacting particles,

$$\mathrm{d}\theta_t^{N} = \gamma_t\frac{1}{N}\sum_{i=1}^N\left[\nabla_\theta L(\theta_t, x_t^{i,N}, \mu_t^N)\mathrm{d}t + \nabla_\theta B(\theta_t, x_t^{i,N}, \mu_t^N)\mathrm{d}w_t^i\right]. \tag{2.9}$$

We can also use these update equations in Case II, in which we instead assume that the N sample paths $(x_t^i)_{t\geq 0}^{i=1,\dots,N}$ correspond to independent replicates of the McKean-Vlasov SDE (1.1). This should not be surprising on the basis of our previous remarks: in particular, the likelihood functions in Cases I and II are identical up to specification of the data. In Case II, we must simply replace $x_t^{i,N}$ by x_t^i, and μ_t^N by $\mu_t^{[N]}$ in (2.8) and (2.9). We will denote the resulting estimates by $(\theta_t^{[i,N]})_{t\geq 0}$ and $(\theta_t^{[N]})_{t\geq 0}$.

Let us briefly remark on these two schemes. The advantage of (2.8) is that the computation can be performed locally at each particle, following a message passing step for retrieving μ_t^N. It is thus convenient for a distributed implementation. On the other hand, (2.9) will typically be more accurate, as we will later demonstrate (see Theorems 3.4 and 3.4*). In Case I, these two schemes can be seen as stochastic gradient descent algorithms for maximising the 'partial' asymptotic log-likelihood of the i^{th} particle in the IPS, or the 'complete' asymptotic log-likelihood of all of the particles, respectively. That is,

$$\tilde{\mathcal{L}}^{i,N}(\theta) = \lim_{t\to\infty}\frac{1}{t}\mathcal{L}_t^{i,N}(\theta) \quad \text{or} \quad \tilde{\mathcal{L}}^{N}(\theta) = \lim_{t\to\infty}\frac{1}{t}\mathcal{L}_t^{N}(\theta).$$

Reasoning as before, we expect that, under suitable conditions on the learning rate, θ_t^N and $\theta_t^{i,N}$ will converge to local maxima of $\tilde{\mathcal{L}}^N(\theta)$ and $\tilde{\mathcal{L}}^{i,N}(\theta)$ as $t\to\infty$. Moreover, assuming uniform-in-time propagation of chaos, we can also now expect that $\tilde{\mathcal{L}}^N(\theta)$ and $\tilde{\mathcal{L}}^{i,N}(\theta)$ will converge to $\tilde{\mathcal{L}}(\theta)$ as $N\to\infty$. Thus, in the joint limit as $t\to\infty$ and $N\to\infty$ it seems reasonable to hypothesise that θ_t^N and $\theta_t^{i,N}$ will in fact converge to local maxima of $\tilde{\mathcal{L}}(\theta)$, the asymptotic log-likelihood of the original McKean-Vlasov SDE. In Theorems 3.3 - 3.4, we will establish rigorously that this is indeed the case.

3. Main Results.

In this section, we present our main results on the asymptotic properties of the offline and online MLEs, as well as our assumptions.

3.1. Assumptions.

Let us begin by stating our basic assumptions.

Assumption B.1. *For all $\theta \in \mathbb{R}^p$, $b(\theta, \cdot) : \mathbb{R}^d \to \mathbb{R}^d$ has the following properties.*

(i) $b(\theta, \cdot)$ is locally Lipschitz. That is, for all $x, x' \in \mathbb{R}^d$ such that $||x||, ||x'|| < R$, there exists $0 < L_1 < \infty$ such that

$$||b(\theta, x) - b(\theta, x')|| \leq L_1 ||x - x'||.$$

(ii) $b(\theta, \cdot)$ is 'monotonic'. That is, for all $x, x' \in \mathbb{R}^d$, there exists $\alpha > 0$ such that

$$\big\langle x - x', b(\theta, x) - b(\theta, x') \big\rangle \leq -\alpha ||x - x'||^2.$$

Assumption B.2. *For all $\theta \in \mathbb{R}^p$, $\phi(\theta, \cdot, \cdot) : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ has the following properties.*

(i) $\phi(\theta, \cdot, \cdot) \in C^2(\mathbb{R}^d, \mathbb{R}^d)$. That is, ϕ is twice continuously differentiable with respect to both of its arguments.

(ii) $\phi(\theta, \cdot, \cdot)$ is globally Lipschitz. In particular, there exists $0 < 2L_2 < \alpha$ such that, for all $x, y, x', y' \in \mathbb{R}^d$,

$$||\phi(\theta, x, y) - \phi(\theta, x', y')|| \leq L_2(||x - x'|| + ||y - y'||).$$

or, in place of (ii),

(ii)' $\phi(\theta, \cdot, \cdot)$ is 'anti-symmetric'. That is, for all $x, y \in \mathbb{R}^d$, $\phi(x, y) = \phi(y, x)$.

(ii)" $\phi(\theta, \cdot, \cdot)$ increases as a function of the distance between its arguments. That is, for all $x, y, x', y' \in \mathbb{R}^d$,

$$\langle (x - y) - (x' - y'), \phi(x, y) - \phi(x', y') \rangle \leq 0.$$

These two conditions are used to establish existence and uniqueness of the strong solution to the McKean-Vlasov SDE, uniform moment bounds, uniform-in-time propagation of chaos, and the existence of, and exponential convergence to, a unique invariant measure (e.g., [16, 83]). We provide a precise statement of these well known results in Appendix A, which we will frequently make use of to prove the main results in this paper.

In the literature on non-linear diffusions, it is typical, as noted previously, to consider the case in which $b(\theta, x) = -\nabla V(\theta, x)$ for some confinement potential V, and $\phi(\theta, x, y) = -\nabla W(\theta, x - y)$ for some interaction potential W. In this context, Condition B.1(ii) is equivalent to the condition that V is strongly convex with parameter α, and Conditions B.2(ii)'-(ii)" are equivalent to the conditions that W is symmetric and convex (see [58]). These are perhaps the simplest and most well established conditions under which the results listed above (uniform-in-time propagation of chaos, exponential convergence to a unique invariant measure) can be obtained; we have thus adopted them here for ease of exposition.

This being said, let us remark briefly upon some weaker conditions under which these results still hold, and therefore under which the main results of our paper will also still hold (albeit with some additional technical overhead). In the case that there is no confinement potential (i.e. $V \equiv 0$), and the interaction potential is uniformly convex with gradient that is locally Lipschitz with polynomial growth, Malrieu established uniform-in-time propagation of chaos and exponential convergence to equilibrium [59]. Cattiaux et al. [21] later established the same results in the case that the interaction potential is degenerately convex. Meanwhile, in [19, 20], the authors establish exponential convergence to equilibrium under the strict convexity condition $\text{Hess}(V+2W) \geq \beta I_d$, for some $\beta > 0$.

In the case that $V+2W$ is not convex, far fewer results are available; indeed, without additional conditions on V and W, even the existence of a unique stationary distribution is not guaranteed (see, e.g., [37, 38, 81, 82]). This being said, Bolley et al. [10] proved uniform exponential convergence to equilibrium in both degenerately convex, and weakly non-convex cases. More recently, [28, 29] have established uniform-in-time propagation of chaos and exponential convergence to equilibrium in the non-convex case, provided the confinement potential V is strictly convex outside a ball, and the interaction potential is globally Lipschitz with sufficiently small Lipschitz constant. For a recent extension of these results, see also [52].

We will also require the following regularity condition.

Assumption C.1. *The functions $b : \mathbb{R}^p \times \mathbb{R}^d \to \mathbb{R}^d$ and $\phi : \mathbb{R}^p \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ have the following properties.*

(i) $\nabla_\theta b(\cdot, x), \nabla_\theta \phi(\cdot, x, y) \in \mathcal{C}^2(\mathbb{R}^p)$ *for all* $x, y \in \mathbb{R}^d$, $\frac{\partial^2}{\partial x^2}\nabla_\theta b \in \mathcal{C}(\mathbb{R}^p, \mathbb{R}^d)$, $\frac{\partial^2}{\partial x^2}\nabla_\theta \phi \in \mathcal{C}(\mathbb{R}^p, \mathbb{R}^d, \mathbb{R}^d)$, *and* $\nabla_\theta^i b(\theta, \cdot) \in \mathcal{C}^{1+\alpha}(\mathbb{R}^d)$, $\nabla_\theta^i \phi(\theta, \cdot, \cdot) \in \mathcal{C}^{1+\alpha}(\mathbb{R}^d, \mathbb{R}^d)$, $i = 1, 2$, *uniformly in* $\theta \in \mathbb{R}^p$ *for some* $\alpha \in (0, 1)$.[1]

(ii) The functions $\nabla_\theta^i b(\theta, \cdot)$ *and* $\nabla_\theta^i \phi(\theta, \cdot, \cdot)$ *are locally Lipschitz with polynomial growth. That is, there exist constants* $q, K < \infty$ *such that, for* $i = 0, 1, 2, 3$,

$$\begin{aligned}\left|\left|\nabla_\theta^i b(\theta, x) - \nabla_\theta^i b(\theta, x')\right|\right| &\leq K||x - x'|| \left[1 + ||x||^q + ||x'||^q\right] \\ \left|\left|\nabla_\theta^i \phi(\theta, x, y) - \nabla_\theta^i \phi(\theta, x', y')\right|\right| &\leq K \left[||x - x'|| + ||y - y'||\right] \\ &\quad \cdot \left[1 + ||x||^q + ||x'||^q + ||y||^q + ||y'||^q\right].\end{aligned}$$

(iii) $b(\theta_0, \cdot) \in C^{2+\alpha}(\mathbb{R}^d)$, $\phi(\theta_0, \cdot, \cdot) \in C^{2+\alpha}(\mathbb{R}^d, \mathbb{R}^d)$ *with* $\alpha \in (0, 1)$. *Namely, these functions have two derivatives, with all partial derivatives Hölder continuous with exponent* α.

In the offline setting, these conditions are required in order to control the growth of the log-likelihood function and its derivatives. In the online setting, they are required in order to control the ergodic behaviour of the solution of the McKean-Vlasov SDE (and the associated IPS), which is central to establishing convergence of the online MLE. In particular, they ensure that fluctuations terms of the form $\int_0^t \gamma_s(\nabla_\theta L(\theta_s, x_s, \mu_s) - \nabla_\theta \tilde{\mathcal{L}}(\theta_s))\mathrm{d}s$, which arise due to the noisy online estimate of the gradient of the asymptotic log-likelihood function $\nabla_\theta \tilde{\mathcal{L}}(\theta_s)$, c.f. (2.7), tend to zero sufficiently quickly as $t \to \infty$. Using an approach which is now well established in the analysis of stochastic approximation algorithms in continuous time (e.g., [7, 74, 75, 77, 78]), we control such terms by rewriting them in terms of the solutions of some related Poisson equations. Condition C.1 ensures that these solutions are unique, and that they grow at most polynomially in a suitable sense (see Lemma D.14 in Appendix D).

[1] In fact, we only require that these properties hold for the function $L(\theta, x, \mu)$, as defined in (2.1) - (2.2). We find it more convenient, however, to specify this condition in terms of the functions $b(\theta, x)$ and $\phi(\theta, x, \mu)$.

We should remark that, for the sake of convenience and to remain in line with much of the recent literature, we have restricted our attention to the case in which the measure enters only linearly in the drift coefficient $B(\theta, x, \mu)$. As such, our main conditions are stated in terms of the functions $b : \mathbb{R}^p \times \mathbb{R}^d \times \mathbb{R}^d$ and $\phi : \mathbb{R}^p \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$. Our main results, however, can be extended straightforwardly to more general choices of interaction function, under suitable conditions on $B : \mathbb{R}^p \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}^d$. In particular, in the online setting, we simple require conditions which guarantee the existence of a unique invariant measure, and uniform-in-time propagation of chaos. As an example, we can replace Condition C.1(ii) by $||\nabla_\theta B(\theta, x, \mu)|| \leq K[1 + ||x||^q + \mu(|| \cdot ||^q)]$. Finally, we will require the following assumption on the initial condition.

Assumption D.1. *The initial law satisfies $\mu_0 \in \mathcal{P}_k(\mathbb{R}^d)$ for all $k \in \mathbb{N}$.*

This condition guarantees that the solutions of the McKean-Vlasov SDE and the associated IPS have bounded moments of all orders (see Proposition A.2), and so do their invariant measures (see Lemma D.1). In turn, this ensures that one can control the polynomial growth of the log-likelihood and its derivatives (in the offline case), and the polynomial growth of the solutions of the relevant Poisson equations (in the online case). We should remark that, in the offline case, we can significantly weaken this assumption: in particular, we only require that $\mu_0 \in \mathcal{P}_q(\mathbb{R}^d)$, where q is the order of the polynomial growth of the functions $b(\theta, \cdot)$ and $\phi(\theta, \cdot, \cdot)$ (see Condition C.1). One can also slightly relax this condition in the online case, though in a much more cumbersome fashion.[2] We note that this condition is trivially satisfied in the case that $x_0 \in \mathbb{R}^d$.

3.2. Offline Parameter Estimation. In the case of offline parameter estimation, we will require the following additional assumptions.

Assumption E.1. *For all $t > 0$, and for all $\theta \in \mathbb{R}^p$, the function $m_t : \mathbb{R}^p \to \mathbb{R}$, defined according to*

$$m_t(\theta) = \int_0^t \int_{\mathbb{R}^d} L(\theta, x, \mu_s)\mu_s(\mathrm{d}x)\mathrm{d}s$$

satisfies $\sup_{||\theta - \theta_0|| > \delta} m_t(\theta) < 0$ a.s. $\forall \delta > 0$.

Assumption E.2. *For all $t > 0$, the matrix $I_t(\theta_0) = [I_t(\theta_0)]_{k,l=1,\dots,p} \in \mathbb{R}^{p \times p}$, defined according to*

$$[I_t(\theta_0)]_{kl} = \int_0^t \int_{\mathbb{R}^d} [\nabla_\theta B(\theta_0, x, \mu_s)]_k [\nabla_\theta B(\theta_0, x, \mu_s)]_l \mu_s(\mathrm{d}x)\mathrm{d}s$$

is positive-definite, with $\lambda^T I_t(\theta_0)\lambda$ increasing for all $\lambda \in \mathbb{R}^p$, and $I_0(\theta_0) = 0$.

The first of these two conditions relates to parameter identifiability, guaranteeing the uniqueness of θ_0 as the optimal parameter in the sense of some asymptotic cost, and is necessary in order to establish consistency of the MLE as $N \to \infty$. It can be seen, in some

[2] In particular, in the online case, one requires $\mu_0 \in \mathcal{P}_k(\mathbb{R}^d)$, where k is the maximum order of polynomial growth of a solution of any of the relevant Poisson equations appearing in the proofs of Theorem 3.3 and Theorem 3.4.

sense, as an analogue of the classical condition used to obtain consistency in the long-time regime (e.g., [12], [53, pp. 137-139], [47, pp. 252 - 253] [69, Condition A_5]). It is also closely related to the so-called 'coercivity condition', introduced in [11], which appears in the study of non-parametric inference for IPSs (see also [48, 54, 55, 56]). Meanwhile, the second condition is necessary in order to establish asymptotic normality, and can be seen as a generalisation of a similar condition introduced in [44] (see also [9]).

We are now ready to state our two main results in the offline case.

Theorem 3.1. *Assume that Conditions A.1, B.1 - B.2, C.1, D.1, and E.1 hold. Let $\Theta \subseteq \mathbb{R}^p$ be a compact set, and suppose $\theta_0 \in \Theta$. Then, for all $t > 0$, $\hat{\theta}_t^N$ is a weakly consistent estimator of θ_0 as $N \to \infty$. That is, as $N \to \infty$,*

$$\hat{\theta}_t^N \xrightarrow{\mathbb{P}} \theta_0.$$

Proof. See Section 4.1. ■

Theorem 3.2. *Assume that Conditions A.1, B.1 - B.2, C.1, D.1, and E.1 - E.2 hold. Let $\Theta \subseteq \mathbb{R}^p$ be a compact set, and suppose $\theta_0 \in \Theta$. Then, for all $t > 0$, $N^{\frac{1}{2}}(\hat{\theta}_t^N - \theta_0)$ is asymptotically normal with mean zero and variance $I_t^{-1}(\theta_0)$. That is, as $N \to \infty$,*

$$N^{\frac{1}{2}}(\hat{\theta}_t^N - \theta_0) \xrightarrow{\mathcal{D}} \mathcal{N}(0, I_t^{-1}(\theta_0)).$$

Proof. See Section 4.2. ■

3.3. Online Parameter Estimation.

In the online case, we will first require the following standard condition on the learning rate.

Assumption F.1. *The learning rate $\gamma_t : \mathbb{R}_+ \to \mathbb{R}_+$ is a positive, non-increasing function such that $\int_0^\infty \gamma_t \mathrm{d}t = \infty$, $\int_0^\infty \gamma_t^2 \mathrm{d}t < \infty$, $\int_0^\infty \gamma_t' \mathrm{d}t < \infty$. Moreover, there exists $p > 0$ such that $\lim_{t\to\infty} \gamma_t^2 t^{2p+\frac{1}{2}} = 0$.*

This condition can be seen as the continuous-time analogue of the standard step-size condition used in the convergence analysis of stochastic approximation algorithms in discrete time (e.g., [72, 75]).

We now proceed with some additional assumptions, which will only be required for our $\mathbb{L}^2$ convergence results (Theorems 3.4, 3.4*, 3.4†, 3.4‡).

Assumption F.2. *Let $\Phi_{s,t} = \exp(-2\eta \int_s^t \gamma_u \mathrm{d}u)$, for the constant η defined below in Condition H.1. The learning rate $\gamma_t : \mathbb{R}_+ \to \mathbb{R}_+$ satisfies $\int_0^t \gamma_s^2 \Phi_{s,t} \mathrm{d}s = O(\gamma_t)$, $\int_0^t \gamma_t' \Phi_{s,t} \mathrm{d}s = O(\gamma_t)$, $\int_0^t \gamma_s \Phi_{s,t} \mathrm{d}s = O(1)$, $\int_0^t \gamma_s \Phi_{s,t} e^{-\lambda s} \mathrm{d}s = O(\gamma_t)$, and $\Phi_{1,t} = O(\gamma_t)$.*

This is another condition on the learning rate, first introduced in [77], and is specific to stochastic gradient descent in continuous time. A standard choice of learning rate which satisfies both of these conditions is $\gamma_t = C_\gamma (C_0 + t)^{-1}$, where $C_\gamma, C_0 > 0$ are positive constants such that $C_\gamma \eta > 1$.

In addition, we introduce the following two assumptions.

Assumption G.1. *There exists a positive constant $R < \infty$, and an almost everywhere positive function $\kappa : \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}$, such that, for all $||\theta|| \geq R$,*

$$\langle \nabla_\theta L(\theta, x, \mu), \theta \rangle \leq -\kappa(x, \mu) ||\theta||^2.$$

Assumption G.2. *Define the function* $\tau : \mathbb{R}^p \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}$ *according to*

$$\tau(\theta, x, \mu) = \left\langle \nabla_\theta B(\theta, x, \mu) \nabla_\theta B^T(\theta, x, \mu) \frac{\theta}{||\theta||}, \frac{\theta}{||\theta||} \right\rangle^{\frac{1}{2}}$$

Then, there exists $0 < q, K < \infty$ *such that, for all* $\theta, \theta' \in \mathbb{R}^p$*, for all* $x, y \in \mathbb{R}^d$*,*

$$|\tau(\theta, x) - \tau(\theta', x)|| \leq K||\theta - \theta'||(1 + ||x||^q + ||\mu(|| \cdot ||^2)||^{\frac{q}{2}})$$

These two conditions ensure, via the comparison theorem (e.g., [41, 87]), that the online parameter estimates generated by the McKean-Vlasov SDE and the IPS, namely $(\theta_t)_{t\geq 0}$ and $(\theta_t^{i,N})_{t\geq 0}$, have uniformly bounded moments (see Lemma 4.1). We refer to [45] for some more general conditions under which this result still holds. The first condition relates to the drift terms in the two parameter update equations, and can be seen as a recurrence condition; the second condition relates to the diffusion terms, and can be seen as an extension of Condition B.2(ii). In particular, in the case that $\theta \in \mathbb{R}$, Condition G.2 essentially reduces to Condition B.2(ii). This condition was introduced in [75], and has since also appeared in [7].

Finally, to establish consistency, we will require the following assumptions on the concavity of the log-likelihood.

Assumption H.1. *The function* $\tilde{\mathcal{L}}(\theta)$ *is strongly concave. That is, there exists* $\eta > 0$ *such that, for all* $\theta, \theta' \in \mathbb{R}^p$*,*

$$\tilde{\mathcal{L}}(\theta') \leq \tilde{\mathcal{L}}(\theta) + \nabla \tilde{\mathcal{L}}(\theta)^T(\theta' - \theta) - \frac{\eta}{2}||\theta' - \theta||^2.$$

Assumption H.1'. *The function* $\tilde{\mathcal{L}}^{i,N}(\theta)$ *is strongly concave, for all* $N \in \mathbb{N}$*, for all* $i = 1, \ldots, N$*. That is, there exists* $\eta^{i,N} > 0$ *such that, for all* $\theta, \theta' \in \mathbb{R}^p$*,*

$$\tilde{\mathcal{L}}^{i,N}(\theta') \leq \tilde{\mathcal{L}}^{i,N}(\theta) + \nabla \tilde{\mathcal{L}}^{i,N}(\theta)^T(\theta' - \theta) - \frac{\eta^{i,N}}{2}||\theta' - \theta||^2.$$

These conditions relate to the properties of the asymptotic log-likelihoods of the McKean Vlasov SDE and the IPS, respectively. They imply, in particular, that $\tilde{\mathcal{L}}(\theta)$ and $\tilde{\mathcal{L}}^{i,N}(\theta)$ have unique maximisers, say θ_* and θ_*^N. Under certain identifiability assumptions, these must in fact be equal to the true parameter θ_0 (e.g., [47]). We note that the first of these conditions is slightly weaker than the second: under the first assumption, we establish that $\theta_t^{i,N} \stackrel{\mathbb{L}^2}{\to} \theta_0$ as $t \to \infty$ and $N \to \infty$ (Theorem 3.4), while under the second assumption, we establish that $\theta_t^{i,N} \stackrel{\mathbb{L}^2}{\to} \theta_0$ as $t \to \infty$ for all $N \in \mathbb{N}$ (Theorem 3.4†). Thus, under the second assumption, there is no requirement to take the limit as $N \to \infty$. We also obtain a sharper $\mathbb{L}^2$ convergence rate.

We are now ready to state our main results in the online case. These results, categorised according to different cases introduced in Section 2, are summarised in Table 2.

Case I. In this case, we assume that we observe the trajectories of N particles $(x_t^i)_{t\geq 0}^{i=1,\ldots,N}$ of the IPS (1.4). We can thus generate online parameter estimates according to (2.8) or (2.9). We here show that, in the limit as $N \to \infty$ and $t \to \infty$, these parameter estimates can maximise $\tilde{\mathcal{L}}(\theta)$, the asymptotic log-likelihood of the McKean-Vlasov SDE. Thus, the proposed approach with finite N can be thought of as a principled approximate method for estimating the unknown parameter θ of the McKean-Vlasov SDE in an online fashion. In our first result, we establish $\mathbb{L}^1$ convergence of (2.8) and (2.9) to the stationary points of $\tilde{\mathcal{L}}(\theta)$.

Case	Theorems	Parameter Estimates	Objective Function		Convergence Rate
Case I	3.3 - 3.4	$\theta_t^{i,N}$ from (2.8)	$\tilde{\mathcal{L}}(\theta)$	MSVDE	$(K_1+K_2)\gamma_t + \frac{K_3}{N^{\frac{1}{2}}}$
		θ_t^{N} from (2.9)			$(K_1+\frac{K_2}{N})\gamma_t + \frac{K_3}{N^{\frac{1}{2}}}$
Case II	3.3* - 3.4*	$\theta_t^{[i,N]}$ from (2.8)	$\tilde{\mathcal{L}}(\theta)$	MSVDE	$(K_1^*+K_2^*)\gamma_t$
		$\theta_t^{[N]}$ from (2.9)			$(K_1^*+\frac{K_2^*}{N})\gamma_t$
Case III	3.3† - 3.4†	θ_t from (2.6)	$\tilde{\mathcal{L}}(\theta)$	MSVDE	$(K_1^\dagger+K_2^\dagger)\gamma_t$
Case I (*finite N*)	3.3‡ - 3.4‡	$\theta_t^{i,N}$ from (2.8)	$\tilde{\mathcal{L}}^{i,N}(\theta)$	IPS (Partial)	$(K_1^\ddagger+K_2^\ddagger)\gamma_t$
		θ_t^{N} from (2.9)	$\tilde{\mathcal{L}}^{N}(\theta)$	IPS (Complete)	$(K_1^\ddagger+\frac{K_2^\ddagger}{N})\gamma_t$

Table 2
Online Parameter Estimation: Summary of Main Results

Theorem 3.3. *Assume that Conditions A.1, B.1 - B.2, C.1, D.1, and F.1 hold. Then, in $\mathbb{L}^1$, it holds that*

$$\lim_{N\to\infty}\lim_{t\to\infty}||\nabla_\theta\tilde{\mathcal{L}}(\theta_t^{i,N})|| = \lim_{t\to\infty}\lim_{N\to\infty}||\nabla_\theta\tilde{\mathcal{L}}(\theta_t^{i,N})|| = 0,$$
$$\lim_{N\to\infty}\lim_{t\to\infty}||\nabla_\theta\tilde{\mathcal{L}}(\theta_t^{N})|| = \lim_{t\to\infty}\lim_{N\to\infty}||\nabla_\theta\tilde{\mathcal{L}}(\theta_t^{N})|| = 0.$$

Proof. See Section 4.3. ■

In our second result, under additional assumptions, we establish $\mathbb{L}^2$ convergence to the unique maximiser of $\tilde{\mathcal{L}}(\theta)$.

Theorem 3.4. *Assume that Conditions A.1, B.1 - B.2, C.1, D.1, F.1 - F.2, G.1 - G.2, and H.1 hold. Then, for sufficiently large t, and for $N \geq 1$, $1 \leq i \leq N$, there exist positive constants K_1, K_2, K_3 such that*

$$\mathbb{E}\left[||\theta_t^{i,N}-\theta_0||^2\right] \leq (K_1+K_2)\,\gamma_t + \frac{K_3}{N^{\frac{1}{2}}}, \tag{3.1}$$

$$\mathbb{E}\left[||\theta_t^{N}-\theta_0||^2\right] \leq (K_1+\frac{K_2}{N})\gamma_t + \frac{K_3}{N^{\frac{1}{2}}}, \tag{3.2}$$

Proof. See Section 4.4. ■

Case II. In this case, we assume that we observe independent sample paths $(x_t^i)_{t\geq 0}^{i=1,\dots,N}$ of the McKean-Vlasov SDE (1.1). We thus generate online parameter estimates according to (2.8) or (2.9), replacing $x_t^{i,N}$ by x_t^i, and μ_t^N by $\mu_t^{[N]}$ where appropriate. In this case, we obtain the following statement of our results, similarly to Case I.

Theorem 3.3*. *Assume that Conditions A.1, B.1 - B.2, C.1, D.1, and F.1 hold. Then, in* $\mathbb{L}^1$, *it holds that*

$$\lim_{N\to\infty}\lim_{t\to\infty}||\nabla_\theta\tilde{\mathcal{L}}(\theta_t^{[i,N]})|| = \lim_{t\to\infty}\lim_{N\to\infty}||\nabla_\theta\tilde{\mathcal{L}}(\theta_t^{[i,N]})|| = 0,$$
$$\lim_{N\to\infty}\lim_{t\to\infty}||\nabla_\theta\tilde{\mathcal{L}}(\theta_t^{[N]})|| = \lim_{t\to\infty}\lim_{N\to\infty}||\nabla_\theta\tilde{\mathcal{L}}(\theta_t^{[N]})|| = 0.$$

Theorem 3.4*. *Assume that Conditions A.1, B.1 - B.2, C.1, D.1, F.1 - F.2, G.1 - G.2, and H.1 hold. Then, for sufficiently large* t, *there exist positive constants* K_1^*, K_2^*, *such that*

$$\mathbb{E}\left[||\theta_t^{[i,N]} - \theta_0||^2\right] \leq (K_1^* + K_2^*)\,\gamma_t,$$
$$\mathbb{E}\left[||\theta_t^{[N]} - \theta_0||^2\right] \leq (K_1^* + \frac{K_2^*}{N})\gamma_t.$$

Proof. See Appendix F. ■

Let us briefly compare the results obtained in Case I (Theorems 3.3 - 3.4) and in Case II (Theorems 3.3* - 3.4*). As remarked previously, the online parameter estimates in both of these cases follow the same parameter update equations; the only difference is the assumed form of the data-generating model. It is thus expected that the results obtained in these two cases will be similar, if not identical. In Theorems 3.3 and 3.3*, this is indeed seen to be the case. These results establish that, regardless of the assumed form of the data-generating mechanism, the online parameter estimates generated via (2.8) or (2.9) converge to the stationary points of $\tilde{\mathcal{L}}(\theta)$ as $N\to\infty$ and $t\to\infty$. On the other hand, in Theorems 3.4 and 3.4*, a difference does arise between the two $\mathbb{L}^2$ convergence rates. In particular, in Case I (Theorem 3.4) there is an additional $O(N^{-\frac{1}{2}})$ term. We can interpret this term as a penalty for the mismatch between the likelihood implied by the assumed data-generating model in Case I, namely the IPS (1.4), and the likelihood implied by the McKean-Vlasov SDE (1.1) - (1.2), which is the function that we are seeking to optimise.

Case III. In this case, we assume that we can observe not only a sample path $(x_t)_{t\geq 0}$ of the non-linear SDE, but also its law $(\mu_t)_{t\geq 0}$. We can thus generate online parameter estimates according to (2.6). As remarked previously, this scenario is mainly of theoretical interest, since in practice it is very rarely possible to measure the law of the non-linear process. Nonetheless, we can still obtain the following statement of our results.

Theorem 3.3†. *Assume that Conditions A.1, B.1 - B.2, C.1, D.1, F.1 hold. Then, in* $\mathbb{L}^1$, *it holds that*

$$\lim_{t\to\infty}||\nabla_\theta\tilde{\mathcal{L}}(\theta_t)|| = 0.$$

Theorem 3.4†. *Assume that Conditions A.1, B.1 - B.2, C.1, D.1, F.1 - F.2, G.1 - G.2 and H.1 hold. Then, for sufficiently large* t, *there exist positive constants* $K_1^\dagger, K_2^\dagger$, *such that*

$$\mathbb{E}\left[||\theta_t - \theta_0||^2\right] \leq (K_1^\dagger + K_2^\dagger)\gamma_t.$$

Proof. See Appendix G. ■

These results represent an extension of [75, Theorem 2.4] and [77, Proposition 2.13], respectively, to the McKean-Vlasov case. We should remark that a rather more direct proof of these results may be possible, which does not require us to pass between the McKean-Vlasov SDE and the IPS, but which instead works directly with the non-linear equation. This approach would require strong regularity results on the solutions of a non-linear, non-local Poisson equation, similar to those recently obtained in [73]. We are not aware, however, of any such results appropriate to our case.

Case I (finite N). For the sake of completeness, we conclude this section by revisiting Case I, now under the condition that the number of particles N if fixed and finite, and that we are only interested in long-time asymptotics. In particular, our objective is now simply to maximise the asymptotic log-likelihood of the IPS, $\tilde{\mathcal{L}}^{i,N}(\theta)$. In this case, we have the following.

Theorem 3.3‡. *Assume that Conditions A.1, B.1 - B.2, C.1, D.1, and F.1 hold. Then, in* $\mathbb{L}^1$*, it holds that*

$$\lim_{t\to\infty}||\nabla_\theta\tilde{\mathcal{L}}^{i,N}(\theta_t^{i,N})||=0.$$
$$\lim_{t\to\infty}||\nabla_\theta\tilde{\mathcal{L}}^{N}(\theta_t^{N})||=0.$$

Theorem 3.4‡. *Assume that Conditions B.1 - B.2, C.1, D.1, F.1 - F.2, G.1 - G.2, and H.1' hold. Then, for sufficiently large t, there exist positive constants $K_1^{\ddagger}, K_2^{\ddagger}$, such that*

$$\mathbb{E}\left[||\theta_t^{i,N}-\theta_0||^2\right]\leq\left(K_1^{\ddagger}+K_2^{\ddagger}\right)\gamma_t. \tag{3.3}$$

$$\mathbb{E}\left[||\theta_t^{N}-\theta_0||^2\right]\leq(K_1^{\ddagger}+\frac{K_2^{\ddagger}}{N})\gamma_t. \tag{3.4}$$

Proof. See Appendix H. ∎

Theorem 3.4‡ demonstrates that, if the asymptotic log-likelihood of the IPS is sufficiently well-behaved (i.e., strongly concave) for finite values of $N\in\mathbb{N}$, then the parameter estimate generated using the IPS is guaranteed to converge to the true parameter value as $t\to\infty$ for all values of $N\in\mathbb{N}$. In particular, it is not necessary to take the limit as $N\to\infty$. This is clear upon comparison of the bounds (3.1) - (3.2) in Theorem 3.4 with the bounds (3.3) - (3.4) in Theorem 3.4†. Broadly speaking, the additional term appearing in the convergence rates in Theorem 3.4 can be regarded as upper bounds on the difference between the maximisers of $\tilde{\mathcal{L}}(\cdot)$ and $\tilde{\mathcal{L}}^{i,N}(\cdot)$, which only arise if $\tilde{\mathcal{L}}(\cdot)$ is strongly concave but $\tilde{\mathcal{L}}^{i,N}(\cdot)$ is not. That is, if Condition H.1 is satisfied but Condition H.1' is not.

4. Proof of Main Results.

In this section, we provide proofs of our main results. Many of these proofs will rely on additional auxiliary lemmas; in the interest of brevity, the statements and proofs of these lemmas have been deferred to the appendices.

4.1. Proof of Theorem 3.1.

We begin by establishing consistency of the offline MLE as $N\to\infty$. We should emphasise that, throughout this proof, the value of t will be fixed and finite. This being said, our method of proof will broadly follow the classical approach for establishing strong consistency of the MLE in a different asymptotic regime, namely, in the limit as $t \to \infty$ (e.g., [12]). Since we consider an entirely different asymptotic regime, however, at times we will need to rely on slightly different arguments (e.g., Lemma C.1), and, of course, different conditions (e.g., Condition E.1).

Proof. Let $\mathbb{P}^{\theta}_{t,N}$ denote the probability measure induced by $(x_s^{\theta,i,N})_{s\in[0,t]}^{i=1,\dots,N}$. We begin with the observation that, since $\Theta \subseteq \mathbb{R}^p$ is compact, for all $t \geq 0$, and for all $N \in \mathbb{N}$, there exists $\hat{\theta}_t^N \in \Theta$ such that

$$\frac{\mathrm{d}\mathbb{P}^{\theta}_{t,N}}{\mathrm{d}\mathbb{P}^{\theta_0}_{t,N}}\bigg|_{\theta=\hat{\theta}_t^N} \geq \frac{\mathrm{d}\mathbb{P}^{\tilde{\theta}}_{t,N}}{\mathrm{d}\mathbb{P}^{\theta_0}_{t,N}} \quad \text{a.s.}$$

for all $\tilde{\theta} \in \Theta$. We thus have, setting $\tilde{\theta} = \theta_0$, that $\mathrm{d}\mathbb{P}^{\theta}_{t,N}/\mathrm{d}\mathbb{P}^{\theta_0}_{t,N}|_{\theta=\hat{\theta}_t^N} \geq 1$ a.s., from which it follows straightforwardly that $\mathcal{L}_t^N(\hat{\theta}_t^N) = \log[\mathrm{d}\mathbb{P}^{\theta}_{t,N}/\mathrm{d}\mathbb{P}^{\theta_0}_{t,N}]_{\theta=\hat{\theta}_t^N} \geq 0$ a.s. It follows, using the definition of the log-likelihood, that, almost surely,

$$\frac{1}{N}\sum_{i=1}^N \int_0^t \langle G(\theta, x_s^{i,N}, \mu_s^N), \mathrm{d}w_s^i\rangle_{\theta=\hat{\theta}_t^N} \geq \frac{1}{2N}\sum_{i=1}^N \int_0^t \left|\left|G(\hat{\theta}_t^N, x_s^{i,N}, \mu_s^N)\right|\right|^2 \mathrm{d}s \geq 0.$$

In addition, by Lemma C.1, we have that $\frac{1}{N}\sum_{i=1}^N \int_0^t \langle G(\theta, x_s^{i,N}, \mu_s^N), \mathrm{d}w_s^i\rangle_{\theta=\hat{\theta}_t^N} \stackrel{\mathbb{P}}{\to} 0$ as $N \to \infty$. It follows straightforwardly that, as $N \to \infty$,

$$\lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^N \int_0^t \left|\left|G(\hat{\theta}_t^N, x_s^{i,N}, \mu_s^N)\right|\right|^2 \mathrm{d}s \stackrel{\mathbb{P}}{\to} 0. \tag{4.1}$$

We next observe, making use of the Cauchy-Schwarz inequality, that

$$\begin{aligned}
&\left|\frac{1}{N}\sum_{i=1}^N \int_0^t ||G(\theta, x_s^{i,N}, \mu_s^N)||^2 \mathrm{d}s - \frac{1}{N}\sum_{i=1}^N \int_0^t ||G(\theta', x_s^{i,N}, \mu_s^N)||^2 \mathrm{d}s\right| \\
&\leq \left[\frac{1}{N}\sum_{i=1}^N \int_0^t ||G(\theta, x_s^{i,N}, \mu_s^N) - G(\theta', x_s^{i,N}, \mu_s^N)||^2 \mathrm{d}s\right]^{\frac{1}{2}} \\
&\quad \cdot \left[\frac{1}{N}\sum_{i=1}^N \int_0^t ||G(\theta, x_s^{i,N}, \mu_s^N) + G(\theta', x_s^{i,N}, \mu_s^N)||^2 \mathrm{d}s\right]^{\frac{1}{2}} \\
&\leq K||\theta - \theta'|| \left[\frac{1}{N}\sum_{i=1}^N \int_0^t ||\frac{1}{N}\sum_{j=1}^N (1 + ||x_s^{i,N}||^q + ||x_s^{j,N}||^q)||^2 \mathrm{d}s\right]^{\frac{1}{2}} \\
&\qquad \cdot \left[\frac{2}{N}\sum_{i=1}^N \left[\int_0^t ||G(\theta, x_s^{i,N}, \mu_s^N)||^2 \mathrm{d}s + \int_0^t ||G(\theta', x_s^{i,N}, \mu_s^N)||^2 \mathrm{d}s\right]\right]^{\frac{1}{2}}
\end{aligned}$$

where in the final line we have used Conditions C.1(i) - C.1(ii). In addition, the uniform moment bounds on the IPS (Proposition A.2), which follow from Condition D.1, together with Condition C.1(ii), imply that all terms on the RHS of this inequality are bounded. It follows immediately that the function $\frac{1}{N}\sum_{i=1}^{N}\int_0^t ||G(\theta,x_s^{i,N},\mu_s^N)||^2\mathrm{d}s$ is Lipschitz continuous in θ, uniformly in N. Combining this with (4.1), we have that, as $N\to\infty$,

$$\hat{\theta}_t^N \xrightarrow{\mathbb{P}} \mathcal{D}_t^N = \left\{\theta\in\Theta : \lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N}\int_0^t ||G(\theta,x_s^{i,N},\mu_s^N)||^2\mathrm{d}s = 0\right\} \tag{4.2}$$

by which we we mean more precisely that $\inf_{\theta\in\mathcal{D}_t}||\hat{\theta}_t^N-\theta|| \xrightarrow{\mathbb{P}} 0$ as $N\to\infty$. It remains to observe that, by a repeated application of the McKean-Vlasov Law of Large Numbers (Proposition A.6), as $N\to\infty$, and for all $t>0$, we have

$$\mathcal{D}_t^N \xrightarrow{\mathbb{P}} \mathcal{D}_t = \left\{\theta\in\Theta : \int_0^t\left[\int_{\mathbb{R}^d}||G(\theta,x,\mu_s)||^2\,\mu_s(\mathrm{d}x)\right]\mathrm{d}s = 0\right\} = \{\theta_0\}, \tag{4.3}$$

where in the second equality we have also made use of the identifiability condition in Condition E.1. It follows immediately, combining (4.2) and (4.3) that, for all fixed $t>0$, as $N\to\infty$, $\hat{\theta}_t^N \xrightarrow{\mathbb{P}} \theta_0$. ■

4.2. Proof of Theorem 3.2. The proof of this theorem, similarly to the previous proof, combines well known techniques used to establishing strong consistency of the MLE as $t\to\infty$ (e.g., [47]) with ideas relevant to the asymptotic regime as $N\to\infty$ (e.g., [44]). Once again, we emphasise that throughout this proof the value of t will be fixed and finite, and we will consider the limit only as $N\to\infty$.

Proof. We begin by considering a Taylor expansion of $\nabla_\theta\mathcal{L}_t^N(\theta)$ around the true value of the parameter $\theta=\theta_0$, viz,

$$0 = \nabla_\theta\mathcal{L}_t^N(\hat{\theta}_t^N) = \nabla_\theta\mathcal{L}_t^N(\theta_0) + (\theta_t^N-\theta_0)\nabla_\theta^2\mathcal{L}_t(\bar{\theta}_t^N)$$

where $\bar{\theta}_t^N$ is point in the segment connecting $\hat{\theta}_t^N$ and θ_0. The validity of this expansion is based on the sample path continuity of the log-likelihood and its derivatives. It follows that

$$N^{\frac{1}{2}}(\hat{\theta}_t^N-\theta_0)\nabla_\theta^2\mathcal{L}_t^N(\bar{\theta}_t^N) = -N^{\frac{1}{2}}\nabla_\theta\mathcal{L}_t^N(\theta_0)$$

To deal with the terms in this equation, we will rely extensively on a multivariate version of Rebolledo's Central Limit Theorem [70], as stated in [44, Corollary to Theorem 2]. Let us begin by considering the RHS. First observe that

$$\begin{aligned} N^{\frac{1}{2}}\nabla_\theta\mathcal{L}_t^N(\theta_0) &= N^{-\frac{1}{2}}\sum_{i=1}^{N}\int_0^t\langle\nabla_\theta B(\theta_0,x_s^{i,N},\mu_s^N),\mathrm{d}w_s^i\rangle \\ &\quad + N^{-\frac{1}{2}}\sum_{i=1}^{N}\int_0^t\nabla_\theta B(\theta_0,x_s^{i,N},\mu_s^N)G(\theta_0,x_s^{i,N},\mu_s^N)\mathrm{d}s \\ &= N^{-\frac{1}{2}}\sum_{i=1}^{N}\int_0^t\langle\nabla_\theta B(\theta_0,x_s^{i,N},\mu_s^N),\mathrm{d}w_s^i\rangle \end{aligned}$$

where in the second line we have used the fact that, by definition, $G(\theta_0, \cdot, \cdot) = 0$ is identically equal to zero. It follows, using also Condition C.1(ii) (the polynomial growth property) and Proposition A.2 (uniform moment bounds for the solutions of the IPS), that for all $t \geq 0$, $(N^{\frac{1}{2}} \nabla_\theta \mathcal{L}_t^N(\theta_0))_{N \in \mathbb{N}}$ is a sequence of local square integrable martingales, which implies that the first condition of [44, Corollary to Theorem 2] is satisfied.

Next, observe that the process $(N^{\frac{1}{2}} \nabla_\theta \mathcal{L}_t^N(\theta))_{t \geq 0}$ is continuous (in time), and thus the second condition of [44, Corollary to Theorem 2] (the Lindenberg condition) is satisfied. Finally, we have that, for all $k, l = 1, \ldots, p$, as $N \to \infty$,

$$
\begin{aligned}
&\left\langle \left[N^{\frac{1}{2}} \nabla_\theta \mathcal{L}_t^N(\theta_0)\right]_k, \left[N^{\frac{1}{2}} \nabla_\theta \mathcal{L}_t^N(\theta_0)\right]_l \right\rangle \\
&\quad = \frac{1}{N} \sum_{i=1}^N \int_0^t [\nabla_\theta B(\theta_0, x_s^{i,N}, \mu_s^N)]_k [\nabla_\theta B(\theta_0, x_s^{i,N}, \mu_s^N)]_l \mathrm{d}s \\
&\quad \xrightarrow{\mathbb{P}} \int_0^t \left[\int_{\mathbb{R}^d} [\nabla_\theta B(\theta_0, x, \mu_s)]_k \, [\nabla_\theta B(\theta_0, x, \mu_s)]_l \, \mu_s(\mathrm{d}x) \right] \mathrm{d}s = [I_t(\theta_0)]_{kl} \,,
\end{aligned} \tag{4.4}
$$

where in the final line, we have used a repeated application of the weak law of large numbers for the empirical distribution of the IPS (Proposition A.6), and the definition of $I_t(\theta)$ (see Condition E.2). Thus, the final condition in [44, Corollary to Theorem 2] is satisfied. It follows from this result that

$$
- N^{\frac{1}{2}} \nabla_\theta \mathcal{L}_t^N(\theta_0) \xrightarrow{\mathcal{D}} \mathcal{N}_p(0, I_t(\theta_0)).
$$

It remains to prove that $\nabla_\theta^2 \mathcal{L}_t^N(\bar{\theta}_t^N) \xrightarrow{\mathbb{P}} -I_t(\theta_0)$. In fact, since $\hat{\theta}_t^N \xrightarrow{\mathbb{P}} \theta_0$ as $N \to \infty$ by Theorem 3.1, the continuity of $\{\nabla_\theta^2 \mathcal{L}_t^N(\cdot)\}_{N \in \mathbb{N}}$ in θ implies that this limit holds provided we can establish that $\nabla_\theta^2 \mathcal{L}_t^N(\theta_0) \xrightarrow{\mathbb{P}} -I_t(\theta_0)$. To do so, let us begin with the observation, via a simple calculation, we have that

$$
\begin{aligned}
\left[\nabla_\theta^2 \mathcal{L}_t^N(\theta_0)\right]_{kl} &= \frac{1}{N} \sum_{i=1}^N \int_0^t \left[\nabla_\theta^2 B(\theta_0, x_s^{i,N}, \mu_s^N)\right]_{kl} \mathrm{d}w_s^i \\
&\quad - \frac{1}{N} \sum_{i=1}^N \int_0^t [\nabla_\theta B(\theta_0, x_s^{i,N}, \mu_s^N)]_k [\nabla_\theta B(\theta_0, x_s^{i,N}, \mu_s^N)]_l \mathrm{d}s
\end{aligned} \tag{4.5}
$$

Arguing as in the proof of Lemma C.1 (see Appendix C), we can show that, as $N \to \infty$, we have

$$
\frac{1}{N} \sum_{i=1}^N \int_0^t \nabla_\theta^2 B(\theta_0, x_s^{i,N}, \mu_s^N) \mathrm{d}w_s^i \xrightarrow{\mathbb{P}} 0. \tag{4.6}
$$

Moreover, we have already established, c.f. (4.4), that, as $N \to \infty$, we have

$$
\frac{1}{N} \sum_{i=1}^N \int_0^t [\nabla_\theta B(\theta_0, x_s^{i,N}, \mu_s^N)]_k [\nabla_\theta B(\theta_0, x_s^{i,N}, \mu_s^N)]_l \mathrm{d}s \xrightarrow{\mathbb{P}} [I_t(\theta_0)]_{kl}. \tag{4.7}
$$

It follows, substituting (4.6) - (4.7) into (4.5), that $\nabla_\theta^2 \mathcal{L}_t^N(\theta_0) \xrightarrow{\mathbb{P}} -I_t(\theta_0)$ as $N \to \infty$. By our previous remarks, this completes the proof. ∎

4.3. Proof of Theorem 3.3. We will prove Theorem 3.3 via a sequence of intermediate Lemmas. In fact, once these lemmas have been established, the proof itself follows straightforwardly.

Before we present this proof, it will first be necessary to introduce some additional notation. Recall from Section 2.1 (e.g., Table 1) that $(x_t^i)_{t\geq 0}$ denotes a solution of the McKean-Vlasov SDE (1.1), where the Brownian motion $(w_t)_{t\geq 0}$ is replaced by $(w_t^i)_{t\geq 0}$. We will now also write $(\mu_t^i)_{t\geq 0}$ to denote the law of this solution,[3] and, for the corresponding log-likelihood function, write

$$\mathcal{L}_t^i(\theta) = \int_0^t L(\theta, x_s^i, \mu_s^i)\mathrm{d}s + \int_0^t \langle G(\theta, x_s^i, \mu_s^i), \mathrm{d}w_s^i\rangle.$$

We can now proceed to the proof of Theorem 3.3.

Proof. Using the triangle inequality, we can decompose the asymptotic log-likelihood of interest as

$$\begin{aligned}||\nabla_\theta\tilde{\mathcal{L}}(\theta_t^{i,N})|| \leq\; & \underbrace{||\nabla_\theta\tilde{\mathcal{L}}(\theta_t^{i,N}) - \tfrac{1}{t}\nabla_\theta\mathcal{L}_t^i(\theta_t^{i,N})||}_{\to 0 \text{ as } t\to\infty \;\; \forall N\in\mathbb{N} \text{ by Lemma 3.4.A}} + \underbrace{||\tfrac{1}{t}\nabla_\theta\mathcal{L}_t^i(\theta_t^{i,N}) - \tfrac{1}{t}\nabla_\theta\mathcal{L}_t^{i,N}(\theta_t^{i,N})||}_{\to 0 \text{ as } N\to\infty \;\; \forall t\in\mathbb{R}_+ \text{ by Lemma 3.4.C}} \\ & + \underbrace{||\tfrac{1}{t}\nabla_\theta\mathcal{L}_t^{i,N}(\theta_t^{i,N}) - \nabla_\theta\tilde{\mathcal{L}}^{i,N}(\theta_t^{i,N})||}_{\to 0 \text{ as } t\to\infty \;\; \forall N\in\mathbb{N} \text{ by Lemma 3.4.B}} + \underbrace{||\nabla_\theta\tilde{\mathcal{L}}^{i,N}(\theta_t^{i,N})||}_{\to 0 \text{ as } t\to\infty \;\; \forall N\in\mathbb{N} \text{ by Lemma 3.4.D}}.\end{aligned}$$

or, almost identically,

$$\begin{aligned}||\nabla_\theta\tilde{\mathcal{L}}(\theta_t^{N})|| \leq\; & \underbrace{||\nabla_\theta\tilde{\mathcal{L}}(\theta_t^{N}) - \tfrac{1}{t}\nabla_\theta\mathcal{L}_t^i(\theta_t^{N})||}_{\to 0 \text{ as } t\to\infty \;\; \forall N\in\mathbb{N} \text{ by Lemma 3.4.A}} + \underbrace{||\tfrac{1}{t}\nabla_\theta\mathcal{L}_t^i(\theta_t^{N}) - \tfrac{1}{t}\nabla_\theta\mathcal{L}_t^{N}(\theta_t^{N})||}_{\to 0 \text{ as } N\to\infty \;\; \forall t\in\mathbb{R}_+ \text{ by Lemma 3.4.C}} \\ & + \underbrace{||\tfrac{1}{t}\nabla_\theta\mathcal{L}_t^{N}(\theta_t^{N}) - \nabla_\theta\tilde{\mathcal{L}}^{N}(\theta_t^{N})||}_{\to 0 \text{ as } t\to\infty \;\; \forall N\in\mathbb{N} \text{ by Lemma 3.4.B}} + \underbrace{||\nabla_\theta\tilde{\mathcal{L}}^{N}(\theta_t^{N})||}_{\to 0 \text{ as } t\to\infty \;\; \forall N\in\mathbb{N} \text{ by Lemma 3.4.D}}.\end{aligned}$$

where $\tilde{\mathcal{L}}^{i,N}(\theta)$ and $\tilde{\mathcal{L}}^{N}(\theta)$ are defined in Lemma 3.4.B. In both of these inequalities, all of the stated limits hold in $\mathbb{L}^1$. This completes the proof. ■

Before we proceed to the proofs of the intermediate Lemmas 3.4.A - 3.4.D, it is instructive to provide a brief high level overview.

(i) In Lemma 3.4.A, we establish the existence of $\tilde{\mathcal{L}}(\theta)$, the asymptotic log-likelihood of the McKean-Vlasov SDE, as well as its derivatives. We provide explicit expressions for these functions in terms of the unique invariant measure of the McKean-Vlasov SDE, prove an appropriate convergence result as $t\to\infty$ (both a.s. and in $\mathbb{L}^1$), and establish convergence rates.

(ii) In Lemma 3.4.B, we establish the existence of $\tilde{\mathcal{L}}^{i,N}(\theta)$ and $\tilde{\mathcal{L}}^{N}(\theta)$, the 'marginal' and 'joint' asymptotic log-likelihoods of the IPS, as well as their derivatives. As above,

[3] We remark that $(\mu_t)_{t\geq 0}^i = (\mu_t)_{t\geq 0}$ for all $i = 1,\ldots,N$. Nonetheless, we will use this notation to emphasise that we are considering solution of the McKean-Vlasov SDE with Brownian motion $(w_t^i)_{t\geq 0}$.

we provide explicit expressions for these functions in terms of the unique invariant measure of the IPS, prove an appropriate convergence result as $t \to \infty$ (both a.s. and in $\mathbb{L}^1$), and establish convergence rates.

(iii) In Lemma 3.4.C, we prove that, for all $t \geq 0$, the gradient of the asymptotic log-likelihood(s) of the IPS converge to the gradient of the asymptotic log-likelihood of the McKean-Vlasov SDE as $N \to \infty$ (in $\mathbb{L}^1$). We also provide $\mathbb{L}^1$ convergence rates. The proof of this result relies on classical uniform-in-time propagation of chaos results.

(iv) In Lemma 3.4.D, we establish that, for all $N \in \mathbb{N}$, the gradient of the asymptotic log-likelihood(s) of the IPS, evaluated at the relevant online parameter updates generated by the IPS, converges to zero as $t \to \infty$ (both a.s. and in $\mathbb{L}^1$). This result can be seen as a generalisation of [75, Theorem 2.4].

4.3.1. Proof of Lemma 3.4.A.

Lemma 3.4.A. *Assume that Conditions B.1 - B.2, C.1, and D.1 hold. Then the processes* $\frac{1}{t}\nabla_\theta^m \mathcal{L}_t^i(\theta)$, $m = 0, 1, 2$, *converge, both a.s. and in* $\mathbb{L}^1$, *to the functions*

$$\nabla_\theta^m \tilde{\mathcal{L}}(\theta) = \int_{\mathbb{R}^d} \nabla_\theta^m L(\theta, x, \mu_\infty)\mu_\infty(\mathrm{d}x).$$

In addition, there exist positive constants K_m^1, K_m^2 *such that*

$$\left|\left|\mathbb{E}\left[\frac{1}{t}\nabla_\theta^m \mathcal{L}_t^i(\theta) - \nabla_\theta^m \tilde{\mathcal{L}}(\theta)\right]\right|\right| \leq \frac{K_m^1(1 - e^{-\lambda t})}{\lambda t} + \frac{K_m^2(1+\sqrt{t})^{\frac{1}{2}}}{t^{\frac{1}{2}}}.$$

Proof. We will prove Lemma 3.4.A for $m = 0$, with $m = 1, 2$ proved similarly. For $m = 1, 2$, we remark only that the processes $\frac{1}{t}\nabla_\theta^m \mathcal{L}_t^i(\theta)$, and hence also $\nabla_\theta^m \tilde{\mathcal{L}}(\theta)$, exist due to Condition C.1. With this established, the proof when $m = 1, 2$ is essentially identical to the proof when $m = 0$. Let us begin by recalling the definition of $\frac{1}{t}\mathcal{L}_t^i(\theta)$, viz

$$\frac{1}{t}\mathcal{L}_t^i(\theta) = \underbrace{\frac{1}{t}\int_0^t L(\theta, x_s^i, \mu_s^i)\mathrm{d}s}_{I_1^N(\theta,t)} + \underbrace{\frac{1}{t}\int_0^t \langle G(\theta, x_s^i, \mu_s^i), \mathrm{d}w_s^i\rangle}_{I_2^N(\theta,t)} \tag{4.8}$$

We first consider the first term on the RHS. We will characterise the asymptotic behaviour of this term via the following decomposition

$$\underbrace{\frac{1}{t}\int_0^t L(\theta, x_s^i, \mu_s^i)\mathrm{d}s}_{I_1^N(\theta,t)} = \underbrace{\frac{1}{t}\int_0^t \left[L(\theta, x_s^i, \mu_s^i) - L(\theta, x_s^i, \mu_\infty)\right]\mathrm{d}s}_{I_{1,1}^N(\theta,t)} + \underbrace{\frac{1}{t}\int_0^t L(\theta, x_s^i, \mu_\infty)\mathrm{d}s}_{I_{1,2}^N(\theta,t)} \tag{4.9}$$

where μ_∞ is the unique invariant measure of the McKean-Vlasov SDE, which exists via Proposition A.3 (see Appendix A). We begin with the observation that, as $t \to \infty$,

$$\frac{1}{t}\int_0^t [L(\theta, x_s^i, \mu_s^i) - L(\theta, x_s^i, \mu_\infty)]\mathrm{d}s \xrightarrow{\text{a.s.}} 0, \tag{4.10}$$

$$\frac{1}{t}\int_0^t L(\theta, x_s^i, \mu_\infty)\mathrm{d}s \xrightarrow[\mathbb{L}^1]{\text{a.s.}} \tilde{\mathcal{L}}(\theta), \tag{4.11}$$

the former by Proposition A.3, and the latter by an appropriate version of the ergodic theorem (e.g., [71, Chapter X]). Let us now demonstrate that $I_{1,1}^N(\theta,t)$ also converges to zero in $\mathbb{L}^1$. Using Lemma D.5, we can write

$$\begin{aligned}\big|\big|L(\theta,x_s^i,\mu_s^i)-L(\theta,x_s^i,\mu_\infty)\big|\big| &\leq K\mathbb{W}_2(\mu_s^i,\mu_\infty)\left[1+||x_s^i||^q+\mu_\infty(||\cdot||^q)+\mu_s^i(||\cdot||^q)\right]\\ &\leq K\big[1+||x_s^i||^q\big]e^{-\lambda s}\end{aligned}$$

where in the second line we have additionally made use of Proposition A.2 (moment bounds for the McKean-Vlasov SDE), Proposition A.3 (the exponential contractivity of the McKean-Vlasov SDE), and Lemma D.1 (moment bounds for the invariant measure of the McKean-Vlasov SDE). It follows straightforwardly, making use once more of Proposition A.2, and allowing the value of K to change from line to line, that

$$\mathbb{E}[|I_{1,1}^N(t)|]\leq\frac{1}{t}\int_0^t K\big(1+\mathbb{E}\big[||x_s^i||^q\big]\big)e^{-\lambda s}\mathrm{d}s\leq\frac{K}{t}\int_0^t e^{-\lambda s}\mathrm{d}s\leq\frac{K(1-e^{-\lambda t})}{\lambda t},\tag{4.12}$$

so that the convergence of $I_{1,1}^N(\theta,t)$ to zero does also hold in $\mathbb{L}^1$. We thus have, substituting (4.10) - (4.11) into (4.9), that $I_1^N(\theta,t)\to\tilde{\mathcal{L}}(\theta)$, both a.s. and in $\mathbb{L}^1$.

Let us now try to establish the convergence rate of this term. We have already established a (non-asymptotic) bound for $I_{1,1}^N(\theta,t)$, so it remains to consider $I_{1,2}^N(\theta,t)$. We can bound the deviation between this term and the asymptotic log-likelihood using arguments similar to those found in, for example, [29]. First note that, using Lemma D.5 and Lemma D.1 (moment bounds for the invariant measure of the McKean-Vlasov SDE), we have

$$|L(\theta,x,\mu_\infty)-L(\theta,y,\mu_\infty)|\leq K||x-y||[1+||x||^q+||y||^q].$$

We can thus utilise Lemma D.3 to obtain

$$\left|\mathbb{E}_{x_0^i}[L(\theta,x,\mu_\infty)]-\int_{\mathbb{R}^d}L(\theta,y,\mu_\infty)\mu_\infty(\mathrm{d}y)\right|\leq K\left[1+||x_0^i||^q\right]e^{-\lambda s}$$

from which, in particular, it follows that

$$\begin{aligned}|\mathbb{E}[I_{1,2}(\theta,t)]| &\leq\left|\mathbb{E}_{x_0^i}\left[\frac{1}{t}\int_0^t L(\theta,x_s^i,\mu_\infty)\mathrm{d}s-\int_{\mathbb{R}^d}L(\theta,y,\mu_\infty)\mu_\infty(\mathrm{d}y)\right]\right| \qquad (4.13)\\ &\leq\frac{1}{t}\int_0^t\left|\mathbb{E}_{x_0^i}[L(\theta,x_s^i,\mu_\infty)]-\int_{\mathbb{R}^d}L(\theta,y,\mu_\infty)\mu_\infty(\mathrm{d}y)\right|\mathrm{d}s\\ &\leq\frac{K(1-e^{-\lambda t})}{\lambda t}\left[1+||x_0^i||^q\right]\leq\frac{K(1-e^{-\lambda t})}{\lambda t}. \qquad (4.14)\end{aligned}$$

where, as previously, we have allowed the value of the constant K to change from line to line. Substituting (4.12) and (4.14) into (4.9), we thus have that, for some $K_0^1>0$,

$$\left|\mathbb{E}\left[\frac{1}{t}\int_0^t L(\theta,x_s^i,\mu_s^i)\mathrm{d}s-\tilde{\mathcal{L}}(\theta)\right]\right|\leq\frac{K_0^1(1-e^{-\lambda t})}{\lambda t}.\tag{4.15}$$

We now turn our attention $I_2^N(\theta,t)$, the second term in (4.8). We begin with the observation that, by the Itô's isometry, Condition C.1(ii) (the polynomial growth of G), Proposition A.2 (the bounded moments of the McKean-Vlasov SDE), and Lemma D.2 (the asymptotic growth rate of the moments of the McKean-Vlasov SDE), we have that

$$\begin{aligned}\mathbb{E}\left[\left|\int_0^t \langle G(\theta,x_s^i,\mu_s^i),\mathrm{d}w_s^i\rangle\right|^2\right] &= \mathbb{E}\left[\int_0^t ||G(\theta,x_s^i,\mu_s^i)||^2\mathrm{d}s\right] \\ &\leq \mathbb{E}\left[\int_0^t K\left(1+||x_s^i||^q+\mathbb{E}\left[||x_s^i||^q\right]\right)\mathrm{d}s\right] \\ &\leq Kt\left[1+\mathbb{E}[\sup_{0\leq s\leq t}||x_s^i||^q]\right] \leq Kt\left[1+\sqrt{t}\right]\end{aligned} \tag{4.16}$$

where the value of the constant K is allowed to change from line to line. It follows, making use of the triangle inequality and the Hölder inequality that, for some $K_0^2>0$, we have

$$\left|\mathbb{E}\left[\frac{1}{t}\langle G(\theta,x_s^i,\mu_s^i),\mathrm{d}w_s^i\rangle\right]\right| \leq \frac{K_0^2(1+\sqrt{t})^{\frac{1}{2}}}{t^{\frac{1}{2}}}, \tag{4.17}$$

so that this term converges in $\mathbb{L}^1$ to zero, and we have the required rate. It remains only to demonstrate a.s. convergence of this term to zero. To do so, consider the local martingale

$$M_t=\int_0^t\frac{1}{s}\langle G(\theta,x_s^i,\mu_s^i),\mathrm{d}w_s^i\rangle = \frac{1}{t}\int_0^t\langle G(\theta,x_s^i,\mu_s^i),\mathrm{d}w_s^i\rangle + \int_0^t\frac{1}{s^2}\left[\int_0^s\langle G(\theta,x_u^i,\mu_u^i),\mathrm{d}w_u^i\rangle\right]\mathrm{d}s,$$

where the second line follows from Itô's Lemma. Using the Itô isometry, Condition C.1(ii) (the polynomial growth of G), and Proposition A.2 (the bounded moments of the McKean-Vlasov SDE), and arguing similarly to above, we have

$$\sup_{t>0}\mathbb{E}\left[|M_t|^2\right]=\mathbb{E}\left[\int_0^\infty\frac{1}{s^2}\mathbb{E}\left[||G(\theta,x_s^i,\mu_s^i)||^2\right]\mathrm{d}s\right]\leq K\left[\int_0^t\frac{1}{s^2}\left(1+\mathbb{E}\left[||x_s^i||^q\right]\right)\mathrm{d}s\right]<\infty. \tag{4.18}$$

By Doob's martingale convergence theorem [27], there thus exists a finite random variable M_∞ such that $M_t\to M_\infty$ a.s. It follows immediately that $\frac{1}{t}\int_0^t\langle G(\theta,x_s^i,\mu_s^i),\mathrm{d}w_s^i\rangle$ also converges to zero a.s., as claimed. Putting everything together, we thus have that $\frac{1}{t}\mathcal{L}_t^i(\theta)$ converges to $\tilde{\mathcal{L}}(\theta)$ both a.s. and in $\mathbb{L}^1$, and, combining (4.8), (4.15) and (4.17), that

$$\begin{aligned}\left|\mathbb{E}\left[\frac{1}{t}\mathcal{L}_t^i(\theta)-\tilde{\mathcal{L}}(\theta)\right]\right| &\leq \left|\mathbb{E}\left[\frac{1}{t}\int_0^t L(\theta,x_s^i,\mu_s^i)\mathrm{d}s-\tilde{\mathcal{L}}(\theta)\right]\right| + \left|\mathbb{E}\left[\frac{1}{t}\int_0^t\langle G(\theta,x_s^i,\mu_s^i),\mathrm{d}w_s^i\rangle\right]\right| \\ &\leq \frac{K_0^1(1-e^{-\lambda t})}{\lambda t}+\frac{K_0^2(1+\sqrt{t})^{\frac{1}{2}}}{t^{\frac{1}{2}}}.\end{aligned}$$

■

4.3.2. Proof of Lemma 3.4.B.

Additional Notation. In order to state and prove the next Lemma, it will be useful to introduce some additional notation. Let $\hat{x}_t^N \in (\mathbb{R}^d)^N$ be the process consisting of the concatenation of the N solutions of the IPS (1.4), viz, $\hat{x}_t^N = (x_t^{1,N}, \ldots, x_t^{N,N})^T$. Observe that this process is the solution of the following SDE on $(\mathbb{R}^d)^N$

$$\mathrm{d}\hat{x}_t^N = \hat{B}(\theta, \hat{x}_t^N)\mathrm{d}t + \mathrm{d}\hat{w}_t^N, \tag{4.19}$$

where $\hat{w}_t^N$ is a $(\mathbb{R}^d)^N$-valued Brownian motion, and the function $\hat{B}(\theta, \cdot) : (\mathbb{R}^d)^N \to (\mathbb{R}^d)^N$ is of the form $\hat{B}(\theta, \hat{x}^N) = (\hat{B}^{1,N}(\theta, \hat{x}^N), \ldots, \hat{B}^{N,N}(\theta, \hat{x}^N))^T$, where, for $i = 1, \ldots, N$, $\hat{B}^{i,N}(\theta, \cdot) : (\mathbb{R}^d)^N \to \mathbb{R}^d$ is defined according to

$$\hat{B}^{i,N}(\theta, \hat{x}^N) = b(\theta, x^{i,N}) + \frac{1}{N}\sum_{j=1}^N \phi(\theta, x^{i,N}, x^{j,N}). \tag{4.20}$$

It will also be useful to define, for $i = 1, \ldots, N$, the functions $\hat{G}^{i,N}(\theta, \cdot) : (\mathbb{R}^d)^N \to \mathbb{R}^d$ and $\hat{L}^{i,N}(\theta, \cdot) : (\mathbb{R}^d)^N \to \mathbb{R}$ according to

$$\hat{G}^{i,N}(\theta, \hat{x}^N) = \hat{B}^{i,N}(\theta, \hat{x}^N) - \hat{B}^{i,N}(\theta_0, \hat{x}^N) \tag{4.21}$$

$$\hat{L}^{i,N}(\theta, \hat{x}) = -\frac{1}{2}||\hat{G}^{i,N}(\theta, \hat{x}^N)||^2. \tag{4.22}$$

Finally, we will write $\hat{\mu}_t^N = \mathcal{L}(\hat{x}_t^N)$ to denote the law of $\hat{x}_t^N = (x_t^{1,N}, \ldots, x_t^{N,N})$. We should be careful not to confuse this with $\mu_t^N = \frac{1}{N}\sum_{i=1}^N \delta_{x_t^{i,N}}$, the empirical measure of the IPS.

Lemma 3.4.B. *Assume that Conditions B.1 - B.2, C.1, and D.1 hold. Then, for all $N \in \mathbb{N}$, the processes $\frac{1}{t}\nabla_\theta^m \mathcal{L}_t^{i,N}(\theta)$ and $\frac{1}{t}\nabla_\theta^m \mathcal{L}_t^N(\theta)$, $m = 0, 1, 2$, converge, both a.s. and in $\mathbb{L}^1$, to the functions*

$$\nabla_\theta^m \tilde{\mathcal{L}}^{i,N}(\theta) = \int_{(\mathbb{R}^d)^N} \nabla_\theta^m \hat{L}^{i,N}(\theta, \hat{x}^N)\hat{\mu}_\infty^N(\mathrm{d}\hat{x}^N) \quad , \quad \nabla_\theta^m \tilde{\mathcal{L}}^N(\theta) = \frac{1}{N}\sum_{i=1}^N \nabla_\theta^m \tilde{\mathcal{L}}^{i,N}(\theta).$$

In addition, there exist positive constants K_m^1, K_m^2, independent of N, such that

$$\left|\left|\mathbb{E}\left[\frac{1}{t}\nabla_\theta^m \mathcal{L}_t^{i,N}(\theta) - \nabla_\theta^m \tilde{\mathcal{L}}^{i,N}(\theta)\right]\right|\right| \leq \frac{K_m^1(1-e^{-\lambda t})}{\lambda t} + \frac{K_m^2(1+\sqrt{t})^{\frac{1}{2}}}{t^{\frac{1}{2}}}$$

and this bound also holds if $\mathcal{L}_t^{i,N}(\cdot)$ and $\tilde{\mathcal{L}}^{i,N}(\cdot)$ are replaced with $\mathcal{L}_t^N(\cdot)$ and $\tilde{\mathcal{L}}^N(\cdot)$.

Proof of Lemma 3.4.B. We will begin by proving that the two statements hold for the function $\mathcal{L}_t^{i,N}(\theta)$. The proof, in this case, is very similar to the proof of Lemma 3.4.A, with some simplifications. We will provide a sketch of the proof, signposting differences with the previous proof where necessary. As previously, we will only consider the case $m = 0$, with the results for $m = 1, 2$ proved analogously. We begin by recalling the definition of the function $\frac{1}{t}\mathcal{L}_t^{i,N}(\theta)$ from (2.4), which we now write in the form

$$\frac{1}{t}\mathcal{L}_t^{i,N}(\theta) = \frac{1}{t}\int_0^t \hat{L}^{i,N}(\theta, \hat{x}_s^N)\mathrm{d}s + \frac{1}{t}\int_0^t \langle \hat{G}^{i,N}(\theta, \hat{x}_s^N), \mathrm{d}w_s^i\rangle \tag{4.23}$$

We begin with the first term on the RHS. By Proposition A.4, the IPS admits a unique invariant measure $\hat{\mu}_{\infty}^{N} \in \mathcal{P}((\mathbb{R}^d)^N)$. Thus, for all $N \in \mathbb{N}$, by the ergodic theorem (e.g., [71, Chapter X]) we have that as $t \to \infty$,

$$\frac{1}{t}\int_0^t \hat{L}^{i,N}\left(\theta, \hat{x}_s^N\right) \mathrm{d}s \xrightarrow[\mathbb{L}^1]{\text{a.s.}} \int_{(\mathbb{R}^d)^N} \hat{L}^{i,N}\left(\theta, \hat{x}^N\right) \hat{\mu}_{\infty}(\mathrm{d}\hat{x}^N) = \tilde{\mathcal{L}}^{i,N}(\theta),$$

Let us now obtain the required convergence rate. By the remark after Lemma D.5, the function $\hat{L}^{i,N}(\theta, \hat{x}^N)$ satisfies the conditions of Lemma D.3. Thus, we can apply Lemma D.3 to obtain

$$\left|\mathbb{E}_{\hat{x}_0}[\hat{L}^{i,N}(\theta, \hat{x}_t^N)] - \int_{(\mathbb{R}^d)^N} \hat{L}^{i,N}(\theta, \hat{y})\hat{\mu}_{\infty}^N(\mathrm{d}\hat{y}^N)\right| \leq K\left[1 + ||x_0^{i,N}||^q + \frac{1}{N}\sum_{j=1}^N ||x_0^{j,N}||^q\right]e^{-\lambda t}$$

and so, arguing as in (4.13) - (4.14) in the proof of Lemma 3.4.A, we have

$$\begin{aligned}
&\left|\mathbb{E}_{\hat{x}_0}\left[\frac{1}{t}\int_0^t \hat{L}^{i,N}(\theta, \hat{x}_s^N)\mathrm{d}s - \int_{(\mathbb{R}^d)^N} \hat{L}^{i,N}(\theta, \hat{y}^N)\hat{\mu}_{\infty}^N(\mathrm{d}\hat{y}^N)\right]\right| \\
&\leq \frac{K(1-e^{-\lambda t})}{\lambda t}\left[1 + ||x_0^{i,N}||^q + \frac{1}{N}\sum_{j=1}^N ||x_0^{j,N}||^q\right] \leq \frac{K_0^1(1-e^{-\lambda t})}{\lambda t}.
\end{aligned}$$

It remains to bound the second term on the RHS of (4.23). We show that this term converges to zero a.s. and in $\mathbb{L}^1$, and satisfies the required convergence rate, using essentially identical arguments to those used in the proof of Lemma 3.4.A, c.f. (4.16) - (4.18). This concludes the proof.

We now turn our attention to the function $\mathcal{L}_t^N(\theta)$. The proof of the statements regarding this function now follows easily. In particular, using the definition of $\mathcal{L}_t^N(\theta)$, c.f. (2.4), and the results above, we have (once more restricting attention to the case $m = 0$)

$$\frac{1}{t}\mathcal{L}_t^N(\theta) = \frac{1}{t}\left[\frac{1}{N}\sum_{i=1}^N \mathcal{L}_t^{i,N}(\theta)\right] = \frac{1}{N}\sum_{i=1}^N\left[\frac{1}{t}\mathcal{L}_t^{i,N}(\theta)\right] \xrightarrow[\mathbb{L}^1]{\text{a.s.}} \frac{1}{N}\sum_{i=1}^N \tilde{\mathcal{L}}^{i,N}(\theta) = \tilde{\mathcal{L}}^N(\theta),$$

and, for the required bound,

$$\begin{aligned}
\left|\left|\mathbb{E}\left[\frac{1}{t}\nabla_\theta^m \mathcal{L}_t^N(\theta) - \nabla_\theta^m \tilde{\mathcal{L}}^N(\theta)\right]\right|\right| &= \left|\left|\frac{1}{N}\sum_{i=1}^N \mathbb{E}\left[\frac{1}{t}\nabla_\theta^m \mathcal{L}_t^{i,N}(\theta) - \nabla_\theta^m \tilde{\mathcal{L}}^{i,N}(\theta)\right]\right|\right| \quad (4.24)\\
&\leq \frac{1}{N}\sum_{i=1}^N \left|\left|\mathbb{E}\left[\frac{1}{t}\nabla_\theta^m \mathcal{L}_t^{i,N}(\theta) - \nabla_\theta^m \tilde{\mathcal{L}}^{i,N}(\theta)\right]\right|\right| \\
&\leq \frac{1}{N}\sum_{i=1}^N \left[\frac{K_m(1-e^{-\lambda t})}{\lambda t} + \frac{K_m(1+\sqrt{t})^{\frac{1}{2}}}{t^{\frac{1}{2}}}\right] \\
&= \frac{K_m(1-e^{-\lambda t})}{\lambda t} + \frac{K_m(1+\sqrt{t})^{\frac{1}{2}}}{t^{\frac{1}{2}}}. \quad (4.25)
\end{aligned}$$

■

4.3.3. Proof of Lemma 3.4.C.

Lemma 3.4.C. *Assume that Conditions B.1 - B.2, C.1, and D.1 hold. Then, for all $\theta \in \mathbb{R}^p$, for all $t \geq 0$, for all $i = 1, \ldots, N$, we have, in $\mathbb{L}^1$, that*

$$\lim_{N\to\infty} ||\tfrac{1}{t}\nabla_\theta \mathcal{L}_t^{i,N}(\theta)|| = ||\tfrac{1}{t}\nabla_\theta \mathcal{L}_t^{i}(\theta)||,$$
$$\lim_{N\to\infty} ||\tfrac{1}{t}\nabla_\theta \mathcal{L}_t^{N}(\theta)|| = ||\tfrac{1}{t}\nabla_\theta \mathcal{L}_t^{i}(\theta)||.$$

In addition, there exists a positive constant K such that, for all $\theta \in \mathbb{R}^p$, for all $N \in \mathbb{N}$,

$$\mathbb{E}\left[\left|\left|\frac{1}{t}\nabla_\theta \mathcal{L}_t^{i}(\theta) - \frac{1}{t}\nabla_\theta \mathcal{L}_t^{i,N}(\theta)\right|\right|\right] \leq \frac{K}{\sqrt{N}}\left(1 + \frac{1}{\sqrt{t}}\right),$$

and this bound also holds if $\mathcal{L}_t^{i,N}(\cdot)$ is replaced by $\mathcal{L}_t^{N}(\cdot)$.

Proof. We begin by proving that the two statements hold for $\mathcal{L}_t^{i,N}(\theta)$. First recall that

$$\frac{1}{t}\nabla_\theta \mathcal{L}_t^{i}(\theta) = \underbrace{-\frac{1}{t}\int_0^t \nabla_\theta G(\theta, x_s^i, \mu_s^i) G(\theta, x_s^i, \mu_s^i)\mathrm{d}s}_{I_1^i(\theta,t)} + \underbrace{\frac{1}{t}\int_0^t \langle \nabla_\theta G(\theta, x_s^i, \mu_s^i), \mathrm{d}w_s^i\rangle}_{I_2^i(\theta,t)}$$

$$\frac{1}{t}\nabla_\theta \mathcal{L}_t^{i,N}(\theta) = \underbrace{-\frac{1}{t}\int_0^t \left[\frac{1}{N}\sum_{j=1}^N \nabla_\theta G(\theta, x_s^{i,N}, x_s^{j,N})\right]\left[\frac{1}{N}\sum_{j=1}^N G(\theta, x_s^{i,N}, x_s^{j,N})\right]\mathrm{d}s}_{I_1^{i,N}(\theta,t)}$$
$$+ \underbrace{\frac{1}{t}\int_0^t \left\langle \frac{1}{N}\sum_{j=1}^N \nabla_\theta G(\theta, x_s^{i,N}, x_s^{j,N}), \mathrm{d}w_s^i\right\rangle}_{I_2^{i,N}(\theta,t)}$$

Let us seek bounds for $\mathbb{E}||I_1^i(\theta,t) - I_1^{i,N}(\theta,t)||$ and $\mathbb{E}||I_2^i(\theta,t) - I_2^{i,N}(\theta,t)||$, starting with the latter. By Lemma D.8 (see Appendix D), for all $s \geq 0$, there exists a positive constant K such that

$$\mathbb{E}\left[||\nabla_\theta G(\theta, x_s^i, \mu_s^i) - \frac{1}{N}\sum_{j=1}^N \nabla_\theta G(\theta, x_s^{i,N}, x_s^{j,N})||^2\right] \leq \frac{K}{N}.$$

Thus, making using of the triangle inequality, the Itô isometry, and Fubini's Theorem, we have that

$$\mathbb{E}\left[||I_2^i(\theta,t) - I_2^{i,N}(\theta,t)||^2\right] \leq \frac{1}{t^2}\mathbb{E}\left[\int_0^t \left|\left|\nabla_\theta G(\theta, x_s^i, \mu_s^i) - \frac{1}{N}\sum_{j=1}^N \nabla_\theta G(\theta, x_s^{i,N}, x_s^{j,N})\right|\right|^2 \mathrm{d}s\right]$$
$$\leq \frac{K}{Nt}.$$

and thus, by the Hölder inequality,

$$\mathbb{E}\big[||I_2^i(\theta,t) - I_2^{i,N}(\theta,t)||\big] \leq \frac{K}{\sqrt{Nt}} \tag{4.26}$$

We will now obtain, in much the same fashion, a bound for $\mathbb{E}[||I_1^i(\theta,t) - I_1^{i,N}(\theta,t)||]$. Once again, by Lemma D.8, for all $s \geq 0$, we have that

$$\mathbb{E}\Big[\big|\big|\nabla_\theta G(\theta, x_s^i, \mu_s^i) - \frac{1}{N}\sum_{j=1}^N \nabla_\theta G(\theta, x_s^{i,N}, x_s^{j,N})\big|\big|^2\Big] \leq \frac{K}{N}, \tag{4.27}$$

$$\mathbb{E}\Big[\big|\big|G(\theta, x_s^i, \mu_s^i) - \frac{1}{N}\sum_{j=1}^N G(\theta, x_s^{i,N}, x_s^{j,N})\big|\big|^2\Big] \leq \frac{K'}{N}. \tag{4.28}$$

To proceed, consider the following inequality, which follows straightforwardly from the triangle inequality and the Cauchy-Schwarz inequality,

$$\mathbb{E}\left[||YZ - Y_N Z_N||\right] \leq \mathbb{E}\left[||Y - Y_N||^2\right]^{\frac{1}{2}} \mathbb{E}\left[||Z||^2\right]^{\frac{1}{2}} + \mathbb{E}\left[||Y_N||^2\right]^{\frac{1}{2}} \mathbb{E}\left[||Z - Z_N||^2\right]^{\frac{1}{2}}$$

Suppose we let $Y = \nabla_\theta G(\theta, x_s^i, \mu_s^i)$, $Y_N = N^{-1}\sum_{j=1}^N \nabla_\theta G(\theta, x_s^{i,N}, x_s^{j,N})$, $Z = G(\theta, x_s^i, \mu_s^i)$, and $Z_N = N^{-1}\sum_{j=1}^N G(\theta, x_s^{i,N}, x_s^{j,N})$. Then, once more allowing the value of the constant K to change from line to line, this inequality yields

$$\begin{aligned}
&\mathbb{E}\Big[\big|\big|\nabla_\theta G(\theta, x_s^i, \mu_s^i) G(\theta, x_s^i, \mu_s^i) - \frac{1}{N}\sum_{j=1}^N \nabla_\theta G(\theta, x_s^{i,N}, x_s^{j,N}) \cdot \frac{1}{N}\sum_{j=1}^N G(\theta, x_s^{i,N}, x_s^{j,N})\big|\big|\Big] \\
&\leq \underbrace{\mathbb{E}\Big[\big|\big|\nabla_\theta G(\theta, x_s^i, \mu_s^i) - \frac{1}{N}\sum_{j=1}^N \nabla_\theta G(\theta, x_s^{i,N}, x_s^{j,N})\big|\big|^2\Big]^{\frac{1}{2}}}_{\leq \frac{K}{\sqrt{N}} \text{ by } (4.27)} \cdot \underbrace{\mathbb{E}\Big[\big|\big|G(\theta, x_s^i, \mu_s^i)\big|\big|^2\Big]^{\frac{1}{2}}}_{\leq K} \\
&+ \underbrace{\mathbb{E}\Big[\big|\big|G(\theta, x_s^i, \mu_s^i) - \frac{1}{N}\sum_{j=1}^N G(\theta, x_s^{i,N}, x_s^{j,N})\big|\big|^2\Big]^{\frac{1}{2}}}_{\leq \frac{K'}{\sqrt{N}} \text{ by } (4.28)} \cdot \underbrace{\mathbb{E}\Big[\big|\big|\frac{1}{N}\sum_{j=1}^N G(\theta, x_s^{i,N}, x_s^{j,N})\big|\big|^2\Big]^{\frac{1}{2}}}_{\leq K'} \\
&\leq \frac{K}{\sqrt{N}},
\end{aligned}$$

where in the penultimate line we have used Condition C.1 (the polynomial growth of G) and Proposition A.2 (the moment bounds for the IPS), to conclude that each of the expectations are bounded above by some positive constants. That is, for example,

$$\mathbb{E}\left[\big|\big|G(\theta, x_s^i, \mu_s^i)\big|\big|^2\right] \leq \mathbb{E}\left[K\left(1 + ||x_s^i||^q + \int_{\mathbb{R}^d} ||y||^q \mu_s^i(\mathrm{d}y)\right)\right] \leq K\left(1 + \mathbb{E}\left[||x_s^i||^q\right]\right) \leq K^2.$$

It follows straightforwardly that

$$\mathbb{E}\left[||I_1^i(\theta,t)-I_1^{i,N}(\theta,t)||\right]\leq\frac{1}{t}\int_0^t\frac{K}{\sqrt{N}}\mathrm{d}s=\frac{K}{\sqrt{N}}. \tag{4.29}$$

Combining inequalities (4.26) and (4.29), and making use of the triangle inequality one final time, we have that

$$\mathbb{E}\left[\left|\left|\frac{1}{t}\nabla_\theta\mathcal{L}_t^i(\theta)-\frac{1}{t}\nabla_\theta\mathcal{L}_t^{i,N}(\theta)\right|\right|\right]\leq\frac{K}{\sqrt{N}}\left(1+\frac{1}{\sqrt{t}}\right)$$

This establishes convergence in $\mathbb{L}^1$ as $N\to\infty$, for all $t\geq 0$. It remains only to establish that the statements of the lemma also hold for $\mathcal{L}_t^N(\theta)$. This is straightforward. Indeed, we omit the calculations, which are essentially identical to those used at the end of the proof of Lemma 3.4.B, c.f. (4.24) - (4.25). ■

4.3.4. Proof of Lemma 3.4.D.

Lemma 3.4.D. *Assume that Conditions B.1 - B.2, C.1, D.1 and F.1 hold. Then, for all $N\in\mathbb{N}$, we have, both almost surely and in $\mathbb{L}^1$, that*

$$\lim_{t\to\infty}||\nabla_\theta\tilde{\mathcal{L}}^{i,N}(\theta^{i,N}(t))||=0,$$
$$\lim_{t\to\infty}||\nabla_\theta\tilde{\mathcal{L}}^{N}(\theta^{N}(t))||=0.$$

Proof. We will prove the first statement of the lemma, with the second proved identically.[4] In particular, we will use a modified version of the approach in [75], which itself is a continuous-time version of the approach first introduced in [6]. In the interest of completeness, we will include the proof in full here, adapted appropriately to the current setting and our notation.

We will require the following additional notation. Define an arbitrary constant $\kappa>0$, with $\lambda=\lambda(\kappa)>0$ to be determined. Set $\sigma=0$, and define the cycle of random stopping times

$$0=\sigma_0\leq\tau_1\leq\sigma_1\leq\tau_2\leq\sigma_2\leq\dots$$

according to

$$\tau_k=\inf\left\{t>\sigma_{k-1}:||\nabla_\theta\tilde{\mathcal{L}}^{i,N}(\theta_t^{i,N})||\geq\kappa\right\} \tag{4.30}$$

$$\sigma_k=\sup\Big\{t>\tau_k:\tfrac{1}{2}||\nabla\tilde{\mathcal{L}}^{i,N}(\theta_{\tau_k}^{i,N})||\leq||\nabla\tilde{\mathcal{L}}^{N}(\theta_s^{i,N})||\leq 2||\nabla\tilde{\mathcal{L}}^{i,N}(\theta_{\tau_k}^{i,N})||\quad\forall s\in[\tau_k,t], \int_{\tau_k}^t\gamma(s)\mathrm{d}s\leq\rho\Big\} \tag{4.31}$$

The purpose of these stopping times is to control the periods of time for which $||\nabla\tilde{\mathcal{L}}^{i,N}(\theta_t^{i,N})||$ is close to zero, and those for which it is away from zero. In addition, let $\eta>0$, and set

[4] We remark that Lemmas D.9 - D.12, which are essential to this proof, all apply to both $\mathcal{L}_t^{i,N}(\theta)$ and $\mathcal{L}_t^N(\theta)$, and thus can still be used to establish the second statement.

$\sigma_{k,\eta} = \sigma_k + \eta$. First consider the case in which there are a finite number of stopping times τ_k. In this case, there exists finite t_0 such that, for all $t \geq t_0$, $[||\nabla_\theta \tilde{\mathcal{L}}^{i,N}(\theta_t^{i,N})||] < \kappa$. Now consider the case in which there are an infinite number of stopping times τ_k. Then, using Lemmas D.11 - D.12 (see Appendix D), there exist $0 < \beta_1 < \beta$, and $k_0 \in \mathbb{N}$, such that for all $k \geq k_0$, almost surely,

$$\tilde{\mathcal{L}}^{i,N}(\theta_{\sigma_k}^{i,N}) - \tilde{\mathcal{L}}^{i,N}(\theta_{\tau_k}^{i,N}) \geq \beta \tag{4.32}$$

$$\tilde{\mathcal{L}}^{i,N}(\theta_{\tau_k}^{i,N}) - \tilde{\mathcal{L}}^{i,N}(\theta_{\sigma_{k-1}}^{i,N}) \geq -\beta_1. \tag{4.33}$$

It follows straightforwardly that

$$\tilde{\mathcal{L}}^{i,N}(\theta_{\tau_{n+1}}^{i,N}) - \tilde{\mathcal{L}}^{i,N}(\theta_{\tau_{k_0}}^{i,N}) = \sum_{k=k_0}^{n}\left[\tilde{\mathcal{L}}^{i,N}(\theta_{\sigma_k}^{i,N}) - \tilde{\mathcal{L}}^{i,N}(\theta_{\tau_k}^{i,N}) + \tilde{\mathcal{L}}^{i,N}(\theta_{\tau_{k+1}}^{i,N}) - \tilde{\mathcal{L}}^{i,N}(\theta_{\sigma_k}^{i,N})\right] \tag{4.34}$$

$$\geq \sum_{k=k_0}^{n} (\beta - \beta_1) = (n+1-k_0)(\beta - \beta_1) \tag{4.35}$$

Since $\beta - \beta_1 > 0$, this implies that $\tilde{\mathcal{L}}^{i,N}(\theta_{\tau_{n+1}}^{i,N}) \to \infty$ as $n \to \infty$. But this is in contradiction with Lemma D.6, which states that $\tilde{\mathcal{L}}^{i,N}(\theta)$ is bounded from above. Thus, there must exist a finite time t_0 such that, for all $t \geq t_0$, $||\tilde{\mathcal{L}}^{i,N}(\theta_t^{i,N})|| < \kappa$. Since our original choice of κ was arbitrary, this completes the proof that, for all $N \in \mathbb{N}$, almost surely,

$$\lim_{t\to\infty} ||\nabla_\theta \tilde{\mathcal{L}}^{i,N}(\theta_t^{i,N})|| = 0.$$

Finally, we observe that, by Lemma D.6, $||\nabla_\theta \tilde{\mathcal{L}}^{i,N}(\theta)||$ is bounded above for all $\theta \in \mathbb{R}^p$. Thus, we also have convergence in $\mathbb{L}^1$ via Lebesgue's dominated convergence theorem (e.g., [85, Chapter 5]). ■

4.4. Proof of Theorem 3.4. Before we proceed to the proof of Theorem 3.4, we state the following Lemma, which provides uniform moment bounds for the online parameter estimate, and will be used frequently in this proof.

Lemma 4.1. *Assume that Conditions B.1 - B.2, C.1, D.1, F.1, and G.1 - G.2 hold. Then, for all $q \geq 1$, for all $i = 1, \ldots, N$, for all $N \in \mathbb{N}$, there exists K such that*

$$\sup_{t>0} \mathbb{E}\left[||\theta_t||^q\right] \leq K$$

$$\sup_{t>0} \mathbb{E}\left[||\theta_t^{i,N}||^q\right] \leq K.$$

Proof. This Lemma follows straightforwardly as an extension of [77, Lemma A.1], making use of the appropriate bounds in Conditions G.1 - G.2. ■

Proof of Theorem 3.4. The proof of this result closely follows the proof of Theorem 2.7 in [77], adapted appropriately to our particular case. We will begin by proving the first statement of the theorem. To begin, let us recall the following form of the parameter update equation

(2.8):

$$\mathrm{d}\theta_t^{i,N} = \gamma_t \nabla_\theta \tilde{\mathcal{L}}^{i,N}(\theta_t^{i,N})\mathrm{d}t + \gamma_t\big(\nabla_\theta L(\theta_t^{i,N}, x_t^{i,N}, \mu_t^N) - \nabla_\theta \tilde{\mathcal{L}}^{i,N}(\theta_t^{i,N})\big)\mathrm{d}t + \gamma_t \nabla_\theta B(\theta_t^{i,N}, x_t^{i,N}, \mu_t^N)\mathrm{d}w_t^i \tag{4.36}$$

$$= \gamma_t \nabla_\theta \tilde{\mathcal{L}}(\theta_t^{i,N})\mathrm{d}t + \gamma_t\big(\nabla_\theta \tilde{\mathcal{L}}^{i,N}(\theta_t^{i,N}) - \nabla_\theta \tilde{\mathcal{L}}(\theta_t^{i,N})\big)\mathrm{d}t + \gamma_t\big(\nabla_\theta L(\theta_t^{i,N}, x_t^{i,N}, \mu_t^N) - \nabla_\theta \tilde{\mathcal{L}}^{i,N}(\theta_t^{i,N})\big)\mathrm{d}t + \gamma_t \nabla_\theta B(\theta_t^{i,N}, x_t^{i,N}, \mu_t^N)\mathrm{d}w_t^i. \tag{4.37}$$

Using a first order Taylor expansion, we have that

$$\nabla_\theta \tilde{\mathcal{L}}(\theta_t^{i,N}) = \nabla_\theta \tilde{\mathcal{L}}(\theta_0) + \nabla^2 \tilde{\mathcal{L}}(\tilde{\theta}_t^{i,N})(\theta_t^{i,N} - \theta_0) = \nabla^2 \tilde{\mathcal{L}}(\tilde{\theta}_t^{i,N})(\theta_t^{i,N} - \theta_0) \tag{4.38}$$

where $\tilde{\theta}_t^{i,N}$ is point in the segment connecting $\theta_t^{i,N}$ and θ_0. Substituting this into (4.37), we obtain the following equation for $Z_t^{i,N} = \theta_t^{i,N} - \theta_0$

$$\mathrm{d}Z_t^{i,N} = \gamma_t \nabla_\theta^2 \tilde{\mathcal{L}}(\tilde{\theta}_t^{i,N}) Z_t^{i,N}\mathrm{d}t + \gamma_t\big(\nabla_\theta \tilde{\mathcal{L}}^{i,N}(\theta_t^{i,N}) - \nabla_\theta \tilde{\mathcal{L}}(\theta_t^{i,N})\big)\mathrm{d}t + \gamma_t\big(\nabla_\theta L(\theta_t^{i,N}, x_t^{i,N}, \mu_t^N) - \nabla_\theta \tilde{\mathcal{L}}^{i,N}(\theta_t^{i,N})\big)\mathrm{d}t + \gamma_t \nabla_\theta B(\theta_t^{i,N}, x_t^{i,N}, \mu_t^N)\mathrm{d}w_t^i.$$

Applying Itô's formula to the function $||\cdot||^2$, we obtain

$$\begin{aligned}\mathrm{d}||Z_t^{i,N}||^2 &= 2\gamma_t\big\langle Z_t^{i,N}, \nabla_\theta^2 \tilde{\mathcal{L}}(\tilde{\theta}_t^{i,N}) Z_t^{i,N}\big\rangle\mathrm{d}t + \gamma_t\big\langle Z_t^{i,N}, \nabla_\theta \tilde{\mathcal{L}}^{i,N}(\theta_t^{i,N}) - \nabla_\theta \tilde{\mathcal{L}}(\theta_t^{i,N})\big\rangle\mathrm{d}t \\ &+ \gamma_t\big\langle Z_t^{i,N}, \nabla_\theta L(\theta_t^{i,N}, x_t^{i,N}, \mu_t^N) - \nabla_\theta \tilde{\mathcal{L}}^{i,N}(\theta_t^{i,N})\big\rangle\mathrm{d}t \\ &+ \gamma_t\big\langle Z_t^{i,N}, \nabla_\theta B(\theta_t^{i,N}, x_t^{i,N}, \mu_t^N)\mathrm{d}w_t^i\big\rangle + \gamma_t^2\big|\big|\nabla_\theta B(\theta_t^{i,N}, x_t^{i,N}, \mu_t^N)\big|\big|_F^2\mathrm{d}t\end{aligned}$$

Due to the strong concavity of $\tilde{\mathcal{L}}(\theta)$ (Condition H.1), it then follows that

$$\begin{aligned}\mathrm{d}||Z_t^{i,N}||^2 + 2\eta\gamma_t||Z_t^{i,N}||^2\mathrm{d}t &\leq \gamma_t\big\langle Z_t^{i,N}, \nabla_\theta \tilde{\mathcal{L}}^{i,N}(\theta_t^{i,N}) - \nabla_\theta \tilde{\mathcal{L}}(\theta_t^{i,N})\big\rangle\mathrm{d}t \\ &+ \gamma_t\big\langle Z_t^{i,N}, \nabla_\theta L(\theta_t^{i,N}, x_t^{i,N}, \mu_t^N) - \nabla_\theta \tilde{\mathcal{L}}^{i,N}(\theta_t^{i,N})\big\rangle\mathrm{d}t \\ &+ \gamma_t\big\langle Z_t^{i,N}, \nabla_\theta B(\theta_t^{i,N}, x_t^{i,N}, \mu_t^N)\mathrm{d}w_t^i\big\rangle \\ &+ \gamma_t^2\big|\big|\nabla_\theta B(\theta_t^{i,N}, x_t^{i,N}, \mu_t^N)\big|\big|_F^2\mathrm{d}t\end{aligned} \tag{4.39}$$

where $||\cdot||_F$ is the Frobenius norm. Now, let us define the function $\Phi_{t,t'} = \exp[-2\eta\int_t^{t'}\gamma_u\mathrm{d}u]$, with $\partial_t\Phi_{t,t'} = 2\eta\gamma_t\Phi_{t,t'}$. Using the product rule, and (4.39), we obtain

$$\begin{aligned}\mathrm{d}\left[\Phi_{t,t'}||Z_t^{i,N}||^2\right] &= \Phi_{t,t'}\left[\mathrm{d}||Z_t^{i,N}||^2 + 2\eta\gamma_t||Z_t^{i,N}||^2\mathrm{d}t\right] \\ &\leq \gamma_t\Phi_{t,t'}\big\langle Z_t^{i,N}, \nabla_\theta \tilde{\mathcal{L}}^{i,N}(\theta_t^{i,N}) - \nabla_\theta \tilde{\mathcal{L}}(\theta_t^{i,N})\big\rangle\mathrm{d}t \\ &+ \gamma_t\Phi_{t,t'}\big\langle Z_t^{i,N}, \nabla_\theta L(\theta_t^{i,N}, x_t^{i,N}, \mu_t^N) - \nabla_\theta \tilde{\mathcal{L}}^{i,N}(\theta_t^{i,N})\big\rangle\mathrm{d}t \\ &+ \gamma_t\Phi_{t,t'}\big\langle Z_t^{i,N}, \nabla_\theta B(\theta_t^{i,N}, x_t^{i,N}, \mu_t^N)\mathrm{d}w_t^i\big\rangle \\ &+ \gamma_t^2\Phi_{t,t'}\big|\big|\nabla_\theta B(\theta_t^{i,N}, x_t^{i,N}, \mu_t^N)\big|\big|_F^2\mathrm{d}t\end{aligned} \tag{4.40}$$

Rewriting this in integral form, setting $t' = t$, and taking expectations, we obtain

$$\begin{aligned}
(4.41)\ \mathbb{E}\left[||Z_t^{i,N}||^2\right] &\leq \mathbb{E}\left[\Phi_{1,t}||Z_1^{i,N}||^2\right] + \mathbb{E}\left[\int_1^t \gamma_s \Phi_{s,t} \langle Z_s^{i,N}, \nabla_\theta \tilde{\mathcal{L}}^{i,N}(\theta_s^{i,N}) - \nabla_\theta \tilde{\mathcal{L}}(\theta_s^{i,N})\rangle \mathrm{d}s\right] \\
&\quad + \mathbb{E}\left[\int_1^t \gamma_s \Phi_{s,t} \langle Z_s^{i,N}, \nabla_\theta L(\theta_s^{i,N}, x_s^{i,N}, \mu_s^N) - \nabla_\theta \tilde{\mathcal{L}}^{i,N}(\theta_s^{i,N})\rangle \mathrm{d}s\right] \\
&\quad + \mathbb{E}\left[\int_1^t \gamma_s^2 \Phi_{s,t} \big|\big|\nabla_\theta B(\theta_s^{i,N}, x_s^{i,N}, \mu_s^N)\big|\big|_F^2 \mathrm{d}s\right] \\
(4.42)\qquad &= \mathbb{E}\left[\Omega_{t,i,N}^{(1)}\right] + \mathbb{E}\left[\Omega_{t,i,N}^{(2)}\right] + \mathbb{E}\left[\Omega_{t,i,N}^{(3)}\right] + \mathbb{E}\left[\Omega_{t,i,N}^{(4)}\right]
\end{aligned}$$

We will deal with each of these terms separately, beginning with $\Omega_{t,i,N}^{(1)}$. For this term, we have that, for sufficiently large t,

$$\mathbb{E}\left[\Omega_{t,i,N}^{(1)}\right] = \Phi_{1,t}\mathbb{E}\left[||Z_1^{i,N}||^2\right] \leq K^{(1)}\gamma_t \tag{4.43}$$

which follows from Lemma 4.1 (the moment bounds for $\theta_s^{i,N}$), and Condition F.1 (the conditions on the learning rate).

We now turn our attention to $\Omega_{t,i,N}^{(2)}$. For this term, substituting the bound in Lemma E.1 into (4.41), we immediately obtain

$$\begin{aligned}
(4.44)\qquad \mathbb{E}\left[\Omega_{t,i,N}^{(2)}\right] &\leq \int_1^t \gamma_s \Phi_{s,t} \mathbb{E}\left[||Z_s^{i,N}|| \sup_{\theta_s^{i,N}} ||\nabla_\theta \tilde{\mathcal{L}}^{i,N}(\theta_s^{i,N}) - \nabla_\theta \mathcal{L}(\theta_s^{i,N})||\right] \mathrm{d}s \\
&\leq K\left[\frac{1}{N^{\frac{1}{2}}}\right] \int_1^t \gamma_s \Phi_{s,t} \mathrm{d}s \leq K^{(2)}\left[\frac{1}{N^{\frac{1}{2}}}\right].
\end{aligned}$$

where in the last line we have used Condition F.1 (the conditions on the learning rate) to bound the integral.

We now turn our attention to $\Omega_{t,i,N}^{(3)}$. We will analyse this term by constructing an appropriate Poisson equation. Let us define

$$R^{i,N}(\theta, \hat{x}^N) = \left\langle \theta - \theta_0, \nabla_\theta \hat{L}^{i,N}(\theta, \hat{x}^N) - \nabla_\theta \tilde{\mathcal{L}}^{i,N}(\theta) \right\rangle,$$

where, as previously, $\hat{x}^N = (x^{1,N}, \ldots, x^{N,N})$.It is straightforward to verify that this function satisfies all of the conditions of Lemma D.13. Thus, by Lemma D.13, the Poisson equation

$$\mathcal{A}_x v^{i,N}(\theta, \hat{x}^N) = R^{i,N}(\theta, \hat{x}^N) \quad , \quad \int_{\mathbb{R}^d} v^{i,N}(\theta, \hat{x}^N) \hat{\mu}_\infty^N(\mathrm{d}\hat{x}^N) = 0$$

has a unique twice differentiable solution which satisfies

$$\sum_{j=0}^2 \left|\frac{\partial^j v^{i,N}}{\partial \theta^i}(\theta, \hat{x}^N)\right| + \left|\frac{\partial^2 v^{i,N}}{\partial \theta \partial x}(\theta, \hat{x}^N)\right| \leq K\bigg(1 + ||x^{i,N}||^q + \frac{1}{N}\sum_{j=1}^N ||x^{j,N}||^q\bigg).$$

Now, by Itô's formula, we have that

$$v^{i,N}(\theta_t^{i,N},\hat{x}_t^N) - v^{i,N}(\theta_s^{i,N},\hat{x}_s^N) = \int_s^t \mathcal{A}_\theta v^{i,N}(\theta_u^{i,N},\hat{x}_u^N)\mathrm{d}u + \int_s^t \mathcal{A}_{\hat{x}^N} v^{i,N}(\theta_u^{i,N},\hat{x}_u^N)\mathrm{d}u + \int_s^t \gamma_u \partial_\theta v^{i,N}(\theta_u^{i,N},\hat{x}_u^N)\nabla_\theta \hat{B}^{i,N}(\theta_u,\hat{x}_u^N)\mathrm{d}w_u^i + \int_s^t \partial_x v^{i,N}(\theta_u^{i,N},\hat{x}_u^N)\mathrm{d}\hat{w}_u^N + \int_s^t \gamma_u \left[\partial_\theta\partial_{\hat{x}} v^{i,N}(\theta_u^{i,N},\hat{x}_u^N)\nabla_\theta \hat{B}^{i,N}(\theta_u^{i,N},\hat{x}_u^N)\right]\mathrm{d}u \tag{4.45}$$

where $\hat{w}_u^N$ was defined in (4.19). It follows, now writing $v_t^{i,N} := v^{i,N}(\theta_t^{i,N},\hat{x}_t^N)$, that

$$\begin{aligned} R^{i,N}(\theta_t^{i,N},\hat{x}_t^N)\mathrm{d}t &= \mathcal{A}_{\hat{x}^N} v^{i,N}(\theta_t^{i,N},\hat{x}_t^N)\mathrm{d}t \\ &= \mathrm{d}v_t^{i,N} - \mathcal{A}_\theta v^{i,N}(\theta_t^{i,N},\hat{x}_t^N)\mathrm{d}t - \gamma_t\partial_\theta v^{i,N}(\theta_t^{i,N},\hat{x}_t^N)\nabla_\theta\hat{B}^{i,N}(\theta_t^{i,N},\hat{x}_t^N)\mathrm{d}w_t^i \\ &\quad - \partial_{\hat{x}} v^{i,N}(\theta_t^{i,N},\hat{x}_t^N)\mathrm{d}\hat{w}_t^N - \gamma_t\left[\partial_\theta\partial_{\hat{x}} v^{i,N}(\theta_t^{i,N},\hat{x}_t^N)\nabla_\theta\hat{B}^{i,N}(\theta_t^{i,N},\hat{x}_t^N)\right]\mathrm{d}t \end{aligned}$$

Thus, we can rewrite $\Omega_{t,i,N}^{(3)}$ as

$$\begin{aligned} \Omega_{t,i,N}^{(3)} &= \int_1^t \gamma_s\Phi_{s,t} \underbrace{\left\langle \theta_s^{i,N}-\theta_0, \nabla_\theta\hat{L}^{i,N}(\theta_s^{i,N},\hat{x}_s^N) - \nabla_\theta\tilde{\mathcal{L}}^{i,N}(\theta_s^{i,N})\right\rangle \mathrm{d}s}_{R^{i,N}(\theta_s^{i,N},\hat{x}_s^N)\mathrm{d}s} \\ &= \int_1^t \gamma_s\Phi_{s,t}\mathrm{d}v_s^{i,N} - \int_1^t \gamma_s\Phi_{s,t}\mathcal{A}_\theta v^{i,N}(\theta_s^{i,N},\hat{x}_s^N)\mathrm{d}s \\ &\quad - \int_1^t \gamma_s^2\Phi_{s,t}\partial_\theta v^{i,N}(\theta_s^{i,N},\hat{x}_s^N)\nabla_\theta\hat{B}^{i,N}(\theta_s^{i,N},\hat{x}_s^N)\mathrm{d}w_s^i \\ &\quad - \int_1^t \gamma_s\Phi_{s,t}\partial_{\hat{x}} v^{i,N}(\theta_s^{i,N},\hat{x}_s^N)\mathrm{d}\hat{w}_s^N \\ &\quad - \int_1^t \gamma_s^2\Phi_{s,t}\partial_\theta\partial_x v^{i,N}(\theta_s,\hat{x}_s^N)\nabla_\theta\hat{B}^{i,N}(\theta_s^{i,N},\hat{x}_s^N)\mathrm{d}s \end{aligned} \tag{4.46}$$

We can rewrite the first term in this expression by applying Itô's formula to the function $f(s,v_s) = \gamma_s\Phi_{s,t}v_s$. This yields

$$\gamma_t\Phi_{t,t}v_t^{i,N} - \gamma_1\Phi_{1,t}v_1^{i,N} = \int_1^t \gamma_s\Phi_{s,t}\mathrm{d}v_s^{i,N} + \int_1^t \dot{\gamma}_s\Phi_{s,t}v_s^{i,N}\mathrm{d}s + \int_1^t 2\eta\gamma_s^2\Phi_{s,t}v_s^{i,N}\mathrm{d}s.$$

Substituting the resulting expression for $\int_1^t \gamma_s\Phi_{s,t}\mathrm{d}v_s^{i,N}$ into (4.46), and taking expectations, we obtain

$$\begin{aligned}\mathbb{E}\left[\Omega^{(3)}_{t,i,N}\right] &= \mathbb{E}\left[\gamma_t\Phi_{t,t}v^{i,N}(\theta^{i,N}_t,\hat{x}^N_t)\right] - \mathbb{E}\left[\gamma_1\Phi_{1,t}v^{i,N}(\theta^{i,N}_1,\hat{x}^N_1)\right] \\ &- \mathbb{E}\left[\int_1^t \dot{\gamma}_s\Phi_{s,t}v^{i,N}(\theta^{i,N}_s,\hat{x}^N_s)\mathrm{d}s\right] - \mathbb{E}\left[\int_1^t 2\eta\gamma_s^2\Phi_{s,t}v^{i,N}(\theta^{i,N}_s,\hat{x}^N_s)\mathrm{d}s\right] \\ &- \mathbb{E}\left[\int_1^t \gamma_s\Phi_{s,t}\mathcal{A}_\theta v^{i,N}(\theta^{i,N}_s,\hat{x}^N_s)\mathrm{d}s\right] \\ &- \mathbb{E}\left[\int_1^t \gamma_s^2\Phi_{s,t}\partial_\theta\partial_x v^{i,N}(\theta^{i,N}_s,\hat{x}^N_s)\nabla_\theta\hat{B}^{i,N}(\theta^{i,N}_s,\hat{x}^N_s)\mathrm{d}s\right] \\ &\leq K\left[\gamma_t + \int_1^t\left(\dot{\gamma}_s+\gamma_s^2\right)\Phi_{s,t}\mathrm{d}s\right] \leq K^{(3)}\gamma_t, \end{aligned} \tag{4.47}$$

where in the penultimate inequality we have used the polynomial growth of $v^{i,N}(\theta,\hat{x}^N)$ and $\partial_\theta\partial_x v^{i,N}(\theta,\hat{x}^N)$, Condition C.1(ii) (which implies the polynomial growth of $\nabla_\theta\hat{B}^{i,N}(\theta,\hat{x}^N)$), Proposition A.2 (the moment bounds for $\hat{x}^N_t$), Lemma 4.1 (the moment bounds for $\theta^{i,N}_s$), and in the final inequality we have used Condition F.1 (the conditions on the learning rate). It remains only to bound $\Omega^{(4)}_{t,i,N}$. For this term, once more making use of the above assumptions, we obtain

$$\mathbb{E}\left[\Omega^{(4)}_{t,i,N}\right] = \mathbb{E}\left[\int_1^t \gamma_s^2\Phi_{s,t}\big|\big|\nabla_\theta B(\theta_s,x_s,\mu_s)\big|\big|_F^2\mathrm{d}s\right] \leq K\int_1^t\gamma_s^2\Phi_{s,t}\mathrm{d}s \leq K^{(4)}\gamma_t. \tag{4.48}$$

Combining inequalities (4.43), (4.44), (4.47), and (4.48), and setting $K_1 = \max\{K^{(1)},K^{(3)}\}$, $K_2 = K^{(4)}$, and $K_3 = K^{(2)}$, we thus have that

$$\mathbb{E}\left[||\theta^{i,N}_t-\theta_0||^2\right] \leq \mathbb{E}\left[\Omega^{(1)}_{t,i,N}\right]+\mathbb{E}\left[\Omega^{(2)}_{t,i,N}\right]+\mathbb{E}\left[\Omega^{(3)}_{t,i,N}\right]+\mathbb{E}\left[\Omega^{(4)}_{t,i,N}\right] \leq (K_1+K_2)\gamma_t + \frac{K_3}{N^{\frac{1}{2}}},$$

which completes the proof of the first statement of the theorem.

Let us now turn our attention to the second statement. The proof of this bound goes through almost verbatim. Let us briefly highlight the main points of difference. To begin, we now have the following decomposition of the parameter update equation

$$\begin{aligned}\mathrm{d}\theta^N_t &= \gamma_t\nabla_\theta\tilde{\mathcal{L}}(\theta^N_t)\mathrm{d}t + \gamma_t\frac{1}{N}\sum_{i=1}^N\big(\nabla_\theta\tilde{\mathcal{L}}^{i,N}(\theta^N_t)-\nabla_\theta\tilde{\mathcal{L}}(\theta^N_t)\big)\mathrm{d}t \\ &+ \gamma_t\frac{1}{N}\sum_{i=1}^N\big(\nabla_\theta L(\theta^N_t,x^{i,N}_t,\mu^N_t)-\nabla_\theta\tilde{\mathcal{L}}^{i,N}(\theta^N_t)\big)\mathrm{d}t + \gamma_t\frac{1}{N}\sum_{i=1}^N\nabla_\theta B(\theta^{i,N}_t,x^{i,N}_t,\mu^N_t)\mathrm{d}w^i_t.\end{aligned}$$

Using a first order Taylor expansion around θ_0, defining $Z^N_t = \theta^N_t-\theta_0$, applying Itô's formula to the function $||Z_t^N||^2$, and using the strong concavity of $\tilde{\mathcal{L}}(\theta)$, as in (4.38) - (4.39), we obtain

$$
\begin{aligned}
\mathrm{d}||Z_t^N||^2 + 2\eta\gamma_t||Z_t^N||^2\mathrm{d}t \leq \gamma_t \frac{1}{N}\sum_{i=1}^N \langle Z_t^N, \nabla_\theta \tilde{\mathcal{L}}^{i,N}(\theta_t^{i,N}) - \nabla_\theta\tilde{\mathcal{L}}(\theta_t^N)\rangle \mathrm{d}t \\
+ \gamma_t \frac{1}{N}\sum_{i=1}^N \langle Z_t^N, \nabla_\theta L(\theta_t^{i,N}, x_t^{i,N}, \mu_t^N) - \nabla_\theta \tilde{\mathcal{L}}^{i,N}(\theta_t^N)\rangle \mathrm{d}t \\
+ \gamma_t \frac{1}{N}\sum_{i=1}^N \langle Z_t^N, \nabla_\theta B(\theta_t^N, x_t^{i,N}, \mu_t^N)\mathrm{d}w_t^i\rangle \\
+ \gamma_t^2 \frac{1}{N^2}\sum_{i=1}^N \big|\big|\nabla_\theta B(\theta_t, x_t, \mu_t)\big|\big|_F^2 \mathrm{d}t
\end{aligned}
$$

where, as previously, $||\cdot||_F$ denotes the Frobenius norm. Continuing to follow our previous arguments, c.f. (4.40) - (4.42), we finally arrive at

$$
\mathbb{E}\left[||Z_t^N||^2\right] \leq \frac{1}{N}\sum_{i=1}^N \left[\mathbb{E}\left[\tilde{\Omega}^{(1)}_{t,i,N}\right] + \mathbb{E}\left[\tilde{\Omega}^{(2)}_{t,i,N}\right] + \mathbb{E}\left[\tilde{\Omega}^{(3)}_{t,i,N}\right]\right] + \frac{1}{N^2}\sum_{i=1}^N \mathbb{E}\left[\tilde{\Omega}^{(4)}_{t,i,N}\right]
$$

where, up to minor modifications, $\tilde{\Omega}^{(1)}_{t,i,N}, \ldots, \tilde{\Omega}^{(4)}_{t,i,N}$ are identical to $\Omega^{(1)}_{t,i,N}, \ldots, \Omega^{(4)}_{t,i,N}$ as defined in (4.41) - (4.42). In particular, all instances of $\theta_s^{i,N}$ have been replaced by θ_s^N. We thus have, using the bounds established previously, c.f. (4.43), (4.44), (4.47), and (4.48), that

$$
\mathbb{E}\left[||Z_t^N||^2\right] \leq \frac{1}{N}\sum_{i=1}^N \left(K_1\gamma_t + K_3\frac{1}{N^{\frac{1}{2}}}\right) + \frac{1}{N^2}\sum_{i=1}^N K_2\gamma_t = (K_1 + \frac{K_2}{N})\gamma_t + \frac{K_3}{N^{\frac{1}{2}}}. \qquad \blacksquare
$$

5. Numerical Examples.

To illustrate the results of Section 3, we now provide two illustrative examples of parameter estimation in McKean-Vlasov SDEs, and the associated systems of interacting particles. In particular, we consider a one-dimensional linear mean-field model with two unknown parameters, and a stochastic opinion dynamics model with a single unknown parameter. In both cases, we simulate sample paths and implement the recursive MLE using a standard Euler-Maruyama scheme with $\Delta t = 0.1$.

5.1. Linear Mean Field Dynamics.

We first consider a one-dimensional linear mean field model, parametrised by $\theta = (\theta_1, \theta_2)^T \in \mathbb{R}^2$, given by

$$
\begin{aligned}
\mathrm{d}x_t &= -\left[\theta_1 x_t + \theta_2 \int_{\mathbb{R}} (x_t - y)\mu_t(\mathrm{d}y)\right] dt + \sigma \mathrm{d}w_t, \\
\mu_t &= \mathcal{L}(x_t).
\end{aligned}
$$

where $\sigma > 0$ and $w = (w_t)_{t\geq 0}$ is a standard Brownian motion. We will assume that $x_0 \in \mathbb{R}$. This is clearly of the form of the McKean-Vlasov SDE (1.1) - (1.2) with $b(\theta, x) = -\theta_1 x$ and $\phi(\theta, x, y) = -\theta_2(x - y)$. The corresponding system of interacting particles is given by

$$
\mathrm{d}x_t^{i,N} = -\left[\theta_1 x_t^{i,N} + \theta_2 \frac{1}{N}\sum_{j=1}^N (x_t^{i,N} - x_t^{j,N})\right] \mathrm{d}t + \sigma \mathrm{d}w_t^{i,N}, \quad i = 1, \ldots, N.
$$

In this model, the parameter θ_1 controls the strength of attraction of the non-linear process (or, in the IPS, of each individual particle) towards zero, while the strength of the parameter θ_2 controls the strength of the attraction of the non-linear process (of each individual particle) towards its mean (the empirical mean). We remark that, in the case $\theta_2 = 0$, the non-linear process reduces to a one-dimensional Orstein-Uhlenbeck (OU) process, and the system of interacting particles reduces to N independent samples of this process. It is straightforward to show that this model satisfies all of the conditions specified in Section 3.1. We defer the details to Appendix I.

5.1.1. Offline Parameter Estimation. We begin by illustrating the performance of the offline MLE. Since this model is linear in both of the parameters, in this case it is possible to obtain the maximum likelihood in closed form as (see also [44])

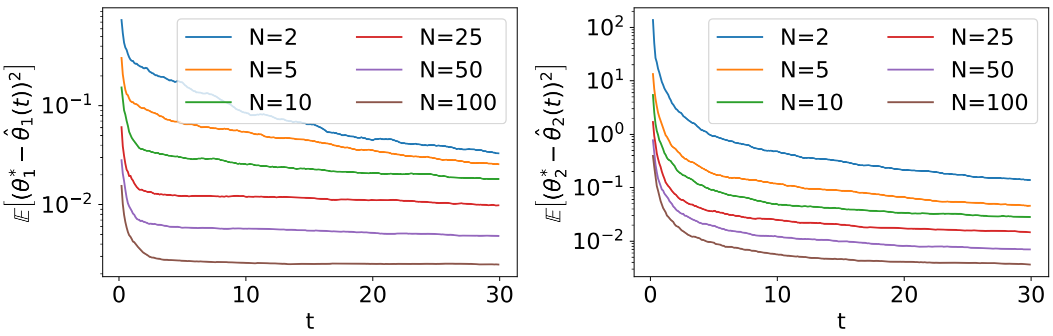
$$\hat{\theta}^N_{1,t} = \frac{A^N_t - B^N_t}{C^N_t - D^N_t} \quad , \quad \hat{\theta}^N_{2,t} = \frac{D^N_t A^N_t - C^N_t B^N_t}{(C^N_t)^2 - C^N_t D^N_t}$$

where we have defined, writing $\bar{x}^N_s = \frac{1}{N}\sum_{j=1}^N x^{j,N}_s$,

$$A^N_t = \int_0^t \sum_{i=1}^N (x^{i,N}_s - \bar{x}^N_s)\mathrm{d}x^{i,N}_s \ , \quad B^N_t = \int_0^t \sum_{i=1}^N x^{i,N}_s \mathrm{d}x^{i,N}_s$$

$$C^N_t = \int_0^t \sum_{i=1}^N (x^{i,N}_s - \bar{x}^N_s)^2 \mathrm{d}s \quad , \quad D^N_t = \int_0^t \sum_{i=1}^N (x^{i,N}_s)^2 \mathrm{d}s.$$

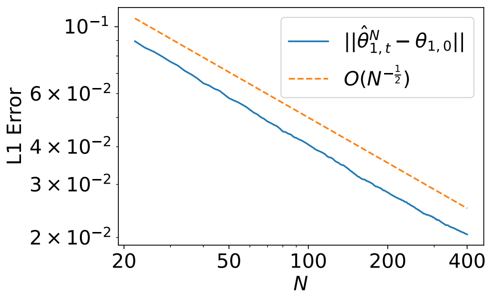
For our first simulation, we assume that the true parameter is given by $\theta^* = (1, 0.5)^T$, and that the diffusion coefficient is equal to the identity, $\sigma = 1$. The performance of the MLE is visualised in Figure 1, in which we plot the mean squared error (MSE) of the offline parameter estimate for $t \in [0, 30]$, and $N \in \{2, 5, 10, 25, 50, 100\}$, averaged over 500 random trials. As expected, the parameter estimates converge to the true parameter values as N increases with t fixed (see Theorem 3.1), and also as t increases with N fixed (see, e.g., [12, 47]).



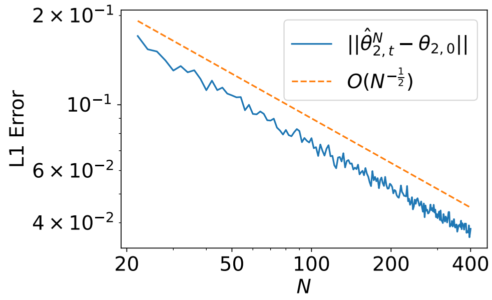
(a) $\hat{\theta}^N_{1,t}$. (b) $\hat{\theta}^N_{2,t}$.

Figure 1. *$\mathbb{L}^2$ error of the offline MLE for $t \in [0, 30]$ and $N = \{2, 5, 10, 25, 50, 100\}$. The $\mathbb{L}^2$ error is plotted on a log-scale.*

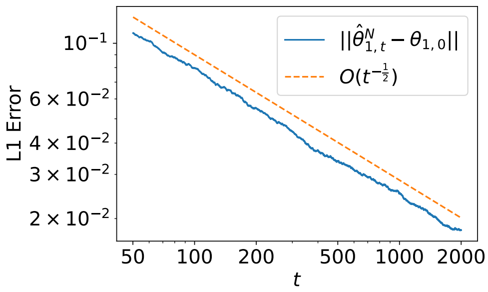
We investigate the convergence rate of the offline MLE further in Figure 2, in which we plot the mean absolute error (MAE) of the offline parameter estimate for $N \in \{20, 21, \dots, 400\}$ with $t = 5$, and also for $t \in [50, 2000]$ with $N = 2$, averaged over 500 random trials. Our results suggest that the offline MLE for this model has an $\mathbb{L}^1$ convergence rate of order $O((Nt)^{-\frac{1}{2}})$. This is rather unsurprising: such a rate was recently established by Chen [24] for a linear mean field model (of arbitrary dimension) in the absence of the global confinement term.



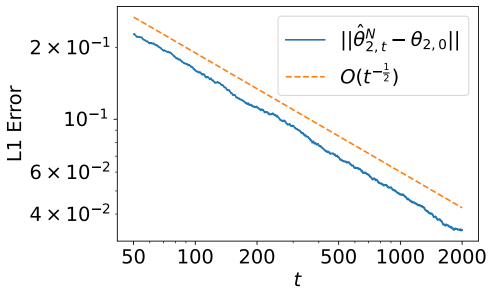
(a) $\hat{\theta}^N_{1,t=5}$.



(b) $\hat{\theta}^N_{2,t=5}$.



(c) $\hat{\theta}^{N=2}_{1,t}$.



(d) $\hat{\theta}^{N=2}_{2,t}$.

Figure 2. *Log-log plot of the $\mathbb{L}^1$ error of the offline MLE for $t = 5$ and $N \in \{20, \dots, 400\}$ (top panel), and for $t \in [50, 2000]$ and $N = 2$ (bottom panel).*

To conclude this section, we provide numerical confirmation of the asymptotic normality of the MLE (Theorem 3.2). For the linear mean field model of interest, it is in fact possible to obtain the asymptotic information matrix in closed form (see also [44]). In particular, it is given by

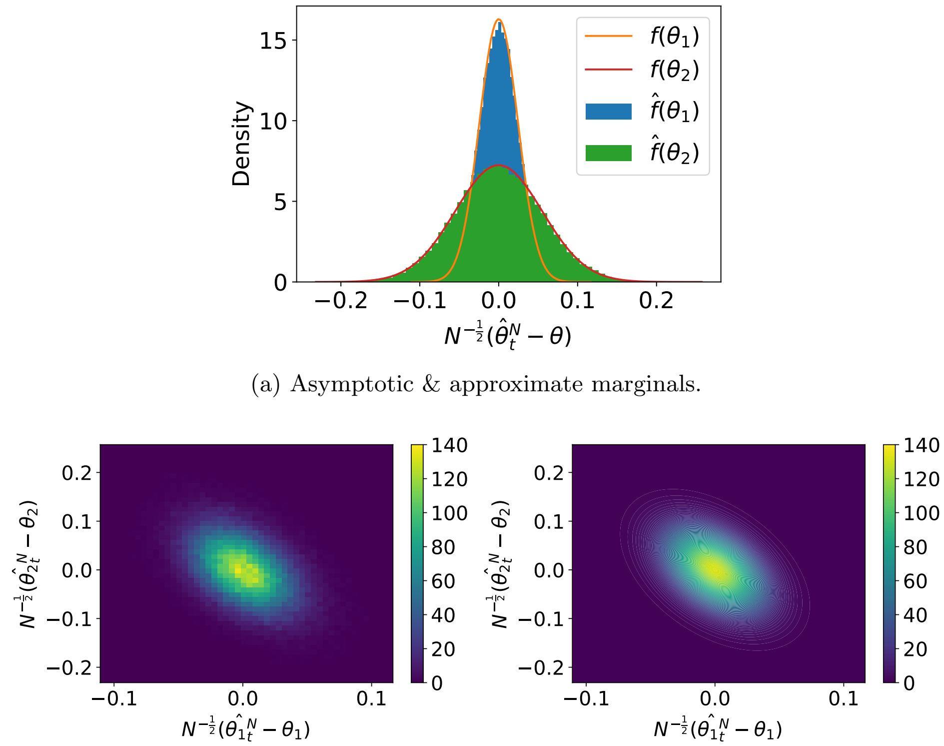
$$I_t(\theta) = \begin{pmatrix} D_t(\theta) & C_t(\theta) \\ C_t(\theta) & C_t(\theta) \end{pmatrix},$$

where, with $\gamma(\theta) = -2(\theta_1 + \theta_2)$,

$$C_t(\theta) = \frac{1}{\gamma^2(\theta)}(e^{\gamma(\theta)t} - 1) - \frac{t}{\gamma(\theta)} + \frac{\sigma_0^2}{\gamma}(e^{\gamma(\theta)t} - 1),$$

$$D_t(\theta) = \frac{1}{\gamma^2(\theta)}(e^{\gamma(\theta)t} - 1) - \frac{t}{\gamma(\theta)} + \frac{\sigma_0^2}{\gamma(\theta)}(e^{\gamma(\theta)t} - 1) - \frac{\mu_0^2}{2\theta_1}(e^{-2\theta_1 t} - 1).$$

As such, in Figure 3, we are able to provide a direct comparison of the asymptotic normal distribution of the MLE, and the approximate normal distribution obtained using a finite number of particles.



(a) Asymptotic & approximate marginals.

(b) Approximate bivariate density. (c) Asymptotic bivariate density.

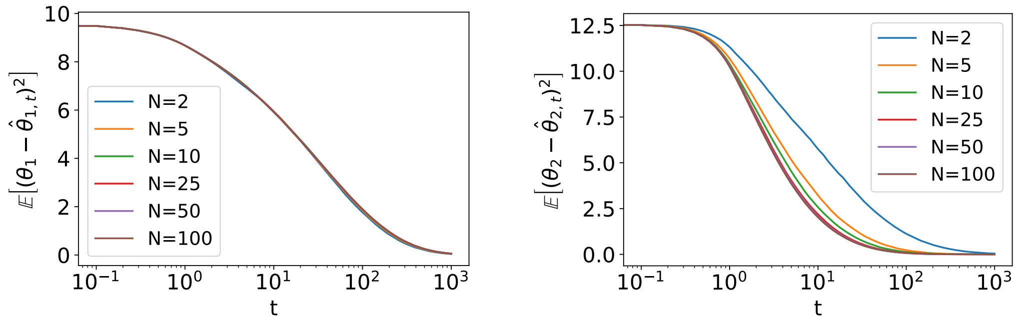
Figure 3. *A comparison between the asymptotic normal distribution and the approximate normal distribution of the MLE for $N = 500$ particles. The histograms were obtained using 10^5 independent runs.*

5.1.2. Online Parameter Estimation. We now turn our attention to the online MLE, which for this model evolves according to

$$\mathrm{d}\theta_{1,t}^N = \frac{\gamma_{1,t}}{N\sigma^2}\sum_{i=1}^N \left[-x_t^{i,N}\mathrm{d}x_t^{i,N} - x_t^{i,N}(\theta_{1,t}^N x_t^{i,N} + \theta_{2,t}^N(x_t^{i,N} - \bar{x}_t^N))\mathrm{d}t\right],$$

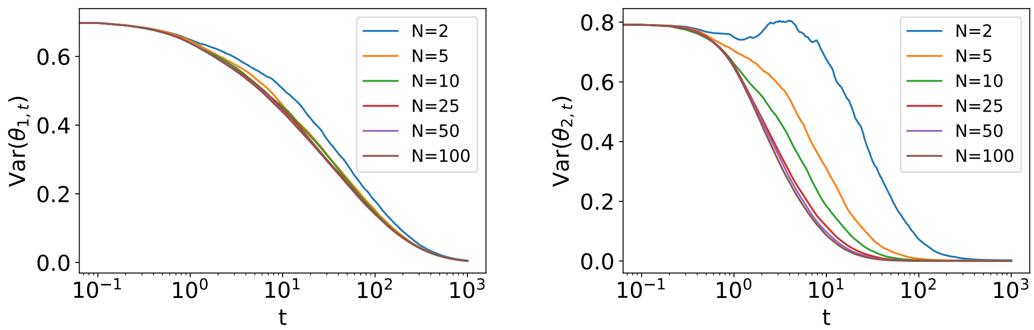
$$\mathrm{d}\theta_{2,t}^N = \frac{\gamma_{2,t}}{N\sigma^2}\sum_{i=1}^N \left[-(x_t^{i,N} - \bar{x}_t^N)\mathrm{d}x_t^{i,N} - (x_t^{i,N} - \bar{x}_t^N)(\theta_{1,t}^N x_t^{i,N} + \theta_{2,t}^N(x_t^{i,N} - \bar{x}_t^N))\mathrm{d}t\right].$$

We will initially assume that one of the parameters is fixed (and equal to the true value), while the other parameter is to be estimated. The true parameters are given by $\theta_1^* = 0.5$ and $\theta_2^* = 0.1$. Meanwhile, the initial parameter estimates are randomly generated according to $\theta_1^0, \theta_2^0 \sim \mathcal{U}([2,5])$. Finally, the learning rates are given by $\gamma_{i,t} = \min\{\gamma_i^0, \gamma_i^0 t^{-\alpha}\}$, $i = 1, 2$, where $\gamma_1^0 = 0.05$, $\gamma_2^0 = 0.30$, and $\alpha = 0.51$. The performance of the stochastic gradient descent algorithm is visualised in Figures 4 and 5, in which we plot the mean squared error (MSE) and the variance of the online parameter estimates for $t \in [0, 1000]$ and $N = \{2, 5, 10, 25, 50, 100\}$. The results are computed over 500 independent random trials. Interestingly, increasing the number of particles can result in a relatively significant reduction in the MSE of the interaction parameter θ_2, but has little consequence for the error of the confinement parameter θ_1. Meanwhile, there is a relatively significant reduction in the variance of both estimates.



(a) $\theta_{1,t}^N$. (b) $\theta_{2,t}^N$.

Figure 4. $\mathbb{L}^2$ *error of the online parameter estimates for* $t \in [0, 1000]$ *and* $N = \{2, 5, 10, 25, 50, 100\}$.



(a) $\theta_{1,t}^N$. (b) $\theta_{2,t}^N$.

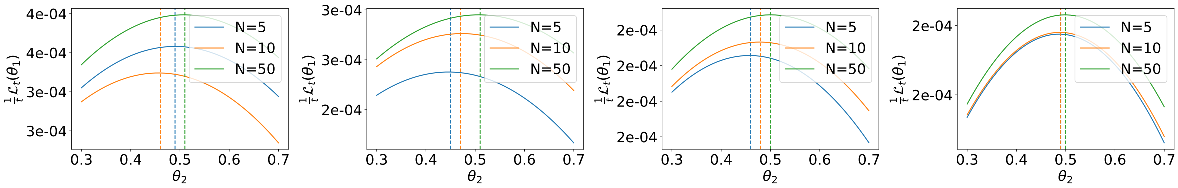
Figure 5. *Variance of the online parameter estimates for* $t \in [0, 1000]$ *and* $N = \{2, 5, 10, 25, 50, 100\}$.

We should remark that, in the linear mean field model, with one parameter fixed, the online parameter estimates generated via the system of interacting particles will converge to the true value of the parameter (which coincides with the global minimum of the asymptotic log-likelihood of the McKean-Vlasov SDE) for all values of N. Indeed, for this model, the (asymptotic) log-likelihood of the IPS is strongly concave for all values of N, with unique global maximum at the true parameter values. This is visualised in Figures 6d and 7d, in which we have plotted approximations of profile asymptotic log-likelihood of the IPS for several values of N. We are thus in the regime of Case I with finite N, meaning θ_t^N will converge to the true parameter as $t \to \infty$, regardless of the value of N.

Figures 6 and 7 also provide a numerical illustration of why the finite-time performance of the online estimator improves with the number of particles (see Theorem 3.4‡), and why this improvement is more pronounced for the interaction parameter θ_2. As N increases, we observe that the time weighted average of the log-likelihood $\mathcal{L}_t^N(\theta)$ (the noisy objective function) much more closely resembles the asymptotic log-likelihood $\tilde{\mathcal{L}}^N(\theta)$ (the true objective function), even for small time values. This means, in particular, that the fluctuations terms appearing in the proof of Theorems 3.3‡ - 3.4‡ of the form

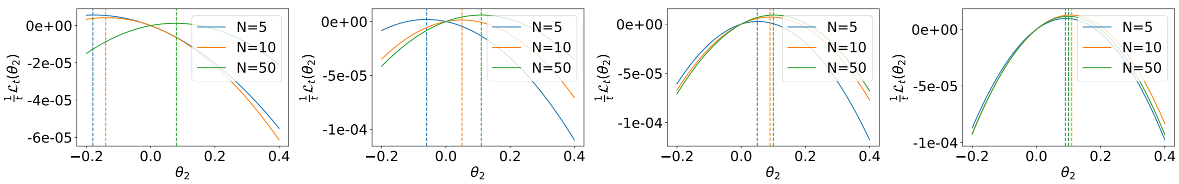
$$\int_0^t \gamma_s(\nabla_\theta \tilde{\mathcal{L}}^N(\theta_s^N) - \frac{1}{N}\sum_{i=1}^N \nabla_\theta L(\theta_s^N, x_s^N, \mu_s^N))\mathrm{d}s,$$

converge more rapidly to zero (as a function of time), for larger values of N. This disparity in the convergence rate of the log-likelihood (as a function of the time), for different values of N, appears to be much more significant for the interaction parameter θ_2 (Figure 7) than it is for the confinement parameter θ_1 (Figure 6). Consequently, the online parameter estimate $\theta_{2,t}^N$ converges more rapidly as N increases, while there is little difference in the convergence rate of $\theta_{1,t}^N$.



(a) $T = 1.0$. (b) $T = 2.5$. (c) $T = 5.0$. (d) $T = 7.5$.

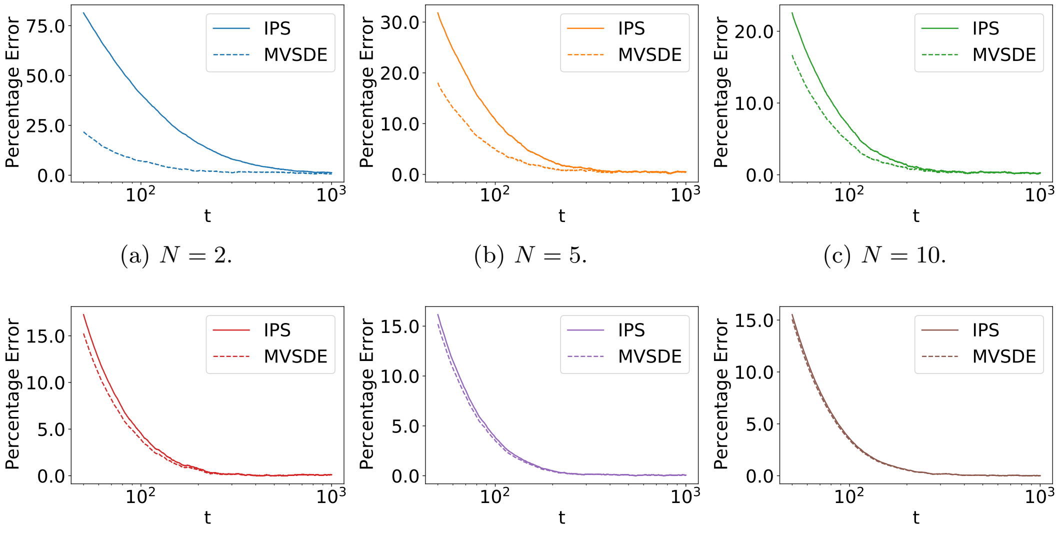
Figure 6. *Plots of the average log-likelihood, $\frac{1}{T}\mathcal{L}_T^N(\theta_1)$, for $T = \{1, 2.5, 5.7.5\}$ and $N = \{5, 10, 50\}$.*



(a) $T = 1.0$. (b) $T = 2.5$. (c) $T = 5.0$. (d) $T = 7.5$.

Figure 7. *Plots of the average log-likelihood, $\frac{1}{T}\mathcal{L}_T^N(\theta_2)$, for $T = \{1, 2.5, 5.7.5\}$ and $N = \{5, 10, 50\}$.*

We conclude this discussion with a comparison between the online parameter estimates generated using N particles from the IPS, and those generated using a single sample path of McKean-Vlasov SDE, and its law. We should emphasise that the latter is only possible when the solution of the non-linear equation is available. Illustrative results are provided in Figure 8, in which we plot the percentage error of the online parameter estimates for the interaction parameter, for several values of N. In each case, the estimate based on the McKean-Vlasov SDE converges more rapidly to the true parameter value. We also note, perhaps unsurprisingly, that this disparity becomes less apparent as the number of particles increases, reflecting the fact that the dynamics of the interacting particles increasingly resemble the dynamics of the solutions of the non-linear equation. Consistent with our previous observations, this disparity is also less apparent for the online estimates of the confinement parameter (results omitted).

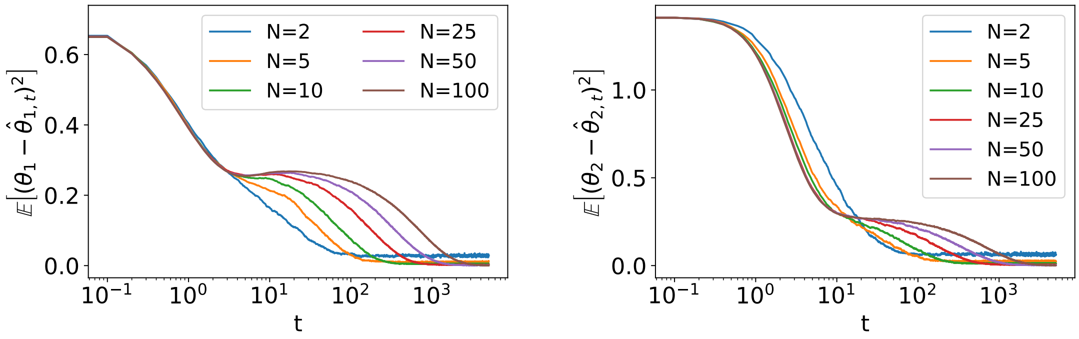


(a) $N = 2$. (b) $N = 5$. (c) $N = 10$.

(d) $N = 25$. (e) $N = 50$. (f) $N = 100$.

Figure 8. *Percentage error of the online maximum likelihood estimates of the interaction parameter* (θ_2) *for* $T \in [0, 1000]$ *and* $N = \{2, 5, 10, 25, 50, 100\}$*, generated using the IPS and the McKean-Vlasov SDE.*

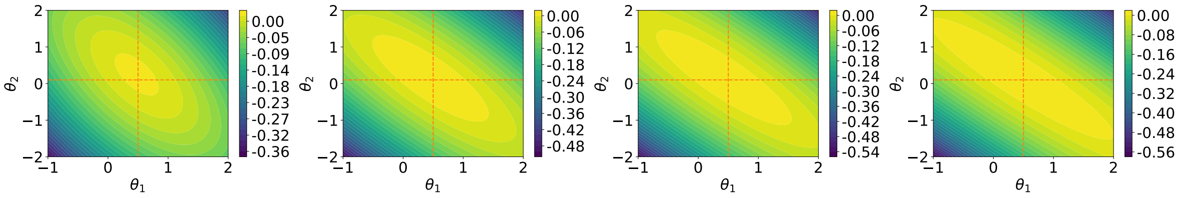
Let us now turn our attention to the case in which both parameters are unknown, and to be estimated from the data. For the sake of comparison, we will once more assume that that the true parameter is given by $\theta^* = (\theta_1^*, \theta_2^*) = (0.5, 0.1)$. The initial parameter estimates are now generated according to $\theta_1^0 \sim \mathcal{U}([-1, 2])$ and $\theta_2^0 \sim \mathcal{U}([-2, 2])$. Finally, we use constant learning rates, with $\gamma_{1,t} = 0.1$ and $\gamma_{2,t} = 0.2$. The performance of the stochastic gradient descent algorithm is illustrated in Figure 9, in which we plot the MSE of the online parameter estimates for both of the unknown parameters, averaged over 500 random trials.



(a) $\theta_{1,t}^N$. (b) $\theta_{2,t}^N$.

Figure 9. $\mathbb{L}^2$ *error of the online MLEs for* $T \in [0, 5000]$ *and* $N = \{2, 5, 10, 25, 50, 100\}$.

In this case, the evolution of the MSE appears to indicate three distinct learning phases. In the initial phase, the performance of the online estimator improves as a function of the number of particles, with this improvement being more noticeable for the interaction parameter θ_2, as observed previously. Conversely, in the middle phase, the online estimator performs (significantly) better for smaller values of N. These observations are readily explained with reference to the asymptotic log-likelihood of the IPS for different values of N, as shown in Figure 10. In particular, far from the global maximum at $\theta^* = (0.5, 0.1)$, the asymptotic log-likelihood decreases more steeply as the value of N increases. Broadly speaking, we can think of this region of the optimisation landscape as responsible for the initial learning phase, hence the improved performance of the estimator for larger values of N. On the other hand, close to the global maximum, the asymptotic log-likelihood exhibits an increasingly large plateau as the value of N increases (i.e., an increasingly flat maximum). This region of the optimisation landscape is largely responsible for the middle learning phase, which explains the slower convergence of the estimator for larger values of N. In the final learning phase, the (steady-state) error of the recursive MLE appears to decrease as a function of the number of particles. This is unsurprising, given the $O(N^{-\frac{1}{2}})$ term appearing in Theorem 3.4.



(a) $N = 2$. (b) $N = 5$. (c) $N = 10$. (d) $N = 100$.

Figure 10. *Contour plots of the asymptotic log-likelihood* $\tilde{\mathcal{L}}^N(\theta)$ *for* $N = \{2, 5, 10, 100\}$.

5.2. Stochastic Opinion Dynamics. We now consider a one-dimensional stochastic opinion dynamics model, parametrised by $\theta = (\theta_1, \theta_2)^T \in \mathbb{R}^2$, of the form

$$\mathrm{d}x_t = -\left[\int_{\mathbb{R}} \varphi_\theta(||x_t - y||)(x_t - y)\mu_t(\mathrm{d}y)\right]\mathrm{d}t + \sigma \mathrm{d}w_t,$$

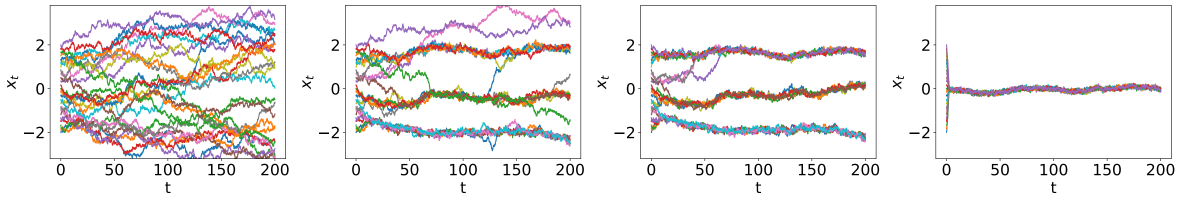
where the *interaction kernel* $\varphi_\theta : \mathbb{R}_+ \to \mathbb{R}_+$ is defined according to

$$\varphi_\theta(r) = \begin{cases} \theta_1 \exp\left[-\dfrac{0.01}{1-(r-\theta_2)^2}\right] & , \quad r > 0 \\ 0 & , \quad r \leq 0. \end{cases}$$

This provides an approximation, infinitely differentiable on $\mathbb{R}_+$, to a scaled indicator function with magnitude θ_1, and support $[0, \theta_2^+] := [0, \theta_2 + 1]$. We can interpret θ_1 as a scale parameter, which controls the strength of the attraction between particles, and θ_2^+ as a range parameter, which determines the distance within which particles must be of one another in order to interact. This model is perhaps more frequently specified in terms of the corresponding system of interacting particles, which is given by

$$\mathrm{d}x_t^{i,N} = -\frac{1}{N}\sum_{j=1}^{N}\varphi_\theta(||x_t^{i,N} - x_t^{j,N}||)(x_t^{i,N} - x_t^{j,N})\mathrm{d}t + \sigma \mathrm{d}w_t.$$

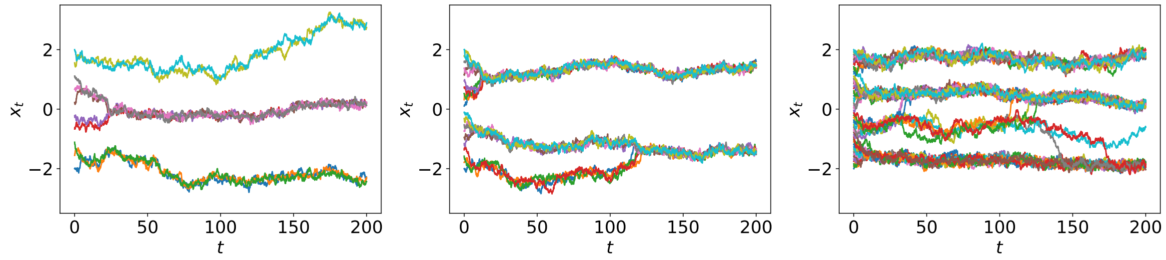
Models of this form arise in various applications, from biology to the social sciences, in which φ_θ determines how the dynamics of one particle (e.g., the opinions of one person) may influence the dynamics of other particles (e.g., the opinions of other people). For a more detailed account of such models, we refer to [13, 65] and references therein. For deterministic models of this type, it is well known that, asymptotically, the particles merge into clusters, the number of which depends both on the interaction kernel (i.e., the range and strength of the interaction between particles) and the initialisation. In the stochastic setting, the random noise prohibits the formation of exact clusters; instead, the particles merge into metastable 'soft clusters' (see also [55]). This is shown in Figure 11.



(a) $\theta_2 = 0.0$. (b) $\theta_2 = 0.3$. (c) $\theta_2 = 0.5$. (d) $\theta_2 = 1.0$.

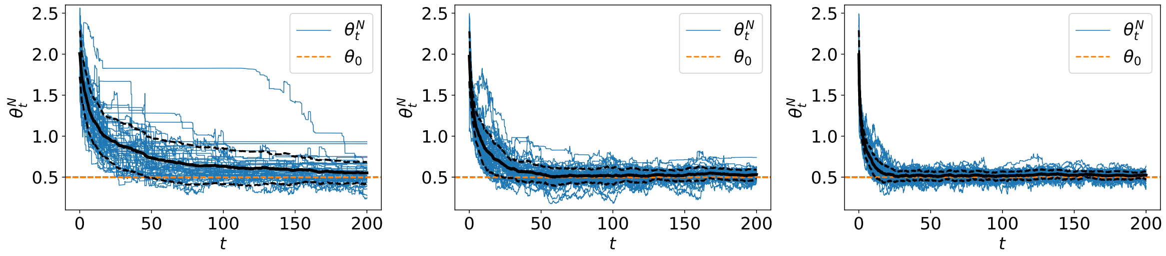
Figure 11. *Sample trajectories of the system of interacting particles for* $\theta_2 = \{0.0, 0.3, 0.5, 1.0\}$.

We will focus on online parameter estimation in the case in which the scale parameter θ_1 is fixed, and the range parameter θ_2^+ is to be estimated. We assume that $\theta_1 = \theta_1^* = 2$, and that $\theta_2^+ = 0.5$. This corresponds to an interaction kernel with with compact support on $[0, 0.5]$. The initial parameter estimates are generated uniformly at random on $[1.5, 2.5]$. Finally, we use constant learning rates with $\gamma_{2,t} = 0.002$. The performance of the recursive MLE is illustrated in Figure 13, in which we plot the sequence of online parameter estimates for θ_2^+, for several values of N, and for 50 different random initialisations. Encouragingly, (almost) all of the online parameter estimates converge to within a small neighbourhood of the true value of the parameter, suggesting that it is indeed possible to estimate the range of the interaction kernel in an online fashion. As in our previous simulations, the performance of the online estimator improves as the number of particles is increased. We should remark that the performance of the online estimator is highly dependent on the initial conditions of the particles. This should not come as a surprise; indeed, if the distance between particles is greater than then support of the interaction kernel, then the interaction kernel (and its gradient) are identically zero, and thus so too are all of the terms in the parameter update equation. Thus, the value of the parameter estimate will remain unchanged. We see this phenomenon in Figure 13, particularly when there are fewer particles.



(a) $N = 10$. (b) $N = 20$. (c) $N = 50$.

Figure 12. *Sample trajectories of the system of interacting particles for* $N = \{10, 20, 50\}$.



(a) $N = 10$. (b) $N = 20$. (c) $N = 50$.

Figure 13. *Sequence on online parameter estimates (blue) for the range parameter* θ_2*, for 50 different random initialisation* $\theta_2^0 \sim \mathcal{U}([1.5, 2.5])$*, and* $N = \{10, 20, 50\}$*. We also plot the true parameter value (orange), the mean online parameter estimate plus/minus one standard deviation (black: solid, dashed).*

6. Conclusions. In this paper, we have considered the problem of parameter estimation for a stochastic McKean-Vlasov equation and the associated system of weakly interacting particles. We established consistency and asymptotic normality of the offline MLE for the IPS as the number of particles $N \to \infty$, extending classical results in [44]. Moreover, we proposed an online estimator for the parameters of the stochastic McKean-Vlasov equation, based on observations of the trajectories of the the associated IPS, multiple independent replicates of the McKean-Vlasov SDE, or a single path of the McKean-Vlasov SDE and its law. We demonstrated $\mathbb{L}^1$ convergence of this estimator to the stationary points of the asymptotic log-likelihood of the McKean-Vlasov SDE as $N \to \infty$ and $t \to \infty$ and, under additional assumptions, obtained an $\mathbb{L}^2$ convergence rate. Finally, we presented two numerical examples as a proof of concept. A more detailed numerical analysis of parameter estimation in several IPSs of practical interest (e.g., stochastic opinion dynamics with heterophilious dynamics as in [65]) will appear in a subsequent paper.

Regarding other interesting directions for future research, in the offline case, it is of interest to establish a non-asymptotic $\mathbb{L}^p$ convergence rate for the MLE in both the mean-field (large N) and long time (large T) regimes, extending the recent results in [24] to a more general class of IPSs. In the online case, a natural extension of our results is to obtain a central limit theorem for the recursive estimator, extending the results in [77] to non-linear McKean-

Vlasov diffusions. Finally, one could aim to extend our results to the case in which the diffusion coefficient is unknown, and must be estimated online (see [75] for online estimation of the diffusion coefficient in the linear case, and [40] for offline estimation of the diffusion coefficient in IPSs). This is a particularly interesting problem given that, for a broad class of McKean-Vlasov SDEs, the uniqueness (or non-uniqueness) of the invariant measure(s) is known to depend on the magnitude of the noise coefficient (e.g., [37, 38, 81]).

Appendix A. Existing Results on the McKean-Vlasov SDE.

Proposition A.1 (Existence and Uniqueness, [16, Theorem 2.2.3]). *Assume that Conditions B.1(i) - B.2(i) hold. If $\mu_0^\theta \in \mathcal{P}_2(\mathbb{R}^d)$, the McKean-Vlasov SDE (1.1) has a unique strong solution $x^\theta = (x_t^\theta)_{t\geq 0}$ for all $t \geq 0$. In addition, the IPS (1.4) has a unique strong solution $x^{\theta,N} = (x_t^{\theta,N})_{t\geq 0}$ for all $t \geq 0$.*

Proposition A.2 (Moment Bounds, [16, Lemma 2.3.1]). *Assume that Conditions B.1(i) - B.2(i) and D.1 hold. Then, for all $k \geq 0$, there exists $C_k > 0$ such that for all $\theta \in \mathbb{R}^p$, and for all $N \in \mathbb{N}$,*

$$\sup_{t\geq 0} \mathbb{E}||x_t^{\theta,i,N}||^k \leq C_k \left(\int_{\mathbb{R}^d} x^k \mu_0(\mathrm{d}x) + 1 \right)$$
$$\sup_{t\geq 0} \mathbb{E}||x_t^{\theta}||^k \leq C_k \left(\int_{\mathbb{R}^d} x^k \mu_0(\mathrm{d}x) + 1 \right)$$

Proposition A.3 (Unique Invariant Measure of the MVSDE [16, Theorem 2.3.3]). *Assume that Conditions B.1 - B.2 hold. Then the McKean-Vlasov SDE admits a unique equilibrium measure μ_∞ which is independent of the initial condition μ_0. Moreover, with $\lambda = \alpha - 2L_2$, the following contraction rate holds*

$$\mathbb{W}_2(\mu_t, \mu_\infty) \leq e^{-\lambda t} \mathbb{W}_2(\mu_0, \mu_\infty)$$

Proposition A.4 (Unique Invariant Measure of the IPS, [Appendix B, Proposition 3.9]). *Assume that Conditions B.1 - B.2 hold. Then the IPS admits a unique equilibrium measure $\hat{\mu}_\infty^N$ which is independent of the initial condition $\hat{\mu}_0^N$. Moreover, with $\lambda = \alpha - 2L_2$, and writing $\hat{\mu}_t^{(k),N}$ for the law of a subset of $1 \leq k \leq N$ interacting particles, the following contraction rate holds*

$$\mathbb{W}_2(\hat{\mu}_t^{(k),N}, \hat{\mu}_\infty^{(k),N}) \leq e^{-\lambda t} \mathbb{W}_2(\mu_0^{\otimes k}, \hat{\mu}_\infty^{(k),N}).$$

Proposition A.5 (Propagation of Chaos, [16, Lemma 2.4.1]). *Let $x^i = (x_t^i)_{t\geq 0}$ be N independent copies of the solutions of (1.1) - (1.2) driven by independent Brownian motions w^i. Assume that Conditions B.1 - B.2 hold. Then there exist $0 < C < \infty$, independent of time, such that*

$$\sup_{t\geq 0} \mathbb{E}\left[||x_t^{i,N} - x_t^i||^2 \right] \leq \frac{C}{N}.$$

Proposition A.6 (A Law of Large Numbers, [25, Theorem 1.2], [66]). *Assume that Conditions B.1(i) - B.2(i) hold. If* $(\mu_0^N)_{N\in\mathbb{N}}$ *converge weakly to* μ_0, *then for all* $g \in \mathcal{C}(\mathbb{R}^d)$ *and for all* $t \geq 0$, *as* $N \to \infty$,

$$\lim_{N\to\infty}\left[\frac{1}{N}\sum_{i=1}^N g(x_t^{i,N})\right] \stackrel{\mathbb{P}}{=} \int_{\mathbb{R}^d} g(x)\mu_t(\mathrm{d}x).$$

Appendix B. Proof of Proposition A.4.

Proof. We will prove the first statement for $k = N$, with the general case proved almost identically. Let $\hat{x}_t^N$ and $\hat{y}_t^N$ be the solutions of (4.19) starting from $\hat{x}_0^N$ and $\hat{y}_0^N$, both driven by the same Brownian motion. We will write $\hat{\mu}_t^N$ and $\hat{\nu}_t^N$ to denote the laws of x_t^N and y_t^N, respectively. Now, by Itô's formula, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}||\hat{x}_t^N - \hat{y}_t^N||^2 = 2\big(\hat{x}_t^N - \hat{y}_t^N\big)\cdot\big(\hat{B}(\theta,\hat{x}_t^N) - \hat{B}(\theta,\hat{y}_t^N)\big). \tag{B.1}$$

Using Conditions B.1 and B.2, it is straightforward to show that, for all $\hat{x}^N, \hat{y}^N \in (\mathbb{R}^d)^N$, we have

$$\begin{aligned}\big(\hat{x}^N - \hat{y}^N\big)\cdot\big(\hat{B}(\hat{x}^N) - \hat{B}(\hat{y}^N)\big) &\leq -\alpha\sum_{i=1}^N||\hat{x}^{i,N} - \hat{y}^{i,N}||^2 + \frac{1}{N}\sum_{i,j=1}^N 2L_2\left(||\hat{x}^{i,N} - \hat{y}^{i,N}||^2\right)\\ &\leq -\left(\alpha - 2L_2\right)||\hat{x}^N - \hat{y}^N||^2.\end{aligned} \tag{B.2}$$

Combining (B.1) and (B.2), and taking expectations, it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\left[||\hat{x}_t^N - \hat{y}_t^N||^2\right] \leq 2\left(\alpha - 2L_2\right)\mathbb{E}\left[||\hat{x}_t^N - \hat{y}_t^N||^2\right].$$

Thus, writing $\lambda = \alpha - 2L_2$, we have that

$$\mathbb{E}\left[||\hat{x}_t^N - \hat{y}_t^N||^2\right] \leq e^{-2\lambda t}||\hat{x}_0^N - \hat{y}_0^N||^2.$$

Let π_0^N be an arbitrary coupling of μ_0^N and ν_0^N. We then have

$$\int_{(\mathbb{R}^d)^N\times(\mathbb{R}^d)^N}\mathbb{E}\left[||\hat{x}_t^N - \hat{y}_t^N||^2\right]\pi_0^N(\mathrm{d}\hat{x}_0^N,\mathrm{d}\hat{y}_0^N) \leq e^{-2\lambda t}\int_{(\mathbb{R}^d)^N\times(\mathbb{R}^d)^N}||\hat{x}_0^N - \hat{y}_0^N||^2\pi_0^N(\mathrm{d}\hat{x}_0^N,\mathrm{d}\hat{y}_0^N).$$

It follows, taking the infimum over all coupling measures π_0, and taking square roots, that

$$\mathbb{W}_2(\hat{\mu}_t^N,\hat{\nu}_t^N) \leq e^{-\lambda t}\mathbb{W}_2(\hat{\mu}_0^N,\hat{\nu}_0^N).$$

Using this inequality, it is now straightforward to demonstrate existence and uniqueness of an invariant measure for the IPS. Indeed, classical arguments used to establish existence and uniqueness of an invariant measure for the corresponding McKean-Vlasov SDE (e.g. [16, Theorem 2.3.3]) go through almost verbatim. The details are omitted here.

Appendix C. Proof of Lemma for Theorem 3.1 and Theorem 3.2.

Lemma C.1. *For all $T \geq 0$, for all $\theta \in \Theta \subseteq \mathbb{R}^p$,*

$$\lim_{N\to\infty} \sup_{0\leq t\leq T} \frac{1}{N}\sum_{i=1}^{N}\int_0^t \langle G(\theta, x_s^{i,N}, \mu_s^N), \mathrm{d}w_s^i\rangle = 0$$

Proof. For ease of notation, let us define

$$M_t^N(\theta) := \frac{1}{N}\sum_{i=1}^{N}\int_0^t \langle G(\theta, x_s^{i,N}, \mu_s^N), \mathrm{d}w_s^i\rangle.$$

Now, for all $N \in \mathbb{N}$, and for all $\theta \in \mathbb{R}^p$, $(M_t^N(\theta))_{t\geq 0}$ is a zero mean continuous square integrable martingale, with quadratic variation

$$\left[M^N(\theta)\right]_t = \frac{1}{N^2}\sum_{i=1}^{N}\int_0^t ||G(\theta, x_s^{i,N}, \mu_s^N)||^2 \mathrm{d}s. \tag{C.1}$$

It follow, using the elementary fact that $\sup_x [f(x) - g(x)] \geq \sup_x f(x) - \sup_x g(x)$, and the martingale inequality [62, page 25], that

$$\mathbb{P}\left(\sup_{0\leq t\leq T} M_t^N(\theta) - \sup_{0\leq t\leq T}\frac{\alpha}{2}\left[M^N(\theta)\right]_t > \beta\right) \leq \mathbb{P}\left(\sup_{0\leq t\leq T}\left\{M_t^N(\theta) - \frac{\alpha}{2}\left[M^N(\theta)\right]_t\right\} > \beta\right)$$
$$< e^{-\alpha\beta}.$$

Thus, substituting (C.1) and using symmetry, we have that

$$\mathbb{P}\left(\sup_{0\leq t\leq T}\left|M_t^N(\theta)\right| > \beta + \frac{\alpha}{2N^2}\sum_{i=1}^{N}\int_0^T ||G(\theta, x_s^{i,N}, \mu_s^N)||^2 \mathrm{d}s\right) < 2e^{-\alpha\beta}.$$

Let $\alpha = N^a$, $\beta = N^{-b}$, for some $0 < a < b < 1$. Then

$$\mathbb{P}\left(\sup_{0\leq t\leq T}|M_t^N(\theta)| > \frac{1}{N^b} + \frac{1}{2N^{1-a}}\frac{1}{N}\sum_{i=1}^{N}\int_0^T ||G(\theta, x_s^{i,N}, \mu_s^N)||^2 \mathrm{d}s\right) < 2e^{-N^{a-b}}.$$

By a repeated application of Proposition A.6 (the McKean-Vlasov Law of Large Numbers), we have that, as $N \to \infty$,

$$\frac{1}{N}\sum_{i=1}^{N}\int_0^T ||G(\theta, x_s^{i,N}, \mu_s^N)||^2 \mathrm{d}s \xrightarrow{\mathbb{P}} \int_0^T\left[\int_{\mathbb{R}^d} ||G(\theta, x, \mu_s)||^2 \mu_s(\mathrm{d}x)\right]\mathrm{d}s.$$

By definition, Condition C.1(ii), and Proposition A.2 the limiting function on the RHS is finite and non-random. Moreover, we have that $\sum_{N=1}^{\infty} e^{-N^{a-b}} < \infty$. The Borel-Cantelli Lemma thus implies

$$\lim_{N\to\infty}\sup_{0\leq t\leq T} M_t^N(\theta) = 0.$$

■

Appendix D. Proof of Lemmas for Theorem 3.3.

D.1. Additional Lemmas for Lemmas 3.4.A and 3.4.B.

Lemma D.1. *Assume that Conditions B.1 - B.2 and D.1 hold. Then, for all $k \in \mathbb{N}$, there exists a positive constant $K > 0$ such that, for all $i = 1, \ldots, N$, and for all $N \in \mathbb{N}$,*

$$\int_{\mathbb{R}^d} ||x||^k \mu_\infty(\mathrm{d}x) \leq K,$$
$$\int_{(\mathbb{R}^d)^N} ||x_i||^k \hat{\mu}_\infty^N(\mathrm{d}\hat{x}^N) \leq K.$$

Proof. By Proposition A.3, the McKean-Vlasov SDE (1.1) - (1.2) admits a unique equilibrium measure μ_∞ which is independent of the initial condition μ_0. By the ergodic theorem (e.g., [71, Chapter X]), we thus have, for all $k \in \mathbb{N}$, that

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t ||x_s||^k \mathrm{d}s = \int_{\mathbb{R}^d} ||x||^k \mu_\infty(\mathrm{d}x) \ , \quad \text{a.s.} \tag{D.1}$$

Using Jensen's inequality and Proposition A.2 (uniform moment bounds for the McKean-Vlasov SDE), we obtain uniform integrability of the family $\{\frac{1}{t}\int_0^t ||x_s||^k \mathrm{d}s\}_{t>0}$. In particular, for all $1 \leq k' < k$, for all $t > 0$, we have, for some $\varepsilon > 0$,

$$\mathbb{E}\left[\frac{1}{t}\int_0^t ||x_s||^{k'}\mathrm{d}s\right]^{1+\varepsilon} \leq \frac{1}{t}\int_0^t \mathbb{E}\left[||x_s||^{k'(1+\varepsilon)}\right]\mathrm{d}s \leq C_k\left(\int_{\mathbb{R}^d} x^k\mu_0(\mathrm{d}x) + 1\right).$$

It follows, taking expectations of (D.1), using uniform integrability in order to interchange the limit and the expectation, and once more making use of Proposition A.2, that

$$\int_{\mathbb{R}^d} ||x||^k \mu_\infty(\mathrm{d}x) = \lim_{t\to\infty}\left[\frac{1}{t}\int_0^t \mathbb{E}\left[||x_s||^k\right]\mathrm{d}s\right] < \infty.$$

The proof of the bound for the IPS is identical, noting that all of the relevant results (Propositions A.2 and A.3) have analogues for for the IPS (Propositions A.2 and A.4). ∎

Lemma D.2. *Assume that Conditions B.1 - B.2 and D.1 hold. Then, for all $k \geq 1$, and for all $t \geq 0$, there exists $K > 0$ such that*

$$\mathbb{E}\left[\sup_{0\leq s\leq t} ||x_s||^k\right] \leq Kt^{\frac{1}{2}},$$
$$\mathbb{E}\left[\sup_{0\leq s\leq t} ||x_s^{i,N}||^k\right] \leq Kt^{\frac{1}{2}} \ , \quad \forall i = 1, \ldots, N.$$

Proof. We will prove the first claim (the proof of the second being essentially identical).

By Itô's Lemma, we have

$$\begin{aligned}||x_t||^{2k} = ||x_0||^{2k} &+ \int_0^t 2k||x_s||^{2k-2}\langle x_s, B(\theta, x_s, \mu_s)\rangle \mathrm{d}s \\ &+ \int_0^t k||x_s||^{2k-2}\mathrm{Tr}[I_d + (k-2)[x_s^i x_s^j]_{i,j=1}^d ||x_s||^{-2}]\mathrm{d}s \\ &+ \int_0^t 2k||x_s||^{2k-2}\langle x_s, \mathrm{d}w_s\rangle\end{aligned}$$

It follows, taking the supremum and taking expectations, that

$$\text{(D.2)}\quad \mathbb{E}\left[\sup_{0\leq s\leq t}||x_t||^{2k}\right] \leq \mathbb{E}\left[||x_0||^{2k}\right] + \underbrace{2k\int_0^t \mathbb{E}\left[\left|||x_s||^{2k-2}\langle x_s, B(\theta, x_s, \mu_s)\rangle\right|\right]\mathrm{d}s}_{\Pi_t^1}$$
$$+ \underbrace{k\int_0^t \mathbb{E}\left[\left|||x_s||^{2k-2}\mathrm{Tr}[I_d + (k-2)[x_s^i x_s^j]_{i,j=1}^d||x_s||^{-2}]\right|\right]\mathrm{d}s}_{\Pi_t^2}$$
$$+ \underbrace{2k\mathbb{E}\left[\sup_{0\leq s\leq t}\int_0^t ||x_s||^{2k-2}\langle x_s, \mathrm{d}w_s\rangle\right]}_{\Pi_t^3}$$

We begin by bounding the first term. First note that, by Conditions B.1 - B.2, there exist positive constants C_0 and C_1 such that

$$\langle x_s, B(\theta, x_s, \mu_s)\rangle \leq -(\alpha - L_2)||x_s||^2 + C||x_s|| + L||x_s||\mathbb{E}\left[||x_s||\right]$$

It then follows that

$$\begin{aligned}\text{(D.3)}\quad \Pi_t^1 &\leq K\int_0^t \mathbb{E}\left[||x_s||^{2k}\right] + \mathbb{E}\left[||x_s||^{2k-1}\right] + \mathbb{E}\left[||x_s||^{2k-1}\right]\mathbb{E}\left[||x_s||\right]\mathrm{d}s \\ &\leq K\int_0^t \mathbb{E}\left[||x_s||^{2k}\right] + \mathbb{E}\left[||x_s||^{2k}\right]^{\frac{2k-1}{2k}} + \mathbb{E}\left[||x_s||^{2k}\right]^{\frac{2k-1}{2k}}\mathbb{E}\left[||x_s||^2\right]^{\frac{1}{2}}\mathrm{d}s \\ &\leq Kt\left[1 + \int_{\mathbb{R}^d} x^{2k}\mu_0(\mathrm{d}x) + \left[\int_{\mathbb{R}^d} x^{2k}\mu_0(\mathrm{d}x)\right]^{\frac{2k-1}{2k}}\left[1 + \left(\int_{\mathbb{R}^d} x^2\mu_0(\mathrm{d}x)\right)^{\frac{1}{2}}\right]\right]\end{aligned}$$

where in the penultimate line we have used Hölder's inequality, and in the final line we have used Proposition A.2 (moment bounds for the McKean-Vlasov SDE). Similarly, for the second term in (D.2), we have

$$\text{(D.4)}\qquad \Pi_t^2 \leq K\int_0^t \mathbb{E}\left[||x_s||^{2k}\right]\mathrm{d}s \leq Kt\left[1 + \int_{\mathbb{R}^d} x^{2k}\mu_0(\mathrm{d}x)\right].$$

It remains to bound the final term in (D.2). For this term, we use the Burkholder-Davis-Gundy inequality and Proposition A.2 to obtain

$$\text{(D.5)}\qquad \Pi_t^3 \leq K\mathbb{E}\left[\int_0^t ||x_s||^{4k-4}x_s^T x_s \mathrm{d}s\right]^{\frac{1}{2}} \leq K\mathbb{E}\left[\int_0^t ||x_s||^{4k-2}\mathrm{d}s\right]^{\frac{1}{2}} \leq Kt^{\frac{1}{2}}.$$

Combining equations (D.2), (D.3), (D.4), and (D.5), and using the Hölder inequality, we conclude that, for all $t > 0$, there exists a positive constant K such that ■

$$\mathbb{E}\left[\sup_{0\leq s\leq t}||x_s||^k\right] \leq \mathbb{E}\left[\sup_{0\leq s\leq t}||x_s||^{2k}\right]^{\frac{1}{2}} \leq Kt^{\frac{1}{2}}.$$

Lemma D.3. *Assume that Conditions B.1 - B.2 and D.1 hold. Suppose that, for all* $\theta \in \mathbb{R}^p$, $f(\theta,\cdot): \mathbb{R}^d \to \mathbb{R}$ *is locally Lipschitz, and satisfies a polynomial growth condition, viz*

$$||f(\theta,x) - f(\theta,y)|| \leq K||x-y||\left[1 + ||x||^q + ||y||^q\right].$$

Then, for all $\theta \in \mathbb{R}^p$, $x,y \in \mathbb{R}^d$, $t \geq 0$, *there exist positive constants* $q, K > 0$ *such that*

$$\left|\mathbb{E}_x\left[f(\theta,x_t)\right] - \int_{\mathbb{R}^d} f(\theta,z)\mu_\infty(\mathrm{d}z)\right| \leq K[1+||x||^q]e^{-\lambda t}.$$

$$\left|\mathbb{E}_x\left[f(\theta,x_t)\right] - \mathbb{E}_y\left[f(\theta,x_t)\right]\right| \leq K[1+||x||^q+||y||^q]e^{-\lambda t}.$$

Alternatively, suppose that, for all $\theta \in \mathbb{R}^p$, $f(\theta,\cdot): (\mathbb{R}^d)^N \to \mathbb{R}$ *is locally Lipschitz and satisfies a polynomial growth condition in the sense that*

$$\left|f(\theta,\hat{x}^N) - f(\theta,\hat{y}^N)\right| \leq K\left[1 + ||x^{i,N}||^q + ||y^{i,N}||^q + \frac{1}{N}\sum_{j=1}^N ||x^{j,N}||^q + \frac{1}{N}\sum_{j=1}^N ||y^{j,N}||^q\right] \cdot \left[||x^{i,N} - y^{i,N}|| + \left(\frac{1}{N}\sum_{j=1}^N ||x^{j,N} - y^{j,N}||^2\right)^{\frac{1}{2}}\right]$$

where $\hat{x}^N = (x^{1,N},\dots,x^{N,N}) \in (\mathbb{R}^d)^N$. *Then, for all* $i = 1,\dots,N$, *and for all* $\theta \in \mathbb{R}^p$, *there exist positive constants* $q, K > 0$ *such that*

$$\left|\mathbb{E}_{\hat{x}^N}[f(\theta,\hat{x}_t^N)] - \int_{(\mathbb{R}^d)^N} f(\theta,\hat{z}^N)\hat{\mu}_\infty^N(\mathrm{d}\hat{z}^N)\right| \leq K\left[1 + ||x^{i,N}||^q + \frac{1}{N}\sum_{j=1}^N ||x^{j,N}||^q\right]e^{-\lambda t}$$

$$\left|\mathbb{E}_{\hat{x}^N}[f(\theta,\hat{x}_t^N)] - \mathbb{E}_{\hat{y}^N}[f(\theta,\hat{x}_t^N)]\right| \leq K\left[1 + ||x^{i,N}||^q + ||y^{i,N}||^q + \frac{1}{N}\sum_{j=1}^N (||x^{j,N}||^q + ||y^{j,N}||^q)\right]e^{-\lambda t}$$

for all $\hat{x}^N, \hat{y}^N \in (\mathbb{R}^d)^N$, *and for all* $t \geq 0$.

Proof. We will focus on the first statement of the first part of the Lemma. Let $\mu,\nu \in \mathcal{P}(\mathbb{R}^d)$, and $\pi \in \Pi(\mu,\nu)$. Then, using the Hölder inequality and the local Lipschitz assumption, it follows that

$$
\begin{aligned}
&\int_{\mathbb{R}^d\times\mathbb{R}^d} |f(\theta,y)-f(\theta,z)|\,\pi(\mathrm{d}y,\mathrm{d}z) \\
&\leq K\left[\int_{\mathbb{R}^d\times\mathbb{R}^d}||y-z||^2\pi(\mathrm{d}y,\mathrm{d}z)\right]^{\frac{1}{2}}\left[1+\left[\int_{\mathbb{R}^d\times\mathbb{R}^d}||y||^{2q}\pi(\mathrm{d}y,\mathrm{d}z)\right]^{\frac{1}{2}}+\left[\int_{\mathbb{R}^d\times\mathbb{R}^d}||y||^{2q}\pi(\mathrm{d}y,\mathrm{d}z)\right]^{\frac{1}{2}}\right] \\
&= K\left[\int_{\mathbb{R}^d\times\mathbb{R}^d}||y-z||^2\pi(\mathrm{d}y,\mathrm{d}z)\right]^{\frac{1}{2}}\left[1+\left[\int_{\mathbb{R}^d}||y||^{2q}\mu(\mathrm{d}y)\right]^{\frac{1}{2}}+\left[\int_{\mathbb{R}^d}||y||^{2q}\mu(\mathrm{d}z)\right]^{\frac{1}{2}}\right]
\end{aligned}
$$

Let x_t be a solution of the McKean-Vlasov SDE starting from $x\in\mathbb{R}^d$. Let μ_t^x denote the law of x_t, and let μ_∞ denote the invariant measure of the McKean-Vlasov SDE. Moreover, let $\pi_t^{x,\infty}$ denote an arbitrary coupling of μ_t^x and μ_∞. It then follows straightforwardly from the previous inequality that

$$
\begin{aligned}
\left|\mathbb{E}_x\left[f(\theta,x_t)\right]-\int_{\mathbb{R}^d}f(\theta,z)\mu_\infty(\mathrm{d}z)\right| &= \left|\int_{\mathbb{R}^d}f(\theta,y)\mu_t^x(\mathrm{d}y)-\int_{\mathbb{R}^d}f(\theta,z)\mu_\infty(\mathrm{d}z)\right| \\
&\leq \int_{\mathbb{R}^d\times\mathbb{R}^d}|f(\theta,y)-f(\theta,z)|\pi_t^{x,\infty}(\mathrm{d}y,\mathrm{d}z) \\
&\leq K\left[\int_{\mathbb{R}^d\times\mathbb{R}^d}||y-z||^2\pi_t^{x,\infty}(\mathrm{d}y,\mathrm{d}z)\right]^{\frac{1}{2}} \\
&\quad\cdot\left[1+\left[\int_{\mathbb{R}^d}||y||^{2q}\mu_t^x(\mathrm{d}y)\right]^{\frac{1}{2}}+\left[\int_{\mathbb{R}^d}||z||^{2q}\mu_\infty(\mathrm{d}z)\right]^{\frac{1}{2}}\right]
\end{aligned}
$$

Finally, using the fact that the chosen coupling was arbitrary, and using Lemma D.1 (the bounded moments of the invariant measure of the McKean-Vlasov SDE), Proposition A.2 (the moment bounds for the McKean-Vlasov SDE), Proposition A.3 (exponential contractivity of the McKean-Vlasov SDE), the previous inequality implies

$$
\begin{aligned}
\left|\mathbb{E}_x\left[f(\theta,x_t)\right]-\int_{\mathbb{R}^d}f(\theta,z)\mu_\infty(\mathrm{d}z)\right| &\leq K\mathbb{W}_2(\mu_t^x,\mu_\infty)\left[1+\left[\int_{\mathbb{R}^d}||y||^{2q}\mu_t^x(\mathrm{d}y)\right]^{\frac{1}{2}}+K'\right] \\
&\leq K\mathbb{W}_2(\mu_0^x,\mu_\infty)\left[1+||x||^q\right]e^{-\lambda t} \\
&\leq K\left[1+||x||^q\right]e^{-\lambda t}
\end{aligned}
$$

This completes the proof of the first statement of the first part of the Lemma. The proof of the second statement is essentially identical, this time considering an arbitrary coupling of μ_t^x and μ_t^y, and making use of the bound $\mathbb{W}_2(\mu_t^x,\mu_t^y)\leq e^{-\lambda t}\mathbb{W}_2(\mu_0^x,\mu_0^y)$. Finally, the proof of the second part of the Lemma follows closely the previous proof, now using the statements in Lemma D.1, Proposition A.2, and Proposition A.4 that are relevant to the IPS. ■

Lemma D.4. *Assume that Condition C.1 holds. Then, for $k=0,1,2,3$, there exist constants $q,K<\infty$, such that $\nabla_\theta^k L(\theta,x,\mu)$, satisfy the following polynomial growth conditions:*

$$
||\nabla_\theta^k L(\theta,x,\mu)||\leq K\left[1+||x||^q+\mu(||\cdot||^q)\right].
$$

Proof. We first observe that, by Condition C.1(ii), there exist constants $q_k, K_k < \infty$ such that $\nabla_\theta^k b(\theta, x) \leq K_j(1 + ||x||^{q_k})$ and $\nabla_\theta^k \phi(\theta, x, y) \leq K_k(1 + ||x||^{q_k} + ||y||^{q_k})$. It follows from the definition of $B(\theta, x, \mu)$, c.f. (1.3), that

$$\begin{aligned}\nabla_\theta^k B(\theta, x, \mu) &= \nabla_\theta^k b(\theta, x, \mu) + \int_{\mathbb{R}^d} \nabla_\theta^k \phi(\theta, x, y)\mu(\mathrm{d}y) \\ &= K_k(1 + ||x||^{q_k}) + K_k \int_{\mathbb{R}^d} (1 + ||x||^{q_k} + ||y||^{q_k})\mu(\mathrm{d}y) \\ &\leq K_k \left[1 + ||x||^{q_k} + \mu(||\cdot||^{q_k})\right]\end{aligned}$$

where we allow the values of q_k, K_k to vary from line to line. Thus, from the definition of $G(\theta, x, \mu)$, c.f. (2.1), we have that

$$||\nabla_\theta^k G(\theta, x, \mu)|| = ||\nabla_\theta^k B(\theta, x, \mu) - \nabla_\theta^k B(\theta_0, x, \mu)|| \leq K_k \left[1 + ||x||^{q_k} + \mu(||\cdot||^{q_k})\right]. \tag{D.6}$$

It now follows, recalling the definition of $L(\theta, x, \mu)$, c.f. (2.2), that

$$||L(\theta, x, \mu)|| = \frac{1}{2}||G(\theta, x, \mu)||^2 \leq K_0^2 \left[1 + ||x||^{q_0} + \mu(||\cdot||^{q_0})\right]^2 \leq K \left[1 + ||x||^q + \mu(||\cdot||^q)\right]$$

where in the final inequality we have set $K = 3K_0^2$ and $q = 2q_0$, after applying Hölder's inequality. The bounds for $\nabla_\theta^k L(\theta, x, \mu)$, for $k = 1, 2, 3$, are obtained in an almost identical fashion. ∎

Remark. This result implies, substituting $x = x^{i,N}$ and $\mu = \mu^N$, and recalling that $\hat{L}^{i,N}(\theta, \hat{x}^N) := L(\theta, x^{i,N}, \hat{\mu}^N)$, that for all $i = 1, \ldots, N$, $N \in \mathbb{N}$, we have

$$||\nabla_\theta^k \hat{L}^{i,N}(\theta, \hat{\mu}^N)|| \leq K \left[1 + ||x^{i,N}||^q + \frac{1}{N}\sum_{j=1}^N ||x^{j,N}||^q\right]$$

Lemma D.5. *Assume that Condition C.1 holds. Then, for $k = 0, 1, 2, 3$, there exist constants $q, K < \infty$ such that $\nabla_\theta^k L(\theta, x, \mu)$ satisfy*

$$\begin{aligned}||\nabla_\theta^k L(\theta, x, \mu) - \nabla_\theta^k L(\theta, x', \mu')|| &\leq K \left[||x - x'|| + \mathbb{W}_2(\mu, \mu')\right] \\ &\quad \cdot \left[1 + ||x||^q + ||x'||^q + \mu(||\cdot||^q) + \mu'(||\cdot||^q)\right]\end{aligned}$$

Proof. We begin by recalling that, from Condition C.1(ii), there exist constants $q, K < \infty$ such that $\nabla_\theta^k \phi(\theta, x, y) \leq K[||x - x'|| + ||y - y'||][1 + ||x||^q + ||x'||^q + ||y||^q + ||y'||^q]$. It follows, letting $\pi \in \Pi(\mu, \mu')$ and using the Hölder inequality, that

$$\begin{aligned}&\left|\left|\int_{\mathbb{R}^d} \nabla_\theta^k \phi(\theta, x, y)\mu(\mathrm{d}y) - \int_{\mathbb{R}^d} \nabla_\theta^k \phi(\theta, x, y')\mu'(\mathrm{d}y')\right|\right| \\ &\leq K\left[\int_{\mathbb{R}^d \times \mathbb{R}^d} ||y - y'|| \left[1 + ||y||^q + ||y'||^q\right] \pi(\mathrm{d}y, \mathrm{d}z)\right] \\ &\leq K\left[\int_{\mathbb{R}^d \times \mathbb{R}^d} ||y - y'||^2 \pi(\mathrm{d}y, \mathrm{d}z)\right]^{\frac{1}{2}} \left[1 + \left[\int_{\mathbb{R}^d} ||y||^{2q}\mu(\mathrm{d}y)\right]^{\frac{1}{2}} + \left[\int_{\mathbb{R}^d} ||y'||^{2q}\mu(\mathrm{d}z)\right]^{\frac{1}{2}}\right] \\ &\leq K\mathbb{W}_2(\mu, \mu') \left[1 + \mu(||\cdot||^{2q})^{\frac{1}{2}} + \mu'(||\cdot||^{2q})^{\frac{1}{2}}\right].\end{aligned} \tag{D.7}$$

We then have, via the triangle inequality, the bound (D.7), and another application of both Condition C.1(ii) and the Hölder inequality, that

$$\begin{aligned}
&\left|\left|\int_{\mathbb{R}^d} \nabla_\theta^k \phi(\theta, x, y)\mu(\mathrm{d}y) - \int_{\mathbb{R}^d} \nabla_\theta^k \phi(\theta, x', y')\mu'(\mathrm{d}y')\right|\right| \\
&\leq \left|\left|\int_{\mathbb{R}^d} \nabla_\theta^k \phi(\theta, x, y)\mu(\mathrm{d}y) - \int_{\mathbb{R}^d} \nabla_\theta^k \phi(\theta, x, y')\mu'(\mathrm{d}y')\right|\right| + \int_{\mathbb{R}^d} \left|\left|\nabla_\theta^k \phi(\theta, x, y') - \nabla_\theta^k \phi(\theta, x', y')\right|\right| \mu'(\mathrm{d}y') \\
&\leq K\mathbb{W}_2(\mu, \mu') \left[1 + \mu(||\cdot||^{2q})^{\frac{1}{2}} + \mu'(||\cdot||^{2q})^{\frac{1}{2}}\right] + K||x - x'|| \left[1 + \mu'(||\cdot||^{2q})^{\frac{1}{2}}\right] \\
&\leq K\left[||x - x'|| + \mathbb{W}_2(\mu, \mu')\right] \left[1 + \mu(||\cdot||^{2q})^{\frac{1}{2}} + \mu'(||\cdot||^{2q})^{\frac{1}{2}}\right].
\end{aligned}$$

Thus, recalling the definition of $B(\theta, x, \mu)$, c.f. (1.3), and once more making use Condition C.1(ii), we obtain

$$\begin{aligned}
||\nabla_\theta^k B(\theta, x, \mu) - \nabla_\theta^k B(\theta, x', \mu')|| &\leq \left|\left|\nabla_\theta^k b(\theta, x) - \nabla_\theta^k b(\theta, x')\right|\right| \\
&\quad + \left|\left|\int_{\mathbb{R}^d} \nabla_\theta^k \phi(\theta, x, y)\mu(\mathrm{d}y) - \int_{\mathbb{R}^d} \nabla_\theta^k \phi(\theta, x', y')\mu'(\mathrm{d}y')\right|\right| \\
&\leq K\left[||x - x'|| + \mathbb{W}_2(\mu, \mu')\right] \\
&\qquad \cdot \left[1 + ||x||^q + ||x'||^q + \mu(||\cdot||^{2q})^{\frac{1}{2}} + \mu'(||\cdot||^{2q})^{\frac{1}{2}}\right].
\end{aligned}$$

From the definition of $G(\theta, x, \mu)$, c.f. (2.1), we trivially then have

$$\begin{aligned}
||\nabla_\theta^k G(\theta, x, \mu) - \nabla_\theta^k G(\theta, x', \mu')|| &\leq K\left[||x - x'|| + \mathbb{W}_2(\mu, \mu')\right] \\
&\qquad \cdot \left[1 + ||x||^q + ||x'||^q + \mu(||\cdot||^{2q})^{\frac{1}{2}} + \mu'(||\cdot||^{2q})^{\frac{1}{2}}\right].
\end{aligned} \tag{D.8}$$

Finally, recalling the definition of $L(\theta, x, \mu)$, c.f. (2.2), and combining (D.6) and (D.8), we obtain

$$\begin{aligned}
||L(\theta, x, \mu) - L(\theta, x', \mu')|| &= \frac{1}{2}\left|\left|G^T(\theta, x, \mu)G(\theta, x, \mu) - G^T(\theta, x', \mu')G(\theta, x', \mu')\right|\right| \\
&\leq \frac{1}{2}||G(\theta, x, \mu) - G(\theta, x', \mu')||\,||G(\theta, x, \mu) + G(\theta, x', \mu')|| \\
&\leq K\left[||x - x'|| + \mathbb{W}_2(\mu, \mu')\right] \\
&\qquad \cdot \left[1 + ||x||^q + ||x'||^q + \mu(||\cdot||^{2q})^{\frac{1}{2}} + \mu'(||\cdot||^{2q})^{\frac{1}{2}}\right]^2 \\
&\leq K\left[||x - x'|| + \mathbb{W}_2(\mu, \mu')\right] \\
&\qquad \cdot \left[1 + ||x||^q + ||x'||^q + \mu(||\cdot||^q) + \mu'(||\cdot||^q)\right].
\end{aligned}$$

where in the final line we have replaced the unimportant constant $q \to 2q$, after applying the Hölder inequality. The bounds for $\nabla_\theta^k L(\theta, x, \mu)$, $k = 0, 1, 2$, follow analogously. ■

Remark. In the case that substituting $x = x^{i,N}$, $x' = y^{i,N}$, $\mu = \mu_x^N$ and $\mu' = \mu_y^N$, and recalling that $\hat{L}^{i,N}(\theta, \hat{x}^N) := L(\theta, x^{i,N}, \hat{\mu}^N)$, that for all $i = 1, \ldots, N$, $N \in \mathbb{N}$, it holds that

$$\left|\nabla_\theta^k L(\theta, \hat{x}^N) - \nabla_\theta^k L(\theta, \hat{y}^N)\right| \leq K\left[||y^{i,N} - z^{i,N}|| + \left(\frac{1}{N}\sum_{j=1}^N ||y^{j,N} - z^{j,N}||^2\right)^{\frac{1}{2}}\right] \cdot \left[1 + ||x^{i,N}||^q + ||y^{i,N}||^q + \frac{1}{N}\sum_{j=1}^N ||x^{j,N}||^q + \frac{1}{N}\sum_{j=1}^N ||y^{j,N}||^q\right]$$

Lemma D.6. *Assume that Conditions B.1 - B.2, C.1 and D.1 hold. Then, for $k = 0, 1, 2$, there exist $K, K' > 0$ such that, for all $\theta \in \mathbb{R}^p$, $||\nabla_\theta^k \tilde{\mathcal{L}}(\theta)|| \leq K$ and $||\nabla_\theta^k \tilde{\mathcal{L}}^{i,N}(\theta)|| \leq K'$.*

Proof. Using the definition of $\nabla_\theta^k \tilde{\mathcal{L}}(\theta)$ (Lemma 3.4.A), the polynomial growth property of $\nabla_\theta^k L(\theta, x, \mu)$ (Lemma D.4), and the finite moments of the invariant measure of the McKean-Vlasov SDE (Lemma D.1), we have that

$$\begin{aligned}
||\nabla_\theta^k \tilde{\mathcal{L}}(\theta)|| &\leq \int_{\mathbb{R}^d} ||\nabla_\theta^k L(\theta, x, \mu_\infty)|| \mu_\infty(\mathrm{d}x) \\
&\leq K \int_{\mathbb{R}^d} \left[1 + ||x||^q + [\int_{\mathbb{R}^d} ||y||^q \mu_\infty(\mathrm{d}y)\right] \mu_\infty(\mathrm{d}y) \\
&\leq K \int_{\mathbb{R}^d} (1 + ||x||^q) \mu_\infty(\mathrm{d}x) \leq K.
\end{aligned}$$

The bound for $\nabla_\theta^k \tilde{\mathcal{L}}^{i,N}(\theta)$ follows identically, this time using the defintion of $\nabla_\theta^k \tilde{\mathcal{L}}^{i,N}(\theta)$ (Lemma 3.4.B), and the finite moments of the invariant measure of the IPS (Lemma D.1). ■

D.2. Additional Lemmas for Lemma 3.4.C.

Lemma D.7. *Assume that Conditions B.1 - B.2 and D.1 hold. For all Lipschitz functions φ, there exists $K > 0$ such that, for all $t \geq 0$, for all $N \in \mathbb{N}$,*

$$\mathbb{E}\left[\left|\left|\int_{\mathbb{R}^d} \varphi(y)\mu_t(\mathrm{d}y) - \frac{1}{N}\sum_{i=1}^N \varphi(x_t^{i,N})\right|\right|^2\right] \leq \frac{K}{N}$$

Proof. Let x_t^i, $i = 1, \ldots, N$ denote independent solutions of the McKean-Vlasov SDE (1.1) - (1.2). We then have, using the elementary inequality $||a + b||^2 \leq 2||a||^2 + 2||b||^2$, that

$$\begin{aligned}
\mathbb{E}\left[||\int_{\mathbb{R}^d} \varphi(y)\mu_t(\mathrm{d}y) - \frac{1}{N}\sum_{i=1}^N \varphi(x_t^{i,N})||^2\right] \leq & 2\mathbb{E}\left[||\int_{\mathbb{R}^d} \varphi(y)\mu_t(\mathrm{d}y) - \frac{1}{N}\sum_{i=1}^N \varphi(x_t^i)||^2\right] \\
&+ 2\mathbb{E}\left[||\frac{1}{N}\sum_{i=1}^N \left(\varphi(x_t^i) - \varphi(x_t^{i,N})\right)||^2\right]
\end{aligned}$$

For the first term, we observe, using the independence of the variables x_t^i, $i = 1, \ldots, N$, that

$$\mathbb{E}\left[||\int_{\mathbb{R}^d} \varphi(y)\mu_t(\mathrm{d}y) - \frac{1}{N}\sum_{i=1}^N \varphi(x_t^i)||^2\right] \leq \frac{1}{N}\mathbb{E}\left[||\varphi(x_t^1) - \mathbb{E}[\varphi(x_t^1)]||^2\right]$$

It is straightforward to show that $\mathbb{E}[(\varphi(x_t^1) - \mathbb{E}[\varphi(x_t^1)])^2] \leq \mathbb{E}[(\varphi(x_t^1) - \varphi(\mathbb{E}[x_t^1]))^2]$. It follows, using also the fact that φ is Lipschitz, and Proposition A.2 (the bounded moments of the McKean-Vlasov SDE), that

$$\mathbb{E}\Big[||\int_{\mathbb{R}^d} \varphi(y)\mu_t(\mathrm{d}y) - \frac{1}{N}\sum_{i=1}^N \varphi(x_t^i)||^2\Big] \leq \frac{1}{N}\mathbb{E}\left[||\varphi(x_t^1) - \varphi(\mathbb{E}[x_t^1])||^2\right] \leq \frac{K}{N}.$$

where, as previously, the value of the constant K is allowed to vary from line to line. For the second term, using the Cauchy-Schwarz inequality, the fact that φ is Lipschitz, and Proposition A.5 (uniform-in-time propagation of chaos), we obtain

$$\mathbb{E}[||\frac{1}{N}\sum_{i=1}^N \left(\varphi(x_t^i) - \varphi(x_t^{i,N})\right)||^2] \leq \frac{K}{N}\sum_{i=1}^N \mathbb{E}[||x_t^i - x_t^{i,N}||^2] \leq \frac{K}{N}.$$

The result follows immediately. ■

Lemma D.8. *Assume that Conditions B.1 - B.2 and D.1 hold. Suppose also that $\mu_0 \in \mathcal{P}_2(\mathbb{R}^d)$. Let x_t^i denote a solution of the McKean-Vlasov SDE, driven by $w^i = (w_t^i)_{t\geq 0}$. Then, for all Lipschitz functions φ, there exists $K > 0$ such that, for all $t \geq 0$, for all $N \in \mathbb{N}$,*

$$\mathbb{E}\left[\left|\left|\int_{\mathbb{R}^d} \varphi(x_t^i, y)\mu_t(\mathrm{d}y) - \frac{1}{N}\sum_{i=1}^N \varphi(x_t^{i,N}, x_t^{j,N})\right|\right|^2\right] \leq \frac{K}{N}$$

Proof. The is an immediate corollary of Lemma D.7. Indeed, using the Hölder inequality, and that φ is Lipschitz, we have

$$\begin{aligned}
&||\int_{\mathbb{R}^d} \varphi(x_t^i, y)\mu_t(\mathrm{d}y) - \frac{1}{N}\sum_{j=1}^N \varphi(x_t^{i,N}, x_t^{j,N})||^2 \\
&\leq 2||\varphi(x_t^i, y)\mu_t(\mathrm{d}y) - \frac{1}{N}\sum_{j=1}^N \varphi(x_t^i, x_t^{j,N})||^2 + 2||\frac{1}{N}\sum_{j=1}^N [\varphi(x_t^i, x_t^{j,N}) - \varphi(x_t^{i,N}, x_t^{j,N})]||^2 \\
&\leq 2||\varphi(x_t^i, y)\mu_t(\mathrm{d}y) - \frac{1}{N}\sum_{j=1}^N \varphi(x_t^i, x_t^{j,N})||^2 + \frac{2K}{N}\sum_{j=1}^N ||x_t^i - x_t^{i,N}||^2
\end{aligned}$$

It follows immediately, as required, that

$$\begin{aligned}
&\mathbb{E}\left[\left|\left|\int_{\mathbb{R}^d} \varphi(x_t^i, y)\mu_t(\mathrm{d}y) - \frac{1}{N}\sum_{i=1}^N \varphi(x_t^{i,N}, x_t^{j,N})\right|\right|^2\right] \\
&\leq K\,\underbrace{\mathbb{E}\left[||\varphi(x_t^i, y)\mu_t(\mathrm{d}y) - \frac{1}{N}\sum_{j=1}^N \varphi(x_t^i, x_t^{j,N})||^2\right]}_{\leq \frac{K}{N} \text{ by Lemma D.7}} + \underbrace{\mathbb{E}\left[\frac{1}{N}\sum_{j=1}^N ||x_t^i - x_t^{i,N}||^2\right]}_{\leq \frac{K}{N} \text{ by Proposition A.5}} \leq \frac{K}{N}.
\end{aligned}$$

■

D.3. Additional Lemmas for Lemma 3.4.D.

D.3.1. Main Lemmas. The lemmas in this section are variations of Lemmas 3.1 - 3.5 in [75]. For convenience, and since we will later also need to prove modified versions of these lemmas (see Appendix G), we provide the proofs of these results in full, appropriately adapted to the current setting.

Lemma D.9. *Assume that Conditions B.1 - B.2, C.1, D.1, and F.1 hold. Define, with* $\hat{x}^N = (x^{1,N}, \ldots, x^{N,N})$*, the function*

$$\Gamma_{k,\eta} = \int_{\tau_k}^{\sigma_{k,\eta}} \gamma_s \left(\nabla_\theta \hat{L}^{i,N}(\theta_s^{i,N}, \hat{x}_s^N) - \nabla_\theta \tilde{\mathcal{L}}^{i,N}(\theta_s^{i,N}) \right) \mathrm{d}s.$$

Then, almost surely, $||\Gamma_{k,\eta}|| \to 0$ *as* $k \to \infty$.

Proof. Let $\hat{x}^N = (x^{1,N}, \ldots, x^{N,N}) \in (\mathbb{R}^d)^N$. Consider the function

$$S^{i,N}(\theta, \hat{x}^N) = \nabla_\theta \hat{L}^{i,N}(\theta, \hat{x}^N) - \nabla_\theta \tilde{\mathcal{L}}^{i,N}(\theta).$$

We begin by noting that this function is 'centred' with respect to the invariant measure $\hat{\mu}_\infty(\cdot)$. using the definition of $\nabla_\theta \tilde{\mathcal{L}}^{i,N}(\cdot)$ from Lemma 3.4.B. In addition, observe that, by Lemma D.15 (see Appendix D.3.3), the function $S^{i,N}(\theta, \hat{x}) \in \mathcal{C}^{2,\alpha}(\mathbb{R}^p, (\mathbb{R}^d)^N)$, and there exist positive constants $q, K > 0$ such that, for $j = 0, 1, 2$,

$$|\partial_\theta^j S^{i,N}(\theta, \hat{x}^N)| \leq K(1 + ||x_i||^q + \frac{1}{N}\sum_{j=1}^N ||x_j||^q),$$

Thus, the function $S^{i,N} : \mathbb{R}^p \times (\mathbb{R}^d)^N \to \mathbb{R}^p$ satisfies the conditions of Lemma D.14. It follows that, for all $i = 1, \ldots, N$, the Poisson equation

$$\mathcal{A}_{\hat{x}} v^{i,N}(\theta, \hat{x}^N) = S^{i,N}(\theta, \hat{x}^N) \quad , \quad \int_{(\mathbb{R}^d)^N} v^{i,N}(\theta, \hat{x}^N) \hat{\mu}_\infty^N(\mathrm{d}\hat{x}^N) = 0$$

has a unique twice differentiable solution which satisfies

$$\sum_{j=0}^2 \left| \frac{\partial^j v^{i,N}}{\partial \theta^i}(\theta, \hat{x}^N) \right| + \left| \frac{\partial^2 v^{i,N}}{\partial \theta \partial x}(\theta, \hat{x}^N) \right| \leq K\left(1 + ||x^{i,N}||^q + \frac{1}{N}\sum_{j=1}^N ||x^{j,N}||^q\right). \tag{D.9}$$

Let $u^{i,N}(t, \theta, \hat{x}^N) = \gamma_t v^{i,N}(\theta, \hat{x}^N)$. Applying Ito's formula to each component of this vectorvalued function, we obtain, for $l = 1, \ldots, p$,

(D.10)

$$
\begin{aligned}
u_l^{i,N}(t_2, \theta_{t_2}^{i,N}, \hat{x}_{t_2}^N) - u_l^{i,N}(t_1, \theta_{t_1}^{i,N}, \hat{x}_{t_1}^N) = &\int_{t_1}^{t_2} \partial_s u_l^{i,N}(s, \theta_s^{i,N}, \hat{x}_s^N)\mathrm{d}s \\
&+ \int_{t_1}^{t_2} \mathcal{A}_{\hat{x}} u_l^{i,N}(s, \theta_s^{i,N}, \hat{x}_s^N)\mathrm{d}s + \int_{t_1}^{t_2} \mathcal{A}_\theta u_l^{i,N}(s, \theta_s^{i,N}, \hat{x}_s^N)\mathrm{d}s \\
&+ \int_{t_1}^{t_2} \gamma_s \mathrm{Tr}\left[\nabla_\theta \hat{B}^{i,N}(\theta_s^{i,N}, \hat{x}_s^N)\partial_\theta\partial_{\hat{x}} u_l^{i,N}(s, \theta_s^{i,N}, \hat{x}_s^N)\right]\mathrm{d}s \\
&+ \int_{t_1}^{t_2} \partial_{\hat{x}} u_l^{i,N}(s, \theta_s^{i,N}, \hat{x}_s^N) \cdot \mathrm{d}\hat{w}_s^N \\
&+ \int_{t_1}^{t_2} \gamma_s \partial_\theta u_l^{i,N}(s, \theta_s^{i,N}, \hat{x}_s^N) \cdot \nabla_\theta \hat{B}^{i,N}(\theta_s^{i,N}, \hat{x}_s^N)\mathrm{d}w_s^i
\end{aligned}
$$

where $\mathcal{A}_{\hat{x}}$ and $\mathcal{A}_\theta$ are the infinitesimal generators of $\hat{x}^N$ and θ, respectively, and we recall from (4.19) that $\hat{w}_t^N = (w_t^1, \ldots, w_t^N)^T$. Rearranging this identity, and also recalling that $v^{i,N}(\theta, \hat{x}^N)$ is the solution of the Poisson equation, we obtain

$$
\begin{aligned}
\Gamma_{k,\eta} &= \int_{\tau_k}^{\sigma_{k,\eta}} \gamma_s \mathcal{A}_{\hat{x}} v^{i,N}(\theta_s^{i,N}, \hat{x}_s^N)\mathrm{d}s \\
&= \gamma_{\sigma_{k,\eta}} v^{i,N}(\theta_{\sigma_{k,\eta}}^{i,N}, \hat{x}_{\sigma_{k,\eta}}^N) - \gamma_{\tau_k} v^{i,N}(\theta_{\tau_k}^{i,N}, \hat{x}_{\tau_k}^N) - \int_{\tau_k}^{\sigma_{k,\eta}} \dot{\gamma}_s v^{i,N}(\theta_s^{i,N}, \hat{x}_s^N)\mathrm{d}s \\
&- \int_{\tau_k}^{\sigma_{k,\eta}} \gamma_s \mathcal{A}_\theta v^{i,N}(\theta_s^{i,N}, \hat{x}_s^N)\mathrm{d}s - \int_{\tau_k}^{\sigma_{k,\eta}} \gamma_s^2 \mathrm{Tr}\left[\nabla_\theta \hat{B}^{i,N}(\theta_s^{i,N}, \hat{s}_x^N)\partial_\theta\partial_{\hat{x}} v^{i,N}(\theta_s^{i,N}, \hat{x}_s^N)\right]\mathrm{d}s \\
&- \int_{\tau_k}^{\sigma_{k,\eta}} \gamma_s \partial_{\hat{x}} v^{i,N}(\theta_s^{i,N}, \hat{x}_s^N) \cdot \mathrm{d}\hat{w}_s^N - \int_{\tau_k}^{\sigma_{k,\eta}} \gamma_s^2 \partial_\theta v^{i,N}(\theta_s^{i,N}, \hat{x}_s^N) \cdot \nabla_\theta \hat{B}^{i,N}(\theta_s^{i,N}, \hat{x}_s^N)\mathrm{d}w_s^i
\end{aligned}
$$

We now prove the convergence of each term on the right hand side of this equation. As previously, we allow the value of K to change from line to line. First define

$$
J_t^{(1)} = \gamma_t ||v^{i,N}(\theta_t^{i,N}, \hat{x}_t^N)||
$$

We have, make use of the polynomial growth of $v^{i,N}(\theta, \hat{x}^N)$, and Proposition A.2 (the bounded moments of the IPS), that

$$
\mathbb{E}[|J_t^{(1)}|^2] \leq K\gamma_t^2 \left(1 + \mathbb{E}[||x_t^{i,N}||^q] + \frac{1}{N}\sum_{j=1}^N \mathbb{E}[||x_t^{j,N}||^q]\right) \leq K\gamma_t^2.
$$

Applying the Borel-Cantelli argument as in [77, Appendix B], it follows that $J_t^{(1)}$ converges to zero with probability one. We next consider the term

$$
\begin{aligned}
J_{0,t}^{(2)} = &\int_0^t \partial_s \dot{\gamma}_s v^{i,N}(\theta_s^{i,N}, \hat{x}_s^N)\mathrm{d}s + \int_0^t \gamma_s \mathcal{A}_\theta v^{i,N}(\theta_s^{i,N}, \hat{x}_s^N)\mathrm{d}s \\
&+ \int_0^t \gamma_s^2 \mathrm{Tr}\left[\nabla_\theta \hat{B}^{i,N}(\theta_s^{i,N}, \hat{x}_s^N)\partial_\theta\partial_x v^{i,N}(\theta_s^{i,N}, \hat{x}_s^N)\right]\mathrm{d}s
\end{aligned}
$$

This term obeys the bound

$$\sup_{t>0} \mathbb{E}|J_{0,t}^{(2)}| \leq K \int_0^\infty (|\dot{\gamma}_s| + \gamma_s^2)(1 + \mathbb{E}[||x_s^{i,N}||^q] + \frac{1}{N}\sum_{j=1}^N \mathbb{E}[||x_s^{j,N}||^q])\mathrm{d}s$$
$$\leq K \int_0^\infty (|\dot{\gamma}_s| + \gamma_s^2)\mathrm{d}s < \infty.$$

Here, the first inequality follows from the growth properties of the $v^{i,N}(\theta, \hat{x}^N)$ in (D.9), the second inequality from Proposition A.2 (the bounded moments of the IPS), and the final inequality from Condition F.1 (the properties of the learning rate). It follows that there exists a finite random variable $J_{0,\infty}^{(2)}$ such that, with probability one,

$$J_{0,t}^{(2)} \to J_{0,\infty}^{(2)}, \quad \text{as } t \to \infty. \tag{D.11}$$

The last term to consider is the stochastic integral

$$J_{0,t}^{(3)} = \int_0^t \gamma_s \partial_{\hat{x}} v^{i,N}(\theta_s^{i,N}, \hat{x}_s^N) \cdot \mathrm{d}\hat{w}_s^N + \int_{\tau_k}^{\sigma_{k,\eta}} \gamma_s^2 \partial_\theta v^{i,N}(\theta_s^{i,N}, \hat{x}_s^N) \cdot \nabla_\theta \hat{B}^{i,N}(\theta_s^{i,N}, \hat{x}_s^N)\mathrm{d}w_s^i$$

In this case, using the BDG inequality, and the same bounds as above, we have

$$\mathbb{E}\left[|J_{0,t}^{(3)}|^2\right] \leq K \int_0^\infty (\gamma_s^2 + \gamma_s^4)\left[1 + \mathbb{E}[||x_s^{i,N}||^q] + \frac{1}{N}\sum_{j=1}^N \mathbb{E}[||x_s^{j,N}||^q]\right]\mathrm{d}s \leq K \int_0^\infty \gamma_s^2 \mathrm{d}s < \infty.$$

Thus, by Doob's martingale convergence theorem, there exists a square integrable random variable $J_{0,\infty}^{(3)}$ such that, both almost surely and in $\mathbb{L}^2$,

$$J_{0,t}^{(3)} \to J_{0,\infty}^{(3)}, \quad \text{as } t \to \infty. \tag{D.12}$$

It remains only to observe, combining (D.11) and (D.12), we have

$$||\Gamma_{k,\eta}|| \leq J_{\sigma_{k,\eta}}^{(1)} + J_{\tau_k}^{(1)} + J_{\tau_k,\sigma_{k,\eta}}^{(2)} + J_{\tau_k,\sigma_{k,\eta}}^{(3)} \overset{k\to\infty}{\to} 0.$$

■

Lemma D.10. *Assume that Conditions B.1 - B.2, C.1, D.1 and F.1 hold. Let $\rho > 0$ be such that, for a given $\kappa > 0$, it is true that $3\rho + \frac{\rho}{4\kappa} = \frac{1}{2L}$, where L denotes the Lipschitz constant of $\nabla_\theta \tilde{\mathcal{L}}^{i,N}(\theta)$. For k large enough, and for $\eta > 0$ small enough (potentially random, and depending on k), one has*

$$\int_{\tau_k}^{\sigma_{k,\eta}} \gamma_s \mathrm{d}s > \rho \quad \textit{and, a.s.,} \quad \frac{\rho}{2} \leq \int_{\tau_k}^{\sigma_k} \gamma_s \mathrm{d}s \leq \rho.$$

Proof. We proceed by contradiction. Let us assume that $\int_{\tau_k}^{\sigma_{k,\eta}} \gamma_s \mathrm{d}s \leq \rho$. Choose arbitrary $\varepsilon > 0$ such that $\varepsilon \leq \frac{\rho}{8}$. We begin with the observation that, via the Itô isometry, we have that

$$\sup_{t\geq 0} \mathbb{E}\Big|\Big| \int_0^t \gamma_s \frac{\kappa}{||\nabla \tilde{\mathcal{L}}^{i,N}(\theta_{\tau_k}^{i,N})||} \nabla_\theta \hat{B}^{i,N}(\theta_s^{i,N}, \hat{x}_s^N) \mathrm{d}w_s^i \Big|\Big|^2 \leq \int_0^t K\gamma_s^2 \left(1 + \mathbb{E}\left[||\hat{x}_s^N||^q\right]\right) \mathrm{d}s < \infty$$

where, we have used the polynomial growth of $\nabla_\theta \hat{B}^{i,N}(\theta, \hat{x})$ (see the proof of Lemma D.15), Proposition A.2 (the bounded moments of the IPS), and Condition F.1 (the properties of the learning rate). Thus, by the Doob's martingale convergence theorem, there exists a finite random variable M such that, both almost surely and in $\mathbb{L}^2$,

$$\int_0^t \gamma_s \frac{\kappa}{||\nabla \tilde{\mathcal{L}}^{i,N}(\theta_{\tau_k}^{i,N})||} \nabla_\theta \hat{B}^{i,N}(\theta_s^{i,N}, \hat{x}_s^N) \mathrm{d}w_s^i \to M$$

It follows that, for the chosen $\varepsilon > 0$, there exists k such that

$$\int_{\tau_k}^{\sigma_{k,\eta}} \gamma_s \frac{\kappa}{||\nabla \tilde{\mathcal{L}}^{i,N}(\theta_{\tau_k}^{i,N})||} \nabla_\theta \hat{B}^{i,N}(\theta_s^{i,N}, \hat{x}_s^N) \mathrm{d}w_s^i < \varepsilon. \tag{D.13}$$

Let us now also assume that, for the given k, η is small enough such that for all $s \in [\tau_k, \sigma_{k,\eta}]$, we have $||\nabla_\theta \tilde{\mathcal{L}}^{i,N}(\theta_s^{i,N})|| \leq 3||\nabla_\theta \tilde{\mathcal{L}}^{i,N}(\theta_{\tau_k}^{i,N})||$. We can then compute

(D.14)
$$\begin{aligned}
||\theta_{\sigma_{k,\eta}}^{i,N} - \theta_{\tau_k}^{i,N}|| &= \Big|\Big| \int_{\tau_k}^{\sigma_{k,\eta}} \gamma_s \nabla_\theta \hat{L}^{i,N}(\theta_s^{i,N}, \hat{x}_s^N) \mathrm{d}s + \int_{\tau_k}^{\sigma_{k,\eta}} \gamma_s \langle \nabla_\theta \hat{B}^{i,N}(\theta_s^{i,N}, \hat{x}_s^N), \mathrm{d}w_s^i \rangle \Big|\Big| \\
&\leq 3||\nabla_\theta \tilde{\mathcal{L}}^{i,N}(\theta_{\tau_k}^{i,N})|| \int_{\tau_k}^{\sigma_{k,\eta}} \gamma_s \mathrm{d}s + \Big|\Big| \int_{\tau_k}^{\sigma_{k,\eta}} \gamma_s [\nabla_\theta \hat{L}^{i,N}(\theta_s^{i,N}, \hat{x}_s^N) - \nabla_\theta \tilde{\mathcal{L}}^{i,N}(\theta_s^{i,N})] \mathrm{d}s \Big|\Big| \\
&\quad + \frac{||\nabla \tilde{\mathcal{L}}^{i,N}(\theta_{\tau_k}^{i,N})||}{\kappa} \Big|\Big| \int_{\tau_k}^{\sigma_{k,\eta}} \gamma_s \frac{\kappa}{||\nabla \tilde{\mathcal{L}}^{i,N}(\theta_{\tau_k}^{i,N})||} \langle \nabla_\theta B(\theta_s^{i,N}, \hat{x}_s^N), \mathrm{d}w_s^i \rangle \Big|\Big| \\
&\leq 3||\nabla_\theta \tilde{\mathcal{L}}^{i,N}(\theta_{\tau_k}^{i,N})||\rho + \varepsilon + \frac{||\nabla \tilde{\mathcal{L}}^{i,N}(\theta_{\tau_k}^{i,N})||}{\kappa} \varepsilon \\
&\leq ||\nabla_\theta \tilde{\mathcal{L}}^{i,N}(\theta_{\tau_k}^{i,N})|| \left[3\rho + \frac{\rho}{4\kappa}\right]
\end{aligned}$$

where in the penultimate line we have used Lemma D.9 and (D.13), and in the final line we have used the fact that our choice of ε satisfies $\varepsilon \leq \frac{\rho}{8}$. We thus obtain

$$||\theta_{\sigma_{k,\eta}}^{i,N} - \theta_{\tau_k}^{i,N}|| \leq ||\nabla_\theta \tilde{\mathcal{L}}^{i,N}(\theta_{\tau_k}^{i,N})|| \left[3\rho + \frac{\rho}{4\kappa}\right] \leq ||\nabla_\theta \tilde{\mathcal{L}}^{i,N}(\theta_{\tau_k}^{i,N})|| \frac{1}{2L}.$$

Thus, using also the definition of the Lipschitz constant L, we obtain

$$||\nabla_\theta \tilde{\mathcal{L}}^{i,N}(\theta_{\sigma_{k,\eta}}^{i,N}) - \nabla_\theta \tilde{\mathcal{L}}^{i,N}(\theta_{\tau_k}^{i,N})|| \leq L||\theta_{\sigma_{k,\eta}}^{i,N} - \theta_{\tau_k}^{i,N}|| \leq \frac{1}{2}||\nabla_\theta \tilde{\mathcal{L}}^{i,N}(\theta_{\tau_k}^{i,N})||$$

which then yields

$$\frac{1}{2}||\nabla_\theta\tilde{\mathcal{L}}^{i,N}(\theta^{i,N}_{\tau_k})|| \leq ||\nabla_\theta\tilde{\mathcal{L}}^{i,N}(\theta^{i,N}_{\sigma_{k,\eta}})|| \leq 2||\nabla_\theta\tilde{\mathcal{L}}^{i,N}(\theta^{i,N}_{\tau_k})||. \tag{D.15}$$

But this implies that $\sigma_{k,\eta} \in [\tau_k, \sigma_k]$, which is a contradiction. Thus we do indeed have $\int_{\tau_k}^{\sigma_{k,\eta}} \gamma_s \mathrm{d}s > \rho$. We now turn our attention to the second part of the Lemma. By definition, we have that $\int_{\tau_k}^{\sigma_k} \gamma_s \mathrm{d}s \leq \rho$. Thus, it remains only to show that $\frac{\rho}{2} \leq \int_{\tau_k}^{\sigma_k} \gamma_s \mathrm{d}s$. From the first part of the Lemma, we have that $\int_{\tau_k}^{\sigma_{k,\eta}} \gamma_s \mathrm{d}s > \rho$. Moreover, for k sufficiently large and η sufficiently small, we must have $\int_{\sigma_k}^{\sigma_{k,\eta}} \gamma_s \mathrm{d}s \leq \frac{\rho}{2}$. We thus obtain

$$\int_{\tau_k}^{\sigma_k} \gamma_s \mathrm{d}s \geq \rho - \int_{\sigma_k}^{\sigma_{k,\eta}} \gamma_s \mathrm{d}s \geq \rho - \frac{\rho}{2} = \frac{\rho}{2}.$$

■

Lemma D.11. *Assume that Conditions B.1 - B.2, C.1, D.1 and F.1 hold. Suppose that there are an infinite number of intervals $[\tau_k, \sigma_k)$. Then there exists a fixed constant $\beta = \beta(\kappa) > 0$ such that, for k large enough, almost surely,*

$$\tilde{\mathcal{L}}^{i,N}(\theta^{i,N}_{\sigma_k}) - \tilde{\mathcal{L}}^{i,N}(\theta^{i,N}_{\tau_k}) \geq \beta.$$

Proof. By Itô's formula, we have that

$$\begin{aligned}\tilde{\mathcal{L}}^{i,N}(\theta^{i,N}_{\sigma_k}) &- \tilde{\mathcal{L}}^{i,N}(\theta^{i,N}_{\tau_k}) \\ &= \underbrace{\int_{\tau_k}^{\sigma_k} \gamma_s ||\nabla_\theta\tilde{\mathcal{L}}^{i,N}(\theta^{i,N}_s)||^2 \mathrm{d}s}_{A^{i,N}_{1,k}} \\ &+ \underbrace{\int_{\tau_k}^{\sigma_k} \gamma_s \langle\nabla_\theta\tilde{\mathcal{L}}^{i,N}(\theta^{i,N}_s), \nabla_\theta\hat{L}^{i,N}(\theta^{i,N}_s, \hat{x}^N_s) - \nabla_\theta\tilde{\mathcal{L}}^{i,N}(\theta^{i,N}_s)\rangle \mathrm{d}s}_{A^{i,N}_{2,k}} \\ &+ \underbrace{\int_{\tau_k}^{\sigma_k} \gamma_s \langle\nabla_\theta\tilde{\mathcal{L}}^{i,N}(\theta^{i,N}_s), \nabla_\theta\hat{B}^{i,N}(\theta^{i,N}_s, \hat{x}^N_s)\mathrm{d}w^i_s\rangle}_{A^{i,N}_{3,k}} \\ &+ \underbrace{\int_{\tau_k}^{\sigma_k} \frac{1}{2}\gamma_s^2 \mathrm{Tr}\left[\nabla_\theta\hat{B}^{i,N}(\theta^{i,N}_s, \hat{x}^N_s)\nabla_\theta\hat{B}^{i,N}(\theta^{i,N}_s, \hat{x}^N_s)^T\nabla^2_\theta\tilde{\mathcal{L}}^{i,N}(\theta^{i,N}_s)\mathrm{d}s\right]}_{A^{i,N}_{4,k}}\end{aligned} \tag{D.16}$$

We will deal with each of these terms individually. First consider $A^{i,N}_{1,k}$. For this term, we have that

$$A^{i,N}_{1,k} = \int_{\tau_k}^{\sigma_k} \gamma_s ||\nabla_\theta\tilde{\mathcal{L}}^{i,N}(\theta^{i,N}_s)||^2 \mathrm{d}s \geq \frac{||\nabla_\theta\tilde{\mathcal{L}}^{i,N}(\theta^{i,N}_{\tau_k})||^2}{4} \int_{\tau_k}^{\sigma_k} \gamma_s \mathrm{d}s \geq \frac{||\nabla_\theta\tilde{\mathcal{L}}^{i,N}(\theta^{i,N}_{\tau_k})||^2}{8}\rho$$

where, in the first inequality, we have used the definition of the $\{\tau_k\}_{k\geq 0}$, namely, that $||\nabla_\theta\tilde{\mathcal{L}}^{i,N}(\theta^{i,N}_s)|| \geq \frac{1}{2}||\nabla_\theta\tilde{\mathcal{L}}^{i,N}(\theta^{i,N}_{\tau_k})||$ for all $s \in [\tau_k, \sigma_k]$, and in the second inequality we have used Lemma D.10.

We now turn our attention to $A_{2,k}^{i,N}$. We will handle this term using a very similar to approach to that used in the proof of Lemma D.9. Let us consider the function

$$T^{i,N}(\theta,\hat{x}^N)=\langle\nabla_\theta\tilde{\mathcal{L}}^{i,N}(\theta),\nabla_\theta\hat{L}^{i,N}(\theta,\hat{x}^N)-\nabla_\theta\tilde{\mathcal{L}}^{i,N}(\theta)\rangle.$$

By Lemma D.16, we have that $T^{i,N}(\theta,\hat{x}^N)\in\mathcal{C}^{2,\alpha}(\mathbb{R}^p,\mathbb{R}^d)$, and that $||\partial_\theta^j T^{i,N}(\theta,\hat{x}^N)|\leq K(1+||x^{i,N}||^q+\frac{1}{N}\sum_{j=1}^N||x_j||^q)$, for $j=0,1,2$. Moreover, it is straightforward to show that this function satisfies $\int_{(\mathbb{R}^d)^N}T^{i,N}(\theta,\hat{x}^N)\hat{\mu}_\infty(\mathrm{d}\hat{x}^N)=0$. Thus, Lemma D.14, the Poisson equation

$$\mathcal{A}_{\hat{x}}v^{i,N}(\theta,\hat{x}^N)=T^{i,N}(\theta,\hat{x}^N)\quad,\quad\int_{(\mathbb{R}^d)^N}v^{i,N}(\theta,\hat{x}^N)\mu_\infty(\mathrm{d}\hat{x}^N)=0$$

has a unique twice differentiable solution which satisfies

$$\sum_{j=0}^{2}\left|\frac{\partial^j v^{i,N}}{\partial\theta^i}(\theta,\hat{x}^N)\right|+\left|\frac{\partial^2 v^{i,N}}{\partial\theta\partial x}(\theta,\hat{x}^N)\right|\leq K\left[1+||x^{i,N}||^q+\frac{1}{N}\sum_{j=1}^N||x^{j,N}||^q\right].$$

and, using the same steps as in the proof of Lemma D.9, we can prove that, a.s.,

$$\left|\left|\int_{\tau_k}^{\sigma_k}\gamma_s\langle\nabla_\theta\tilde{\mathcal{L}}^{i,N}(\theta_s^{i,N}),\nabla_\theta\hat{L}^{i,N}(\theta_s^{i,N},\hat{x}_s^N)-\nabla_\theta\tilde{\mathcal{L}}^{i,N}(\theta_s^{i,N})\rangle\mathrm{d}s\right|\right|\overset{k\to\infty}{\to}0.$$

We next consider $A_{3,k}^{i,N}$. Using Itô's isometry, Lemma D.6, the polynomial growth of the function $\nabla_\theta\hat{B}^{i,N}(\theta,\hat{x})$ (see the proof of Lemma D.15), Proposition A.2 (the moment bounds for solutions of the IPS) and Condition F.1 (the square summability of the learning rate), we have that

$$\begin{aligned}&\sup_{t\geq0}\mathbb{E}\left[\left|\int_0^t\gamma_s\langle\nabla_\theta\tilde{\mathcal{L}}^{i,N}(\theta_s^{i,N}),\nabla_\theta\hat{B}^{i,N}(\theta_s^{i,N},\hat{x}_s^N)\mathrm{d}w_s^i\rangle\right|^2\right]\\&\qquad\leq K\mathbb{E}\int_0^\infty\gamma_s^2||\nabla_\theta\hat{B}^{i,N}(\theta_s^{i,N},\hat{x}_s^N)||^2\mathrm{d}s\\&\qquad\leq K\int_0^\infty\gamma_s^2(1+\mathbb{E}\left[||x_s^{i,N}||^q\right]+\frac{1}{N}\sum_{j=1}^N\mathbb{E}\left[||x_s^{j,N}||^q\right]\mathrm{d}s<\infty.\end{aligned}$$

Thus, by Doob's martingale convergence theorem, there exists a finite random variable $A_{3,\infty}^{i,N}$ such that, both almost surely and in $\mathbb{L}^2$,

$$\int_0^t\gamma_s\langle\nabla_\theta\tilde{\mathcal{L}}^{i,N}(\theta_s^{i,N}),\nabla_\theta\hat{B}^{i,N}(\theta_s^{i,N},\hat{x}_s^N)\mathrm{d}w_s^i\rangle\to A_{3,\infty}^{i,N}.$$

as $t\to\infty$. It follows that $A_{3,k}^{i,N}\to0$ a.s. as $k\to\infty$. Finally, we turn our attention to $A_{4,k}^{i,N}$.

For this term, we observe that

$$\sup_{t\geq 0}\mathbb{E}\left|\left|\int_0^t \frac{1}{2}\gamma_s^2 \mathrm{Tr}\left[\nabla_\theta \hat{B}^{i,N}(\theta_s^{i,N},\hat{x}_s^N)\nabla_\theta \hat{B}^{i,N}(\theta_s^{i,N},\hat{x}_s^N)^T\nabla_\theta^2\tilde{\mathcal{L}}^{i,N}(\theta_s^{i,N})\right]\mathrm{d}s\right|\right|$$
$$\leq K\int_0^\infty \gamma_s^2(1+\mathbb{E}\left[||x_s^{i,N}||^q\right]+\frac{1}{N}\sum_{j=1}^N\mathbb{E}\left[||\hat{x}_s^{j,N}||^q\right])\mathrm{d}s<\infty,$$

where, as above, we have used Lemma D.6, the polynomial growth of $\nabla_\theta\hat{B}^{i,N}(\theta,\hat{x})$, Proposition A.2, and Condition F.1. It follows that the random variable

$$\int_0^\infty \frac{1}{2}\gamma_s^2 \mathrm{Tr}\left[\nabla_\theta \hat{B}^{i,N}(\theta_s^{i,N},\hat{x}_s^N)\nabla_\theta \hat{B}^{i,N}(\theta_s^{i,N},\hat{x}_s^N)^T\nabla_\theta^2\tilde{\mathcal{L}}^{i,N}(\theta_s^{i,N})\right]\mathrm{d}s$$

is finite a.s., which in turn implies that there exists a finite random variable $A_{4,\infty}^{i,N}$ such that

$$\int_0^t \frac{1}{2}\gamma_s^2 \mathrm{Tr}\left[\nabla_\theta \hat{B}^{i,N}(\theta_s^{i,N},\hat{x}_s^N)\nabla_\theta \hat{B}^{i,N}(\theta_s^{i,N},\hat{x}_s^N)^T\nabla_\theta^2\tilde{\mathcal{L}}^{i,N}(\theta_s^{i,N})\right]\mathrm{d}s\rightarrow A_4^\infty$$

almost surely. It follows, in particular, that $A_{4,k}^{i,N}\rightarrow 0$ a.s. as $k\rightarrow\infty$. Summarising, we thus have that, for all $\varepsilon>0$, there exists k such that

$$\begin{aligned}\tilde{\mathcal{L}}^{i,N}(\theta_{\sigma_k}^{i,N})-\tilde{\mathcal{L}}^{i,N}(\theta_{\tau_k}^{i,N}) &= A_{1,k}^{i,N}+A_{2,k}^{i,N}+A_{3,k}^{i,N}+A_{4,k}^{i,N}\\ &\geq A_{1,k}^{i,N}-||A_{2,k}^{i,N}||-||A_{3,k}^{i,N}||-||A_{4,k}^{i,N}||\\ &= \frac{||\nabla_\theta\tilde{\mathcal{L}}^{i,N}(\theta_{\tau_k}^{i,N})||^2}{8}\rho-3\varepsilon\end{aligned}$$

The claim follows by setting $\varepsilon=\frac{\rho(\kappa)\kappa^2}{32}$ and $\beta=\frac{\rho(\kappa)\kappa^2}{32}$. ■

Lemma D.12. *Assume that Conditions B.1 - B.2, C.1, D.1 and F.1 hold. Suppose that there are an infinite number of intervals $[\tau_k,\sigma_k)$. Then there exists a fixed constant $0<\beta_1<\beta$ such that, for k large enough,*

$$\tilde{\mathcal{L}}^{i,N}(\theta_{\tau_k}^{i,N})-\tilde{\mathcal{L}}^{i,N}(\theta_{\sigma_{k-1}}^{i,N})\geq-\beta_1.$$

Proof. Using Itô's formula, we have that

$$\begin{aligned}\tilde{\mathcal{L}}^{i,N}(\theta_{\tau_k}^{i,N})-\tilde{\mathcal{L}}^{i,N}(\theta_{\sigma_{k-1}}^{i,N}) \geq& \underbrace{\int_{\sigma_{k-1}}^{\tau_k}\gamma_s\langle\nabla_\theta\tilde{\mathcal{L}}^{i,N}(\theta_s^{i,N}),\nabla_\theta\hat{L}^{i,N}(\theta_s^{i,N},\hat{x}_s^N)-\nabla_\theta\tilde{\mathcal{L}}^{i,N}(\theta_s^{i,N})\rangle\mathrm{d}s}_{B_{1,k}^{i,N}}\\ &+\underbrace{\int_{\sigma_{k-1}}^{\tau_k}\gamma_s\langle\nabla_\theta\tilde{\mathcal{L}}^{i,N}(\theta_s^{i,N}),\nabla_\theta\hat{B}^{i,N}(\theta_s^{i,N},\hat{x}_s^N)\mathrm{d}w_s^i\rangle}_{B_{2,k}^{i,N}}\\ &+\underbrace{\int_{\sigma_{k-1}}^{\tau_k}\frac{1}{2}\gamma_s^2\mathrm{Tr}\left[\nabla_\theta \hat{B}^{i,N}(\theta_s^{i,N},\hat{x}_s^N)\nabla_\theta \hat{B}^{i,N}(\theta_s^{i,N},\hat{x}_s^N)^T\nabla_\theta^2\tilde{\mathcal{L}}^{i,N}(\theta_s^{i,N})\mathrm{d}s\right]}_{B_{3,k}^{i,N}}.\end{aligned}$$

Arguing as in the proof of Lemma D.11, the magnitude of each of the terms converges to zero a.s. as $k \to \infty$. This is sufficient for the conclusion. ■

D.3.2. Technical Lemmas: On A Related Poisson Equation.

Lemma D.13. *Assume that Conditions B.1 - B.2 and D.1 hold. Suppose that, for all $\theta \in \mathbb{R}^p$, $f(\theta, \cdot) : (\mathbb{R}^d)^N \to \mathbb{R}$ satisfies a polynomial growth condition of the form*

$$||f(\theta, \hat{x}^N)|| \leq K\left(1 + ||x^{i,N}||^q + \frac{1}{N}\sum_{j=1}^N ||x^{j,N}||^q\right)$$

Moreover, suppose that $f(\theta, \cdot)$ is centred, in the sense that $\int_{(\mathbb{R}^d)^N} f(\theta, \hat{x}^N) \mathrm{d}\hat{\mu}_\infty^N(\mathrm{d}\hat{x}^N) = 0$. Then, for all $N \in \mathbb{N}$, the function

$$F(\theta, \hat{x}^N) = \int_0^\infty \mathbb{E}_{\hat{x}^N, \theta_0}\left[f(\theta, \hat{x}_t^N)\right] \mathrm{d}t \tag{D.17}$$

is a well defined, continuous function of Sobolev class $\cap_{p \geq 1} W^2_{p,\mathrm{loc}}$, which satisfies the Poisson equation

$$\mathcal{A}_{\hat{x}^N, \theta^*} F(\theta, \hat{x}^N) = -f(\theta, \hat{x}^N). \tag{D.18}$$

Moreover, F is centred, in the sense that $\int_{(\mathbb{R}^d)^N} F(\theta, \hat{x}^N) \hat{\mu}_\infty^N(\mathrm{d}\hat{x}^N) = 0$, and there exist constants $q, K > 0$ such that

$$|F(\theta, \hat{x}^N)| \leq K\left[1 + ||x^{i,N}||^q + \frac{1}{N}\sum_{j=1}^N ||x^{j,N}||^q\right] \tag{D.19}$$

$$|\nabla_{\hat{x}^N} F(\theta, \hat{x})^N| \leq K\left[1 + ||x^{i,N}||^q + \frac{1}{N}\sum_{j=1}^N ||x^{j,N}||^q\right] \tag{D.20}$$

Remark. This is essentially a statement of [68, Theorem 1], adapted appropriately to the current statement. In our case, however, since we are interested in the solution of the Poisson equation associated with the generator of the IPS $\hat{x}^N = (x^{1,N}, \ldots, x^{N,N}) \in (\mathbb{R}^d)^N$ for any $N \in \mathbb{N}$, a little care is needed in places to ensure that arguments in the proof of [68, Theorem 1], in particular those used to establish that the solution is well defined, and that it satisfies the bounds in (D.19) - (D.20), are independent of N. Indeed, we are interested in the solution of this Poisson equation for arbitrarily large N, since we will later take the limit as $N \to \infty$. As an example, if we were to use [68, Theorem 1] directly, we would only have, in place of (D.19), the bound $|F(\theta, \hat{x})| \leq K(1 + ||\hat{x}^N||^q)$, which, due to the $||\hat{x}^N||^q$ term, is unbounded in the limit as $N \to \infty$.

Proof. We begin by showing that the function $F(\theta, \hat{x}^N)$ is well defined, and that it satisfies (D.19). Let $\hat{x}_t^N$ denote a solution of the IPS starting from $\hat{x}^N \in (\mathbb{R}^d)^N$. Let $\hat{\mu}_t^N$ denote the law of $\hat{x}_t^N$. Using the bounds in Lemma D.3, and that f is centred, we have

$$\left|\mathbb{E}_{\hat{x}^N}\left[f(\theta,\hat{x}_t^N)\right]\right| = \left|\mathbb{E}_{\hat{x}^N}\left[f(\theta,\hat{x}_t^N)\right] - \int_{(\mathbb{R}^d)^N} f(\theta,\hat{z}^N)\hat{\mu}_\infty^N(\mathrm{d}\hat{z}^N)\right| \tag{D.21}$$

$$\leq K\left[1+||x^{i,N}||^q + \frac{1}{N}\sum_{j=1}^N ||x^{j,N}||^q\right]e^{-\lambda t} \tag{D.22}$$

We remark that, crucially, the constants $q, K, \lambda > 0$ are independent of N. Thus, for all $N \in \mathbb{N}$, the function F, as defined in (D.17), is absolutely integrable, and thus well defined. Moreover, via the triangle inequality, we immediately obtain the bound in (D.19).

The remaining statements in Lemma D.13 now follow directly from [68, Theorem 1]. In particular, the arguments in the proof of [68, Theorem 1(b), 1(c), 1(d), 1(f)] show that (D.17) defines a continuous, centred solution, unique in the class of solutions belonging to $\cap_{p\geq 1} W^2_{p,\mathrm{loc}}$, of the Poisson equation (D.18).

Finally, we can obtain the bound in (D.20) using the argument in the proof of [68, Theorem 1(e)], replacing the intermediate bound on $||F(\theta,\cdot,\cdot)||$ by (D.19), and the intermediate bound on $||f(\theta,\cdot,\cdot)||$ by our condition on the polynomial growth of $f(\cdot)$.[5]. This completes the proof.■

Lemma D.14. *Assume that Conditions B.1 - B.2 and D.1 hold. Suppose that the function $f(\theta,\hat{x}^N) \in \mathcal{C}^{\alpha,2}(\mathbb{R}^p,(\mathbb{R}^d)^N)$, for some $\alpha > 0$, is centred in the same sense as Lemma D.13, and satisfies*

$$|f(\theta,\hat{x}^N)| + |\partial_\theta f(\theta,\hat{x}^N)| + |\partial_\theta^2 f(\theta,\hat{x}^N)| \leq K\left[1+||x^{i,N}||^q + \frac{1}{N}\sum_{j=1}^N ||x^{j,N}||^q\right]$$

where $\hat{x} = (x^{1,N},\ldots,x^{N,N})$. Then the solution (D.17) of the Poisson equation (D.18) satisfies $F(\cdot,\hat{x}^N) \in \mathcal{C}^2$ for all $\hat{x}^N \in (\mathbb{R}^d)^N$. Moreover, there exist $q', K' > 0$ such that

$$\sum_{k=0}^2 \left|\frac{\partial^k F}{\partial\theta^k}\right| + \left|\frac{\partial^2 F}{\partial\hat{x}\partial\theta}\right| \leq K\left[1+||x^{i,N}||^q + \frac{1}{N}\sum_{j=1}^N ||x^{j,N}||^q\right]$$

Proof. The first statement of the Theorem follows directly from [67, Theorem 3]. Now, observe that, since $\partial_\theta^k f$, $k = 0,1,2$, satisfies a polynomial growth condition in the required sense, $\partial_\theta^k f^{i,N}$ can be shown to satisfy bounds of the form given in Lemma D.3. It follows, arguing as in (D.21) - (D.22), that

$$\left|\mathbb{E}_{\hat{x}^N}\left[\frac{\partial^k f}{\partial\theta^k}(\theta,\hat{x}_t^N)\right]\right| \leq K\left[1+||x^{i,N}||^q + \frac{1}{N}\sum_{j=1}^N ||x^{j,N}||^q\right]e^{-\lambda t}$$

[5] In the original notation, these are the bounds on $||u||$ and $||Lu||$, respectively. See [68, pg. 1070]

We thus have that, allowing the value of the constant K to change from line to line, that

$$\left|\frac{\partial^k F}{\partial\theta^k}(\theta,\hat{x}^N)\right| \leq \int_0^\infty \left|\mathbb{E}_{\hat{x}^N}\left[\frac{\partial^k f}{\partial\theta^k}(\theta,\hat{x}^N)\right]\right|\mathrm{d}t \leq K\int_0^\infty \left[1+||x^{i,N}||^q+\frac{1}{N}\sum_{j=1}^N ||x^{j,N}||^q\right]e^{-\lambda t}\mathrm{d}t$$
$$\leq K\left[1+||x^{i,N}||^q+\frac{1}{N}\sum_{j=1}^N ||x^{j,N}||^q\right].$$

Finally, the bound on the mixed derivative follows from (D.20) in Lemma D.13. ■

D.3.3. Technical Lemmas: Miscellaneous.

Lemma D.15. *Assume that Conditions B.1 - B.2, C.1 and D.1 hold. Then, for all $i = 1,\dots,N$, $N\in\mathbb{N}$, the function $S^{i,N}(\theta,\hat{x}^N) = \nabla_\theta \hat{L}^{i,N}(\theta,\hat{x}^N) - \nabla_\theta\tilde{\mathcal{L}}^{i,N}(\theta)$ is in $\mathcal{C}^{2,\alpha}(\mathbb{R}^p,(\mathbb{R}^d)^N)$. Moreover, for $k=0,1,2$, there exists q and K such that*

$$||\nabla_\theta^k S^{i,N}(\theta,\hat{x}^N)|| \leq K\left[1+||x^{i,N}||^q+\frac{1}{N}\sum_{j=1}^N ||x^{j,N}||^q\right]. \tag{D.23}$$

Proof. By definition, we have that, for $k=0,1,2$,

$$\nabla_\theta^k S^{i,N}(\theta,\hat{x}^N) = \nabla_\theta^{k+1}\hat{L}^{i,N}(\theta,\hat{x}^N) - \nabla_\theta^{k+1}\tilde{\mathcal{L}}^{i,N}(\theta).$$

By Condition C.1(i), $\nabla_\theta b(\theta,x)\in\mathcal{C}^{2,\alpha}(\mathbb{R}^p,\mathbb{R}^d)$, and $\nabla_\theta\phi(\theta,x,y)\in\mathcal{C}^{2,\alpha,\alpha}(\mathbb{R}^p,\mathbb{R}^d,\mathbb{R}^d)$. It follows from the definitions, c.f. (4.20), (4.21) and (4.22), that $\nabla_\theta\hat{B}^{i,N}(\theta,\hat{x}^N)$, $\nabla_\theta\hat{G}^{i,N}(\theta,\hat{x}^N)$, and $\nabla_\theta\hat{L}^{i,N}(\theta,\hat{x}^N)$ are in $\mathcal{C}^{2,\alpha}(\mathbb{R}^p,(\mathbb{R}^d)^N)$. It also follows from the definition (Lemma 3.4.B) that $\nabla_\theta\tilde{\mathcal{L}}^{i,N}(\theta)$ is in $\mathcal{C}^2(\mathbb{R}^p)$. Thus, as claimed, $S^{i,N}(\theta,\hat{x})$ is in $\mathcal{C}^{2,\alpha}(\mathbb{R}^p,(\mathbb{R}^d)^N)$. It remains to note that the bound (D.23) follows immediately from Lemma D.4 and Lemma D.6 ■

Lemma D.16. *Assume that Conditions B.1 - B.2, C.1 and D.1 hold. Then, for all $i = 1,\dots,N$, $N\in\mathbb{N}$, the function $T^{i,N}(\theta,\hat{x}^N) = \langle\nabla_\theta\tilde{\mathcal{L}}^{i,N}(\theta),\nabla_\theta\hat{L}^{i,N}(\theta,\hat{x}^N)-\nabla_\theta\tilde{\mathcal{L}}^{i,N}(\theta)\rangle$ is in $\mathcal{C}^{2,\alpha}(\mathbb{R}^p,(\mathbb{R}^d)^N)$. Moreover, for $k=0,1,2$, there exists q, K such that*

$$||\nabla_\theta^k T^{i,N}(\theta,\hat{x}^N)|| \leq K\left[1+||x^{i,N}||^q+\frac{1}{N}\sum_{j=1}^N ||x^{j,N}||^q\right]. \tag{D.24}$$

Proof. This lemma follows almost immediately from Lemma D.15. First note that, by definition, we can write $T^{i,N}(\theta,\hat{x}^N) = \langle\nabla_\theta\tilde{\mathcal{L}}^{i,N}(\theta),S^{i,N}(\theta,\hat{x}^N)\rangle$. By Lemma D.15, $S^{i,N}(\theta,\hat{x}^N)$ is in $\mathcal{C}^{2,\alpha}(\mathbb{R}^p,(\mathbb{R}^d)^N)$ and $\nabla_\theta\tilde{\mathcal{L}}^{i,N}(\theta)$ is in $\mathcal{C}^2(\mathbb{R}^p)$. It follows immediately that also $T^{i,N}(\theta,\hat{x}^N)\in\mathcal{C}^{2,\alpha}(\mathbb{R}^p,(\mathbb{R}^d)^N)$. Finally, the bound (D.24) follows from Lemma D.6 and Lemma D.15, via an application of Holdër's inequality. ■

Appendix E. Proof of Lemma for Theorem 3.4.

Lemma E.1. *Assume that Conditions A.1, B.1 - B.2, C.1, and D.1 hold. Let $i=1,\dots,N$, and $N\in\mathbb{N}$. Then, for all $\theta\in\mathbb{R}^p$, there exists $K<\infty$ such that*

$$||\nabla_\theta\tilde{\mathcal{L}}(\theta)-\nabla_\theta\tilde{\mathcal{L}}^{i,N}(\theta)|| \leq \frac{K}{N^{\frac{1}{2}}}\ , \quad a.s.$$

Proof. Let us define $g : \mathbb{R}^p \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ as the function which satisfies $G(\theta, x, \mu) = \int_{\mathbb{R}^d} g(\theta, x, y)\mu(\mathrm{d}y)$, where $G(\theta, x, \mu)$ is defined in (2.1). Thus, in particular,

$$g(\theta, x, y) = [b(\theta, x) + \phi(\theta, x, y)] - [b(\theta_0, x) + \phi(\theta_0, x, y)]$$

From the definition of $L(\theta, x, \mu)$, c.f. (2.2), we have that

$$\nabla_\theta L(\theta, x, \mu) = -\nabla_\theta^T G(\theta, x, \mu)G(\theta, x, \mu) = -\int_{\mathbb{R}^d\times\mathbb{R}^d} \nabla_\theta^T g(\theta, x, y)g(\theta, x, z)\mu(\mathrm{d}y)\mu(\mathrm{d}z)$$

We can thus define $l : \mathbb{R}^p \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^p$ as the function which satisfies $\nabla_\theta L(\theta, x, \mu) = \int_{\mathbb{R}^d\times\mathbb{R}^d} l(\theta, x, y, z)\mu(\mathrm{d}y)\mu(\mathrm{d}z)$. In particular, we identify

$$l(\theta, x, y, z) = -\nabla_\theta^T g(\theta, x, y)g(\theta, x, z).$$

We note that, via Condition C.1(ii), $l(\theta, \cdot, \cdot, \cdot)$ is locally Lipschitz with polynomial growth. That is, for all $x, x', y, y', z, z' \in \mathbb{R}^d$, we have

$$\begin{aligned} ||l(\theta, x, y, z) - l(\theta, x', y', z')|| \leq K &\left[[1 + ||x||^q + ||x'||^q + ||y||^q + ||y'||^q + ||z||^q + ||z'||^q\right] \\ &\cdot \left[||x - x'|| + ||y - y'|| + ||z - z'||\right] \end{aligned} \tag{E.1}$$

In terms of this function, we can now write

$$\begin{aligned} \nabla_\theta L(\theta, x^i, \mu_\infty) &= \int_{\mathbb{R}^d\times\mathbb{R}^d} \varphi(\theta, x^i, x^j, x^k)\mu_\infty(\mathrm{d}x^j)\mu_\infty(\mathrm{d}x^k) \\ &= \frac{1}{N^2}\sum_{j=1}^N\sum_{k=1}^N \int_{\mathbb{R}^d\times\mathbb{R}^d} \varphi(\theta, x^i, x^j, x^k)\mu_\infty(\mathrm{d}x^j)\mu_\infty(\mathrm{d}x^k) \\ \nabla_\theta L(\theta, x^{i,N}, \mu^N) &= \frac{1}{N^2}\sum_{j=1}^N\sum_{k=1}^N \varphi(\theta, x^{i,N}, x^{j,N}, x^{k,N}). \end{aligned}$$

where, in the second line, we have simply summed over the dummy variables x^j and x^k. and thus, from the definitions (see Lemmas 3.4.A - 3.4.B),

$$\begin{aligned} \nabla_\theta \tilde{\mathcal{L}}(\theta) &= \frac{1}{N^2}\sum_{j=1}^N\sum_{k=1}^N \int_{\mathbb{R}^d}\left[\int_{\mathbb{R}^d\times\mathbb{R}^d} \varphi(\theta, x^i, x^j, x^k)\mu_\infty(\mathrm{d}x^j)\mu_\infty(\mathrm{d}x^k)\right]\mu_\infty(\mathrm{d}x^i) \\ &= \frac{1}{N^2}\sum_{j=1}^N\sum_{k=1}^N \int_{(\mathbb{R}^d)^N} \varphi(\theta, x^i, x^j, x^k)\mu_\infty(\mathrm{d}x^1)\cdots\mu_\infty(\mathrm{d}x^N) \\ \nabla_\theta \tilde{\mathcal{L}}^{i,N}(\theta) &= \frac{1}{N^2}\sum_{j=1}^N\sum_{k=1}^N \int_{(\mathbb{R}^d)^N} \varphi(\theta, x^{i,N}, x^{j,N}, x^{k,N})\hat{\mu}_\infty^N(\mathrm{d}\hat{x}). \end{aligned}$$

where in the second line we have simply integrated with respect to the invariant probability measure μ_∞ over additional dummy variables, which does not change the value of the integral.

Let $\hat{x}_t^N$ denote a solution of the IPS starting from $\hat{x}_0^N = (x_0^{1,N}, \ldots, x_0^{N,N})$, and let $x_t^{[N]}$ denote N independent solutions of the McKean-Vlasov SDE starting from $x_0^{[N]} = (x_0^1, \ldots, x_0^N)$. Then, using the definition of an invariant measure, we can write

$$\nabla_\theta \tilde{\mathcal{L}}(\theta) = \frac{1}{N^2}\sum_{j=1}^N\sum_{k=1}^N \int_{(\mathbb{R}^d)^3} \mathbb{E}_{(x_0^i,x_0^j,x_0^k)}\left[\varphi(\theta, x^i, x^j, x^k)\right]\mu_\infty(\mathrm{d}x^1)\cdots\mu_\infty(\mathrm{d}x^N) \tag{E.2}$$

$$\nabla_\theta \tilde{\mathcal{L}}^{i,N}(\theta) = \frac{1}{N^2}\sum_{j=1}^N\sum_{k=1}^N \int_{(\mathbb{R}^d)^N} \mathbb{E}_{(x_0^{i,N},x_0^{j,N},x_0^{k,N})}\left[\varphi(\theta, x^{i,N}, x^{j,N}, x^{k,N})\right]\hat{\mu}_\infty^N(\mathrm{d}\hat{x}). \tag{E.3}$$

Let $\pi^\infty \in \Pi(\hat{\mu}_\infty^N, \mu_\infty^{\otimes N})$ denote an arbitrary coupling of $\hat{\mu}_\infty^N$ and $\mu_\infty^{\otimes N}$. Then, using (E.2) - (E.3), it follows straightforwardly that

$$\begin{aligned}
&||\nabla_\theta \tilde{\mathcal{L}}(\theta) - \nabla_\theta \tilde{\mathcal{L}}^{i,N}(\theta)|| \\
&\leq \frac{1}{N^2}\sum_{j=1}^N\sum_{k=1}^N \int_{(\mathbb{R}^d)^N\times(\mathbb{R}^d)^N} \mathbb{E}_{(x_0^i,x_0^j,x_0^k,x_0^{i,N},x_0^{j,N},x_0^{k,N})} \\
&\qquad\qquad [||\varphi(\theta, x_s^i, x_s^j, x_s^k) - \varphi(\theta, x_s^{i,N}, x_s^{j,N}, x_s^{k,N})||]\pi^\infty(\mathrm{d}\hat{x}^N, \mathrm{d}x^{[N]})
\end{aligned} \tag{E.4}$$

Now, using the growth property (E.1) and Hölder's inequality, we obtain (now suppressing dependence of the expectation on the initial conditions)

$$\mathbb{E}\left[||\varphi(\theta, x_s^i, x_s^j, x_s^k) - \varphi(\theta, x_s^{i,N}, x_s^{j,N}, x_s^{k,N})||\right] \tag{E.5}$$

$$\begin{aligned}
&\leq \left[1 + \mathbb{E}\left[||x_s^i||^{2q}\right]^{\frac{1}{2}} + \cdots + \left[\mathbb{E}||x_s^{k,N}||^{2q}\right]^{\frac{1}{2}}\right] \\
&\quad\cdot\left[\mathbb{E}\left[||x_s^i - x_s^{i,N}||^2\right]^{\frac{1}{2}} + \mathbb{E}\left[||x_s^j - x_s^{j,N}||^2\right]^{\frac{1}{2}} + \mathbb{E}\left[||x_s^k - x_s^{k,N}||^2\right]^{\frac{1}{2}}\right]
\end{aligned} \tag{E.6}$$

$$\leq \frac{K}{N^{\frac{1}{2}}}, \tag{E.7}$$

where in the final line we have used Proposition A.2 (the bounded moments of the McKean-Vlasov SDE and the IPS) and Proposition A.5 (uniform in time propagation of chaos). Finally, substituting (E.5) - (E.7) into (E.4), the result follows. ∎

Appendix F. Proof of Theorem 3.3* and Theorem 3.4*.

F.1. Proof of Theorem 3.3*.

Proof. The proof of Theorem 3.3* is similar to the proof of Theorem 3.3, with several small modifications. Firstly, we replace all instances of $\theta_t^{i,N}$ and $\theta_t^{[N]}$ (the parameter estimates generated by the IPS) with $\theta_t^{[i,N]}$ and $\theta_t^{[N]}$ (the parameter estimates generate by partial observations of the McKean-Vlasov SDE). In addition, we will now utilise a slightly different decomposition of the asymptotic log-likelihood, namely

$$||\nabla_\theta \tilde{\mathcal{L}}(\theta_t^{[i,N]})|| \leq \underbrace{||\nabla_\theta \tilde{\mathcal{L}}(\theta_t^{[i,N]}) - \tfrac{1}{t}\nabla_\theta \mathcal{L}_t^i(\theta_t^{[i,N]})||}_{\to 0 \text{ as } t\to\infty \ \forall N\in\mathbb{N} \text{ by Lemma 3.4.A}} + \underbrace{||\tfrac{1}{t}\nabla_\theta \mathcal{L}_t^i(\theta_t^{[i,N]}) - \tfrac{1}{t}\nabla_\theta \mathcal{L}_t^{[i,N]}(\theta_t^{[i,N]})||}_{\to 0 \text{ as } N\to\infty \ \forall t\in\mathbb{R}_+ \text{ by Lemma 3.4.C}^*}$$
$$+ \underbrace{||\tfrac{1}{t}\nabla_\theta \mathcal{L}_t^{[i,N]}(\theta_t^{[i,N]}) - \nabla_\theta \tilde{\mathcal{L}}^{[i,N]}(\theta_t^{[i,N]})||}_{\to 0 \text{ as } t\to\infty \ \forall N\in\mathbb{N} \text{ by Lemma 3.4.B}^*} + \underbrace{||\nabla_\theta \tilde{\mathcal{L}}^{[i,N]}(\theta_t^{[i,N]})||}_{\to 0 \text{ as } t\to\infty, N\to\infty \text{ by Lemma 3.4.D}^*}.$$

where $\mathcal{L}_t^{[i,N]}(\theta)$ is defined in (2.5), and $\tilde{\mathcal{L}}^{[i,N]}(\theta) = \lim_{t\to\infty}\frac{1}{t}\mathcal{L}_t^{[i,N]}(\theta)$. In particular, we have replaced all instance the 'partial' log-likelihood of the i^{th} particle in the original IPS, $\mathcal{L}_t^{i,N}(\theta)$, with the 'partial' log-likelihood of the i^{th} particle in the 'IPS' consisting of N independent solutions of the McKean-Vlasov SDE, $\mathcal{L}_t^{[i,N]}(\theta)$. The result of the theorem now follows, using the Lemma 3.4.A and the modified Lemmas 3.4.B* - 3.4.D* (see below). ∎

We remark that we can still use Lemma 3.4.A to show that the first term converges to zero in $\mathbb{L}^1$, since this lemma applies for all values of $\theta \in \mathbb{R}^p$, and thus replacing $\theta_t^{i,N}$ by $\theta_t^{[i,N]}$ is irrelevant. We do, however, require modified versions of Lemmas 3.4.B, 3.4.C, and 3.4.D. We state and sketch the proofs of these below.

Lemma 3.4.B*. *Assume that Conditions B.1 - B.2, C.1, and D.1 hold. Then, for all $N \in \mathbb{N}$, the processes $\frac{1}{t}\nabla_\theta^m \mathcal{L}_t^{[i,N]}(\theta)$ and $\frac{1}{t}\nabla_\theta^m \mathcal{L}_t^{N}(\theta)$, $m = 0,1,2$, converge, both a.s. and in $\mathbb{L}^1$, to the functions*

$$\nabla_\theta^m \tilde{\mathcal{L}}^{[i,N]}(\theta) = \int_{(\mathbb{R}^d)^N} \nabla_\theta^m \hat{L}^{i,N}(\theta, \hat{x}^N)\mu_\infty^{\otimes N}(\mathrm{d}\hat{x}^N) \quad , \quad \nabla_\theta^m \tilde{\mathcal{L}}^{[N]}(\theta) = \frac{1}{N}\sum_{i=1}^N \nabla_\theta^m \tilde{\mathcal{L}}^{[i,N]}(\theta).$$

In addition, there exist positive constants K_m^1, K_m^2, independent of N, such that

$$\left|\left|\mathbb{E}\left[\frac{1}{t}\nabla_\theta^m \mathcal{L}_t^{[i,N]}(\theta) - \nabla_\theta^m \tilde{\mathcal{L}}^{[i,N]}(\theta)\right]\right|\right| \leq \frac{K_m^1(1-e^{-\lambda t})}{\lambda t} + \frac{K_m^2(1+\sqrt{t})^{\frac{1}{2}}}{t^{\frac{1}{2}}}$$

and this bound also holds if $\mathcal{L}_t^{[i,N]}(\cdot)$ and $\tilde{\mathcal{L}}^{[i,N]}(\cdot)$ are replaced with $\mathcal{L}_t^{[N]}(\cdot)$ and $\tilde{\mathcal{L}}^{[N]}(\cdot)$.

Proof. The proof is essentially identical to the proof of Lemma 3.4.B. On this occasion, we use Proposition A.3 (existence of a unique invariant measure for the McKean-Vlasov SDE) in the place of Proposition A.4 (existence of a unique invariant measure for the IPS) to establish almost sure convergence. The remainder of the proof goes through almost verbatim, replacing all instance of $x_t^{i,N}$ with x_t^i, applying a modified version of Lemma D.3,[6] and noting that all of the required bounds hold for both the McKean-Vlasov SDE and the IPS. ∎

Lemma 3.4.C*. *Assume that Conditions B.1 - B.2, C.1, and D.1 hold. Then, for all $\theta \in \mathbb{R}^p$, for all $t \geq 0$, for all $i = 1,\dots,N$, we have, in $\mathbb{L}^1$, that*

$$\lim_{N\to\infty} ||\tfrac{1}{t}\nabla_\theta \mathcal{L}_t^{[i,N]}(\theta)|| = ||\tfrac{1}{t}\nabla_\theta \mathcal{L}_t^{i}(\theta)||,$$
$$\lim_{N\to\infty} ||\tfrac{1}{t}\nabla_\theta \mathcal{L}_t^{[N]}(\theta)|| = ||\tfrac{1}{t}\nabla_\theta \mathcal{L}_t^{i}(\theta)||.$$

[6] In particular, it is straightforward to verify that Lemma D.3 still holds if the original IPS is replaced by the IPS consisting of N independent copies of the McKean-Vlasov SDE.

In addition, there exists a positive constant K such that, for all $\theta \in \mathbb{R}^p$, for all $N \in \mathbb{N}$,

$$\mathbb{E}\left[\left|\left|\frac{1}{t}\nabla_\theta \mathcal{L}_t^i(\theta) - \frac{1}{t}\nabla_\theta \mathcal{L}_t^{[i,N]}(\theta)\right|\right|\right] \leq \frac{K}{\sqrt{N}}\left(1 + \frac{1}{\sqrt{t}}\right),$$

and this bound also holds if $\mathcal{L}_t^{[i,N]}(\cdot)$ is replaced by $\mathcal{L}_t^{[N]}(\cdot)$.

Proof. The proof of Lemma 3.4.C goes through almost unchanged. In addition to the modifications outlined above, we also now apply a modified (simpler) version of Lemma D.8. Indeed, several terms in the proof of Lemma D.8 (and theauxiliary Lemma D.7) vanish when $x_t^{i,N}$ is replaced by x_t^i. In addition, where appropriate, we now make use of the moment bounds for the solutions of the McKean-Vlasov SDE, rather than the IPS (Proposition A.2).■

Lemma 3.4.D*. *Assume that Conditions B.1 - B.2, C.1, D.1, and F.1 hold. Then, for all $N \in \mathbb{N}$, we have, both almost surely and in $\mathbb{L}^1$, that*

$$\lim_{t\to\infty} ||\nabla_\theta \tilde{\mathcal{L}}^{[i,N]}(\theta^{[i,N]}(t))|| = 0,$$
$$\lim_{t\to\infty} ||\nabla_\theta \tilde{\mathcal{L}}^{[N]}(\theta^{[N]}(t))|| = 0.$$

Proof. The proof is essentially the same as the proof of Lemma 3.4.D, with all instances of $\theta_t^{i,N}$ and $\mathcal{L}_t^{i,N}(\cdot)$ replaced by $\theta_t^{[i,N]}$ and $\mathcal{L}_t^{[i,N]}(\cdot)$, respectively. This is also the case for the auxiliary Lemmas D.9 - D.12. In these lemmas, we must additionally replace all instances of $x_t^{i,N}$, $\hat{x}_t^N = (x_t^{1,N}, \ldots, x_t^{N,N})$ by x_t^i, $\hat{x}_t^{[N]} = (x_t^1, \ldots, x_t^N)$, respectively. Moreover, as above, we now make use of the appropriate bounds for the solutions of the McKean-Vlasov SDE, rather than those for IPS (Proposition A.2). ■

F.2. Proof of Theorem 3.4*.

Proof. The proof of Theorem 3.4 goes through almost verbatim, using the same modifications outlined above. With these modifications, the analogue of $\mathbb{E}[\Omega_{t,i,N}^{(2)}]$ in (4.42), which yields the $N^{-\frac{1}{2}}$ term in the final bound, vanishes. In particular, if one follows the arguments in Lemma E.1 (see Appendix E), the expectations involving squared differences in (E.5) - (E.6) are identically zero. Moreover, due to these modifications, the constants appearing in the bounds for the analogues of $\mathbb{E}[\Omega_{t,i,N}^{(1)}]$, $\mathbb{E}[\Omega_{t,i,N}^{(3)}]$ and $\mathbb{E}[\Omega_{t,i,N}^{(4)}]$, c.f. (4.43), (4.47), and (4.48), may differ. These observations explain the differences between the rates in Theorem 3.4 and Theorem 3.4*. ■

Appendix G. Proof of Theorem 3.3† and Theorem 3.4†.

G.1. Proof of Theorem 3.3†.

Proof. The proof of Theorem 3.3† is very similar to the proof of Theorem 3.3, with $\theta_t^{i,N}$ (the parameter estimate generated by the IPS) is replaced by θ_t^i (the parameter estimate generated by the McKean-Vlasov SDE). In particular, we once more make use of the decomposition

$$||\nabla_\theta \tilde{\mathcal{L}}(\theta_t^i)|| \leq \underbrace{||\nabla_\theta \tilde{\mathcal{L}}(\theta_t^i) - \tfrac{1}{t}\nabla_\theta \mathcal{L}_t^i(\theta_t^i)||}_{\to 0 \text{ as } t\to\infty \;\; \forall N\in\mathbb{N} \text{ by Lemma 3.4.A}} + \underbrace{||\tfrac{1}{t}\nabla_\theta \mathcal{L}_t^i(\theta_t^i) - \tfrac{1}{t}\nabla_\theta \mathcal{L}_t^{i,N}(\theta_t^i)||}_{\to 0 \text{ as } N\to\infty \;\; \forall t\in\mathbb{R}_+ \text{ by Lemma 3.4.C}}$$
$$+ \underbrace{||\tfrac{1}{t}\nabla_\theta \mathcal{L}_t^{i,N}(\theta_t^i) - \nabla_\theta \tilde{\mathcal{L}}^{i,N}(\theta_t^i)||}_{\to 0 \text{ as } t\to\infty \;\; \forall N\in\mathbb{N} \text{ by Lemma 3.4.B}} + \underbrace{||\nabla_\theta \tilde{\mathcal{L}}^{i,N}(\theta_t^i)||}_{\to 0 \text{ as } t\to\infty, N\to\infty \text{ by Lemma 3.4.D}^\dagger}.$$

The result then follows immediately from Lemmas 3.4.A - 3.4.C and Lemma 3.4.D† (see below). ■

It is still possible to use Lemmas 3.4.A, 3.4.B, and 3.4.C to show that the first three terms converge to zero in $\mathbb{L}^1$. In particular, these lemmas apply for all values of $\theta \in \mathbb{R}^p$, and so it is irrelevant that $\theta_t^{i,N}$ has been replaced by θ_t^i. This modification is, however, relevant to Lemma 3.4.D. We must therefore establish the following modified version of this lemma.

Lemma 3.4.D†. *Assume that Conditions B.1 - B.2, C.1, D.1 and F.1 hold. Then, in $\mathbb{L}^1$, we have*

$$\lim_{t\to\infty}\lim_{N\to\infty} ||\nabla_\theta \tilde{\mathcal{L}}^{i,N}(\theta_t^i)|| = 0,$$

Proof. The proof of this lemma largely follows the proof of Lemma 3.4.D. Let us briefly outline the main changes. The first modification, of course, is to replace all instances of $\theta_t^{i,N}$ with θ_t^i. In addition, we must slightly modify the definition of the original stopping times, c.f. (4.30) - (4.31), which will now be given by

$$\text{(G.1)} \;\; \tau_k = \inf\left\{t > \sigma_{k-1} : ||\nabla_\theta \tilde{\mathcal{L}}^{i,N}(\theta_t^i)|| \geq \kappa\right\}$$

$$\text{(G.2)} \;\; \sigma_k = \sup\Big\{t > \tau_k : \tfrac{1}{2}||\nabla \tilde{\mathcal{L}}^{i,N}(\theta_{\tau_k}^i)|| \leq ||\nabla \tilde{\mathcal{L}}^{i,N}(\theta_s^i)|| \leq 2||\nabla \tilde{\mathcal{L}}^{i,N}(\theta_{\tau_k}^i)|| + \frac{1}{N} \;\forall s\in[\tau_k, t], \int_{\tau_k}^t \gamma(s)\mathrm{d}s \leq \rho\Big\}$$

In addition, we must now rely on slightly modified versions of Lemma D.11 (namely, Lemma D.11†) and Lemma D.12 (namely, Lemma D.12†) which yield, instead of (4.32) - (4.33), the following inequalities

$$\tilde{\mathcal{L}}^{i,N}(\theta_{\sigma_k}^i) - \tilde{\mathcal{L}}^{i,N}(\theta_{\tau_k}^i) \geq \beta - \frac{K}{N^{\frac{1}{2}}}\int_{\tau_k}^{\sigma_k} \gamma_s \mathrm{d}s$$
$$\tilde{\mathcal{L}}^{i,N}(\theta_{\tau_k}^i) - \tilde{\mathcal{L}}^{i,N}(\theta_{\sigma_{k-1}}^i) \geq -\beta_1 - \frac{K}{N^{\frac{1}{2}}}\int_{\sigma_{k-1}}^{\tau_k} \gamma_s \mathrm{d}s.$$

Using these inequalities, it follows, arguing as in (4.34) - (4.35), that

$$\begin{aligned}\tilde{\mathcal{L}}^{i,N}(\theta^i_{\tau_{n+1}}) - \tilde{\mathcal{L}}^{i,N}(\theta^i_{\tau_{k_0}}) &= \sum_{k=k_0}^{n}\left[\tilde{\mathcal{L}}^{i,N}(\theta^{i,N}_{\sigma_k}) - \tilde{\mathcal{L}}^{i,N}(\theta^i_{\tau_k}) + \tilde{\mathcal{L}}^{i,N}(\theta^i_{\tau_{k+1}}) - \tilde{\mathcal{L}}^{i,N}(\theta^i_{\sigma_k})\right] \\ &\geq \sum_{k=k_0}^{n}(\beta-\beta_1) - \frac{1}{N^{\frac{1}{2}}}\int_{\tau_0}^{\tau_{n+1}}\gamma_s \mathrm{d}s \\ &= (n+1-k_0)(\beta-\beta_1) - \frac{K}{N^{\frac{1}{2}}}\int_{\tau_0}^{\tau_{n+1}}\gamma_s\mathrm{d}s\end{aligned}$$

Suppose we let $N = \mathcal{O}((\int_0^{\tau_{n+1}}\gamma_s \mathrm{d}s)^2)$. Then, using also the fact that $\beta - \beta_1 > 0$, it follows that $\tilde{\mathcal{L}}^{i,N}(\theta^i_{\tau_{n+1}}) \to \infty$ as $n \to \infty$, $N \to \infty$. But this is in contradiction with Lemma D.6, which states that $\tilde{\mathcal{L}}^{i,N}(\theta)$ is bounded from above. The remainder of the proof of Lemma 3.4.D now goes through unchanged. ∎

This proof relies on modified versions of Lemmas D.11 and D.12 (namely, Lemmas D.11† and D.12†), which themselves rely on modified versions of Lemmas D.9 and D.10 (namely, Lemmas D.9† and D.10†). Note that we can still use the original version of Lemma D.6, since it applies for all $\theta \in \mathbb{R}^p$. It remains, therefore, to state and prove the modified versions of these lemmas. We state these lemmas in full, and sketch any changes to the original proofs, below.

G.1.1. Additional Lemmas for Lemma 3.4.D†.

Lemma D.9†. *Assume that Conditions B.1 - B.2, C.1, D.1, and F.1 hold. Define, with $\hat{x}^N = (x^{1,N}, \ldots, x^{N,N})$, the function*

$$\Gamma_{k,\eta} = \int_{\tau_k}^{\sigma_{k,\eta}} \gamma_s \left(\nabla_\theta \hat{L}^{i,N}(\theta^i_s, \hat{x}^N_s) - \nabla_\theta\tilde{\mathcal{L}}^{i,N}(\theta^i_s)\right)\mathrm{d}s.$$

Then, almost surely, $||\Gamma_{k,\eta}|| \to 0$ as $k \to \infty$.

Proof. The proof follows, essentially verbatim, the proof of Lemma D.9, with $\theta^{i,N}_t$ replaced by θ^i_t throughout. In addition, when we apply Itô's formula to the solution of the appropriate Poisson equation, c.f. (D.10), the third, fourth, and sixth terms will have minor modifications, due to the differences between the dynamics of $\theta^{i,N}_t$ and θ^i_t. Since all of the bounds used in the subsequent arguments apply both to the solutions of the McKean-Vlasov SDE and the IPS, the remainder of the argument goes through unchanged. ∎

Lemma D.10†. *Assume that Conditions B.1 - B.2, C.1, D.1 and F.1 hold. Let $\rho > 0$ be such that, for a given $\kappa > 0$, it is true that $3\rho + \frac{\rho}{4\kappa} = \frac{1}{2L}$, where L denotes the Lipschitz constant of $\nabla_\theta\tilde{\mathcal{L}}^{i,N}(\theta)$. For k large enough, and for $\eta > 0$ small enough (potentially random, and depending on k), one has*

$$\int_{\tau_k}^{\sigma_{k,\eta}}\gamma_s\mathrm{d}s > \rho \quad \textit{and, a.s.,} \quad \frac{\rho}{2} \leq \int_{\tau_k}^{\sigma_k}\gamma_s\mathrm{d}s \leq \rho.$$

Proof. The proof is more or less identical to the proof of Lemma D.10, with $\theta_t^{i,N}$ replaced by θ_t^i throughout. In addition, due to the different dynamics, instead of (D.14), we now have that

$$\begin{aligned}||\theta^i_{\sigma_{k,\eta}} - \theta^i_{\tau_k}|| = \Big|\Big| \int_{\tau_k}^{\sigma_{k,\eta}} \gamma_s \nabla_\theta L(\theta_s^i, x_s^i, \mu_s)\mathrm{d}s + \int_{\tau_k}^{\sigma_{k,\eta}} \gamma_s \langle \nabla_\theta B(\theta_s^i, x_s^i, \mu_s), \mathrm{d}w_s^i\rangle \Big|\Big| \\ \leq \int_{\tau_k}^{\sigma_{k,\eta}} \gamma_s \nabla_\theta ||\tilde{\mathcal{L}}^{i,N}(\theta_s^i)||\mathrm{d}s + \Big|\Big| \int_{\tau_k}^{\sigma_{k,\eta}} \gamma_s [\nabla_\theta \hat{L}^{i,N}(\theta_s^i, \hat{x}_s^N) - \nabla_\theta \tilde{\mathcal{L}}^{i,N}(\theta_s^{i,N})]\mathrm{d}s \Big|\Big| \\ + \frac{||\nabla \tilde{\mathcal{L}}^{i,N}(\theta^i_{\tau_k})||}{\kappa} \Big|\Big| \int_{\tau_k}^{\sigma_{k,\eta}} \gamma_s \frac{\kappa}{||\nabla \tilde{\mathcal{L}}^{i,N}(\theta^i_{\tau_k})||} \langle \nabla_\theta B(\theta_s^i, x_s^i, \mu_s), \mathrm{d}w_s^i\rangle \Big|\Big|. \\ + \Big|\Big| \int_{\tau_k}^{\sigma_{k,\eta}} \gamma_s [\nabla_\theta L(\theta_s^i, x_s^i, \mu_s) - \nabla_\theta \hat{L}^{i,N}(\theta_s^i, \hat{x}_s^N)\mathrm{d}s \Big|\Big|\end{aligned}$$

This is identical to (D.14), aside from the presence of the additional final term. Under the condition that $\int_{\tau_k}^{\sigma_{k,\eta}} \gamma_s \mathrm{d}s < \rho$, we can bound this term in $\mathbb{L}^1$ by $K\rho N^{-\frac{1}{2}}$. Then, following the remaining arguments in (D.14) - (D.15), we arrive at

$$\frac{1}{2}||\nabla_\theta \tilde{\mathcal{L}}^{i,N}(\theta^i_{\tau_k})|| \leq ||\nabla_\theta \tilde{\mathcal{L}}^{i,N}(\theta^i_{\sigma_{k,\eta}})|| \leq 2||\nabla_\theta \tilde{\mathcal{L}}^{i,N}(\theta^i_{\tau_k})|| + \frac{1}{N^{\frac{1}{2}}}.$$

Using our modified definition of the stopping times $\{\sigma_k\}_{k\geq 1}$, c.f. (G.1) - (G.2), this implies that $\sigma_{k,\eta} \in [\tau_k, \sigma_k]$, which is a contradiction, as in the proof of Lemma D.10. The remainder of the proof goes through unchanged. ■

Lemma D.11†. *Assume that Conditions B.1 - B.2, C.1, D.1 and F.1 hold. Suppose that there are an infinite number of intervals* $[\tau_k, \sigma_k)$. *Then there exists a fixed constant* $\beta = \beta(\kappa) > 0$ *such that, for* k *large enough, almost surely,*

$$\tilde{\mathcal{L}}^{i,N}(\theta^i_{\sigma_k}) - \tilde{\mathcal{L}}^{i,N}(\theta^i_{\tau_k}) \geq \beta \left[1 - \frac{1}{N^{\frac{1}{2}}} \int_{\tau_k}^{\sigma_k} \gamma_s \mathrm{d}s\right].$$

Proof. Comparing to the proof of Lemma D.11, we once more replace $\theta_t^{i,N}$ by θ_t^i throughout. Moreover, instead of (D.16), we now have

$$\begin{aligned}\tilde{\mathcal{L}}^{i,N}(\theta^i_{\sigma_k}) - \tilde{\mathcal{L}}^{i,N}(\theta^i_{\tau_k}) = \int_{\tau_k}^{\sigma_k} \gamma_s ||\nabla_\theta \tilde{\mathcal{L}}^{i,N}(\theta_s^i)||^2 \mathrm{d}s \\ + \int_{\tau_k}^{\sigma_k} \gamma_s \langle \nabla_\theta \tilde{\mathcal{L}}^i(\theta_s^{i,N}), \nabla_\theta \hat{L}^{i,N}(\theta_s^i, \hat{x}_s^N) - \nabla_\theta \tilde{\mathcal{L}}^{i,N}(\theta_s^i)\rangle \mathrm{d}s \\ + \int_{\tau_k}^{\sigma_k} \gamma_s \langle \nabla_\theta \tilde{\mathcal{L}}^{i,N}(\theta_s^{i,N}), \nabla_\theta B(\theta_s^i, \hat{x}_s^i, \mu_s)\mathrm{d}w_s^i\rangle \\ + \int_{\tau_k}^{\sigma_k} \frac{1}{2}\gamma_s^2 \mathrm{Tr}\left[\nabla_\theta B(\theta_s^i, \hat{x}_s^i, \mu_s) \nabla_\theta B(\theta_s^i, \hat{x}_s^i, \mu_s)^T \nabla_\theta^2 \tilde{\mathcal{L}}^{i,N}(\theta_s^i)\mathrm{d}s\right] \\ + \int_{\tau_k}^{\sigma_k} \gamma_s \langle \nabla_\theta \tilde{\mathcal{L}}^{i,N}(\theta_s^{i,N}), \nabla_\theta L(\theta_s^i, x_s^i, \mu_s) - \nabla_\theta \hat{L}^{i,N}(\theta_s^i, \hat{x}_s^N)\rangle \mathrm{d}s\end{aligned}$$

We can deal with the first four terms using the same method as in the proof of Lemma D.11, with similar modifications to those required for the extension of Lemma D.9. We can bound the final term above in $\mathbb{L}^1$ by $\frac{K}{N^{\frac{1}{2}}}\int_{\tau_k}^{\sigma_k}\gamma_s \mathrm{d}s$. Following the arguments in the proof of Lemma D.11, we arrive, as required, at ■

$$\tilde{\mathcal{L}}^{i,N}(\theta^i_{\sigma_k}) - \tilde{\mathcal{L}}^{i,N}(\theta^i_{\tau_k}) \geq \beta\left[1 - \frac{1}{N^{\frac{1}{2}}}\int_{\tau_k}^{\sigma_k}\gamma_s \mathrm{d}s\right].$$

Lemma D.12†. *Assume that Conditions B.1 - B.2, C.1, D.1 and F.1 hold. Suppose that there are an infinite number of intervals $[\tau_k,\sigma_k)$. Then there exists a fixed constant $0<\beta_1<\beta$ such that, for k large enough,*

$$\tilde{\mathcal{L}}^{i,N}(\theta^{i,N}_{\tau_k}) - \tilde{\mathcal{L}}^{i,N}(\theta^{i,N}_{\sigma_{k-1}}) \geq -\beta_1\left[1 - \frac{1}{N^{\frac{1}{2}}}\int_{\sigma_{k-1}}^{\tau_k}\gamma_s \mathrm{d}s\right].$$

Proof. The modifications to the proof of Lemma D.12 are identical to the modifications to the proof of Lemma D.11 (see above). ■

G.2. Proof of Theorem 3.4†.

Proof. The proof of Theorem 3.4† proceeds in much the same way as the proof of Theorem 3.4. To begin, let us consider the parameter update equation in the form

$$\begin{aligned}
\mathrm{d}\theta^i_t &= \gamma_t\nabla_\theta L(\theta^i_t,x^i_t,\mu_t)\mathrm{d}t + \gamma_t\nabla_\theta B(\theta^i_t,x^i_t,\mu_t)\mathrm{d}w^i_t \\
&= \gamma_t\nabla_\theta\tilde{\mathcal{L}}(\theta^i_t) + \gamma_t(\nabla_\theta\tilde{\mathcal{L}}^{i,N}(\theta^i_t) - \nabla_\theta\tilde{\mathcal{L}}(\theta^i_t))\mathrm{d}t + \gamma_t(\nabla_\theta L(\theta^i_t,x^i_t,\mu^i_t) - \nabla_\theta L(\theta^i_t,x^{i,N}_t,\mu^N_t))\mathrm{d}t \\
&\quad + \gamma_t(\nabla_\theta L(\theta^i_t,x^{i,N}_t,\mu^N_t) - \nabla_\theta\tilde{\mathcal{L}}^{i,N}(\theta^i_t))\mathrm{d}t + \gamma_t\nabla_\theta B(\theta^i_t,x^i_t,\mu_t)\mathrm{d}w^i_t.
\end{aligned}$$

Following almost verbatim the arguments in (4.38) - (4.42), we can obtain

$$\begin{aligned}
\text{(G.3)}\qquad \mathbb{E}\left[||Z^i_t||^2\right] &\leq \mathbb{E}\left[\Phi_{1,t}||Z^i_1||^2\right] \\
&+ \mathbb{E}\left[\int_1^t \gamma_s\Phi_{s,t}\langle Z^i_s, \nabla_\theta\tilde{\mathcal{L}}^{i,N}(\theta^i_s) - \nabla_\theta\tilde{\mathcal{L}}(\theta^i_s))\rangle\mathrm{d}s\right] \\
&+ \mathbb{E}\left[\int_1^t \gamma_s\Phi_{s,t}\langle Z^i_s, \nabla_\theta L(\theta^i_s,x^{i,N}_s,\mu^N_s) - \nabla_\theta\tilde{\mathcal{L}}^{i,N}(\theta^i_s))\rangle\mathrm{d}s\right] \\
&+ \mathbb{E}\left[\int_1^t \gamma_s^2\Phi_{s,t}||\nabla_\theta B(\theta^i_s,x^i_s,\mu_s)||^2_F\mathrm{d}s\right] \\
&+ \mathbb{E}\left[\int_1^t \gamma_s\Phi_{s,t}\langle Z^i_s, \nabla_\theta L(\theta^i_s,x^i_s,\mu^i_s) - \nabla_\theta L(\theta^i_s,x^{i,N}_s,\mu^N_s))\rangle\mathrm{d}s\right] \\
\text{(G.4)}\qquad &= \mathbb{E}\left[\bar{\Omega}^{(1)}_{t,i,N}\right] + \mathbb{E}\left[\bar{\Omega}^{(2)}_{t,i,N}\right] + \mathbb{E}\left[\bar{\Omega}^{(3)}_{t,i,N}\right] + \mathbb{E}\left[\bar{\Omega}^{(4)}_{t,i,N}\right] + \mathbb{E}\left[\bar{\Omega}^{(5)}_{t,i,N}\right]
\end{aligned}$$

where we have defined $Z^i_t = \theta^i_t - \theta_0$. It remains to bound each of these terms. We first note that $\bar{\Omega}^{(1)}_{t,i,N},\dots,\bar{\Omega}^{(4)}_{t,i,N}$ in (G.4) are essentially identical to $\Omega^{(1)}_{t,i,N},\dots,\Omega^{(4)}_{t,i,N}$ in (4.42) , up to the fact that $Z^{i,N}_s$ and $\theta^{i,N}_s$ have been replaced by Z^i_s and θ^i_s, respectively. Thus, noting that all relevant bounds apply both to the IPS and the McKean-Vlasov SDE, we can use almost identical arguments to obtain

$$\mathbb{E}\left[\bar{\Omega}^{(1)}_{t,i,N}\right]+\mathbb{E}\left[\bar{\Omega}^{(2)}_{t,i,N}\right]+\mathbb{E}\left[\bar{\Omega}^{(3)}_{t,i,N}\right]+\mathbb{E}\left[\bar{\Omega}^{(4)}_{t,i,N}\right] \leq \bar{K}^{(1)}\gamma_t+\bar{K}^{(2)}\left[\frac{1}{N^{\frac{1}{2}}}\right]+\bar{K}^{(3)}\gamma_t+\bar{K}^{(4)}\gamma_t \tag{G.5}$$

The only term which requires some additional care here is $\bar{\Omega}^{(3)}_{t,i,N}$. In particular, since $\theta_t^{i,N}$ has now been replaced by θ_t^i, when applying Itô's formula to the solution of the relevant Poisson equation, c.f. (4.45), the first, third, and fifth terms have minor modifications.

It remains only to deal with the additional term $\bar{\Omega}^{(5)}_{t,i,N}$. Let $p_1,p_2,q_1,q_2 \in (1,\infty)$, with $\frac{1}{p_1}+\frac{1}{p_2}=\frac{1}{q_1}+\frac{1}{q_2}=1$. Then, using Lemma D.5, and a repeated application of the Hölder inequality, we obtain

$$\begin{aligned}
&\mathbb{E}\left[\langle Z_s^i, \nabla_\theta L(\theta_s^i, x_s^i, \mu_s) - \nabla_\theta L(\theta_s^i, x_s^{i,N}, \mu_s^N)\rangle\right] \\
&\leq K\left[\mathbb{E}\left[||Z_s^i||^{p_1}\right]\right]^{\frac{1}{p_1}}\left[\left[\mathbb{E}\left[||x_s^i - x_s^{i,N}||^{p_2q_1}\right]\right]^{\frac{1}{q_1}}\left[\mathbb{E}\left[\left[1+\cdots+||\mu_s^N([\cdot]^2)||^{\frac{q}{2}}\right]^{p_2q_2}\right]\right]^{\frac{1}{q_2}}\right. \\
&\qquad\left.+\left[\mathbb{E}\left[\left[\mathbb{W}_2(\mu_s,\mu_s^N)\right]^{p_2q_1}\right]^{\frac{1}{q_1}}\left[\mathbb{E}\left[\left[1+\cdots+||\mu_s^N([\cdot]^2)||^{\frac{q}{2}}\right]^{p_2q_2}\right]\right]^{\frac{1}{q_2}}\right]^{\frac{1}{p_2}}
\end{aligned}$$

Let $p_1=3$, $p_2=\frac{3}{2}$, $q_1=\frac{4}{3}$, and $q_2=4$. Then $p_2q_1=2$ and $p_2q_2=6$, and the previous inequality yields

$$\mathbb{E}\left[\langle Z_s^i, \nabla_\theta L(\theta_s^i, x_s^i, \mu_s) - \nabla_\theta L(\theta_s^i, x_s^{i,N}, \mu_s^N)\rangle\right] \leq K\left[\frac{1}{N^{\frac{3}{4}}}+\frac{1}{N^{\frac{3}{4}}}\right]^{\frac{2}{3}} \leq K\left[\frac{1}{N^{\frac{1}{2}}}\right],$$

where we have also used Lemma 4.1 (uniform moment bounds for Z_s^i), Proposition A.2 (uniform moment bounds for the McKean-Vlasov SDE and the IPS), Proposition A.5 (uniform-in-time propagation of chaos), and [30, Theorem 1] (bound on the Wasserstein distance between the law of the McKean-Vlasov SDE and the empirical law of the IPS). We thus obtain

$$\mathbb{E}\left[\bar{\Omega}^{(5)}_{t,i,N}\right] \leq K\left[\frac{1}{N^{\frac{1}{2}}}\right]\int_1^t \gamma_s\Phi_{s,t}\mathrm{d}s \leq \bar{K}^{(5)}\left[\frac{1}{N^{\frac{1}{2}}}\right]. \tag{G.6}$$

Substituting (G.5) and (G.6) into (G.3) - (G.4), setting $K_1^\dagger=\max\{\bar{K}^{(1)},\bar{K}^{(3)}\}$, $K_2^\dagger=\bar{K}^{(4)}$, and taking the limit as $N\to\infty$, we finally obtain

$$\mathbb{E}\left[||\theta_t-\theta_0||^2\right] \leq (K_1^\dagger+K_2^\dagger)\gamma_t.$$

■

Appendix H. Proof of Theorem 3.3‡ and Theorem 3.4‡.

H.1. Proof of Theorem 3.3‡.

Proof. This theorem is identical to Lemma 3.4.D, which we have already proved in Section 4.3.4. ■

H.2. Proof of Theorem 3.4‡.

Proof. The proof of this result is a simplified version of the proof of Theorem 3.4. We now assume that $\tilde{\mathcal{L}}^{i,N}(\cdot)$ is strongly concave (Condition H.1') rather than that $\tilde{\mathcal{L}}(\cdot)$ is strongly concave (Condition H.1). We thus do not require the decomposition in (4.37), and can consider the decomposition in (4.36) directly, viz

$$\begin{aligned}\mathrm{d}\theta_t^{i,N} &= \gamma_t \nabla_\theta \tilde{\mathcal{L}}^{i,N}(\theta_t^{i,N})\mathrm{d}t + \gamma_t\big(\nabla_\theta L(\theta_t^{i,N}, x_t^{i,N}, \mu_t^N) - \nabla_\theta \tilde{\mathcal{L}}^{i,N}(\theta_t^{i,N})\big)\mathrm{d}t \\ &\quad + \gamma_t \nabla_\theta B(\theta_t^{i,N}, x_t^{i,N}, \mu_t^N)\mathrm{d}w_t^i.\end{aligned}$$

The arguments in (4.36) - (4.42) are essentially unchanged, with two minor modifications to (4.42). In particular, the second term is identically zero, while $\Phi_{s,t} = \exp(-2\eta\int_s^t \gamma_u \mathrm{d}u)$ is replaced by $\Phi_{s,t}^{i,N} = \exp(-2\eta^{i,N}\int_s^t \gamma_u \mathrm{d}u)$ in the remaining terms, where $\eta^{i,N}$ is the concavity constant for $\tilde{\mathcal{L}}^{i,N}(\cdot)$. The remainder of the proof goes through verbatim. The result is that the $N^{-\frac{1}{2}}$ is absent from the final bound (this arises from the second term in (4.42)), while the remaining terms are the same up to possibly different constants. This result, and its proof, can be seen as a modified version of their counterparts in [77, Proposition 2.13]. ■

Appendix I. Verification of Conditions for the Linear Mean Field Model. In this Appendix, we verify explicitly that the conditions of Theorems 3.1 - 3.4 hold for the linear one-dimensional mean field model studied in Section 5.1.

I.1. Main Assumptions.

Assumption A.1. This condition follows directly from Proposition A.2, which can be applied once we have verified Assumptions B.1 - B.2 and D.1 (see below).

Assumption B.1. For this model, we have $b(\theta,\cdot):\mathbb{R}\to\mathbb{R}$, with $b(\theta,x) = -\theta_1 x$. This function is Lipschitz continuous with constant θ_1, and satisfies $\langle x-x', b(\theta,x)-b(\theta,x')\rangle = \langle x-x', -\theta_1(x-x')\rangle = -\theta_1||x-x'||^2$. This verifies Condition B.1, provided $\theta_1 > 0$.

Assumption B.2. For this model, we have $\phi(\theta,\cdot,\cdot):\mathbb{R}\times\mathbb{R}\to\mathbb{R}$, with $\phi(\theta,x,y) = -\theta_2(x-y)$. This function is twice differentiable with respect to both of its arguments, and is globally Lipschitz with constant $|\theta_2|$. This verifies Condition B.2, provided $|\theta_2| \leq \frac{1}{2}\theta_1$.

Assumption C.1. The functions $b:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ and $\phi:\mathbb{R}\times\mathbb{R}\times\mathbb{R}$ are infinitely differentiable with respect to all of their arguments. Moreover, we have that $||\nabla_\theta^i b(\theta,x)|| = ||\nabla_\theta^i\phi(\theta,x,y)|| = 0$ for $i = 1,2,3$. Finally, $||b(\theta,x)-b(\theta',x)|| \leq ||\theta-\theta'||||x||$, $||\phi(\theta,x,y)-\phi(\theta',x,y)|| \leq ||\theta-\theta'||(||x||+||y||)$.This verifies Condition C.1.

Assumption D.1. We assume that $x_0\in\mathbb{R}$ and thus this condition is trivially satisfied. We note that this condition would also be satisfied if $x_0 \sim \mathcal{N}(\mu,\sigma^2)$ for some $\mu,\sigma\in\mathbb{R}$.

I.2. Offline Parameter Estimation.

Assumption E.1. For this model, we have $B(\theta,x,\mu_s) = -\theta_1 x - \theta_2(x-\mathbb{E}[x_s])$ and thus $G(\theta,x,\mu_s) = -(\theta_1-\theta_{1,0})x - (\theta_2-\theta_{2,0})(x-\mathbb{E}[x_s])$, where $\theta_0 = (\theta_{1,0},\theta_{2,0})\in\mathbb{R}^2$ denotes the true value of the parameter. We can then compute

(I.1)
$$L(\theta, x, \mu_s) = -\frac{1}{2}\left[(\theta_1-\theta_{1,0})x + (\theta_2-\theta_{2,0})(x-\mathbb{E}[x_s])\right]^2$$
$$= -\frac{1}{2}\left[(\theta_1-\theta_{1,0})^2x^2 + 2(\theta_1-\theta_{1,0})(\theta_2-\theta_{2,0})x(x-\mathbb{E}[x_s]) + (\theta_2-\theta_{2,0})^2(x-\mathbb{E}[x_s])^2\right]$$

and thus

$$\begin{aligned}
m_t(\theta) &= \int_0^t \int_{\mathbb{R}^d} L(\theta, x, \mu_s)\mu_s(\mathrm{d}x)\mathrm{d}s \\
&= -\frac{1}{2}\int_0^t (\theta_1-\theta_{1,0})^2\mathbb{E}\left[x_s^2\right] + 2(\theta_1-\theta_{1,0})(\theta_2-\theta_{2,0})\mathrm{Var}(x_s) + (\theta_2-\theta_{2,0})^2\mathrm{Var}(x_s)\mathrm{d}s \\
&= -\frac{1}{2}\int_0^t \left[(\theta_1-\theta_{1,0}) + (\theta_2-\theta_{2,0})\right]^2\mathrm{Var}(x_s) + (\theta_1-\theta_{1,0})^2\mathbb{E}\left[x_s\right]^2\mathrm{d}s.
\end{aligned}$$

Let $\mathbb{E}\left[x_0\right] = \mu_0$ and $\mathrm{Var}(x_0) = \sigma_0^2 > 0$. It is then relatively straightforward to compute (e.g., [44]), defining $\gamma(\theta) = -2(\theta_1+\theta_2)$,

$$\mathbb{E}_\theta\left[x_s\right]^2 = \mu_0^2 e^{-2\theta_1 s} \tag{I.2}$$

$$\mathrm{Var}_\theta(x_s) = \sigma_0^2 e^{\gamma(\theta)s} + \frac{e^{\gamma(\theta)s}-1}{\gamma(\theta)} \tag{I.3}$$

We thus have $\inf_{s\geq 0}\mathbb{E}\left[x_s\right]^2 > 0$ provided $\mu_0 \neq 0$, and $\inf_{s\geq 0}\mathrm{Var}(x_s) > 0$. It follows that $m_t(\theta) \leq 0$, with equality if and only if $\theta_1 = \theta_{1,0}$ and $\theta_2 = \theta_{2,0}$.[7] That is, equivalently, $\inf_{||\theta-\theta_0||>\delta} m_t(\theta) < 0$ a.s. $\forall \delta > 0$. This verifies Condition E.1.

Assumption E.2. For this model, as noted above, we have $B(\theta, x, \mu_s) = -\theta_1 x - \theta_2(x - \mathbb{E}[x_s])$, and thus $\nabla_\theta B(\theta, x, \mu_s) = [-x, -(x-\mathbb{E}[x_s])]$. It follows that

$$\begin{aligned}
I_t(\theta) &= \int_0^t \int_{\mathbb{R}^d} \nabla_\theta B(\theta, x, \mu_s) \otimes \nabla_\theta B(\theta, x, \mu_s)\mu_s(\mathrm{d}x)\mathrm{d}s \\
&= \int_0^t \begin{pmatrix} \mathbb{E}_\theta[x_s^2] & \mathrm{Var}_\theta(x_s) \\ \mathrm{Var}_\theta(x_s) & \mathrm{Var}_\theta(x_s)\end{pmatrix}\mathrm{d}s = \begin{pmatrix} D_t(\theta) & C_t(\theta) \\ C_t(\theta) & C_t(\theta)\end{pmatrix}
\end{aligned}$$

where, using (I.2) - (I.3), and integrating, we can obtain $C_t(\theta)$ and $D_t(\theta)$ explicitly as

$$\begin{aligned}
C_t(\theta) &= \frac{1}{\gamma^2(\theta)}(e^{\gamma(\theta)t}-1) - \frac{t}{\gamma(\theta)} + \frac{\sigma_0^2}{\gamma(\theta)}(e^{\gamma(\theta)t}-1), \\
D_t(\theta) &= \frac{1}{\gamma^2(\theta)}(e^{\gamma(\theta)t}-1) - \frac{t}{\gamma(\theta)} + \frac{\sigma_0^2}{\gamma(\theta)}(e^{\gamma(\theta)t}-1) - \frac{\mu_0^2}{2\theta_1}(e^{-2\theta_1 t}-1),
\end{aligned}$$

[7] We remark that, if $\mu_0 = 0$, then $\mathbb{E}\left[x_s\right]^2 = 0$ for all $s \geq 0$. Thus, while we certainly still have $m_t(\theta) \leq 0$, we now have equality whenever $(\theta_1-\theta_{1,0}) + (\theta_2-\theta_{2,0}) = 0$. That is, whenever $\theta_1+\theta_2 = \theta_{1,0}+\theta_{2,0}$. Thus, in this case, θ_1 and θ_2 are no longer jointly identifiable.

It remains to show that this matrix is positive-definite, and that for all $\lambda = (\lambda_1, \lambda_2) \in \mathbb{R}^2$, $\lambda^T I_t(\theta_0)\lambda$ is increasing as a function of t. Observe that

$$\begin{aligned}\lambda^T I_t(\theta)\lambda &= \lambda_1^2 D_t(\theta) + 2\lambda_1\lambda_2 C_t(\theta) + \lambda_2^2 C_t(\theta) \\ &= \lambda_1^2(D_t(\theta) - C_t(\theta)) + (\lambda_1+\lambda_2)^2 C_t(\theta) > 0\end{aligned}$$

where, to obtain the final inequality, we have use the fact that $C_t(\theta) = \int_0^t \mathrm{Var}_\theta(x_s)\mathrm{d}s > 0$ and $D_t(\theta) - C_t(\theta) = \int_0^t [\mathbb{E}_\theta[x_s^2] - \mathrm{Var}_\theta(x_s)]\mathrm{d}s = \int_0^t \mathbb{E}_\theta\left[x_s\right]^2 \mathrm{d}s > 0$ for all $s \geq 0$. Thus, $I_t(\theta)$ is positive definite. Finally, it is straightforward to see that $\lambda^T I_t(\theta_0)\lambda$ is increasing as a function of t, and that $I_t(0) = 0$. This verifies Condition E.2.

I.3. Online Parameter Estimation.

Assumption F.1 - F.2. Conditions F.1 - F.2 are satisfied by $\gamma_t = \min\{\gamma^0, \gamma^0 t^{-\delta}\}$, where $\gamma^0 \in [0,\infty)$, and $\delta \in (\frac{1}{2}, 1]$. The straightforward calculations are omitted.

Assumption G.1. For this model, we recall from (I.1) that $L(\theta, x, \mu) = -\frac{1}{2}[(\theta_1 - \theta_{1,0})x + (\theta_2 - \theta_{2,0})(x - \mathbb{E}_\mu[x])]^2$. For simplicity, let us focus on the 'pure interaction' case, in which $\theta_1 = \theta_{1,0} = 0$. In this case, we have $L(\theta, x, \mu) = -\frac{1}{2}(\theta_2 - \theta_{2,0})^2(x - \mathbb{E}_\mu[x])^2$, and thus $\nabla_\theta L(\theta, x, \mu) = -(\theta_2 - \theta_{2,0})(x - \mathbb{E}_\mu[x])^2$. It is then straightforward to compute

$$\langle \nabla_\theta L(\theta, x, \mu), \theta\rangle = -\theta_2(\theta_2 - \theta_{2,0})(x - \mathbb{E}_\mu[x])^2 = -\big(1 - \frac{\theta_{2,0}}{\theta_2}\big)(x - \mathbb{E}_\mu[x])^2\theta_2^2.$$

It follows that, for all $||\theta_2|| \geq ||\theta_{2,0}||$, we have

$$\langle \nabla_\theta L(\theta, x, \mu), \theta\rangle \leq -2(x - \mathbb{E}_\mu[x])^2||\theta_2||^2 = -\kappa(x,\mu)||\theta_2||^2.$$

This verifies Condition G.1.

Assumption G.2. For this model, we recall that $\nabla_\theta B(\theta, x, \mu) = [-x, -(x - \mathbb{E}_\mu[x])]$. We thus have

$$\tau(\theta, x, \mu) = \big\langle \nabla_\theta B(\theta, x, \mu)\nabla_\theta B^T(\theta, x, \mu)\frac{\theta}{||\theta||}, \frac{\theta}{||\theta||}\big\rangle^{\frac{1}{2}} = \left[x^2 + (x - \mathbb{E}_\mu(x))^2\right]^{\frac{1}{2}}.$$

Thus implies, in particular, that $|\tau(\theta, x, \mu) - \tau(\theta', x, \mu)| = 0$, which verifies Condition G.2.

Assumption H.1 - H.2. For this model, defining $m_\infty = \mathbb{E}_{\mu_\infty}[\mathrm{d}x]$, the unique invariant measure $\mu_\infty(\cdot)$ can be obtained as (e.g., [58])

$$\begin{aligned}\mu_\infty(\mathrm{d}x) &= \frac{1}{Z(\mu_\infty)}e^{-\left(\theta_{1,0}x^2 + \theta_{2,0}(x - m_\infty)^2\right)}\mathrm{d}x \\ Z(\mu_\infty) &= \int_{\mathbb{R}} e^{-\left(\theta_{1,0}y^2 + \theta_{2,0}(y - m_\infty)^2\right)}\mathrm{d}y\end{aligned}$$

We can thus compute the asymptotic log-likelihood, up to a constant of proportionality, as

$$\begin{aligned}\tilde{\mathcal{L}}(\theta) &= \int_{\mathbb{R}} L(\theta, x, \mu_\infty)\mu_\infty(\mathrm{d}x) \\ &= -\frac{1}{2}\frac{1}{Z_\infty(\mu_\infty)}\int_{\mathbb{R}}[(\theta_1-\theta_{1,0})x + (\theta_2-\theta_{2,0})(x-m_\infty)]^2 e^{-\left(\theta_{1,0}x^2+\theta_{2,0}(x-m_\infty)^2\right)}\mathrm{d}x \\ &\propto -\frac{1}{2}\left[A_{1,1}(\theta_1-\theta_{1,0})^2 + 2A_{1,2}(\theta_1-\theta_{1,0})(\theta_2-\theta_{2,0}) + A_{2,2}(\theta_2-\theta_{2,0})^2\right]\end{aligned}$$

where in the final line we have computed (omitting the tedious calculations need to compute the relevant Gaussian integrals) the constants

$$A_{1,1} = \theta_{1,0} + \theta_{2,0} + 2m_\infty^2\theta_{2,0}^2 \tag{I.4}$$

$$A_{1,2} = \theta_{1,0} + \theta_{2,0} - 2m_\infty^2\theta_{1,0}\theta_{2,0}$$

$$A_{2,2} = \theta_{1,0} + \theta_{2,0} + 2m_\infty^2\theta_{1,0}^2. \tag{I.5}$$

It follows straightforwardly that the Hessian is given by

$$\nabla_\theta^2\tilde{\mathcal{L}}(\theta) = -\begin{pmatrix} A_{1,1} & A_{1,2} \\ A_{1,2} & A_{2,2}\end{pmatrix}$$

In order to establish global strong concavity, we are required to show that, for all $\theta \in \mathbb{R}^2$, $\nabla_\theta^2\tilde{\mathcal{L}}(\theta) \preceq -\eta I$ for some $\eta > 0$, that is, the matrix $\nabla_\theta^2\tilde{\mathcal{L}}(\theta) + \eta I$ is negative semi-definite. We will show, equivalently, that $\mathrm{trace}(\nabla_\theta^2\tilde{\mathcal{L}}(\theta) + \eta I) < 0$ and $\det(\nabla_\theta^2\tilde{\mathcal{L}}(\theta) + \eta I) > 0$. In particular, we will demonstrate that this holds whenever the true parameter θ_0 satisfies $\theta_{1,0}+\theta_{2,0} > \eta > 0$, where η can be chosen arbitrarily close to zero. We begin with the observation that

$$\begin{aligned}\mathrm{trace}(\nabla_\theta^2\tilde{\mathcal{L}}(\theta) + \eta I) &= 2\eta - (A_{1,1} + A_{2,2}) \\ &= 2\left[\eta - (\theta_{1,0}+\theta_{2,0})\right] - 2m_\infty^2(\theta_{1,0}^2 + \theta_{2,0}^2) \\ &< -2m_\infty^2(\theta_{1,0}^2+\theta_{2,0}^2) < 0,\end{aligned}$$

as required, where in the penultimate line we have used our condition on θ_0. We now turn our attention to

$$\begin{aligned}\det(\nabla_\theta^2\tilde{\mathcal{L}}(\theta) + \eta I) &= (\eta - A_{1,1})(\eta - A_{2,2}) - A_{1,2}^2 \\ &= \eta^2 - (A_{1,1}+A_{2,2})\eta + (A_{1,1}A_{2,2} - A_{1,2}^2) > 0\end{aligned}$$

for $\eta \in (-\infty, \eta_-) \cup (\eta_+, \infty)$, where

$$\eta_\pm = \frac{(A_{1,1}+A_{1,2}) \pm \sqrt{(A_{1,1}+A_{2,2})^2 - 4(A_{1,1}A_{2,2} - A_{1,2}^2)}}{2}.$$

We first remark that both of these roots are real. Indeed a simple calculation shows that the discriminant is positive: $(A_{1,1}+A_{2,2})^2 - 4(A_{1,1}A_{2,2}-A_{1,2}^2) = (A_{1,1}-A_{2,2})^2 + 2A_{1,2}^2 > 0$. We now claim that η_- is positive. To see this, we can compute, using (I.4) - (I.5),

$$A_{1,1}A_{2,2} - A_{1,2}^2 = 2m_\infty^2\left[\theta_{1,0}+\theta_{2,0}\right]\left[\theta_{1,0}+\theta_{2,0}\right]^2 > 2m_\infty^2\eta\left[\theta_{1,0}+\theta_{2,0}\right]^2 > 0,$$

where, to obtain the inequality, we have once more used our condition on θ_0. It follows straightforwardly that $(A_{1,1}+A_{2,2})^2 - 4(A_{1,1}A_{2,2} - A_{1,2}^2) < (A_{1,1}+A_{2,2})^2$, and so $\eta_- > 0$. We thus have, for arbitrary $\eta \in (0, \eta_-)$,

$$\det(\nabla_\theta^2 \tilde{\mathcal{L}}(\theta) + \eta I) > 0.$$

We thus have shown that, for all $\theta \in \mathbb{R}^2$, $\nabla_\theta^2 \tilde{\mathcal{L}}(\theta) \preceq -\eta I$ for some $\eta > 0$. This verifies Condition H.1. For the one-dimensional linear mean field model, one can verify Condition H.1' using almost identical arguments. The details are omitted.

REFERENCES

[1] J. Baladron, D. Fasoli, O. Faugeras, and J. Touboul, *Mean-field description and propagation of chaos in networks of Hodgkin-Huxley and FitzHugh-Nagumo neurons*, The Journal of Mathematical Neuroscience, 2 (2012), p. 10, https://doi.org/10.1186/2190-8567-2-10.

[2] F. Ball and D. Sirl, *Stochastic SIR in Structured Populations*, in Stochastic Epidemic Models with Inference, Part II, T. Britton and E. Pardoux, eds., Springer, 2020, pp. 123–240.

[3] M. Bauer, T. Meyer-Brandis, and F. Proske, *Strong solutions of mean-field stochastic differential equations with irregular drift*, Electronic Journal of Probability, 23 (2018), pp. 1–35, https://doi.org/10.1214/18-EJP259.

[4] S. Benachour, B. Roynette, D. Talay, and P. Vallois, *Nonlinear self-stabilizing processes – I Existence, invariant probability, propagation of chaos*, Stochastic Processes and their Applications, 75 (1998), pp. 173–201, https://doi.org/10.1016/S0304-4149(98)00018-0.

[5] D. Benedetto, E. Caglioti, and M. Pulvirenti, *A kinetic equation for granular media*, Mathematical Modelling and Numerical Analysis, 31 (1997), pp. 615–641, https://doi.org/10.1051/m2an/1997310506151.

[6] D. P. Bertsekas and J. N. Tsitsiklis, *Gradient Convergence in Gradient methods with Errors*, SIAM Journal on Optimization, 10 (2000), pp. 627–642, https://doi.org/10.1137/S1052623497331063.

[7] T. Bhudisaksang and Á. Cartea, *Online Drift Estimation for Jump-Diffusion Processes*, SSRN, (2020), https://doi.org/10.2139/ssrn.3540252.

[8] J. P. N. Bishwal, *Parameter Estimation in Stochastic Differential Equations*, Springer, Berlin, Heidelberg, 2008, https://doi.org/10.1007/978-3-540-74448-1.

[9] J. P. N. Bishwal, *Estimation in Interacting Diffusions: Continuous and Discrete Sampling*, Applied Mathematics, 2 (2011), pp. 1154–1158, https://doi.org/10.4236/am.2011.29160.

[10] F. Bolley, I. Gentil, and A. Guillin, *Uniform Convergence to Equilibrium for Granular Media*, Archive for Rational Mechanics and Analysis, 208 (2013), pp. 429–445, https://doi.org/10.1007/s00205-012-0599-z.

[11] M. Bongini, M. Fornasier, M. Hansen, and M. Maggioni, *Inferring interaction rules from observations of evolutive systems I: The variational approach*, Mathematical Models and Methods in Applied Sciences, 27 (2017), pp. 909–951, https://doi.org/10.1142/S0218202517500208.

[12] V. Borkar and A. Bagchi, *Parameter estimation in continuous-time stochastic processes*, Stochastics, 8 (1982), pp. 193–212, https://doi.org/10.1080/17442508208833238.

[13] C. Brugna and G. Toscani, *Kinetic models of opinion formation in the presence of personal conviction.*, Physical review. E, Statistical, nonlinear, and soft matter physics, 92 (2015), p. 52818, https://doi.org/10.1103/PhysRevE.92.052818.

[14] R. Buckdahn, J. Li, and J. Ma, *A mean-field stochastic control problem with partial observations*, The Annals of Applied Probability, 27 (2017), pp. 3201–3245, https://doi.org/10.1214/17-AAP1280.

[15] M. Burger, V. Capasso, and D. Morale, *On an aggregation model with long and short range interactions*, Nonlinear Analysis: Real World Applications, 8 (2007), pp. 939–958, https://doi.org/10.1016/j.nonrwa.2006.04.002.

[16] A. G. Bustos, *Approximations des distributions d'equilibre de certains systems stochastiques avec interactions McKean-Vlasov*, phd, Universite de Nice-Sophia Antipolis, 2008.

[17] C. Canuto, F. Fagnani, and P. Tilli, *An Eulerian Approach to the Analysis of Krause's Consensus Models*, SIAM Journal on Control and Optimization, 50 (2012), pp. 243–265, https://doi.org/10.1137/100793177.
[18] R. Carmona and F. Delarue, *Probabilistic Theory of Mean Field Games with Applications I*, Springer International Publishing, 2018.
[19] J. Carrillo, R. McCann, and C. Villani, *Kinetic equilibration rates for granular media and related equations: entropy dissipation and mass transportation estimates.*, Revista Matematica Iberoamericana, 19 (2003), pp. 971–1018, https://doi.org/10.4171/RMI/376.
[20] J. A. Carrillo, R. J. McCann, and C. D. Villani, *Contractions in the 2-Wasserstein length space and thermalization of granular media*, Archive for Rational Mechanics and Analysis, 179 (2006), pp. 217–263, https://doi.org/10.1007/s00205-005-0386-1.
[21] P. Cattiaux, A. Guillin, and F. Malrieu, *Probabilistic approach for granular media equations in the non-uniformly convex case*, Probability Theory and Related Fields, 140 (2008), pp. 19–40, https://doi.org/10.1007/s00440-007-0056-3.
[22] P. E. Chaudru de Raynal, *Strong well posedness of McKean–Vlasov stochastic differential equations with Hölder drift*, Stochastic Processes and their Applications, 130 (2020), pp. 79–107, https://doi.org/10.1016/j.spa.2019.01.006.
[23] B. Chazelle, Q. Jiu, Q. Li, and C. Wang, *Well-posedness of the limiting equation of a noisy consensus model in opinion dynamics*, Journal of Differential Equations, 263 (2017), pp. 365–397, https://doi.org/10.1016/j.jde.2017.02.036.
[24] X. Chen, *Maximum Likelihood Estimation of Potential Energy in Interacting Particle Systems From Single Trajectory Data*, arXiv preprint, (2020), https://arxiv.org/abs/2007.11048.
[25] F. Coppini, *Weakly interacting diffusions on graphs*, phd thesis, Universite de Paris, 2020.
[26] D. Crisan and J. Xiong, *Approximate McKean–Vlasov representations for a class of SPDEs*, Stochastics, 82 (2010), pp. 53–68, https://doi.org/10.1080/17442500902723575.
[27] J. L. Doob, *Stochastic Processes*, Wiley, New York, 1953.
[28] A. Durmus, A. Eberle, A. Guillin, and R. Zimmer, *An elementary approach to uniform in time propagation of chaos*, Proceedings of the American Mathematical Society, 148 (2020), pp. 5387–5398, https://doi.org/10.1090/proc/14612.
[29] A. Eberle, A. Guillin, and R. Zimmer, *Quantitative Harris-type theorems for diffusions and McKean-Vlasov processes*, Transactions of the American Mathematical Society, 371 (2019), pp. 7135–7173, https://doi.org/10.1090/tran/7576.
[30] N. Fournier and A. Guillin, *On the rate of convergence in Wasserstein distance of the empirical measure*, Probability Theory and Related Fields, 162 (2015), pp. 707–738, https://doi.org/10.1007/s00440-014-0583-7.
[31] T. Funaki, *A certain class of diffusion processes associated with nonlinear parabolic equations*, Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete, 67 (1984), pp. 331–348, https://doi.org/10.1007/BF00535008.
[32] V. Genon-Catalot and J. Jacod, *Estimation of the Diffusion Coefficient for Diffusion Processes: Random Sampling*, Scandinavian Journal of Statistics, 21 (1994), pp. 193–221.
[33] V. Genon-Catalot and C. Laredo, *Parametric Inference for Small Variance and Long Time Horizon McKean-Vlasov Diffusion Models*. 2021, https://hal.archives-ouvertes.fr/hal-03095560.
[34] V. Genon-Catalot and C. Laredo, *Probabilistic Properties and Parametric Inference of Small Variance Nonlinear Self-Stabilising Stochastic Differential Equations*. 2021, https://hal.archives-ouvertes.fr/hal-02955171.
[35] K. Giesecke, G. Schwenkler, and J. A. Sirignano, *Inference for large financial systems*, Mathematical Finance, 30 (2020), pp. 3–46, https://doi.org/10.1111/mafi.12222.
[36] S. N. Gomes, A. M. Stuart, and M.-T. Wolfram, *Parameter estimation for macroscopic pedestrian dynamics models from microscopic data*, arXiv preprint, (2021), https://arxiv.org/abs/1809.08046.
[37] S. Herrmann and J. Tugaut, *Non-uniqueness of stationary measures for self-stabilizing processes*, Stochastic Processes and their Applications, 120 (2010), pp. 1215–1246, https://doi.org/10.1016/j.spa.2010.03.009.
[38] S. Herrmann and J. Tugaut, *Self-stabilizing processes: uniqueness problem for stationary measures and convergence rate in the small-noise limit*, ESAIM: Probability and Statistics, 16 (2012), pp. 277–305,

https://doi.org/10.1051/ps/2011152.
[39] C. C. Heyde, *A Quasi-Likelihood Approach to Estimating Parameters in Diffusion-Type Processes*, Journal of Applied Probability, 31 (1994), pp. 283–290, https://doi.org/10.2307/3214962.
[40] X. Huang and F.-Y. Wang, *Distribution dependent SDEs with singular coefficients*, Stochastic Processes and their Applications, 129 (2019), pp. 4747–4770, https://doi.org/10.1016/j.spa.2018.12.012.
[41] N. Ikeda and S. Watanabe, *A comparison theorem for solutions of stochastic differential equations and its applications*, Osaka Journal of Mathematics, 14 (1977), pp. 619–633, https://doi.org/ojm/1200770674.
[42] B. Jourdain, S. Meleard, and W. A. Woyczynski, *Nonlinear SDEs driven by Levy processes and related PDEs*, Alea, 4 (2008), pp. 1–29.
[43] M. Kac, *Foundations of Kinetic Theory*, in Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, Volume 3: Contributions to Astronomy and Physics, Third Berkeley Symposium on Mathematical Statistics and Probability, Berkeley, Calif., 1956, University of California Press, pp. 171–197.
[44] R. A. Kasonga, *Maximum Likelihood Theory for Large Interacting Systems*, SIAM Journal on Applied Mathematics, 50 (1990), pp. 865–875, https://doi.org/10.1137/0150050.
[45] R. Khasminskii, *Stochastic Stability of Differential Equations*, Springer, Berlin, Heidelberg, 2 ed., 2012, https://doi.org/10.1007/978-3-642-23280-0.
[46] Y. A. Kutoyants, *Parameter estimation for stochastic processes*, Heldermann, 1984.
[47] D. Levanony, A. Shwartz, and O. Zeitouni, *Recursive identification in continuous-time stochastic processes*, Stochastic Processes and their Applications, 49 (1994), pp. 245–275, https://doi.org/10.1016/0304-4149(94)90137-6.
[48] Z. Li, F. Lu, M. Maggioni, S. Tang, and C. Zhang, *On the identifiability of interaction functions in systems of interacting particles*, Stochastic Processes and their Applications, 132 (2021), pp. 135–163, https://doi.org/10.1016/j.spa.2020.10.005.
[49] R. S. Liptser and A. N. Shiryaev, *Statistics of Random Processes*, Springer, Berlin, Heidelberg, 2 ed., 2001, https://doi.org/10.1007/978-3-662-13043-8.
[50] M. Liu and H. Qiao, *Parameter Estimation of Path-Dependent McKean-Vlasov Stochastic Differential Equations*, arXiv preprint, (2020), https://arxiv.org/abs/2004.09580.
[51] Q. Liu, *Stein Variational Gradient Descent as Gradient Flow*, in Proceedings of the 31st International Conference on Neural Information Processing Systems, NIPS'17, Red Hook, NY, USA, 2017, pp. 3118–3126.
[52] W. Liu, L. Wu, and C. Zhang, *Long-time behaviors of mean-field interacting particle systems related to McKean-Vlasov equations*, arXiv preprint, (2020), https://arxiv.org/abs/2007.09462.
[53] L. Ljung, *On The Consistency of Prediction Error Identification Methods*, in System Identification Advances and Case Studies, R. K. Mehra, D. G. B. T. M. i. S. Lainiotis, and Engineering, eds., vol. 126, Elsevier, 1976, pp. 121–164, https://doi.org/https://doi.org/10.1016/S0076-5392(08)60871-1.
[54] F. Lu, M. Maggioni, and S. Tang, *Learning interaction kernels in stochastic systems of interacting particles from multiple trajectories*, arXiv preprint, (2020), https://arxiv.org/abs/2007.15174.
[55] F. Lu, M. Maggioni, and S. Tang, *Learning interaction kernels in heterogeneous systems of agents from multiple trajectories*, Journal of Machine Learning Research, 22 (2021), pp. 1–67.
[56] F. Lu, M. Zhong, S. Tang, and M. Maggioni, *Nonparametric inference of interaction laws in systems of agents from trajectory data*, Proceedings of the National Academy of Sciences, 116 (2019), pp. 14424 LP – 14433, https://doi.org/10.1073/pnas.1822012116.
[57] L. D. Maestra and M. Hoffmann, *Nonparametric Estimaton for Interacting Particle Systems: McKean-Vlasov Models*, arXiv preprint, (2020), https://arxiv.org/abs/2011.03762.
[58] F. Malrieu, *Logarithmic Sobolev inequalities for some nonlinear PDE's*, Stochastic Processes and their Applications, 95 (2001), pp. 109–132, https://doi.org/10.1016/S0304-4149(01)00095-3.
[59] F. Malrieu, *Convergence to equilibrium for granular media equations and their Euler schemes*, Annals of Applied Probability, 13 (2003), pp. 540–560, https://doi.org/10.1214/aoap/1050689593.
[60] H. P. McKean, *A class of Markov processes associated with nonlinear parabolic equations*, Proceedings of the National Academy of Sciences of the United States of America, 56 (1966), pp. 1907–1911, https://doi.org/10.1073/pnas.56.6.1907.
[61] H. P. McKean, *Propagation of chaos for a class of non-linear parabolic equations*, vol. 7, Catholic

University, Washington, D.C., 1967.
[62] H. P. McKean, *Stochastic Integrals*, Academic Press, New York, 1969.
[63] S. Méléard, *Asymptotic behaviour of some interacting particle systems; McKean-Vlasov and Boltzmann models*, Probabilistic Models for Nonlinear Partial Differential Equations: Lectures (Montecatini Terme, 1995), 1627 (1996), pp. 42–95, https://doi.org/10.1007/BFb0093177.
[64] Y. Mishura and A. Y. Veretennikov, *Existence and uniqueness theorems for solutions of McKean–Vlasov stochastic equations*, arXiv preprint, (2020), https://arxiv.org/abs/1603.02212.
[65] S. Motsch and E. Tadmor, *Heterophilious Dynamics Enhances Consensus*, SIAM Review, 56 (2014), pp. 577–621, https://doi.org/10.1137/120901866.
[66] K. Oelschlager, *A Martingale Approach to the Law of Large Numbers for Weakly Interacting Stochastic Processes*, The Annals of Probability, 12 (1984), pp. 458–479, https://doi.org/https://doi.org/10.1214/aop/1176993301.
[67] E. Pardoux and A. Y. Veretennikov, *On Poisson equation and diffusion approximation 2*, Annals of Probability, 31 (2003), pp. 1166–1192, https://doi.org/10.1214/aop/1015345596.
[68] E. Pardoux and Y. Veretennikov, *On the Poisson Equation and Diffusion Approximation. I*, Annals of Probability, 29 (2001), pp. 1061–1085, https://doi.org/10.1214/aop/1015345596.
[69] B. L. S. P. Rao and H. Rubin, *Asymptotic Theory of Estimation in Nonlinear Stochastic Differential Equations*, Sankhyā: The Indian Journal of Statistics, Series A, 43 (1981), pp. 170–189.
[70] R. Rebolledo, *Central limit theorems for local martingales*, Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete, 51 (1980), pp. 269–286, https://doi.org/10.1007/BF00587353.
[71] D. Revuz and M. Yor, *Continuous Martingales and Brownian Motion*, Springer-Verlag, Berlin, Heidelberg, 3 ed., 1999, https://doi.org/10.1007/978-3-662-06400-9.
[72] H. Robbins and S. Monro, *A Stochastic Approximation Method*, Ann. Math. Stat., 22 (1951), pp. 400–407.
[73] M. Röckner, X. Sun, and Y. Xie, *Strong convergence order for slow–fast McKean–Vlasov stochastic differential equations*, Annales de l'Institut Henri Poincaré, Probabilités et Statistiques, 57 (2021), pp. 547–576, https://doi.org/10.1214/20-AIHP1087.
[74] L. Sharrock and N. Kantas, *Two-Timescale Stochastic Gradient Descent in Continuous Time with Application to Joint Online Parameter Estimation and Optimal Sensor Placement*, arXiv preprint, (2020), https://arxiv.org/abs/2007.15998.
[75] J. Sirignano and K. Spiliopoulos, *Stochastic Gradient Descent in Continuous Time*, SIAM Journal on Financial Mathematics, 8 (2017), pp. 933–961, https://doi.org/10.1137/17M1126825.
[76] J. Sirignano and K. Spiliopoulos, *Mean Field Analysis of Neural Networks: A Law of Large Numbers*, SIAM Journal on Applied Mathematics, 80 (2020), pp. 725–752, https://doi.org/10.1137/18M1192184.
[77] J. Sirignano and K. Spiliopoulos, *Stochastic Gradient Descent in Continuous Time: A Central Limit Theorem*, Stochastic Systems, 10 (2020), pp. 124–151, https://doi.org/10.1287/stsy.2019.0050.
[78] S. C. Surace and J. Pfister, *Online Maximum-Likelihood Estimation of the Parameters of Partially Observed Diffusion Processes*, IEEE Transactions on Automatic Control, 64 (2019), pp. 2814–2829, https://doi.org/10.1109/TAC.2018.2880404.
[79] A.-S. Sznitman, *Topics in propagation of chaos, Ecole d'Eté de Probabilités de Saint-Flour XIX - 1989*, Lecture Notes in Mathematics, 1464 (1991), pp. 165–251.
[80] Y. Tamura, *On asymptotic behaviors of the solution of a nonlinear diffusion equation*, Journal of the Faculty of Science, the University of Tokyo. Sect. 1 A, Mathematics, 31 (1984), pp. 195–221.
[81] J. Tugaut, *Convergence to the equilibria for self-stabilizing processes in double-well landscape*, The Annals of Probability, 41 (2013), pp. 1427–1460, https://doi.org/10.1214/12-AOP749.
[82] J. Tugaut, *Phase transitions of McKean–Vlasov processes in double-wells landscape*, Stochastics, 86 (2014), pp. 257–284, https://doi.org/10.1080/17442508.2013.775287.
[83] A. Y. Veretennikov, *On Ergodic Measures for McKean-Vlasov Stochastic Equations*, in Monte Carlo and Quasi-Monte Carlo Methods 2004, H. Niederreiter and D. Talay, eds., Springer Berlin Heidelberg, Berlin, Heidelberg, 2006, pp. 471–486, https://doi.org/10.1007/3-540-31186-6_29.
[84] A. A. Vlasov, *The vibrational properties of an electron gas*, Soviet Physics Uspekhi, 10 (1968), pp. 721–733, https://doi.org/10.1070/pu1968v010n06abeh003709.
[85] A. J. Weir, *Lebesgue Integration and Measure*, Cambridge University Press, 1973, https://doi.org/10.

1017/9781139171458.
[86] J. Wen, X. Wang, S. Mao, and X. Xiao, *Maximum likelihood estimation of McKean–Vlasov stochastic differential equation and its application*, Applied Mathematics and Computation, 274 (2016), pp. 237–246, https://doi.org/10.1016/j.amc.2015.11.019.
[87] T. Yamada, *On a comparison theorem for solutions of stochastic differential equations and its applications*, Journal of Mathematics of Kyoto University, 13 (1973), pp. 497–512, https://doi.org/10.1215/kjm/1250523321.