

# A random journey through dynamics and finance: pullback attractors, price impact, nonlinear valuation and FX market.

by

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A thesis presented for the degree of  
Doctor of Philosophy of Imperial College London  
and the  
Diploma of Imperial College

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November 2021

# Content of the thesis

## Abstract

The main objective of this thesis is to explore several areas of Random Dynamical Systems and Mathematical Finance. We start by considering random dynamical systems with two different sources of noise, which we call common and intrinsic. We study the interplay between these two sources of randomness from a novel point of view, going beyond the usual statistical approach. We determine the stochastic Fokker-Planck equation describing the system and prove that such equation has a pullback attractor for almost all realizations of the common noise. On the mathematical finance side, we start by discussing consistency properties of jump-diffusion models with respect to inversion, with applications to the Foreign Exchange market. We first solve the constant jump size case, and then analyze the more involved case of the compound Poisson process. We determine a fairly general class of admissible densities for the jump size in the domestic measure. Then, we delve into the nonlinear valuation framework under credit risk, collateral and funding costs, generalizing the mathematical framework of [39] for what concerns in particular the filtrations and the default times. Finally, we propose a first theory of price impact in presence of an interest-rates term structure. We formulate an instantaneous and transient price impact model for zero-coupon bond, defining a cross price impact that is endogenous to the term structure. We extend this setup to coupon-bearing bonds, HJM framework and conclude by solving an optimal execution problem in interest rates market.

## Statement of Originality

I certify that this thesis, and the research to which it refers, are the product of my own work, and that any ideas or quotations from the work of other people, published or otherwise, are fully acknowledged in accordance with the standard referencing practices of the discipline.

Chapter 2 is the result of a fruitful collaboration with Jeroen S.W. Lamb. This collaboration resulted in a paper submitted to the *Journal of Differential Equations* [101].

Chapter 3 is the result of a fruitful collaboration with Damiano Brigo and Andrea Pallavicini. This collaboration resulted in a paper published by the *International Journal of Financial Engineering* [99].

Chapter 4 is the result of a fruitful collaboration with Damiano Brigo and Alexander Kalinin. The content of this Chapter is currently in preparation to be submitted for publication.

Chapter 5 is the result of a fruitful collaboration with Damiano Brigo and Eyal Neuman. This collaboration resulted in a paper published by *Quantitative Finance* [40].

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# 1

## Introduction

*Not all those who wonder are lost.*

– J.R.R. Tolkien

This Thesis presents several results across Random Dynamical Systems and Mathematical Finance. Although the chapters are substantially independent from each other, they all share the common language of probability modelling. Throughout the whole Thesis, the reader will often encounter stochastic differential equations (SDEs), Brownian motions, conditional expectations, probability measures and filtrations. This is not surprising, since SDEs are nowadays ubiquitous in applications, from physical sciences to social sciences and finance.

Nevertheless, the questions that we investigate chapter after chapter are radically different. One of the main reasons for this is the evident rift that has been growing in the last decades between the Random Dynamical Systems and the Mathematical Finance communities. Lots of results in finance heavily rely on the statistical and probabilistic properties of the processes involved. Very rarely do finance papers investigate dynamic questions and concepts such as attractors, bifurcation, synchronization or stability.

Interestingly enough, it also happens that the same mathematical object is interpreted differently, depending on whether we are working in random dynamics or in stochastic analysis. As a prototypical example, we can think about the concept of time in a SDE. As remarked, for instance, by Arnold [11], in stochastic analysis time is generally one-sided, the whole positive real line or some finite interval. The information available at each point in time is collected in what we call a *filtration* and this object is what allows to distinguish between past, present and future. Time is a *physical* quantity and the system evolves from the past to the future. From the viewpoint of dynamical systems, instead, time may be two-sided and it is an *algebraic* quantity. The sequence of noise realizations driving the system is given and the evolution translates into the so-called *shift map*.

Another significant aspect that is worth mentioning is the fundamentally different use of mathematical models in the two communities. In physics, Newton's, Maxwell's and Einstein's equations have been successfully employed to pursue the laws of the universe, to explain observed phenomena that could not be explained with the current theory of the time, or to potentially predict new observations. In mathematical finance, and particularly in the option pricing theory, instead, models have been used mainly as interpolation tools. Given some prices of liquid financial instruments, models are typically employed to determine the fair price of more illiquid products in a consistent manner.

At contrast with over-specialization and compartmentalized knowledge, which seem to be more and more present in nowadays research, this PhD has been a wonderful opportunity to explore a variety of topics in applied and financial mathematics, always reasoning from different perspectives. The wide range of research interests notwithstanding, we investigate fundamental questions in each topic as described in the following sections.

## 1.1 What a random world

Since the pioneering work of Poincaré's on the three-body problem, the main objective of Dynamical Systems theory has been to provide a qualitative picture of differential equations that cannot be solved analytically. Conventional approaches involve the study of topological properties of the system, as well as stability of solutions, existence of attractors and forward convergence to the ergodic invariant measure characterizing the statistical properties of long time behaviour. Adding an element of randomness to the equations of motion led to the development of the modern theory of Random Dynamical Systems (RDSs).

A RDS consists in general of two elements: a model of the noise and a model of the system which is influenced by the noise, usually assumed to be a difference or differential equation, depending on whether the setting is discrete or continuous. RDSs are characterized by having a *skew product structure*, meaning that the dynamics on the space of all noise realizations, called *base*, drives the dynamics on the phase space, but it is not influenced by it. Random dynamical systems can be viewed as non-autonomous dynamical systems, where time dependence enters through the noise. As explained, for example, by Arnold [11], the non-autonomous nature of a RDS has deep consequences. In particular, we are interested in the fact that letting the system evolve from time 0 to time  $t$  is going to be different from moving the points from time  $-t$  to time 0. In the first case, there is no hope for convergence in the limit  $t \rightarrow \infty$ , since the behaviour of the system depends on the future realization of the noise. In the second scenario, though, the asymptotic result may converge. This has motivated the definition of *pullback attractor*.

This interesting concept plays a key role in Chapter 2, where we will delve into dynamical systems characterized by having two different kinds of random drivers. To gain some intuition, one might think of a system with identical non-interacting particles (or agents) that are subject to independent noise at the level of each particle and a common, or environmental, noise that is equal to all. This kind of framework arises in a variety of contexts, from biology to genetics, from social networks to pattern formation and avalanching phenomena.

The interplay between these two sources of noise is often analysed by means of statistical tools, such as the law of total variation, correlations and similar. From a dynamics perspective, this leads to the study of the stationary measure of the system, obtained by averaging over all possible common and intrinsic noise realizations. Although quite useful in some circumstances, the stationary measure may not carry relevant information to answer certain types of questions. The stationary measure provides us with an average description of the system, hence its application inevitably entails loss of information. For instance, in Chapter 2 we show an example of a distribution which is localized at each given point in time but the average of which is very broad. In such context, the stationary measure would potentially indicate a lack of localization. Furthermore, there are scenarios where intrinsic noise is not observable whereas common noise may well be. For instance, in a particle systems context, where it's not possible to keep track of the motion of each particle and macroscopic features like synchronization are of interest, or in a fast-slow system, where the intrinsic noise is fast compared to the measurement time-scale.

In light of these considerations, we study an ensemble of intrinsic noise realizations, while fixing one common noise realization only. In Chapter 2 we consider a class of SDEs with gradient-like dynamics and where the intrinsic and common noise are additive Brownian motions. We establish almost sure existence and uniqueness of pullback attractors with respect to realisations of the common noise only. These pullback attractors are smooth probability densities that depend only on the past of a common noise realization (and not on the initial conditions) and to which the pullback evolution of a corresponding stochastic Fokker-Planck equation converges.

## 1.2 Money never sleeps

In the second part of our journey we will enter the realm of Mathematical Finance. The first financial topic we will discuss is the Foreign Exchange (FX) market. This is the market where currencies are traded and where the foreign exchange rates at which individuals, financial companies, central banks convert one currency into another are determined. This market is quite special, in that it distinguishes itself from other markets, such as equities, commodities or fixed-income products, in a number of ways. First, the FX market is global, decentralized and over-the counter (OTC), meaning that trades occur not through a venue where contracts are sold and bought,

the so-called *exchange*, but directly between the two parties. Moreover, due to decentralisation, trading is allowed 24 hours a day. The feature we are mostly interested in is, however, the presence of *symmetries*. Consider, for example, the GBP-USD spot exchange rate. In the usual convention, the US dollar is called *quote currency*, and represents the currency the trader is selling, while the Pound sterling is called *base currency*, and represents the currency the trader is buying. The exchange rate GBP-USD equals to the amount US dollars needed to buy 1 pound. Then, its reciprocal USD-GBP will be again an exchange rate. This is true for any exchange rate, of course. The point we would like to make is that the same is *not* true with other financial instruments such as equities. Given the Google share price, for instance, its reciprocal has no financial meaning and doesn't correspond to any other share price. We call this property symmetry with respect to inversion. The second symmetry is with respect to multiplication and involves three exchange rates. Take the USD-GBP and the GBP-EUR exchange rates. Then, their product will be the cross rate USD-EUR. More generally, given two exchange rates such that the foreign currency of the first one corresponds to the domestic currency of the other, then their product will be another exchange rate. These two stylized facts suggest that any mathematical equation aiming at modelling an exchange rate ought to fulfil some kind of *consistency conditions*. In Chapter 3 we will in particular investigate the consistency under inversion and address the following question: if an exchange rate follows a given type of dynamics under the domestic measure, under which conditions will its reciprocal follow the same kind of dynamics under the foreign measure, that is, the same dynamics up to a reparametrization? Notice that, theoretically speaking, an exchange rate and its reciprocal are fundamentally the same process. They are describing the same physical quantity, just from two different points of view. Hence, it looks quite natural to ask they are described by the same kind of process. From more a practical point of view, instead, it might be quite convenient and efficient to have a consistent dynamics for all exchange rates when designing libraries. Our research will be devoted to the study of jump-diffusion models and we will discuss both scenarios where the jump size is constant (Poisson process) and random (compound Poisson process). The main result of Chapter 4 will be determining a fairly large class of densities for the jump size that are not affected by the measure change, modulo a reparametrization. Interestingly, it will turn out that a power law distribution with an exponential cutoff satisfies such condition.

## 1.3 Nonlinearities all around

After discussing consistency in the Foreign Exchange markets, the landscape changes considerably. In Chapter 4 we will delve into the so called nonlinear valuation framework. After the huge financial crisis of 2007-2008, it became clear to market participants and regulators that the classic framework for pricing derivatives was somehow

incomplete, in the sense that it was overlooking several sources of risk that are extremely relevant in practice. This resulted in a significant shift of focus from pricing financial products with complicated payoffs and simple sources of risk, to pricing financial products with simple payoffs and complicated sources of risk. People started to take into consideration *funding costs*, meaning the amount of money needed in order to fund a trade and, above all, *default risk*, that is, the possibility that the counterpart might default. A considerable amount of effort was therefore put in place by banks and other financial institutions to account and mitigate such risk. This led to the introduction of the so-called *valuation adjustments*, i.e. little corrections to be added to the classic price. The research on this topic is quite extended and broadly speaking two general, equivalent, approaches can be followed. One approach consists in writing down all cash flows involved in the financial derivative (payoff, funding cost, collateral cost, default cost, etc.) and defining the fair price as the expectation of the discounted cashflow with respect to the filtration representing the overall information available in the market. Then, the idea is to change filtration to the one representing the information before default. This, in turn, leads to a forward backward stochastic differential equation (FBSDE), to which one can associate a nonlinear PDE, the solution of which represents the price. The whole analysis therefore boils down to determining the conditions on the dynamics of the underlying asset and cashflows under which such PDE admits a unique solution, either viscous or classical. This approach, called *adjusted cash flow approach*, was followed, for example, by Brigo, Francischello, Pallavicini and co-authors [39, 140, 139, 41]. Alternatively, one might follow the so-called *replication approach*, investigated by Bielecki, Rutkowski, Crepey and co-authors (see for example [20, 19, 63, 64, 38]). The idea is to adapt classic concepts such as replication and self-financing portfolio to a collateralized contract. As we shall see in the devoted chapter, we will derive two implicit representations for the pre-default value process that hedges the derivative by following the ideas in [39]. Such work will be generalized for what concerns default times and filtrations. The stopping times at which either the investor or the counterparty might default will be arbitrary, up to a mild distribution condition. More importantly, the filtration modelling the whole available information in the market may provide no, some or full insight into the default times of the parties involved in the contract. Our work extends [39] in that full insight was assumed there. In practice, relaxing such assumption could be helpful in dealing with *fraud risk*, meaning a situation where a company's books are fraudulent and we can only suspect, but have no certainty, that its default has already occurred. Future research will be devoted to the model specification for the underlying asset and to the discussion of mild solutions of the valuation partial differential equation, thereby further generalizing [39].

## 1.4 Bridging price impact and interest rates

We study the interface between classic interest rate theory and price impact theory. As we will explain in detail in Chapter 5, these two research areas are usually treated separately. Our work is the first attempt to merge the two frameworks and build a theory of market microstructure for interest rates. With the introduction of such a novel framework, we hope to shed some light in this long-lasting problem.

Term-structure of interest rates, generally speaking, aims at modelling the time evolution of interest rates and fixed-income securities, that is, those debt instruments that pay a fixed amount of interest to investors. The most common of these securities is the zero-coupon bond, which pays one unit of currency at expiration. A lot of research has been devoted to the pricing of so-called interest rate derivatives, that is, those contracts whose value depends on some underlying interest rate. Common examples are futures, Interest Rate Swaps (IRS), swaptions and forward rate agreements (FRAs). Similarly to option pricing for equities, the idea is to write the price as a conditional expectation, under an artificial risk-neutral probability measure, of the cash flows generated by the contract. The key difference, compared to equities, is that the term-structure of interest rates is infinite dimensional. The bond market consists of infinitely many traded assets. Consequently, the market will be incomplete, meaning that the price will not be unique. Once we specify the dynamics of the short rate under the real world measure, we also know the dynamics of the money market account. This, however, is not sufficient to replicate the payoff of a bond. To solve this issue, one has to specify in addition to the model also the so-called *market price of risk*, notion that constitutes a bridge between the objective world, where data is observed, and the risk-neutral world, where expectations are computed. For pricing purposes, one typically defines the interest rate model directly under the risk neutral measure, thereby defining the market price of risk implicitly.

Price impact describes the fact that trading a significant amount of a certain asset affects the price of that asset in a detrimental way for the agent who is trading. To put it simply, buying an asset induces the price of the asset to increase, selling that asset induces its price to decrease. Price impact becomes particularly significant when orders are large in comparison to the liquidity available in the market. In practice, trading costs are minimized by splitting large trades into a sequence of smaller trades which are spread out over a certain time interval. The stochastic control problem is typically formulated as cost minimization, where additional terms accounting for the risk of holding the asset for too long can be added. In other words, there is an interesting trade-off the trader faces, where on the one hand they wish to trade slowly in order not to create price impact and, on the other hand, they can't trade too slowly because having the asset in the inventory entails the risk of adverse price movement. *Cross impact* describes the price impact interactions among different assets when trading portfolios, for example. This effect leads to additional trading costs that should be taken into account, as discussed, for instance, by Benzaquen and

co-authors [17, 131, 158]. At the same time, Cont and Capponi [49] raised doubts regarding the theoretical and empirical evidence of cross impact. They argued that its existence is not necessarily implied by the presence of positive covariation between the returns of an asset and the order flow of another asset, simply describable as plain price correlations.

As we shall see more in detail in Chapter 5, our fundamental idea to combine interest rate theory and price impact will be to formulate price impact for zero-coupon bonds, thereby introducing an impacted version of the notion of market price of risk, which was mentioned above. This will have several important consequences, such as preservation of no-arbitrage and the possibility of defining a new risk-neutral measure under which prices of interest rates derivatives can be computed. Cross impact between bonds with different maturities will naturally emerge from our model. It will be specific for the term-structure and *endogenous*, meaning that it will refer to bonds belonging to the same yield curve. Another contribution of our work will be the incorporation of price impact into the HJM framework. This will allow to understand how the forward curve is affected when trading zero-coupon bonds. After we are done with pricing under price impact effects, we will address an optimal execution problem of zero-coupon bonds. We start by specifying the risk-cost functional to be minimized over a suitable class of execution strategies. This functional will be the expectation of transactions costs due to price impact and risk terms accounting for adverse price changes of the asset which is held in the portfolio. By following a variational approach, we will study the unique critical point at which the Gateaux derivative of the cost functional vanishes. Then, we will derive the corresponding system of forward-backward stochastic differential equations (FBSDEs) and derive the optimal trading speed.

## 1.5 Structure of the thesis

This Thesis is structured as follows. Chapter 2 is devoted to the analysis of random dynamical systems driven by two different sources of randomness. In Chapter 3 we discuss consistency properties of jump-diffusion models with applications to the Foreign Exchange market. In Chapter 4 we study nonlinear pricing equations of financial contracts under credit risk, collateral and funding costs. In Chapter 5 we incorporate price impact into the classic term structure theory of interest rates.

## 2

# Common noise pullback attractors for stochastic dynamical systems

## 2.1 Introduction and summary of the main results

### 2.1.1 Motivation

In the theory of dynamical systems, broadly speaking, two dominant points of view may be distinguished: the topological point of view (understanding of the dynamics at the level of (typical) individual trajectories) and the probabilistic point of view (understanding of the dynamics at the level of average statistical properties, e.g. through Ergodic Theory) [117]. Dynamical systems in the presence of noise (such as stochastic dynamical systems defined by SDEs) are predominantly approached from the latter point of view, with powerful analytical techniques from stochastic analysis and Markov processes [138]. The alternative *random dynamical systems* approach considers dynamical systems with noise as skew-product systems, where noise drives an otherwise deterministic dynamical system and the noise driving process admits a pathwise and probabilistic (in terms of ergodic theory) description. The latter allows a blend of the traditional topological and probabilistic approaches to achieve probabilistic results about the behaviour of trajectories of the (non-autonomous) noise-driven system. For instance, Arnold and co-workers [11] have established the existence of random generalisations of attractors, as well as stable, unstable and centre manifolds.

In this Chapter we develop a random dynamical systems point of view for SDEs with two distinguished sources of noise, which we refer to as *intrinsic* and *common* in view of the motivating example of a system with identical non-interacting particles (or agents) that are subject to intrinsic noise at the level of each particle and a common noise that is equal to all. Such settings naturally arise in a broad range of applications, for instance in genetics [71, 81, 156, 157], neuroscience [2, 124], epidemics [110], pattern formation [147] and financial mathematics [95].

The aim of this Chapter is to study a stochastic dynamical system with intrinsic and common noise, conditioned on the past of the common noise realisation. In the context of the motivating particle system, this yields a description of the probability distribution of the particle system, the evolution of which is described by a stochastic Fokker-Planck equation, subject to the past of the realisation of the common noise.<sup>1</sup> We establish the existence and uniqueness of a corresponding *common noise pullback attractor* for a specific class of SDEs where the intrinsic and common noises are additive Brownian motions. Our approach is not limited to this special class, but the main aim of this Chapter is to develop the concept of common noise pullback attractors in this specific transparent setting.

Common noise pullback attractors facilitate the study of time-averaged properties of the distribution describing the particle system, through the application of ergodic theory, cf. equation (2.1.5) in Section 2.1.2, for example observing the variance of the distribution as a measure of synchronisation. Synchronisation is a widely studied dynamical phenomenon in complex systems with ramifications in a wide range of applications [143]. In addition, from a modelling perspective, our point of view is natural where the intrinsic noise is inherently or practically not observable, while the common noise can in principle be observed. Examples include the sentiment of traders in markets and voters under the influence of mass media, where the latter can be treated as a stochastic process or as a deterministic signal, leading to the consideration of dynamics with common noise or more general non-autonomous dynamics. In fact, the random dynamical systems approach taken in this Chapter in principle allows us to address both settings at the same time.

The notion of pullback attractors is well-established in non-autonomous and random dynamical systems, see for instance [11, 121] and [15, 50, 62, 88, 130, 164, 166] in the context of SPDEs. The analysis of pullback attractors in applications of complex nonlinear systems is gaining popularity in recent years, for instance in the context of climate science and turbulence [76, 94]. This Chapter is a further contribution in this direction.

### 2.1.2 Common noise pullback attractor

We consider SDEs in  $\mathbb{R}^d$  of the form

$$dx(t) = -\nabla V(x(t))dt + \sigma dW(t) + \eta dB(t), \quad (2.1.1)$$

where  $V$  represents a smooth potential,  $\sigma, \eta$  are positive definite matrices,  $W$  is a Brownian motion and  $B$  represents another source of noise, such as another Brownian motion or related process. This equation can be looked at from different points of view. On the one hand, one could fix a single path of  $B$  and a single path  $W$ ,

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<sup>1</sup>Bressloff calls this the *population* or *SPDE perspective*, in contrast with the *particle perspective*, where one averages over both the intrinsic and common noise [37].

hence analysing individual trajectories. Alternatively,  $W$  and  $B$  could be treated both as distributions, in which case one is led to the study of the associated well known Fokker Planck equation. In this work we will instead follow an intermediate approach, whereby we single out a realization of  $B$ , while viewing  $W$  as a distribution. As we shall see more in detail below, this idea leads to the stochastic Fokker-Planck equation

$$dp = \left[ \Delta V(x)p + \nabla V(x) \frac{\partial p}{\partial x} + \frac{1}{2} \sigma^2 \frac{\partial^2 p}{\partial x^2} \right] dt - \eta \frac{\partial p}{\partial x} \circ dB(t), \quad (2.1.2)$$

where, as usual,  $\circ$  refers to the Stratonovich convention of stochastic integration. We refer to  $W$  as *intrinsic* and  $B$  as *common* noise. Such a mixed approach is naturally motivated by the setting of a system of identical particles with states  $x_i \in \mathbb{R}^d$ ,  $i \in \{1, \dots, N\}$ , each subject to an independent noise  $W_i$  and an identical common noise  $B$

$$dx_i(t) = -\nabla V(x_i(t))dt + \sigma dW_i(t) + \eta dB(t). \quad (2.1.3)$$

In the limit of large  $N$ , the evolution of this particle system is described by the evolution of a measure on  $\mathbb{R}^d$ . This measure has a Lebesgue density  $p$  whose evolution is governed precisely by (2.1.2).

Similar equations have been derived earlier by Giles and Reisinger [95] in the context of pricing baskets of financial derivatives, Bressloff [37] in the context of neuronal dynamics, Bain and Crisan [14] in the context of stochastic filtering and Carmona and Delarue [51] in the context of mean field games.<sup>2</sup>

We aim to employ (2.1.2), like Bressloff [37], to study the evolution of population densities as a function of the common noise. Thereto, we approach the stochastic Fokker-Planck equation (2.1.2) from a random dynamical systems point of view, considering the non-autonomous evolution of the density  $p$  as a function of the realisation  $\beta$  of the common noise  $B$ . In Section 2.2 and in particular Theorem 2.2.2, we provide a detailed discussion on the existence, uniqueness and regularity of solutions to the initial value problem of (2.1.2) as a deterministic non-autonomous PDE, for a sufficiently regular common noise realisation. It is shown that the flow evolves initial conditions in  $L^1$  to the Schwartz space of smooth rapidly decaying Lebesgue densities. In Section 2.4, we establish that the SPDE (2.1.2) is a random dynamical system in the sense that it admits a description in skew-product form with ergodic base dynamics generating the common noise  $B(t)$ , see Lemma 2.4.1.

Traditionally, dynamical systems theory focuses mostly on the long-term behaviour of solutions. In the non-autonomous setting, as it is rare to have convergence in forward time (since the equations of motion vary with time), it is natural to consider the asymptotic behaviour of *pullback dynamics* instead, which has better prospects of convergence and reveals important aspects of the dynamics. Let  $\Phi(t, \beta)$  represent the

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<sup>2</sup>Often, (2.1.2) is written in Itô form, resulting in an additional term  $\eta^2$  added to  $\sigma^2$  in the diffusion term.

time- $t$  flow of (2.1.2) with common noise realisation  $\beta$ . Instead of studying the behaviour of initial conditions with fixed noise realisation in the limit  $t \rightarrow \infty$ , pullback dynamics considers the behaviour of initial conditions of this flow, fixing the end-time, say at  $t = 0$ , in the limit of the starting time  $\tau \rightarrow -\infty$ . A pullback attractor describes the state of the system, conditioned on the past of the time-dependent input (noise realisation). In the context of particle dynamics with intrinsic and common noise, the objective is to describe the distribution of the particle system with intrinsic noise, subject to the past realisation of the common noise.

Under a natural assumption on the potential  $V$  that guarantees the existence of a unique stationary density for (2.1.1) in the absence of common noise, the main result of this Chapter is that the stochastic Fokker-Planck equation (2.1.2) has a unique pullback attractor that is a random equilibrium, i.e. for almost all common noise realisations  $\beta$ , the limit

$$p_\beta := \lim_{\tau \rightarrow \infty} \Phi(\tau, \theta_{-\tau}\beta)p \quad (2.1.4)$$

exists in the Schwartz space and is independent of the initial probability density  $p \in L^1$ , see Theorem 2.4.2. This result relies on the fact, obtained in Section 2.3, that for almost every noise realisation the non-autonomous evolution is a contraction. Moreover, the convergence is uniform in the common noise realisation. We refer to  $p_\beta$  as the *common noise* pullback attractor of the SDE (2.1.1). It turns out that  $p_\beta$  is the density of the measure obtained by averaging, for a fixed common noise realisation  $\beta$ , the canonical  $(\omega, \beta)$ -dependent pullback measures of SDE (2.1.1) over all intrinsic noise relations  $\omega$ , cf. Proposition 2.4.4.

From a random dynamical systems point of view,  $p_\beta$  in (2.1.4) is called a globally attracting random equilibrium of the stochastic Fokker-Planck equation (2.1.2). This is the simplest type of attractor one may encounter in a random dynamical system. In general, random (pullback) attractors may display more complicated behaviour, cf. [62].

Finally, by virtue of ergodicity we find (in Proposition 2.4.5) that if  $g$  is a continuous observable on the relevant solution space of densities for (2.1.2),  $\mathbb{P}_B$ -almost surely,

$$\lim_{\tau \rightarrow \infty} \frac{1}{\tau} \int_0^\tau g(\Phi(\tau, \beta)p) dt = \mathbb{E}^{\mathbb{P}_B}[g(p)], \quad (2.1.5)$$

with  $\mathbb{E}^{\mathbb{P}_B}$  denoting the expectation with respect to the probability measure  $\mathbb{P}_B$  of the common noise. For special types of observables, the expectation (2.1.5) is related to an expectation with respect to the stationary measure  $\rho$  of the SDE (2.1.1). In particular, when the observable  $g$  is a  $p_\beta$ -expectation of a continuous observable  $h : \mathbb{R}^d \rightarrow \mathbb{R}$ , i.e.

$g(p_\beta) = \int_{\mathbb{R}^d} h(x)p_\beta(x)dx$ , then

$$\begin{aligned}\mathbb{E}^{\mathbb{P}_B}[g(p_\beta)] &:= \int_{\Omega_B} g(p_\beta) \mathbb{P}_B(d\beta) = \int_{\Omega_B} \int_{\mathbb{R}^d} h(x)p_\beta(x)dx \mathbb{P}_B(d\beta) \\ &= \int_{\mathbb{R}^d} h(x) \int_{\Omega_B} p_\beta(x) \mathbb{P}_B(d\beta) dx = \int_{\mathbb{R}^d} h(x)\rho(dx) =: \mathbb{E}^\rho[h]\end{aligned}\tag{2.1.6}$$

However, in general the expectation (2.1.5) is not expressible in terms of the stationary measure  $\rho$  of (2.1.1). For instance, the variance

$$\text{Var}(p_\beta) := \mathbb{E}^{p_\beta}[x^2] - (\mathbb{E}^{p_\beta}[x])^2$$

which is an indicator of synchronisation (of the particle system), is an observable whose  $\mathbb{P}_B$ -expectation cannot be deduced from  $\rho$ , cf. the examples discussed in Section 2.1.3.

### 2.1.3 Examples

#### Ornstein-Uhlenbeck process with intrinsic and common noise

Our results are well-illustrated in the elementary example of an Ornstein-Uhlenbeck process with intrinsic and common noise

$$dx(t) = -ax(t)dt + \sigma dW(t) + \eta dB(t),\tag{2.1.7}$$

with  $x \in \mathbb{R}$ ,  $a > 0$  and  $W$  and  $B$  are independent Brownian motions.<sup>3</sup> Due to the linearity of (2.1.7) the solution of (2.1.2) with initial condition  $\delta_{x(s)}$  and common noise realisation  $\beta$  can be explicitly calculated (for all  $t > s$ ) to have the form<sup>4</sup>

$$p(x, t) = \sqrt{\frac{a}{\pi\sigma^2(1 - e^{-2a(t-s)})}} \exp\left(-\frac{a}{\sigma^2(1 - e^{-2a(t-s)})}(x - m_\beta(t, s))^2\right),$$

where

$$m_\beta(t, s) := x(s)e^{-a(t-s)} + \eta \int_s^t e^{-a(t-u)} d\beta(u),$$

and the latter integral is  $\mathbb{P}_B$ -almost surely finite. Indeed, by Theorem 2.4.2, the unique common noise pullback attractor of (2.1.7) is independent of the initial condition (in  $L^1$ , cf. Section 2.2) and thus equals  $\mathbb{P}_B$ -almost surely the normal distribution

$$p_\beta(x) = \lim_{s \rightarrow -\infty} p(x, s) = \sqrt{\frac{a}{\pi\sigma^2}} \exp\left(-\frac{a}{\sigma^2} \left(x - \eta \int_{-\infty}^0 e^{au} d\beta(u)\right)^2\right).\tag{2.1.8}$$

<sup>3</sup>Most of the results here do not require  $B$  to be a Brownian motion, cf. also footnote 9.

<sup>4</sup>For details, see Section 2.8.

This example illustrates how the exact synchronisation of solutions of (2.1.7) in the absence of intrinsic noise ( $\sigma = 0$ ) turns into an approximate synchronisation of the corresponding particle system in the presence of small intrinsic noise ( $\sigma \ll 1$ ), characterized by small  $\text{Var}(p_\beta)$ . Namely, in the absence of intrinsic noise,  $\mathbb{P}_B$ -almost surely all pairs of initial conditions  $x, y \in \mathbb{R}^d$  pathwise converge, i.e.  $x_\beta(t), y_\beta(t)$  of (2.1.7) with noise realisation  $\beta$  satisfy  $\lim_{t \rightarrow \infty} |x_\beta(t) - y_\beta(t)| = 0$  [61], while in the presence of intrinsic noise the distribution converges to a normal distribution with variance  $\text{Var}(p_\beta) = \frac{\sigma^2}{2a}$ .

The location of this normal distribution depends on (the past of) the common noise realisation  $\beta$ , i.e. the mean  $m(p_\beta) = m_\beta(0, -\infty)$  and is independent of the intrinsic noise strength  $\sigma$ . In view of (2.1.5), this implies for the time-averages of the variance and mean of the (particle) distribution that,  $\mathbb{P}_B$ -almost surely,

$$\lim_{\tau \rightarrow \infty} \frac{1}{\tau} \int_0^\tau \text{Var}(\Phi(\tau, \beta)p) dt = \frac{\sigma^2}{2a} \text{ and } \lim_{\tau \rightarrow \infty} \frac{1}{\tau} \int_0^\tau m(\Phi(\tau, \beta)p) dt = \mathbb{E}^\rho(x) = 0,$$

where  $\rho$  denotes the stationary measure of (2.1.7).

We contrast the average of the observed variance along trajectories of (2.1.2) with the fact that the stationary density  $p_\rho$  of (2.1.7),  $p_\rho(x) = \int p_\beta(x) \mathbb{P}(d\beta)$ , has a different variance. In particular,

$$p_\rho(x) = \sqrt{\frac{a}{\pi(\sigma^2 + \eta^2)}} \exp\left(-\frac{a}{(\eta^2 + \sigma^2)} x^2\right)$$

is a normal distribution with mean 0 and variance  $\frac{\sigma^2 + \eta^2}{2a}$ . Indeed, synchronisation of the particle system corresponds to localisation of the pullback measure, rather than to localisation of the stationary measure. If  $\sigma$  is small and  $\eta$  is large, the particle distribution is asymptotically strongly localized, while the stationary distribution is not.

We note that the Ornstein-Uhlenbeck example (2.1.7) is very special, in particular the fact that the shape of the density  $p_\beta$  does not depend on the noise realisation  $\beta$ . This is a consequence of the linearity of this example, which also yields it exactly solvable.

### Bi-stable dynamics with intrinsic and common noise

We next consider the less degenerate, nonlinear, example of (2.1.1) with  $x \in \mathbb{R}$  and  $V(x) = \frac{1}{4}x^4 - \frac{a}{2}x^2$  is a double-well potential

$$dx(t) = x(t)(a - x(t)^2)dt + \sigma dW(t) + \eta dB(t), \quad (2.1.9)$$

with  $a > 0$  and  $\sigma, \eta$  constants as above and  $W(t), B(t)$  are Brownian motions. In this case, the stationary probability density of the SDE (2.1.9) admits the explicit

expression

$$p_\rho(x) = \frac{1}{N} \exp \left( -\frac{2}{\sigma^2 + \eta^2} \left( \frac{x^4}{4} - a \frac{x^2}{2} \right) \right), \quad (2.1.10)$$

where  $N := \int \exp \left( -\frac{2}{\sigma^2 + \eta^2} \left( \frac{x^4}{4} - a \frac{x^2}{2} \right) \right) dx$  is a normalization constant

An important difference with (2.1.7) is that (2.1.9) is nonlinear. To our best knowledge, in this case, the common noise pullback attractor  $p_\beta$  of (2.1.9) does not admit a comprehensive analytical expression, but it can be approximated numerically (for instance, by means of Monte Carlo methods, cf. [120]). In Figure 2.1 some numerically obtained examples of densities for common noise pullback attractors are presented, illustrating how the stationary density (depicted in the background in grey) may differ substantially from the densities of individual pullback attractors  $p_\beta$  which depend on the common noise realisation  $\beta$ . This figure illustrates some of the limitations in dynamical information that a stationary measure of a stochastic dynamical system provides.

There are various important questions concerning common noise pullback attractors that we have not addressed here, but which deserve further attention. For instance, it would be of interest to determine the support of the stationary measure of (2.1.2), i.e. the range of possible densities of common noise pullback attractors, in particular also as a function of system parameters. In the exactly solvable Ornstein-Uhlenbeck example of Section 2.1.3, the range consists of a one-parameter family of normal distributions, with identical variance depending on the strength of the intrinsic noise. In the double-well example of Section 2.1.3, Figure 2.1 suggests that the range is also limited but with a more complicated dependence on system parameters.

## 2.2 The non-autonomous Fokker Planck equation and its initial value problem

In this section we consider the derivation and analysis of equation (2.1.2), as a non-autonomous Fokker-Planck equation. This forms the basis of our discussion of (2.1.2) in Section 2.4 in the stochastic setting, as a random dynamical system. In Section 2.2.1, we discuss the derivation of the non-autonomous Fokker-Planck equation (2.1.2) from two points of view: as the Fokker-Planck equation for a non-autonomous SDE and from a particle system approximation, which motivates the choice of terminology *common noise*. In Section 2.2.2 we establish the existence and uniqueness of the solution for the non-autonomous Fokker-Planck equation (2.1.2) within a suitable setting and discuss how this solution smoothenes when  $t > 0$ , given an initial condition  $q_0 \in L^1(\mathbb{R}^d)$  at  $t = 0$ . The techniques employed are in principle deterministic and well-established, see eg [59, 84, 159], but as our specific non-autonomous setting is not normally addressed, we present a self-contained technical discussion in

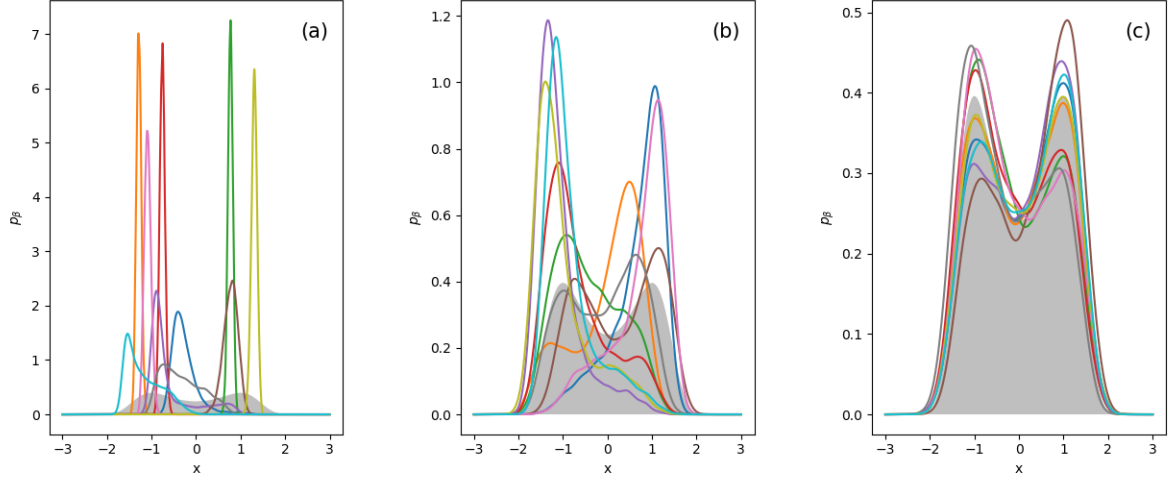


Figure 2.1: Densities of common noise pullback attractors  $p_\beta$  of the SDE with double well potential with intrinsic and common additive noise (2.1.9), with  $a = 1, \sigma^2 + \eta^2 = 1$  and (a)  $\sigma \ll \eta = 0.99$ , (b)  $\sigma = \eta = \frac{1}{2}\sqrt{2}$  and (c)  $\sigma \gg \eta = 0.15$ . Pullback attractors for different common noise realisations  $\beta$  are represented by graphs with different colors. The stationary density  $p_\rho$  (2.1.10) of the SDE (2.1.9), plotted in grey in the background, is identical in all three cases. Different scales on the  $p_\beta$ -axis have been chosen so as to achieve similar resolutions in the graphs of the densities of the common noise pullback attractors. When common noise dominates intrinsic noise,  $\sigma \ll \eta$  (a), one predominantly observes localized pullback attractors, corresponding to approximate synchronisation. When intrinsic noise dominates common noise,  $\sigma \gg \eta$  (c), the densities of the common noise pullback attractors tend to be less localized and relatively close to the stationary density. To obtain an objective quantification of the degree of synchronisation, we have numerically approximated the time-averaged variance of (particle) distributions, as  $\mathbb{E}^{\mathbb{P}^B} [\text{Var}(p)]$  by virtue of (2.1.5), yielding the values (a) 0.04, (b) 0.53 and (c) 0.90 (the latter being close to  $\text{Var}(p_\rho)$ ), in accordance with the perceived degrees of localisation in the density graphs.

Section 2.5. The choice of initial conditions in  $L^1$  (rather than in  $L^2$ , as is commonly found in the literature) is to cater for natural densities relevant to the particle system interpretation, such as Dirac's delta, representing a system with all particles in the same initial state. It turns out that initial conditions in  $L^1$  evolve immediately into  $L^2$ .

### 2.2.1 Derivation of the non-autonomous Fokker-Planck equation

#### The Fokker-Planck equation of the non-autonomous SDE

Let us consider (2.1.1) as a non-autonomous SDE where  $B(t) = \beta(t)$  is deterministic. We let  $\beta \in C^{1/2}(\mathbb{R}, \mathbb{R}^d)$ , defined as the set of functions from  $\mathbb{R}$  to  $\mathbb{R}^d$  that are everywhere locally  $\alpha$ -Hölder continuous for any  $\alpha < \frac{1}{2}$ .<sup>5</sup> Writing  $y(t) := x(t) - \eta\beta(t)$ , the SDE (2.1.1) can be written as

$$dy(t) = -\nabla V(y(t) + \eta\beta(t))dt + \sigma dW(t). \quad (2.2.1)$$

With  $U(y, t; \beta) := V(y(t) + \eta\beta(t))$ , we find

$$\nabla U(y, t; \beta) = \nabla V(y + \eta\beta(t)), \quad \Delta U(y, t; \beta) = \Delta V(y + \eta\beta(t))$$

where  $\nabla$  and  $\Delta$  denote the gradient and Laplacian with respect to the first argument. The Fokker-Planck equation describing the annealed evolution of Lebesgue probability densities  $q$  associated with the SDE (2.2.1) is given by [142]

$$\partial_t q = \nabla(\nabla U(y, t; \beta)q) + \frac{1}{2}\sigma^2 \Delta q. \quad (2.2.2)$$

Transforming variables  $y$  back to  $x$  in (2.2.2), with densities  $p(x) := q(y)$ , yields (2.1.2).

#### The Fokker-Planck equation of the non-autonomous particle system

The Fokker-Planck equation (2.1.2) can also be motivated directly from a particle system point of view, following for instance [123]. Let us consider a system of particles  $x_i$ ,  $i = 1, \dots, N$  satisfying (2.1.3)

$$dx_i(t) = -\nabla V(x_i(t))dt + \sigma dW_i(t) + \eta d\beta(t),$$

with  $W_i$  independent Brownian motions representing the intrinsic noise and  $\beta \in C^{1/2}(\mathbb{R}, \mathbb{R}^d)$  a deterministic common driving. Let  $\varphi \in C_b^2(\mathbb{R}^d)$  be an observable which

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<sup>5</sup>The choice of this regularity is motivated by the fact that pathwise realisations of the Brownian motion possess this regularity almost surely, see e.g. [116]

is a bounded function of compact support with bounded first and second derivative. Its evolution is given by

$$\begin{aligned} \varphi(x_i(t)) = & \varphi(x_i(0)) + \int_0^t \left( -\nabla^T \varphi(x_i(s)) \nabla V(x_i(s)) + \frac{1}{2} \sigma^2 \Delta \varphi(x_i(s)) \right) ds \\ & + \int_0^t \sigma \nabla^T \varphi(x_i(s)) dW_i(s) + \int_0^t \eta \nabla^T \varphi(x_i(s)) d\beta(s). \end{aligned} \quad (2.2.3)$$

It is crucial to recognise that the stochastic integral with respect to the intrinsic noise  $W_i$  represents a distribution, while the other integrals yield scalars. The empirical measure for a particle distribution may be defined, as usual, as

$$\nu(t) := \lim_{N \rightarrow +\infty} \frac{1}{N} \sum_{i=1}^N \delta_{x_i(t)},$$

where  $\delta_x$  denotes the Dirac measure at  $x$  and we consider convergence in the weak topology of the collection of all finite signed Borel probability measures on  $\mathbb{R}^d$ .<sup>6</sup> Integrating both sides of (2.2.3) with respect to this empirical measure, denoting  $\nu_t(\varphi) := \int \varphi d\nu_t$ , yields the SPDE

$$\nu_t(\varphi) = \nu_0(\varphi) + \int_0^t \nu_s(A_1\varphi) ds + \int_0^t \nu_s(A_2\varphi) d\beta(s), \quad (2.2.4)$$

where  $A_1, A_2$  are the differential operators

$$\begin{aligned} A_1\varphi &:= \frac{1}{2} \sigma^2 \Delta \varphi - \nabla^T \varphi \nabla V \\ A_2\varphi &:= \eta \nabla^T \varphi. \end{aligned}$$

In particular, the term containing the stochastic integral vanishes, see [123, Proof of Theorem 3.1]. Following [14, Chapter 7.3], assuming that the empirical measure  $\nu$  has a sufficiently smooth Lebesgue density  $p(t)$ , one may reformulate (2.2.4) as

$$\begin{aligned} \nu_t(\varphi) &= \int_{\mathbb{R}} \varphi(x) p(t, x) dx \\ &= \int_{\mathbb{R}} \varphi(x) \left( p(0, x) + \int_0^t A_1^* p(s, x) ds + \int_0^t A_2^* p(s, x) d\beta(s) \right) dx, \end{aligned}$$

where  $A_1^*, A_2^*$  are given by

$$\begin{aligned} A_1^* \psi &= \frac{1}{2} \sigma^2 \Delta \psi + \nabla(\nabla V \psi) = \frac{1}{2} \sigma^2 \Delta \psi + \Delta V \psi + \nabla \psi \nabla V \\ A_2^* \psi &= -\eta \nabla^T \psi. \end{aligned}$$

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<sup>6</sup>The Glivenko-Cantelli theorem [161] provides more detail on the convergence in the weak topology of the empirical measure. See Boissard and Le Gouic [29], for a discussion of the convergence of the empirical measure with respect to the Wasserstein distance.

We thus obtain

$$p(t, x) = p(0, x) + \int_0^t A_1^* p(s, x) ds + \int_0^t A_2^* p(s, x) d\beta(s),$$

which is the integral form of (2.1.2).

### 2.2.2 The initial value problem: existence, uniqueness and regularity

We denote by  $C_c^\infty(\mathbb{R}^d)$  the space of all smooth and compactly supported functions on  $\mathbb{R}^d$  and

$$\langle q, \varphi \rangle := \int_{\mathbb{R}^d} \varphi q dx.$$

We further denote by  $\mathcal{P}(\mathbb{R}^d)$  the space of all Borel probability measures on  $\mathbb{R}^d$ . Our notion of a weak solution is as follows:

**Definition 2.2.1** (Weak  $L^1$ -probability solution). *A function  $q \in C([0, T]; L^1(\mathbb{R}^d); C^{1/2}(\mathbb{R}, \mathbb{R}^d))$  is a weak  $L^1$ -probability solution of the initial value problem for the non-autonomous Fokker-Planck equation (2.2.2) if  $q$  solves this equation with initial condition  $q_0 \in \mathcal{P}(\mathbb{R}^d)$  and  $\beta \in C^{1/2}(\mathbb{R}, \mathbb{R}^d)$ , such that  $q(t) \in \mathcal{P}(\mathbb{R}^d)$  for all  $t > 0$  and  $\langle q(t), \varphi \rangle \rightarrow \langle q_0, \varphi \rangle$  as  $t \rightarrow 0$  for all  $\varphi \in C_c^\infty(\mathbb{R}^d)$ .*

Our main result establishes the existence and uniqueness of probability solutions of the  $L^1$  initial value problem and the fact that such solutions are smooth and their derivatives of any order are rapidly decreasing after any finite time, i.e. they belong to the Schwartz space

$$\mathcal{S} := \mathcal{S}(\mathbb{R}^d) := \{f \in C^\infty(\mathbb{R}^d) : \forall n, m \in \mathbb{N}^d, \|f\|_{n,m} < \infty\},$$

where  $C^\infty(\mathbb{R}^d)$  denotes the space of infinitely differentiable functions on  $\mathbb{R}^d$  and <sup>7</sup>

$$\|f\|_{n,m} := \sup_{x \in \mathbb{R}^d} |x^n (D^m f)(x)|. \quad (2.2.5)$$

**Theorem 2.2.2** (Existence, uniqueness and regularity). *Let  $q_0 \in \mathcal{P}(\mathbb{R}^d)$  and  $\beta \in C^{1/2}(\mathbb{R}, \mathbb{R}^d)$ . Let us assume the potential  $U$  is  $C^\infty$  in  $x$  and satisfies the dissipation condition*

$$\nabla U(x, t; \beta) \cdot x \|x\|^2 \geq \frac{1}{2} \|x\|^6 - C |\beta|^6 \quad (2.2.6)$$

*for some constant  $C > 0$ . Then, the non-autonomous Fokker-Planck equation (2.2.2) with initial condition  $q(0) = q_0$  admits a unique weak probability solution  $q(t) \in \mathcal{P}(\mathbb{R}^d)$  for all  $t > 0$  which is everywhere locally  $\alpha$ -Hölder continuous in time for any  $\alpha < \frac{1}{2}$ , such that  $q(\cdot, t; \beta) \in \mathcal{S}$  for  $t > 0$ .*

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<sup>7</sup>Here we use the multi-index notation, as in Evans [83].

We defer the proof of this result to Section 2.5.

It turns out that,  $L^1$  solutions are not unique in general. In Section 2.5, it is shown that uniqueness is ensured by a weighted integrability condition, expressing the fact that no mass can come from infinity, nor disappears through infinity, in finite time. It turns out (Lemma 2.5.1) that this weighted integrability condition is equivalent to the conservation of mass, hence ensuring existence and uniqueness in the context of probability solutions. In view of the particle systems motivation of Section 2.2.1, this setting is natural as it concerns the conservation of particles.

**Remark 2.2.3.** *Note that the dissipation condition (2.2.6) also ensures the existence of a unique stationary solution of (2.1.1). This condition is fulfilled, for example, in the case of the double well potential  $-\nabla V(x) = x(a - \|x\|^2)$ , cf. the example discussed in Section 2.1.3.*

## 2.3 Contraction property of the non-autonomous Fokker-Planck equation

In this section we establish that the time- $t$  evolution operator  $\Phi(t, \beta)$  of the non-autonomous Fokker-Planck equation (2.2.2) is a contraction for all  $t > 0$  and  $\beta \in C^{1/2}(\mathbb{R}, \mathbb{R}^d)$ . Technical proofs are deferred to Section 2.6.

In the autonomous setting ( $\eta = 0$ ), if  $V$  is strictly convex,  $\Phi$  is known to be a contraction for all  $t > 0$  if the metric on the solution space is chosen to be the usual Wasserstein distance  $W^p$  for any  $p \geq 1$  [30]. Moreover, strict convexity of  $V$  is also a necessary condition [163]. Under the milder assumption that the potential  $V$  is strictly convex outside a given ball in  $\mathbb{R}^d$ , with is a less restrictive and more realistic condition, Eberle [78] established contractivity of the evolution (again in the autonomous setting) for an appropriately chosen Kantorovich-Rubinstein metric. In this section, we adapt the results from [78] for autonomous Fokker-Planck equations to the non-autonomous setting.

Let us consider again (2.1.1) as a non-autonomous SDE where  $B(t) = \beta(t)$  at all times  $t$  for some  $\beta \in C^{1/2}(\mathbb{R}, \mathbb{R}^d)$ ,

$$dx(t) = -\nabla V(x(t))dt + \sigma dW(t) + \eta d\beta(t), \quad (2.3.1)$$

where  $W$  is a  $d$ -dimensional Brownian motion,  $\sigma, \eta \in \mathbb{R}^{d \times d}$  are constant matrices with positive determinants and the potential  $V$  satisfies the same assumptions as in Theorem 2.2.2. We denote by  $\mu_{t,\beta}$  and  $\nu_{t,\beta}$  the time- $t$  evolved probability measures with respect to a given input  $\beta$  and initial conditions  $\mu, \nu$  respectively. In other words, with  $p_\mu$  denoting the Lebesgue density of  $\mu$ , we have

$$\mu_{t,\beta}(A) := \int_A \Phi(t, \beta) p_\mu(x) dx, \quad \forall A \in \mathcal{B}(\mathbb{R}^d), \quad t \geq 0, \quad (2.3.2)$$

and similarly for  $\nu_{t,\beta}$ . We employ a reflection coupling method to determine a bound for the distance between  $\mu_{t,\beta}$  and  $\nu_{t,\beta}$  with respect to some appropriately chosen metric. This method entails the introduction of an additional auxiliary process  $y(t)$  such that  $x(t) = y(t)$  for  $t \geq T$ , for some  $T$ , adapting [78, Eqs. (2)-(3)] to the non-autonomous setting:

$$\begin{cases} dy(t) = -\nabla V(y(t))dt + \sigma(I - 2e(t)e^\top(t))dW(t) + \eta d\beta(t), & \text{for } t < T \\ y(t) = x(t), & \text{for } t \geq T \end{cases}$$

where  $T := \inf \{t \geq 0 : x(t) = y(t)\}$  is the coupling time and  $ee^\top$  is the orthogonal projection onto the unit vector

$$e(t) := \frac{\sigma^{-1}(x(t) - y(t))}{|\sigma^{-1}(x(t) - y(t))|}.$$

The general aim is to construct a function  $f$  such that the process  $e^{ct}f(|x(t) - y(t)|)$  is a (local) supermartingale for  $t < T$ , with a constant  $c > 0$  that is maximized by choosing  $f$  appropriately. This ensures uniform, exponential contraction with respect to a Kantorovich-Rubinstein metric  $\mathcal{W}_f$ .

**Definition 2.3.1** (Kantorovich-Rubinstein distance). *Let  $f \in C^2([0, \infty))$  be concave and increasing with  $f(0) = 0, f'(0) = 1$ . The  $\mathcal{W}_f$ -distance between two Borel probability measures  $\mu, \nu \in \mathcal{P}(\mathbb{R}^d)$  is defined by*

$$\mathcal{W}_f(\mu, \nu) := \inf_{\gamma} \mathbb{E}^{\gamma}[d_f(x, y)] = \inf_{\gamma} \int d_f(x, y) \gamma(dxdy)$$

with  $\mathbb{R}^d$  distance  $d_f(x, y) := f(\|x - y\|)$ , where  $\|\cdot\|$  is a norm on  $\mathbb{R}^d$ , and the infimum is taken over all couplings  $\gamma$  of  $\mu$  and  $\nu$ .<sup>8</sup>

In this Chapter, we always choose the norm to be  $\|\cdot\| = |\sigma^{-1} \cdot|$ , with  $|\cdot|$  denoting the Euclidean norm in  $\mathbb{R}^d$  and  $\sigma$  the nondegenerate diffusion matrix from (2.3.1).

We adapt [78, Theorem 1 and Corollary 2] to obtain:

**Proposition 2.3.2** (Kantorovich-Rubinstein contraction). *Consider the non-autonomous stochastic differential equation (2.3.1) and the setting of Theorem 2.2.2. Let  $\mu_{t,\beta}, \nu_{t,\beta}$  be time- $t$  evolved probability measures, as defined in (2.3.2). Then, there exist a constant  $c > 0$  and a convex and increasing function  $f$  such that for any  $\beta \in C^{1/2}(\mathbb{R}, \mathbb{R}^d)$ ,  $t > 0$  and initial conditions  $\mu, \nu \in \mathcal{P}(\mathbb{R}^d)$ ,*

$$\mathcal{W}_f(\mu_{t,\beta}, \nu_{t,\beta}) \leq e^{-ct} \mathcal{W}_f(\mu, \nu).$$

---

<sup>8</sup>Recall that a Borel measure  $\gamma$  on  $X \times X$  is called a coupling of Borel measures  $\mu$  and  $\nu$  on  $X$  if  $\gamma(A \times X) = \mu(A)$  and  $\gamma(X \times B) = \nu(B)$  for all  $A, B \in \mathcal{B}(X)$ .

Note that the function  $f$  in this proposition can be determined constructively. It turns out that convergence in the chosen Kantorovich-Rubinstein metric implies convergence in  $L^1$ .

**Proposition 2.3.3** (Convergence in  $L^1$ ). *Consider the non-autonomous stochastic differential equation (2.3.1) and the setting of Theorem 2.2.2. Let  $\mu_{t,\beta}$  be the time- $t$  evolved probability measure, as defined in (2.3.2). Assume that  $\beta \in C^{1/2}(\mathbb{R}, \mathbb{R}^d)$  is such that  $(\mu_{t,\beta})_{t>0}$  is a Cauchy sequence with respect to  $\mathcal{W}_f$ . Then, the sequence  $(p_{t,\beta})_{t>0}$  of the associated Lebesgue densities converges in  $L^1$ .*

At this point it is important to note that forward convergence, as obtained in Proposition 2.3.2, does not necessarily imply pullback convergence. While the contraction property ensures that all solutions approach each other as time progresses forwards, in order to guarantee pullback convergence additional conditions (on  $\beta$ ) must be satisfied. For instance, boundedness of  $\beta \in C^{1/2}(\mathbb{R}, \mathbb{R}^d)$  would suffice. In Section 2.4 we consider the stochastic setting of (2.1.2) and obtain in Theorem 2.4.2 pullback convergence for  $\mathbb{P}_B$ -almost all Brownian paths  $\beta$ .

## 2.4 The stochastic Fokker-Planck equation as a random dynamical system

In Sections 2.2 and 2.3, we have considered the Fokker Planck equation (2.1.2) as a non-autonomous PDE. In this Section we consider the stochastic setting with  $B(t)$  a Brownian motion and show that the resulting stochastic Fokker-Planck equation (2.1.2) is a random dynamical system.<sup>9</sup> We establish almost sure pullback attractors, using the contractivity obtained in the non-autonomous analysis in Section 2.3, noting that sample paths of the Brownian motion  $B(t)$  are  $\mathbb{P}_B$ -almost surely in  $C^{1/2}(\mathbb{R}, \mathbb{R}^d)$  [116].

The main results of this section concern the fact that (2.1.2) is a random dynamical system (Proposition 2.4.1) which (almost surely) possesses a unique pullback attractor (Theorem 2.4.2) and the correspondence between the common noise pullback attractor of (2.1.2) and a partial disintegration of the invariant Markov measure of (2.1.1) (Proposition 2.4.4). Technical proofs are deferred to Section 2.7. We first recall briefly some preliminaries from the random dynamical system approach towards stochastic differential equations, involving the sample path space of Brownian motions [11, Chapters 1,2 and Appendix A].

Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space and  $X$  be a metric space with Borel  $\sigma$ -algebra  $\mathcal{B}(X)$ . We consider the situation with two-sided continuous time  $t \in \mathbb{R}$ . A

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<sup>9</sup>Our results extend naturally to other common noise processes  $B(t)$ ; for instance, those described by an SDE of the form  $dB(t) = f(B(t))dt + \eta d\tilde{W}(t)$  for some  $f \in C^1$  and Brownian motion  $\tilde{W}(t)$ , such as the Ornstein-Uhlenbeck process.

*random dynamical system* on  $X$  consists of two components. The first is a *metric dynamical system* modelling the noise. This is a  $(\mathcal{B}(\mathbb{R}) \otimes \mathcal{F}, \mathcal{F})$ -measurable function  $\theta : \mathbb{R} \times \Omega \rightarrow \Omega$  such that

- (i)  $\theta(0, \omega) = \omega$  and  $\theta(t + s, \omega) = \theta(t, \theta(s, \omega))$  for all  $t, s \in \mathbb{R}, \omega \in \Omega$ ,
- (ii) the measure is preserved, i.e.  $\mathbb{P}(\theta(t, A)) = \mathbb{P}(A)$  for all  $t \in \mathbb{R}$  and  $A \in \mathcal{F}$ .

Moreover,  $\theta$  is said to be *ergodic* if for any  $A \in \mathcal{F}, \theta_t A = A$  for all  $t \in \mathbb{R}$  implies  $\mathbb{P}(A) \in \{0, 1\}$ . The second component is a mapping that models the dynamics of the system. This is a  $(\mathcal{B}(\mathbb{R}) \otimes \mathcal{F} \otimes \mathcal{B}(X), \mathcal{B}(X))$ -measurable function  $\phi : \mathbb{R} \times \Omega \times X \rightarrow X$  such that<sup>10</sup>

- (i)  $\phi(0, \omega, x) = x$  for all  $x \in X$ ,
- (ii)  $\phi(t + s, \omega, x) = \phi(t, \theta_s \omega, \phi(s, \omega, x))$  for all  $t, s \in \mathbb{R}$  and  $x \in X$ ,  $\mathbb{P}$ -almost surely (*cocycle property*).<sup>11</sup>

The skew-product structure  $\Theta : \mathbb{R} \times \Omega \times X \rightarrow \Omega \times X$  characterizing the random dynamical system  $(\theta, \phi)$  can be succinctly written as

$$\Theta(t)(\omega, x) := (\theta_t \omega, \phi(t, \omega, x)).$$

A probability measure  $\mu$  on  $(\Omega \times X, \mathcal{F} \otimes \mathcal{B}(X))$  is said to be *invariant* if

- (i)  $\mu(\Theta_t A) = \mu(A)$  for all  $t \in \mathbb{R}$  and  $A \in \mathcal{F} \otimes \mathcal{B}(X)$
- (ii) The marginal of  $\mu$  on  $(\Omega, \mathcal{F})$  is  $\mathbb{P}$ .

The canonical construction of the sample path space of Brownian motions can be briefly outlined as follows. Let  $\Omega := C_0(\mathbb{R}, \mathbb{R}^{2d})$  be the space of all continuous functions  $\xi : \mathbb{R} \rightarrow \mathbb{R}^{2d}$  such that  $\xi(0) = 0$ , endowed with the compact open topology. Let  $\mathcal{F} = \mathcal{B}(\Omega)$  denote the Borel  $\sigma$ -algebra on  $\Omega$ . Then, there exists the so-called *Wiener probability measure*  $\mathbb{P}$  on  $(\Omega, \mathcal{F})$  ensuring that the processes  $(B(t))_{t \in \mathbb{R}}$  and  $(W(t))_{t \in \mathbb{R}}$  are independent  $d$ -dimensional Brownian motions, with corresponding sample paths  $(\omega, \beta) := \xi \in \Omega = \Omega_W \times \Omega_B$  where  $\Omega_W$  and  $\Omega_B$  denote the intrinsic and common noise sample spaces. The natural filtration is the  $\sigma$ -algebra  $\mathcal{F}_{s,t}$  generated by  $\xi(u) - \xi(v)$  for  $s \leq v \leq u \leq t$ . The Wiener measure  $\mathbb{P}$  is ergodic with respect to the Wiener shift map  $\theta_t : \Omega \rightarrow \Omega$  defined by

$$(\theta_t \xi)(s) := \xi(s + t) - \xi(t), \quad s \in \mathbb{R}.$$

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<sup>10</sup>Here and throughout the Chapter we will use both the equivalent notations  $\phi(t, \omega, x)$  and  $\phi(t, \omega)x$ .

<sup>11</sup>This definition of random cocycle follows the convention in e.g. [15, 85]. In Arnold [11], the cocycle property is required to hold for *all*  $\omega \in \Omega$ , instead of almost surely. In case the cocycle exists for almost all  $\omega \in \Omega$  only,  $\phi$  is called a *crude cocycle* and through a *perfection procedure* it is possible to define an indistinguishable RDS for which the cocycle property is fulfilled for all noise realizations, see e.g. [77, Chapter 4.10] and references therein, most notably [87, 12].

Therefore,  $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in \mathbb{R}})$  is an ergodic random dynamical system. With this sample path evolution as explicit representation of the noise, the SDE (2.1.1) is a random dynamical system on the product  $\Omega \times \mathbb{R}^d$ . We find that the stochastic Fokker-Planck equation (2.1.2) is a random dynamical system on  $\Omega_B \times \mathcal{S}$ , where  $\mathcal{S}$  is the Schwartz space of solutions of (2.1.2) identified in Theorem 2.2.2.

**Proposition 2.4.1.** *The stochastic Fokker-Planck equation (2.1.2) is a random dynamical system.*

Next, we show that (2.1.2) possesses a unique global pullback attractor in the Schwartz space  $\mathcal{S}$  of rapidly decaying functions for  $\mathbb{P}_B$ -almost all  $\beta \in \Omega_B$ .

**Theorem 2.4.2** (Pullback attractor). *Let  $\Phi$  be the random dynamical system associated to (2.1.2). Then, for  $\mathbb{P}_B$ -almost all  $\beta \in \Omega_B$  (2.1.2) has a unique pullback attractor defined by*

$$p_\beta := \lim_{\tau \rightarrow \infty} \Phi(\tau, \theta_{-\tau}\beta)p$$

*which is independent of  $p \in L^1$ . Moreover,  $p_\beta \in \mathcal{S}$  and convergence is with respect to the semi-norm (2.2.5) on  $\mathcal{S}$ .*

By the *Correspondence Theorem* (see e.g. Arnold [11, Remark 1.4.2 and Proposition 1.4.3] and Crauel and Flandoli [62, Section 4]), if (2.1.1) has a unique stationary measure, then the corresponding random dynamical system has a unique invariant *Markov* measure, i.e. an invariant measure that is measurable with respect to the past,<sup>12</sup> and pullback attractors are identified with disintegrations of this Markov measure. We show in Proposition 2.4.4 that the common noise pullback attractors of (2.1.2) are equal to the expectation of the pullback attractors of (2.1.1) with respect to the intrinsic noise  $W$ , for a single fixed common noise realisation  $\beta$ . We summarize some well-established results on Markov measures and their disintegration [11] in the context of our setting.

**Proposition 2.4.3** (Markov measure and its disintegration). *Let  $\rho$  be the (unique) stationary measure of the random dynamical system  $\phi$  associated to (2.1.1) and*

$$\mu_{\omega, \beta} := \lim_{\tau \rightarrow \infty} \phi(\tau, \theta_{-\tau}\omega, \theta_{-\tau}\beta)_* \rho. \quad (2.4.1)$$

*Then  $\{\mu_{\omega, \beta}\}_{(\omega, \beta) \in \Omega}$  is  $\mathbb{P}$ -a.e. unique on  $\mathcal{B}(\mathbb{R}^d)$  and*

- (i) *for all  $C \in \mathcal{B}(\mathbb{R}^d)$ ,  $(\omega, \beta) \rightarrow \mu_{\omega, \beta}(C)$  is  $\mathcal{F}^-$ -measurable, where  $\mathcal{F}^- = \sigma(\cup_{s \leq t} \mathcal{F}_{s, t})$ .*
- (ii) *for  $\mathbb{P}$ -a.e.  $(\omega, \beta) \in \Omega_W \times \Omega_B$ ,  $\mu_{\omega, \beta}$  is a probability measure on  $(X, \mathcal{B}(\mathbb{R}^d))$ .*

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<sup>12</sup>See Proposition 2.4.3(i).

(iii) for all  $A \in \mathcal{F} \otimes \mathcal{B}(\mathbb{R}^d)$

$$\begin{aligned}\mu(A) &:= \int_{\Omega_B} \int_{\Omega_W} \int_X \mathbb{1}_A(\omega, \beta, x) \mu_{\omega, \beta}(dx) \mathbb{P}_W(d\omega) \mathbb{P}_B(d\beta) \\ &= \int_{\Omega_B} \int_{\Omega_W} \mu_{\omega, \beta}(A_{\omega, \beta}) \mathbb{P}_W(d\omega) \mathbb{P}_B(d\beta),\end{aligned}$$

where

$$A_{\omega, \beta} := \{x : (\omega, \beta, x) \in A\},$$

is an invariant probability measure of  $\phi$ . The measure  $\mu$  is known as the Markov measure associated to  $\rho$  and it is the unique invariant probability measure of  $\phi$  that is measurable with respect to the past, cf. (i), such that

$$\int_{\Omega_B} \int_{\Omega_W} \mu_{\omega, \beta} \mathbb{P}_W(d\omega) \mathbb{P}_B(d\beta) = \rho.$$

Against this background, we now prove that the common noise pullback attractor of (2.1.2) is the expectation of the pullback attractor of the underlying SDE (2.1.1) with respect to the intrinsic noise.

**Proposition 2.4.4** (Common noise pullback attractor). *Let  $\Phi$  be the random dynamical system associated to (2.1.2),  $p_\rho$  be the Lebesgue density of the stationary measure  $\rho$  of (2.1.1), and*

$$\mu_\beta := \int_{\Omega_W} \mu_{\omega, \beta} \mathbb{P}_W(d\omega).$$

*Then,  $\mathbb{P}_B$ -a.s.,  $\mu_\beta$  is a probability measure on  $\mathcal{B}(\mathbb{R}^d)$  with Lebesgue density  $p_\beta$ , where*

$$p_\beta = \lim_{\tau \rightarrow \infty} \Phi(\tau, \theta_{-\tau} \beta) p_\rho$$

*and*

$$\int_{\Omega_B} p_\beta \mathbb{P}_B(d\beta) = p_\rho.$$

We finally stipulate a direct consequence of ergodicity for observables  $g : \mathcal{S} \rightarrow \mathbb{R}$ .

**Proposition 2.4.5.** *Let  $g : \mathcal{S} \rightarrow \mathbb{R}$  be continuous and integrable, then*

$$\lim_{\tau \rightarrow \infty} \frac{1}{\tau} \int_0^\tau g(\Phi(t, \beta) p) dt = \int_{\Omega_B} g(p_\beta) \mathbb{P}_B(d\beta), \quad (2.4.2)$$

*$\mathbb{P}_B$ -almost surely, independent of  $p \in \mathcal{S}$ .*

## 2.5 Proof of Theorem 2.2.2

We start this section by demonstrating the equivalence between total mass conservation for measures and a weighted integrability condition to be fulfilled by a weak solution  $q(t)$  of the initial value problem (2.2.2). As we shall discuss below, this condition will be employed to prove an important  $L^1$  estimate (Lemma 2.5.6) which, in turn, will be crucial for establishing uniqueness of solutions (Lemma 2.5.7).

**Lemma 2.5.1** (Equivalence between mass conservation and the weighted integrability condition). *A weak solution  $q$  of the non-autonomous Fokker-Planck equation (2.2.2) with initial condition  $q_0 \in \mathcal{P}(\mathbb{R}^d)$  is a probability solution for any given  $\beta \in C^{1/2}(\mathbb{R}, \mathbb{R}^d)$ , i.e.*

$$q(t, x; \beta) \geq 0, \quad \int_{\mathbb{R}^d} q(t, x; \beta) dx = 1 \text{ for all } t \geq 0,$$

if and only if the weighted integrability condition

$$\lim_{N \rightarrow \infty} \int_0^T \int_{N < \|x\| < 2N} \|x\|^{-1} \|\nabla U\| \|q(t, x; \beta)\| dx dt = 0 \quad (\text{W.I.C.})$$

holds.

*Proof.* Let us define the test function  $\varphi_N := \vartheta(\frac{x}{N})$  for any  $N \in \mathbb{N}$ , where  $\vartheta \in C_c^\infty(\mathbb{R}^d, \mathbb{R}_+)$  is a cut-off function such that  $\vartheta' \leq 0$  and

$$\vartheta(z) = \begin{cases} 1 & \text{if } z \in [0, 1]^d \\ 2 - z & \text{if } z \in [1, 2]^d. \end{cases}$$

$\vartheta$  is constructed to be at least  $C^2$  at  $z = 1$  and  $z = 2$ . It is assumed to be equal to 0 for  $z \in [2, \infty)^d$  and extended evenly for  $z \in (-\infty, 0]^d$ . Testing the non-autonomous Fokker Planck equation (2.2.2) with  $\varphi_N$ , integrating by parts and rearranging terms we obtain

$$\langle q(0), \varphi_N \rangle - \int_0^T \int_{\mathbb{R}^d} \nabla U \nabla \varphi_N q dx dt = \langle q(T), \varphi_N \rangle - \int_0^T \langle q, \Delta \varphi_N \rangle dt,$$

where we suppressed the dependence on  $(x, \beta)$  in order to simplify the notation. Since in the limit  $N \rightarrow \infty$   $\Delta \varphi_N \sim N^{-2}$  and

$$\begin{aligned} \lim_{N \rightarrow \infty} \langle q(T), \varphi_N \rangle &= \|q(T)\|_{L^1} \\ \lim_{N \rightarrow \infty} \langle q(0), \varphi_N \rangle &= \|q(0)\|_{L^1}, \end{aligned}$$

we immediately obtain

$$\lim_{N \rightarrow \infty} \int_0^T \int_{N < \|x\| < 2N} \|x\|^{-1} \nabla U q dx dt = \left| \|q(T)\|_{L^1} - \|q(0)\|_{L^1} \right|$$

and therefore we conclude.  $\square$

**Remark 2.5.2** (Probability and bounded measures). *The problems of uniqueness in the class of probability measures and in the class of all bounded measures are not equivalent. Consider for simplicity the one dimensional case, the autonomous scenario  $\beta \equiv 0$  and the potential*

$$V'(x) = 4x^3 + 16x^3(1 + 4x^4)^{-1}.$$

*Then, the Fokker-Planck equation (2.2.2) admits the following stationary, bounded sign-changing solution*

$$q(x) = x(1 + 4x^4)^{-1}.$$

*The weighted integrability condition is violated, since*

$$\int_{\mathbb{R}} x^2 |q(x)| dx = +\infty.$$

*At the same time, (2.2.2) admits the well known stationary probability solution*

$$\rho(x) = e^{-V(x)} = (1 + 4x^4)^{-1} e^{-x^4}.$$

*Therefore, there exists a unique solution in the class of probability measures, but there are also nonzero signed solutions in the class of bounded measures. For further details, see [25], Example 4.1.3.*

Theorem 2.2.2 is proved by combining a series of energy-type estimates. We remark that, although the focus of this theorem is on probability measures, from here onwards we consider more broadly the evolution of signed measures. This is needed for the proof of uniqueness of the weak solution in Lemma 2.5.7, where the evolution of the difference between two probability solutions is considered.

We make the following key assumptions:

**Assumption 2.5.3.**

- (I) *Weak  $L^1$  solutions of the Fokker Planck equation (2.2.2) are required to satisfy the weighted integrability condition (W.I.C.).*
- (II) *The initial condition  $q_0$  is a signed measure.*
- (III) *The potential  $U$  is infinitely differentiable and satisfies the dissipation condition (2.2.6).*

Restricted to the setting of probability measures, Assumption 2.5.3 boils down to the setting of Theorem 2.2.2.

Next, we note that the space  $L^1(\mathbb{R}^d)$  can be interpreted as regular measures and embedded isometrically into the space of signed Borel measures  $\mathcal{M}(\mathbb{R}^d)$ . Although this result is already known, for the sake of having a self-contained discussion, we provide below an explicit proof, adapting the one in [144], Proposition 2.7.

**Lemma 2.5.4** (Approximation of signed measures). *For every measure  $\mu \in \mathcal{M}(\mathbb{R}^d)$ , there exists a sequence of (signed) measures  $(\mu_n)_{n \in \mathbb{N}} \in L^1(\mathbb{R}^d)$  such that*

$$\|\mu_n\|_{L^1} \leq \|\mu\|_{\mathcal{M}(\mathbb{R}^d)}$$

for any  $n \in \mathbb{N}$  and

$$\lim_{n \rightarrow \infty} \langle \mu_n, \varphi \rangle = \langle \mu, \varphi \rangle$$

for all test functions  $\varphi \in C_0(\mathbb{R}^d)$ , where  $C_0(\mathbb{R}^d)$  denotes the space of all continuous functions with compact support on  $\mathbb{R}^d$ .

*Proof.* Let  $(\varrho_n)_{n \in \mathbb{N}}$  be a sequence of mollifiers, that is, for every  $n \in \mathbb{N}$ ,  $\varrho_n \in C_0^\infty(\mathbb{R}^d)$ ,  $\varrho_n$  is nonnegative, such that

$$\int_{\mathbb{R}^d} \varrho_n = 1$$

and for every  $\delta > 0$

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^d \setminus B(0, \delta)} \varrho_n = 0,$$

where  $B(0, \delta)$  denotes the open ball centred at 0 with radius  $\delta$ . Then, we consider the convolution  $\mu_n := \varrho_n * \mu$  between the mollifier and the measure  $\mu \in \mathcal{M}(\mathbb{R}^d)$ . We immediately have  $\mu_n \in L^1(\mathbb{R}^d)$ . From the convolution definition and thanks to Fubini's theorem, we deduce

$$\begin{aligned} \int_{\mathbb{R}^d} \varphi d(\varrho_n * \mu) &= \int_{\mathbb{R}^d} \varphi(x) \int_{\mathbb{R}^d} \varrho_n(x-y) d\mu(y) dx \\ &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \varrho_n(x-y) \varphi(x) dx d\mu(y) \\ &= \int_{\mathbb{R}^d} \varrho_n * \varphi d\mu. \end{aligned}$$

Since by construction  $\varphi \in C_0(\mathbb{R}^d)$ , the sequence  $(\varrho_n * \varphi)_{n \in \mathbb{N}}$  converges uniformly to  $\varphi$  on  $\mathbb{R}^d$ . Hence,  $\mu_n \rightarrow \mu$  weakly. Finally, we observe that

$$|\mu_n| \leq \int_{\mathbb{R}^d} \varrho_n(x-y) d|\mu|(y).$$

Applying again Fubini's theorem we obtain

$$\|\mu_n\|_{L^1} \leq \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} \varrho_n(x-y) dx \right) d|\mu|(y) \leq \int_{\mathbb{R}^d} d|\mu|(y) = \|\mu\|_{\mathcal{M}(\mathbb{R}^d)}$$

and we conclude. □

Since our result will imply  $q(t) \in L^1(\mathbb{R}^d)$  for all  $t > 0$ , thanks to Lemma (2.5.4), in the proof of Theorem 2.2.2 we can restrict ourselves to the case  $q_0 \in L^1(\mathbb{R}^d)$ .

Our strategy proceeds as follows. First, we prove the  $L^1$  estimate (2.5.10) by employing the weighted integrability condition (W.I.C.). Thanks to (2.5.10), we establish uniqueness in Lemma 2.5.7. Next, we prove the weighted  $L^1$ -estimate (2.5.12), which controls the behaviour of the tails of the solution. Following the same ideas, we prove the  $L^1$ -localization estimate (2.5.15) and the two smoothing estimates (2.5.17) and (2.5.21). These results establish that the unique solution of the Fokker Planck equation (2.2.2) is smooth in the space variable  $x$ , as regular in time as the function  $\beta$  and belongs to  $L^2(\mathbb{R}^d)$  at all times  $t > 0$ . Exploiting the structure of the equation and the infinite differentiability of the potential, we iterate the argument and achieve infinite differentiability and rapidly decreasing behaviour of all derivatives.

The  $L^1$ -estimate (2.5.10) will be proved with the help of inequality (2.5.5) below. Given a weak solution  $q$  of the non-autonomous Fokker Planck equation (2.2.2), the fundamental idea to prove this inequality consists in regularizing the equation with a mollifier  $\vartheta_\delta$  first and then taking the limit  $\delta \rightarrow 0$ . Let  $\vartheta \in C_0^\infty(\mathbb{R}^d, \mathbb{R}_+)$  denote a non-negative mollification kernel satisfying  $\int_{\mathbb{R}^d} \vartheta(x) dx = 1$  and define the standard Dirac's delta approximation

$$\vartheta_\delta(x) := \frac{1}{\delta} \vartheta\left(\frac{x}{\delta}\right).$$

Observe that, since  $\vartheta$  has a compact support,

$$\nabla_x \vartheta_\delta(x - y) \neq 0 \quad \text{if } |x - y| \leq C\delta \tag{2.5.1}$$

for some positive constant  $C$ . Moreover, using integration by parts,

$$\int_{\mathbb{R}^d} \vartheta'(s) s ds = - \int_{\mathbb{R}^d} \vartheta(s) ds = -1. \tag{2.5.2}$$

We denote by  $S_\delta$  the mollification operator

$$(S_\delta(q))(x) := \int_{\mathbb{R}^d} \vartheta_\delta(x - y) q(y) dy \tag{2.5.3}$$

and by  $\text{sgn}_\gamma(x)$  the standard smooth and monotone approximation of  $\text{sgn}(x)$ , i.e.

$$\text{sgn}_\gamma(x) := \frac{x}{\sqrt{x^2 + \gamma^2}}. \tag{2.5.4}$$

Finally, we define

$$|z|_\gamma := \int_0^z \text{sgn}_\gamma(s) ds.$$

and notice that, by construction,  $\lim_{\gamma \rightarrow 0} |z|_\gamma = |z|$ .

**Lemma 2.5.5** (Weak solution inequality). *Any weak solution of (2.2.2) satisfies the inequality*

$$\frac{d}{dt} \langle \|q(\cdot, t; \beta)\|, \varphi \rangle \leq \langle \partial_t \varphi - \nabla U \nabla \varphi + \Delta \varphi, \|q(\cdot, t; \beta)\| \rangle \quad (2.5.5)$$

for almost all  $t \geq 0$ , any  $\beta \in C^{1/2}(\mathbb{R}, \mathbb{R}^d)$  and  $\varphi \in C_0^\infty((0, T) \times \mathbb{R}^d)$  such that  $\varphi \geq 0$ .

*Proof.* We apply the mollification operator  $S_\delta$  defined in (2.5.3) to both sides of the non-autonomous Fokker-Planck equation (2.2.2), thereby obtaining

$$\partial_t(S_\delta(q)) = \nabla(S_\delta(\nabla U q)) + \Delta(S_\delta(q)), \quad S_\delta(q)|_{t=0} = S_\delta(q_0).$$

We define the test function

$$\psi(t, x) := \varphi(t, x) \operatorname{sgn}_\gamma(S_\delta(q)(x)),$$

where  $\varphi$  is the test function in (2.5.5) and  $\operatorname{sgn}_\gamma$  is the smooth approximation of the sgn function defined in (2.5.4). We obtain

$$\frac{d}{dt} \langle |S_\delta(q)|_\gamma, \psi \rangle = \underbrace{\langle \partial_t \psi, |S_\delta(q)|_\gamma \rangle}_a + \underbrace{\langle \nabla(S_\delta(\nabla U q)), \psi \rangle}_b + \underbrace{\langle \Delta S_\delta(q), \psi \rangle}_c, \quad (2.5.6)$$

The idea now is to write each term on the RHS of (2.5.6) in a convenient form by means of integration by parts and take the limit  $\delta \rightarrow 0$ . Firstly, integration by parts implies term (a) can be rewritten as

$$\langle \Delta S_\delta(q), \psi \rangle = -\langle \nabla S_\delta(q), \operatorname{sgn}_\gamma'(S_\delta(q)) \nabla S_\delta(q) \varphi \rangle - \langle \nabla S_\delta(q), \operatorname{sgn}_\gamma(S_\delta(q)) \nabla \varphi \rangle, \quad (2.5.7)$$

where  $'$  denotes the derivative, since boundary terms vanish thanks to the dissipation condition (2.2.6). Dropping the first term on the RHS of (2.5.7) and applying integration by parts again yields

$$\langle S_\delta(q), \psi \rangle \leq \langle \Delta \varphi, |S_\delta(q)|_\gamma \rangle.$$

Taking the limit,

$$\lim_{(\gamma, \delta) \rightarrow (0, 0)} \langle \Delta \varphi, |S_\delta(q)|_\gamma \rangle = \langle |q|, \Delta \varphi \rangle.$$

Next, term (b) in (2.5.6) reads as

$$\begin{aligned} \langle \nabla(S_\delta(\nabla U q)), \varphi \operatorname{sgn}_\gamma(S_\delta(q)) \rangle &= \langle \nabla(\nabla U S_\delta(q)), \varphi \operatorname{sgn}_\gamma(S_\delta(q)) \rangle \\ &\quad + \langle \nabla(S_\delta(\nabla U q) - \nabla U S_\delta(q)), \varphi \operatorname{sgn}_\gamma(S_\delta(q)) \rangle. \end{aligned} \quad (2.5.8a)$$

$$(2.5.8b)$$

The first term (2.5.8a) can be written as

$$\begin{aligned}
\langle \nabla(\nabla U S_\delta(q)), \varphi \operatorname{sgn}_\gamma(S_\delta(q)) \rangle &= \langle \Delta U \varphi, S_\delta(q) \operatorname{sgn}_\gamma(S_\delta(q)) \rangle + \langle \nabla U \psi, \nabla |S_\delta(q)|_\gamma \rangle \\
&= \langle \Delta U \varphi, S_\delta(q) \operatorname{sgn}_\gamma(S_\delta(q)) \rangle - \langle \nabla(\nabla U \varphi), |S_\delta(q)|_\gamma \rangle \\
&\rightarrow \langle \Delta U \varphi, |q| \rangle - \langle \nabla(\nabla U \varphi), |q| \rangle \\
&= -\langle \nabla U \nabla \varphi, |q| \rangle
\end{aligned}$$

as  $(\gamma, \delta) \rightarrow (0, 0)$ . In order to write more explicitly the term (2.5.8b), we recall that

$$S_\delta(\nabla U q)(x) - \nabla U S_\delta(q)(x) = \int_{\mathbb{R}^d} \vartheta_\delta(x - y) [\nabla_y U - \nabla_x U] q(y) dy$$

and therefore

$$\begin{aligned}
&\langle \nabla(S_\delta(\nabla U q) - \nabla U S_\delta(q)), \varphi \operatorname{sgn}_\gamma(S_\delta(q)) \rangle \\
&= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \nabla \vartheta_\delta(x - y) (\nabla_y U - \nabla_x U) q(y) \varphi(x) \operatorname{sgn}_\gamma(S_\delta(q)(x)) dy dx \quad (2.5.9a)
\end{aligned}$$

$$- \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \vartheta_\delta(x - y) \Delta U(x) q(y) \varphi(x) \operatorname{sgn}_\gamma(S_\delta(q)(x)) dy dx. \quad (2.5.9b)$$

Taking the limit  $\delta \rightarrow 0$  in (2.5.9b) yields

$$\lim_{\delta \rightarrow 0} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \vartheta_\delta(x - y) \Delta U(x) q(y) \varphi(x) \operatorname{sgn}_\gamma(S_\delta(q)(x)) dy dx = \langle \Delta U \varphi, \operatorname{sgn}_\gamma(q) \rangle.$$

For what concerns the integral term (2.5.9a), instead, we use (2.5.1). Consequently, we may write

$$\nabla_y U - \nabla_x U = -\Delta U(x)(x - y) + O(\delta^2(\|x\| + 1)).$$

Thus, at first order approximation, (2.5.9a) reduces to

$$- \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \nabla \vartheta_\delta(x - y) \Delta U(x)(x - y) q(y) \varphi(x) \operatorname{sgn}_\gamma(S_\delta(q)(x)) dy dx.$$

Thanks to (2.5.2), we deduce

$$\begin{aligned}
&\lim_{\delta \rightarrow 0} - \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \nabla \vartheta_\delta(x - y) \Delta U(x)(x - y) q(y) \varphi(x) \operatorname{sgn}_\gamma(S_\delta(q)(x)) dy dx \\
&= \langle \Delta U \varphi, \operatorname{sgn}_\gamma(q) \rangle.
\end{aligned}$$

Hence, (2.5.8b) vanishes as  $\delta \rightarrow 0$ . Putting everything together, we let  $(\gamma, \delta) \rightarrow (0, 0)$ , observe that  $\psi \rightarrow \varphi$  and  $|S_\delta(q)|_\gamma \rightarrow |q|$  and finally obtain the inequality (2.5.5).  $\square$

We are now ready to prove the following  $L^1$ -estimate, the proof of which heavily relies on the weighted integrability condition (W.I.C.).

**Lemma 2.5.6** ( $L^1$ -estimate). *Any weak solution of the non-autonomous Fokker Planck equation (2.2.2) satisfies for any given  $\beta \in C^{1/2}(\mathbb{R}, \mathbb{R}^d)$  the  $L^1$ -estimate*

$$\|q(\cdot, t; \beta)\|_{L^1} \leq \|q(\cdot, 0; \beta)\|_{L^1}, \quad t > 0. \quad (2.5.10)$$

*Proof.* Let us consider a cut-off function  $\vartheta \in C_0^\infty(\mathbb{R}^d, [0, 1])$  such that

$$\vartheta(z) = \begin{cases} 1 & \text{if } z \in [-1, 1]^d \\ 0 & \text{if } z \in [2, \infty)^d \end{cases}$$

and the test function  $\varphi_N := \vartheta\left(\frac{x}{N}\right)$  for some  $N \in \mathbb{N}$ . Then, using the fact that the function  $t \rightarrow \langle |q(t)|, \varphi_N \rangle$  is continuous in time, inequality (2.5.5) implies

$$\langle |q(T)|, \varphi_N \rangle \leq \langle |q(\tau)|, \varphi_N \rangle + \int_\tau^T \langle \Delta \varphi_N - \nabla U \nabla \varphi_N, \|q\| \rangle dt, \quad (2.5.11)$$

where we suppressed the dependence on  $x$  and  $\beta$  in order to simplify the notation. The weighted integrability condition (W.I.C.) ensures that the integral on the RHS of (2.5.11) tends to 0 as  $N \rightarrow +\infty$ . Indeed,

$$\begin{aligned} |\langle \nabla U \nabla \varphi_N, \|q\| \rangle| &\leq \frac{1}{N} \int_{N < \|x\| < 2N} \|\nabla U\| \left| \vartheta'\left(\frac{x}{N}\right) \right| \|q\| dx \\ &\leq C \int_{N < \|x\| < 2N} \|x\|^{-1} \|\nabla U\| \|q\| dx \end{aligned}$$

and

$$\int_\tau^T \langle \nabla U \nabla \varphi_N, \|q\| \rangle dt \leq \int_\tau^T \int_{N < \|x\| < 2N} \|x\|^{-1} \|\nabla U\| \|q\| dx dt \rightarrow 0,$$

as  $N \rightarrow \infty$ . Passing to this limit in (2.5.11) therefore yields

$$\|q(\tau)\|_{L^1} \leq \|q(0)\|_{L^1}.$$

Finally, we let  $\tau \rightarrow 0$ , exploit continuity and conclude.  $\square$

Uniqueness of the weak solution for the initial value problem (2.2.2) is an immediate consequence of Lemma 2.5.6.

**Lemma 2.5.7** (Uniqueness). *The initial value problem (2.2.2) admits a unique weak solution for any given  $\beta \in C^{1/2}(\mathbb{R}, \mathbb{R}^d)$ .*

*Proof.* Let us denote by  $q_1, q_2$  two distinct weak solutions of (2.2.2) with the same initial condition  $q_0$ . Define  $q(x, t; \beta) := q_1(x, t; \beta) - q_2(x, t; \beta)$ . Then, thanks to linearity,  $q$  will be a weak solution of (2.2.2) and

$$\|q(\cdot, t; \beta)\|_{L^1} \leq \|q(\cdot, 0; \beta)\|_{L^1} = 0$$

thanks to the  $L^1$ -estimate (2.5.10).  $\square$

Next, we proceed with proving the following weighted  $L^1$ -estimate, which describes the global behaviour of the tails of the weak solution.

**Lemma 2.5.8** (Weighted  $L^1$ -estimate). *The unique weak solution of the non-autonomous Fokker Planck equation (2.2.2) satisfies for any given  $\beta \in C^{1/2}(\mathbb{R}, \mathbb{R}^d)$  the weighted  $L^1$ -estimate*

$$\|(1 + x^n)q(\cdot, t; \beta)\|_{L^1} \leq C(\beta)\|(1 + x^n)q(\cdot, 0; \beta)\|_{L^1} \quad (2.5.12)$$

for any  $n \in \mathbb{N}$  and some constant  $C = C(\beta) < \infty$ .<sup>13</sup>

*Proof.* We multiply the non-autonomous Fokker Planck equation (2.2.2) by  $(1 + x^n)\text{sgn}(q)$  for any  $n \in \mathbb{N}$  and integrate over  $\mathbb{R}^d$ :

$$\begin{aligned} \frac{d}{dt}\|(1 + x^n)q\|_{L^1} &= \int_{\mathbb{R}^d} (1 + x^n)\text{sgn}(q)\partial_t q dx \\ &= \underbrace{\int_{\mathbb{R}^d} (1 + x^n)\text{sgn}(q)\Delta q dx}_a + \underbrace{\int_{\mathbb{R}^d} (1 + x^n)\text{sgn}(q)\nabla(\nabla U q) dx}_b. \end{aligned}$$

Integration by parts applied to the term (b) yields

$$\begin{aligned} \int_{\mathbb{R}^d} (1 + x^n)\text{sgn}(q)\nabla(\nabla U q) dx &= \nabla U q (1 + x^n)\text{sgn}(q) \Big|_{-\infty}^{+\infty} - \int_{\mathbb{R}^d} \nabla U q \text{sgn}(q) n x^{n-1} dx \\ &\leq -\gamma \|(1 + x^{3+(n-1)})q\|_{L^1} + C(\beta)\|q\|_{L^1}, \end{aligned}$$

for some constants  $\gamma > 0$  and  $C(\beta) > 0$  depending on the modulus of  $\beta$ , where we used the dissipation condition for the shifted potential (2.2.6). Similarly, integration by parts applied to the term (a) yields

$$\begin{aligned} \int_{\mathbb{R}^d} (1 + x^n)\text{sgn}(q)\Delta q dx &= - \int_{\mathbb{R}^d} (\nabla q)\text{sgn}(q) n x^{n-1} dx \\ &\leq -n \int_{\mathbb{R}^d} \nabla \|q\| x^{n-1} dx \\ &= n(n-1) \int_{\mathbb{R}^d} \|q\| x^{n-2} dx = n(n-1)\|x^{n-2}q\|_{L^1}. \end{aligned}$$

Hence,

$$\frac{d}{dt}\|(1 + x^n)q\|_{L^1} + \gamma\|(1 + x^{3+(n-1)})q\|_{L^1} \leq C(\beta)\|q\|_{L^1}$$

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<sup>13</sup>In this and subsequent lemmas time-dependence of the constant  $C$  is not problem. Since we are interested in local regularity for  $t > 0$ , we are considering  $t \in [0, T]$ . Without loss of generality, we might set  $T = 1$ . For what concerns the  $\beta$ -dependence instead, the constant  $C$  in general will not be uniform with respect to  $\beta$ . To gain uniformity, additional assumptions on  $\beta$  would be required (such as boundedness), but in our context this is not needed.

Setting  $\gamma = 0$ , integrating with respect to time, using the  $L^1$ -estimate (2.5.10) and noticing that

$$\|(1 + x^n)q(\cdot, 0; \beta)\|_{L^1} \geq \|q(\cdot, 0; \beta)\|_{L^1},$$

we finally obtain inequality (2.5.12).  $\square$

Next, given a weighted weak solution of our non-autonomous Fokker-Planck equation, we establish a  $L^1$ -localization estimate.

**Lemma 2.5.9** ( $L^1$  localization estimate). *The unique weak solution of the non-autonomous Fokker Planck equation (2.2.2) satisfies for any given  $\beta \in C^{1/2}(\mathbb{R}, \mathbb{R}^d)$  the  $L^1$  localization estimate*

$$\|(1 + x^n)q(\cdot, t; \beta)\|_{L^1} \leq C(\beta) \frac{1 + t^N}{t^N} \|q(\cdot, 0; \beta)\|_{L^1}, \quad (2.5.15)$$

for any  $n \in \mathbb{N}, t > 0$ , some  $N \in \mathbb{N}$  and some constant  $C = C(\beta) < \infty$ .

*Proof.* Let us consider again the inequality

$$\frac{d}{dt} \|(1 + x^n)q\|_{L^1} + \gamma \|(1 + x^{3+(n-1)})q\|_{L^1} \leq C(\beta) \|q\|_{L^1}$$

for any  $n \in \mathbb{N}$  and some constant  $\gamma > 0$ , as derived in the proof of Lemma 2.5.8. We multiply both sides by  $t^N$  for some  $N \in \mathbb{N}$ :

$$t^N \frac{d}{dt} \|(1 + x^n)q\|_{L^1} + t^N \gamma \|(1 + x^{3+(n-1)})q\|_{L^1} \leq t^N C(\beta) \|q\|_{L^1}.$$

Using the Hölder inequality

$$t^{N-1} \|(1 + x^n)q\|_{L^1} \leq \|q\|_{L^1}^{\frac{1}{N}} (t^N \|(1 + x^{n \frac{N}{N-1}})q\|_{L^1})^{\frac{N-1}{N}}, \quad (2.5.16)$$

we obtain

$$\frac{d}{dt} (t^N \|(1 + x^n)q\|_{L^1}) + \gamma t^N \|(1 + x^{3+(n-1)})q\|_{L^1} \leq C(\beta) (1 + t^N) \|q\|_{L^1},$$

which implies

$$\frac{d}{dt} (t^N \|(1 + x^n)q\|_{L^1}) \leq C(\beta) (1 + t^N) \|q\|_{L^1} \leq C(\beta) (1 + t^N) \|q_0\|_{L^1},$$

thanks to the  $L^1$ -estimate (2.5.10). Integrating on both sides, we conclude.  $\square$

Next, we prove two smoothing estimates which ensure the unique weak solution of the initial value problem (2.2.2) belongs to  $L^2$  and, in fact, to the Sobolev space  $H^1 := W^{1,2}$ , for all  $t > 0$ , that is, its spatial derivative belongs to  $L^2$  as well.

**Lemma 2.5.10** (First smoothing estimate). *The unique weak solution of the non-autonomous Fokker Planck equation (2.2.2) satisfies for any given  $\beta \in C^{1/2}(\mathbb{R}, \mathbb{R}^d)$*

$$\|q(\cdot, t; \beta)\|_{L^2}^2 + \int_t^{t+1} \|\nabla_s q(s, t; \beta)\|_{L^2}^2 ds \leq C(\beta) \frac{t^N + 1}{t^N} \|q(\cdot, 0; \beta)\|_{L^1}^2, \quad (2.5.17)$$

for some  $N \in \mathbb{N}$  and some constant  $C = C(\beta) < \infty$ .

*Proof.* Let us multiply the non-autonomous Fokker Planck equation (2.2.2) by  $q$  and integrate over  $\mathbb{R}^d$ :

$$\int_{\mathbb{R}^d} q \partial_t q dx = \int_{\mathbb{R}^d} q \Delta q dx + \int_{\mathbb{R}^d} q \nabla(\nabla U q) dx.$$

Using integration by parts we obtain

$$\begin{aligned} \int_{\mathbb{R}^d} q \Delta q dx &= \nabla q \nabla q \Big|_{-\infty}^{+\infty} - \int_{\mathbb{R}^d} (\nabla q)^2 dx = -\|\nabla q\|_{L^2}^2 \\ \int_{\mathbb{R}^d} q \nabla(\nabla U q) dx &= \frac{1}{2} \int_{\mathbb{R}^d} \|q\|^2 \Delta U dx. \end{aligned}$$

Putting everything together,

$$\frac{1}{2} \frac{d}{dt} \|q\|_{L^2}^2 + \|\nabla q\|_{L^2}^2 = \frac{1}{2} \langle \Delta U, \|q\|^2 \rangle.$$

Thanks to the dissipation condition (2.2.6) on the potential  $U$ , we obtain

$$\frac{1}{2} \langle \Delta U, q^2 \rangle \leq C(\beta) \langle 1 + \|x\|^2, q^2 \rangle \leq C(\beta) \|(1 + \|x\|^2)q\|_{L^1} \|q\|_{L^\infty}.$$

Next, we use the inequality

$$\|q\|_{L^\infty}^2 \leq \|q\|_{L^2} \|\nabla q\|_{L^2} \quad (2.5.19)$$

and deduce

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|q\|_{L^2}^2 + \|\nabla q\|_{L^2}^2 &\leq C(\beta) \|(1 + \|x\|^2)q\|_{L^1} \|q\|_{L^\infty} \\ &\leq C(\beta) \|(1 + \|x\|^2)q\|_{L^1} \|q\|_{L^2}^{1/2} \|\nabla q\|_{L^2}^{1/2}. \end{aligned}$$

Using also  $ab \leq \frac{1}{2}(\epsilon^2 a^2 + \epsilon^{-2} b^2)$  for  $\epsilon$  small enough, we obtain

$$\frac{d}{dt} \|q\|_{L^2}^2 + \|\nabla q\|_{L^2}^2 \leq C(\beta) \|q\|_{L^2}^2 + C(\beta) \|(1 + \|x\|^2)q\|_{L^1}^2.$$

Using inequality (2.5.19), we also have

$$\begin{aligned}\|q\|_{L^2}^2 &\leq \|q\|_{L^1}\|q\|_{L^\infty} \\ &\leq \|q\|_{L^1}\|q\|_{L^2}^{1/2}\|\nabla q\|_{L^2}^{1/2},\end{aligned}$$

from which we deduce

$$\|q\|_{L^2} \leq \|q\|_{L^1}^{2/3}\|\nabla q\|_{L^1}^{1/3}. \quad (2.5.20)$$

Putting everything together, we obtain the relationship

$$\frac{d}{dt}\|q\|_{L^2}^2 + \|\nabla q\|_{L^2}^2 \leq C(\beta)\|(1+x^2)q\|_{L^1}.$$

Multiplying by  $t^N$  for some  $N \in \mathbb{N}$ , using again inequality (2.5.20) and integrating with respect to time from  $t$  to  $t+1$ , we conclude.  $\square$

**Lemma 2.5.11** (Second smoothing estimate). *The unique weak solution of the non-autonomous Fokker Planck equation (2.2.2) satisfies for any given  $\beta \in C^{1/2}(\mathbb{R}, \mathbb{R}^d)$*

$$\|q(\cdot, t; \beta)\|_{H^1} \leq C(\beta) \frac{1+t^N}{t^N} \|q(\cdot, 0; \beta)\|_{L^1} \quad (2.5.21)$$

for some  $N \in \mathbb{N}$  and some constant  $C = C(\beta) < \infty$ .

*Proof.* Let us multiply the non-autonomous Fokker Planck equation (2.2.2) by  $\Delta q$  and integrate over  $\mathbb{R}^d$ :

$$\underbrace{\int_{\mathbb{R}^d} \partial_t q \Delta q dx}_a = \underbrace{\int_{\mathbb{R}^d} (\Delta q)^2 dx}_b + \underbrace{\int_{\mathbb{R}^d} \Delta q \nabla(\nabla U q) dx}_c. \quad (2.5.22)$$

Terms (a) and (b) in (2.5.22) can be rewritten respectively as

$$\begin{aligned}\int_{\mathbb{R}^d} \partial_t q \Delta q dx &= -\frac{1}{2} \frac{d}{dt} \|\nabla q\|_{L^2}^2 \\ \int_{\mathbb{R}^d} (\Delta q)^2 dx &= \|\Delta q\|_{L^2}^2.\end{aligned}$$

Regarding term (c), using again integration by parts and vanishing at the boundary due to the dissipation condition, we deduce

$$\begin{aligned}\int_{\mathbb{R}^d} \Delta q \nabla(\nabla U q) dx &= - \int_{\mathbb{R}^d} \nabla q \nabla(\nabla(\nabla U q)) dx \\ &= - \int_{\mathbb{R}^d} \nabla q (\nabla^3 U q + 2\Delta U \nabla q + \nabla U \Delta q) dx \\ &= - \int_{\mathbb{R}^d} \nabla q \nabla^3 U q dx - 2 \int_{\mathbb{R}^d} \nabla q \nabla q \Delta U dx - \int_{\mathbb{R}^d} \nabla q \nabla U \Delta q dx.\end{aligned}$$

Using

$$\int_{\mathbb{R}^d} \Delta U \nabla q \nabla q dx = - \int_{\mathbb{R}^d} q \nabla^3 U \nabla q dx - \int_{\mathbb{R}^d} \Delta q \Delta U q dx,$$

we obtain

$$\int_{\mathbb{R}^d} \Delta q \nabla (\nabla U q) dx = - \int_{\mathbb{R}^d} \Delta U \nabla q \nabla q dx + \int_{\mathbb{R}^d} \Delta q \Delta U q dx - \int_{\mathbb{R}^d} \nabla q \nabla U \Delta q dx.$$

We further notice that

$$\int_{\mathbb{R}^d} \nabla q \Delta q \nabla U dx = -\frac{1}{2} \int_{\mathbb{R}^d} \nabla q \nabla q \Delta U dx.$$

Putting everything together, we find

$$\begin{aligned} \int_{\mathbb{R}^d} \nabla (\nabla U q) \Delta q dx &= -\frac{1}{2} \int_{\mathbb{R}^d} \Delta U \nabla q \nabla q dx + \int_{\mathbb{R}^d} \Delta q \Delta U q dx \\ &= \frac{3}{2} \int_{\mathbb{R}^d} \Delta q \Delta U q dx + \frac{1}{2} \int_{\mathbb{R}^d} \nabla^3 U \nabla q q dx. \end{aligned}$$

Concisely, the equality above can be written as

$$\langle \nabla (\nabla U q), \Delta q \rangle_{L^2} = \frac{3}{2} \langle \Delta q, \Delta U q \rangle_{L^2} + \frac{1}{2} \langle \nabla^3 U \nabla q, q \rangle_{L^2}.$$

Therefore, using the inequality

$$\|\nabla q\|_{L^2}^2 \leq \|\Delta q\|_{L^2} \|q\|_{L^2}^2 \quad (2.5.26)$$

we deduce

$$\begin{aligned} |\langle \nabla (\nabla U q), \Delta q \rangle_{L^2}| &\leq \epsilon \|\Delta q\|_{L^2}^2 + C(\beta) \|\nabla q\|_{L^2}^2 + C(\beta) \|(1+x^2)q\|_{L^2}^2 \\ &\leq \epsilon \|\Delta q\|_{L^2}^2 + C(\beta) \|\nabla q\|_{L^2}^2 + C(\beta) \|(1+x^4)q\|_{L^1} \|q\|_{L^\infty} \\ &\leq \epsilon \|\Delta q\|_{L^2}^2 + C(\beta) (\|\nabla q\|_{L^2}^2 + \|q\|_{L^2}^2 + \|(1+x^4)q\|_{L^1} \|q\|_{L^\infty}^2) \\ &\leq 2\epsilon \|\Delta q\|_{L^2}^2 + C(\beta) (\|\nabla q\|_{L^2}^2 + \|q\|_{L^2}^2 + \|(1+x^4)q\|_{L^1} \|q\|_{L^\infty}^2) \end{aligned}$$

for  $\epsilon > 0$  small enough and some positive constant  $C = C(\beta)$ . This implies

$$\frac{d}{dt} \|\nabla q\|_{L^2}^2 + \gamma \|\Delta q\|_{L^2}^2 + \|\nabla q\|_{L^2}^2 \leq C(\beta) (\|\nabla q\|_{L^2}^2 + \|q\|_{L^2}^2 + \|(1+x^4)q\|_{L^1} \|q\|_{L^\infty}),$$

for some constant  $\gamma > 0$ . Multiplying by  $t^N$  for some  $N \in \mathbb{N}$ , using again inequality (2.5.26), integrating with respect to time and, finally, employing the first smoothing estimate (2.5.17), we conclude.  $\square$

Finally, we are ready to prove the main theorem of this section. Having established existence, uniqueness and all estimates above for signed measures, we now restrict ourselves to probability measures. We denote by  $H^k := W^{k,2}$  the Hilbert space of all functions  $f \in L^2$  such that their weak derivatives up to order  $k$  have finite  $L^2$  norm. We further denote by  $L^1_{1+x^n}$ , for any  $n \in \mathbb{N}$ , the weighted space of measurable functions  $f$  such that

$$\|f\|_{L^1_{1+x^n}} := \int_{\mathbb{R}^d} (1+x^n) \|f(x)\| dx < \infty.$$

*Proof of Theorem 2.2.2.* Combining the  $L^1$  estimates (2.5.10), (2.5.12), (2.5.15) and the smoothing estimates (2.5.17), (2.5.21), we conclude that for any given  $\beta \in C^{1/2}(\mathbb{R}, \mathbb{R}^d)$ ,  $q(t) \in L^1_{(1+x^n)} \cap H^1$  for any  $n \in \mathbb{N}$  and  $t > 0$ . In order to gain more regularity, we differentiate the Fokker-Planck equation (2.2.2) with respect to the space variable  $x$  iteratively, via a standard bootstrapping procedure. After one step, we obtain  $\partial_x q(t) \in L^1_{(1+x^n)} \cap H^1$ , which implies in particular  $q(t) \in H^2$  for  $t > 0$ . After two steps, we obtain  $\partial_x^2 q(t) \in L^1_{(1+x^n)} \cap H^1$ , which implies  $q(t) \in H^3$  for  $t > 0$  and so on. This shows that, at any time  $t > 0$ ,  $q(t) \in H^\infty$  and any spatial derivative  $\partial_x^m q(t)$  decays faster than any polynomial as  $|x| \rightarrow \infty$ , at any time  $t > 0$ . Finally, we recall that the Sobolev space  $H^s(\mathbb{R}^d)$  can be continuously embedded into  $C^k(\mathbb{R}^d)$  for any  $k \in \mathbb{N}$  and  $s > k + \frac{n}{2}$  [55]. This readily implies  $H^\infty$  can be continuously embedded in  $C^\infty$ . Putting everything together, we conclude that  $q(t)$  belongs to the Schwarz space  $\mathcal{S}$  for any time  $t > 0$ .  $\square$

## 2.6 Proofs of Propositions 2.3.2 and 2.3.3

**Lemma 2.6.1** (Dissipation condition and strict convexity). *Assume the potential  $V$  satisfies the dissipation condition (2.2.6), that is,*

$$\nabla V(x) \cdot x \|x\|^2 \geq \frac{1}{2} \|x\|^6 - C$$

*for some  $C > 0$ . Then  $V$  is strictly convex outside a given ball in  $\mathbb{R}^d$ .*

*Proof.* We rewrite the dissipation condition as

$$\nabla V(x) = Mx \|x\|^2 + h(x), \tag{2.6.1}$$

with  $M \geq \frac{1}{2}$ ,  $h(x) = O(\|x\|^2)$  and such that

$$f(x) := \left(M - \frac{1}{2}\right) \|x\|^6 + h(x) \cdot x \|x\|^2$$

is lower bounded. Differentiating (2.6.1) yields

$$\Delta V(x) = 3M\|x\|^2 + g(x),$$

with  $g(x) = \nabla \cdot h(x) = O(\|x\|)$ , meaning that there exists  $L > 0$  and  $x_0 \in \mathbb{R}$  such that for any  $\|x\| \geq x_0$

$$|g(x)| \leq L\|x\|.$$

Then, letting  $N := 3M$ , we have

$$\Delta V(x) = N\|x\|^2 + g(x) \geq N\|x\|^2 - L\|x\|$$

and so the potential  $V$  is strictly convex outside the ball centered in  $\frac{LN}{2}$  with radius  $\frac{LN}{2}$ .  $\square$

We illustrate this result in the context of the example of Section 2.1.3.

**Example 2.6.2** (Double well potential). *We consider the one-dimensional double-well potential  $V'(x) = x(x^2 - a)$ , with  $a > 0$ . Then, the dissipation condition (2.2.6) is fulfilled if and only if*

$$x^4 \left( \frac{1}{2}x^2 - a \right) \geq -C$$

for some constant  $C > 0$ . Let  $f(x) := \frac{1}{2}x^6 - x^4a$ . Then, we set

$$f'(x) = 3x^5 - 4ax^3 = x^3(3x^2 - 4a) = 0$$

and we find the local extrema  $x = 0$  and  $x \pm \sqrt{\frac{4a}{3}}$ . Hence,  $f(0) = 0$  and

$$f \left( \pm \sqrt{\frac{4a}{3}} \right) = a^3 \left( -\frac{16}{27} \right) \geq -C$$

for any constant  $C \geq \frac{16}{27}a^3$ . The dissipation condition is therefore satisfied. Moreover,

$$V''(x) = 3x^2 - a > 0 \iff x^2 > \frac{a}{3}$$

and so we immediately deduce the potential  $V$  is strictly convex outside the ball centred at 0 with radius  $\sqrt{\frac{a}{3}}$ .

The converse implication of Lemma (2.6.1) does not hold, as the following example shows.

**Example 2.6.3** (Strict convexity outside a ball does not imply the dissipation condition). *We consider the one-dimensional potential*

$$V(x) = \frac{x^4}{8} - a\frac{x^2}{2}, \quad a > 0.$$

Then,

$$V'(x) = \frac{x^3}{2} - ax = x \left( \frac{x^2}{2} - a \right)$$

$$V''(x) = \frac{3}{2}x^2 - a > 0 \iff x^2 > \frac{2a}{3}.$$

We immediately see that  $V$  is strictly convex outside the ball centred at 0 with radius  $\sqrt{\frac{2a}{3}}$ . However, the dissipation condition reads as

$$V'(x)x^3 = \frac{1}{2}x^6 - ax^4 \geq \frac{1}{2}x^6 - C$$

which is equivalent to  $C \geq ax^4$  for some positive constant  $C$  and this is clearly not possible.

*Proof of Proposition 2.3.2.* We extend results by Eberle [78] to a non-autonomous setting. Consider the difference process  $z(t) := x(t) - y(t)$ . Then,

$$dz(t) = (\nabla V(y) - \nabla V(x))dt + 2|\sigma^{-1}z(t)|^{-1}z(t)d\tilde{W}(t), \quad t < T$$

and  $z(t) = 0$  for  $t \geq T$ , where

$$\tilde{W}(t) := \int_0^t e^\top(s)dW(s)$$

is a new Brownian motion by Levy's characterization,  $e(t)$  is the unit vector defined by

$$e(t) := \frac{\sigma^{-1}(x(t) - y(t))}{|\sigma^{-1}(x(t) - y(t))|}$$

and  $T$  is the coupling time. Next, we define  $r(t) := \|z(t)\| := |\sigma^{-1}z(t)|$ . By application of Ito's formula we find

$$dr(t) = 2|\sigma^{-1}z(t)|^{-1}r(t)d\tilde{W}(t) + r^{-1}(t)z(t)(\sigma\sigma^T)^{-1}(-\nabla V(x(t)) + \nabla V(y(t)))dt.$$

Given a smooth function  $f \in C^1(\mathbb{R}^d)$ , this implies

$$\begin{aligned} df(r(t)) &= 2|\sigma^{-1}z(t)|^{-1}r(t)f'(r(t))d\tilde{W}(t) \\ &\quad + r^{-1}(t)z(t)(\sigma\sigma^T)^{-1}(-\nabla V(x(t)) + \nabla V(y(t)))f'(r(t))dt \\ &\quad + 2|\sigma^{-1}z(t)|^{-2}r^2(t)f''(r(t))dt. \end{aligned} \tag{2.6.2}$$

We also define for any  $r > 0$  the function

$$\begin{aligned} k(r) &:= \inf_{x, y \in \mathbb{R}^d, \|x-y\|=r} \left\{ -2 \frac{|\sigma^{-1}(x-y)|^2}{\|x-y\|^2} \frac{(x-y) \cdot (\sigma\sigma^T)^{-1}(-\nabla V(x) + \nabla V(y))}{\|x-y\|^2} \right\} \\ &= \inf_{x, y \in \mathbb{R}^d, \|x-y\|=r} \left\{ -2 \frac{(x-y) \cdot (\sigma\sigma^T)^{-1}(-\nabla V(x) + \nabla V(y))}{\|x-y\|^2} \right\}. \end{aligned} \tag{2.6.3}$$

Indeed,  $k(r)$  is the largest positive real number such that

$$(x-y) \cdot (\sigma\sigma^T)^{-1}(-\nabla V(x) + \nabla V(y)) \leq -\frac{1}{2}k(r)\|x-y\|^2$$

for any  $x, y \in \mathbb{R}^d$  with  $\|x-y\| = r$ . Let us denote by  $m(t)$  the drift on the right hand side of (2.6.2). By definition of  $k$ ,

$$m(t) \leq \Gamma(t) := 2|\sigma^{-1}z(t)|^{-2}r^2(t) \cdot \left( f''(r(t)) - \frac{1}{4}k(r(t))f'(r(t)) \right).$$

Hence, the process  $e^{ct}f(r(t))$  is a supermartingale for  $t < T$  if  $\Gamma(t) \leq -cf(r(t))$ . We aim to find a constant  $c$  and function  $f$  such that this inequality holds. Let

$$\alpha := \sup \{ |\sigma^{-1}z|^2 : z \in \mathbb{R}^d \text{ with } \|z\| = 1 \}.$$

Since for any  $z \in \mathbb{R}^d$

$$|\sigma^{-1}z|^2 \leq \alpha\|z\|^2,$$

it suffices for  $f$  to satisfy

$$f''(r) - \frac{1}{4}rk(r)f'(r) \leq -\frac{\alpha c}{2}f(r) \tag{2.6.4}$$

for all  $r > 0$ , cf. [78, eq. 63]. We observe this equation holds with  $c = 0$  in case

$$f'(r) = \varphi(r) := \exp \left( -\frac{1}{4} \int_0^r sk^-(s)ds \right),$$

where  $k^- := \max \{-k, 0\}$  denotes the negative part of the function  $k$ . Next, following [78], we make the ansatz

$$f'(r) = \varphi(r)g(r), \tag{2.6.5}$$

where  $g \geq \frac{1}{2}$  is a decreasing absolutely continuous function satisfying  $g(0) = 1$ . Notice that the condition  $g \geq 0$  is necessary to ensure that  $f$  is non-decreasing. The condition

$$\frac{1}{2} \geq g \geq 1$$

ensures

$$\frac{\Phi}{2} \leq f \leq \Phi, \quad \Phi(r) := \int_0^r \varphi(s) ds.$$

The ansatz (2.6.5) yields

$$f''(r) = -\frac{1}{4}k^-(r)f(r) + \varphi(r)g(r) \leq \frac{1}{4}rk(r)f(r) + \varphi(r)g'(r).$$

In turn, condition (2.6.4) is satisfied if

$$g'(r) \leq -\frac{\alpha c}{2} \frac{f(r)}{\varphi(r)}. \quad (2.6.6)$$

Next, we define two constants  $R_0, R_1 \geq 0$ , with  $R_0 \leq R_1$ :

$$\begin{aligned} R_0 &:= \inf \{R \geq 0 : k(r) \geq 0, \forall r \geq R\} \\ R_1 &:= \inf \{R \geq R_0 : k(r)R(R - R_0) \geq 8, \forall r \geq R\}. \end{aligned}$$

As remarked in [78], we can rewrite  $k$  as

$$k(r) = \inf \left\{ 2 \int_0^1 \partial_{(x-y)/|x-y|}^2 (\sigma \sigma^\top)^{-1} V((1-t)x + ty) dt : x, y \in \mathbb{R}^d \text{ s.t. } |x - y| = r \right\}.$$

Thanks to lemma 2.6.1, the potential  $V$  is strictly convex outside a given ball in  $\mathbb{R}^d$  and this, in turn, ensures  $k$  is continuous on  $(0, \infty)$  and such that

$$\lim_{r \rightarrow +\infty} \inf k(r) > 0, \quad \int_0^1 rk^-(r) dr < \infty.$$

Thanks to this result, both constants  $R_0, R_1$  are finite. For  $r \geq R_1$ , condition (2.6.4) is satisfied since  $k$  is sufficiently positive. It is then enough to assume condition (2.6.6) holds on the open interval  $(0, R_1)$ . Under this assumption,

$$g(R_1) \leq 1 - \frac{\alpha c}{2} \int_0^{R_1} f(s) \varphi^{-1}(s) ds \leq 1 - \frac{\alpha c}{4} \int_0^{R_1} \Phi(s) \varphi^{-1}(s) ds. \quad (2.6.7)$$

Condition (2.6.7), in turn, is satisfied if

$$\alpha c \leq \frac{2}{\int_0^{R_1} \Phi(s) \varphi^{-1}(s) ds}.$$

So, by choosing, for  $r < R_1$ ,

$$g'(r) = -\frac{\Phi(r)}{2\varphi(r)} \bigg/ \int_0^{R_1} \frac{\Phi(s)}{\varphi(s)} ds,$$

condition (2.6.6) is fulfilled if we choose the constant as

$$\alpha c = 1 / \int_0^{R_1} \Phi(s) \varphi^{-1}(s) ds.$$

At this point, we can show that the quantity  $\Gamma$  is smaller than  $-cf(r)$ , with our choices of  $f$  and  $c$ . Consider the scenario  $r < R_1$ . Then, we have (see [78, eq. (68)])

$$f''(r) \leq \frac{1}{4}rk(r)f'(r) - \frac{1}{2}f(r) \Big/ \int_0^{R_1} \Phi(s) \varphi^{-1}(s) ds. \quad (2.6.8)$$

Consider, now, the scenario  $r \geq R_0$ . Then,

$$f'(r) = \frac{\varphi(r)}{2} = \frac{\varphi(R_0)}{2},$$

and  $k(r)R_1(R_1 - R_0) \geq 8$  by construction of  $R_1$ . Moreover, we know that  $r \geq R_0$ , the function  $\varphi$  is constant and, therefore  $\Phi(r) = \Phi(R_0) + (r - R_0)\varphi(R_0)$ . Also,

$$\int_{R_0}^{R_1} \Phi(s) \varphi^{-1}(s) ds \geq (R_1 - R_0)\Phi(R_1)\varphi^{-1}(R_0)/2.$$

This implies (see [78, eq. (69)])

$$f''(r) - \frac{1}{4}rk(r)f'(r) \leq -\frac{1}{2}f(r) \Big/ \int_0^{R_1} \Phi(s) \varphi^{-1}(s) ds. \quad (2.6.9)$$

Putting together equations (2.6.8) and (2.6.9), we conclude the key relationship

$$\Gamma(t) \leq -cf(r(t))$$

at all times  $t < T$ . For any coupling  $\gamma_t$  of the process  $(x(t), y(t))$ , we take the expectation on both sides of (2.6.2) and obtain

$$\mathbb{E}^{\gamma_t}[f(r(t))] = \mathbb{E}^{\gamma_t}[f(r(s))] + \int_s^t \mathbb{E}^{\gamma_t}[m(u)] du \quad (2.6.10)$$

for any  $s \leq t < T$ . Let  $\Upsilon(t) := \mathbb{E}^{\gamma_t}[f(r(t))]$ . Then, differentiating (2.6.10) with respect to time yields

$$\Upsilon'(t) = \mathbb{E}^{\gamma_t}[m(t)].$$

Since we have proved that  $m(t) \leq \Gamma(t) \leq -cf(r(t))$  for  $t < T$ , we deduce

$$\Upsilon'(t) \leq -c\Upsilon(t). \quad (2.6.11)$$

Thanks to standard Gronwall's lemma, we deduce

$$\Upsilon(t) \leq \Upsilon(s)e^{-c(t-s)} \quad (2.6.12)$$

for all  $s \leq t \leq T$ . Hence,  $t \rightarrow e^{ct} \mathbb{E}^{\gamma_t} [d_f(x(t), y(t))]$  is a decreasing function of time. This key result implies

$$\mathcal{W}_f(\mu_{t,\beta}, \nu_{t,\beta}) \leq \mathbb{E}^{\gamma_{t,\beta}} [d_f(x(t), y(t))] \leq e^{-ct} \mathbb{E}^{\gamma_{t,\beta}} [d_f(x_0, y_0)],$$

where  $\mu_{t,\beta}$  and  $\nu_{t,\beta}$  denote the time- $t$  evolved probability measures of the process  $x(t)$  with respect to the initial distributions  $\mu$  and  $\nu$  respectively, and  $\gamma_{t,\beta}$  denotes their coupling, given a realization of  $\beta$ . Taking the infimum over all couplings  $\gamma_{t,\beta}$ , we conclude.  $\square$

*Proof of Proposition 2.3.3.* By construction, the function  $f$  in Proposition 2.3.2 is concave, increasing and satisfies  $f(0) = 1, f'(0) = 1$ . This implies that  $f'(x)x \leq f(x) \leq x$ . Moreover,  $\frac{\varphi(R_0)}{2} \leq f' \leq 1$  thanks to the properties of  $\varphi$  and  $g$ . Hence,

$$\frac{\varphi(R_0)}{2} \|x - y\| \leq d_f(x, y) \leq \|x - y\|$$

for any  $x, y \in \mathbb{R}^d$ . For any coupling  $\gamma_{t,\beta}$  of  $\mu_{t,\beta}$  and  $\nu_{t,\beta}$ ,

$$\begin{aligned} \frac{\varphi(R_0)}{2} \mathbb{E}^{\gamma_{t,\beta}} [\|x(t) - y(t)\|] &\leq \mathbb{E}^{\gamma_{t,\beta}} [d_f(x(t), y(t))] \leq e^{-ct} \mathbb{E}^{\gamma_{t,\beta}} [d_f(x_0, y_0)] \\ &\leq e^{-ct} \mathbb{E}^{\gamma_{t,\beta}} [\|x_0 - y_0\|]. \end{aligned}$$

Let  $K := 2\varphi(R_0)^{-1}$ . Taking the infimum over all couplings  $\gamma_{t,\beta}$  yields

$$K\mathcal{W}^1(\mu_{t,\beta}, \mu_{t+\tau,\beta}) \leq \mathcal{W}_f(\mu_{t,\beta}, \mu_{t+\tau,\beta})$$

for all  $t > 0$ . Hence, if  $(\mu_{t,\beta})_{t>0}$  is a Cauchy sequence with respect to  $\mathcal{W}_f$ , it will be a Cauchy sequence with respect to  $\mathcal{W}^1$  as well. Moreover, with  $p_{t,\beta}$  denoting the Lebesgue density of the measure  $\mu_{t,\beta}$ , the Hardy-Landau-Littlewood inequality [26, 28, 27] entails

$$\|p_{t+\tau,\beta} - p_{t,\beta}\|_{L^1}^2 \leq C \|\nabla(p_{t+\tau,\beta} - p_{t,\beta})\|_{L^1} \mathcal{W}^1(\mu_{t,\beta}, \mu_{t+\tau,\beta})$$

for some constant  $C > 0$ . Since the  $L^1$  norm of the gradient is bounded (see Section 2.5, Lemma 2.5.11), we have

$$\|p_{t+\tau,\beta} - p_{t,\beta}\|_{L^1}^2 \leq \bar{C} \mathcal{W}^1(\mu_{t,\beta}, \mu_{t+\tau,\beta})$$

for some constant  $\bar{C} > 0$ . Hence,  $(p_{t,\beta})_{t>0}$  is a Cauchy sequence in  $L^1$ . Since  $L^1$  is complete, the sequence converges in  $L^1$ , that is, for any initial condition  $\mu \in \mathcal{P}(\mathbb{R}^d)$ ,

$$p_\beta = \lim_{t \rightarrow \infty} \Phi(t, \beta) p_\mu \in L^1.$$

$\square$

## 2.7 Proofs of results in Section 2.4

*Proof of Proposition 2.4.1.* In the stochastic setting, the non-autonomous Fokker-Planck equation (2.2.2) naturally extends to a *random* Fokker Planck equation in terms of common noise sample paths  $\beta$ , which we here write in compact form as

$$\partial_t q = F(\theta_t \beta, q) \quad (2.7.1)$$

for some appropriate functional  $F$ . Subsequently, in analogy to the discussion in Section 2.2.1, the stochastic Fokker Planck equation (2.1.2) is obtained via the transformation  $y = x - \eta\beta$ , yielding the analogous form when choosing the stochastic integral to be of Stratonovich type. The cocycle property is obtained from the existence and uniqueness of solutions of (2.2.2) for almost all sample paths, as established in Section 2.2. The evolution operator  $\Phi$  of the random Fokker Planck equation (2.7.1) is given by

$$\Phi(t, \beta, q) = q + \int_0^t F(\theta_s \beta, \Phi(s, \beta, q)) ds.$$

Following closely the argument in Arnold [11, Proof of Theorem 2.2.1], we prove the cocycle property (for almost all  $\beta \in \Omega_B$ ). Let  $s, t \in \mathbb{R}$  and assume  $s > 0, t > 0$  (the remaining cases are analogous). Then,

$$\begin{aligned} \Phi(t, \theta_s \beta, \Phi(s, \beta, q)) &= \Phi(s, \beta, q) + \int_0^t F(\theta_{u+s} \beta, \Phi(u, \theta_s \beta, \Phi(s, \beta, q))) du \\ &= q + \int_0^t F(\theta_s \beta, \Phi(s, \beta, q)) ds \\ &\quad + \int_s^{t+s} F(\theta_z \beta, \Phi(z - s, \theta_s \beta, \Phi(s, \beta, q))) dz, \end{aligned}$$

where  $z = u + s$ . Therefore, the function

$$\tilde{\Phi}(u, \beta, q) := \begin{cases} \Phi(u, \beta, q) & \text{if } 0 \leq u \leq s \\ \Phi(u - s, \theta_s \beta, \Phi(s, \beta, q)) & \text{if } s \leq u \leq s + t \end{cases}$$

satisfies

$$\tilde{\Phi}(t + s, \beta, q) = q + \int_0^{t+s} F(\theta_s \beta, \tilde{\Phi}(u, \beta, q)) du.$$

By uniqueness, for  $\mathbb{P}$ -a.e.  $\beta \in \Omega_B$ ,

$$\Phi(t + s, \beta, q) = \tilde{\Phi}(t + s, \beta, q) = \Phi(t, \theta_s \beta, \tilde{\Phi}(s, \beta, q)).$$

□

*Proof of Theorem 2.4.2.* From proposition 2.3.2 we deduce that there exist a constant  $c > 0$  and an increasing and convex function  $f$  such that for any  $t > 0, \beta \in C^{1/2}(\mathbb{R}, \mathbb{R}^d)$  and initial probability measures  $\mu, \nu \in \mathcal{M}(\mathbb{R}^d)$ ,

$$\mathcal{W}_f(\mu_{t,\beta}\nu_{t,\beta}) \leq e^{-ct}\mathcal{W}_f(\mu, \nu),$$

with  $\mu_{t,\beta} := \Psi(t, \beta)\mu$  and similarly for  $\nu_{t,\beta}$ , where  $\Psi$  denotes the time- $t$  evolution operator for the measure  $\mu$ , associated to the time- $t$  evolution operator  $\Phi$  of the random Fokker Planck equation 2.7.1. We show that  $(\mu_{t,\beta})_{t>0}$  is a Cauchy sequence in a pullback sense with respect to the  $\mathcal{W}_f$  metric, that is,  $\forall \epsilon > 0 \exists t > 0 : \forall \tau > 0$

$$\mathcal{W}_f(\Psi(t, \theta_{-t}\beta)\mu, \Psi(t + \tau, \theta_{-(t+\tau)}\beta)\mu) < \epsilon.$$

Exploiting the pullback operator and the fact that we have a contraction, we deduce

$$\begin{aligned} \mathcal{W}_f(\Psi(t, \theta_{-t}\beta)\mu, \Psi(t + \tau, \theta_{-(t+\tau)}\beta)\mu) &= \mathcal{W}_f(\Psi(t, \theta_{-t}\beta)\mu, \Psi(t, \theta_{-t}\beta) \circ \Psi(\tau, \theta_{-(t+\tau)}\beta)\mu) \\ &\leq e^{-ct}\mathcal{W}_f(\mu, \Psi(\tau, \theta_{-(t+\tau)}\beta)\mu). \end{aligned}$$

Then, since  $f$  is concave and increasing by construction,

$$\mathcal{W}_f(\mu, \Psi(\tau, \theta_{-(t+\tau)}\beta)\mu) \leq f(W^1(\mu, \Psi(\tau, \theta_{-(t+\tau)}\beta)\mu)) \leq W^1(\mu, \Psi(\tau, \theta_{-(t+\tau)}\beta)\mu).$$

We observe that  $C^{1/2}(\mathbb{R}, \mathbb{R}^d)$  is a subset of full Wiener measure  $\mathbb{P}_\beta$  of the sample path space  $\Omega_B$ . Taking the expectation  $\mathbb{E}^{\mathbb{P}_\beta}$  with respect to  $\mathbb{P}_\beta$  implies

$$\mathbb{E}^{\mathbb{P}_\beta}[\mathcal{W}_f(\mu, \Psi(\tau, \theta_{-(t+\tau)}\beta)\mu)] \leq \mathbb{E}^{\mathbb{P}_\beta}[W^1(\mu, \Psi(\tau, \theta_{-(t+\tau)}\beta)\mu)]$$

Let  $p_\mu$  and  $p_{\tau, t+\tau; \beta}$  denote the Lebesgue densities of  $\mu$  and  $\Psi(\tau, \theta_{-(t+\tau)}\beta)\mu$  respectively. Their product will be the density of the product measure, which is a simple example of a coupling measure. Therefore,

$$\begin{aligned} \mathbb{E}^{\mathbb{P}_\beta}[W^1(\mu, \Psi(\tau, \theta_{-(t+\tau)}\beta)\mu)] &\leq \mathbb{E}^{\mathbb{P}_\beta} \left[ \iint_{\mathbb{R}^{2d}} \|x - y\| p_{\tau, t+\tau; \beta}(x) p_\mu(y) dx dy \right] \\ &= \iint_{\mathbb{R}^{2d}} \|x - y\| \mathbb{E}^{\mathbb{P}_\beta}[p_{\tau, t+\tau; \beta}(x)] p_\mu(y) dx dy. \end{aligned}$$

Let us define  $p_\tau := \mathbb{E}^{\mathbb{P}_\beta}[p_{\tau, t+\tau; \beta}]$ . We notice that this expectation does not depend on  $t$  since we are integrating over all  $\Omega_B$  and  $\theta_{-(t+\tau)}\beta = \theta_{-\tau}\tilde{\beta}$  for some  $\tilde{\beta} \in \Omega_B$ . Then

$$\begin{aligned} \iint_{\mathbb{R}^{2d}} \|x - y\| p_\tau(x) p_\mu(y) dx dy &= \int_{\mathbb{R}^d} p_\mu(y) \int_{\mathbb{R}^d} \|x - y\| p_\tau(x) dx dy \\ &\leq \int_{\mathbb{R}^d} p_\mu(y) \left[ \int_{\mathbb{R}^d} \|x\| p_\tau(x) + \|y\| p_\tau(x) dx \right] dy \\ &= \int_{\mathbb{R}^d} p_\mu(y) \left( \int_{\mathbb{R}^d} \|x\| p_\tau(x) dx + \|y\| \right) dy \\ &= \int_{\mathbb{R}^d} p_\tau(x) \|x\| dx + \int_{\mathbb{R}^d} p_\mu(y) \|y\| dy. \end{aligned}$$

The second term on the RHS of the equation above is bounded. For what concerns the first term, we further notice that  $p_\tau$  is the forward solution at time  $t = \tau$ , with initial condition  $p_\mu$  at time  $t = 0$ , of the autonomous Fokker Planck equation for the SDE (2.1.1)

$$\frac{\partial}{\partial t} p = \Delta V(x)p + \nabla V(x) \frac{\partial p}{\partial x} + \frac{1}{2}(\sigma^2 + \eta^2) \frac{\partial^2 p}{\partial x^2}.$$

Applying the results from Sections 2.2 and 2.3 to the autonomous setting, we deduce this equation admits a unique attractor and, in particular,

$$\lim_{t \rightarrow \infty} p_t = p_\rho \text{ in } L^1,$$

where  $p_\rho$  denotes the density of the stationary measure. In fact, we observe that the fixed point  $p_\rho$  is invariant under the autonomous evolution operator  $\tilde{\Phi} = \Phi(\cdot, 0)$ , i.e.

$$\tilde{\Phi} p_\rho = p_\rho.$$

Thanks to the discussion in Section 2.5, we deduce that  $\tilde{\Phi}$  maps  $L^1$  functions into the Schwartz space  $\mathcal{S}$  of rapidly decreasing functions. Therefore,  $p_t$  converges exponentially fast to  $p_\rho$  as  $t \rightarrow \infty$  in  $\mathcal{S}$ . Next, we consider

$$\begin{aligned} \int_{\mathbb{R}^d} p_\tau(x) \|x\| dx &= \int_{\mathbb{R}^d} (p_\tau(x) - p_\rho(x)) \|x\| dx + \int_{\mathbb{R}^d} p_\rho(x) \|x\| dx \\ &\leq \int_{\mathbb{R}^d} |p_\tau(x) - p_\rho(x)| \|x\| dx + \int_{\mathbb{R}^d} p_\rho(x) \|x\| dx. \end{aligned}$$

For any  $\epsilon > 0$ , there exists  $T > 0$  such that for any  $\tau > T$ ,

$$\int_{\mathbb{R}^d} |p_\tau(x) - p_\rho(x)| \|x\| dx < \epsilon$$

Let us define

$$C := \sup_{\tau \in [0, T]} \int_{\mathbb{R}^d} p_\tau(x) \|x\| dx < \infty,$$

which is finite since  $\int_{\mathbb{R}^d} p_\tau(x) \|x\| dx$  is finite for any  $\tau$  and the supremum is taken over a finite time interval. Then,

$$\begin{aligned} \int_{\mathbb{R}^d} p_\tau(x) \|x\| dx &< \mathbb{1}_{\tau \leq T} C + \mathbb{1}_{\tau > T} \left( \epsilon + \int_{\mathbb{R}^d} p_\rho(x) \|x\| dx \right) \\ &\leq \max \left\{ C, \epsilon + \int_{\mathbb{R}^d} p_\rho(x) \|x\| dx \right\}. \end{aligned}$$

Consequently, there exists a constant  $D < \infty$  such that for any  $\tau > 0$

$$\mathbb{E}^{\mathbb{P}^\beta} [\mathcal{W}_f(\mu, \Psi(\tau, \theta_{-(t+\tau)} \beta) \mu)] < D,$$

hence  $\mathcal{W}_f(\mu, \Psi(\tau, \theta_{-(t+\tau)}\beta)\mu) < D$  for  $\mathbb{P}_B$ -almost all  $\beta \in \Omega_B$ . This shows that

$$\lim_{t \rightarrow \infty} \mathcal{W}_f(\Psi(t, \theta_{-t}\beta)\mu, \Psi(t + \tau, \theta_{-(t+\tau)}\beta)\mu) = 0$$

for all  $\tau > 0$  and  $\mathbb{P}_B$ -almost all  $\beta \in \Omega_B$ . Therefore,  $(\Psi(t, \theta_{-t}\beta)\mu)_{t>0}$  is  $\mathbb{P}_B$ -almost surely a Cauchy sequence with respect to the  $W_f$  metric. Let  $p_{t,\beta}$  denote the Lebesgue density of  $\Psi(t, \theta_{-t}\beta)\mu$ . Thanks to Proposition 2.3.3,

$$\lim_{t \rightarrow \infty} p_{t,\beta} = p_\beta \in L^1$$

for all  $t > 0$  and  $\mathbb{P}_B$ -almost all  $\beta \in \Omega_B$ . Finally, we remark that the limit point  $p_\beta \in L^1$  is invariant under the pullback flow. We have

$$p_\beta = \Phi(t, \beta)p_{\theta_{-t}\beta}$$

for all  $t > 0$ . In light of the discussion in Section 2.5,  $\Phi$  maps  $L^1$  functions to  $\mathcal{S}$  functions. In other words,  $p_\beta \in \mathcal{S}$ . We conclude that for  $\mathbb{P}_B$ -almost all  $\beta \in \Omega_B$  and initial probability density  $p \in L^1$ , there exists a unique pullback attractor for (2.2.2)

$$p_\beta = \lim_{t \rightarrow \infty} \Phi(t, \theta_{-t}\beta)p \in \mathcal{S}.$$

□

*Proof of Proposition 2.4.4.* By construction, for every  $C \in \mathcal{B}(\mathbb{R}^d)$ , any Borel measure  $\nu$  on  $\mathbb{R}^d$  with Lebesgue density  $p_\nu \in \mathcal{S}$  and for  $\mathbb{P}_B$ -almost all  $\beta \in \Omega_B$ , we have for all  $s \leq t$

$$\nu_t(C) := \int_{\Omega_W} \phi_*(t - s; \theta_s \omega, \theta_s \beta) \nu(C) \mathbb{P}_W(d\omega) = \int_C \Phi(t - s; \theta_s \beta) p_{\nu_s} dx = \int_C p_{\nu_t} dx, \quad (2.7.2)$$

and, using the results from proposition 2.4.3 and (2.7.2),

$$\begin{aligned} \mu_\beta(C) &= \int_{\Omega_W} \mu_{\omega,\beta}(C) \mathbb{P}_W(d\omega) = \int_{\Omega_W} \lim_{\tau \rightarrow \infty} \phi(\tau, \theta_{-\tau}\omega, \theta_{-\tau}\beta)_* \rho(C) \mathbb{P}_W(d\omega) \\ &= \lim_{\tau \rightarrow \infty} \int_{\Omega_W} \phi(\tau, \theta_{-\tau}\omega, \theta_{-\tau}\beta)_* \rho(C) \mathbb{P}_W(d\omega) = \lim_{\tau \rightarrow \infty} \int_C \Phi(\tau, \theta_{-\tau}\beta) p_\rho dx \\ &= \int_C \lim_{\tau \rightarrow \infty} \Phi(\tau, \theta_{-\tau}\beta) p_\rho dx \end{aligned}$$

by which the result follows from the fact that  $\mu_\beta(C) = \int_C p_\beta dx$ , for all  $C \in \mathcal{B}(\mathbb{R}^d)$ . □

*Proof of Proposition 2.4.5.* The Dirac measure  $\delta_{p_\beta}$  is the disintegration of a Markov measure of the random dynamical system  $\Phi$  on  $\Omega_B \times \mathcal{S}$ , associated to the stochastic Fokker-Planck (2.1.2). The  $\mathcal{S}$ -marginal of this Markov measure

$$P := \int_{\Omega_B} \delta_{p_\beta} \mathbb{P}_B(d\beta) \quad (2.7.3)$$

is the corresponding stationary measure of (2.1.2). Application of Birkhoff's Ergodic Theorem then yields that time-averages of ( $P$ -integrable) observables  $g : \mathcal{S} \rightarrow \mathbb{R}$  satisfy

$$\lim_{\tau \rightarrow \infty} \frac{1}{\tau} \int_0^\tau g(\Phi(\tau, \beta)p) dt = \int_{\mathcal{A}} g(p) P(dp) = \int_{\Omega_B} g(p_\beta) \mathbb{P}_B(d\beta),$$

$\mathbb{P}_B \times P$ -almost surely. By Elton's Ergodic Theorem [82], this relation holds in fact  $\mathbb{P}_B$ -almost surely if  $g$  is continuous.  $\square$

## 2.8 Exact solutions for the Ornstein-Uhlenbeck SDE with intrinsic and common additive noise

In this Section, we present the calculations of the closed expressions for common noise pullback attractors of the one-dimensional Ornstein-Uhlenbeck SDE with additive intrinsic and common noise, discussed in Section 2.1.3. The flow of the SDE (2.1.7) from time  $s$  to  $t$  for fixed noise realisations  $\beta$  and  $\omega$  is explicitly given by

$$\phi(t-s, \theta_s \omega, \theta_s \beta)x(s) = x(s)e^{-a(t-s)} + \eta \int_s^t e^{-a(t-u)} d\beta(u) + \sigma \int_s^t e^{-a(t-u)} d\omega(u).$$

Averaging this equation over the intrinsic noise yields

$$\begin{aligned} \int_{\Omega_W} \phi(t-s, \theta_s \omega, \theta_s \beta)x(s) \mathbb{P}_W(d\omega) &= x(s)e^{-a(t-s)} + \eta \int_s^t e^{-a(t-u)} d\beta(u) \\ &\quad + \sigma \int_s^t e^{-a(t-u)} dW(u). \end{aligned} \quad (2.8.1)$$

It's important to emphasize that the integral with respect to the single path  $\beta$  is a real number while the integral with respect to the intrinsic noise  $W$  is a Gaussian distribution. The density of the distribution in (2.8.1) is given by

$$p(x, t) = \sqrt{\frac{a}{\pi \sigma^2 (1 - e^{-2a(t-s)})}} \exp \left( -\frac{a}{\sigma^2 (1 - e^{-2a(t-s)})} (x - m_\beta(t, s))^2 \right), \quad (2.8.2)$$

where

$$m_\beta(t, s) := x(s)e^{-a(t-s)} + \eta \int_s^t e^{-a(t-u)} d\beta(u).$$

It is readily checked that indeed the density (2.8.2) is a solution of the stochastic Fokker-Planck equation (2.1.2) with  $V(x) = \frac{a}{2}x^2$ .

Averaging (2.8.1) over the common noise yields

$$\begin{aligned} \int_{\Omega_B} \int_{\Omega_W} \phi(t-s, \theta_s \omega, \theta_s \beta) x(s) \mathbb{P}_W(d\omega) \mathbb{P}_B(d\beta) &= x(s) e^{-a(t-s)} + \eta \int_s^t e^{-a(t-u)} dB(u) \\ &\quad + \sigma \int_s^t e^{-a(t-u)} dW(u), \end{aligned} \quad (2.8.3)$$

where now both integrals represent Gaussian distributions. The density of the distribution (2.8.3) is

$$\begin{aligned} \bar{p}(x, t) &= \int_{\Omega_B} p(x, t) \mathbb{P}_B(d\beta) \\ &= \sqrt{\frac{a}{\pi(\eta^2 + \sigma^2)(1 - e^{-2a(t-s)})}} \exp \left\{ -\frac{a}{(\eta^2 + \sigma^2)(1 - e^{-2a(t-s)})} x^2 \right\}, \end{aligned} \quad (2.8.4)$$

which in turn is a solution of the Fokker-Planck equation of the SDE (2.1.1) with  $V(x) = \frac{a}{2}x^2$ :

$$\frac{\partial \bar{p}}{\partial t} = a\bar{p} + ax \frac{\partial \bar{p}}{\partial x} + \frac{1}{2}(\sigma^2 + \eta^2) \frac{\partial^2 \bar{p}}{\partial x^2}.$$

We conclude this section with a discussion on pullback attractors. The pullback attractor of the SDE (2.1.7) with respect to both intrinsic and common noise is

$$\alpha(\omega, \beta) := \lim_{s \rightarrow -\infty} \phi(t-s, \theta_s \omega, \theta_s \beta) x(s) = \eta \int_{-\infty}^0 e^{au} d\beta(u) + \sigma \int_{-\infty}^0 e^{au} d\omega(u). \quad (2.8.5)$$

This is a point attractor, confirming that, at the SDE level, the system is synchronizing. The fiberwise measures resulting from disintegration (see Section 2.4) are therefore

$$\mu_{\omega, \beta} = \delta_{\alpha(\omega, \beta)}.$$

Integrating with respect to the intrinsic noise yields

$$\mu_\beta := \int_{\Omega_W} \mu_{\omega, \beta} \mathbb{P}_W(d\omega) = \eta \int_{-\infty}^0 e^{au} d\beta(u) + \sigma \int_{-\infty}^0 e^{au} dW(u).$$

This is normally distributed with variance depending on the intensity of the intrinsic noise  $\sigma$  and mean depending on the intensity of the common noise  $\eta$ . Its density is

$$p_\beta(x) = \sqrt{\frac{a}{\pi\sigma^2}} \exp \left\{ -\frac{a}{\sigma^2} \left( x - \eta \int_{-\infty}^0 e^{au} d\beta(u) \right)^2 \right\}. \quad (2.8.6)$$

Finally, integrating over all common noise realizations we obtain the stationary measure

$$\begin{aligned}\rho &= \int_{\Omega_B} \mu_\beta \mathbb{P}_B(d\beta) = \int_{\Omega_B} \int_{\Omega_W} \mu_{\omega,\beta} \mathbb{P}_W(d\omega) \mathbb{P}_B(d\beta) \\ &= \eta \int_{-\infty}^0 e^{au} dB(u) + \sigma \int_{-\infty}^0 e^{au} dW(u),\end{aligned}$$

with density

$$p_\rho(x) = \sqrt{\frac{a}{\pi(\sigma^2 + \eta^2)}} \exp \left\{ -\frac{a}{(\eta^2 + \sigma^2)} x^2 \right\}.$$

Of course, we also have  $p_\rho(x) = \int_{\Omega_B} p_\beta(x) \mathbb{P}_B(d\beta)$  and  $p_\rho(x) = \lim_{(t-s) \rightarrow \infty} \bar{p}(x, t)$  confirming global convergence of solutions of (2.8.5) to the stationary measure in forward and pullback sense.

## Acknowledgements

We are grateful to Sergey Zelik for extended discussions and input on the material in Section 2.2.2 and Section 2.5 and to Martin Hairer for pointing out [78].

# 3

## On the consistency of jump-diffusion dynamics for FX rates under inversion

### 3.1 Introduction

The foreign exchange (FX) market has peculiar *symmetries* that distinguish it from other markets. The first is the symmetry with respect to inversion: given an exchange rate, its reciprocal is again an exchange rate. For example, the USD-GBP is the reciprocal of the GBP-USD exchange rate. The other "symmetry" is what we could term *triangular consistency* and is with respect to multiplication: given two exchange rates such that the domestic currency of one corresponds to the foreign currency of the other, their product is another exchange rate. For example, the product of USD-GBP and GBP-EUR is the cross rate USD-EUR. Triangular consistency requires that if USD-GBP and GBP-EUR are in the same model class up to reparametrization, so is USD-EUR. For an example and a related discussion with multivariate mixture models and an application involving China's FX rates see for instance Brigo et al. [47]. These two stylized facts have motivated research in understanding which mathematical models fulfil some kind of *consistency conditions* which make them compatible with such empirical facts.

Let us start with a definition

**Definition 3.1.1.** *A model for  $S(t)$  is said to be consistent under inversion if the dynamics of  $S(t)$  under the domestic measure is the same as the dynamics of  $1/S(t)$  under the foreign measure, up to a reparametrization. Expressing  $S(t)$  as a Ito stochastic differential equation (SDE), both the finite variation drift and the diffusive part will be required to have the same functional form. In case there are hidden sources of randomness, such as stochastic volatility or random jumps, consistency will be said*

to hold if also the description of such sources is invariant modulo reparametrization.

Such a requirement can be justified from different points of view. Firstly, in principle there is no reason why an exchange rate and its inverse should be described in substantially different ways. They are actually the same entity, just seen from two different perspectives. Furthermore, in terms of design of libraries, it is helpful to have a consistent dynamics for all FX rates involved in transactions.

The issue of consistency with respect to inversion was raised, for instance, by Brigo et al. [46] in the context of multi currency CDSs and FX rate devaluation in conjunction with default events. In Section 2.2. the authors explain that when pricing quanto CDS one might be interested in pricing either under the liquid-currency measure or the contractual-currency measure. FX symmetry plays a role in that the measure change affects all risk factors whose dynamics is defined under a measure different from the one in which they were calibrated.

The Heston model is certainly one of the most widespread [111, 146]. Its consistency with respect to inversion was first addressed by Del Baño Rollin [70], who showed that the Heston model is indeed well behaving (see also [69, 96]). On the other hand, inconsistent models are numerous: for instance, the Garch stochastic volatility model [96], the SABR model [96, 69], the Hull-White stochastic volatility model [96], and the Scott model [69]. By following the *intrinsic currency* framework introduced by Doust [75, 74], De Col et al. [69] presented a multi-factor SV model of Heston type which remains invariant under a risk-neutral measure change. This approach was later generalized by Gnoatto [96], who introduced a consistent affine stochastic volatility model. The intrinsic currency approach was employed by Gnoatto and Grasselli as well [97], who extended the model presented previously in [69] to the case where the stochastic factors driving the volatilities of the exchange rates belong to the cone of positive semidefinite  $d \times d$  matrices  $S_d^+$ . Specifically, they showed that their model is at the same time an affine multifactor stochastic volatility model for the FX rate where the instantaneous variance is driven by a Wishart process, and a Wishart affine short-rate model. Recently, Graceffa et al. [100] studied consistency with respect to inversion of fairly general classes of local stochastic volatility (LSV) models, determining general conditions that a LSV model has to satisfy in order to be consistent. Finally, it is worth mentioning that this problem was also discussed in the context of semimartingales, see for instance the works of Eberlein and Papapantoleon [80] and Eberlein et al. [79] who discussed the so-called duality principle.

This Chapter aims at including jumps into the analysis and discussing how Poisson and compound Poisson processes behave under inversion in the FX rate. It is structured as follows. In Section 3.2 we present a fairly general jump-diffusion model with local-stochastic volatility, and illustrate some numerical results highlighting the consistency and inconsistency of the Heston model and SABR model respectively. In section 3.3 we discuss consistency for a general local volatility structure, identifying some suitable functional forms satisfying the required property. In section 3.4 we

focus without loss of generality on the jump component, and discuss the case where jump size is constant. Here consistency with respect to inversion turns out to be automatically satisfied, since the jump size is a constant as well. Finally, in Section 3.5 we analyze the more complicated case of compound Poisson processes. We identify a fairly general class of jump size distributions which are invariant, up to a reparametrization, under the transformation from domestic to foreign measure.

## 3.2 General model and numerics

Consider the jump-diffusion with local stochastic model

$$\begin{aligned} dS(t) &= \Delta r S(t) dt + \eta(t, V(t)) \sigma(t, S(t)) S(t) dW_1^{\mathbb{Q}_d}(t) + S(t_-) J^d dN^{\mathbb{Q}_d}(t) \\ dV(t) &= m(t, V(t)) dt + \xi(t, V(t)) dW_2^{\mathbb{Q}_d}(t) \end{aligned} \quad (3.2.1)$$

where the process  $S(t)$  denotes the exchange rate,  $\Delta r = r^d - r^f$  the differential of the domestic and foreign risk free interest rates,  $m, \eta, \sigma, \xi : [0, T] \rightarrow \mathbb{R}$  measurable functions,  $\mathbb{Q}_d$  denotes the risk neutral domestic measure,  $W_1^{\mathbb{Q}_d}(t)$ ,  $W_2^{\mathbb{Q}_d}(t)$  standard Brownian motions under the domestic measure,  $N^{\mathbb{Q}_d}(t)$  a Poisson process under the domestic measure with intensity  $\lambda^d$ , and  $J^d$  the size of the relative jump of the exchange rate. The Poisson process will be assumed to be independent from the Brownian motions while the two Brownian motions will in general be correlated.

As mentioned in the introduction, a model  $S(t)$  is said to be consistent under inversion if the SDE describing  $S$  in the domestic measure and the SDE describing  $1/S$  in the foreign measure are the same, up to a reparametrization. Furthermore, any hidden source of randomness, such as stochastic volatility or stochastic jumps, must be described by the same kind of SDE/distribution. To give a numerical measure of the inconsistency, we can consider, for the sake of simplicity, the Heston model (which is consistent) and the SABR model (which is not) [100]. The Heston model is

$$\begin{aligned} dS(t) &= \Delta r S(t) dt + \sqrt{V(t)} S(t) dW_1^{\mathbb{Q}_d}(t) \\ dV(t) &= \kappa(\bar{V} - V(t)) dt + \sigma \sqrt{V(t)} dW_2^{\mathbb{Q}_d}(t). \end{aligned}$$

This model is well known to be consistent. Its inverse is

$$\begin{aligned} dY(t) &= -\Delta r Y(t) dt + \sqrt{V(t)} Y(t) dW_1^{\mathbb{Q}_f}(t) \\ dV(t) &= (\kappa - \rho\sigma) \left( \frac{\kappa}{\kappa - \rho\sigma} \bar{V} - V(t) \right) dt + \sigma \sqrt{V(t)} dW_2^{\mathbb{Q}_f}(t). \end{aligned}$$

Indeed, the model dynamics followed by the inverse FX rate is again of Heston type. It is important to point out, though, that the term  $\kappa - \rho\sigma$  should be positive. The reason being that, otherwise, the volatility model is not mean reverting anymore.

Such a condition is easily fulfilled in case, for example, the correlation between the asset and volatility processes is negative.

The SABR model reads as

$$\begin{aligned} dS(t) &= \Delta r S(t) dt + S^\beta(t) dW_1^{\mathbb{Q}_d}(t) \\ dV(t) &= \nu V(t) dW_2^{\mathbb{Q}_d}(t). \end{aligned}$$

Unlike Heston, the SABR model is inconsistent. The inverse of the SABR is

$$\begin{aligned} dY(t) &= -\Delta r Y(t) dt + v(t) Y^{2-\beta}(t) dW_1^{\mathbb{Q}_f}(t) \\ dV(t) &= \nu \rho Y^{1-\beta}(t) V^2(t) dt + \nu V(t) dW_2^{\mathbb{Q}_f}(t). \end{aligned}$$

The model dynamics is indeed not a SABR model anymore. In order to provide the reader with a clearer understanding of what consistency means in practice, we fit both models and their inverses to market data. As we shall see, consistency will imply that the smile of a model and the smile of the inverse model will match almost perfectly. Using an inconsistent model, instead, will cause the two smiles to be markedly different.

In Figure 3.1 we calibrate Heston model and SABR model to the market data, and show the resulting smiles.

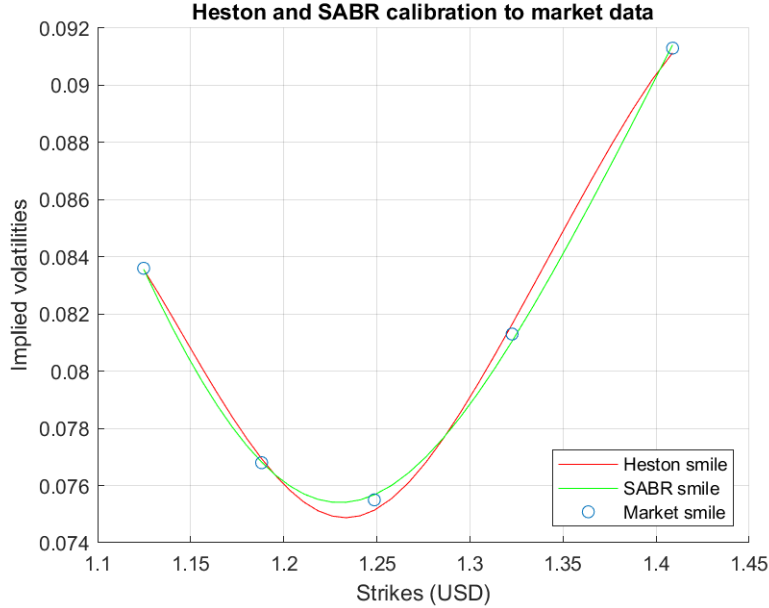


Figure 3.1: Calibration Heston and SABR model to EUR/USD market volatility, 30-th January 2018, 3 months maturity.

The underlying asset is the EUR/USD exchange rate as of 30-th January 2018, with spot  $S_0 = 1.24122$ . We used five market volatilities: 10-delta put, 25-delta put,

ATM, 25-delta call, 10-delta call. The maturity is  $T = 3$  months. The scattered strikes are those from market data, and were computed via (see [58], Eq. (3.8))

$$K_{\text{market}} := F_{0,T} \exp \left( \frac{1}{2} \sigma_{ATM}^2 T \right),$$

$F_{0,T}$  denoting the current forward price. In our specific case,  $F_{0,T} = 1.2478$  and  $\sigma_{ATM} = 0.0755$ . As usual, calibration was carried out by minimizing the sum of squared differences between market volatilities and model implied volatilities. In both cases, results are quite satisfactory. Heston calibration gives the parameters

$$v_0^* = 0.0025, \quad \theta^* = 0.0287, \quad k^* = 1.1718, \quad \sigma^* = 0.1720, \quad \rho^* = 0.0952. \quad (3.2.2)$$

The condition ensuring consistency is indeed satisfied. On the other hand, calibrating the SABR model gives us the parameters

$$\alpha^* = 0.0748, \quad \rho^* = 0.1435, \quad \nu^* = 0.7330,$$

with  $\alpha_{shift} = 9.8986 \times 10^{-8}$ , where, we recall,  $\alpha$  denotes the current SABR volatility and it is shifted so as to match the ATM volatility, and  $\nu$  denotes the volatility of volatility. The parameter  $\beta$ , instead, is chosen a priori to be 0.50. In order to make Heston and SABR smiles look smooth, we built a denser strikes vector and then performed a spline interpolation. Next, we illustrate the consistency of the Heston model. Using the calibrated parameters (3.2.2) of the Heston model, we priced options on the reciprocal exchange rate using the reciprocal of the Heston model, which is known to be a Heston model as well. Then, given these prices, we employed a standard numerical routine and obtained the corresponding implied volatilities, coherent with the Heston reciprocal. Finally, we plotted the smile of these implied volatilities against the one of the original model.

Visualizing both smiles in the same plot, we see that they are almost overlapping. More precisely, the norm of the difference between the two is of order  $10^{-14}$ . Such an overlap indicates the model and its reciprocal have the same volatility, that is, the volatility in the models is described by the same kind of stochastic dynamics. This, in turn, is a clear sign confirming model consistency.

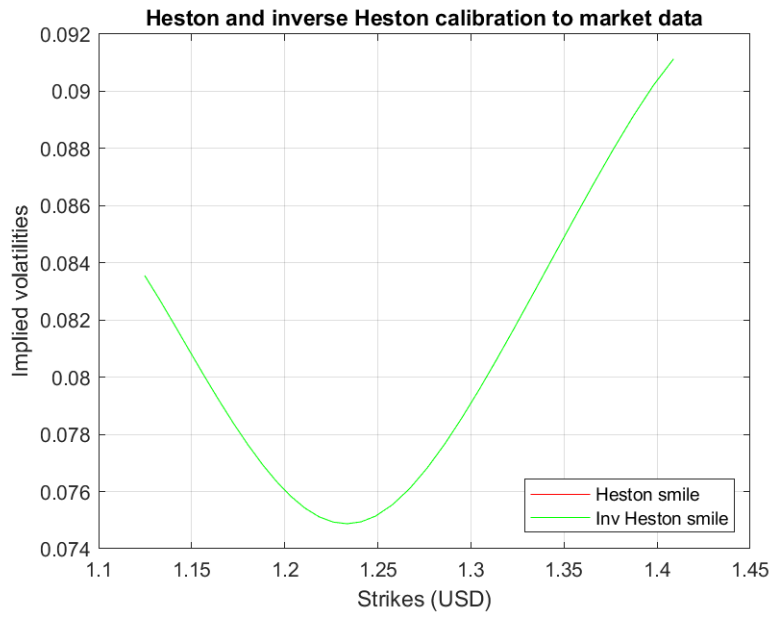


Figure 3.2: Consistency Heston model with respect to inversion: Heston smile and inverse smile are almost exactly overlapping.

Repeating the same procedure with the SABR model and its reciprocal, we see that the result is considerably different.

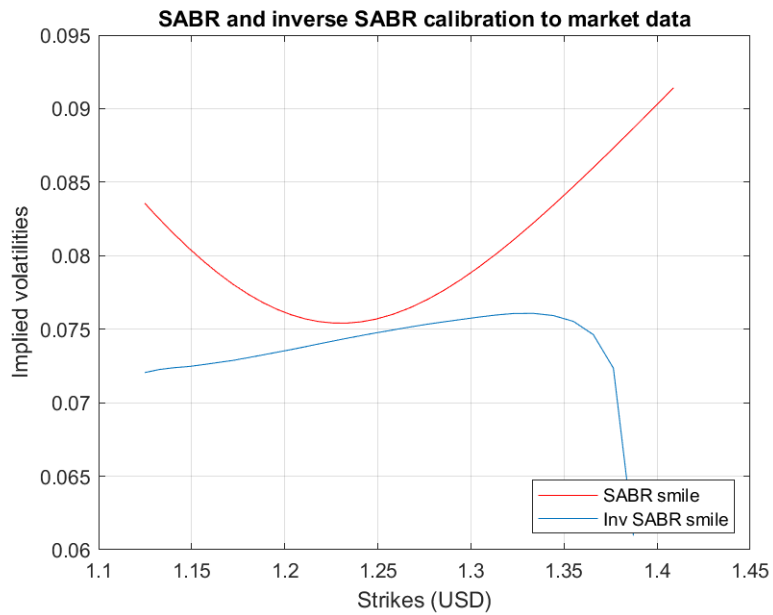


Figure 3.3: Inconsistency SABR model with respect to inversion

Unlike what happens in Figure 3.2, in Figure 3.3 we observe that the two smiles do not overlap at all. This clearly suggests this model is not consistent under inversion.

### 3.3 Inversion with local volatility structure

After showing a few numerical examples, we proceed by analysing more complex models. We start by investigating the scenario where no jumps nor stochastic volatility are present, that is we specify the model (3.2.1) with  $J^d = 0$  and  $\eta = 1$ . Inversion of local stochastic volatility was discussed in [100]. The authors determined an affine condition for the local volatility component, and a relationship linking the functions  $m, \xi, \eta$ . Here we propose a further viable specification for the general volatility. Let us consider the model

$$dS(t) = \Delta r S(t) dt + \sigma(S, t) S(t) dW^{\mathbb{Q}_d}(t).$$

Then, by Ito's formula, the inverted dynamics in the domestic measure reads as

$$d\left(\frac{1}{S(t)}\right) = \left[-\frac{1}{S(t)}\Delta r + \sigma^2(S, t)\frac{1}{S(t)}\right] dt - \frac{1}{S(t)}\sigma(S, t)dW^{\mathbb{Q}_d}(t).$$

Implementing the change of measure from domestic to foreign

$$dW^{\mathbb{Q}_f}(t) = dW^{\mathbb{Q}_d}(t) - \sigma(S(t), t)dt,$$

implies

$$d\left(\frac{1}{S(t)}\right) = -\Delta r \frac{1}{S(t)}dt - \frac{1}{S(t)}\sigma(S(t), t)dW^{\mathbb{Q}_f}(t).$$

Therefore, the dynamics of the inverted exchange rate  $Y(t) := 1/S(t)$  in the foreign measure becomes

$$dY(t) = -\Delta r Y(t)dt - \sigma\left(\frac{1}{Y(t)}, t\right)Y(t)dW^{\mathbb{Q}_f}(t).$$

This means that in order to ensure consistency, we will require

$$\sigma\left(\frac{1}{Y}, t\right) \sim \sigma(Y, t),$$

where by  $\sim$  we mean the same functional form. A non trivial function satisfying this is, for example, the logarithm, since

$$\log\left(\frac{1}{x}\right) = -\log(x).$$

More generally, we might consider any *polynomial of logarithms*. Another class is given by

$$\sigma(x, t) = \frac{1}{x^k} + \frac{1}{x^{k-1}} + \cdots + 1 + x^{k-1} + x^k.$$

Interestingly, this last expression is a local volatility which is useful in practice thanks to its flexibility in the parametrization. Generally, we could consider

$$\sigma(x) = f(\log(x)),$$

with function  $f$  having some sort of symmetry around the  $x$ -axis

$$f(x) \sim f(-x).$$

### 3.4 Inversion of jump diffusion with constant jump size

By virtue of the independence of the Brownian motion from the Poisson process, we can, without loss of generality, set the volatility structure to be constant, and focus on the jump component of our model. In the current section we will assume the jump size to be constant,

$$\begin{aligned} S(t) &= S(t_-) + \Delta S(t) = S(t_-) + S(t_-)\gamma^d \\ &= S(t_-)(1 + \gamma^d). \end{aligned}$$

Specifying  $\eta = 1$ ,  $\sigma(S(t), t) = \sigma$  in (3.2.1), our model becomes

$$dS(t) = (\Delta r - \gamma^d \lambda^d) S(t) dt + \sigma S(t) dW^{\mathbb{Q}_d}(t) + S(t_-) \gamma^d dN^{\mathbb{Q}_d}(t),$$

In order to determine whether, and under which conditions, consistency is fulfilled, we define, as above, the inverse exchange rate  $Y(t) := \frac{1}{S(t)}$ . Applying Ito's formula for jump-diffusion processes (see Cont and Tankov (2004) [60], Prop 8.14,) yields

$$dY(t) = (\Delta r + \gamma^d \lambda^d + \sigma^2) Y(t) dt - \sigma Y(t) dW^{\mathbb{Q}_d}(t) + Y(t_-) \left( -\frac{\gamma^d}{1 + \gamma^d} \right) dN^{\mathbb{Q}_d}(t).$$

Next, we perform a change of measure so as to express  $Y$  in the foreign measure. As it is well known (see e.g. Brigo and Mercurio [42]), the change of measure is defined via the Radon-Nikodym derivative

$$L(t) := \frac{S(t)B^f(t)}{S(0)B^d(t)}, \tag{3.4.1}$$

$B^d(t), B^f(t)$  denoting the domestic and foreign bank accounts respectively. In general, this can be rewritten in closed form as (see e.g. Shreve [153])

$$L(t) = L_1(t)L_2(t) \quad (3.4.2)$$

with

$$L_1(t) = \exp \left\{ \sigma W(t) - \frac{1}{2} \sigma^2 t \right\}$$

responsible for the Brownian motion and

$$L_2(t) = e^{(\lambda^f - \lambda^d)t} \left( \frac{\lambda^f}{\lambda^d} \right)^{N(t)} \quad (3.4.3)$$

responsible for the Poisson process,  $\lambda^f$  denoting the intensity of the Poisson process in the foreign measure. In differential form, we might write

$$\begin{aligned} dL_1(t) &= \sigma L_1(t) dW^{\mathbb{Q}^d}(t) \\ dL_2(t) &= \frac{\lambda^f - \lambda^d}{\lambda^d} L_2(t) dM^{\mathbb{Q}^d}(t) = \gamma^d L_2(t) dM^{\mathbb{Q}^d}(t), \end{aligned}$$

with  $dM^{\mathbb{Q}^d}(t) := dN^{\mathbb{Q}^d}(t) - \lambda^d dt$  a martingale. In the second expression, the first equality is due to (3.4.3), while the second is due to (3.4.1). More compactly, applying formula (3.4.2) and noting that Brownian motion and Poisson process are independent:

$$dL(t) = \sigma L(t) dW^{\mathbb{Q}^d}(t) + \gamma^d L(t) dM^{\mathbb{Q}^d}(t).$$

Hence, we deduce that

$$\frac{\lambda^f - \lambda^d}{\lambda^d} = \gamma^d,$$

that is

$$\lambda^f = \lambda^d(1 + \gamma^d). \quad (3.4.4)$$

The new Brownian motion is given by

$$dW^{\mathbb{Q}^f}(t) := dW^{\mathbb{Q}^d}(t) - \sigma dt.$$

**Remark 3.4.1.** *It could be interesting to notice that equation (3.4.4) can be deduced heuristically as follows (see also [23])*

$$\begin{aligned} \lambda^f dt &= \lambda^d dt + \frac{\mathbb{E}[dL(t)dN(t)|\mathcal{F}_t]}{L(t)} \\ &= \lambda^d dt + \frac{\mathbb{E}[\gamma^d L(t)(dN(t))^2|\mathcal{F}_t]}{L(t)} \\ &= \lambda^d dt + \gamma^d \lambda^d dt \\ &= \lambda^d(1 + \gamma^d) dt. \end{aligned}$$

Let us now notice that there appear to be two choices for the definition of the new jump size  $\gamma^f$ : we can define it as the whole term multiplying the Poisson process or that term with a minus in front. We opt for the first choice, that is

$$\gamma^f := -\frac{\gamma^d}{1 + \gamma^d}.$$

The reason for doing so is that in this way both the two jumps sizes in the different measures have domain  $D = (-1, +\infty)$ . Indeed,

$$\begin{aligned}\gamma^d \rightarrow -1^+ &\implies \gamma^f \rightarrow +\infty \\ \gamma^d \rightarrow +\infty &\implies \gamma^f \rightarrow -1^+.\end{aligned}$$

Therefore, the dynamics of  $Y$  under the foreign measure reads as

$$dY(t) = Y(t)(\Delta r - \gamma^f \lambda^f)dt - \sigma Y(t)dW^{\mathbb{Q}^f}(t) + Y(t_-)\gamma^f dN^{\mathbb{Q}^f}(t).$$

Since  $\gamma^d$  is constant, so is  $\gamma^f$ . Hence, consistency is readily fulfilled. It is also easy to check that  $Y$  is correctly compensated. This happens when

$$-\lambda^d \gamma^d + \lambda^f \gamma^f = 0$$

and this is satisfied in view of (3.4.4) and the definition of  $\gamma^f$ . In absolute values,

$$\frac{\lambda^f}{\lambda^d} = \left| \frac{\gamma^d}{\gamma^f} \right|.$$

This means that the higher the jump size in the domestic measure, the higher the jump frequency in the foreign measure. Since the foreign jump size is decreasing as a function of the domestic jump size, the foreign intensity must somehow compensate this effect and then increase. In other words, *the Poisson process in the foreign measure is expected to have a higher number jumps, but with a lower size.*

### 3.5 Inversion of jump diffusion with compound Poisson process

Finally, we discuss the case where the jump size is random, that is when the exchange rate is driven by a compound Poisson process. Consistency will now be more restrictive, as we will require the distribution of jump sizes not to be affected by the measure change. The aim of this section will be to determine a fairly general class of densities for the jump size in the domestic measure for which such condition will be satisfied.

$$dS(t) = (\Delta r - \beta^d \lambda^d)S(t)dt + \sigma S(t)dW^{\mathbb{Q}^d}(t) + S(t_-)dK^{\mathbb{Q}^d}(t),$$

where  $K^{\mathbb{Q}_d}(t)$  is a compound Poisson process (under the domestic measure)

$$K^{\mathbb{Q}_d}(t) := \sum_{i=1}^{N^{\mathbb{Q}_d}(t)} J_i^d,$$

with the jump sizes  $J_i^d$  are i.i.d. (independent of the processes  $W$  and  $N$ ) and

$$\beta^d := \mathbb{E}^d[J^d]$$

is the expectation of the domestic jump size under the domestic measure. The jump part might be conveniently rewritten as

$$\begin{aligned} dK^{\mathbb{Q}_d}(t) &= K^{\mathbb{Q}_d}(t+dt) - K^{\mathbb{Q}_d}(t) \\ &= \sum_{i=1}^{N_{t+dt}^{\mathbb{Q}_d}} J_i^d - \sum_{i=1}^{N_t^{\mathbb{Q}_d}} J_i^d \\ &= \sum_{i=1}^{N_t^{\mathbb{Q}_d} + dN_t^{\mathbb{Q}_d}} J_i^d - \sum_{j=1}^{N_t^{\mathbb{Q}_d}} J_j^d \\ &= J_1^d dN_t^{\mathbb{Q}_d}. \end{aligned}$$

Hence, our model can be rewritten also as

$$dS(t) = (\Delta r - \beta^d \lambda^d) S(t) dt + \sigma S(t) dW^{\mathbb{Q}_d}(t) + S(t_-) J_1^d dN^{\mathbb{Q}_d}(t).$$

For the sake of clarity, we remark that the compensator  $\beta^d \lambda^d$  guarantees absence of arbitrage. Since  $S(t_-)$  is  $\mathcal{F}_{t-}$ -measurable,

$$\mathbb{E}_{t-}^d [S(t_-) dK^{\mathbb{Q}_d}(t)] = S(t_-) \mathbb{E}_{t-}^d [dK^{\mathbb{Q}_d}(t)] = S(t_-) \mathbb{E}^d[J^d] \lambda^d dt.$$

In close form, this is (see Shreve [153])

$$S(t) = S(0) \exp \left\{ \sigma W^{\mathbb{Q}_d}(t) + (\Delta r - \beta^d \lambda^d - \frac{1}{2} \sigma^2) t \right\} \prod_{i=1}^{N^{\mathbb{Q}_d}(t)} (J_i^d + 1).$$

We can readily see that the domain  $D$  of the density of the jump size must be contained in  $(-1, +\infty)$ . Performing the inversion and changing measure yields

$$dY(t) = (-\Delta r + \beta^d \lambda^d) Y(t) dt - \sigma Y(t) dW^{\mathbb{Q}_f}(t) + Y(t) \left( -\frac{J^d}{1 + J^d} \right) dN^{\mathbb{Q}_f}(t).$$

Analogously to the constant scenario, we might define

$$J^f := -\frac{J^d}{1 + J^d}.$$

In order to determine expressions for  $\lambda^f$  and  $f^f$ , we look at the Radon-Nykodim derivative

$$L(t) := \frac{S(t)B^f(t)}{S(0)B^d(t)}.$$

Its differential is

$$dL(t) = \sigma L(t)dW^{\mathbb{Q}_d}(t) + L(t)dM^{\mathbb{Q}_d}(t),$$

with  $M$  being a martingale defined via

$$dM(t)^{\mathbb{Q}_d} := dK^{\mathbb{Q}_d}(t) - \beta^d \lambda^d dt.$$

In general, the RD derivative describing the joint change of measure of a Brownian motion and compound Poisson process is (see Shreve [153])

$$L(t) = L_1(t)L_2(t),$$

with

$$L_1(t) = \exp \left\{ \sigma W(t) - \frac{\sigma^2}{2} t \right\}$$

and

$$L_2(t) = e^{(\lambda^f - \lambda^d)t} \prod_{i=1}^{N(t)} \frac{\lambda^f f^f(J_i)}{\lambda^d f^d(J_i)}.$$

In differential form, we have (see Shreve [153])

$$dL_2(t) = L_2(t-)d(H(t) - \lambda^f t) - L_2(t-)d(N(t) - \lambda^d t)$$

with

$$H(t) = \sum_{i=1}^{N(t)} \frac{\lambda^f f^f(J_i)}{\lambda^d f^d(J_i)}.$$

Also,

$$dH = \Delta H(t) = \frac{\lambda^f f^f(Y_1)}{\lambda^d f^d(Y_1)} \Delta N(t).$$

Therefore,

$$\begin{aligned} dL_2(t) &= L_2(t) [dH(t) - \lambda^f dt - dN(t) + \lambda^d dt] \\ &= L_2(t) \left[ \left( \frac{\lambda^f f^f(Y_1)}{\lambda^d f^d(Y_1)} - 1 \right) dN(t) + (\lambda^d - \lambda^f) dt \right]. \end{aligned}$$

In our context of FX measure change, we have

$$L(t) = \exp \left\{ \sigma W^{\mathbb{Q}_d}(t) + (-\beta^d \lambda^d - \frac{1}{2} \sigma^2) t \right\} \prod_{i=1}^{N^{\mathbb{Q}_d}(t)} (Y_i + 1).$$

The differential of the jump part can be written as

$$dL_2(t) = L_2(t) [J_1^d dN^{\mathbb{Q}_d} - \beta^d \lambda^d dt].$$

It is now evident that, comparing the  $dt$  and  $dN$  terms, we obtain

$$\lambda^d - \lambda^f = -\beta^d \lambda^d,$$

that is, the intensity in the foreign measure is

$$\lambda^f = (1 + \beta^d) \lambda^d.$$

Moreover,

$$\frac{\lambda^f f^f(J_1)}{\lambda^d f^d(J_1)} - 1 = J_1,$$

that is

$$\frac{f^f(J_1)}{f^d(J_1)} = (1 + J_1) \frac{\lambda^d}{\lambda^f}.$$

Therefore, the probability distribution function of the jump size in the foreign measure is

$$f^f(x) = f^d(x)(1 + x) \frac{\lambda^d}{\lambda^f}. \quad (3.5.1)$$

For the sake of completeness, we check that the inverted process  $Y(t)$  is free of arbitrage. This is true if and only if

$$\mathbb{E}^d[J^d] \lambda^d + \mathbb{E}^f \left[ -\frac{J^d}{1 + J^d} \right] \lambda^f = 0.$$

In terms of densities, this is equivalent to

$$\lambda^d \int_{-1}^{+\infty} x f^d(x) dx + \lambda^f \int_{-1}^{+\infty} -\frac{x}{1 + x} f^f(x) dx = 0.$$

and this is readily satisfied in view of (3.5.1).

As far as consistency is concerned, we require *the density  $f^d$  of the jump size  $J^d$  under the domestic measure to belong to the same class as the density of  $J^f$  under the foreign measure*. Notice that  $f^f$  is the density of  $J^d$  under the foreign measure.

**Remark 3.5.1.** *For small  $J$ , at first order we have*

$$-\frac{J}{1 + J} = \frac{1}{1 + J} - 1 \approx 1 - J - 1 = -J,$$

*that is*

$$J^f = -J^d.$$

*This means that for small jumps, consistency is automatically satisfied.*

**Lemma 3.5.2.** *Let  $X$  be a generic random variable and define*

$$Y = -\frac{X}{1+X}.$$

*Let  $f_X$  be the density of  $X$  and  $f_Y$  the density of  $Y$ . Then, for any measure  $\mathbb{P}$ ,*

$$f_Y(y) = f_X\left(-\frac{y}{1+y}\right) \frac{1}{(1+y)^2}.$$

*Proof.* We have

$$\begin{aligned} \mathbb{P}(Y \leq y) &= \mathbb{P}\left(-\frac{X}{1+X} \leq y\right) \\ &= \mathbb{P}(-X \leq y + Xy) \\ &= \mathbb{P}(X(1+y) \geq -y) \\ &= \mathbb{P}\left(X \geq -\frac{y}{y+1}\right) \\ &= 1 - \mathbb{P}\left(X < -\frac{y}{y+1}\right). \end{aligned}$$

So,

$$F_Y(y) = 1 - F_X\left(-\frac{y}{1+y}\right).$$

Differentiating, we conclude. □

Let us denote by  $f_{J^d}^f$  the density of  $J^d$  under the foreign measure  $\mathbb{Q}_f$ , by  $f_{J^d}^d$  the density of  $J^d$  under the domestic measure  $\mathbb{Q}_d$ , and by  $f_{J^f}^f$  the density of  $J^f$  under the foreign measure  $\mathbb{Q}_f$ . Then, in view of equation (3.5.1), we write

$$f_{J^d}^f(x) = f_{J^d}^d(x)(1+x) \frac{\lambda^d}{\lambda^f}.$$

Moreover, in light of the lemma above, it holds

$$f_{J^f}^f(y) = f_{J^d}^f\left(-\frac{y}{1+y}\right) \frac{1}{(1+y)^2}.$$

Combining the two equations yields the relationship

$$f_{J^f}^f(y) = f_{J^d}^d\left(-\frac{y}{1+y}\right) \frac{1}{(1+y)^3} \frac{\lambda^d}{\lambda^f}.$$

At this point we might ask ourselves which kind of densities are appropriate. Keep in mind the domain  $D = (-1, +\infty)$ . Let us consider a power law distribution with cutoff function

$$f_{J^d}^d(x) = c \frac{1}{(1+x)^\alpha} e^{g(x)},$$

where  $c$  is the normalization constant. Then,

$$\begin{aligned} f_{Jf}^f(y) &= c \frac{1}{\left(1 - \frac{y}{1+y}\right)^\alpha} e^{g\left(-\frac{y}{1+y}\right)} \frac{1}{(1+y)^3} \frac{\lambda^d}{\lambda^f} \\ &= c \frac{\lambda^d}{\lambda^f} e^{g\left(-\frac{y}{1+y}\right)} \frac{1}{(1+y)^{3-\alpha}}. \end{aligned}$$

We might define the new scaling parameter  $\beta := 3 - \alpha$ . Consistency is then fulfilled only for those cut-off functions such that

$$g(y) \sim g\left(-\frac{y}{1+y}\right).$$

Notice that in this way the cut-off function ensures convergence at both  $-1$  and  $+\infty$ , since

$$\lim_{x \rightarrow +\infty} g(x) = \lim_{x \rightarrow -1^+} g(x) = -\infty.$$

A possible guess for  $g$  is

$$g(x) = -q \frac{x^2}{1+x},$$

with  $q$  positive constant. Indeed,

$$\frac{-\left(\frac{y}{1}\right)^2}{1 - \frac{y}{1+y}} = -\frac{y^2}{(1+y)^2} (1+y) = -\frac{y^2}{1+y}.$$

Summing up, a good candidate for the jump size density is

$$f_{Jd}^d(x) = c \frac{1}{(1+x)^\alpha} e^{-q \frac{x^2}{1+x}},$$

This density is defined on the interval  $[-1, +\infty)$ , and it is made of two components: a power law part depending on the scaling parameter  $\alpha$ , and an exponential cutoff depending on a positive parameter  $q$ .

## 4

# XVA-related global valuation equations

## 4.1 Introduction

In this chapter we consider the problem of formulating the most general mathematical equations for dealing with global valuation of derivatives contracts under credit risk, collateral modeling and funding costs. Valuation with credit, collateral and funding effects has become key after the credit crisis started in 2007-2008. Following the bankruptcy of big financial institutions like Lehman Brothers and the significant increment in the spread between the overnight indexed swap (OIS) rate and the LIBOR rate, practitioners realized several other sources of risk had to be taken into account when pricing illiquid contracts.

Both researchers in the industry and in the academia started to devote a lot of efforts to the study and management of the *funding costs*, representing the interest rate paid by financial institutions for the funds needed to carry on their business, and of the *default risk*, meaning the possibility that one of the parties involved in the contract will default. This resulted in the introduction of the so called *valuation adjustments* (XVA). These are correction terms to be added, or subtracted, from the default-free fair price in order to account for these risks. As illustrated in [39] and references therein, the most common adjustment is the so called *credit valuation adjustment* (CVA), which reflects losses due to the possible default of the counterparty. CVA is positive, increasing as a function of the counterparty default probability and the moneyness of the option, and has to be subtracted from the default-free price. Vice versa, there is also the possibility for the investor to default. From the point of view of the counterparty, the adjustment accounting for this risk is called *debit valuation adjustment* (DVA). This quantity has proven to be controversial for a series of reasons, e.g. it can give the dealer a positive mark to market as their credit quality worsens and it is particularly hard to hedge. One way out is to interpret DVA as a

funding benefit rather than a debit adjustment, as discussed in [44]. Another commonly studied adjustment is the *funding valuation adjustment* (FVA), which accounts for the cost of funding an uncollateralized derivative. Such derivative will be hedged by the trader via a portfolio consisting of the underlying asset and cash. In order to hold these positions as well as feed the collateral account, the trader will need funds from the bank treasury, which in turn will have to raise these funds by means of external funders. The quantity FVA will consist of all interests charged resulting from borrowing and lending activities. As remarked, for example, in [39, 9, 114], existence of funding costs is actually a matter of perspective. They are real if we consider the shareholder perspective, while they vanish once we take the whole bank's point of view.

The simplest idea is to compute all these adjustments separately and then adding them to the default-free price. However, such a naive approach can only work if the pricing equation is linear, and this is not in general the case. The problem is therefore to understand how these adjustments actually affect the price, in light of all the nonlinearities in place. Generally speaking, two approaches have been discussed in the literature. On the one hand, one might state explicitly all cash flows involved in the contract and define the fair price as the expectation, under the risk neutral measure, of the discounted cash flows. This idea, which we might call *adjusted cash flow approach*, was followed, for instance, by Brigo and co-authors [39, 140, 139, 41]. Alternatively, one could follow the so-called *replication approach*, which consists in reformulating classic notions such as replication and self-financing portfolio for a collateralized contract, as investigated for instance by Bielecki, Rutkowski, Crepey and co-authors [20, 19, 63, 64, 38].

In this chapter we generalize and improve the valuation equations obtained in [39] in two ways. First, we relax their assumptions on the default times. The default times of the investor and the counterparty, provided they are conditionally independent and satisfy a mild distribution condition, are arbitrary. Secondly, we relax their assumptions on the filtrations. The available market information may provide no, some or full insight into the default times of the investor and the counterparty. In [39], instead, the authors assumed full insight. Financially speaking, this means that we might not know when CVA and DVA cash flows will be triggered. In practice, it means that, as we monitor the market, we may have a situation where we suspect a company has defaulted, but we have no way to make sure this is actually the case. This framework could be used, for instance, to deal with fraud risk. If a company reports or balance books are fraudulent, as in the Parmalat default of 2003 with misreporting [43], the default time has not been observed fully, but it is already there *de facto*. Moreover, the filtration representing the default-free market information is arbitrary and does not need to coincide with the augmented filtration of some Brownian motion, as in [39]. This chapter constitutes just a first step towards a more general valuation equation. In future research based on this chapter, we will specify a stochastic volatility model for the underlying asset and discuss the derivation of mild

solutions to the valuation PDE, thereby generalizing previous results on classical or viscosity solutions.

The chapter is structured as follows. Section 4.2 sets up the notation and discusses the required probabilistic methods to deal with the market model. Namely, after introducing the notation in Section 4.2.1, a variety of representations for conditional expectations are derived in Section 4.2.2. In this context, we provide a class of conditionally independent default times in Section 4.2.3. Section 4.3 is devoted to the generalisation of the financial market model proposed in [39]. First, in Section 4.3.1 we explain the framework and the parameters of the model in detail. Then, all cash flows and costs involved in the derivative contract are quantified in Section 4.3.2. Finally, we provide two implicit representations for the pre-default value process that hedges the derivative. While the first valuation equation involves conditional expectations, the second is formulated in terms of a Lebesgue-Stieltjes and a stochastic integral. Finally, Sections 4.4 and 4.5 will be devoted to the proofs of the results from Sections 4.2 and 4.3 respectively.

## 4.2 Preliminaries

Throughout the chapter, let  $(\Omega, \mathcal{F}, P)$  denote a probability space,  $T > 0$  and  $(\mathcal{F}_t)_{t \in [0, T]}$ ,  $(\tilde{\mathcal{F}}_t)_{t \in [0, T]}$  be two filtrations of  $\mathcal{F}$ .

### 4.2.1 Notation and basic concepts

We recall that the extended non-negative real line  $[0, \infty]$  is metrizable in such a way that the resulting trace topology of  $\mathbb{R}_+$  agrees with the topology on  $\mathbb{R}_+$  induced by the absolute value function. For instance, take the metric given by

$$d_\infty(x, y) = |f_\infty(x) - f_\infty(y)|$$

for any  $x, y \in [0, \infty]$  with the strictly increasing homeomorphism  $f_\infty : \mathbb{R}_+ \rightarrow [0, 1)$  given by  $f_\infty(x) := x/(1+x)$  that satisfies  $f_\infty(\infty) = 1$ , where we set  $f(\infty) := \lim_{x \uparrow \infty} f(x)$  for any real-valued monotone function  $f$  defined on some interval. We shall use the induced topology of  $d_\infty$  in Sections 4.2.2 and 4.2.3.

### 4.2.2 Conditional expectations relative to different filtrations

In this section let  $\mathcal{T}$  be a non-empty finite set of  $[0, T] \cup \{\infty\}$ -valued random variables. Each  $\tau \in \mathcal{T}$  defines the smallest filtration  $(\mathcal{H}_t^\tau)_{t \in [0, T]}$  under which it becomes a stopping time. Namely,

$$\mathcal{H}_t^\tau = \sigma(\mathbb{1}_{\{\tau \leq s\}} : s \in [0, t]) \quad \text{for all } t \in [0, T]. \quad (4.2.1)$$

By setting  $\mathcal{H}_t := \bigvee_{\tau \in \mathcal{T}} \mathcal{H}_t^\tau$  for any  $t \in [0, T]$ , we obtain the smallest filtration under which any  $\tau \in \mathcal{T}$  is a stopping time. Then the  $(\mathcal{H}_t)_{t \in [0, T]}$ -stopping time  $\rho := \min_{\tau \in \mathcal{T}} \tau$  gives rise to the filtration defined via

$$\mathcal{F}_t^{\mathcal{T}} := \{\tilde{A} \in \mathcal{F} \mid \exists A \in \mathcal{F}_t : \{\rho > t\} \cap A = \{\rho > t\} \cap \tilde{A}\}$$

and which satisfies  $\mathcal{F}_t \vee \mathcal{H}_t \subset \mathcal{F}_t^{\mathcal{T}}$  for any  $t \in [0, T]$ . For this reason, let  $(\tilde{\mathcal{F}}_t)_{t \in [0, T]}$  denote another filtration satisfying  $\mathcal{F}_t \subset \tilde{\mathcal{F}}_t \subset \mathcal{F}_t \vee \mathcal{H}_t$  for all  $t \in [0, T]$ . These concepts generalise the framework in [[21], Section 5.1.1] and allow for the so-called *key lemma*, which relates conditional expectations (Cfr. Lemma 3.1 in [39])

**Lemma 4.2.1.** *Any  $[0, \infty]$ -valued random variable  $X$  satisfies*

$$E[X \mathbb{1}_{\{\rho > t\}} | \tilde{\mathcal{F}}_s] P(\rho > s | \mathcal{F}_s) = E[X \mathbb{1}_{\{\rho > t\}} | \mathcal{F}_s] P(\rho > s | \tilde{\mathcal{F}}_s) \quad a.s.$$

for all  $s, t \in [0, T]$  with  $s \leq t$ .

We notice that any decreasing sequence  $(A_t)_{t \in [0, T]}$  in  $\mathcal{F}$  satisfies

$$P(A_s | \mathcal{F}_s) \geq P(A_t | \mathcal{F}_s) = E[P(A_t | \mathcal{F}_t) | \mathcal{F}_s] \quad a.s.$$

for all  $s, t \in [0, T]$  with  $s \leq t$ . In particular, for every random variable  $\tau$  with values in  $[0, T] \cup \{\infty\}$  we have

$$P(\tau > t | \mathcal{F}_t) = G_t(\tau) \quad a.s. \quad \text{for any } t \in [0, T] \quad (4.2.2)$$

and some  $[0, 1]$ -valued  $(\mathcal{F}_t)_{t \in [0, T]}$ -supermartingale  $G(\tau)$ , called a survival process of  $\tau$  relative to this filtration. This fact implies an identification of random variables before  $\rho$  occurs.

**Corollary 4.2.2.** *For  $t \in [0, T]$  let  $X$  and  $\tilde{X}$  be two  $\mathbb{R}_+$ -valued random variables that are measurable relative to  $\mathcal{F}_t$  and  $\tilde{\mathcal{F}}_t$ , respectively. Then  $X = \tilde{X}$  a.s. on  $\{\rho > t\}$  if and only if*

$$X G_t(\rho) = E[\tilde{X} \mathbb{1}_{\{\rho > t\}} | \mathcal{F}_t] \quad a.s.$$

In this case,  $X$  is a.s. uniquely determined as soon as  $G_t(\rho) > 0$  a.s.

We remark that Corollary 4.2.2 allows for the survival process  $G(\tau)$  in the model to reach the point zero, which is more realistic, as the conditional probability of an event may take zero values even if the event has positive probability. In the case  $P(G_t(\rho) = 0 > 0)$ , the pre-default value process is not uniquely determined, yet it is still characterized in the pre-default valuation equation of Theorem 4.3.2 below. Now we rewrite a conditional expectation of a stopped integral relative to  $(\mathcal{F}_t)_{t \in [0, T]}$  by means of the associated survival process (Cfr. Lemma 3.2 in [39]).

**Lemma 4.2.3.** *Let  $s \in [0, T]$  and  $G(\rho)$  be measurable. If  $X$  and  $\tilde{X}$  are two  $[0, \infty]$ -valued measurable processes such that  $X_t$  is  $\mathcal{F}_t$ -measurable and  $X_t = \tilde{X}_t$  a.s. on  $\{\rho > t\}$  for all  $t \in [s, T]$ . Then*

$$E \left[ \int_s^{T \wedge \rho} \tilde{X}_t dt \middle| \mathcal{F}_s \right] = E \left[ \int_s^T X_t G_t(\rho) dt \middle| \mathcal{F}_s \right] \quad a.s.$$

To consider conditional expectations of processes combined with stopping times, we require a generalized concept of conditional independence.

**Definition 4.2.4.** *Let  $n \in \mathbb{N}$  and  $\tau_1, \dots, \tau_n$  be  $[0, T] \cup \{\infty\}$ -valued random variables. Then  $\tau_1, \dots, \tau_n$  are called  $(\mathcal{F}_t)_{t \in [0, T]}$ -conditionally independent if*

$$P(\tau_1 > s_1, \dots, \tau_n > s_n | \mathcal{F}_t) = P(\tau_1 > s_1 | \mathcal{F}_t) \cdots P(\tau_n > s_n | \mathcal{F}_t) \quad a.s.$$

for each  $t \in [0, T]$  and any  $s_1, \dots, s_n \in [0, t]$ .

If two  $[0, T] \cup \{\infty\}$ -valued random variables are  $(\mathcal{F}_t)_{t \in [0, T]}$ -conditionally independent, then their joint conditional distribution relative to  $\mathcal{F}_T$  is completely determined.

**Lemma 4.2.5.** *Let  $\sigma, \tau$  be two  $[0, T] \cup \{\infty\}$ -valued  $(\mathcal{F}_t)_{t \in [0, T]}$ -conditionally independent random variables. Then*

$$P((\sigma, \tau) \in C | \mathcal{F}_T)(\omega) = K(\omega, \cdot) \otimes L(\omega, \cdot)(C) \quad \text{for } P\text{-a.e. } \omega \in \Omega, \quad (4.2.3)$$

all  $C \in \mathcal{B}([0, T] \cup \{\infty\})^2$  and any two respective regular conditional probabilities  $K$  and  $L$  of  $\sigma$  and  $\tau$  given  $\mathcal{F}_T$ .

We conclude with the following integral representation within conditional expectations, which extends Proposition 5.11 in [21] (Cfr. Lemma 3.3 in [39]).

**Proposition 4.2.6.** *Let  $s \in [0, T]$ ,  $\sigma, \tau \in \mathcal{T}$  and  $\tilde{X}$  be an  $(\tilde{\mathcal{F}}_t)_{t \in [0, T]}$ -adapted càdlàg process for which the following three conditions hold:*

- (i)  *$G(\sigma)$  is right-continuous and of finite variation, the paths of  $G(\tau)$  are left-continuous except at countably many points and  $\sigma, \tau$  are  $(\mathcal{F}_t)_{t \in [0, T]}$ -conditionally independent.*
- (ii) *There exists an  $(\mathcal{F}_t)_{t \in [0, T]}$ -adapted càdlàg process  $X$  such that  $X_t = \tilde{X}_t$  a.s. on  $\{t < \sigma \leq T \wedge \tau\}$  for each  $t \in ]s, T]$ .*

If

$$\begin{aligned} & \sup_{t \in ]s, T]} |\tilde{X}_t^\sigma| \mathbb{1}_{\{s < \sigma \leq T \wedge \tau\}} \\ & \sup_{t \in ]s, T]} |X_t| G_t(\tau) V_T(\sigma) \end{aligned}$$

are integrable, where  $V(\sigma)$  is the variation process of  $G(\sigma)$ , then

$$E[\tilde{X}_T^\sigma \mathbb{1}_{\{s < \sigma \leq T \wedge \tau\}} | \mathcal{F}_s] = -E \left[ \int_{]s, T]} X_t G_t(\tau) dG_t(\sigma) \middle| \mathcal{F}_s \right] \quad a.s.$$

### 4.2.3 Construction of conditionally independent stopping times

The aim of this section is to provide a class of  $(\mathcal{F}_t)_{t \in [0, T]}$ -conditionally independent hitting times. We fix  $n \in \mathbb{N}$  and suppose that  $\xi_i$  is an  $\mathbb{R}_+$ -valued  $\tilde{\mathcal{F}}_0$ -measurable random variable that is independent of  $\mathcal{F}_T$  with survival function  $G_i$  for any  $i \in \{1, \dots, n\}$ . Based on the metrization discussed in Section 4.2.1, let  ${}_1X, \dots, {}_nX$  be  $[0, \infty]$ -valued  $(\mathcal{F}_t)_{t \in [0, T]}$ -adapted increasing right-continuous processes and define a function on  $\Omega$  with values in  $[0, T] \cup \{\infty\}$  by

$$\tau_i := \inf\{t \in [0, T] \mid {}_iX_t \geq \xi_i\}$$

for any  $i \in \{1, \dots, n\}$ . Then the hitting time  $\tau_i$  does not need to be an  $(\mathcal{F}_t)_{t \in [0, T]}$ -stopping time, as  $\xi_i$  may fail to be  $\mathcal{F}_0$ -measurable. However, the following facts hold.

**Lemma 4.2.7.** *For each  $i \in \{1, \dots, n\}$  the function  $\tau_i$  is an  $(\tilde{\mathcal{F}}_t)_{t \in [0, T]}$ -stopping time such that  $\{\tau_i > s\} = \{\xi_i > {}_iX_s\}$  and*

$$P(\tau_i > s \mid \mathcal{F}_t) = G_i({}_iX_s) \quad a.s.$$

for all  $s, t \in [0, T]$  with  $s \leq t$ . In particular,  $P(\tau_i > s) = E[G_i({}_iX_s)]$ .

Now we derive properties of the  $(\tilde{\mathcal{F}}_t)_{t \in [0, T]}$ -stopping time  $\rho := \min_{i \in \{1, \dots, n\}} \tau_i$  when  $\tau_1, \dots, \tau_n$  are conditionally independent relative to  $(\mathcal{F}_t)_{t \in [0, T]}$ .

**Proposition 4.2.8.** *Suppose that  $\xi_1, \dots, \xi_n$  are independent. Then the  $(\tilde{\mathcal{F}}_t)_{t \in [0, T]}$ -stopping times  $\tau_1, \dots, \tau_n$  are  $(\mathcal{F}_t)_{t \in [0, T]}$ -conditionally independent and*

$$P(\rho > s \mid \mathcal{F}_t) = \prod_{i=1}^n G_i({}_iX_s) \quad a.s.$$

for any  $s, t \in [0, T]$  with  $s \leq t$ . Moreover, the following three assertions hold:

- (i)  $P(\rho > s \mid \mathcal{F}_s) > 0$  a.s. if and only if  ${}_iX_s < \text{ess sup } \xi_i$  a.s. for all  $i \in \{1, \dots, n\}$ .
- (ii)  $\rho > s$  a.s. if and only if  ${}_iX_s \leq \text{ess inf } \xi_i$  a.s. for any  $i \in \{1, \dots, n\}$ , and  $\rho < \infty$  a.s. if and only if  $\text{ess sup } \xi_i \leq {}_iX_T$  for some  $i \in \{1, \dots, n\}$  a.s.
- (iii)  $P(\rho = s) = 0$  whenever  $s > 0$ ,  $G_1, \dots, G_n$  are continuous and  ${}_1X, \dots, {}_nX$  are a.s. continuous.

**Remark 4.2.9.** *For  $i \in \{1, \dots, n\}$  let  $\hat{x}_i \geq 0$  and  ${}_i\lambda$  be an  $[0, \infty]$ -valued process that is progressively measurable relative to  $(\mathcal{F}_t)_{t \in [0, T]}$  such that*

$${}_iX = \hat{x}_i + \int_0^t {}_i\lambda_s ds \quad \text{for all } t \in [0, T]. \quad (4.2.4)$$

Then  ${}_iX$  is left-continuous, by monotone convergence. The assumed right-continuity of  ${}_iX$  holds if and only if for any  $\omega \in \Omega$  there is  $t_\omega \in ]0, T]$  such that

$$\int_0^{t_\omega - \varepsilon} {}_i\lambda_s(\omega) ds < \infty \quad \text{and} \quad \int_0^{t_\omega} {}_i\lambda_s(\omega) ds = \infty \quad (4.2.5)$$

for all  $\varepsilon \in ]0, t_\omega[$  or the path  ${}_i\lambda(\omega)$  is Lebesgue-integrable.

For any  $t \in [0, T]$  we infer from Lemma 4.2.7 that the event

$$\Lambda_t := \bigcap_{i=1}^n \{ {}_iX_t < \text{ess sup } \xi_i \},$$

which lies in  $\mathcal{F}_t$ , includes  $\{\rho > t\}$ , and in the setting of Proposition 4.2.8 we have  $\rho < \infty$  a.s. if and only if  $\Lambda_T^c$  is a null set. Hence, the next representation implies an explicit formula for the density function of  $\rho$  when  $\{\rho = 0\}$  and  $\Lambda_T^c$  are null sets.

**Proposition 4.2.10.** *For any  $i \in \{1, \dots, n\}$  let  ${}_iX$  be of the form (4.2.4) for some  $\hat{x}_i \geq 0$  and a  $[0, \infty]$ -valued  $(\mathcal{F}_t)_{t \in [0, T]}$ -progressively measurable process  ${}_i\lambda$  satisfying (4.2.5). Then*

$$P(\{\rho > s\} \cap \Lambda_t) = P(\rho > t) - \sum_{j=1}^n \int_s^t E \left[ {}_j\lambda_{\tilde{s}} \left( \frac{G'_j}{G_j} \right) ({}_jX_{\tilde{s}}) \prod_{i=1}^n G_i({}_iX_{\tilde{s}}); \Lambda_t \right] d\tilde{s}$$

for any  $s, t \in [0, T]$  with  $s \leq t$  as soon as  $\xi_1, \dots, \xi_n$  are independent,  $\text{ess inf } \xi_i \leq \hat{x}_i$  and  $G_i$  is continuously differentiable on  $] \text{ess inf } \xi_i, \text{ess sup } \xi_i[$  for any  $i \in \{1, \dots, n\}$ .

To conclude our analysis, let us impose the gamma distribution on  $\xi_1, \dots, \xi_n$ . This includes the hitting times considered in Brigo et al. [39] as special case, by choosing an exponential distribution with mean one.

**Example 4.2.11.** *For each  $i \in \{1, \dots, n\}$  let  $\xi_i$  be gamma distributed with shape  $\alpha_i > 0$  and rate  $\beta_i > 0$ . That is, its survival function and the gamma function  $\Gamma$  satisfy*

$$G_i(x) = \frac{\beta_i^{\alpha_i}}{\Gamma(\alpha_i)} \int_x^\infty y^{\alpha_i-1} e^{-\beta_i y} dy \quad \text{for all } x \geq 0.$$

Let  $\hat{x}_i \geq 0$  and  ${}_i\lambda$  be an  $[0, \infty]$ -valued  $(\mathcal{F}_t)_{t \in [0, T]}$ -progressively measurable process so that (4.2.4) and (4.2.5) hold, and suppose that  $\xi_1, \dots, \xi_n$  are independent. Then it follows from Proposition 4.2.8 that

$$P(\rho > s | \mathcal{F}_t) = \prod_{i=1}^n \frac{\gamma(\alpha_i, \beta_i {}_iX_s)}{\Gamma(\alpha_i)} \quad \text{a.s.}$$

for any  $s, t \in [0, T]$  with  $s \leq t$ , where  $\gamma : ]0, \infty[ \rightarrow ]0, \infty[$ ,  $\gamma(\alpha, x) := \int_x^\infty y^{\alpha-1} e^{-y} dy$  is the upper incomplete gamma function. Moreover, the following properties hold:

- (i)  $P(\rho = t) = 0$  for any  $t \in [0, T]$ , and we have  $P(\rho > s) < 1$  if and only if  $\int_0^t \lambda_s ds > 0$  a.s. for any  $t \in ]0, T]$  with

$$\lambda := \sum_{i=1}^n \lambda_i.$$

- (ii)  $P(\rho > t | \mathcal{F}_t) > 0$  a.s. if and only if  $\int_0^t \lambda_s ds < \infty$  a.s. for all  $t \in [0, T]$ , and it holds that  $\rho < \infty$  a.s. if and only if  $\int_0^T \lambda_s ds = \infty$  a.s.

- (iii) For any  $s, t \in [0, T]$  with  $s \leq t$  we have

$$\Lambda_t = \left\{ \int_0^t \lambda_s ds < \infty \right\}$$

and the difference between  $P(\{\rho > s\} \cap \Lambda_t)$  and  $P(\rho > t)$  coincides with

$$- \sum_{j=1}^n \int_s^t E \left[ j \lambda_{\tilde{s}} \frac{\beta_j^{\alpha_j} X_{\tilde{s}}^{\alpha_j-1}}{\gamma(\alpha_j, \beta_j X_{\tilde{s}})} e^{-\beta_j X_{\tilde{s}}} \prod_{i=1}^n \frac{\gamma(\alpha_i, \beta_i X_{\tilde{s}})}{\Gamma(\alpha_i)}; \Lambda_t \right] d\tilde{s}.$$

In particular, if we have both  $\int_0^t \lambda_s ds < \infty$  for all  $t \in ]0, T[$  and  $\int_0^T \lambda_s ds = \infty$  a.s., then  $\tau$  is a.s. finite and continuously distributed.

## 4.3 A general market model with default

Our aim is to evaluate a derivative contract with maturity  $T$  between an *investor*  $I$  and a *counterparty*  $C$ , both considered as financial entities, with a special focus on the case that  $I$  stands for an *investment bank*  $B$ .

### 4.3.1 Model specifications

In the sequel, we interpret the two filtrations  $(\mathcal{F}_t)_{t \in [0, T]}$  and  $(\tilde{\mathcal{F}}_t)_{t \in [0, T]}$  as the *temporal developments of the default-free information* and the *whole available information* on an underlying financial market, respectively. We use two  $[0, T] \cup \{\infty\}$ -valued random variables  $\tau_I$  and  $\tau_C$  to model the respective *default times* of the investor and the counterparty. Then  $\tau := \tau_I \wedge \tau_C$  stands for the *time of a party to default first*. By using the notation (4.2.1), we require that

$$\mathcal{F}_t \vee \mathcal{H}_t^\tau \subset \tilde{\mathcal{F}}_t \subset \mathcal{F}_t \vee \mathcal{H}_t^{\tau_I} \vee \mathcal{H}_t^{\tau_C} \quad \text{for all } t \in [0, T]. \quad (4.3.1)$$

Thus, the available market information gives full knowledge about the first time of default, but it may fail to provide full insight into the respective default times of  $I$  and  $C$ . This generalizes [39], where full insight was assumed.

Next, for any  $(\mathcal{F}_t)_{t \in [0, T]}$ -progressively measurable process  $\gamma$  with Lebesgue-integrable paths we introduce an  $]0, \infty[$ -valued function  $D(\gamma)$  on  $[0, T]^2 \times \Omega$  by

$$D_{s,t}(\gamma) := \exp \left( - \int_s^t \gamma_{\tilde{s}} d\tilde{s} \right), \quad \text{if } s \leq t,$$

and  $D_{s,t}(\gamma) := 1$ , otherwise. Then the function  $[0, T]^2 \rightarrow ]0, \infty[$ ,  $(s, t) \mapsto D_{s,t}(\gamma)(\omega)$  is continuous for any  $\omega \in \Omega$  and  $D_{s,t}(\gamma)$  is  $\mathcal{F}_t$ -measurable for all  $s, t \in [0, T]$  with  $s \leq t$ . Moreover,  $D(\gamma)$  is bounded as soon as  $\gamma$  is bounded from below. Let  $r$  be an  $(\mathcal{F}_t)_{t \in [0, T]}$ -progressively measurable process with Lebesgue-integrable paths that represents the *instantaneous risk-free interest rate*. Then  $D_{s,t}(r)$  is the *discount factor* from time  $s \in [0, T]$  to  $t \in [s, T]$ . Put differently,  $D_{s,t}(r)$  specifies the required amount to invest risk-free at time  $s$ , in order to receive 1 unit of cash at time  $t$ . Finally, let  $\tilde{P}$  be a *martingale measure* in the sense that each discounted price process of a non-dividend-paying traded risky asset is an  $(\tilde{\mathcal{F}}_t)_{t \in [0, T]}$ -martingale. That is, there is a non-empty set of processes  $U$  representing the price processes of all such assets for which the process  $[0, T] \times \Omega \rightarrow \mathbb{R}$ ,  $(t, \omega) \mapsto D_{0,t}(r)(\omega)U_t(\omega)$  is an  $(\tilde{\mathcal{F}}_t)_{t \in [0, T]}$ -martingale under  $\tilde{P}$ . At all times,  $\tilde{P}$  is ought to be equivalent to  $P$ .

In our continuous-time setting we assume that the distributions of  $\tau_I$  and  $\tau_C$  admit at most one atom, which is at infinity, and both parties *cannot default simultaneously*. That is, for any  $t \in [0, T]$  we have

$$P(\tau_I = t) = P(\tau_C = t) = 0 \quad \text{and} \quad P(\tau_I = \tau_C, \tau < \infty) = 0. \quad (4.3.2)$$

The condition on the distributions implies that  $\tau \neq t$  a.s. for all  $t \in [0, T]$ . However, as  $\{\tau_I = \tau_C, \tau = \infty\} = \{\tau = \infty\}$  and we have made no restrictions on  $\tilde{P}(\tau_I = \infty)$  and  $\tilde{P}(\tau_C = \infty)$ , both entities may not default at all. So, we allow for  $\tilde{P}(\tau = \infty) \in [0, 1]$ . Moreover, we stress the fact that the event  $\{\tau_I = \tau_C, \tau < \infty\}$  of simultaneous default is automatically a null set if  $\tau_I$  and  $\tau_C$  are  $(\mathcal{F}_t)_{t \in [0, T]}$ -conditionally independent under  $\tilde{P}$ . In fact, in this case Lemma 4.2.5 gives

$$\tilde{P}(\tau_I = \tau_C, \tau < \infty) = \tilde{P}((\tau_I, \tau_C) \in \Delta) = \tilde{E} \left[ \int_0^T K(\cdot, \{t\}) L(\cdot, dt) \right] = 0$$

for any two regular conditional probabilities  $K$  and  $L$  of  $\tau_I$  and  $\tau_C$  under  $\tilde{P}$  given  $\mathcal{F}_T$ , where  $\Delta := \{(s, t) \in [0, T] \mid s = t\}$ . Indeed, as  $\tilde{P}(\tau_I = t | \mathcal{F}_t) = 0$  a.s. for any  $t \in [0, T]$  and  $\mathcal{B}([0, T] \cup \{\infty\})$  is countably generated, there is a null set  $N \in \mathcal{F}_T$  such that  $K(\omega, \{t\}) = 0$  for all  $\omega \in N^c$ .

### 4.3.2 Incorporation of all relevant cash flows and costs

First, let us summarize all cash flows and costs that have an impact on the value of the contract between  $I$  and  $C$ . These quantities are the *contractual derivative*

*cash flows* (4.3.3), the *costs of a collateral account* (4.3.4), the *funding costs* (4.3.5), the *hedging costs* (4.3.6) and the *cash flows arising on the default of one of the two parties* (4.3.7).

- (i) **The contractual derivative cash flows** with  $C$  are supposed to depend on the path of a dividend-paying risky asset and a payoff functional.
- The *underlying asset* and its *instantaneous dividend process* are modelled by an  $(\mathcal{F}_t)_{t \in [0, T]}$ -adapted càdlàg process  $S$  and some  $(\mathcal{F}_t)_{t \in [0, T]}$ -progressively measurable process  $\pi$  with Lebesgue-integrable paths, respectively.
  - The  $\mathbb{R}_+$ -valued Borel measurable functional  $\Phi$  on the Banach space  $D([0, T])$  of all real-valued càdlàg functions on  $[0, T]$ , equipped with the supremum norm, represents the *payoff functional*.
  - The cash flows consist of the amount  $\Phi(S)$  paid at maturity and dividends according to the rate  $\pi$ . The continuous process  ${}_{\text{con}}\text{CF}$  representing the discounted future cash flows at any time point is given by

$${}_{\text{con}}\text{CF}_s := D_{s, T}(r)\Phi(S)\mathbb{1}_{\{\tau > T\}} + \int_s^{T \wedge \tau} D_{s, t}(r)\pi_t dt. \quad (4.3.3)$$

- (ii) **The costs of a collateral account** that arise from the *collateralisation procedure* to mitigate the default risk which are subject to the *collateral remuneration rate*.
- Namely, the collateral serves as guarantee in case of default and the party receiving it will have to remunerate it at a certain interest rate, called the *collateral rate*, determined by the contract. We assume that the assets received as collateral can be re-hypotecated and do not have to be kept segregated.
  - The *cash flows of the collateral procedure* and the *two collateral rates of each party* are given by an  $(\mathcal{F}_t)_{t \in [0, T]}$ -adapted càdlàg process  $C$  and two  $(\mathcal{F}_t)_{t \in [0, T]}$ -progressively measurable processes  ${}_+c$  and  ${}_ -c$  with Lebesgue-integrable paths, respectively.
  - The processes are modelled so that  $I$  is a collateral receiver remunerating the assets at the rate  ${}_+c_t$  on  $\{C_t > 0\}$  and a collateral provider investing at the rate  ${}_ -c_t$  on  $\{C_t < 0\}$  for any  $t \in [0, T]$ . The process  $c$  representing the *respective collateral rate* is given by

$$c_t := {}_+c_t\mathbb{1}_{\{C_t > 0\}} + {}_-c_t\mathbb{1}_{\{C_t < 0\}}.$$

- The continuous process  ${}_{\text{col}}C$  that stands for the time evolution of the discounted future cash flows is specified by

$${}_{\text{col}}C_s := \int_s^{T \wedge \tau} D_{s,t}(r)(c_t - r_t)C_t dt. \quad (4.3.4)$$

- (iii) **The costs due to a funding account** that may accrue, since  $I$  is supposed to have access to an account for borrowing or investing money at two respective risk-free interest rates.

- The *funding amount* and the two *interest rates for borrowing and lending* are given by an  $(\tilde{\mathcal{F}}_t)_{t \in [0, T]}$ -adapted càdlàg process  $\tilde{F}$  and two  $(\tilde{\mathcal{F}}_t)_{t \in [0, T]}$ -progressively measurable processes  ${}_+\tilde{f}$  and  ${}_-\tilde{f}$  with Lebesgue-integrable paths, respectively.
- More precisely,  $I$  is borrowing the amount  $\tilde{F}_t$  at the interest rate  ${}_+\tilde{f}_t$  on  $\{\tilde{F}_t > 0\}$  and she is lending the amount  $-\tilde{F}_t$  at the rate  ${}_-\tilde{f}_t$  on  $\{\tilde{F}_t < 0\}$  for each  $t \in [0, T]$ . Thus, the *respective funding rate*  $\tilde{f}$  is given by

$$\tilde{f}_t := {}_+\tilde{f}_t \mathbb{1}_{\{\tilde{F}_t > 0\}} + {}_-\tilde{f}_t \mathbb{1}_{\{\tilde{F}_t < 0\}}.$$

- Hence, the continuous process  ${}_{\text{fun}}C$  representing the temporal development of the present value of these *funding costs* is defined via

$${}_{\text{fun}}C_s := \int_s^{T \wedge \tau} D_{s,t}(r)(\tilde{f}_t - r_t)\tilde{F}_t dt. \quad (4.3.5)$$

- (iv) As  $I$  may stand for a bank, we assume that she may enter repurchase agreements to hedge its exposure. For this reason, **the costs that result from hedging the derivative** should be taken into account.

- The *value of the risky asset position that I has via the repo* and the two *repo rates* are given by an  $(\tilde{\mathcal{F}}_t)_{t \in [0, T]}$ -adapted càdlàg process  $\tilde{H}$  and two  $(\tilde{\mathcal{F}}_t)_{t \in [0, T]}$ -progressively measurable processes  ${}_+\tilde{h}$  and  ${}_-\tilde{h}$  with Lebesgue-integrable paths.
- Regarding the interpretation,  $I$  borrows a risky asset with the repo rate  ${}_+\tilde{h}_t$  on  $\{\tilde{H}_t > 0\}$  and lends a risky asset with the rate  ${}_-\tilde{h}_t$  on  $\{\tilde{H}_t < 0\}$  for all  $t \in [0, T]$ . Hence, the *respective repo rate*  $\tilde{h}$  is given by

$$\tilde{h}_t := {}_+\tilde{h}_t \mathbb{1}_{\{\tilde{H}_t > 0\}} + {}_-\tilde{h}_t \mathbb{1}_{\{\tilde{H}_t < 0\}}.$$

- We implicitly suppose that  $I$  continuously rolls over repo contracts and that at each point  $t \in [0, T]$  she receives in the repo the exact value of the assets she is lending. Thus, the gain of the repo position is given by the growth of the assets that are being repoed minus  $\tilde{h}_t(-\tilde{H}_t)$ , the repo rate times the amount of cash received.

- In consequence, the continuous process  ${}_{\text{hed}}C$  that stands for the discounted future cash flows is given by

$${}_{\text{hed}}C_s := \int_s^{T \wedge \tau} D_{s,t}(r)(r_t - \tilde{h}_t)\tilde{H}_t dt. \quad (4.3.6)$$

- (v) **The cash flows arising on the default of one of the two parties** that can be computed with the *residual value of the claim*, the *net exposure*, the *losses given default* and the *funding amount*.

- The time evolution of the *close-out value* is given by an  $(\mathcal{F}_t)_{t \in [0, T]}$ -càdlàg process  $\varepsilon$ . Then we interpret  $\varepsilon_\tau$  as the *residual value of the claim* at the time  $\tau$  of a party to default first on  $\{\tau < \infty\}$ , since  $\tau_I \neq \tau_C$  a.s. on this event.
- On  $\{\tau_C < \tau_I\}$  we specify the following: If the *net exposure*  $\varepsilon_\tau - C_\tau$  at the moment of default is non-positive, then  $I$  is a net debtor and repays  $\varepsilon_\tau$  to  $C$ . If instead  $\varepsilon_\tau - C_\tau > 0$ , then  $I$  is a net creditor and recovers a fraction  $1 - \text{LGD}_I$  of its credits, in which case it receives  $C_\tau + (1 - \text{LGD}_C)(\varepsilon_\tau - C_\tau)$ .
- In this context, we implicitly assume that the loss fractions  $\text{LGD}_I, \text{LGD}_C \in [0, 1]$ , which denote the *losses given defaults* of  $I$  and  $C$ , respectively, are deterministic exogenous quantities.
- The case in which  $I$  is a bank and defaults before  $C$  is symmetrical. If, however,  $I \neq B$ , then merely  $\varepsilon_\tau$  is being considered on  $\{\tau_I < \tau_C\}$ .
- This shows that the *cash flows on default due to the contract* can be modelled by a càdlàg process  ${}_{\text{def},c}\text{CF}$  specified via

$$\begin{aligned} {}_{\text{def},c}\text{CF}_s &:= D_{s,\tau}(r)(\varepsilon_\tau - \text{LGD}_C(\varepsilon_\tau - C_\tau)^+ \mathbb{1}_{\{\tau_C < \tau_I\}}) \\ &\quad + D_{s,\tau}(r)\text{LGD}_I(\varepsilon_\tau - C_\tau)^- \mathbb{1}_{\{B\}}(I) \mathbb{1}_{\{\tau_I < \tau_C\}} \end{aligned}$$

on  $\{s < \tau < T\}$  and  ${}_{\text{def},c}\text{CF}_s := 0$  on the complement of this set.

- As we suppose that if  $I$  is a bank and has a cash surplus, then it may invest into risk-free assets, we also consider the *cash flows on the bank's default due to funding*. The time evolution of the corresponding net present value is represented by a càdlàg process  ${}_{\text{def},f}\text{CF}$  defined by

$${}_{\text{def},f}\text{CF}_s := D_{s,\tau}(r)\text{LGD}_I \tilde{F}_{\tau_I}^+ \mathbb{1}_{\{B\}}(I) \mathbb{1}_{\{\tau_I < \tau_C\}}$$

on  $\{s < \tau < T\}$  and  ${}_{\text{def},f}\text{CF}_s := 0$  on its complement. The process  ${}_{\text{def}}\text{CF}$  summing up both sources of default risk is given by

$${}_{\text{def}}\text{CF}_s := {}_{\text{def},c}\text{CF}_s + {}_{\text{def},f}\text{CF}_s. \quad (4.3.7)$$

### 4.3.3 Pre-default representations of the value process

For a valuation of the underlying derivative contract we first ensure the integrability of the net present values of all the cash flows and costs associated to the contract. Thus, throughout the section  $(\Omega, \mathcal{F}, \tilde{P})$  serves as underlying probability space. For an  $[0, T] \cup \{\infty\}$ -valued random variable  $\sigma$  let  $\tilde{\mathcal{L}}(r, \sigma)$  be the linear space of all random variables  $X$  for which  $D_{s,T}(r)|X|\mathbb{1}_{\{\sigma > T\}}$  is  $\tilde{P}$ -integrable for any  $s \in [0, T]$  and  $\tilde{\mathcal{P}}(r, \sigma)$  be the linear space of all measurable processes  $X$  so that

$$\tilde{E} \left[ \int_s^{T \wedge \sigma} D_{s,t}(r) |X_t| dt \right] < \infty \quad \text{for all } s \in [0, T].$$

Furthermore, by  $\tilde{\mathcal{D}}(r, \sigma)$  we denote the linear space of all càdlàg processes  $X$  such that  $\sup_{t \in ]s, T[} D_{s,t}(r) |X_t| \mathbb{1}_{\{s < \sigma < T\}}$  is integrable for each  $s \in [0, T[$  and we set

$$\tilde{\mathcal{L}}(r) := \tilde{\mathcal{L}}(r, \infty) \quad \text{and} \quad \tilde{\mathcal{P}}(r) := \tilde{\mathcal{P}}(r, \infty),$$

by convention. Within this context, a process  $X$  will be called *integrable up to time*  $\sigma$  if  $X_t \mathbb{1}_{\{\sigma > t\}}$  is integrable for any  $t \in [0, T]$ . Now we may introduce the following *integrability conditions*:

- (M.1) The amount  $\Phi(S)$  paid at maturity lies in  $\tilde{\mathcal{L}}(r, \tau)$  and the dividend rate  $\pi$  together with the coupled processes  $(c - r)C$ ,  $(f - r)\tilde{F}$  and  $(r - h)\tilde{H}$  belong to  $\tilde{\mathcal{P}}(r, \tau)$ .
- (M.2) The cash flows  $(\varepsilon + \text{LGD}_I((\varepsilon - C)^- + \tilde{F}^+) \mathbb{1}_{\{B\}}(I) \mathbb{1}_{\{\tau_I < \tau_C\}})$  on default of  $I$  and the cash flows  $(\varepsilon - \text{LGD}_C(\varepsilon - C)^+) \mathbb{1}_{\{\tau_C < \tau_I\}}$  on default of  $C$  lie in  $\tilde{\mathcal{D}}(r, \tau)$ .

We are then able to prove the following result.

**Lemma 4.3.1.** *Under (M.1) and (M.2), the processes  ${}_{\text{con}}\text{CF}$ ,  ${}_{\text{col}}\text{C}$ ,  ${}_{\text{fun}}\text{C}$ ,  ${}_{\text{hed}}\text{C}$  and  ${}_{\text{def}}\text{CF}$ , defined by (4.3.3)-(4.3.7), are  $\tilde{P}$ -integrable.*

Let us give a necessary requirement of the *value process*  $\tilde{\mathcal{V}}$  of any trading strategy that hedges the contract under the martingale measure  $\tilde{P}$  that leads to no arbitrage under the available market information. For this purpose, we require that (M.1) and (M.2) hold and  $\tilde{\mathcal{V}}$  is an  $(\tilde{\mathcal{F}}_t)_{t \in [0, T]}$ -adapted càdlàg process satisfying  $\tilde{\mathcal{V}}_T = \Phi(S)$  a.s. We stipulate that  $\tilde{\mathcal{V}}_s$  coincides with the conditional expectation of the sum of the net present values of all cash flows and costs relative to the current available market information under  $\tilde{P}$ . Namely,

$$\tilde{\mathcal{V}}_s = \tilde{E} [ {}_{\text{con}}\text{CF}_s - {}_{\text{col}}\text{C}_s - {}_{\text{fun}}\text{C}_s - {}_{\text{hed}}\text{C}_s + {}_{\text{def}}\text{CF}_s | \tilde{\mathcal{F}}_s ] \quad \text{a.s.} \quad (4.3.8)$$

for all  $s \in [0, T]$  and  $\tilde{\mathcal{V}}$  is necessarily  $\tilde{P}$ -integrable, by the preceding lemma. This imposed conditional representation refines the *valuation equation* (1) in Brigo et al. [39],

which is built on the valuation problems in Pallavicini et al. [140, 139]. As we seek a valuation that does not require any knowledge of the default of any of the two parties, let  $G(\sigma)$  denote an  $(\mathcal{F}_t)_{t \in [0, T]}$ -survival process of any  $[0, T] \cup \{\infty\}$ -valued random variable  $\sigma$ . This is an  $[0, 1]$ -valued  $(\mathcal{F}_t)_{t \in [0, T]}$ -supermartingale under  $\tilde{P}$  so that

$$\tilde{P}(\sigma > t | \mathcal{F}_t) = G_t(\sigma) \quad \text{a.s. for all } t \in [0, T],$$

as introduced at (4.2.2). Then Corollary 4.2.2 entails that to any  $(\tilde{\mathcal{F}}_t)_{t \in [0, T]}$ -adapted process  $\tilde{X}$  that is integrable up to time  $\tau$  there is an  $(\mathcal{F}_t)_{t \in [0, T]}$ -adapted process  $X$  so that  $XG(\tau)$  is integrable and  $\tilde{X}_s = X_s$  a.s. on  $\{\tau > s\}$  for all  $s \in [0, T]$ . Namely,

$$X_s G_s(\tau) = \tilde{E}[\tilde{X}_s \mathbb{1}_{\{\tau > s\}} | \mathcal{F}_s] \quad \text{a.s. for any } s \in [0, T]. \quad (4.3.9)$$

In what follows, we shall call  $X$  a *pre-default version* of  $\tilde{X}$ . In particular, if  $G_s(\tau) > 0$  a.s. for all  $s \in [0, T]$ , which implies that the probability that neither  $I$  nor  $C$  defaults is positive, then  $X$  is unique up to a modification.

In this spirit we introduce valuation based on default-free information only. That is, we aim to characterise any *pre-default value process*  $\mathcal{V}$  defined as pre-default version of  $\tilde{\mathcal{V}}$ . To this end, as direct consequence of (4.3.8) and (4.3.9) we see that

$$\mathcal{V}_s G_s(\tau) = \tilde{E}[\text{conCF}_s - \text{colC}_s - \text{funC}_s - \text{hedC}_s + \text{defCF}_s | \mathcal{F}_s] \quad \text{a.s.} \quad (4.3.10)$$

for any  $s \in [0, T]$ , since  $\text{conCF}_s$ ,  $\text{colC}_s$ ,  $\text{funC}_s$ ,  $\text{hedC}_s$  and  $\text{defCF}_s$  vanish on  $\{\tau \leq s\}$ . This in turn leads to the *pre-default valuation equation* (4.3.10) that characterises the process  $\mathcal{V}$ . This refines the valuation equation (3) in Brigo et al. [39]. To replace all the  $\tilde{\mathcal{F}}_T$ -measurable random variables in the conditional expectation in (4.3.10) by  $\mathcal{F}_T$ -measurable ones, we will use the probabilistic results from Section 4.2.2 and require a set of conditions:

- (M.3) The rates  ${}_+\tilde{f}$ ,  ${}_-\tilde{f}$ ,  ${}_+\tilde{h}$ ,  ${}_-\tilde{h}$  are integrable up to time  $\tau$  and admit pre-default versions  ${}_+f$ ,  ${}_-f$ ,  ${}_+h$ ,  ${}_-h$ , respectively, that are  $(\mathcal{F}_t)_{t \in [0, T]}$ -progressively measurable with Lebesgue-integrable paths.
- (M.4) The funding amount  $\tilde{F}$  and the hedging process  $\tilde{H}$  are integrable up to time  $\tau$  and possess respective càdlàg pre-default versions  $F$  and  $H$ .

Under (M.3) and (M.4), it readily follows that the  $(\mathcal{F}_t)_{t \in [0, T]}$ -progressively measurable processes  $f$  and  $h$  defined via

$$f_t := {}_+f_t \mathbb{1}_{\{F_t > 0\}} + {}_-f_t \mathbb{1}_{\{F_t < 0\}} \quad \text{and} \quad h_t := {}_+h_t \mathbb{1}_{\{H_t > 0\}} + {}_-h_t \mathbb{1}_{\{H_t < 0\}}$$

have Lebesgue-integrable paths and serve as pre-default versions of the rates  $\tilde{f}$  and  $\tilde{h}$ , respectively. We continue with the following coupled regularity conditions:

(M.5)  $G(\tau_I)$  and  $G(\tau_C)$  are continuous and of finite variation,  $\tau_I$  and  $\tau_C$  are conditionally independent relative to  $(\mathcal{F}_t)_{t \in [0, T]}$  under  $\tilde{P}$  and  $G(\tau) = G(\tau_I)G(\tau_C)$ .

(M.6) If  $V(\tau_I)$  and  $V(\tau_C)$  are the respective variation processes of  $G(\tau_I)$  and  $G(\tau_C)$ , then

$$\begin{aligned} & \sup_{t \in ]s, T[} D_{s,t}(r) |\varepsilon_t + \text{LGD}_I((\varepsilon_t - C_t)^- + F_t^+) \mathbb{1}_{\{B\}}(I) | G_t(\tau_C) V_T(\tau_I) \quad \text{and} \\ & \sup_{t \in ]s, T[} D_{s,t}(r) |\varepsilon_t - \text{LGD}_C(\varepsilon_t - C_t)^+ | G_t(\tau_I) V_T(\tau_C) \end{aligned}$$

are  $\tilde{P}$ -integrable for any  $s \in [0, T[$ .

The last condition in (M.5) simply means that  $G(\tau)$  and  $G(\tau_I)G(\tau_C)$  are not only modifications of each other, but in fact equal. This ensures that all the paths of  $G(\tau)$  are continuous and of finite variation. The integrability condition (M.6) on the discounted cash flows on default is necessary to apply Proposition 4.2.6, as we shall see. And if  $G(\tau_I)$  and  $G(\tau_C)$  are decreasing, as in Example (4.3.5) below, then  $V_T(\tau_i) = 1 - G_T(\tau_i) \in [0, 1]$  for both  $i \in \{I, C\}$ .

Provided all these conditions hold, we define an  $(\mathcal{F}_t)_{t \in [0, T]}$ -progressively measurable process  ${}_0B$  and two  $(\mathcal{F}_t)_{t \in [0, T]}$ -adapted càdlàg processes  ${}_IB$  and  ${}_CB$  by

$$\begin{aligned} {}_0B_t &:= \pi_t - (c_t - r_t)C_t - (f_t - r_t)F_t - (r_t - h_t)H_t, \\ {}_IB_t &:= \varepsilon_t + \text{LGD}_I((\varepsilon_t - C_t)^- + F_t^+) \mathbb{1}_{\{B\}}(I) \quad \text{and} \\ {}_CB_t &:= \varepsilon_t - \text{LGD}_C(\varepsilon_t - C_t)^+. \end{aligned}$$

Then  ${}_0B_t G(\tau) \in \mathcal{P}(r)$  and the preliminary pre-default valuation equation (4.3.10) yields an implicit representation for  $\mathcal{V}$  relative to conditional expectations of  $\mathcal{F}_T$ -measurable random variables, as our main result of this section shows.

**Theorem 4.3.2.** *Let (M.1)-(M.6) hold. Then  $\Phi(S)G_T(\tau) \in \tilde{\mathcal{L}}(r)$ , the product of  $G(\tau)$  with any of the processes  $\pi$ ,  $(c - r)C$ ,  $(f - r)F$  or  $(r - h)H$  belongs to  $\tilde{\mathcal{P}}(r)$  and the integrals*

$$\int_s^T D_{s,t}(r) |{}_IB_t| G_t(\tau_C) dV_t(\tau_I) \quad \text{and} \quad \int_s^T D_{s,t}(r) |{}_CB_t| G_t(\tau_I) dV_t(\tau_C)$$

are  $\tilde{P}$ -integrable for any  $s \in [0, T]$ . Moreover,  $\mathcal{V}$  satisfies the representation

$$\mathcal{V}_s G_s(\tau) = \tilde{E} \left[ D_{s,T}(r) \Phi(S) G_T(\tau) + \int_s^T D_{s,t}(r) d{}_0A_t \middle| \mathcal{F}_s \right] \quad (4.3.11)$$

a.s. for each  $s \in [0, T]$  and the  $(\mathcal{F}_t)_{t \in [0, T]}$ -adapted continuous process  ${}_0A$  of finite variation given by

$${}_0A_t := \int_0^t {}_0B_s G_s(\tau) ds - \int_0^t {}_IB_s G_s(\tau_C) dG_s(\tau_I) - \int_0^t {}_CB_s G_s(\tau_I) dG_s(\tau_C).$$

**Corollary 4.3.3.** *Under (M.1)-(M.6), the  $(\mathcal{F}_t)_{t \in [0, T]}$ -adapted continuous process  $A$  of finite variation given by*

$$A_t := \int_0^t D_{0,s}(r) d_0 A_s$$

*is integrable and the process  ${}_{\mathcal{V}}M$  defined via*

$${}_{\mathcal{V}}M_t := D_{0,t}(r) \mathcal{V}G_t(\tau) + A_t$$

*is an  $(\mathcal{F}_t)_{t \in [0, T]}$ -martingale under  $\tilde{P}$ .*

For a backward stochastic integral representation of the pre-default value process we assume until the end of this section that  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \tilde{P})$  satisfies the usual conditions. That is,  $(\Omega, \mathcal{F}, \tilde{P})$  is complete and  $(\mathcal{F}_t)_{t \in [0, T]}$  is right-continuous. The next result generalizes Proposition 3.1 in Brigo et al. [39].

**Corollary 4.3.4.** *Suppose that (M.1)-(M.6) are valid and  ${}_{\mathcal{V}}M$  admits continuous paths. Then*

$$\mathcal{V}_s G_s(\tau) = \Phi(S) G_T(\tau) + \int_s^T (d_0 A_t - r_t \mathcal{V}_t G_t(\tau) dt) - \int_s^T D_{0,t}(-r) d_{\mathcal{V}} M_t$$

*for any  $s \in [0, T]$  a.s. Moreover, if  $G(\tau) > 0$ , then  ${}_{\mathcal{V}}M$  is, up to indistinguishability, the unique continuous  $(\mathcal{F}_t)_{t \in [0, T]}$ -local martingale such that*

$$\begin{aligned} \mathcal{V}_s = & \Phi(S) + \int_s^T \left( ({}_0B_t - r_t \mathcal{V}_t) dt - \frac{{}_I B_t - \mathcal{V}_t}{G_t(\tau_I)} dG_t(\tau_I) - \frac{{}_C B_t - \mathcal{V}_t}{G_t(\tau_C)} dG_t(\tau_C) \right) \\ & - \int_s^T \frac{D_{0,t}(-r)}{G_t(\tau)} d_{\mathcal{V}} M_t \end{aligned} \quad (4.3.12)$$

*for all  $s \in [0, T]$  a.s.*

We suppose in the setting of the preceding corollary that  $G(\tau_I)$  and  $G(\tau_C)$  are not only continuous and of finite variation, but actually absolutely continuous. Then the same holds for  $G(\tau)$  and

$$\frac{\dot{G}_t(\tau)}{G_t(\tau)} = \frac{\dot{G}_t(\tau_I)}{G_t(\tau_I)} + \frac{\dot{G}_t(\tau_C)}{G_t(\tau_C)} \quad \text{for a.e. } t \in [0, T] \text{ a.s.}$$

In this case, we can readily rewrite the representations (4.3.11) and (4.3.12) in the form

$$\begin{aligned} \mathcal{V}_s = & \tilde{E} \left[ D_{s,T}(r) \Phi(S) \frac{G_T(\tau)}{G_s(\tau)} \middle| \mathcal{F}_s \right] \\ & + \tilde{E} \left[ \int_s^T D_{s,t}(r) \frac{G_t(\tau)}{G_s(\tau)} \left( B_t + \left( r_t - \frac{\dot{G}_t(\tau)}{G_t(\tau)} \right) \mathcal{V}_t \right) dt \middle| \mathcal{F}_s \right] \\ = & \Phi(S) + \int_s^T B_t dt - \int_s^T \frac{D_{0,t}(-r)}{G_t(\tau)} d_{\mathcal{V}} M_t \quad \text{a.s.} \end{aligned} \quad (4.3.13)$$

for any  $s \in [0, T]$  with the  $(\mathcal{F}_t)_{t \in [0, T]}$ -progressively measurable integrable process  $B$  defined by

$$B_t := {}_0B_t - r_t \mathcal{V}_t - \frac{\dot{G}_t(\tau_I)}{G_t(\tau_I)}({}_IB_t - \mathcal{V}_t) - \frac{\dot{G}_t(\tau_C)}{G_t(\tau_C)}({}_CB_t - \mathcal{V}_t). \quad (4.3.14)$$

We note that the first component of this process is independent of all the survival processes and the second and third component depend on the survival processes of the times of default of  $I$  and  $C$ , respectively. Finally, we show that if we impose a gamma distribution for the random variables  $\xi_i$ , as done in Section 4.2.3, then the term  $B_t$  in (4.3.14) boils down to the term  $B$  defined in Brigo et al. [39], equation (7).

**Example 4.3.5.** For  $i \in \{I, C\}$  let  $\xi_i$  be an  $\tilde{\mathcal{F}}_0$ -measurable gamma distributed random variable with shape  $\alpha_i > 0$  and rate  $\beta_i > 0$  and  ${}_i\lambda$  be an  $\mathbb{R}_+$ -valued  $(\mathcal{F}_t)_{t \in [0, T]}$ -progressively measurable process with Lebesgue-integrable paths such that

$$\tau_i = \inf \left\{ t \in [0, T] \mid \int_0^t {}_i\lambda_s ds \geq \xi_i \right\}.$$

We suppose that  $\xi_I$  and  $\xi_C$  are independent. Then, by using the upper incomplete gamma function  $\gamma$ , it follows for  $i \in \{I, C\}$  from Example 4.2.11 that

$$-\frac{\dot{G}_t(\tau_i)}{G_t(\tau_i)} = \frac{\beta_i^{\alpha_i} {}_i\lambda_t}{\gamma(\alpha_i, \beta_i \int_0^t {}_i\lambda_s ds)} \left( \int_0^t {}_i\lambda_s ds \right)^{\alpha_i-1} \exp \left( -\beta_i \int_0^t {}_i\lambda_s ds \right)$$

for a.e.  $t \in [0, T]$ . In particular, for  $\alpha_i = \beta_i = 1$  this reduces to  $-\dot{G}_t(\tau_i)/G_t(\tau_i) = {}_i\lambda$  a.e. Thus, if  $\xi_I$  and  $\xi_C$  are in fact exponentially distributed with mean one, then for  $\lambda := {}_I\lambda + {}_C\lambda$  we obtain that

$$\begin{aligned} B_t &= \pi_t - (c_t - f_t)C_t - (f_t + \lambda_t)\mathcal{V}_t - (r_t - h_t)H_t + \lambda_t \varepsilon_t \\ &\quad + {}_I\lambda_t \text{LGD}_I((\varepsilon_t - C_t)^- + F_t^+) \mathbb{1}_{\{B\}}(I) - {}_C\lambda_t \text{LGD}_C(\varepsilon_t - C_t)^+ \end{aligned}$$

for all  $t \in [0, T]$  under the financing hypothesis that  $\mathcal{V}_t = C_t + F_t$  a.s. for any  $t \in [0, T]$ . Hence, from (4.3.13) we recover the representations (5) and (6) for the pre-default value process in Brigo et al. [39].

## 4.4 Proofs of results from Section 4.2

*Proof of Lemma 4.2.1.* For  $\tilde{A} \in \tilde{\mathcal{F}}_s$  there is  $A \in \mathcal{F}_s$  such that  $A \cap \{\rho > s\} = \tilde{A} \cap \{\rho > s\}$ . So,

$$\begin{aligned} E[X \mathbb{1}_{\{\rho > t\}} P(\rho > s | \mathcal{F}_s) \mathbb{1}_{\tilde{A}}] &= E[E[X \mathbb{1}_{\{\rho > t\}} | \mathcal{F}_s] P(\rho > s | \mathcal{F}_s) \mathbb{1}_A] \\ &= E[E[X \mathbb{1}_{\{\rho > t\}} | \mathcal{F}_s] \mathbb{1}_{\{\rho > s\} \cap \tilde{A}}]. \end{aligned}$$

This implies the assertion.  $\square$

*Proof of Corollary 4.2.2.* If  $X = \tilde{X}$  a.s. on  $\{\rho > t\}$ , then the  $\mathcal{F}_t$ -measurability of  $X$  yields that

$$XG_t(\rho) = E[X\mathbb{1}_{\{\rho>t\}}|\mathcal{F}_t] = E[\tilde{X}\mathbb{1}_{\{\rho>t\}}|\mathcal{F}_t] \quad \text{a.s.}$$

Conversely, suppose that  $X$  satisfies the asserted almost sure representation. Then

$$\tilde{X}P(\rho > t|\tilde{\mathcal{F}}_t) = E[\tilde{X}\mathbb{1}_{\{\rho>t\}}|\tilde{\mathcal{F}}_t] = XP(\rho > t|\tilde{\mathcal{F}}_t)$$

a.s. on the event  $A := \{G_t(\rho) > 0\}$ , by Lemma 4.2.1. Thus,

$$E[\tilde{X}\mathbb{1}_{\{\rho>t\}\cap A\cap\tilde{A}}] = E[\tilde{X}P(\rho > t|\tilde{\mathcal{F}}_t)\mathbb{1}_{A\cap\tilde{A}}] = E[X\mathbb{1}_{\{\rho>t\}\cap A\cap\tilde{A}}]$$

for any  $\tilde{A} \in \tilde{\mathcal{F}}_t$ . We first choose  $\tilde{A} = \{n \geq \tilde{X} > X\}$  and then  $\tilde{A} = \{\tilde{X} \leq X \leq n\}$  in this identity for each  $n \in \mathbb{N}$  to infer that  $X\mathbb{1}_{\{\rho>t\}} = \tilde{X}\mathbb{1}_{\{\rho>t\}}$  a.s., since

$$P(\{\rho > t\} \cap A^c) = E[G_t(\rho)\mathbb{1}_{\{G_t(\rho)=0\}}] = 0.$$

□

*Proof of Lemma 4.2.3.* Since  $E[\tilde{X}_t\mathbb{1}_{\{\rho>t\}}|\mathcal{F}_t] = X_tG_t(\rho)$  a.s. for every  $t \in [s, T]$ , Fubini's theorem directly yields that

$$E\left[\int_s^{T\wedge\rho} \tilde{X}_t dt \mathbb{1}_A\right] = \int_s^T E[\tilde{X}_t\mathbb{1}_{\{\rho>t\}}\mathbb{1}_A] dt = E\left[\int_s^T X_tG_t(\rho) dt \mathbb{1}_A\right]$$

for each  $A \in \mathcal{F}_s$ . Thus, the claim holds. □

*Proof of Lemma 4.2.5.* As  $[0, T] \cup \{\infty\}$  endowed with its Borel  $\sigma$ -field is a standard Borel space, any  $[0, T] \cup \{\infty\}$ -valued random variable admits a regular conditional probability given  $\mathcal{F}_T$ , which is a Markovian kernel from  $(\Omega, \mathcal{F}_T)$  to  $[0, T] \cup \{\infty\}$ .

We readily observe that  $\mathcal{E} := \{[s, t] \cup \{\infty\} \mid s, t \in [0, T] : s \leq t\}$  is an  $\cap$ -stable generator of  $\mathcal{B}([0, T] \cup \{\infty\})$  and  $\sigma, \tau$  satisfy

$$P(s_1 < \sigma \leq t_1, s_2 \leq \tau \leq t_2|\mathcal{F}_t) = P(s_1 < \sigma \leq t_1|\mathcal{F}_t)P(s_2 < \tau \leq t_2|\mathcal{F}_t) \quad \text{a.s.}$$

for all  $t \in [0, T]$  and any  $s_1, t_1, s_2, t_2 \in [0, t] \cup \{\infty\}$  with  $s_1 \leq t_1$  and  $s_2 \leq t_2$ . In particular, the  $d$ -system of all  $C \in \mathcal{B}([0, T] \cup \{\infty\})^2$  for which (4.2.3) holds includes  $\mathcal{E} \times \mathcal{E}$ . Hence, the claim follows from the monotone class theorem. □

*Proof of Proposition 4.2.6.* For fixed  $\tilde{s} \in ]s, T[$  and any  $n \in \mathbb{N}$  let  $\mathbb{T}_n$  be a partition of  $[\tilde{s}, T]$  that we write in the form

$$\mathbb{T}_n = \{t_{0,n}, \dots, t_{k_n,n}\}$$

for some  $k_n \in \mathbb{N}$  and  $t_{0,n}, \dots, t_{k_n,n} \in [\tilde{s}, T]$  with  $\tilde{s} = t_{0,n} < \dots < t_{k_n,n} = T$  and mesh denoted by

$$|\mathbb{T}_n| := \max_{i \in \{0, \dots, k_n-1\}} (t_{i+1,n} - t_{i,n}).$$

We assume that the sequence  $(\mathbb{T}_n)_{n \in \mathbb{N}}$  is refining, which means that  $\mathbb{T}_n \subset \mathbb{T}_{n+1}$  for all  $n \in \mathbb{N}$ , and satisfies  $\lim_{n \uparrow \infty} |\mathbb{T}_n| = 0$ . Then the sequences  $({}_nX)_{n \in \mathbb{N}}$  and  $({}_nG(\tau))_{n \in \mathbb{N}}$  of left-continuous  $(\mathcal{F}_t)_{t \in [0, T]}$ -adapted processes given by

$${}_nX_t := \sum_{i=0}^{k_n-1} X_{t_{i,n}} \mathbb{1}_{]t_{i,n}, t_{i+1,n}]}(t) \quad \text{and} \quad {}_nG_t(\tau) := \sum_{i=0}^{k_n-1} G_{t_{i,n}}(\tau) \mathbb{1}_{]t_{i,n}, t_{i+1,n}]}(t)$$

satisfy  $\lim_{n \uparrow \infty} {}_nX_t(\omega) = X_t(\omega)$  and  $\lim_{n \uparrow \infty} {}_nG_t(\tau)(\omega) = G_t(\tau)(\omega)$  for a.e.  $t \in ]\tilde{s}, T]$  for each  $\omega \in \Omega$ . For the decreasing sequence  $(\tau_n)_{n \in \mathbb{N}}$  of  $[0, T] \cup \{\infty\}$ -valued random variables defined via

$$\tau_n(\omega) := \sum_{i=0}^{k_n-1} t_{i+1,n} \mathbb{1}_{\{t_{i,n} < \tau \leq t_{i+1,n}\}}(\omega), \quad \text{if } \tau(\omega) < \infty,$$

and  $\tau_n(\omega) := \infty$ , if  $\tau(\omega) = \infty$ , we have  $\inf_{n \in \mathbb{N}} \tau_n = \tau$  on  $\{\tilde{s} < \tau\}$ . For given  $n \in \mathbb{N}$  we define a left-continuous  $(\tilde{\mathcal{F}}_t)_{t \in [0, T]}$ -adapted process  ${}_n\tilde{X}$  by using the definition of  ${}_nX$  when  $X$  is replaced by  $\tilde{X}$  and compute that

$$\begin{aligned} E[{}_n\tilde{X}_T^\sigma \mathbb{1}_{\{\tilde{s} < \sigma \leq T \wedge \tau_n\}} | \mathcal{F}_s] &= \sum_{i=0}^{k_n-1} E[X_{t_{i,n}} P(t_{i,n} < \sigma \leq t_{i+1,n}, \sigma \leq \tau_n | \mathcal{F}_{t_{i,n}}) | \mathcal{F}_s] \\ &= - \sum_{i=0}^{k_n-1} E[X_{t_{i,n}} G_{t_{i,n}}(\tau) (G_{t_{i+1,n}}(\sigma) - G_{t_{i,n}}(\sigma)) | \mathcal{F}_s] \\ &= -E \left[ \int_{]\tilde{s}, T]} {}_nX_t {}_nG_t(\tau) dG_t(\sigma) \middle| \mathcal{F}_s \right] \quad \text{a.s.} \end{aligned}$$

Indeed, the  $(\mathcal{F}_t)_{t \in [0, T]}$ -conditional independence of  $\sigma$  and  $\tau$  gives

$$\begin{aligned} P(t_{i,n} < \sigma \leq t_{i+1,n}, \sigma \leq \tau_n | \mathcal{F}_{t_{i,n}}) &= P(t_{i,n} < \sigma \leq t_{i+1,n}, \tau = \infty | \mathcal{F}_{t_{i,n}}) \\ &\quad + \sum_{j=i}^{k_n-1} P(t_{i,n} < \sigma \leq t_{i+1,n}, t_{j,n} < \tau \leq t_{j+1,n} | \mathcal{F}_{t_{i,n}}) \\ &= -E[G_{t_{i,n}}(\tau) (G_{t_{i+1,n}}(\sigma) - G_{t_{i,n}}(\sigma)) | \mathcal{F}_{t_{i,n}}] \quad \text{a.s.} \end{aligned}$$

for each  $i \in \{0, \dots, k_n - 1\}$ . By construction,  $|{}_nX_t| \leq \sup_{\tilde{t} \in ]s, T]} |X_{\tilde{t}}|$  for every  $n \in \mathbb{N}$  and all  $t \in ]\tilde{s}, T]$ . Therefore, dominated convergence yields that

$$\lim_{n \uparrow \infty} \int_{]\tilde{s}, T]} {}_nX_t {}_nG_t(\tau) dG_t(\sigma) = \int_{]\tilde{s}, T]} X_t G_t(\tau) dG_t(\sigma).$$

Since  $|\int_{]\tilde{s}, T]} {}_nX_t {}_nG_t(\tau) dG_t(\sigma)|$  does not exceed  $\sup_{t \in ]s, T]} |X_t| G_t(\tau) V_T(\sigma)$  for each  $n \in \mathbb{N}$ , dominated convergence also implies that

$$E[\tilde{X}_T^\sigma \mathbb{1}_{\{\tilde{s} < \sigma \leq T \wedge \tau\}} | \mathcal{F}_s] = -E \left[ \int_{]\tilde{s}, T]} X_t G_t(\tau) dG_t(\sigma) \middle| \mathcal{F}_s \right] \quad \text{a.s.}$$

Finally, for any sequence  $(s_n)_{n \in \mathbb{N}}$  in  $]s, T[$  that converges to  $s$ , we have

$$\lim_{n \uparrow \infty} \mathbb{1}_{]s_n, T]}(\sigma) = \mathbb{1}_{]s, T]}(\sigma)$$

and

$$\lim_{n \uparrow \infty} \int_{]s, s_n]} X_t G_t(\tau) dG_t(\sigma) = 0.$$

Hence, the claim follows from a final application of the Dominated Convergence Theorem.  $\square$

*Proof of Lemma 4.2.7.* For any  $\omega \in \Omega$  we have  $\tau_i(\omega) \leq s$  if and only if  ${}_iX_s(\omega) \geq \xi_i(\omega)$ , as the increasing function  ${}_iX(\omega)$  is right-continuous. So,  $\{\tau_i \leq s\} \in \tilde{\mathcal{F}}_s$  and

$$P(\tau_i > s | \mathcal{F}_t) = P(\xi_i > {}_iX_s | \mathcal{F}_t) = P(\xi_i > x)_{|x = {}_iX_s} \quad \text{a.s.},$$

by the independence of  $\xi_i$  and  $\mathcal{F}_t$ , which completes the verification.  $\square$

*Proof of Proposition 4.2.8.* As the  $\mathbb{R}^n$ -valued random vector  $(\xi_1, \dots, \xi_n)$  is independent of  $\mathcal{F}_t$ , we obtain that

$$\begin{aligned} P(\tau_1 > s_1, \dots, \tau_n > s_n | \mathcal{F}_t) &= P(\xi_1 > x_1, \dots, \xi_n > x_n)_{|(x_1, \dots, x_n) = ({}_1X_{s_1}, \dots, {}_nX_{s_n})} \\ &= G_1({}_1X_{s_1}) \cdots G_n({}_nX_{s_n}) \quad \text{a.s.} \end{aligned}$$

for every  $s_1, \dots, s_n \in [0, t]$ , which gives the  $(\mathcal{F}_t)_{t \in [0, T]}$ -conditional independence. Since

$$\{\rho > s\} = \bigcap_{i=1}^n \{\tau_i > s\}$$

for any  $s \in [0, T]$ , the a.s. representation for the conditional probability  $P(\rho > s | \mathcal{F}_t)$  follows as well.

(i) We have  $\prod_{i=1}^n G_i({}_iX_s) > 0$  a.s. if and only if  $G_i({}_iX_s) > 0$  a.s. for any  $i \in \{1, \dots, n\}$ , which yields the claim, by the definition of essential supremum.

(ii) As

$$P(\rho > s) = E\left[\prod_{i=1}^n G_i({}_iX_s)\right]$$

and

$$\prod_{i=1}^n G_i({}_iX) \leq 1,$$

we see that  $\rho > s$  a.s. if and only if  $G_i({}_iX_s) = 1$  a.s. for all  $i \in \{1, \dots, n\}$ . Moreover, we note that

$$P(\rho = \infty) = P(\rho > T) = E\left[\prod_{i=1}^n G_i({}_iX_T)\right]$$

and for any  $x_1, \dots, x_n \in \mathbb{R}$  the product  $\prod_{i=1}^n x_i$  vanishes exactly if  $x_i$  does for at least one  $i \in \{1, \dots, n\}$ . So,  $P(\rho = \infty) = 0$  if only if  $P(\exists i \in \{1, \dots, n\} : G_i(iX_T) = 0) = 1$  and the claimed equivalences follow from the definitions of the essential infimum and supremum.

(iii) Let  $(s_n)_{n \in \mathbb{N}}$  be an increasing sequence in  $[0, T]$  converging to some  $s \in ]0, T]$ . Then the a.s. left-continuity of  $G_i(iX)$  yields that  $\lim_{n \uparrow \infty} G_i(iX_{s_n}) = G_i(iX_s)$  a.s. for any  $i \in \{1, \dots, n\}$ . Thus,

$$\lim_{n \uparrow \infty} P(\rho > s_n) = \lim_{n \uparrow \infty} E \left[ \prod_{i=1}^n G_i(iX_{s_n}) \right] = E \left[ \prod_{i=1}^n G_i(iX_s) \right] = P(\rho > s),$$

by monotone convergence. If for any  $n \in \mathbb{N}$  we also suppose that  $s_n < s$ , then

$$P(\rho = s) \leq P(\rho > s_n) - P(\rho > s) = 0,$$

which gives  $P(\rho = s) = 0$  by taking the limit  $n \uparrow \infty$ .  $\square$

*Proof of Proposition 4.2.10.* We set  $a_i := \text{ess inf } \xi_i$  and  $b_i := \text{ess sup } \xi_i$  for all  $i \in \{1, \dots, n\}$ . If  $\xi_i$  is a.s. constant for some  $i \in \{1, \dots, n\}$ , then

$$P(\tau > 0) \leq P(\xi_i > \hat{x}_i) = 0$$

and  $\Lambda_0 = \emptyset$ , since  $\xi_i = a_i$  a.s. Thus,  $\tau = 0$  a.s. and the asserted identity holds. We may now assume that  $a_i < b_i$  for each  $i \in \{1, \dots, n\}$  and define an  $[0, 1]$ -valued continuous function  $\varphi$  on the closed  $n$ -dimensional rectangle

$$R := \bigtimes_{i=1}^n [a_i, b_i] \cap \mathbb{R}$$

via

$$\varphi(x) := \prod_{i=1}^n G_i(x_i).$$

Then  $\varphi$  is continuously differentiable on the interior  $\bigtimes_{i=1}^n ]a_i, b_i[$  of  $R$  and we see that for the  $[0, \infty]^n$ -valued process  $X := ({}_1X, \dots, {}_nX)$  the path  $[s, t] \rightarrow [0, \infty]^n$ ,  $\tilde{s} \mapsto X_{\tilde{s}}(\omega)$  is absolutely continuous for any  $\omega \in \Lambda_t$ . Hence,

$$\varphi(X_s) - \varphi(X_t) = \sum_{j=1}^n \int_s^t \frac{\partial \varphi}{\partial x_j}(X_{\tilde{s}}) d_j X_{\tilde{s}} = - \sum_{j=1}^n \int_s^t {}_j \lambda_{\tilde{s}} \left( \frac{G'_j}{G_j} \right) ({}_j X_{\tilde{s}}) \varphi(X_{\tilde{s}}) d\tilde{s}$$

on  $\Lambda_t$ , by the Fundamental Theorem of Calculus for Lebesgue-Stieltjes integrals [22], Chapter 17. Finally, from Proposition 4.2.8 and Fubini's theorem we infer that

$$\begin{aligned} P(\{\rho > s\} \cap \Lambda_t) &= E[P(\rho > s | \mathcal{F}_t) \mathbb{1}_{\Lambda_t}] = E[\varphi(X_s); \Lambda_t] \\ &= P(\rho > t) - \sum_{j=1}^n \int_s^t E \left[ {}_j \lambda_{\tilde{s}} \left( \frac{G'_j}{G_j} \right) ({}_j X_{\tilde{s}}) \varphi(X_{\tilde{s}}); \Lambda_t \right] d\tilde{s}, \end{aligned}$$

because  $E[\varphi(X_t); \Lambda_t] = E[P(\rho > t | \mathcal{F}_t) \mathbb{1}_{\Lambda_t}] = P(\rho > t)$ .  $\square$

## 4.5 Proofs of results from Section 4.3

*Proof of Lemma 4.3.1.* For fixed  $s \in [0, T]$  the  $\tilde{P}$ -integrability of  $\text{conCF}_s$ ,  $\text{colC}_s$ ,  $\text{funC}_s$  and  $\text{hedC}_s$  follows from that of  $D_{s,T}(r)\Phi(S)\mathbb{1}_{\{\tau > T\}}$  and

$$\int_s^{T \wedge \tau} D_{s,t}(r) (|\pi_t| + |(c_t - r_t)C_t| + |(\tilde{f}_t - r_t)\tilde{F}_t| + |(r_t - \tilde{h}_t)\tilde{H}_t|) dt,$$

by (M.1). As condition (4.3.2) states that  $\tau_I \neq \tau_C$  a.s. on  $\{\tau < \infty\}$ , we readily check for  $s < T$  that

$$\begin{aligned} |\text{defCF}_s| &\leq \sup_{t \in ]s, T[} |\varepsilon_t + \text{LGD}_I((\varepsilon_t - C_t)^- + \tilde{F}_t^+) \mathbb{1}_{\{B\}}(I)| \mathbb{1}_{\{\tau_I < \tau_C\}} \\ &\quad + \sup_{t \in ]s, T[} |\varepsilon_t - \text{LGD}_C(\varepsilon_t - C_t)^+| \mathbb{1}_{\{\tau_C < \tau_I\}} \end{aligned}$$

a.s. on  $\{s < \tau < T\}$ . Hence, (M.2) entails the integrability of  $\text{defCF}_s$ .  $\square$

*Proof of Theorem 4.3.2.* Let us first regard the amount paid at maturity. Since  $\Phi(S)$  is  $\mathcal{F}_T$ -measurable, from  $\Phi(S) \in \tilde{\mathcal{L}}(r, \tau)$  we directly get that  $\Phi(S)G_T(\tau) \in \tilde{\mathcal{L}}(r)$  and

$$\tilde{E}[D_{s,T}(r)\Phi(S)\mathbb{1}_{\{\tau > T\}} | \mathcal{F}_s] = \tilde{E}[D_{s,T}(r)\Phi(s)G_T(\tau) | \mathcal{F}_s] \quad \text{a.s.}$$

for any fixed  $s \in [0, T]$ , by the tower property of the conditional expectation. Now we turn to the dividend cash flows and the collateral, funding and hedging costs. The processes  $\pi$ ,  $r$ ,  $c$  and  $C$  are  $(\mathcal{F}_t)_{t \in [0, T]}$ -progressively measurable and we have  $(\tilde{f}_t - r_t)\tilde{F}_t = (f_t - r_t)F_t$  and  $(r_t - \tilde{h}_t)\tilde{H}_t = (r_t - h_t)H_t$  a.s. on  $\{\tau > t\}$  for all  $t \in [0, T]$ . Thus,

$$\pi G(\tau), (c - r)CG(\tau), (f - r)FG(\tau), (r - h)HG(\tau) \in \tilde{\mathcal{P}}(r)$$

is another direct consequence of (M.1), due to Fubini's theorem. In particular, the process  ${}_0BG(\tau)$  also belongs to  $\tilde{\mathcal{P}}(r)$  and Lemma 4.2.3 entails that

$$\begin{aligned} \tilde{E}[\text{conCF}_s - \text{colC}_s - \text{funC}_s - \text{hedC}_s | \mathcal{F}_s] &= \tilde{E}\left[D_{s,T}(r)\Phi(S)G_T(\tau) \middle| \mathcal{F}_s\right] \\ &\quad + \tilde{E}\left[\int_s^T D_{s,t}(r) {}_0B_t G_t(\tau) dt \middle| \mathcal{F}_s\right] \quad \text{a.s.} \end{aligned}$$

Now let us turn to the cash flows on defaults. The integrals

$$\begin{aligned} &\int_s^T D_{s,t}(r) |{}_IB_t| G_t(\tau_C) dV_t(\tau_I), \\ &\int_s^T D_{s,t}(r) |{}_CB_t| G_t(\tau_I) dV_t(\tau_C) \end{aligned}$$

are finite, as  ${}_IB$  and  ${}_CB$  have càdlàg paths and their integrability is ensured by (M.6). We recall the definition of  ${}_{\text{def}}CF$  at (4.3.7) and observe that  $\{\tau_i < \tau_j\} \cap \{s < \tau < T\} = \{s < \tau_i < \tau_j \wedge T\}$  for both  $i, j \in \{I, C\}$  with  $i \neq j$ . Thus, thanks to (M.2), we obtain

$$\begin{aligned} \tilde{E}[{}_{\text{def}}CF_s | \mathcal{F}_s] = \\ - \tilde{E} \left[ \int_s^T D_{s,t}(r) \left( {}_IB_t G_t(\tau_C) dG_t(\tau_I) + {}_CB_t G_t(\tau_I) dG_t(\tau_C) \right) \middle| \mathcal{F}_s \right] \quad \text{a.s.} \end{aligned}$$

from two applications of Proposition 4.2.6, since (4.3.2) ensures  $\tau_I \neq \tau_C$  a.s. on  $\{\tau < \infty\}$ . Now the claimed representation follows from the pre-default valuation equation (4.3.10).  $\square$

*Proof of Corollary 4.3.3.* By the definition of  $\tilde{\mathcal{P}}(r)$  at the beginning of Section 4.3.3, we see that Theorem 4.3.2 implies the integrability of

$$\int_0^T D_{0,t}(r) (|{}_0B_t| G_t(\tau) dt + |{}_IB_t| G_t(\tau_C) dV_t(\tau_I) + |{}_CB_t| G_t(\tau_I) dV_t(\tau_C)).$$

At the same time, the representation (4.3.11) entails that the process  $[0, T] \times \Omega \times \mathbb{R}, (t, \omega) \rightarrow D_{0,t}(r)(\omega) \mathcal{V}_t(\omega) G_t(\tau)(\omega)$  is integrable. Hence,  $A$  and  ${}_{\mathcal{V}}M$  are integrable and the claimed martingale property follows from

$$\tilde{E}[{}_{\mathcal{V}}M_T | \mathcal{F}_s] = \tilde{E} \left[ D_{0,T}(r) \Phi(S) G_T(\tau) + \int_s^T D_{0,t}(r) d_0 A_t, \middle| \mathcal{F}_s \right] + A_s = {}_{\mathcal{V}}M_s$$

a.s. for any  $s \in [0, T]$ , since we may use that  $D_{0,t}(r) = D_{0,s}(r) D_{s,t}(r)$  for all  $t \in [s, T]$ .  $\square$

*Proof of Corollary 4.3.4.* Since

$$\mathcal{V}_s G_s(\tau) = D_{0,s}(-r)({}_{\mathcal{V}}M_s - A_s)$$

for any  $s \in [0, T]$ , Ito's product rule yields for each fixed  $t \in [0, T]$  that

$$\mathcal{V}_t G_t(\tau) - \mathcal{V}_s G_s(\tau) = \int_s^t D_{0,u}(-r) d({}_{\mathcal{V}}M_u - A_u) + \int_s^t r_u \mathcal{V}_u G_u(\tau) du$$

for all  $u \in [0, t]$  a.s., which gives the first identity. In the case that  $G(\tau) > 0$ , another application of Ito's product rule implies that

$$\mathcal{V}_s - \mathcal{V}_s = \int_s^t \frac{1}{G_u(\tau)} d\mathcal{V}_u G_u(\tau) - \int_s^t \frac{\mathcal{V}_u}{G_u(\tau)} dG_u(\tau)$$

for all  $s \in [0, t]$  a.s. Thus, the second identity (4.3.12) follows from the first and the definition of  ${}_0A$ . Finally, if  $M$  is a continuous  $(\mathcal{F}_t)_{t \in [0, T]}$ -local martingales satisfying (4.3.12) when  ${}_{\gamma}M$  is replaced by  $M$ , then

$$\int_0^T \frac{D_{0,t}(-r)}{G_t(\tau)} d\langle {}_{\gamma}M - M \rangle_t = 0 \quad \text{a.s.}$$

As the continuous process  $[0, T] \times \Omega \times \mathbb{R}, (t, \omega) \rightarrow D_{0,t}(-r)(\omega)/G_t(\tau)(\omega)$  admits only positive paths, we conclude that  $\langle {}_{\gamma}M - M \rangle_T = 0$  a.s., which shows that  ${}_{\gamma}M$  and  $M$  are indistinguishable.  $\square$

# 5

## Price impact on Term Structure

### 5.1 Introduction

The main aim of this work is to present a combined theory of the term structure of interest rates and of price impact, with applications to optimal execution. This objective entails the inclusion of a specific type of cross price impact that is specific to fixed income.

Term structure modeling with a view to derivatives valuation and hedging has been developed over several decades. For the purposes of this work and what we might call the classic theory we refer to the monographs by Bjork [24], Brigo and Mercurio [42] and Filipovic [86]. After the crisis that started in 2007, the gap between two of the rates that were used as benchmarks for risk free rates, namely interbank and overnight rates, widened considerably, peaking in October 2008, following the defaults of several financial institutions in the space of one month [45]. This highlighted the fact that interbank rates could no longer be used as benchmarks for risk free rates and neither could they be used to derive zero-coupon curves that were not contaminated by credit and liquidity risk. This led to the necessity to model multiple interest rate curves, treating interbank rates as risky rates affected by credit and liquidity risk and adopting overnight based rates as new risk free rates.

This multiple interest rate curve academic literature was initiated by practitioners, see in particular the monograph by Henrard [109]. Substantial contributions were made later by numerous academics, where we refer to a monograph by Grbac and Runggaldier [103], Crepey et al. [65], Grbac et al. [102], Cuchiero et al. [66, 67], Nguyen and Seifried [136] and finally to Bormetti et al. [31], for multiple curves in conjunction with valuation adjustments and credit risk.

Further recent developments include the presence of negative interest rates in many currencies, see for example the BIS report [165], and the ongoing project of eliminating current interbank rates like the London Interbank Offer Rate (LIBOR),

replacing them with new types of risk free rates inspired by overnight rates. This puts the multiple-curve area in a state of uncertainty, while negative rates prompted the mainstream resurgence of Gaussian models that were previously justified only in very special economies exhibiting negative rates, such as for example Switzerland in the seventies.

Given the state of market uncertainty on benchmark interest rates, products and markets, we will not consider these recent developments in this work, except for allowing for negative rates in our formulation. We are interested in developing a combination of term structure modeling and price impact in the classic theory of interest rates. We are confident that multiple curves and further discussion of negative rates, if still present in the market after reforms and updated central bank policies, can be incorporated in further work after the classic theory has been developed.

Despite the fact that the bond market size is considerably larger than the equity market size, relevance of liquidity risk in the context of bonds was pointed out in several papers. A report from the Federal Reserve Bank of New York from 2003 [57], for instance, showed that bond and stock markets have common factors driving their liquidity. A strong relationship between liquidity in the Treasury bond market and in the stock market was later highlighted also by Goyenko and Ukhov [98]. Another significant contribution appears in a recent paper by Schneider and Lillo [151], which studied cross price impact among sovereign bonds.

The financial crisis of 2008-2009 have also raised concerns about the inventories kept by intermediaries. Regulators and policy makers took advantage of two main regulatory changes (Reg NMS in the US and MiFID in Europe) and enforced more transparency on the transactions and hence on market participants positions, which pushed the trading processes toward electronic platforms [125]. Simultaneously, consumers and producers of financial products asked for less complexity and more transparency.

This tremendous pressure on the business habits of the financial system, shifted it from a customized and high margins industry, in which intermediaries could keep large (and potentially risky) inventories, to a mass market industry where logistics have a central role. As a result, investment banks nowadays unwind their risks as fast as possible. In the context of small margins and high velocity of position changes, trading costs are of paramount importance. A major factor of the trading costs is the price impact: the faster the trading rate, the more the buying or selling pressure will move the price in a detrimental way.

Academic efforts to quantify and reduce the transaction costs of large trades trace back to the seminal papers of Almgren and Chriss [8] and Bertsimas and Lo [18]. In both models one large market participant (for instance an asset manager or a bank) would like to buy or sell a large amount of shares or contracts during a specified duration. The cost minimization problem turned out to be quite involved, due to multiple constraints on the trading strategies. On one hand, the price impact demands

to trade slowly, or at least at a pace which takes into account the available liquidity (see [13] and references therein). On the other hand, traders have an incentive to trade rapidly, because they do not want to carry the risk of an adverse price move far away from their decision price. The importance of optimal trading in the industry generated a lot of variations for the initial mean-variance minimization of the trading costs (see [125, 53, 104] for details). These type of problems are usually formulated as optimisation problems in the context of stochastic control where the agent tries to minimize the transaction costs which result by the price impact and to reduce the risk associated with holding the assets for too long (see e.g. [118, 105, 134]).

In this spirit, we will follow the term structure theory as developed by Bjork [24], modifying it to allow for the inclusion of price impact. We will start from the simplest possible dynamics, namely one factor short rate models, and extend it later to instantaneous forward rate models. We will introduce price impact formulated on zero-coupon bonds, since they are a possible choice of building blocks for the term structure. We will later connect this with impact on coupon bearing bonds that are more commonly traded.

We assume that an agent who is executing a large order of bonds is creating two types of price impact which are extensively used in the literature. The first one is an instantaneous (or temporary) price impact, which affects the asset price only while trading, and fades away immediately after. This type of price impact occurs due to the fact that a large buy (sell) trade consumes the liquidity which is available in the market by “walking through” the first few price levels of the limit order book (see e.g. [8] and Chapter 6.3 of [53]). However, empirical studies have shown that price impact also has a *transient* effect. A short of liquidity due to a large trade creates an imbalance between supply and demand, which in turn pushes the price in a detrimental direction. This effect decays within a short time period after each trade (see [13] and [137]). Execution in presence of transient price impact was studied extensively in the context of optimal control problems (see e.g. [92, 93, 3, 135, 16]). In this work we incorporate these two types of price impact models into a bonds trading framework, as was done in [135] for equities.

Applying these price impact models to the term structure will be challenging. At every point in time the term structure of interest rates is a high dimensional object, or even an infinite dimensional one when considering all possible maturities for interest rates or zero-coupon bonds at a given time. This is a unique feature of term structure modeling, where differently from FX or equity modeling for example, we model a whole curve dynamics rather than a point dynamics. We can expect that trading a bond with a specific maturity may impact the price of bonds with different maturities *on the same currency curve*. In this sense, the cross price impact is endogenous to the same underlying asset, differently from what happens in other markets. We will investigate how price impact interacts with no-arbitrage dynamics, and we will encapsulate the effect of price impact in the definition of a new no-arbitrage pricing

measure embedding impact itself. This will be done by extending the market price of risk to an impacted version embedding the bond price impact speed. The impacted zero-coupon bond dynamics will then be written as the unimpacted bond dynamics but under a different measure. We will also introduce an impacted physical measure that could be useful for risk management and risk analysis. Finally, we will illustrate our theory by proposing an application to optimal execution.

We will not limit ourselves to short rate models. We will also see how in the Heath-Jarrow-Morton model the no-arbitrage drift condition for instantaneous forward rate dynamics can be maintained under price impact by resorting to the modified pricing measure.

The Chapter is structured as follows. In Section 5.2 we introduce the short rate models setup and the main theoretical results. In particular, we introduce price impact for zero-coupon bonds, and we look at the impacted market price of risk, absence of arbitrage and the impacted risk neutral measure. We define the impacted yield curve and extend impact to coupon bearing bonds. We further show how to use the impact setup in a HJM framework. Section 5.3 features a few examples including valuation of impacted Eurodollar futures with the Hull and White model. Section 5.4 presents some numerical results illustrating how the yield curve behaves under impact. Finally, we introduce a result on optimal execution with impacted bonds in Section 5.5. Sections 5.6 and 5.7 include proofs that have not been included in the main text.

## 5.2 Model setup and main results

### 5.2.1 Impacted market price of risk, impacted risk neutral measure and absence of arbitrage

We introduce our initial assumption on the interest rate dynamics, assuming it is a one-dimensional short rate model dynamics. This will be relaxed later with a Heath-Jarrow-Morton setting in Section 5.2.6.

Let us fix a maturity  $T > 0$  and let  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$  be a filtered probability space satisfying the usual conditions, on which there is a standard  $(\mathcal{F}_t)_{t \in [0, T]}$ -Brownian motion  $W^{\mathbb{P}}$ . We consider the following dynamics of the short rate under the real world measure  $\mathbb{P}$

$$dr(t) = \mu(t, r(t))dt + \sigma(t, r(t))dW^{\mathbb{P}}(t), \quad (5.2.1)$$

where  $\mu(t, r), \sigma(t, r)$  are given real valued functions, assumed to be regular enough to ensure the SDE has a unique strong solution. For example one can assume that both  $\mu$  and  $\sigma$  are Lipschitz continuous in the  $r$  coordinate, and has at most linear growth in  $r$  uniformly in  $t \in [0, T]$ . We moreover assume that  $\sigma(t, r(t))$  is  $\mathbb{P}$ -a.s. strictly positive for any  $t > 0$ .

Assume that the dynamics of a zero-coupon bond with maturity at time  $T$ , under the real world measure, is given by

$$dP(t, T) = \mu_T(t, r(t))dt + \sigma_T(t, r(t))dW^\mathbb{P}(t), \quad (5.2.2)$$

with  $\mu_T, \sigma_T$  depending on the maturity  $T$  and regular enough as in (5.2.1). Typically, one might assume the price process of the  $T$ -bond to be of the form  $P(t, T) = F(t, r(t); T)$  for some function  $F$  smooth in three variables. Then, under suitable assumptions (see Assumption 3.2 in Chapter 3.2 of [24]) one can define for any finite maturity  $T > 0$  the stochastic process

$$\lambda(t) = \frac{\mu_T(t, r(t)) - r(t)P(t, T)}{\sigma_T(t, r(t))}, \quad (5.2.3)$$

and show that  $\lambda$  may depend on  $r$  but it does not actually depend on  $T$ . Such process is called *market price of risk*. Provided the Novikov condition holds, this process can be used to define a change of measure from the real world measure  $\mathbb{P}$  to the risk neutral measure  $\mathbb{Q}$ :

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp \left( \int_0^t \lambda(s) dW^\mathbb{P}(s) - \frac{1}{2} \int_0^t \lambda^2(s) ds \right). \quad (5.2.4)$$

The dynamics of the short rate under  $\mathbb{Q}$  becomes

$$dr(t) = [\mu(t, r(t)) - \lambda(t)\sigma(t, r(t))]dt + \sigma(t, r(t))dW^\mathbb{Q}(t).$$

The model will be fully specified once the stochastic process  $\lambda$  is defined. Our strategy for establishing a mathematical framework that encompasses both risk neutral pricing and price impact in the context of interest rates derivatives consists, first of all, in specifying the dynamics for an impacted bond with maturity  $T$  under the real world measure  $\mathbb{P}$ .

We consider a trader with an initial position of  $x_T > 0$  zero-coupon bonds with maturity  $T$ . Let  $0 < \tau \leq T$  denote some finite deterministic time horizon. In an optimal execution problem, the objective of the trader would be to complete her transaction by time  $\tau$ , starting from the  $x_T$  position at time 0. In this sense, we should avoid confusion between  $T$ , which is the traded bond maturity, and  $\tau$ , which is the trading horizon of the  $T$ -maturity bond. The number of bonds the trader holds at time  $t \in [0, \tau]$  is given by

$$X_T(t) = x_T - \int_0^t v_T(s) ds. \quad (5.2.5)$$

where the function  $v_T$  denotes the trader's selling rate, which takes negative values in case of a buy strategy. In what follows we assume that  $v_T = \{v_T(t)\}_{0 \leq t \leq \tau}$  is

progressively measurable and has a  $\mathbb{P}$ -a.s. bounded derivative (in the  $t$ -variable), that is, there exists  $M > 0$  such that

$$\sup_{0 \leq t \leq \tau^+} |\partial_t v_T(t)| < M, \quad \mathbb{P} - \text{a.s.}, \quad (5.2.6)$$

where  $0 \leq t \leq \tau$ . After the trading stops, we assume that  $v_T(t) = 0$ . We denote the class of such trading speeds as  $\mathcal{A}_T$ .

The main idea behind the assumption of the differentiability of  $v_T$  is that the overall impact we add to the zero-coupon bond should affect the drift only (see (5.2.12)). Moreover due to price impact effect, we allow bond price which are large than 1 for some time intervals but we do need to control their upper bound.

We consider a price impact model with both transient and instantaneous impact, which is a slight generalization of the model which was considered in [135]. The impacted bond price is therefore given by

$$\tilde{P}(t, T) = P(t, T) - l(t, T)v_T(t) - K(t, T)\Upsilon_T^v(t). \quad (5.2.7)$$

Here  $\Upsilon_T^v$  represents the transient impact effect and it has the form

$$\Upsilon_T^v(t) := ye^{-\rho t} + \gamma \int_0^t e^{-\rho(t-s)} v_T(s) ds, \quad (5.2.8)$$

where  $y, \rho$  and  $\gamma$  are positive constants. The term  $v_T(t)$  in (5.2.7) represents the instantaneous price impact, where we absorb in the function  $l$  any constants that should factor it. Lastly,  $l$  and  $K$  are differentiable functions with respect to both variables  $(t, T)$  which take positive values on  $0 \leq t < T$  and for any  $0 \leq \tau < T$  we have

$$\inf_{0 \leq t \leq \tau} l(t, T) > 0. \quad (5.2.9)$$

Moreover we assume that

$$\begin{aligned} \sup_{0 \leq t \leq \tau} |\partial_t l(t, T)| &< \infty, & \lim_{t \rightarrow T} l(t, T) &= 0, \\ \sup_{0 \leq t \leq \tau} |\partial_t K(t, T)| &< \infty, & \lim_{t \rightarrow T} K(t, T) &= 0. \end{aligned} \quad (5.2.10)$$

While the assumption on boundedness of the derivatives of functions  $K$  and  $l$  arise from technical reasons which has similar motivation as the reason for (5.2.6), the assumptions on the behaviour at expiration is meant to enforce the boundary condition on the price of the impacted bond at expiration, which is  $\tilde{P}(T, T) = 1$ . Note that  $K$  and  $l$  are time-dependent versions of the parameters  $\lambda, k$  in [135]. A prominent example of such functions is

$$l(t, T) = \kappa \left(1 - \frac{t}{T}\right)^\alpha, \quad K(t, T) = \left(1 - \frac{t}{T}\right)^\beta,$$

for some constants  $\alpha, \beta \geq 1$  and  $\kappa > 0$ .

We define for convenience the overall price impact:

$$I_T(t) := l(t, T)v_T(t) + K(t, T)\Upsilon_T^v(t). \quad (5.2.11)$$

Then, since  $v_T$  is in  $\mathcal{A}_T$  we can rewrite (5.2.7) as follows:

$$d\tilde{P}(t, T) = dP(t, T) - J_T(t)dt, \quad \tilde{P}(T, T) = 1, \quad (5.2.12)$$

with

$$\begin{aligned} J_T(t) &:= \partial_t I_T(t) \\ &= \partial_t l(t, T)v_T(t) + l(t, T)\partial_t v_T(t) + \partial_t K(t, T)\Upsilon_T^v(t) + K(t, T)[- \rho \Upsilon_T^v(t) + v_T(t)]. \end{aligned} \quad (5.2.13)$$

Our model so far describes how trading a  $T$ -bond affects its price. Next, we show the existence of an *impacted market price of risk process* which will be a generalization of (5.2.3). Using this process we will define an equivalent martingale measure, under which bonds and derivatives prices can be computed. Such a measure will be called an *impacted risk neutral measure*. It is important to remark that, as in the classic case, this change of measure will be unique for all bond maturities.

Before stating the main theorem of this section, let us first introduce a few important definitions.

**Definition 5.2.1** (Impacted portfolio). *Let  $\hat{T} < +\infty$  be some finite time horizon. An impacted portfolio is a  $(n+1)$ -dimensional, bounded progressively measurable process  $\tilde{h} = (\tilde{h}_t)_{t \in [0, \hat{T}]}$  with  $\tilde{h}_t = (\tilde{h}_t^0, \tilde{h}_t^1, \dots, \tilde{h}_t^n)$ , where  $\tilde{h}_t^i$  represents the number of shares in the impacted bond  $\tilde{P}(t, T_i)$  held in the portfolio at time  $t$ . The liquidation value at time  $t$  of such a portfolio  $\tilde{h}$  is defined as*

$$\tilde{V}(t) \equiv \tilde{V}(t, \tilde{h}) := \sum_{i=0}^n \tilde{h}^i(t) \tilde{P}(t, T_i).$$

We shall stress the fact that  $\tilde{V}$  denotes the *liquidation* value of the portfolio, that is, the total value of the portfolio when the agent sells it and she has to take impact into account, indeed. In contrast, the current book value of the portfolio is known as *mark-to-market* value and can be defined as

$$V(t) := \sum_{i=0}^n h^i(t) P(t, T_i),$$

i.e. as the counterpart of  $\tilde{V}$  with no impact. Here  $h^i$ , similarly to  $\tilde{h}^i$ , represents the number of shares in the unimpacted bond  $P(t, T_i)$  at all times. From now on, we

will always work with the liquidation value  $\tilde{V}$ . We also point out that  $\tilde{h}$  is related to the trader's selling rate  $v_T$  via  $\tilde{h}_t^i = X_{T_i}(t)$  for each  $i = 1, \dots, n$ , where  $X_T(t)$  is the inventory, at time  $t$ , relative to the bond with a given maturity  $T$ , as defined in (5.2.5). Clearly, in a context where only one bond is traded, there will be only one index  $i$  such that  $v_{T_i} \neq 0$ .

**Definition 5.2.2** (Self-financing). *Let  $\hat{T} < +\infty$  be some finite time horizon and let  $\tilde{h}$  be an impacted portfolio as in Definition 5.2.1. We say that  $\tilde{h}$  is self-financing if its liquidation value  $\tilde{V}$  is such that*

$$d\tilde{V}(t, \tilde{h}) = \sum_{i=0}^n \tilde{h}^i(t) d\tilde{P}(t, T_i), \quad \text{for all } 0 \leq t \leq \hat{T}. \quad (5.2.14)$$

**Definition 5.2.3** (Locally risk free). *Let  $\tilde{h}$  be an impacted portfolio as in Definition 5.2.1 and let  $\tilde{V}$  be its liquidation value. Let also  $\alpha = (\alpha_t)_{t \in [0, \hat{T}]}$  be an adapted process. We say that  $\tilde{h}$  is locally risk-free if, for almost all  $t$ ,*

$$d\tilde{V}(t) = \alpha(t)\tilde{V}(t)dt \implies \alpha(t) = r(t),$$

where  $r(t)$  is the risk-free interest rate introduced in (5.2.1).

Here is the main result of this section.

**Theorem 5.2.4** (Impacted market price of risk). *Let  $\hat{T} < +\infty$  be some finite time horizon and let  $\mathbb{T} := (0, \hat{T}]$ . Let  $J_T$  be the impact density defined in (5.2.13). Given an impacted portfolio  $\tilde{h}$  as in (5.2.14), we assume that it is self-financing and locally risk-free, as in Definitions (5.2.2) and (5.2.3), respectively. Then, there exists a progressively measurable stochastic process  $\tilde{\lambda}(t)$  such that*

$$\tilde{\lambda}(t) = \frac{\mu_{T_i}(t, r(t)) - r(t)\tilde{P}(t, T_i) - J_{T_i}(t)}{\sigma_{T_i}(t, r(t))}, \quad t \geq 0, \quad (5.2.15)$$

for each maturity  $T_i$ ,  $i = 1, \dots, n$ , with  $\tilde{\lambda}$  depending on the short rate  $r$  but not on  $T_i$ .

The proof of Theorem 5.2.4 is given in Section 5.6.

**Remark 5.2.5** (Self-financing in presence of price impact). *In presence of price impact it is of course not obvious that the self-financing condition should hold. Adjusted self-financing conditions have been proposed, for instance, by Carmona and Webster [52]. We notice that, in their work, the adjustment consists of two parts: the covariation between the inventory and the price process, and the bid-ask spread. In our work we will assume the inventory is a finite variation process and that the bid ask spread is negligible, thereby obtaining the classic self-financing condition.*

A generalized notion of self-financing trading strategy has been proposed also by Cetin et al. [54] in order to account for liquidity risk. Similarly to before, we notice that, since our inventory is continuous and with bounded variation, their definition of self-financing reduces to the classical one (see in particular [54], Section 2). Finally, we remark that here we are not constrained by any self-financing condition on the unimpacted portfolio, hence there is no consistency to enforce from this point of view.

**Remark 5.2.6** (Intrinsic price impact). From Theorem 5.2.4 it follows that endogenous cross price impact naturally emerges in our framework. Indeed, once an agent trades a bond with maturity  $T_1$ , the process  $\tilde{\lambda}$  is uniquely determined. Note that  $\tilde{\lambda}$  does not depend on the maturity. For any bond with maturity  $T_2 \in \mathbb{T}$ , which is not traded, we have  $J_{T_2} \equiv 0$  but by (5.2.15), the price  $\tilde{P}(t, T_2)$  will be affected by the trade on the bond with maturity  $T_1$ . We remark that, by endogenous, we mean that the bonds with different maturities  $T_1$  and  $T_2$  are thought of as belonging to the same currency curve. If we were to discuss multiple interest rate curves, then exogenous cross price impact should be taken into account as well.

## 5.2.2 Impacted risk-neutral measure

We previously introduced two measures: the real world measure  $\mathbb{P}$  and the classic risk neutral measure  $\mathbb{Q}$ , as defined in (5.2.4). Now we use the result of Theorem 5.2.4 to define a third measure, which we call *impacted risk neutral measure* and denote by  $\tilde{\mathbb{Q}}$ . This is defined as follows:

$$\frac{d\tilde{\mathbb{Q}}}{d\mathbb{P}} = \exp \left\{ \int_0^t \tilde{\lambda}(s) dW^{\mathbb{P}}(s) - \frac{1}{2} \int_0^t \tilde{\lambda}^2(s) ds \right\}. \quad (5.2.16)$$

The well posedness of  $\tilde{\mathbb{Q}}$  can be checked via the Novikov condition. It might be useful to recall that the usual approach does not consist in determining the conditions on  $\mu_T, \sigma_T$  under which the Novikov condition is fulfilled. Rather, one chooses a specific short rate model to begin with. Then, one can specify the market price of risk process, exploiting the fact that it depends on  $t$  and  $r$ , but not on  $T$ . For example, in the case of Vasicek model, the market price of risk is assumed to be  $\lambda(t) = \lambda r(t)$ , for some constant  $\lambda$ . At this point, Novikov condition can be checked much more easily. Since we proved that  $\tilde{\lambda}$  depends on  $t$  and  $r$  only, we can assume the two processes to have the same structure and follow the same idea. In the case of Vasicek model, for example, we can assume  $\tilde{\lambda}(t) = \tilde{\lambda} r(t)$ , for some constant  $\tilde{\lambda}$  incorporating the impact. Consequently, determining the existence and well-posedness of  $\tilde{\mathbb{Q}}$  is fundamentally equivalent to determining the existence and well-posedness of  $\mathbb{Q}$ .

The Girsanov change of measure from the classic risk neutral measure to the impacted one is given by

$$\frac{d\tilde{\mathbb{Q}}}{d\mathbb{Q}} = \frac{d\tilde{\mathbb{Q}}}{d\mathbb{P}} \frac{d\mathbb{P}}{d\mathbb{Q}}.$$

with

$$\frac{d\mathbb{P}}{d\mathbb{Q}} = \exp \left\{ - \int_0^t \lambda(s) dW^{\mathbb{Q}}(s) - \frac{1}{2} \int_0^t \lambda^2(s) ds \right\}.$$

where  $\lambda$  was defined in (5.2.3). Hence,

$$\frac{d\tilde{\mathbb{Q}}}{d\mathbb{Q}} = \exp \left\{ \int_0^t \tilde{\lambda}(s) dW^{\mathbb{P}}(s) - \frac{1}{2} \int_0^t \tilde{\lambda}^2(s) ds + \int_0^t \lambda(s) dW^{\mathbb{Q}}(s) - \frac{1}{2} \int_0^t \lambda^2(s) ds \right\}.$$

Since  $W^{\mathbb{P}}(t) := W^{\mathbb{Q}}(t) + \int_0^t \lambda(s) ds$  is a Brownian motion under the measure  $\mathbb{P}$ , we have

$$\frac{d\tilde{\mathbb{Q}}}{d\mathbb{Q}} = \exp \left\{ \int_0^t (\tilde{\lambda}(s) - \lambda(s)) dW^{\mathbb{Q}}(s) - \frac{1}{2} \int_0^t \left( \lambda^2(s) + \tilde{\lambda}^2(s) - 2\lambda(s)\tilde{\lambda}(s) \right) ds \right\}.$$

In other words,

$$W^{\tilde{\mathbb{Q}}}(t) := W^{\mathbb{Q}}(t) - \int_0^t (\tilde{\lambda}(s) - \lambda(s)) ds,$$

is a Brownian motion under the measure  $\tilde{\mathbb{Q}}$ . It is then straightforward to notice that the impacted zero-coupon bond under the impacted measure  $\tilde{\mathbb{Q}}$  will be described by the dynamics

$$d\tilde{P}(t, T) = r(t)\tilde{P}(t, T)dt + \sigma_T(t, r(t))dW^{\tilde{\mathbb{Q}}}(t). \quad (5.2.17)$$

We further remark that, in principle, we could start by defining a new measure  $\tilde{\mathbb{P}}$  to get rid of the additional drift due to impact. Just rewrite the dynamics of the impacted zero-coupon bond as

$$d\tilde{P}(t, T) = \mu_T(t, r(t))dt + \sigma_T(t, r(t)) \left( \frac{J_T(t)}{\sigma_T(t, r(t))} dt + dW^{\mathbb{P}}(t) \right).$$

This suggests to define

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = \exp \left\{ \int_0^t \frac{J_T(s)}{\sigma_T(s, r(s))} dW^{\mathbb{P}}(s) - \frac{1}{2} \int_0^t \left( \frac{J_T(s)}{\sigma_T(s, r(s))} \right)^2 ds \right\}.$$

The impacted bond under this measure would follow the dynamics

$$d\tilde{P}(t, T) = \mu_T(t, r(t))dt + \sigma_T(t, r(t))dW^{\tilde{\mathbb{P}}}(t). \quad (5.2.18)$$

At this point,  $\tilde{\mathbb{Q}}$  can be defined from  $\tilde{\mathbb{P}}$  by using the classic market price of risk  $\lambda(t)$ . In other words,

$$\frac{d\tilde{\mathbb{Q}}}{d\tilde{\mathbb{P}}} = \frac{d\tilde{\mathbb{Q}}}{d\mathbb{Q}} \frac{d\mathbb{Q}}{d\tilde{\mathbb{P}}} = \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} \frac{d\mathbb{Q}}{d\tilde{\mathbb{P}}} = \frac{d\mathbb{Q}}{d\mathbb{P}}.$$

Putting everything together, we have the following commuting diagram

$$\begin{array}{ccc} \mathbb{P} & \xrightarrow{\lambda} & \mathbb{Q} \\ \tilde{\lambda}-\lambda \downarrow & \searrow \tilde{\lambda} & \downarrow \tilde{\lambda}-\lambda \\ \tilde{\mathbb{P}} & \xrightarrow{\lambda} & \tilde{\mathbb{Q}} \end{array}$$

By (5.2.17) and usual arguments it follows that discounted impacted traded prices, that is  $\{\tilde{P}(\cdot, T)/B(t)\}_{t \geq 0}$ , are martingales for any  $0 \leq T \leq \hat{T}$  under  $\tilde{\mathbb{Q}}$ . Here  $B$  is the usual money market account at time  $t$  given by

$$B(t) = e^{\int_0^t r(s) ds}. \quad (5.2.19)$$

We therefore have

$$\frac{\tilde{P}(t, T)}{B(t)} = \mathbb{E}^{\tilde{\mathbb{Q}}} \left[ \frac{\tilde{P}(T, T)}{B(T)} \middle| \mathcal{F}_t \right]. \quad (5.2.20)$$

Multiplying both sides by  $B(t)$  and exploiting the boundary condition  $\tilde{P}(T, T) = 1$ , we obtain the fundamental equation

$$\tilde{P}(t, T) = \mathbb{E}^{\tilde{\mathbb{Q}}} \left[ e^{-\int_t^T r(s) ds} \middle| \mathcal{F}_t \right]. \quad (5.2.21)$$

**Remark 5.2.7** (Interpretation impacted real world measure). *From (5.2.18) we observe that, financially speaking, under the impacted real world measure  $\tilde{\mathbb{P}}$ , impacted bond dynamics  $\tilde{P}(\cdot, T)$  has the same dynamics as the classic bond (without price impact modeling) in (5.2.2) under  $\mathbb{P}$ . In particular we have that  $\tilde{P}(T, T) = 1$  for all maturities  $T$ .*

### 5.2.3 Applications to pricing of interest rate derivatives

We start this section by remarking that the notion of arbitrage we use in our work is the classic one (see e.g. Harrison and Kreps [106] or Harrison and Pliska [107]), adjusted with impacted portfolios. We stress the fact that arbitrage is defined using the *liquidation* value and not the *mark-to-market* value of a portfolio.

**Definition 5.2.8** (Arbitrage portfolio). *An arbitrage portfolio is an impacted self-financing portfolio  $\tilde{h}$  such that its corresponding liquidation value process  $\tilde{V}$  satisfies*

1.  $\tilde{V}(0) = 0$ .
2.  $\tilde{V}(T) \geq 0$   $\mathbb{P}$ -a.s.
3.  $\mathbb{P}(\tilde{V}(T) > 0) > 0$

Using such definition, we show that our term structure model with price impact is free of arbitrage.

**Theorem 5.2.9** (Absence of arbitrage). *Assume that there exists an impacted equivalent martingale measure  $\tilde{\mathbb{Q}}$  as in (5.2.16). Then, our impacted model is arbitrage free.*

The proof of Theorem 5.2.9 is given in Section 5.6.

This result allows to price interest rate derivatives by taking the expectation of discounted payoffs under the impacted risk neutral measure  $\tilde{\mathbb{Q}}$ . As a benchmark example, we consider the price of an impacted Eurodollar future. In the classic context, a Eurodollar-futures contract provides its owner with the payoff (see Chapter 13.12 of [42])

$$N(1 - L(S, T)),$$

where  $N$  denotes the notional and  $L(S, T)$  is the LIBOR rate, defined as (see Chapter 1 of [42], Definition 1.2.4)

$$L(S, T) := \frac{1 - P(S, T)}{\tau(S, T)P(S, T)}, \quad (5.2.22)$$

with  $\tau(S, T)$  denoting the year fraction between  $S$  and  $T$ . Motivated by this, we introduce the impacted counterpart of the LIBOR rate in (5.2.22), i.e.

$$\tilde{L}(S, T) := \frac{1 - \tilde{P}(S, T)}{\tau(S, T)\tilde{P}(S, T)}, \quad (5.2.23)$$

with  $\tau$  defined as above and the impacted zero-coupon bond in place of the classic one. This new rate  $\tilde{L}$  is interpreted as the simply-compounded rate which is consistent with the impacted bond. This corresponds to the classic LIBOR rate which is the constant rate at which one needs to invest  $P(t, T)$  units of currency at time  $t$  in order to get an amount of one unit of currency at maturity  $T$ . Then, the fair price of an impacted Eurodollar future at time  $t$  is (see [42], Chapter 13, eq. (13.19))

$$\begin{aligned} \tilde{C}_t &= \mathbb{E}_t^{\tilde{\mathbb{Q}}}[N(1 - \tilde{L}(S, T))], \\ &= N \left( 1 + \frac{1}{\tau(S, T)} - \frac{1}{\tau(S, T)} \mathbb{E}_t^{\tilde{\mathbb{Q}}} \left[ \frac{1}{\tilde{P}(S, T)} \right] \right), \end{aligned} \quad (5.2.24)$$

where the discounting was left out due to continuous rebalancing (see again Chapter 13.12 of [42]). We will demonstrate in Section 5.3 how such expectation can be computed analytically provided the short rate model is simple enough as in Vasicek and Hull-White models.

**Remark 5.2.10** (Linear and nonlinear pricing equations). *Our success in retaining analytical tractability and linearity in the pricing equation may look surprising at first. In the context of equities, pricing derivatives in presence of price impact typically leads to nonlinear PDEs. This, in turn, motivated the study of super-replicating strategies and the so-called gamma constrained strategies. Several works provide also necessary and sufficient conditions ensuring the parabolicity of the pricing equation, hence the existence and uniqueness of a self-financing, perfectly replicating strategy. We refer, for example, to Abergel and Loeper [1], Bourchard, Loeper et al. [33, 32] and Loeper [128]. The point we would like to stress here is that the nonlinearity of the pricing equation is a consequence of the trading strategy having a diffusion term, or a consequence of the presence of transaction costs. In other words, under the assumption that trading strategies have bounded variation and no transaction costs are present, the pricing PDE becomes linear again. Hence, our work is actually in agreement to what can be found in the context of equities.*

#### 5.2.4 Cross price impact and impacted yield curve

In this section we discuss how trading a bond  $P(t, T)$  impacts the yield curve. For the sake of analytical tractability, we will consider affine short-rate models, that is, those models where bond prices are of the form

$$P(t, T) = A(t, T)e^{-B(t, T)r(t)}, \quad 0 \leq t \leq T, \quad (5.2.25)$$

for some deterministic, smooth functions  $A$  and  $B$  and  $r$  is given by (5.2.1). The remarkable property of these models is that they can be completely characterized as in the following theorem (see, e.g., Filipovic [86], Section 5.3, Brigo and Mercurio [42], Section 3.2.4, Bjork [24] Section 3.4 and references therein).

**Lemma 5.2.11** (Characterization affine short-rate models). *The short rate model (5.2.1) is affine if and only if there exist deterministic, continuous functions  $a, \alpha, b, \beta$  such that the diffusion and the drift terms in (5.2.1) are of the form*

$$\begin{aligned} \sigma^2(t, r) &= a(t) + \alpha(t)r, \\ \mu(t, r) &= b(t) + \beta(t)r, \end{aligned}$$

and the functions  $A, B$  satisfy the following system of ODEs

$$\begin{aligned} -\frac{\partial}{\partial t} \ln A(t, T) &= \frac{1}{2}a(t)B^2(t, T) - b(t)B(t, T), \quad A(T, T) = 1, \\ \frac{\partial}{\partial t} B(t, T) &= \frac{1}{2}\alpha(t)B^2(t, T) - \beta(t)B(t, T) - 1, \quad B(T, T) = 0, \end{aligned}$$

for all  $t \leq T$ .

As explained in [86], the functions  $a, \alpha, b, \beta$  can be further specified by observing that any non-degenerate short rate affine model, that is a model with  $\sigma(t, r) \neq 0$  for all  $t > 0$ , can be transformed, by means of an affine transformation, in two cases only, depending on whether the state space of the short rate  $r$  is the whole real line  $\mathbb{R}$  or only the positive part  $\mathbb{R}_+$ . In the first case, it must hold  $\alpha(t) = 0$  and  $a(t) \geq 0$ , with  $b, \beta$  arbitrary. In the second case, it must hold  $a(t) = 0, \alpha(t), b(t) \geq 0$  and  $\beta$  arbitrary.

Let  $\hat{T} < +\infty$  be some finite time horizon. The yield curve at a pre-trading time  $t_0$  (i.e. before price impact effects kick in) according to classic theory of interest rates is defined by

$$Y(t, T) := P(t, T)^{-1/T} - 1, \quad 0 \leq t \leq t_0, \quad (5.2.26)$$

for all maturities  $0 \leq T \leq \hat{T}$ . Next, we consider the impacted bond dynamics

$$d\tilde{P}(t, T) = dP(t, T) - J_T(t)dt,$$

where  $J_T$  was defined in (5.2.13). Recall that the dynamics of  $r(t)$  is given in (5.2.1). Applying Ito's formula on  $P(t, T)$  in (5.2.25) we get

$$\begin{aligned} dP(t, T) = e^{-B(t, T)r(t)} & \left[ \frac{\partial A}{\partial t} - A(t, T) \frac{\partial B}{\partial t} r(t) + \frac{1}{2} A(t, T) B^2(t, T) \sigma^2(t, r(t)) + \right. \\ & \left. - A(t, T) B(t, T) \mu(t, r(t)) \right] dt - \sigma(t, r(t)) B(t, T) A(t, T) e^{-B(t, T)r(t)} dW^{\mathbb{P}}(t). \end{aligned}$$

From this equation, we readily extract the drift and the diffusion of the zero-coupon bond with maturity  $T$ :

$$\begin{aligned} \mu_T(t, r(t)) &:= e^{-B(t, T)r(t)} \left[ \frac{\partial A}{\partial t} - A(t, T) \frac{\partial B}{\partial t} r(t) + \frac{1}{2} A(t, T) B^2(t, T) \sigma^2(t, r(t)) \right. \\ & \quad \left. - A(t, T) B(t, T) \mu(t, r(t)) \right], \\ \sigma_T(t, r(t)) &:= -\sigma(t, r) A(t, T) B(t, T) e^{-B(t, T)r(t)}. \end{aligned} \quad (5.2.27)$$

Next, we consider the effect of an agent trading on the bond with maturity  $T$  on a bond which is not traded by the agent with maturity  $S$ . We call this effect the *endogenous cross-impact* on the bond with maturity  $S$ . Recall that in this case the dynamics of the  $S$ -bond is given by

$$d\tilde{P}(t, S) = \mu_S(t, r(t))dt + \sigma_S(t, r(t))dW^{\mathbb{P}}(t), \quad (5.2.28)$$

where the coefficients  $\mu_S$  and  $\sigma_S$  are given by analogous formulas to (5.2.27). Since we are trading the  $T$ -bond only,  $J_S$  in (5.2.13) will be identically equal to zero. Hence, the definition of the impacted market price of risk (5.2.15) implies the following relationship

$$\frac{\mu_T(t, r(t)) - r(t)\tilde{P}(t, T) - J_T(t)}{\sigma_T(t, r(t))} = \frac{\mu_S(t, r(t)) - r(t)\tilde{P}(t, S)}{\sigma_S(t, r(t))}.$$

This equation tells us how the drift the  $S$ -bond has to change in order to avoid arbitrage. That is, this equation describes the *cross-price impact*. Specifically we have

$$\mu_S(t, r(t)) = \frac{\sigma_S(t, r(t))}{\sigma_T(t, r(t))} \left[ \mu_T(t, r(t)) - r(t)\tilde{P}(t, T) - J_T(t) \right] + r(t)\tilde{P}(t, S).$$

Substituting this drift in (5.2.28) we get

$$\begin{aligned} d\tilde{P}(t, S) &= r(t)\tilde{P}(t, S)dt + \frac{\sigma_S(t, r(t))}{\sigma_T(t, r(t))} \left[ \mu_T(t, r(t)) - r(t)\tilde{P}(t, T) - J_T(t) \right] dt \\ &\quad + \sigma_S(t, r(t))dW^{\mathbb{P}}(t). \end{aligned} \quad (5.2.29)$$

Finally, we define the impacted yield curve for all  $t_0 \leq T \leq \hat{T}$  as follows:

$$\tilde{Y}(t, T) := \tilde{P}(t, T)^{-1/T} - 1. \quad (5.2.30)$$

**Remark 5.2.12** (Cross impacted bonds at maturity). *We have shown in (5.2.12) that according to our model  $\tilde{P}(T, T) = 1$ . However, we should also ensure that all cross-impacted bonds with maturity  $S \neq T$  reach value 1 at their maturities. This of course, would make the model much more involved and we may lose tractability.*

### 5.2.5 Coupon bonds

It is worth recalling that the zero-coupon bond  $P(t, T)$  is rarely traded. In practice, its price is derived using some bootstrapping procedure applied, for instance, to coupon bonds. In the classic theory, coupon bonds are defined as

$$B(t, T) = \sum_{i=1}^n c_i P(t, T_i) + NP(t, T_n),$$

where  $N$  denotes the reimbursement notional,  $(c_i, T_i)_{i=1}^n$  are the coupons and the maturities at which the coupons are paid, respectively. In order to determine an expression for the impacted coupon bond, we start from its cash flow

$$C(t) := \sum_{i=1}^n c_i D(t, T_i) + ND(t, T_n),$$

where  $D(t, T)$  is the stochastic discount factor defined by

$$D(t, T) := e^{-\int_t^T r(s)ds},$$

where  $r$  is given by (5.2.1). Then, we define the impacted coupon bond as the expectation of this cash flow under the impacted risk neutral measure  $\tilde{\mathbb{Q}}$  (see (5.2.16)):

$$\tilde{B}(t, T) := \mathbb{E}^{\tilde{\mathbb{Q}}} [C(t)].$$

Substituting the expression of  $C$  immediately yields

$$\begin{aligned} \tilde{B}(t, T) &= \sum_{i=1}^n c_i \mathbb{E}^{\tilde{\mathbb{Q}}} [D(t, T_i)] + N \mathbb{E}^{\tilde{\mathbb{Q}}} [D(t, T_n)] \\ &= \sum_{i=1}^n c_i \tilde{P}(t, T_i) + N \tilde{P}(t, T_n), \end{aligned} \tag{5.2.31}$$

where  $\tilde{P}(\cdot, T_i)$  is the (directly) impacted price of a zero-coupon bond as defined in (5.2.7). Note that (5.2.31) gives the price of the impacted coupon bond in terms of impacted zero-coupon bonds. Since zero-coupon bonds are not always traded, we would like to get a direct pricing formula for impacted coupon bonds. Let  $\{v_{T_i}\}_{i=1}^n$  be admissible trading speeds on zero-coupon bonds with maturities  $\{T_i\}_{i=1}^n$  as defined in Section 5.2, that is  $v_{T_i} \in \mathcal{A}_{T_i}$  for any  $i = 1, \dots, n$ . From (5.2.7) and (5.2.31) we get

$$\begin{aligned} \tilde{B}(t, T) &= \sum_{i=1}^n c_i \tilde{P}(t, T_i) + N \tilde{P}(t, T_n) \\ &= B(t, T) - \sum_{i=1}^n c_i l(t, T_i) v_{T_i}(t) - N l(t, T_n) v_{T_n}(t) \\ &\quad - \sum_{i=1}^n c_i K(t, T_i) y e^{-\rho t} - N K(t, T_n) y e^{-\rho t} \\ &\quad - \gamma \int_0^t e^{-\rho(t-s)} \left( \sum_{i=1}^n c_i K(t, T_i) v_{T_i}(s) + N K(t, T_n) v_{T_n}(s) \right) ds. \end{aligned}$$

Let us now assume that  $l = \kappa K$  at all times and for all maturities, where  $\kappa > 0$  is a constant. Then, the impacted coupon bond dynamics can be written as

$$\begin{aligned}\tilde{B}(t, T) = & B(t, T) - ye^{-\rho t} \left[ \sum_{i=1}^n c_i K(t, T_i) + NK(t, T_n) \right] \\ & - \int_0^t e^{-\rho(t-s)} \kappa \delta(s-t) \left[ \sum_{i=1}^n c_i K(t, T_i) v_{T_i}(s) + NK(t, T_n) v_{T_n}(s) \right] ds \\ & - \gamma \int_0^t e^{-\rho(t-s)} \left[ \sum_{i=1}^n c_i K(t, T_i) v_{T_i}(s) + NK(t, T_n) v_{T_n}(s) \right] ds,\end{aligned}\tag{5.2.33}$$

where  $\delta$  denotes the Dirac delta. Notice that under this assumption the impacted zero-coupon bond dynamics defined in (5.2.7) boils down to

$$\tilde{P}(t, T) = P(t, T) - K(t, T) \left[ ye^{-\rho t} + \int_0^t e^{-\rho(t-s)} v_T(s) (\gamma + \kappa \delta(s-t)) ds \right]. \tag{5.2.34}$$

This suggest we can define

$$K^B(t, T) := \sum_{i=1}^n c_i K(t, T_i) + NK(t, T_n).$$

and the trading speed relative to the coupon bond as

$$v^B(t, s) := \frac{1}{K^B(t, T)} \left[ \sum_{i=1}^n c_i K(t, T_i) v_{T_i}(s) + NK(t, T_n) v_{T_n}(s) \right]. \tag{5.2.35}$$

Therefore, we obtain the following price impact model for the coupon bond:

$$\tilde{B}(t, T) = B(t, T) - K^B(t, T) \left[ ye^{-\rho t} + \int_0^t e^{-\rho(t-s)} v^B(t, s) (\gamma + \kappa \delta(s-t)) ds \right]. \tag{5.2.36}$$

Interestingly, under the simplifying assumption that the functions  $l$  and  $K$  are equal up to some constant, we observe that the impacted zero-coupon bond  $\tilde{P}(t, T)$  in

(5.2.34) and the impacted coupon bond  $\tilde{B}(t, T)$  in (5.2.36) are described by the same kind of dynamics.

This is particularly useful because, provided enough data on traded coupon bonds are available, one might attempt to use (5.2.35) and (5.2.36) to bootstrap the trading speeds  $v_{T_i}$  relative to the zero-coupon bonds. Using the price impact model (5.2.7), it would be then possible to price impacted zero-coupon bonds consistently with market data. Finally, using these impacted zero-coupon bonds as building blocks, it would be possible to price, consistently with market data, more complicated and less liquid impacted interest rate derivatives, as discussed in Section 5.2.3.

### 5.2.6 HJM framework

In this section we turn our discussion to incorporating price impact into the Heath, Jarrow and Morton framework [108], in order to model the forward curve. Notice that this approach, although it may look different, has some common aspects to the framework developed in Section 5.2.1. Namely, we start by adding artificially a price impact term to the forward rate dynamics. This corresponds to adding price impact to zero-coupon bonds in Section 5.2.1. The important difference is that, here, we are creating an impacted interest rate, which was not done in Section 5.2.1. Then we will develop the connection between the price impact of zero-coupon bonds and the price impact term incorporated into the forward rate, in order to reveal the financial interpretation of the latter. Note that both the zero-coupon bonds and the forward rate can be used as building blocks for the whole interest rates theory. We are therefore interested in showing the connection between the two in the presence of price impact. For a thorough discussion on the HJM framework in the classic interest rate theory, we refer to Chapter 6 of the book by Filipovic [86].

Given an integrable initial forward curve  $T \rightarrow \tilde{f}(0, T)$ , we assume that the impacted forward rate process  $\tilde{f}(\cdot, T)$  is given by

$$\tilde{f}(t, T) = \tilde{f}(0, T) + \int_0^t (\alpha(s, T) + J^f(s, T)) ds + \int_0^t \sigma(s, T) dW^{\mathbb{P}}(s), \quad (5.2.37)$$

for any  $0 \leq t \leq T$  and each maturity  $T > 0$ . Here  $W^{\mathbb{P}}$  is a Brownian motion under the measure  $\mathbb{P}$  and  $\alpha(\cdot, T)$ ,  $J^f(\cdot, T)$  and  $\sigma(\cdot, T)$  are assumed to be progressively measurable processes and satisfy for any  $T > 0$

$$\int_0^T \int_0^T (|\alpha(s, t)| + |J^f(s, t)|) ds dt < \infty, \\ \sup_{s, t \leq T} |\sigma(s, t)| < \infty.$$

While the roles of  $\alpha(\cdot, T)$  and  $\sigma(\cdot, T)$  above are as in standard HJM model, the stochastic process  $J^f$  represents the impact density relative to the forward rate and

accounts for the fact that the forward curve is affected by the trading activity. From a modelling perspective, it plays a completely analogous role as the quantity  $J_T$  defined in (5.2.13) for the impacted zero-coupon bond. In fact, in Proposition (5.2.14) we will determine the mathematical relationship linking these two quantities. Such relationship will allow us to understand how the forward curve is impacted by trading zero-coupon bonds.

In this framework, the impacted short rate model is given by

$$\tilde{r}(t) := \tilde{f}(t, t) = \tilde{f}(0, t) + \int_0^t (\alpha(s, t) + J^f(s, t)) ds + \int_0^t \sigma(s, t) dW^\mathbb{P}(s), \quad (5.2.38)$$

and the impacted zero-coupon bond is defined as follows

$$\tilde{P}(t, T) = e^{-\int_t^T \tilde{f}(t, u) du}. \quad (5.2.39)$$

Next we derive the explicit dynamics of  $\{\tilde{P}(t, T)\}_{0 \leq t \leq T}$ . The following corollary is an impacted version of Lemma 6.1 in [86].

**Corollary 5.2.13** (Impacted zero-coupon bond in HJM framework). *For every maturity  $T$  the impacted zero-coupon bond defined in (5.2.39) follows the dynamics*

$$\tilde{P}(t, T) = \tilde{P}(0, T) + \int_0^t \tilde{P}(s, T) \left( \tilde{r}(s) + \tilde{b}(s, T) \right) ds + \int_0^t \tilde{P}(s, T) \nu(s, T) dW^\mathbb{P}(s), \quad (5.2.40)$$

for  $t \leq T$ , where  $\tilde{r}$  is the impacted short rate defined in (5.2.38) and

$$\begin{aligned} \nu(s, T) &:= - \int_s^T \sigma(s, u) du, \\ \tilde{b}(s, T) &:= - \int_s^T \alpha(s, u) du - \int_s^T J^f(s, u) du + \frac{1}{2} \nu^2(s, T). \end{aligned} \quad (5.2.41)$$

We now show that the impact  $J^f$  can be expressed in terms of the impact relative to the zero-coupon bond, and vice versa. In order to show this correspondence in terms of agent's trading speed, we need to make an additional assumption on the trading speeds on zero-coupon bonds. We assume that the price impact in the forward curve is a result of trading by one or many agents over a continuum of zero-coupon bonds with maturities  $T$  and trading speeds  $\{T \geq 0 : v_T \in \mathcal{A}_T\}$  so that

$$|\partial_T v_T(t)| < \infty, \quad \text{for all } 0 \leq t \leq T, \quad \mathbb{P} - \text{a.s.} \quad (5.2.42)$$

Note that this assumption in fact makes sense in bond trading, which has discrete maturities, as it claims that when there is a highly traded  $T_i$ -bond, you would find that also the neighbouring  $T_{i-1}$ ,  $T_{i+1}$  are liquid. Assumption (5.2.42) implies that  $\partial_T I_T(t)$  is well defined as needed in the following Proposition. We recall that  $f$  represents the unimpacted forward rate which is given by setting  $J^f \equiv 0$  in (5.2.37).

**Proposition 5.2.14** (Forward rate and zero-coupon bond price impact relation). *Let  $I_T(t)$  be the overall impact defined in (5.2.11) and  $\tilde{P}(\cdot, T)$  the impacted zero-coupon bond price in (5.2.39). Assume  $\tilde{f}(0, t) = f(0, t)$ , meaning that the initial value of the forward curve is not affected by trading. Then, the forward rate impact  $J^f$  introduced in (5.2.37) is given by*

$$J^f(t, T) = -\frac{\partial}{\partial T} \log \left( 1 - \frac{I_T(t)}{P(t, T)} \right), \quad \text{for all } 0 \leq t \leq T \text{ such that } \tilde{P}(t, T) > 0. \quad (5.2.43)$$

The proof of Proposition 5.2.14 is given in Section 5.6.

**Remark 5.2.15.** *Note that the requirement that  $\tilde{P}(t, T) > 0$  ensures that the logarithm on the right-hand side of (5.2.43) is well defined, as (5.6.9) in the proof suggests. The proof also gives another relation between  $J^f(\cdot, T)$  and  $I_T$  which always holds but is perhaps not as direct.*

A well known feature of the classic HJM framework is that, under the risk neutral measure, the drift of the forward rate is completely specified by the volatility through the so called *HJM condition*. In order to understand how this condition is affected by the introduction of price impact, we will follow Theorem 6.1 of [86]. In particular, we have the following key result.

**Theorem 5.2.16** (HJM condition with price impact). *Let  $\mathbb{P}$  be the real world measure under which the impacted forward rate as in (5.2.37). Let  $\tilde{\mathbb{Q}} \sim \mathbb{P}$  be an equivalent probability measure of the form*

$$\frac{d\tilde{\mathbb{Q}}}{d\mathbb{P}} = \exp \left\{ \int_0^t \tilde{\gamma}(s) dW^{\mathbb{P}}(s) - \frac{1}{2} \int_0^t \tilde{\gamma}^2(s) ds \right\}, \quad (5.2.44)$$

for some progressively measurable stochastic process  $\tilde{\gamma} = \{\tilde{\gamma}(t)\}_{t \geq 0}$  such that

$$\int_0^t \tilde{\gamma}^2(s) ds < \infty, \quad t > 0, \mathbb{P} - a.s.$$

Then,  $\tilde{\mathbb{Q}}$  is an equivalent (local) martingale measure if and only if

$$\tilde{b}(t, T) = -\nu(t, T)\tilde{\gamma}(t), \quad \text{for all } 0 \leq t \leq T. \quad (5.2.45)$$

with  $\tilde{b}(\cdot, T)$  and  $\nu(\cdot, T)$  defined as in (5.2.41). In this case, the dynamics of the impacted forward rate under the measure  $\tilde{\mathbb{Q}}$  is given by

$$\tilde{f}(t, T) = \tilde{f}(0, T) + \int_0^t \left( \sigma(s, T) \int_s^T \sigma(s, u) du \right) ds + \int_0^t \sigma(s, T) dW^{\tilde{\mathbb{Q}}}(s). \quad (5.2.46)$$

Moreover, the prices of impacted zero-coupon bonds are

$$\tilde{P}(t, T) = \tilde{P}(0, T) + \int_0^t \tilde{P}(s, T) \tilde{r}(s) ds + \int_0^t \tilde{P}(s, T) \nu(s, T) dW^{\tilde{\mathbb{Q}}}(s). \quad (5.2.47)$$

The proof of Theorem 5.2.16 is given in Section 5.6.

In our context such a measure  $\tilde{\mathbb{Q}}$  would be clearly interpreted as an impacted risk-neutral measure, completely analogous to the measure defined in (5.2.16). In fact, the stochastic process  $\tilde{\gamma}$  in the HJM condition (5.2.45) is the counterpart in the HJM framework, of the impacted market price of risk  $\tilde{\lambda}$  defined in Section 5.2. Indeed, combining equations (5.2.3) and (5.2.40) we obtain for  $0 \leq t \leq T$ ,

$$\tilde{\lambda}(t) = \frac{\tilde{P}(t, T) \left( r(t) - \tilde{b}(t, T) \right) - r(t) \tilde{P}(t, T)}{\tilde{P}(t, T) \nu(t, T)} = -\frac{-\tilde{b}(t, T)}{\nu(t, T)} = \tilde{\gamma}(t).$$

The HJM framework adjusted with price impact discussed in this section is therefore perfectly consistent with the price impact model for zero-coupon bonds introduced in Section 5.2.1.

We remark once again that, in the classic theory of interest rates, the meaning of the HJM condition lies in the fact that the drift of the forward rate is constrained under the risk neutral measure. Similarly, looking at the market price of risk, we notice that a constraint is present for the drift of the zero-coupon bond. In particular, its drift, under the risk neutral measure, has to be precisely the risk free interest rate. The interesting point we would like to make here is that, once we incorporate price impact, the same kind of constraints holds, only under the newly defined impacted measure  $\tilde{\mathbb{Q}}$ .

We conclude this section by making two remarks. We first address the question of when the measure defined in Theorem 5.2.16 is an equivalent martingale measure, instead of just local martingale measure. The second remark concerns the Markov property of the impacted short rate. In both cases, we see that the classic results carry over to the price impact framework, thanks to the key fact that the impact component affects only the drift of the forward rate.

**Remark 5.2.17** (Impacted risk neutral measure is an EMM). *Let  $\nu(t, T)$  be defined as in (5.2.41). From Corollary 6.2 of [86] it follows that the measure  $\tilde{\mathbb{Q}}$  defined in Theorem 5.2.16 is an equivalent martingale measure if either*

$$\mathbb{E}^{\tilde{\mathbb{Q}}} \left[ e^{\frac{1}{2} \int_0^T \nu^2(t, T) dt} \right] < \infty, \quad \text{for all } T \geq 0,$$

or

$$f(t, T) \geq 0, \quad \text{for all } 0 \leq t \leq T.$$

**Remark 5.2.18** (Markov property of the short rate). *As pointed out in Chapter 5 of [42], one of the main drawbacks of HJM theory is that the implied short rate dynamics is usually not Markovian. Here we simply remark that, since the volatility of the forward rate  $\sigma(t, T)$  is not affected by price impact, if the Markov property of the short rate  $r(t)$  is ensured under the measure  $\mathbb{Q}$  when there is no trading, hence no price impact, then it is also preserved in the presence of price impact under the  $\tilde{\mathbb{Q}}$ .*

## 5.3 Examples

### 5.3.1 Pricing impacted Eurodollar futures with Vasicek model

In this section we illustrate the argument outlined in Section 5.2.3 by computing the explicit price of a Eurodollar-futures contract when the underlying short rate follows an Ornstein-Uhlenbeck process [162]. The dynamics under the risk neutral measure  $\mathbb{Q}$  is given by

$$dr(t) = k(\theta - r(t))dt + \sigma dW^{\mathbb{Q}}(t), \quad (5.3.1)$$

with  $k, \theta, \sigma$  positive parameters. The dynamics of the short rate under the real world measure  $\mathbb{P}$  can be expressed as

$$dr(t) = k(\theta - r(t))dt + \sigma(dW^{\mathbb{P}}(t) - \lambda(t)dt), \quad (5.3.2)$$

where we highlight the classic market price of risk process  $\lambda$  defined in (5.2.3). Another representation for  $r(t)$  under  $\mathbb{P}$  is

$$dr(t) = \tilde{k}(\tilde{\theta} - r(t))dt + \sigma(dW^{\mathbb{P}}(t) - \tilde{\lambda}(t)dt), \quad (5.3.3)$$

where  $\tilde{\lambda}$  is the impacted market price of risk defined in (5.2.15) and  $\tilde{k}, \tilde{\theta}$  are positive constants. Combining the two equivalent representations (5.3.2) and (5.3.3), we see that the following holds for any  $t \geq 0$

$$k\theta - kr(t) - \sigma\lambda(t) = \tilde{k}\tilde{\theta} - \tilde{k}r(t) - \sigma\tilde{\lambda}(t). \quad (5.3.4)$$

Similarly to what is done in the standard theory (see Brigo and Mercurio [42], section 3.2.1), we assume the short rate  $r(t)$  has the same kind of dynamics under the measures  $\mathbb{P}$ ,  $\mathbb{Q}$  and  $\tilde{\mathbb{Q}}$ , that is

$$\lambda(t) = \lambda r(t), \quad \tilde{\lambda}(t) = \tilde{\lambda} r(t), \quad (5.3.5)$$

with  $\lambda, \tilde{\lambda}$  constants. The whole impact is then encapsulated in the constant  $\tilde{\lambda}$ . By plugging (5.3.5) into (5.3.4), we deduce

$$\begin{aligned} \tilde{k} &= k - \sigma(\tilde{\lambda} - \lambda), \\ \tilde{\theta} &= \frac{k\theta}{k - \sigma(\tilde{\lambda} - \lambda)}. \end{aligned} \quad (5.3.6)$$

Clearly, in order to ensure all parameters are positive, we must require

$$k > \sigma(\tilde{\lambda} - \lambda).$$

In this way, the short rate  $r(t)$  is normally distributed under all three measures. In particular, plugging the Girsanov transformation from the measure  $\mathbb{P}$  to the measure  $\tilde{\mathbb{Q}}$ , defined in (5.2.16), into equation (5.3.3), the short rate dynamics under  $\tilde{\mathbb{Q}}$  can be conveniently rewritten as

$$dr(t) = \tilde{k}(\tilde{\theta} - r(t))dt + \sigma dW^{\tilde{\mathbb{Q}}}(t). \quad (5.3.7)$$

Since the short rate under  $\tilde{\mathbb{Q}}$  is Gaussian,  $\{\int_t^T r(s)ds\}_{t \geq 0}$  is also a Gaussian process. At the same time, we recall the well known fact that if  $X$  is a normal random variable with mean  $\mu_X$  and variance  $\sigma_X^2$ , then  $\mathbb{E}(\exp(X)) = \exp(\mu_X + \frac{1}{2}\sigma_X^2)$ . Following the same argument as in (Brigo and Mercurio [42], Chapters 3.2.1, 3.3.2 and Chapter 4), we can use (5.3.7) in order to express the impacted zero-coupon bond price as follows

$$\tilde{P}(t, T) = A(t, T)e^{-B(t, T)r(t)}$$

where

$$\begin{aligned} A(t, T) &= \exp \left\{ \left( \tilde{\theta} - \frac{\sigma^2}{2\tilde{k}^2} \right) [B(t, T) - T + t] - \frac{\sigma^2}{4\tilde{k}} B^2(t, T) \right\}, \\ B(t, T) &= \frac{1}{\tilde{k}} \left( 1 - e^{-\tilde{k}(T-t)} \right). \end{aligned} \quad (5.3.8)$$

Hence, the key expectation needed to compute the impacted Eurodollar future fair price in equation (5.2.24) is equal to

$$\mathbb{E}^{\tilde{\mathbb{Q}}} \left[ \frac{1}{\tilde{P}(t, T)} \right] = \frac{1}{A(t, T)} \mathbb{E}^{\tilde{\mathbb{Q}}} [e^{B(t, T)r(t)}]. \quad (5.3.9)$$

Since  $r(t)$  is normally distributed,  $B(t, T)r(t)$  will be normally distributed as well with mean and variance respectively equal to (see Brigo and Mercurio [42], Eq. (3.7))

$$\begin{aligned} \mathbb{E}^{\tilde{\mathbb{Q}}}[B(t, T)r(t)] &= B(t, T) \left[ r(0)e^{-\tilde{k}t} + \theta(1 - e^{-\tilde{k}t}) \right], \\ \text{Var}^{\tilde{\mathbb{Q}}}[B(t, T)r(t)] &= B^2(t, T) \left[ \frac{\sigma^2}{2\tilde{k}} (1 - e^{-2\tilde{k}t}) \right]. \end{aligned}$$

Therefore in order to get the impacted price of a Eurodollar-future contract we need to compute the expectation in the right hand side of (5.3.9) which can be written explicitly as

$$\begin{aligned} \mathbb{E}^{\tilde{\mathbb{Q}}} \left[ \frac{1}{\tilde{P}(t, T)} \right] &= \frac{1}{A(t, T)} \times \\ &\times \exp \left\{ B(t, T)[r(0)e^{-\tilde{k}t} + \theta(1 - e^{-\tilde{k}t})] + \frac{1}{2} B^2(t, T) \left[ \frac{\sigma^2}{2\tilde{k}} (1 - e^{-2\tilde{k}t}) \right] \right\}. \end{aligned}$$

The main conclusion here is that defining the short rate under the impacted risk neutral measure preserves analytical tractability of interest rate derivatives prices.

### 5.3.2 Pricing impacted Eurodollar futures with Hull White model

In this section we compute the explicit price of a Eurodollar-futures contract when the underlying short rate follows a Hull White model [115]. We start with the classic framework where there is not price impact. In this case the short rate is given by

$$dr(t) = [\theta(t) - ar(t)] dt + \sigma dW^{\mathbb{Q}}(t),$$

where  $a$  and  $\sigma$  are positive constants and the function  $\theta$  is chosen in order to fit exactly the term structure of interest rates being currently observed in the market. Denoting by  $P^M(0, T)$  the unimpacted market discount factor for the maturity  $T$  and defining the (unimpacted) market instantaneous forward rate at time 0 for the maturity  $T$

$$f^M(0, T) := -\frac{\partial}{\partial T} \ln P^M(0, T),$$

the function  $\theta$  is given by (see e.g. Brigo and Mercurio [42], Chapter 3, Eq. (3.34))

$$\theta(t) = \frac{\partial f^M(0, t)}{\partial T} + af^M(0, t) + \frac{\sigma^2}{2a} (1 - e^{-2at}),$$

where  $\frac{\partial f^M(0, t)}{\partial T}$  denotes the partial derivative of  $f^M$  with respect to its second variable. We start by computing the price under the classic risk neutral measure  $\mathbb{Q}$ . According to eq. (3.36)–(3.37) in Chapter 3 of [42], the short rate is normally distributed with mean and variance respectively equal to

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}}[r(t)|\mathcal{F}_s] &= r(s)e^{-a(t-s)} + \alpha(t) - \alpha(s)e^{-a(t-s)} \\ \text{Var}^{\mathbb{Q}}[r(t)|\mathcal{F}_s] &= \frac{\sigma^2}{2a} [1 - e^{-2a(t-s)}], \end{aligned}$$

where

$$\alpha(t) := f^M(0, t) + \frac{\sigma^2}{2a^2} (1 - e^{-at})^2.$$

As before, we notice that the integral of the short rate will be normally distributed as well, hence the price of a zero-coupon bond under the classic risk neutral measure is given by (see eq. (3.39) in Chapter 3 of [42]),

$$P(t, T) = A(t, T)e^{-B(t, T)r(t)},$$

where

$$A(t, T) = \frac{P^M(0, T)}{P^M(0, t)} \exp \left\{ B(t, T) f^M(0, t) - \frac{\sigma^2}{4a} (1 - e^{-2at}) B^2(t, T) \right\},$$

$$B(t, T) = \frac{1}{a} [1 - e^{-a(T-t)}].$$

Moreover, the term  $B(t, T)r(t)$  is still normally distributed and we immediately have

$$\mathbb{E}^{\mathbb{Q}}[B(t, T)r(t)|\mathcal{F}_s] = B(t, T) (r(s)e^{-a(t-s)} + \alpha(t) - \alpha(s)e^{-a(t-s)}),$$

$$\text{Var}^{\mathbb{Q}}[B(t, T)r(t)|\mathcal{F}_s] = B^2(t, T) \frac{\sigma^2}{2a} [1 - e^{-2a(t-s)}].$$

This implies that the expectation we are interested in, under the classic risk neutral measure  $\mathbb{Q}$ , can be written explicitly as (see Section 13.12.1 in [42])

$$\mathbb{E}^{\mathbb{Q}} \left[ \frac{1}{P(t, T)} \right] = \frac{1}{A(t, T)} \exp \left\{ B(t, T) \mathbb{E}^{\mathbb{Q}}[r(t)] + \frac{1}{2} B^2(t, T) \text{Var}^{\mathbb{Q}}[r(t)] \right\}.$$

Next we derive the corresponding expression under the impacted risk neutral measure  $\tilde{\mathbb{Q}}$  in (5.2.16). We assume as in Section 5.3.1 that the market price of risk and impacted market price of risk are given by

$$\lambda(t) = \lambda r(t), \quad \tilde{\lambda}(t) = \tilde{\lambda} r(t),$$

for some constants  $\lambda, \tilde{\lambda}$ . Using the Girsanov change of measure from  $\mathbb{Q}$  to  $\tilde{\mathbb{Q}}$  defined in Section 5.2.2, it follows that the short rate under  $\tilde{\mathbb{Q}}$  is given by

$$\begin{aligned} dr(t) &= [\theta(t) - ar(t)] dt + \sigma dW^{\mathbb{Q}}(t) \\ &= [\theta(t) - ar(t)] dt + \sigma dW^{\tilde{\mathbb{Q}}}(t) + \sigma(\tilde{\lambda} - \lambda)r(t)dt \\ &= \left[ \theta(t) - (a - \sigma(\tilde{\lambda} - \lambda))r(t) \right] dt + \sigma dW^{\tilde{\mathbb{Q}}}(t). \end{aligned}$$

Hence, we can define the impacted parameter

$$\tilde{a} := a - \sigma(\tilde{\lambda} - \lambda).$$

The pricing formula for  $\mathbb{E}^{\tilde{\mathbb{Q}}} \left[ \frac{1}{P(t, T)} \right]$  is then derived by following the same steps as in the classic case. Similarly to the Vasicek model, analytical tractability is preserved.

## 5.4 Numerical results

In this section we give a few numerical examples for the behaviour of the yield curve under price impact in the framework of short-rate affine models, which was described

in Section 5.2.4. In order to compute the cross price impact, we need the drift and the volatility of the zero-coupon bond. For the sake of simplicity, we assume the short rate is described by a Vasicek model (5.3.1)

$$dr(t) = k(\theta - r(t))dt + \sigma dW^{\mathbb{Q}}(t),$$

with  $k, \theta, \sigma$  positive parameters. Then, the drift and the diffusion coefficients of the unimpacted zero-coupon bond are given by (5.2.27):

$$\begin{aligned} \mu_T(t, r(t)) &= e^{-B(t,T)r(t)} \left[ \frac{\partial A}{\partial t} - A(t, T) \frac{\partial B}{\partial t} r(t) + \frac{1}{2} A(t, T) B^2(t, T) \sigma^2 \right. \\ &\quad \left. - A(t, T) B(t, T) k(\theta - r(t)) \right], \\ \sigma_T(t, r(t)) &= -\sigma B(t, T) A(t, T) e^{-B(t,T)r(t)}, \end{aligned}$$

where the functions  $A, B$  are given as in (5.3.8) and their derivatives are given by

$$\frac{\partial B}{\partial t} = -e^{-k(T-t)}, \quad \frac{\partial A}{\partial t} = A(t, T) \left[ \left( \theta - \frac{\sigma^2}{2k^2} \right) \left( \frac{\partial B}{\partial t} + 1 \right) - \frac{\sigma^2}{2k} B(t, T) \frac{\partial B}{\partial t} \right].$$

We can then plug all these quantities in equation (5.2.29) to determine the dynamics of the cross-impacted zero-coupon bond and therefore the corresponding impacted yield. We set the following values for the parameters in (5.3.1):

$$k = 0.20, \quad \theta = 0.10, \quad \sigma = 0.05, \quad r_0 = 0.01.$$

We consider zero-coupon bonds with maturities  $\mathbb{T} := \{1, 2, 5, 10, 15\}$  years and assume that an agent is trading on the bond with maturity  $T = 5$  years. All the other zero-coupon bonds experience cross price impact during the trading period. We fix the execution time horizon to be  $\tau = 10$  days. All bonds are simulated over the time interval  $[0, 9 \text{ months}]$ , discretized in  $N = 365$  subintervals with time step  $\Delta t = 1/365$ . The short rate  $r$  defined in (5.3.1) is simulated via Euler-Maruyama scheme. Since we are going to describe the average behaviour of the yield curve under market impact, we also set the number of Monte Carlo simulations to  $M = 10.000$ . As we shall see below in the detailed algorithm, for each realization of the short rate, we will have a corresponding impacted yield curve. The idea is then to plot the average of such curves.

For the sake of simplicity, we discuss the benchmark trading strategy

$$v_T(s) := \begin{cases} c, & \text{if } s \leq \tau \\ 0, & \text{otherwise} \end{cases} \quad (5.4.1)$$

with  $c$  some positive constant if we buy, negative if we sell. In our simulations we choose  $c = 2$ . The transient impact defined in (5.2.8) reads as

$$\Upsilon_T^v(t) = ye^{-\rho t} + \gamma e^{-\rho t} \int_0^t e^{\rho s} c \mathbb{1}_{s \leq \tau} ds, \quad (5.4.2)$$

where the parameters are set to

$$\rho = 2, \quad \gamma = 1, \quad y = 0.01.$$

The functions  $l, K$  introduced in (5.2.7) are assumed to be of the form

$$l(t, T) = \kappa \left(1 - \frac{t}{T}\right)^\alpha, \quad K(t, T) = \left(1 - \frac{t}{T}\right)^\beta$$

with  $\kappa \geq 0, \alpha, \beta \geq 1$ . In particular, we choose

$$\alpha = 1, \quad \beta = 1, \quad \kappa = 0.01.$$

Following (5.2.12), the price of the impacted bond in  $T = 5$ y is

$$\tilde{P}(t, T) = P(t, T) + \int_0^t J_T(s) ds,$$

where  $J_T$ , which was defined in (5.2.13), is specified to be

$$J_T(t) = -\frac{\kappa}{T} v_T(t) + \left(1 - \frac{t}{T}\right) [-\rho \Upsilon_T^v + v_T(t)] - \Upsilon_T^v(t)$$

The algorithm we implemented to simulate the impacted yield curve consists of the following steps.

- Step 1:* Simulate a path of the short rate  $r(t)$  given in (5.3.1) for  $t \in [0, 9 \text{ months}]$ .
- Step 2:* Compute the unimpacted zero-coupon bond price  $P(t, T)$  for the trading maturity  $T = 5$  years using equation (5.2.25) for  $t \in [0, 9 \text{ months}]$ .
- Step 3:* Compute the unimpacted yield  $Y(t, T)$  by plugging  $P(t, T)$  in (5.2.26) for  $t \in [0, 9 \text{ months}]$ .
- Step 4:* Compute the (directly) impacted zero-coupon bond  $\tilde{P}(t, T)$  using (5.2.12) for  $t \in [0, 9 \text{ months}]$ .
- Step 5:* Compute the (directly) impacted yield  $\tilde{Y}(t, T)$  by plugging  $\tilde{P}(t, T)$  into (5.2.30) for  $t \in [0, 9 \text{ months}]$ .
- Step 6:* For all other maturities  $S = 1, 2, 10, 15$  years, compute the cross impacted zero-coupon bond price  $\tilde{P}(t, S)$  using equation (5.2.29) for  $t \in [0, 9 \text{ months}]$ .
- Step 7:* Compute the cross impacted yield  $\tilde{Y}(t, S)$  by plugging  $\tilde{P}(t, S)$  into (5.2.30) for  $t \in [0, 9 \text{ months}]$ .
- Step 8:* Repeat these steps  $M = 10.000$  times and compare the average of  $Y(t, T)$  with the average of  $\tilde{Y}(t, T)$ .

In Figure 5.1 we visualize for all maturities the average classic yield  $\mathbb{E}[Y(t, T)]$  versus the average impacted yield  $\mathbb{E}[\tilde{Y}(t, T)]$  at times  $t = 5$  days (middle of trading),  $t = 11$  days (right after trading is ended) and  $t = 270$  days (after 9 months).

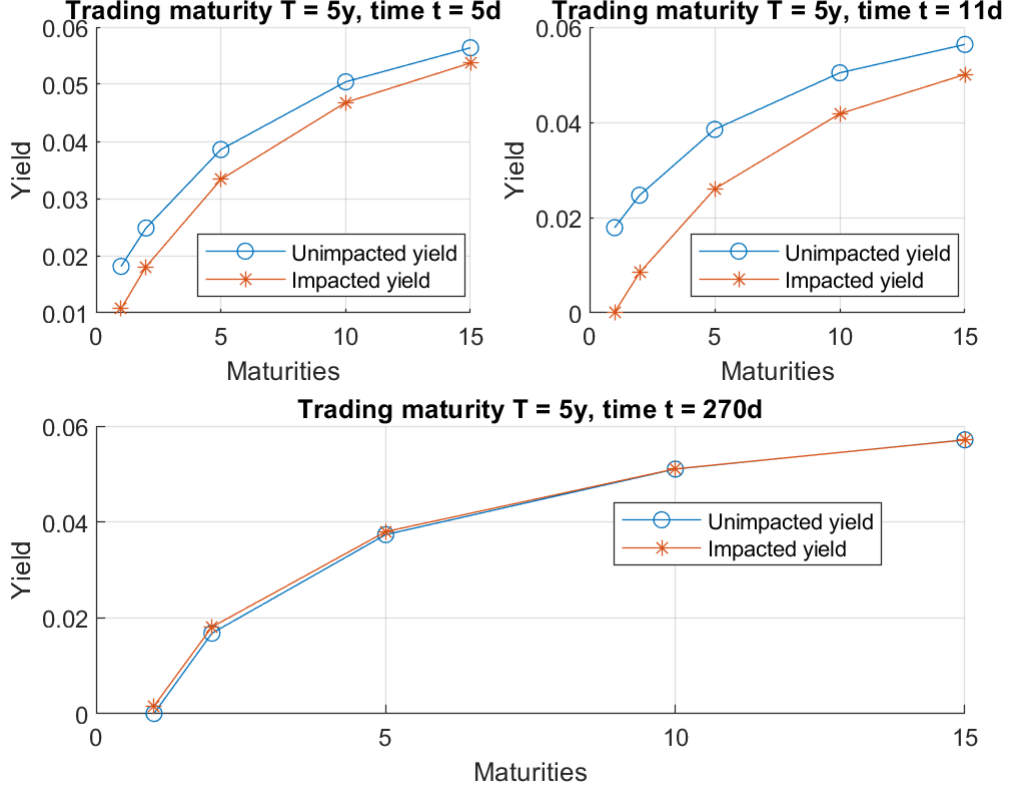


Figure 5.1: Trading zero-coupon bond with maturity  $T = 5$  years. Average unimpacted yield curve and average impacted yield curve in the middle of trading (top left panel), right after trading is concluded (top right panel) and after nine months (bottom panel).

In the top panels we see that the yield has decreased over all maturities as result of trading. This is consistent with the fact that a buy strategy of bonds pushes their prices up due to price impact, hence the yield decreases. Clearly, the almost parallel shift of the yield curve is a just a consequence of the very simple (constant) trading strategy we defined in equation (5.4.1). We expect to observe much more complicated behaviours when implementing more sophisticated strategies. In the bottom panel, instead, we observe that, roughly nine months after performing the trades, the two yield curves pretty much coincide. This is due to the transient component in the price impact model, which induces impacted yield curve to converge to its classic counterpart as time goes by. When analysing price impact due to zero-coupon bond trading, one aspect that certainly can't be ignored is the special nature of the assets

we are trading. Unlike what happens with stocks, the time evolution of zero-coupon bonds is constrained, specifically by the fact that they must reach value 1 at maturity. It therefore appears that two fundamental forces are in play: the intrinsic *pull to par* effect, which makes both the impacted and unimpacted bond price go to 1, hence the corresponding yields to 0, and the *price impact* effect, which induces the bond price to first increase (if we buy) or decrease (if we sell), and then revert back to its unimpacted value. Interestingly, when trading stocks, it will take the impacted asset forever to converge to the unimpacted counterpart, as the transient impact converges to 0 as time  $t$  goes to infinity. When trading bonds, though, this convergence occurs in finite time. In order to better understand the role played by price impact, in Figure 5.2 we compare directly the behaviour of the impacted bond  $\tilde{P}(t, T)$  and of the classic bond  $P(t, T)$  for different maturities  $T$ .

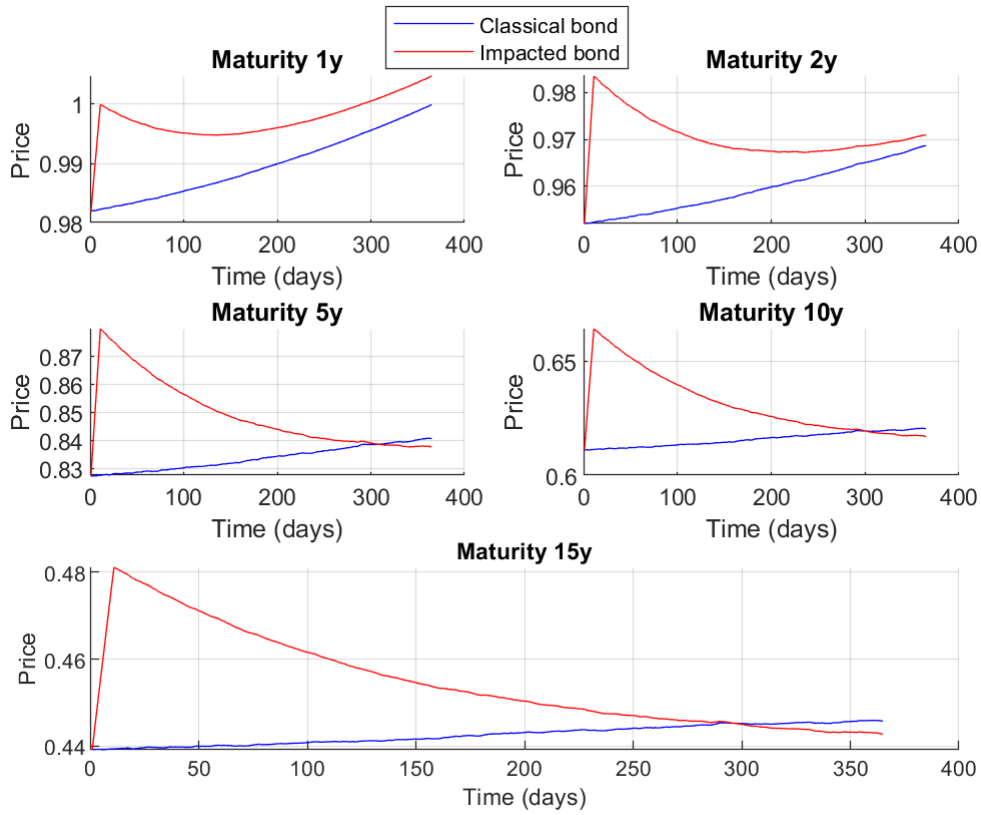


Figure 5.2: Trading bond with maturity  $T = 5$  years. Averaged impacted zero-coupon bond vs averaged classic zero-coupon bond for maturities 1 year (top left panel), 2 years (top right panel), 5 years (middle left panel), 10 years (middle right panel), 15 (bottom panel). All curves are observed over the interval  $[0, 1]$  year.

We observe that, over one year, the pull-to-par effect is somehow stronger than the transient impact effect in bonds with short maturity ( $T = 1, 2$ ). By this, we mean

that the unimpacted and impacted bonds meet at, or very close to, maturity.<sup>1</sup> For bonds with long maturity ( $T = 5, 10, 15$ ), instead, the transient effect is prominent. This causes the impacted bond curve and the unimpacted bond curve to cross each other significantly before their maturity. In fact, we can numerically compute the first instant the two curves meet and we observe that the longer the maturity, the sooner this happens. This is illustrated in Figure 5.3.

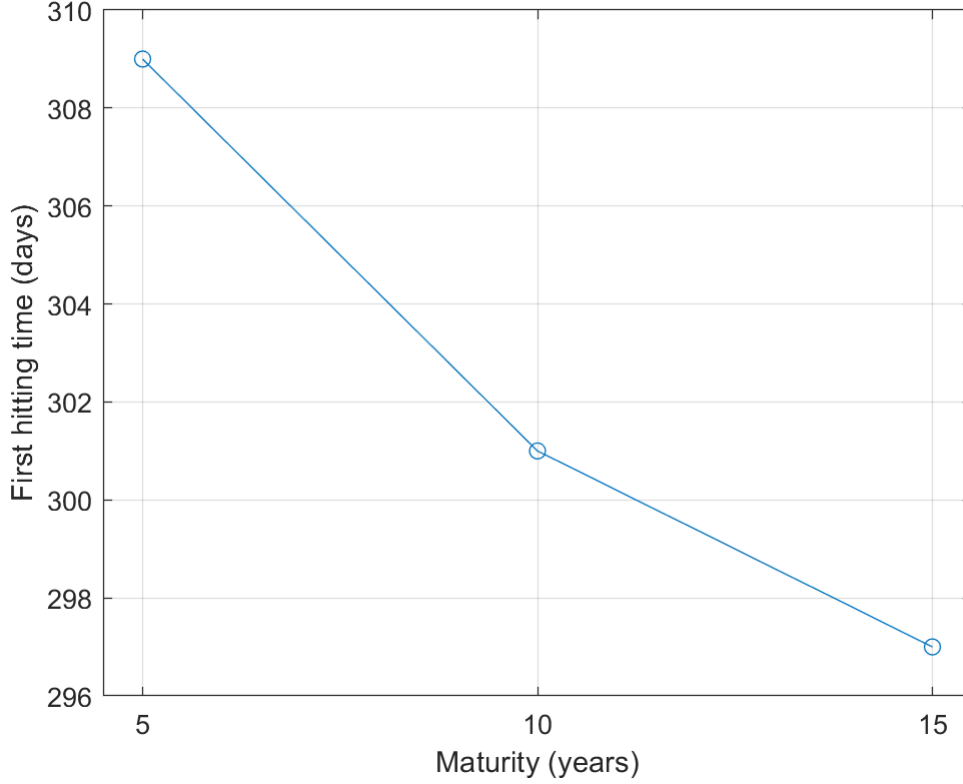


Figure 5.3: Trading bond with maturity  $T = 5$  years. First instant (in days) impacted bond curve and unimpacted bond curve cross for maturities 5, 10, 15 years. All curves are observed over the interval  $[0, 1]$  year].

The interplay between the cross price impact effect, averaged over 10.000 realizations, and the pull to par effect is demonstrated in Figure 5.4 for the price of a zero-coupon bond with maturity  $S = 1$  year when trading a bond with maturity  $T = 5$  years. Trading takes place on the first 10 days of the year, while the time scale

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<sup>1</sup>It can be observed that the price of the cross-impacted zero-coupon bond with maturity  $S = 1$  year is not 1 at expiration, as it should be, but slightly higher (top left panel). This is not a numerical error, but rather a consequence of our model not being able to ensure the cross-impacted bonds reach value precisely 1 at their respective maturities. See Remark 5.2.12.

in the graph is of one year. We illustrate this effect for various values of the transient impact parameter  $\rho$  in equation (5.4.2).

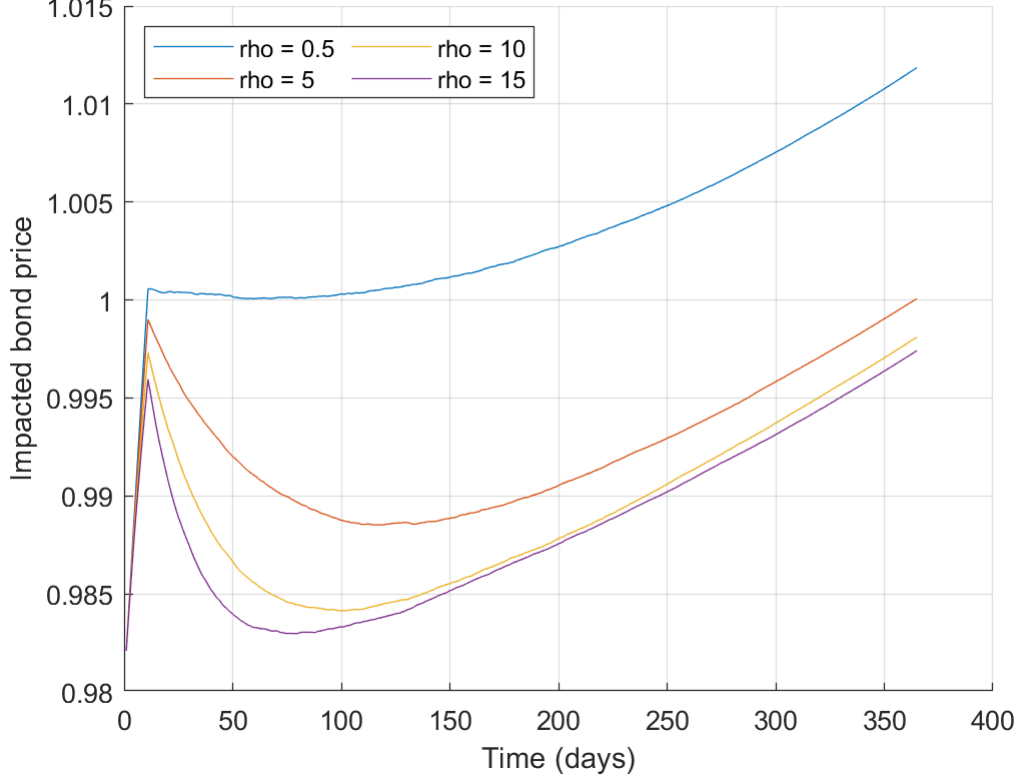


Figure 5.4: Averaged cross price impact effect vs. pull to par effect over 10.000 realizations is demonstrated for the price of a bond with maturity  $S = 1$  year when trading a bond with maturity  $T = 5$  years, for various values of  $\rho$ . Trading takes place on the first 10 days of the year, while the time scale in the graph is of one year.

It can be observed that the higher  $\rho$ , the more aggressively the price is "pulled down" close to the original price before the trades. At the beginning, far from maturity, the transient impact component dominates and the price decreases. After some time, though, the bond intrinsic nature takes over and the price starts to increase.

Another phenomenon which is revealed in our framework is the interplay between the mean reversion of the short rate model and the price impact. Recall that in Section 5.3.1 we found that the mean reversion speed  $k$  under the measure  $\mathbb{Q}$  and the mean reversion  $\tilde{k}$  under the price-impacted measure  $\tilde{\mathbb{Q}}$  are linked by (5.3.4) as follows

$$\tilde{k} = k - \sigma(\tilde{\lambda} - \lambda),$$

with  $\tilde{\lambda}, \lambda$  representing the impacted market price of risk and the classic market price of risk respectively. We stress that the higher  $k$ , the faster the short rate  $r$  under

$\mathbb{Q}$  and its counterpart under  $\tilde{\mathbb{Q}}$  converge to their respective stationary distributions. At the same time, since the variance of the stationary distribution is  $\sigma^2/(2k)$ , large values of  $k$  reduce the overall variance of the model, thereby making the two types of rates that we consider closer to each other. This, in turn, implies that after a long time ( $T = 10, 15$  years) the zero-coupon bond  $P$  and impacted zero-coupon bond  $\tilde{P}$ , hence their yields, will be closer to each other. Conversely, if  $k$  is small, the two short rates are quite far from each other and the overall variance of the model is large. Furthermore, looking again at (5.3.4), we notice that the larger  $k$ , the less significant the impact component  $-\sigma(\tilde{\lambda} - \lambda)$ , and vice versa. In a way, the speed of mean reversion works in an opposite direction to the price impact. We demonstrate this in Figure 5.5 for  $k = 0.01$  (top panel) and for  $k = 0.20$  (bottom panel). As above, we trade the zero-coupon bond with maturity  $T = 5$  years, trading occurs for the first 10 days and the yields are observed after 9 months. The difference in behaviour is evident for long maturities ( $T = 5, 10, 15$ ). While in the bottom panel unimpacted yield and impacted yield are really close to each other (as in Figure (5.1), right panel), in the top panel the distance between the two yields is rather significant.

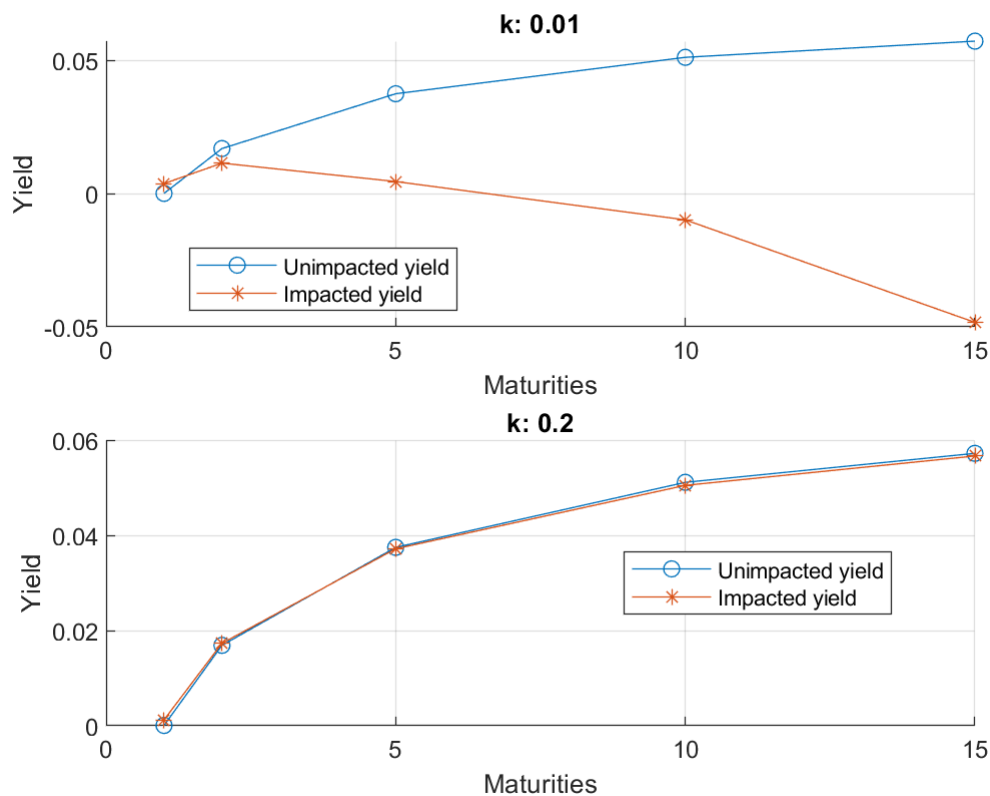


Figure 5.5: Impacted and unimpacted yield curves for  $k = 0.01$  (top panel) and for  $k = 0.20$  (bottom panel) when trading zero-coupon bond with maturity  $T = 5$  years. Trading occurs during the first 10 days. Yield curves are observed after nine months.

## 5.5 Optimal execution of bonds in presence of price impact

In this section we consider a problem of an agent who tries to liquidate a large inventory of  $T$ -bonds within a finite time horizon  $[0, \tau]$  where  $\tau < T$ . We assume that the agent's transactions create both temporary and transient price impact and that the performance of the agent is measured by a revenue-cost functional that captures the transaction costs which result by price impact, and the risk of holding inventory for long time periods. Our optimal execution framework is closely related to the framework which was proposed for execution of equities in Section 5.2.1 of [135]. The main difference between the two frameworks is that in our framework the price impact has to vanish at the bond's maturity in order to satisfy the boundary condition  $\tilde{P}(T, T) = 1$ .

Let  $T > 0$  denote the bond's maturity. We assume that the unimpacted bond price  $P(\cdot, T)$  is given by (5.2.2) and we consider the canonical decomposition  $P(\cdot, T) = A(\cdot, T) + \bar{M}(\cdot, T)$ , where

$$A(t, T) := \int_0^t \mu_T(s, r(s)) ds, \quad 0 \leq t \leq T,$$

is a predictable finite-variation process and

$$\bar{M}(t, T) := \int_0^t \sigma_T(s, r(s)) dW^\mathbb{P}(s), \quad 0 \leq t \leq T,$$

local martingale. We assume that the coefficients  $\sigma_T, \mu_T$  in (5.2.2) are such that we have

$$\mathbb{E}[\langle \bar{M}(\cdot, T) \rangle_\tau] + \mathbb{E} \left[ \left( \int_0^\tau |dA(\cdot, T)| \right)^2 \right] < \infty. \quad (5.5.1)$$

In this case we say that a bond price  $\{P(t, T)\}_{t \in [0, T]}$  is in  $\mathcal{H}^2$ .

The initial position of the agent's inventory is denoted by  $x > 0$  and the number of shares the agent holds at time  $t \in [0, \tau]$  is given by

$$X^{v_T}(t) \triangleq x - \int_0^t v_T(s) ds \quad (5.5.2)$$

where  $\{v_T(t)\}_{t \in [0, \tau]}$  denotes the trading speed. We say that the trading speed is admissible if it belongs to the following class of admissible strategies

$$\mathcal{A} \triangleq \left\{ v_T : v_T \text{ progressively measurable s.t. } \mathbb{E} \left[ \int_0^\tau v_T^2(s) ds \right] < \infty \right\}. \quad (5.5.3)$$

We assume that the trader's trading activity causes price impact on the bond's price as described by  $\{\tilde{P}(t, T)\}_{t \in [0, T]}$  in (5.2.7).

As in Section 2 of [135], we now suppose that the trader's optimal trading objective is to unwind her initial position  $x > 0$  in the presence of temporary and transient price impact through maximizing the following performance functional

$$\begin{aligned} \mathcal{J}(v) := \mathbb{E} \Bigg[ & \int_0^\tau \left( P(t, T) - K(t, T) \Upsilon_T^v(t) \right) v_T(t) dt - \int_0^\tau l(t, T) v_T^2(t) dt + X_T^v(\tau) P(\tau, T) \\ & - \phi \int_0^\tau (X_T^v(t))^2 dt - \varrho (X_T^v(\tau))^2 \Bigg]. \end{aligned} \quad (5.5.4)$$

The first, second and third terms in  $\mathcal{J}$  represent the trader's terminal wealth, meaning the final cash position including the accrued trading costs induced by temporary and transient price impact, as well as the remaining final risky asset position's book value. The fourth and fifth terms, instead, account for the penalties  $\phi, \varrho > 0$  on the trader's running penalty (i.e. the risk aversion term) and the penalty of holding any terminal inventory, respectively.

Since  $T$  is fixed, for the sake of readability we will omit the subscripts  $T$  for the rest of this section. The main result of this section is the derivation of the unique optimal admissible strategy, namely

$$\mathcal{J}(v) \rightarrow \max_{v \in \mathcal{A}}. \quad (5.5.5)$$

and exhibiting an explicit expression for the optimal trading strategy. We define

$$A(t) := \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & -\rho & \gamma & 0 \\ -2\phi\Lambda(t) & \rho K(t, T)\Lambda(t) & 0 & \Lambda'(t) + \rho\Lambda(t) \\ 0 & -\Lambda'(t)K(t, T) - \Lambda(t)\partial_t K(t, T) & K(t, T)\gamma & \rho \end{pmatrix}, \quad (5.5.6)$$

where

$$\Lambda(t) := \frac{1}{2l(t, T)}. \quad (5.5.7)$$

Note that  $\Lambda(t)$  is well defined for  $0 \leq t \leq \tau$  since  $l(t, T) > 0$  on this interval by (5.2.9). Let  $\Phi$  be the fundamental solution of the matrix-valued ordinary differential

equation

$$\begin{aligned}\frac{d}{dt}\Phi(t) &= A(t)\Phi(t), \\ \Phi(0) &= \text{Id}.\end{aligned}\tag{5.5.8}$$

Let us define the matrix

$$\Psi(t, \tau) := \Phi^{-1}(\tau)\Phi(t).\tag{5.5.9}$$

We also define the vector  $G$ :

$$\begin{aligned}G^1(t, \tau) &:= \frac{\varrho}{l(\tau, T)}\Psi^{11}(t, \tau) - \frac{K(\tau, T)}{2l(\tau, T)}\Psi^{21}(t, \tau) - \Psi^{31}(t, \tau), \\ G^2(t, \tau) &:= \frac{\varrho}{l(\tau, T)}\Psi^{12}(t, \tau) - \frac{K(\tau, T)}{2l(\tau, T)}\Psi^{22}(t, \tau) - \Psi^{32}(t, \tau), \\ G^3(t, \tau) &:= \frac{\varrho}{l(\tau, T)}\Psi^{13}(t, \tau) - \frac{K(\tau, T)}{2l(\tau, T)}\Psi^{23}(t, \tau) - \Psi^{33}(t, \tau), \\ G^4(t, \tau) &:= \frac{\varrho}{l(\tau, T)}\Psi^{14}(t, \tau) - \frac{K(\tau, T)}{2l(\tau, T)}\Psi^{24}(t, \tau) - \Psi^{34}(t, \tau).\end{aligned}\tag{5.5.10}$$

Next, we define the process

$$\Gamma^{\hat{v}}(t) := \frac{\Lambda'(t)}{\Lambda(t)} \left( P(t, T) + \tilde{M}(t) - 2\phi \int_0^t X^{\hat{v}}(u) du \right),\tag{5.5.11}$$

where  $\tilde{M}$  is the square integrable martingale

$$\tilde{M}(s) := \mathbb{E}_s \left[ 2\phi \int_0^\tau X^{\hat{v}}(u) du + 2\varrho X^{\hat{v}}(\tau) - P(\tau, T) \right],\tag{5.5.12}$$

and  $\mathbb{E}_t$  denotes the expectation conditioned on the filtration  $\mathcal{F}_t$  for all  $t \in [0, \tau]$ . Finally we define the following functions on  $0 \leq t \leq \tau$ ,

$$\begin{aligned}v_0(t, \tau) &:= \left( 1 - \frac{G^4(t, \tau)\Psi^{43}(t, \tau)}{G^3(t, \tau)\Psi^{44}(t, \tau)} \right)^{-1}, \\ v_1(t, \tau) &:= \left( \frac{G^4(t, \tau)\Psi^{41}(t, \tau)}{G^3(t, \tau)\Psi^{44}(t, \tau)} - \frac{G^1(t, \tau)}{G^3(t, \tau)} \right), \\ v_2(t, \tau) &:= \left( \frac{G^4(t, \tau)\Psi^{42}(t, \tau)}{G^3(t, \tau)\Psi^{44}(t, \tau)} - \frac{G^2(t, \tau)}{G^3(t, \tau)} \right), \\ v_3(t, \tau) &:= \frac{G^4(t, \tau)}{G^3(t, \tau)}.\end{aligned}\tag{5.5.13}$$

In order for the optimal strategy to be well defined, we will need additional assumptions. Note that if  $l, K$  are positive constants these assumptions translate to Assumption 3.1 and Lemma 5.5 in [135].

**Assumption 5.5.1.** *We assume that the following hold:*

(A.1)

$$\inf_{0 \leq t \leq \tau} |G^4(t, \tau) \Psi^{43}(t, \tau) - G^3(t, \tau) \Psi^{44}(t, \tau)| > 0,$$

(A.2)

$$\sup_{0 \leq t \leq \tau} |\Psi^{4j}(t, \tau)| < \infty, \quad \sup_{0 \leq t \leq \tau} |G^j(t, \tau)| < \infty, \quad j \in \{1, 2, 3, 4\}$$

(A.3)

$$\inf_{0 \leq t \leq \tau} |\Psi^{44}(t, \tau)| > 0, \quad \inf_{0 \leq t \leq \tau} |G^3(t, \tau)| > 0.$$

**Remark 5.5.2.** *At this point we stress the fact that the conditions in Assumption 5.5.1 are not very general, however the purpose of this section is to show how to incorporate optimal execution into the impacted bonds framework. Future work may improve the theoretical results on this topic.*

Next we present the main result of this section, which derives the unique optimal trading speed.

**Theorem 5.5.3** (Optimal trading strategy). *Under Assumption 5.5.1, there exists a unique optimal strategy  $\hat{v} \in \mathcal{A}$  which maximises (5.5.5) and it is given by the following feedback form*

$$\begin{aligned} v(t) = v_0(t, \tau) & \left( v_1(t, \tau) X^v(t) + v_2(t, \tau) \Upsilon^v(t) \right. \\ & + v_3(t, \tau) \mathbb{E}_t \left[ \int_t^\tau \frac{\Lambda(s) \Psi^{43}(s, \tau)}{\Psi^{44}(t, \tau)} (\mu(s) + \Gamma^v(s)) ds \right] \\ & \left. - \mathbb{E}_t \left[ \int_t^\tau \Lambda(s) \frac{G^3(s, \tau)}{G^3(t, \tau)} (\mu(s) + \Gamma^v(s)) ds \right] \right), \end{aligned} \quad (5.5.14)$$

for all  $t \in (0, \tau)$ .

The proof Theorem 5.5.3 is given in Section 5.7.

## 5.6 Proofs of the results from Section 5.2

*Proof of Theorem 5.2.4.* We adapt the argument by Bjork in Section 3.2 of [24] to our case. We fix two maturities  $T$  and  $S$ , and we consider a portfolio  $V$  consisting of  $S$ -bonds and  $T$ -bonds. We further assume that both bonds are traded with admissible trading speeds  $v_T$  and  $v_S$  which correspond by (5.2.13) to impact densities  $J_T$  and  $J_S$ .

From (5.2.2) and (5.2.12) we can write the dynamics of the impacted bonds as follows:

$$\begin{aligned} d\tilde{P}(t, T) &= \mu_T(t, r(t))dt + J_T(t)dt + \sigma_T(t, r(t))dW^\mathbb{P}(t), \\ d\tilde{P}(t, S) &= \mu_S(t, r(t))dt + J_S(t)dt + \sigma_S(t, r(t))dW^\mathbb{P}(t). \end{aligned} \quad (5.6.1)$$

Let  $\tilde{h}_T, \tilde{h}_S$  by locally bounded predictable processes representing the weights of the  $T$  and  $S$  bonds, respectively. We denote by  $\tilde{V}(t)$  the portfolio value process, i.e.

$$\tilde{V}(t) \equiv \tilde{V}(t; \tilde{h}) := \tilde{h}_T(t)\tilde{P}(t, T) + \tilde{h}_S(t)\tilde{P}(t, S).$$

Since, by assumption, the impacted-portfolio is self-financing, it holds at any time  $t$  (see Definition 5.2.2)

$$d\tilde{V}(t; \tilde{h}) = \tilde{h}_T(t)d\tilde{P}(t, T) + \tilde{h}_S(t)d\tilde{P}(t, S).$$

It is convenient to define the relative (impacted) weights

$$\alpha_{T_i}(t) := \frac{\tilde{h}_{T_i}(t)\tilde{P}(t, T_i)}{\tilde{V}(t; \tilde{h})}, \quad T_i \in \{T, S\},$$

We conclude that if the impacted portfolio is self financing, then

$$\frac{d\tilde{V}(t)}{\tilde{V}(t)} = \alpha_T(t)\frac{d\tilde{P}(t, T)}{\tilde{P}(t, T)} + \alpha_S(t)\frac{d\tilde{P}(t, S)}{\tilde{P}(t, S)}. \quad (5.6.2)$$

In order to ease the notation, we suppress the dependence on  $r(t)$  in the drift and volatility. Substituting the dynamics (5.6.1) into (5.6.2), we have

$$\begin{aligned} \frac{d\tilde{V}(t)}{\tilde{V}(t)} &= \frac{\alpha_T(t)}{\tilde{P}(t, T)}(\mu_T(t) - J_T(t))dt + \frac{\alpha_S(t)}{\tilde{P}(t, S)}(\mu_S(t) - J_S(t))dt + \\ &\quad + \left( \alpha_S(t)\frac{\sigma_S(t)}{\tilde{P}(t, S)} + \alpha_T(t)\frac{\sigma_T(t)}{\tilde{P}(t, T)} \right) dW^\mathbb{P}(t). \end{aligned} \quad (5.6.3)$$

At this point, we choose the relative weights so that the diffusive part of the equation above vanishes, that is,

$$\begin{aligned} \alpha_T(t) + \alpha_S(t) &= 1, \\ \alpha_T(t)\frac{\sigma_T(t)}{\tilde{P}(t, T)} + \alpha_S(t)\frac{\sigma_S(t)}{\tilde{P}(t, S)} &= 0. \end{aligned} \quad (5.6.4)$$

Solving this system gives

$$\begin{aligned} \alpha_S(t) &= \frac{\sigma_T(t)/\tilde{P}(t, T)}{\sigma_T(t)/\tilde{P}(t, T) - \sigma_S(t)/\tilde{P}(t, S)}, \\ \alpha_T(t) &= -\frac{\sigma_S(t)/\tilde{P}(t, S)}{\sigma_T(t)/\tilde{P}(t, T) - \sigma_S(t)/\tilde{P}(t, S)}. \end{aligned} \quad (5.6.5)$$

Notice that the above expressions are well defined. Indeed, if the denominator was approaching zero, then the sum of the two weights would be zero and this would contradict (5.6.4). Next, we substitute (5.6.5) into (5.6.3). Following again Bjork's argument, we use the fact that our impacted portfolio is locally risk-free (as in Definition 5.2.3) by assumption and deduce the following relationship must hold:

$$\begin{aligned} \frac{\mu_T(t) - J_T(t)}{\tilde{P}(t, T)} \left( -\frac{\sigma_S(t)/\tilde{P}(t, S)}{\sigma_T(t)/\tilde{P}(t, T) - \sigma_S(t)/\tilde{P}(t, S)} \right) + \\ + \frac{\mu_S(t) - J_S(t)}{\tilde{P}(t, S)} \left( \frac{\sigma_T(t)/\tilde{P}(t, T)}{\sigma_T(t)/\tilde{P}(t, T) - \sigma_S(t)/\tilde{P}(t, S)} \right) = r(t). \end{aligned}$$

Multiplying both sides by the term

$$\frac{\sigma_T(t)}{\tilde{P}(t, T)} - \frac{\sigma_S(t)}{\tilde{P}(t, S)},$$

we obtain

$$\left( \frac{\mu_S(t) - J_S(t)}{\tilde{P}(t, S)} - r(t) \right) \left( \frac{\sigma_T(t)}{\tilde{P}(t, T)} \right) = \left( \frac{\mu_T(t) - J_T(t)}{\tilde{P}(t, T)} - r(t) \right) \left( \frac{\sigma_S(t)}{\tilde{P}(t, S)} \right).$$

It follows that,

$$\left( \frac{\mu_S(t) - J_S(t)}{\tilde{P}(t, S)} - r(t) \right) \left( \frac{\tilde{P}(t, S)}{\sigma_S(t)} \right) = \left( \frac{\mu_T(t) - J_T(t)}{\tilde{P}(t, T)} - r(t) \right) \left( \frac{\tilde{P}(t, T)}{\sigma_T(t)} \right),$$

and rearranging we deduce

$$\frac{\mu_S(t) - J_S(t) - r(t)\tilde{P}(t, S)}{\sigma_S(t)} = \frac{\mu_T(t) - J_T(t) - r(t)\tilde{P}(t, T)}{\sigma_T(t)}. \quad (5.6.6)$$

Notice that the left hand side of (5.6.6) depends on  $S$  but not on  $T$ , while the right hand side of (5.6.6) depends on  $T$ , but not on  $S$ . Since  $S$  and  $T$  are arbitrary, we conclude that both sides of (5.6.6) depend only on  $t$  and  $r(t)$ .  $\square$

*Proof of Theorem 5.2.9.* The proof is similar to the proof of Proposition 1.1 in Chapter 1.2 of [24] (see also Harrison and Kreps [106] Theorem 2 and relative Corollary in Section 3 and Harrison and Pliska [107], Theorem 2.7, Section 2). For the sake of completeness, we give the proof here, translated in our price impact environment. Let  $T < +\infty$  be some finite maturity. Let  $\tilde{h}$  be an arbitrage portfolio and  $\tilde{V}$  the corresponding liquidation value process. Then, given the positivity of the discount factor (bank account) defined in (5.2.19) and the equivalence between the real world measure  $\mathbb{P}$  and the impacted risk neutral measure  $\tilde{\mathbb{Q}}$ , we immediately deduce

$$\tilde{\mathbb{Q}} \left( \frac{\tilde{V}(T)}{B(T)} \geq 0 \right) = 1, \quad \tilde{\mathbb{Q}} \left( \frac{\tilde{V}(T)}{B(T)} > 0 \right) > 0. \quad (5.6.7)$$

Moreover we have

$$0 = \tilde{V}(0) = \frac{\tilde{V}(0)}{B(0)} = \mathbb{E}^{\tilde{\mathbb{Q}}} \left[ \frac{\tilde{V}(T)}{B(T)} \right] > 0,$$

where the first equality comes from the definition of arbitrage, the second from the fact that  $B(0) = 1$  and the third from the fact that  $\tilde{V}(t)/B(t)$  is a martingale under  $\tilde{\mathbb{Q}}$ . Finally, the positivity of the expectation is a consequence of (5.6.7). We get a contradiction so we conclude that absence of arbitrage must hold.  $\square$

*Proof of Proposition 5.2.14.* We start by writing the impacted forward rate defined in (5.2.37) as

$$\tilde{f}(t, T) = f(t, T) + \int_0^t J^f(s, T) ds,$$

where  $f$  represents the unimpacted forward rate (see e.g. Chapter 6, of [86]) and we used the assumption  $\tilde{f}(0, t) = f(0, t)$ . Then, using (5.2.39), we deduce

$$\begin{aligned} \tilde{P}(t, T) &= \exp \left\{ - \int_t^T \tilde{f}(t, u) du \right\} \\ &= \exp \left\{ - \int_t^T f(t, u) du - \int_t^T J^f(s, u) du \right\} \\ &= P(t, T) \exp \left\{ - \int_t^T J^f(s, u) du \right\}, \end{aligned} \tag{5.6.8}$$

where  $P$  denotes the unimpacted zero-coupon bond and we used the well known relation between  $P(t, T)$  and  $f(t, T)$ . From (5.2.7) and (5.2.11) we have

$$\tilde{P}(t, T) = P(t, T) - I_T(t). \tag{5.6.9}$$

Substituting this last expression into (5.6.8) and rearranging, we obtain

$$\exp \left\{ - \int_t^T J^f(s, u) du \right\} = \frac{\tilde{P}(t, T)}{\tilde{P}(t, T) + I_T(t)}.$$

By taking logarithms on both sides yields and using (5.6.9) we get

$$\begin{aligned} \int_t^T J^f(s, u) du &= - \log \left( \frac{\tilde{P}(t, T)}{\tilde{P}(t, T) + I_T(t)} \right) \\ &= - \log \left( 1 - \frac{I_T(t)}{P(t, T)} \right). \end{aligned}$$

Differentiating with respect to maturity, we get (5.2.43).  $\square$

*Proof of Theorem 5.2.16 .* Let  $B(t)$  be the bank account defined in (5.2.19) and let the impacted zero-coupon bond  $\tilde{P}$  follow the dynamics (5.2.40). By applying Ito's formula to the discounted impacted zero-coupon bond price, we immediately find

$$d\frac{\tilde{P}(t, T)}{B(t)} = \frac{\tilde{P}(t, T)}{B(t)}\tilde{b}(t, T)dt + \frac{\tilde{P}(t, T)}{B(t)}\nu(t, T)dW^{\mathbb{P}}(t),$$

with  $\tilde{b}$  and  $\nu$  defined as in (5.2.41). Changing measure from the real world  $\mathbb{P}$  to the impacted risk neutral  $\tilde{\mathbb{Q}}$  as in (5.2.44) implies

$$d\frac{\tilde{P}(t, T)}{B(t)} = \frac{\tilde{P}(t, T)}{B(t)}\left(\tilde{b}(s, T) + \nu(t, T)\tilde{\gamma}(t)\right)dt + \frac{\tilde{P}(t, T)}{B(t)}\nu(t, T)dW^{\tilde{\mathbb{Q}}}(t).$$

Therefore, we clearly see that

$$\frac{\tilde{P}(t, T)}{B(t)} \text{ local martingale under } \tilde{\mathbb{Q}} \iff \tilde{b}(s, T) = -\nu(t, T)\tilde{\gamma}(t).$$

This is our new HJM condition. Notice also that since both functions  $\nu$  and  $\tilde{b}$  are continuous with respect to  $T$ , this condition is equivalent to saying that the impacted measure  $\tilde{\mathbb{Q}}$  is an equivalent local martingale measure. Following Theorem 6.1 in [86], Chapter 6, we can plug in the explicit expression for  $\tilde{b}$  in (5.2.41) and write the HJM condition (5.2.45) as

$$-\int_s^T \alpha(s, u)du - \int_s^T J^f(s, u)du + \frac{1}{2}\nu^2(s, T) = -\nu(t, T)\tilde{\gamma}(t). \quad (5.6.10)$$

Differentiating both sides with respect to the maturity  $T$  yields the equation

$$-\alpha(t, T) + \sigma(t, T) \int_t^T \sigma(t, u)du - J^f(t, T) = \sigma(t, T)\tilde{\gamma}(t),$$

that is

$$\alpha(t, T) + J^f(t, T) = \sigma(t, T) \int_t^T \sigma(t, u)du - \sigma(t, T)\tilde{\gamma}(t). \quad (5.6.11)$$

Substituting (5.6.11) in the dynamics of the forward rate (5.2.37) and using Girsanov yields (5.2.46). Using (5.2.45) along with (5.2.40) and Girsanov gives (5.2.47).  $\square$

## 5.7 Proof of Theorem 5.5.3

The uniqueness of the optimal strategy follows by a standard convexity argument for the performance functional (5.5.4). Hence we only need to derive the optimal strategy.

We start by deriving a system of coupled forward-backward stochastic differential equations (FBSDEs) which is satisfied by the solution to the stochastic control problem.

**Lemma 5.7.1** (FBSDE system). *A control  $\hat{v} \in \mathcal{A}$  solves the optimization problem (5.5.5) if and only if the processes  $(X^{\hat{v}}, \Upsilon^{\hat{v}}, \hat{v}, Z^{\hat{v}})$  satisfy the coupled forward-backward stochastic differential equations*

$$\left\{ \begin{array}{l} dX^{\hat{v}}(t) = -\hat{v}(t)dt, \quad X^{\hat{v}}(0) = x, \\ d\Upsilon^{\hat{v}}(t) = -\rho\Upsilon^{\hat{v}}(t)dt + \gamma\hat{v}(t)dt, \quad \Upsilon^{\hat{v}}(0) = y, \\ d\hat{v}(t) = \Lambda(t)dP(t, T) - 2\Lambda(t)\phi X^{\hat{v}}(t)dt \\ \quad + \Upsilon^{\hat{v}}(t) [-\Lambda'(t)K(t, T) - \Lambda(t)\partial_t K(t, T) + \rho K(t, T)\Lambda(t)] dt \\ \quad + Z^{\hat{v}}(t) [\Lambda'(t) + \rho\Lambda(t)] dt + \Lambda(t)\Gamma^{\hat{v}}(t)dt + dM(t), \\ \quad \hat{v}(\tau) = \frac{\varrho}{l(\tau, T)}X^{\hat{v}}(\tau) - \frac{K(\tau, T)}{2l(\tau, T)}\Upsilon^{\hat{v}}(\tau), \\ dZ^{\hat{v}}(t) = (\rho Z^{\hat{v}}(t) + K(t, T)\gamma\hat{v}(t)) dt + dN(t), \quad Z^{\hat{v}}(\tau) = 0, \end{array} \right. \quad (5.7.1)$$

for two suitable square integrable martingales  $M = (M(\cdot, T))_{0 \leq t \leq \tau}$  and  $N = (N(\cdot, T))_{0 \leq t \leq \tau}$ , where the  $\Lambda, \Gamma^{\hat{v}}$  and  $\tilde{M}$  are defined in (5.5.7), (5.5.11) and (5.5.12) respectively.

*Proof.* The proof follows the same lines as Lemmas 5.1 and 5.2 in [135]. Since for all  $v \in \mathcal{A}$  the map  $v \rightarrow \mathcal{J}(v)$  is strictly concave, we can study the unique critical point at which the Gateaux derivative of  $\mathcal{J}$ , which is defined as

$$\langle \mathcal{J}'(v), \alpha \rangle := \lim_{\epsilon \rightarrow 0} \frac{\mathcal{J}(v + \epsilon\alpha) - \mathcal{J}(v)}{\epsilon},$$

is equal to 0 for any  $\alpha \in \mathcal{A}$ . This derivative can be computed analytically as follows. Let  $\epsilon > 0$  and  $v, \alpha \in \mathcal{A}$ . Since for all  $t \in [0, \tau]$ ,

$$\begin{aligned} X^{v+\epsilon\alpha}(t) &= x - \int_0^t (v(s) + \epsilon\alpha(s))ds = X^v(t) - \epsilon \int_0^t \alpha(s)ds \\ \Upsilon^{v+\epsilon\alpha}(t) &= \Upsilon^v(t) + \epsilon\gamma \int_0^t e^{-\rho(t-s)}\alpha(s)ds, \end{aligned} \quad (5.7.2)$$

From (5.5.4) and (5.7.2) we have

$$\begin{aligned}
& \mathcal{J}(v + \epsilon\alpha) = \\
& = \mathbb{E} \left[ \int_0^\tau \left( P(t, T) - K(t, T)\Upsilon^v(t) - K(t, T)\epsilon\gamma \int_0^t e^{-\rho(t-s)}\alpha(s)ds \right) (v(t) + \epsilon\alpha(t)) dt \right. \\
& \quad - \int_0^\tau l(t, T)v^2(t) + \epsilon^2 l(t, T)\alpha_t^2 + 2l(t, T)v(t)\epsilon\alpha(t)dt + X^v(\tau)P(\tau, T) \\
& \quad - \epsilon P(\tau, T) \int_0^\tau \alpha(s)ds - \phi \int_0^\tau (X^v(t))^2 + \epsilon \left( \int_0^t \alpha(s)ds \right)^2 - 2X^v(t)\epsilon \int_0^t \alpha(s)dsdt \\
& \quad \left. - \varrho \left( (X^v(\tau))^2 + \epsilon^2 \left( \int_0^\tau \alpha(s)ds \right)^2 - 2X^v(\tau)\epsilon \int_0^\tau \alpha(s)ds \right) \right].
\end{aligned}$$

It follows that

$$\begin{aligned}
\mathcal{J}(v + \epsilon\alpha) - \mathcal{J}(v) &= \epsilon \mathbb{E} \left[ \int_0^\tau (P(\tau, T) - K(t, T)\Upsilon^v(t)) \alpha(t)dt \right. \\
& \quad - \int_0^\tau K(t, T)v(t) \int_0^t \gamma e^{-\rho(t-s)}\alpha(s)dsdt - 2 \int_0^\tau l(t, T)v(t)\alpha(t)dt \\
& \quad + 2\phi \int_0^\tau X^v(t) \int_0^t \alpha(s)dsdt + 2\varrho X^v(\tau) \int_0^\tau \alpha(s)ds - P(\tau, T) \int_0^\tau \alpha(s)ds \left. \right] \\
& \quad + \epsilon^2 \mathbb{E} \left[ \gamma \int_0^\tau K(t, T)\alpha(t) \int_0^t e^{-\rho(t-s)}\alpha(s)dsdt - \int_0^\tau l^2(t, T)\alpha^2(t)dt \right. \\
& \quad \left. - \phi \int_0^\tau \left( \int_0^t \alpha(s)ds \right)^2 dt - \varrho \left( \int_0^\tau \alpha(s)ds \right)^2 \right].
\end{aligned}$$

Note that all the terms above are finite since  $\ell$  and  $K$  are bounded functions and since  $\alpha, v \in \mathcal{A}$ . Applying Fubini's theorem twice, we obtain

$$\begin{aligned}
\langle \mathcal{J}'(v), \alpha \rangle &= \mathbb{E} \left[ \int_0^\tau \alpha(s) \left( P(s, T) - K(s, T)\Upsilon^v(s) - \int_s^\tau K(t, T)e^{-\rho(t-s)}\gamma v(t)dt + \right. \right. \\
& \quad \left. \left. - 2l(s, T)v(s) + 2\phi \int_s^\tau X^v(t)dt + 2\varrho X^v(\tau) - P(\tau, T) \right) ds \right],
\end{aligned}$$

for any  $\alpha \in \mathcal{A}$ . We get the following condition on the optimal strategy

$$\begin{aligned}
& \mathbb{E} \left[ \int_0^\tau \alpha(s) \left( P(s, T) - K(s, T)\Upsilon^v(s) - \int_s^\tau K(t, T)e^{-\rho(t-s)}\gamma v(t)dt \right. \right. \\
& \quad \left. \left. - 2l(s, T)v(s) + 2\phi \int_s^\tau X^v(t)dt + 2\varrho X^v(\tau) - P(\tau, T) \right) ds \right] = 0.
\end{aligned} \tag{5.7.3}$$

Next we show that given the optimal strategy  $\hat{v} \in \mathcal{A}$ , the vector  $(X^{\hat{v}}, \Upsilon^{\hat{v}})$  satisfies the first order condition (5.7.3) if and only if the vector  $(X^{\hat{v}}, \Upsilon^{\hat{v}}, \hat{v}, Z^{\hat{v}})$  solves a FBSDE system, for some auxiliary process  $Z$ .

For any  $s > 0$  we denote by  $\mathbb{E}_s$  the conditional expectation with respect to the filtration  $\mathcal{F}_s$ . Assume  $\hat{v} \in \mathcal{A}$  maximizes the functional  $\mathcal{J}$ . Applying the optional projection theorem we obtain

$$\mathbb{E} \left[ \int_0^\tau \alpha(s) \left( P(s, T) - K(s, T) \Upsilon^{\hat{v}}(s) - \mathbb{E}_s \left[ \int_s^\tau K(t, T) e^{-\rho(t-s)} \gamma \hat{v}(t) dt \right] - 2l(s, T) \hat{v}(s) \right. \right. \\ \left. \left. + \mathbb{E}_s \left[ 2\phi \int_s^\tau X^{\hat{v}}(t) dt + 2\varrho X^{\hat{v}}(\tau) - P(\tau, T) \right] \right) ds \right] = 0,$$

for all  $\alpha \in \mathcal{A}$ . This implies

$$\begin{aligned} & P(s, T) - K(s, T) \Upsilon^{\hat{v}}(s) - e^{\rho s} \mathbb{E}_s \left[ \int_s^\tau K(t, T) e^{-\rho t} \gamma \hat{v}(t) dt \right] - 2l(s, T) \hat{v}(s) \\ & + \mathbb{E}_s \left[ 2\phi \int_s^\tau X^{\hat{v}}(t) dt + 2\varrho X^{\hat{v}}(\tau) - P(\tau, T) \right] \\ & = P(s, T) - K(s, T) \Upsilon^{\hat{v}}(s) \\ & - e^{\rho s} \left( \mathbb{E}_s \left[ \int_0^\tau K(t, T) e^{-\rho t} \gamma \hat{v}(t) dt \right] - \int_0^s K(t, T) e^{-\rho t} \gamma \hat{v}(t) dt \right) \\ & - 2l(s, T) \hat{v}(s) + \mathbb{E}_s \left[ 2\phi \int_0^\tau X^{\hat{v}}(t) dt + 2\varrho X^{\hat{v}}(\tau) - P(\tau, T) \right] - 2\phi \int_0^s X^{\hat{v}}(t) dt \\ & = 0, \quad d\mathbb{P} \otimes ds \text{ a.e. on } \Omega \times [0, \tau]. \end{aligned} \tag{5.7.4}$$

Next, we define the square-integrable martingale

$$\tilde{N}(s) := \mathbb{E}_s \left[ \int_0^\tau K(t, T) e^{-\rho t} \gamma \hat{v}(t) dt \right] \tag{5.7.5}$$

and the auxiliary square-integrable process

$$Z^{\hat{v}}(s) := e^{\rho s} \left( \int_0^s K(t, T) e^{-\rho t} \gamma \hat{v}(t) dt - \tilde{N}(s) \right),$$

for all  $s \in [0, \tau]$ . Note that since both  $l$  and  $K$  are assumed to be uniformly bounded and  $v \in \mathcal{A}$ , we have that  $P(\tau, T) \in L^2(\Omega, \mathcal{F}_\tau, \mathbb{P})$ . Therefore, we obtain

$$P(s, T) - K(s, T) \Upsilon^{\hat{v}}(s) + Z^{\hat{v}}(s) - 2l(s, T) \hat{v}(s) + \tilde{M}(s) - 2\phi \int_0^s X^{\hat{v}}(t) dt = 0 \quad (5.7.6)$$

almost everywhere on  $\Omega \times [0, \tau]$ , where  $\tilde{M}$  is the square-integrable martingale defined in (5.5.12), and we immediately see that the process  $Z^{\hat{v}}$  satisfies the BSDE

$$dZ^{\hat{v}}(t) = (\rho Z^{\hat{v}}(t) + K(t, T) \gamma \hat{v}(t)) dt - e^{\rho t} d\tilde{N}(t), \quad Z^{\hat{v}}(\tau) = 0.$$

From (5.2.8) we get that  $\Upsilon^{\hat{v}}$  satisfies

$$d\Upsilon^{\hat{v}}(t) = -\rho \Upsilon^{\hat{v}}(t) dt + \gamma \hat{v}(t) dt, \quad \Upsilon^{\hat{v}}(0) = y.$$

Recall that  $\Lambda$  was defined in (5.5.7). From (5.7.6) it follows that  $\hat{v}$  satisfies the backward stochastic differential equation

$$\begin{aligned} d\hat{v}(s) &= \Lambda'(s) \left( P(s, T) - K(s, T) \Upsilon^{\hat{v}}(s) + Z^{\hat{v}}(s) + \tilde{M}(s) - 2\phi \int_0^s X^{\hat{v}}(u) du \right) ds \\ &\quad + \Lambda(s) \left( dP(s, T) - \partial_s K(s, T) \Upsilon^{\hat{v}}(s) ds - K(s, T) d\Upsilon^{\hat{v}}(s) + dZ^{\hat{v}}(s) \right. \\ &\quad \left. + d\tilde{M}(s) - 2\phi X^{\hat{v}}(s) ds \right) \\ &= \Lambda'(s) \left( P(s, T) - K(s, T) \Upsilon^{\hat{v}}(s) + Z^{\hat{v}}(s) + \tilde{M}(s) - 2\phi \int_0^s X^{\hat{v}}(u) du \right) ds \\ &\quad + \Lambda(s) dP(s, T) - \Lambda(s) \partial_s K(s, T) \Upsilon^{\hat{v}}(s) ds + \rho K(s, T) \Upsilon^{\hat{v}}(s) \Lambda(s) ds \\ &\quad + \Lambda(s) \rho Z^{\hat{v}}(s) ds - 2\Lambda(s) \phi X^{\hat{v}}(s) ds + \Lambda(s) d\tilde{M}(s) - \Lambda(s) e^{\rho s} d\tilde{N}(s) \\ \hat{v}(\tau) &= \frac{\varrho}{l(\tau, T)} X^{\hat{v}}(\tau) - \frac{K(\tau, T)}{2l(\tau, T)} \Upsilon^{\hat{v}}(\tau), \end{aligned}$$

Putting these equations together with (5.2.5), we obtain the FBSDE system (5.7.1) with  $M, N$  square-integrable martingales defined as

$$\begin{aligned} M(t) &:= \int_0^t \Lambda(s) d\tilde{M}(s) - \int_0^t \Lambda(s) e^{\rho s} d\tilde{N}(s) \\ N(t) &:= - \int_0^t e^{\rho s} d\tilde{N}(s). \end{aligned}$$

In order to check the integrability of  $M$ , recall that  $\Lambda$  was defined in (5.5.7). Since  $l$  is bounded away from 0 on  $[0, \tau]$  (see (5.2.9)) we have

$$\sup_{0 \leq t \leq \tau} |\Lambda(t)| < \infty.$$

Then, it holds

$$\begin{aligned} \mathbb{E}[M^2(t)] &\leq \mathbb{E} \left[ \int_0^t \Lambda^2(s) d[\tilde{M}]_s \right] + \mathbb{E} \left[ \int_0^t \Lambda^2(s) e^{2\rho s} d[\tilde{N}]_s \right] \\ &\leq C_1 \mathbb{E}[\tilde{M}]_T + C_2 \mathbb{E}[\tilde{N}]_T \\ &< \infty \end{aligned}$$

for some constants  $C_1, C_2$ , where in the last inequality we used the fact that both  $\tilde{M}$  and  $\tilde{N}$  are square integrable martingales.

Next, assume that  $(\hat{v}, X^{\hat{v}}, \Upsilon^{\hat{v}}, Z^{\hat{v}})$  is a solution to the FBSDE system (5.7.1) and  $\hat{v} \in \mathcal{A}$ . We will show that  $\hat{v}$  satisfies the first order condition (5.7.3), hence it maximizes the cost functional (5.5.4). First, note that the BSDE for  $\hat{v}$  can be solved explicitly and the solution is indeed given in (5.7.4)

$$\begin{aligned} \hat{v}(s) &= \frac{1}{2l(s, T)} \left( P(s, T) - K(s, T) \Upsilon^{\hat{v}}(s) + Z^{\hat{v}}(s) + \tilde{M}(s) - 2\phi \int_0^s X^{\hat{v}}(t) dt \right) \\ &= \frac{1}{2l(s, T)} \left( P(s, T) - K(s, T) \Upsilon^{\hat{v}}(s) - e^{\rho s} \left( \tilde{N}(s) - \int_0^s K(t, T) e^{-\rho t} \gamma \hat{v}(t) dt \right) \right. \\ &\quad \left. + \tilde{M}(s) - 2\phi \int_0^s X^{\hat{v}}(u) du \right), \end{aligned}$$

with  $\tilde{N}, \tilde{M}$  defined in (5.7.5) and (5.5.12), respectively. Plugging this into the first order condition (5.7.3) yields

$$\begin{aligned} &\mathbb{E} \left[ \int_0^\tau \left( e^{\rho s} \left( \tilde{N}(s) - \int_0^s K(t, T) e^{-\rho t} \gamma \hat{v}(t) dt \right) - \tilde{M}(s) \right. \right. \\ &\quad \left. \left. + 2\phi \int_0^s X(t) dt + 2\rho X^{\hat{v}}(s) - P(s, T) \right) ds \right] \\ &= \mathbb{E} \left[ \int_0^\tau \alpha(s) \left( e^{\rho s} (\tilde{N}(s) - \tilde{N}(\tau)) - \tilde{M}(s) + \tilde{M}(\tau) \right) ds \right] \\ &= \mathbb{E} \left[ \int_0^\tau \alpha(s) \left( e^{\rho s} (\tilde{N}(s) - \mathbb{E}_s[\tilde{N}(\tau)]) - \tilde{M}(s) + \mathbb{E}_s[\tilde{M}(\tau)] \right) ds \right] \\ &= 0, \end{aligned}$$

for all  $\alpha \in \mathcal{A}$ . Since  $\tilde{N}, \tilde{M}$  are martingales, hence the first order condition (5.7.3) is satisfied and  $\hat{v} \in \mathcal{A}$  is the optimal strategy.  $\square$

Before giving the proof of our main theorem, we will need the following Lemma, which will help us to show the optimal strategy in (5.5.3) is indeed admissible.

**Lemma 5.7.2.** *Let  $\Gamma^{\hat{v}}$  be defined as in (5.5.11). Then, there exist constants  $C_1, C_2 > 0$  such that*

$$\mathbb{E} \left[ \int_0^\tau (\Gamma^{\hat{v}}(s))^2 ds \right] \leq C_1 + C_2 \mathbb{E} \left[ \int_0^\tau v^2(s) ds \right].$$

*Proof.* Firstly, by the assumptions on  $l$  (see (5.2.9) and (5.2.10)) it follows that

$$\sup_{0 \leq t \leq \tau} \left| \frac{\Lambda'(t)}{\Lambda(t)} \right| = \sup_{0 \leq t \leq \tau} \left| \frac{\partial_t l(t, T)}{l(t, T)} \right| < \infty, \quad (5.7.7)$$

where  $\Lambda$  is given in (5.5.7). Therefore, from (5.7.7), (5.5.11) and Jensen's inequality we get that there exist constants  $C_1, C_2 > 0$  such that

$$\begin{aligned} \mathbb{E} \left[ \int_0^\tau \Gamma^{\hat{v}}(s)^2 ds \right] &\leq \mathbb{E} \left[ \int_0^\tau \left( \frac{\Lambda'(s)}{\Lambda(s)} \right)^2 \left( P^2(s, T) + \tilde{M}^2(s) + 4\phi^2 \left( \int_0^s X^{\hat{v}}(u) du \right)^2 \right) ds \right] \\ &\leq C_1 \mathbb{E} \left[ \int_0^\tau \left( P^2(s, T) + \tilde{M}^2(s) + 4\phi^2 \left( \int_0^s X^{\hat{v}}(u) du \right)^2 \right) ds \right] \\ &\leq C_2 + 4\phi^2 \mathbb{E} \left[ \int_0^\tau \left( \int_0^s X^{\hat{v}}(u) du \right)^2 ds \right], \end{aligned}$$

where we used (5.5.1) and the fact that the martingale  $\tilde{M}$  defined in (5.5.12) is square-integrable.

Next, using the definition of  $X^{\hat{v}}$  in (5.5.2) and Jensen's inequality twice, we deduce

$$\begin{aligned} \mathbb{E} \left[ \int_0^\tau \left( \int_0^s X^{\hat{v}}(u) du \right)^2 ds \right] &= \mathbb{E} \left[ \int_0^\tau \left( \int_0^s \left( x - \int_0^u \hat{v}(y) dy \right) du \right)^2 ds \right] \\ &\leq C_1 + C_2 \mathbb{E} \left[ \int_0^\tau \int_0^s \int_0^u \hat{v}^2(y) dy du ds \right] \\ &\leq C_1 + C_2 \mathbb{E} \left[ \int_0^\tau \int_0^\tau \int_0^\tau \hat{v}^2(y) dy du ds \right] \\ &\leq C_1 + C_2 \mathbb{E} \left[ \int_0^\tau \hat{v}^2(y) dy \right], \end{aligned}$$

for some constants  $C_1, C_2$ , and we are done.  $\square$

We are now ready to prove Theorem 5.5.3.

*Proof of Theorem 5.5.3.* Define

$$\mathbf{X}^v(t) := \begin{pmatrix} X^{\hat{v}}(t) \\ \Upsilon^{\hat{v}}(t) \\ \hat{v}(t) \\ Z^{\hat{v}}(t) \end{pmatrix}, \quad \mathbf{M}(t) := \begin{pmatrix} 0 \\ 0 \\ P(t, T) + \int_0^t \Gamma^{\hat{v}}(s) ds + \int_0^t \Lambda^{-1}(s) dM(s) \\ \int_0^t \Lambda^{-1}(s) dN(s) \end{pmatrix},$$

where  $\Lambda$  and  $\Gamma^{\hat{v}}$  are defined in (5.5.7) and (5.5.11) respectively. The FBSDE system (5.7.1) can be written as

$$d\mathbf{X}_t^{\hat{v}} = A(t)\mathbf{X}_t^{\hat{v}}dt + \Lambda(t)d\mathbf{M}(t), \quad 0 \leq t \leq \tau,$$

where the matrix  $A(t)$  is defined in (5.5.6), with initial conditions

$$\mathbf{X}^{\hat{v},1}(0) = x, \quad \mathbf{X}^{\hat{v},2}(0) = y,$$

and terminal conditions

$$\left( \frac{\varrho}{l(\tau, T)}, -\frac{K(\tau, T)}{2l(\tau, T)}, -1, 0 \right) \mathbf{X}^{\hat{v}}(\tau) = 0, \quad (0, 0, 0, 1) \mathbf{X}^{\hat{v}}(\tau) = 0. \quad (5.7.8)$$

Exploiting linearity, the unique solution can be expressed as

$$\mathbf{X}^{\hat{v}}(\tau) = \Phi(\tau)\Phi^{-1}(t)\mathbf{X}^{\hat{v}}(t) + \int_t^\tau \Phi(\tau)\Phi^{-1}(s)\Lambda(s)d\mathbf{M}(s),$$

where  $\Phi$  solves the ODE (5.5.8). Moreover, it can be immediately seen that the first terminal condition in (5.7.8) yields

$$\begin{aligned} 0 &= G^1(t, \tau)X^{\hat{v}}(t) + G^2(t, \tau)\Upsilon^{\hat{v}}(t) + G^3(t, \tau)\hat{v}(t) + G^4(t, \tau)Z^{\hat{v}}(t) \\ &\quad + \int_t^\tau \Lambda(s) \left( G^3(s, \tau) (dP(s, T) + \Gamma^{\hat{v}}(s)ds + \Lambda^{-1}(s)dM(s)) + G^4(s, \tau)\Lambda^{-1}(s)dN(s) \right) \end{aligned}$$

with  $G = (G^1, G^2, G^3, G^4)$  defined in (5.5.10). Solving for the trading speed  $v$ , taking expectations and using that  $P \in \mathcal{H}^2$ , together with the fact that both  $M$  and  $N$  are square integrable martingales, implies

$$\begin{aligned} \hat{v}(t) &= -\frac{G^1(t, \tau)}{G^3(t, \tau)}X^{\hat{v}}(t) - \frac{G^2(t, \tau)}{G^3(t, \tau)}\Upsilon^{\hat{v}}(t) - \frac{G^4(t, \tau)}{G^3(t, \tau)}Z^{\hat{v}}(t) \\ &\quad - \mathbb{E}_t \left[ \int_t^\tau \Lambda(s) \frac{G^3(s, \tau)}{G^3(t, \tau)} (\mu(s) + \Gamma^{\hat{v}}(s)) ds \right]. \end{aligned} \quad (5.7.9)$$

Recall that  $\Psi$  was defined in (5.5.9). Then the second terminal condition in (5.7.8) implies

$$\begin{aligned} 0 &= (0, 0, 0, 1)\Psi(t, \tau)\mathbf{X}^{\hat{v}}(t) + (0, 0, 0, 1) \int_t^\tau \Psi(s, \tau)\Lambda(s)d\mathbf{M}(s) \\ &= \Psi^{41}(t, \tau)X^{\hat{v}}(t) + \Psi^{42}(t, \tau)\Upsilon^{\hat{v}}(t) + \Psi^{43}(t, \tau)\hat{v}(t) + \Psi^{44}(t, \tau)Z^{\hat{v}}(t) \\ &\quad + \int_t^\tau \Lambda(s) \left( \Psi^{43}(s, \tau) (dP(s, T) + \Gamma^{\hat{v}}(s)ds + \Lambda^{-1}(s)dM(s)) + \Psi^{44}(s, \tau)\Lambda^{-1}(s)dN(s) \right). \end{aligned}$$

Hence, taking expectation and solving for  $Z^u$  yields

$$\begin{aligned} Z^{\hat{v}}(t) = & -\frac{\Psi^{41}(t, \tau)}{\Psi^{44}(t, \tau)} X^{\hat{v}}(t) - \frac{\Psi^{42}(t, \tau)}{\Psi^{44}(t, \tau)} \Upsilon^{\hat{v}}(t) - \frac{\Psi^{43}(t, \tau)}{\Psi^{44}(t, \tau)} \hat{v}(t) \\ & - \mathbb{E}_t \left[ \int_t^\tau \frac{\Lambda(s) \Psi^{43}(s, \tau)}{\Psi^{44}(t, \tau)} (\mu(s) + \Gamma^{\hat{v}}(s)) ds \right]. \end{aligned} \quad (5.7.10)$$

Therefore, plugging (5.7.10) into (5.7.9) gives

$$\begin{aligned} \hat{v}(t) = & -\frac{G^1(t, \tau)}{G^3(t, \tau)} X^{\hat{v}}(t) - \frac{G^2(t, \tau)}{G^3(t, \tau)} \Upsilon^{\hat{v}}(t) + \frac{G^4(t, \tau) \Psi^{41}(t, \tau)}{G^3(t, \tau) \Psi^{44}(t, \tau)} X^{\hat{v}}(t) \\ & + \frac{G^4(t, \tau) \Psi^{42}(t, \tau)}{G^3(t, \tau) \Psi^{44}(t, \tau)} \Upsilon^{\hat{v}}(t) + \frac{G^4(t, \tau) \Psi^{43}(t, \tau)}{G^3(t, \tau) \Psi^{44}(t, \tau)} \hat{v}(t) \\ & + \frac{G^4(t, \tau)}{G^3(t, \tau)} \mathbb{E}_t \left[ \int_t^\tau \frac{\Lambda(s) \Psi^{43}(s, \tau)}{\Psi^{44}(t, \tau)} (\mu(s) + \Gamma^{\hat{v}}(s)) ds \right] \\ & - \mathbb{E}_t \left[ \int_t^\tau \Lambda(s) \frac{G^3(s, \tau)}{G^3(t, \tau)} (\mu(s) + \Gamma^{\hat{v}}(s)) ds \right]. \end{aligned}$$

Rearranging and using the Definitions 5.5.13, we obtain the linear feedback form (5.5.14). Finally, we prove that the optimal trading strategy is admissible, that is,  $\hat{v} \in \mathcal{A}$ , as defined in (5.5.3). Thanks to assumptions (A.1) and (A.2), we immediately see that

$$\sup_{0 \leq t \leq \tau} |v_0(t, \tau)| < \infty.$$

Similarly, from assumptions (A.1)-(A.3) we deduce that  $v_1$  and  $v_2$  are both bounded on  $[0, \tau]$ . Exploiting again assumptions (A.1)-(A.3), together with (5.5.1) we get that

$$\begin{aligned} & \sup_{0 \leq t \leq \tau} \left| \mathbb{E}_t \left[ \int_t^\tau \frac{\Lambda(s) \Psi^{43}(s, \tau)}{\Psi^{44}(t, \tau)} (\mu(s) + \Gamma^{\hat{v}}(s)) ds - \mathbb{E}_t \int_t^\tau \Lambda(s) \frac{G^3(s, \tau)}{G^3(t, \tau)} (\mu(s) + \Gamma^{\hat{v}}(s)) ds \right] \right| \\ & \leq C \mathbb{E} \left[ \int_0^\tau (|\mu(s)| + |\Gamma^{\hat{v}}(s)|) ds \right] \\ & \leq \tilde{C}_1 + \tilde{C}_2 \left( \mathbb{E} \left[ \int_0^\tau \Gamma^{\hat{v}}(s)^2 ds \right] \right)^{1/2} \\ & \leq \tilde{C}_1 + \tilde{C}_2 \left( \mathbb{E} \left[ \int_0^\tau \hat{v}^2(s) ds \right] \right)^{1/2}, \end{aligned}$$

where we have used Jensen's inequality and Lemma 5.7.2 in the last two inequalities. Using the above bound, together with equations (5.5.2) and (5.2.8) we get from (5.5.14) that

$$\mathbb{E}[\hat{v}^2(t)] \leq C_1 + C_2 \int_0^\tau \mathbb{E}[\hat{v}^2(s)] ds, \quad 0 \leq t \leq \tau,$$

for some positive constants  $C_1, C_2$ , where we used again Jensen's inequality. Thanks to Gronwall's lemma, we get that

$$\sup_{0 \leq t \leq \tau} \mathbb{E}[\hat{v}^2(t)] < \infty,$$

which implies

$$\int_0^\tau \mathbb{E}[\hat{v}^2(s)] ds < \infty.$$

Hence Fubini's theorem, we conclude that  $\hat{v} \in \mathcal{A}$ . □

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