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THE SIZE RAMSEY NUMBER OF GRAPHS WITH BOUNDED TREEWIDTH*

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Abstract. A graph G is *Ramsey* for a graph H if every 2-coloring of the edges of G contains a monochromatic copy of H. We consider the following question: if H has bounded treewidth, is there a "sparse" graph G that is Ramsey for H? Two notions of sparsity are considered. Firstly, we show that if the maximum degree and treewidth of H are bounded, then there is a graph G with $O(|V(H)|)$ edges that is Ramsey for H. This was previously only known for the smaller class of graphs H with bounded bandwidth. On the other hand, we prove that in general the treewidth of a graph G that is Ramsey for H cannot be bounded in terms of the treewidth of H alone. In fact, the latter statement is true even if the treewidth is replaced by the degeneracy and H is a tree.

Key words. size Ramsey number, bounded treewidth, bounded-degree trees, Ramsey number

AMS subject classifications. 05C55, 05D10, 05C05

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1. Introduction.

A graph G is *Ramsey* for a graph H, denoted by $G \to H$, if every 2-coloring of the edges of G contains a monochromatic copy of H. In this paper we are interested in how sparse G can be in terms of H if $G \to H$. The two measures of sparsity that we consider are the number of edges in G and the treewidth of G.

The *size Ramsey number* of a graph H, denoted by $\widehat{r}(H)$, is the minimum number of edges in a graph G that is Ramsey for H. The notion was introduced by Erdős et al. [19]. Beck [3] proved $\widehat{r}(P_n) \leqslant 900n$, answering a question of Erdős [18]. The constant 900 was subsequently improved by Bollobás [7] and by Dudek and Prałat [16]. In these proofs the host graph G is random. Alon and Chung [2] provided an explicit construction of a graph with $O(n)$ edges that is Ramsey for P_n.

Beck [3] also conjectured that the size Ramsey number of bounded-degree trees is linear in the number of vertices and noticed that there are trees (for instance, double stars) for which it is quadratic. Friedman and Pippenger [25] proved Beck's conjecture. The implicit constant was subsequently improved by Ke [32] and by Haxell and Kohayakawa [28]. Finally, Dellamonica, Jr. [13] proved that the size Ramsey number of a tree T is determined by a simple structural parameter $\beta(T)$ up to a constant factor, thus establishing another conjecture of Beck [4].

In the same paper, Beck asked whether all bounded-degree graphs have a linear size Ramsey number, but this was disproved by Rödl and Szemerédi [40]. They

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constructed a family of graphs of maximum degree 3 with superlinear size Ramsey number.

In 1995, Haxell, Kohayakawa, and Łuczak showed that cycles have linear size Ramsey number [29]. Conlon [11] asked whether, more generally, the kth power of the path P_n has size Ramsey number at most cn, where the constant c only depends on k. Here the kth power of a graph G is obtained by adding an edge between every pair of vertices at distance at most k in G. Conlon's question was recently answered in the affirmative by Clemens et al. [9].

Their result is equivalent to saying that graphs of bounded bandwidth have linear size Ramsey number. We show that the same conclusion holds in the following more general setting. The treewidth of a graph G, denoted by $\mathrm{tw}(G)$, can be defined to be the minimum integer w such that G is a subgraph of a chordal graph with no $(w+2)$-clique. While this definition is not particularly illuminating, the intuition is that the treewidth of G measures how "tree-like" G is. For example, trees have treewidth 1. Treewidth is of fundamental importance in the graph minor theory of Robertson and Seymour and in algorithmic graph theory; see [6, 27, 39] for surveys on treewidth. For the purposes of this paper the only property of treewidth that we need is Lemma 2.1 below.

THEOREM 1.1. *For all integers k, d there exists $c = c(k, d)$ such that if H is a graph of maximum degree d and treewidth at most k, then*

$$\widehat{r}(H) \leqslant c|V(H)|.$$

Theorem 1.1 implies the above $O(|V(H)|)$ bounds on the size Ramsey number from [9], since powers of paths have bounded treewidth and bounded degree. Powers of complete binary trees are examples of graphs covered by our theorem but not covered by any previous results in the literature. Note that the assumption of bounded degree in Theorem 1.1 cannot be dropped in general since, as mentioned above, there are trees of superlinear size Ramsey number [4]. Furthermore, the lower bound from [40] implies that an additional assumption on the structure of H, such as bounded treewidth, is also necessary. We prove Theorem 1.1 in section 3.

We actually prove an off-diagonal strengthening of Theorem 1.1. For graphs H_1 and H_2, the *size Ramsey number* $\widehat{r}(H_1, H_2)$ is the minimum number of edges in a graph G such that every red/blue-coloring of the edges of G contains a red copy of H_1 or a blue copy of H_2. We prove that if H_1 and H_2 both have n vertices, bounded degree, and bounded treewidth, then $\widehat{r}(H_1, H_2) \leqslant cn$. Moreover, we show that there is a host graph that works simultaneously for all such pairs H_1 and H_2 and that has bounded degree.

THEOREM 1.2. *For all integers $k, d \geqslant 1$ there exists $c = c(k, d)$ such that for every integer $n \geqslant 1$ there is a graph G with cn vertices and maximum degree c, such that for all graphs H_1 and H_2 with n vertices, maximum degree d, and treewidth k, every red/blue-coloring of the edges of G contains a red copy of H_1 or a blue copy of H_2.*

The second contribution of this paper fits into the framework of *parameter Ramsey numbers*: for any monotone graph parameter ρ, one may ask whether $\min\{\rho(G) : G \to H\}$ can be bounded in terms of $\rho(H)$. This line of research was conceived in the 1970s by Burr, Erdős, and Lovász [8]. The usual Ramsey number and the size Ramsey number (where $\rho(G) = |V(G)|$ and $\rho(G) = |E(G)|$, respectively) are classical topics. Furthermore, the problem has been studied when ρ is the clique number [21, 36], chromatic number [8, 44], maximum degree [30, 31], and minimum degree [8, 22, 23, 42] (the latter requires the additional assumption that the host graph G is minimal with respect to subgraph inclusion; otherwise the problem is trivial).

It is therefore interesting to ask whether $\min\{\text{tw}(G) : G \to H\}$ can be bounded in terms of $\text{tw}(H)$. Our next theorem shows that the answer is negative, even when replacing treewidth by the weaker notion of degeneracy. For an integer d, a graph G is *d-degenerate* if every subgraph of G has minimum degree at most d. The *degeneracy* of G is the minimum integer d such that G is d-degenerate. It is well known and easily proved that every graph with treewidth w is w-degenerate, but treewidth cannot be bounded in terms of degeneracy (for example, the 1-subdivision of K_n is 2-degenerate but has treewidth $n-1$).

THEOREM 1.3. *For every $d \geqslant 1$ there is a tree T such that if G is d-degenerate, then $G \nrightarrow T$.*

A positive restatement of Theorem 1.3 is that the edges of every d-degenerate graph can be 2-colored with no monochromatic copy of a specific tree T (depending on d). This is a significant strengthening of a theorem by Ding et al. [15, Theorem 3.9], who proved that the edges of every graph with treewidth at most k can be k-colored with no monochromatic copy of a certain tree T. We also note that a statement similar to Theorem 1.3 does not hold in the online Ramsey setting; see section 4 in [12] for more details.

Furthermore, Theorem 1.3 is tight in the following sense. If $\mathcal{G}$ is a monotone graph class with unbounded degeneracy, then for every tree T, there is a graph $G \in \mathcal{G}$ such that $G \to T$. Indeed, for a given tree T, let G be a graph in $\mathcal{G}$ with average degree at least $4|V(T)|$, which exists since $\mathcal{G}$ is monotone with unbounded degeneracy. In any 2-coloring of $E(G)$, one color class has average degree at least $2|V(T)|$. Thus there is a monochromatic subgraph of G with minimum degree at least $|T|$, which contains T as a subgraph by a folklore greedy algorithm.

2. Tools. Our proof of Theorem 1.2 relies on the following characterization of graphs with bounded treewidth and bounded degree. The *strong product* of graphs G and H, denoted by $G \boxtimes H$, is the graph with vertex set $V(G) \times V(H)$, where (v_1, u_1) is adjacent to (v_2, u_2) in $G \boxtimes H$ if $v_1 = v_2$ and $u_1u_2 \in E(H)$, or $v_1v_2 \in E(G)$ and $u_1 = u_2$, or $v_1v_2 \in E(G)$ and $u_1u_2 \in E(H)$. Note that $T \boxtimes K_k$ is obtained from T by replacing each vertex by a clique and replacing each edge by a complete bipartite graph.

LEMMA 2.1 ([14, 43]). *Every graph with treewidth w and maximum degree d is a subgraph of $T \boxtimes K_{18wd}$ for some tree T of maximum degree at most $18wd^2$.*

Our host graph G in the proof of Theorem 1.2 is obtained from a random D-regular graph H on $O(n)$ vertices for a suitable constant D. We then take the third power of H and replace every vertex by a clique of bounded size and every edge by a complete bipartite graph. To show that G has the desired Ramsey properties we will exploit certain expansion properties of H.

An (N, D, λ)*-graph* is a D-regular N-vertex graph in which every eigenvalue except the largest one is at most λ in absolute value. The existence of graphs with $\lambda = O(\sqrt{D})$ is shown, for instance, by considering a random D-regular graph on N vertices, denoted by $G(N, D)$.

LEMMA 2.2 ([24]). *Let $D \geqslant 3$ be an integer, and let ND be even. With probability tending to 1 as $N \to \infty$, every eigenvalue of $G(N, D)$ except the largest one is at most $2\sqrt{D}$ in absolute value.*

For a graph G and sets $U, W \subseteq V(G)$, let $e(U,W)$ be the number of edges with one endpoint in U and the other one in W. Each edge with both endpoints in $U \cap W$ is counted twice. We will use the following well-known estimate on the edge distribution of a graph in terms of its eigenvalues; see, e.g., [34] for a proof.

LEMMA 2.3 ([34]). *For every (N, D, λ)-graph G and for all sets $S, T \subseteq V(G)$,*

$$\left| e(S,T) - \frac{D|S||T|}{N} \right| \leqslant \lambda \sqrt{|S||T| \left(1 - \frac{|S|}{N}\right)\left(1 - \frac{|T|}{N}\right)}.$$

The key tool that we use is the following implicit result of Friedman and Pippenger [25], which shows that every (N, D, λ)-graph with the appropriate parameters is "robustly universal" for bounded-degree trees. Let $\mathcal{T}_{n,d}$ be the set of all trees with n vertices and maximum degree at most d. The next lemma follows implicitly from the proofs of Theorems 2 and 3 in [25].

LEMMA 2.4 ([25]). *Let $\varepsilon > 0$ and d, n be integers. Let D and N be integers such that $D > 100d^2/\varepsilon^4$ and $N > 10d^2n/\varepsilon^2$, and let G be an (N, D, λ)-graph with $\lambda = 2\sqrt{D}$. Then every induced subgraph of G on at least εN vertices contains every tree in $\mathcal{T}_{n,d}$.*

We summarize the above results in the following lemma.

LEMMA 2.5. *For every integer d, every $\varepsilon > 0$, and all even $D > 100d^2/\varepsilon^4$ there exists c such that for all integers n, N with $N \geqslant cn$ there exists an N-vertex D-regular graph H with the following properties:*

(1) *For every pair of disjoint sets $S, T \subseteq V(H)$ with $|S|, |T| \geqslant 2N/\sqrt{D}$ we have $e(S,T) > 0$.*
(2) *Every induced subgraph of H on at least εN vertices contains every tree in $\mathcal{T}_{n,d}$.*

Proof. Let $D > 100d^2/\varepsilon^4$ be an even integer and $N > 10d^2n/\varepsilon^2$. Let H be an (N, D, λ)-graph where $\lambda = 2\sqrt{D}$, which exists by Lemma 2.2. Property (2) follows from Lemma 2.4. Moreover, for all sets $S, T \subseteq V(H)$ with $|S|, |T| \geqslant 2N/\sqrt{D}$ we have $\lambda\sqrt{|S||T|} < \frac{D|S||T|}{N}$, which implies $e(S,T) > 0$ by Lemma 2.3, as desired. ∎

We also need the following lemma of Friedman and Pippenger [25]. For a graph H and $X \subseteq V(H)$, let $\Gamma_H(X)$ be the set of vertices in $V(H)$ adjacent to some vertex in X.

LEMMA 2.6 (Theorem 1 of [25]). *If H is a nonempty graph such that for each $X \subseteq V(H)$ with $1 \leqslant |X| \leqslant 2n - 2$,*

$$|\Gamma_H(X)| \geqslant (d+1)|X|,$$

then H contains every tree in $\mathcal{T}_{n,d}$.

Finally, we need the following standard tools.

LEMMA 2.7 (Kövari, Sós, and Turán [33]). *Every graph with n vertices and no $K_{s,s}$ subgraph has at most $(s-1)^{1/s}n^{2-1/s} + (s-1)$ edges.*

LEMMA 2.8 (Lovàsz local lemma [20]). *Let $\mathcal{E}$ be a set of events in a probability space, each with probability at most p, and each mutually independent of all except at most d other events in $\mathcal{E}$. If $4pd \leqslant 1$, then with positive probability no event in $\mathcal{E}$ occurs.*

3. Proof of Theorem 1.2. We start with the following lemma that states that if a graph does not contain all trees in $\mathcal{T}_{n,d}$, then its complement contains a complete multipartite subgraph where the parts have "large" size. In fact, our proof shows that if the second assertion does not hold, (i.e. there is no complete multipartite graph with large parts in the complement), then the graph contains a "large" expander as a subgraph. The containment of every tree in $\mathcal{T}_{n,d}$ then follows from Lemma 2.6. Statements of similar flavor are also proved and utilized in [17, 38, 37].

LEMMA 3.1. *Fix integers n, d, q, and let $N \geqslant 20ndq$. In every red/blue-coloring of $E(K_N)$ there is either a blue copy of every tree in $\mathcal{T}_{n,d}$ or a red copy of a complete q-partite graph in which every part has size at least $\frac{N}{5dq}$.*

Proof. Let G be the spanning subgraph of K_N consisting of all the blue edges. We may assume that G does not contain every tree in $\mathcal{T}_{n,d}$. By Lemma 2.6, for every nonempty set $S \subseteq V(G)$, there exists $X \subseteq S$ such that $1 \leqslant |X| \leqslant 2n-2$ and $|\Gamma_{G[S]}(X)| < (d+1)|X|$. Note that for such S and X, all the edges of K_N between X and $S \setminus (X \cup \Gamma_{G[S]}(X))$ must be red. Let $S_1, S_2, \ldots, S_{m+1}$ and $X_1, X_2, \ldots, X_m$ be sets of vertices in G such that $S_1 = V(G)$ and, for $1 \leqslant i \leqslant m$,

- $X_i \subseteq S_i$ with $1 \leqslant |X_i| \leqslant 2n-2$ and $|\Gamma_{G[S_i]}(X_i)| < (d+1)|X_i|$ and
- $S_{i+1} = S_i \setminus (X_i \cup \Gamma_{G[S_i]}(X_i))$.

We stop when $S_{m+1} = \emptyset$. Note that $X_1, X_2, \ldots, X_m$ are pairwise disjoint. Since all the edges of K_N between X_i and S_{i+1} are red, all the edges between distinct X_i and X_j are red. Let $X = \bigcup_{i=1}^m X_i$. Note that

$$N = \sum_{i=1}^m |X_i \cup \Gamma_{G[S_i]}(X_i)| < \sum_{i=1}^m (d+2)|X_i| = (d+2)|X|.$$

Thus $|X| > \frac{N}{d+2}$.

We now combine the parts X_i to reach the required size. Let $Y_1 = X_1 \cup X_2 \cup \cdots \cup X_j$, where j is the minimal index such that $|X_1 \cup X_2 \cup \cdots \cup X_j| \geqslant \frac{N}{5dq}$. Since $|X_i| \leqslant 2n-2 < \frac{N}{10dq}$, we have the upper bound, $|Y_1| < \frac{3N}{10dq}$. Repeating the same argument, starting at X_{j+1} and noting that $|X| > \frac{N}{d+2} \geqslant q \cdot \frac{3N}{10dq}$, we construct $Y_1, Y_2, \ldots Y_q$, satisfying $|Y_i| \geqslant \frac{N}{5dq}$ and such that all edges between any distinct Y_i and Y_j are red. ∎

Let T be a rooted tree with root r. For each vertex v of T, let $p_T(v)$ denote the *parent* of v, where for convenience we let $p_T(r) = r$. Let $p_T^2(v)$ denote the *grandparent* of v; that is, $p_T^2(v) = p_T(p_T(v))$. We denote the set of *children* of v by $C_T(v)$, and define $C_T^2(v) = C_T(v) \cup (\bigcup_{x \in C_T(v)} C_T(x))$ to be the set of children and *grandchildren* of v. Let $d_T(v)$ be the distance between r and v, that is, the number of edges on the path from r to v. For each integer i, let $L_i(T)$ be the set of vertices v with $d_T(v) = i$. In the above definitions, we may omit the subscript T if T is clear from the context.

Given a tree T rooted at r, define another tree T' rooted at r as follows. The vertex set of T' is defined to be $\{r\} \cup \bigcup_{i \geqslant 0} L_{2i+1}(T)$. A pair vw with $v, w \in V(T')$ is an edge of T' if $p_T^2(v) = w$ or $p_T^2(w) = v$. In particular, $C_{T'}(r) = C_T(r)$. We call T' the *truncation* of T. An illustration of T and its truncation can be found in Figure 1. Note that if T has maximum degree d, then T' has maximum degree at most d^2.

Let s and m be integers. Suppose we are given a graph G, a vertex partition $(V_1, V_2, \ldots, V_m)$ of G, and an edge-coloring $\psi : E(G) \to \{\text{red}, \text{blue}\}$. Define an auxiliary coloring of the complete graph K_m with vertex set $[m]$ as follows. For distinct $i, j \in [m]$, color the edge ij blue if there is a blue $K_{s,s}$ between V_i and V_j in G, and red

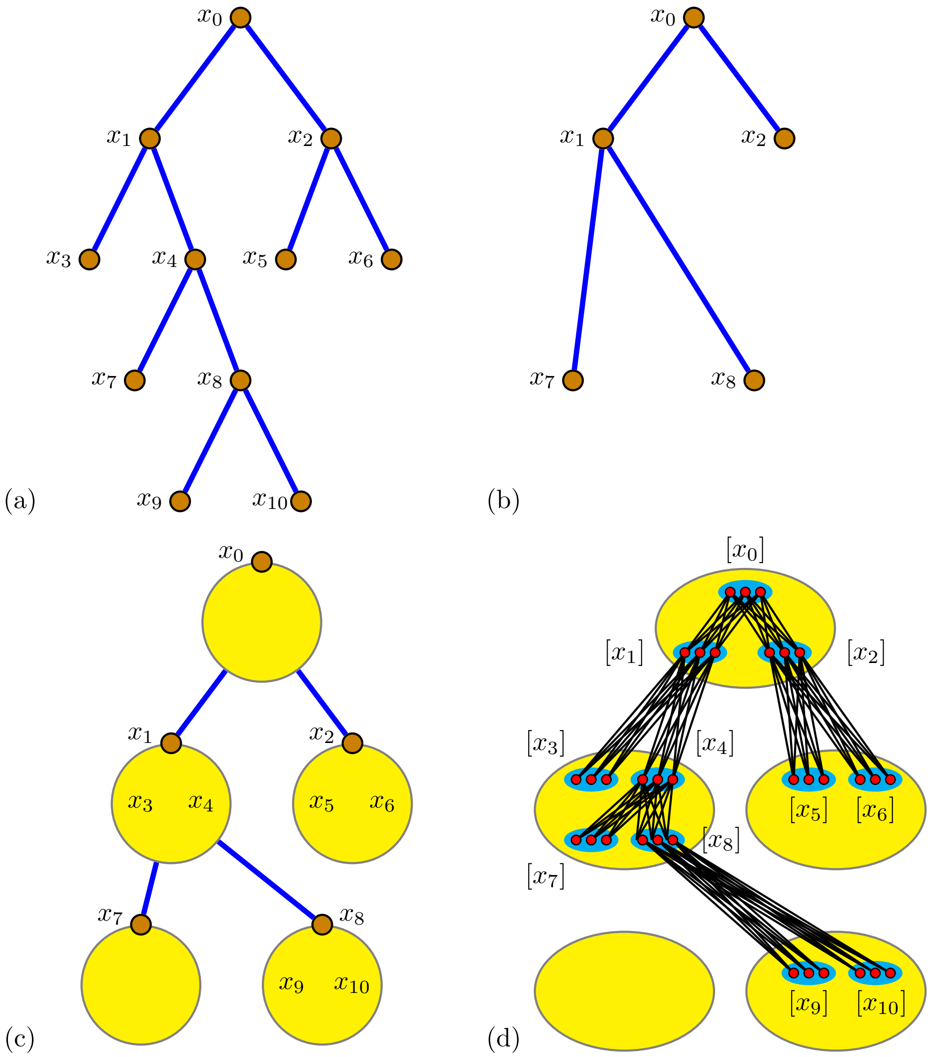

FIG. 1. (a) *Tree* T*,* (b) *truncation* T'*,* (c) *the corresponding bags,* (d) *embedding of* $T \boxtimes K_k$ *where* $[x_i]$ *means* $\{x_i\} \times K_k$*.*

otherwise. We call this edge-coloring the (G, ψ, s)*-coloring of* K_m. This auxiliary coloring also appears in [1] and subsequently in [9]. The lemma below demonstrates the importance of this auxiliary coloring; for any bounded-degree tree T and any k there is some s such that under certain conditions we can effectively "lift" a monochromatic copy of T' in the (G, ψ, s)-coloring of K_m to a monochromatic copy of $T \boxtimes K_k$ in G, with respect to the coloring ψ.

LEMMA 3.2. *Fix integers* n*,* d*,* k*,* m*. Let* T *be a tree in* $\mathcal{T}_{n,d}$ *rooted at* x_0*, and let* T' *be the truncation of* T*. Let* $s = (d + d^2)k$*. Suppose we are given a graph* G*, a vertex partition* $(V_1, V_2, \ldots, V_m)$ *of* G*, and an edge-coloring* $\psi : E(G) \to \{\text{red}, \text{blue}\}$ *such that, for all* $i \in [m]$*, all the edges of* $G[V_i]$ *are present and are blue, and* $|V_i| \geqslant s$*. If there exists a blue copy of* T' *in the* (G, ψ, s)*-coloring of* K_m*, then there exists a blue copy of* $T \boxtimes K_k$ *in* G*.*

Proof. Let φ be the (G, ψ, s)-coloring of K_m, and suppose $g : V(T') \to [m]$ is an embedding of T' in the blue subgraph of K_m. Let $x_0, x_1, x_2, \dots, x_{m'}$ be the vertices of $V(T')$ ordered by their distance from the root x_0 in T'. We will find a blue copy of $T \boxtimes K_k$ whose vertices are in $V_{g(x_i)}$ for $i = 0, \dots, m'$. We warn the reader that in this proof we often use notation $f(S \boxtimes K_k)$ to denote the image of $S \boxtimes K_k$ for some subset $S \subseteq V(T)$, under some embedding f into G, without precisely defining how f acts on each vertex of $S \boxtimes K_k$ but rather claiming that such an embedding exists. This is done for brevity and to keep the proof intuitive.

We define a collection $\{B_x : x \in V(T')\}$ of subsets of $V(T)$ as follows. Let $B_{x_0} = \{x_0\}$, and for each $x \in V(T') \setminus \{x_0\}$, let $B_x = \{x\} \cup C_T(x)$. We call B_x the *bag* of the vertex x. Observe that the bags are pairwise disjoint, and they partition the entire vertex set $V(T)$. They will help us keep track of the embedding of $T \boxtimes K_k$ in G. Our goal is to find an embedding f of $T \boxtimes K_k$ in G satisfying the properties (P1)–(P4) below.

(P1) $f(T \boxtimes K_k) \subseteq \bigcup_{x \in V(T')} V_{g(x)}$,
(P2) $f((\{x_0\} \cup C_T(x_0)) \boxtimes K_k) \subseteq V_{g(x_0)}$,
(P3) for every $x \in V(T') \setminus \{x_0\}$, $f(C_T^2(x) \boxtimes K_k) \subseteq V_{g(x)}$,
(P4) every edge of $f(T \boxtimes K_k)$ will be colored blue.

We will proceed iteratively, starting from the root x_0 and following the order of the vertices x_i we fixed earlier. At each step i, we will have a partial embedding f_i of $T_i \boxtimes K_k$ in G, where T_i is the subtree $T[\cup_{j \leqslant i} B_{x_j}]$. Our final embedding will be $f = f_{m'}$. At step 0 we will embed $B_{x_0} \boxtimes K_k$ in some way; this will define f_0. At step $i \geqslant 1$, f_i will be defined as an extension of f_{i-1}, and the extension will be defined only on $B_{x_i} \boxtimes K_k$ so that the image of the latter "links" back appropriately to the embedding of $T_{i-1} \boxtimes K_k$. Note that (P2) implies that at most $(d+1)k$ vertices are embedded in $V_{g(x_0)}$, and every other $V_{g(x)}$ (with $x \neq x_0$) will contain at most $(d + d^2)k$ embedded vertices by (P3). Moreover, (P4) will be satisfied for edges of $f(T \boxtimes K_k)$ embedded inside one partition class V_j. To guarantee that those edges of $f(T \boxtimes K_k)$ that go between distinct partition classes V_j and V_k are blue, we will make use of the properties of the auxiliary coloring φ. Finally, we define our iterative embedding scheme from which properties (P1)–(P4) can be easily read out, thus completing the proof.

Step 0: Let $T_0 = \{x_0\}$, and embed $T_0 \boxtimes K_k$ into $V_{g(x_0)}$ by picking any k vertices in $V_{g(x_0)}$; this determines f_0. Recall that all edges inside $V_{g(x_0)}$ are blue; hence indeed this is a valid embedding of $T_0 \boxtimes K_k$.

Step $i \geqslant 1$: Having defined f_{i-1}, we now show how to extend it to f_i. Recall that $B_{x_i} = \{x_i\} \cup C_T(x_i)$. Let y be the grandparent of x_i. Since there is an edge $x_i y$ in T' and since g is a blue embedding of T' in K_m, there is a blue $K_{s,s}$ between $V_{g(x_i)}$ and $V_{g(y)}$. Let L be any such copy of $K_{s,s}$. Define f_i on $\{x_i\} \boxtimes K_k$ to be a set of any k vertices in $V_{g(y)} \cap V(L)$ disjoint from the image of f_{i-1}. Define f_i on $C_T(x_i) \boxtimes K_k$ to be any set of $k|C_T(x_i)|$ vertices in $V_{g(x_i)} \cap V(L)$ disjoint from the image of f_{i-1}. This is possible since $|V_{g(x_i)} \cap V(L)| \geqslant s = (d^2 + d)k$, and the total number of vertices embedded into $V_{g(x_i)}$ during the procedure is at most $(d^2 + d)k$. ∎

The next lemma is a standard application of the Lovàsz local lemma. Given a graph F let $F(t)$ denote the *blowup* of F, where each vertex v is replaced by an independent set $I(v)$ of size t and each edge uv is replaced by a complete bipartite graph between $I(u)$ and $I(v)$.

LEMMA 3.3. *Fix $t \geqslant 1$. Let F be a graph with maximum degree Δ. Let F' be a spanning subgraph of $F(t)$ such that for every edge $vw \in E(F)$ there are at least $(1 - \frac{1}{8\Delta})t^2$ edges in F' between $I(v)$ and $I(w)$. Then $F \subseteq F'$.*

Proof. For each vertex v of F, independently choose a random vertex v' in $I(v)$. For each edge vw of F, let E_{vw} be the event that $v'w'$ is not an edge of F'. Since there are at least $(1-\frac{1}{8\Delta})t^2$ edges between $I(v)$ and $I(w)$, the probability of E_{vw} is at most $\frac{1}{8\Delta}$. Each event E_{vw} is mutually independent of all other events, except for the at most 2Δ events corresponding to edges incident to v or w. Since $4(\frac{1}{8\Delta})(2\Delta) \leqslant 1$, by Lemma 2.8, the probability that some event $E_{v,w}$ occurs is strictly less than 1. Thus, there exist choices for v' for all $v \in V(F)$ such that $v'w'$ is an edge of F' for every edge vw of F. This yields a subgraph of F' isomorphic to F. ◻

Both Theorem 1.1 and Theorem 1.2 are implied by Lemma 2.1 and the following result.

THEOREM 3.4. *For all integers k,d there exists $c = c(k,d)$ such that for all n there is a graph G with cn vertices and maximum degree c, such that for all trees T_1 and T_2 with n vertices and maximum degree d, every red/blue-coloring of $E(G)$ contains a red copy of $T_1 \boxtimes K_k$ or a blue copy of $T_2 \boxtimes K_k$.*

Proof. Let $\varepsilon = (d^2(2k+1)2^{2k+4})^{-1}$. Let D be the smallest even number larger than $100d^2/\varepsilon^4$. Let c be derived from Lemma 2.5 applied with this choice of ε, d, and D. Let $N = \max\{cn, 40nd^2(2k+1)\}$, and let H be any N-vertex D-regular graph derived from Lemma 2.5. Set $s = (d^2+d)k$ and $t = (64kd)^s$. Denote the Ramsey number of t by $r(t)$. Recall that H^3 is a graph on the same vertex set as H where uv is an edge in H^3 whenever u and v are at distance at most three in H. Let $G = H^3 \boxtimes K_{r(t)}$.

Since H is D-regular, H^3 has maximum degree at most D^3, and G has maximum degree at most $D^3 r(t) + r(t) - 1$. Let $A(v)$ denote the copy of $K_{r(t)}$ corresponding to $v \in V(H)$. Let $\psi : E(G) \to \{\text{red}, \text{blue}\}$ be any edge-coloring of G. We will show that it must contain either a red copy of $T_1 \boxtimes K_k$ or a blue copy of $T_2 \boxtimes K_k$.

By definition of $r(t)$, for each vertex $v \in V(H)$, $A(v)$ contains a monochromatic copy of K_t, say, on vertex set $B(v)$. Let W be the set of vertices $v \in V(H)$ for which $B(v)$ induces a blue K_t. By symmetry between T_1 and T_2, we may assume that $|W| \geqslant \frac{1}{2}|V(H)|$. Let $N' = |W| \geqslant \frac{N}{2}$.

Let $B(W) = \bigcup_{v\in W} B(v)$, and let φ be the $(G[B(W)], \psi, s)$-coloring of $K_{N'}$. Root T_2 at an arbitrary vertex. Let T_2' be the truncation of T_2. If there is a blue copy of T_2' in $K_{N'}$ with respect to the coloring φ, then Lemma 3.2 implies that $G[B(W)]$ contains a blue copy of $T_2 \boxtimes K_k$ with respect to ψ.

We henceforth assume that there is no blue copy of T_2' in $K_{N'}$. Since T_2' has maximum degree at most d^2 and $N' \geqslant 20nd^2(2k+1)$ there are sets $V_0, V_1, \dots, V_{2k} \subseteq V(K_{N'})$ of size at least $\frac{N'}{5d^2(2k+1)}$ such that all the edges in $K_{N'}$ between two distinct parts V_i and V_j are red, by Lemma 3.1.

For $i \in [2k]$, define an *i-matching* to be a matching of edges in H with one endpoint in V_0 and the other in V_i. (Note that we are now considering the original graph H, not $K_{N'}$.) We will find a set $S \subseteq V_0$ satisfying $|S| > 2^{-2k}|V_0|$, and a collection of i-matchings $\{M_i\}_{i=1}^{2k}$ such that each M_i covers S. We proceed by induction on i. Assume at the end of step $j \leqslant 2k-1$ we have found a set $S_j \subseteq V_0$ with $|S_j| > 2^{-j}|V_0|$ and a collection of i-matchings $\{M_i\}_{i=1}^{j}$, where each M_i covers S_j. At step $j+1$, let M_{j+1} be a maximum matching between S_j and V_{j+1}. If M_{j+1} consists of fewer than $|S_j|/2$ edges, then, by Kőnig's theorem, the bipartite graph between S_j and V_{j+1} has a vertex cover of order at most $|S_j|/2$. But then we can find sets $X \subset S_j$ and $Y \subset V_{j+1}$ with $e_H(X,Y) = 0$ and $|X|, |Y| \geqslant |S_j|/2 \geqslant 2^{-2k-2}|V_0| > \varepsilon N$. This contradicts property (1) from Lemma 2.5. Hence M_{j+1} covers at least $|S_j|/2 \geqslant |V_0| \cdot 2^{-(j+1)}$

vertices of S_j. We set $S_{j+1} = V(M_{j+1}) \cap S_j$ and proceed. After $2k$ steps, we reach the desired set S_{2k}, which we call S.

For each vertex $v \in S$, for $i \in [2k]$, let $v_i \in V_i$ be the unique neighbor of v in M_i. Since $|S| > 2^{-2k}|V_0| > \varepsilon N$, $H[S]$ contains a copy $\widetilde{T}_1$ of T_1 on some vertex set U by property (2) from Lemma 2.5. Next we show that there is a red (with respect to φ) copy of $T_1 \boxtimes K_k$ in $K_{N'}$ contained in the vertex set of $\widetilde{T}_1 \cup \{M_i\}_{i=1}^{2k}$ and use this copy to find a red (with respect to ψ) copy of $T_1 \boxtimes K_k$ in $G[B(W)]$ via Lemma 3.3.

Root $\widetilde{T}_1$ at any vertex $\widetilde{r}$. For every vertex $v \in V(\widetilde{T}_1)$ let $S(v) = \{v_1, v_2, \dots, v_k\}$ if v is at even distance from $\widetilde{r}$ and $S(v) = \{v_{k+1}, v_{k+2}, \dots, v_{2k}\}$ otherwise. Note that for any $u, v \in V(\widetilde{T}_1)$, the sets $S(u)$ and $S(v)$ are disjoint. Moreover, for every $v \in V(\widetilde{T}_1)$, $S(v)$ induces a red clique in $K_{N'}$ because the vertices of $S(v)$ are elements of distinct partition classes V_i. If u and v are adjacent in $\widetilde{T}_1$, then also edges between $S(u)$ and $S(v)$ are red in $K_{N'}$ since all the vertices of $S(u) \cup S(v)$ lie in distinct partition classes V_i. So this shows that the vertex set $\bigcup_{v \in U} S(v)$ induces a red copy $\widetilde{T_1 \boxtimes K_k}$ of $T_1 \boxtimes K_k$ in $K_{N'}$. It remains to "lift" this copy to the graph $G[B(W)]$ with the coloring ψ. First we observe that every edge in $\widetilde{T_1 \boxtimes K_k}$ is in fact an edge of H^3. Indeed, for any $u \in V(\widetilde{T}_1)$ and any $i \neq j$, $u_i, u_j \in S(u)$ are at distance at most two in H; hence $u_i u_j$ is an edge in H^3. Now if u and v are adjacent in $\widetilde{T}_1$, then for any $u_i \in S(u)$ and $v_j \in S(v)$, the distance between u_i and v_j in H is at most 3, so u_i and v_j are also adjacent in H^3.

Recall that if uv is an edge of H^3 and $\varphi(uv)$ is red in $K_{N'}$, then the complete bipartite graph G_{uv} between $B(u)$ and $B(v)$ in G contains no blue copy of $K_{s,s}$. Lemma 2.7 implies that G_{uv} has at most $(s-1)^{1/s} t^{2-1/s} + (s-1) \leqslant 4t^{2-1/s}$ blue edges. Note that $4t^{2-1/s} \leqslant \frac{t^2}{16dk}$. Let $F = \widetilde{T_1 \boxtimes K_k}$, and let F' be the subgraph of G consisting of all the red edges of G_{uv} over all $uv \in E(F)$. It is now easy to see that F and F' satisfy the assumptions of Lemma 3.3. Therefore G contains a red copy of $T_1 \boxtimes K_k$ which finishes the proof. ∎

4. Proof of Theorem 1.3. Let $T_{d,h}$ be the complete d-ary tree of height h with a root vertex r; that is, every nonleaf vertex has exactly d children, and every leaf is at distance h from r. Theorem 1.3 is implied by the following. Recall that, for a rooted tree T, $d_T(v)$ denotes the number of edges of the path from the root to v in T.

THEOREM 4.1. *For every integer $i \geqslant 1$, every $(2^i - 1)$-degenerate graph G is not Ramsey for the tree $T_{2^{i+1}, 2^i}$.*

Proof. We proceed by induction on i. For $i = 1$, G is a tree, so fix an arbitrary vertex to be the root of G and color the edges of G by their distance to the root modulo 2 (where the distance of an edge uv to the root r is $\min\{d_G(u), d_G(v)\}$). There is no monochromatic path of length 3, and in particular no monochromatic copy of $T_{4,2}$.

Now let $i \geqslant 2$, and set $d = 2^i$ and $h = 2^{i-1}$ for brevity. Let G be a $(d-1)$-degenerate graph. It follows from the definition of degeneracy that G has a vertex-ordering $v_1, v_2, \dots, v_n$, such that each vertex v_j has at most $d-1$ neighbors v_k with $k < j$. Form an oriented graph $\vec{G}$ by choosing the orientation (v_j, v_k) for an edge $v_j v_k \in E(G)$ if $j < k$. Then each vertex has in-degree at most $d-1$.

We now partition $V(G)$ into sets V_r and V_b such that both $G[V_r]$ and $G[V_b]$ are $(d/2-1)$-degenerate. Start by assigning v_1 to V_r. For $j = 2, 3, \dots, n$, assume that $v_1, v_2, \dots, v_{j-1}$ have been assigned to V_r or V_b. Add v_j to V_r if V_r contains at most $d/2-1$ of the in-neighbors of v_j. Otherwise add it to V_b. Note that in the latter case, V_b contains at most $d/2-1$ of the in-neighbors of v_j, since v_j has at most $d-1$

in-neighbors in $\vec{G}$. Clearly, this does not affect the in-degree of $v_1, v_2, \ldots, v_{j-1}$ in $\vec{G}[V_r]$ and $\vec{G}[V_b]$. Thus, this process produces the desired sets V_r and V_b.

By induction, there is a red/blue-coloring ψ' of the edges in $E_G(V_r) \cup E_G(V_b)$ not containing a monochromatic copy of $T_{d,h}$. We extend ψ' to a red/blue-coloring ψ of $E(G)$ in the following way. For an edge $uv \in E_G(V_r, V_b)$ assume without loss of generality that it is directed from u to v in $\vec{G}$, that is, $(u,v) \in \vec{G}$. Then color uv red if $u \in V_r$, and blue if $u \in V_b$. In other words, the edge uv "inherits" the color from its source vertex in $\vec{G}$.

We claim that there is no monochromatic copy of $T_{2d,2h}$ in this coloring of $E(G)$. Assume the opposite, and let $\widetilde{T}_{2d,2h}$ be a monochromatic copy of $T_{2d,2h}$ in G. For each vertex v in $T_{2d,2h}$, we denote its copy in $\widetilde{T}_{2d,2h}$ by $\widetilde{v}$. Without loss of generality we may assume that $\widetilde{T}_{2d,2h}$ is red.

CLAIM 4.2. *If $\widetilde{v}$ is a nonleaf vertex of $\widetilde{T}_{2d,2h}$ that lies in V_b, then there are at least d children $\widetilde{u}_1, \ldots, \widetilde{u}_d$ of $\widetilde{v}$ in $\widetilde{T}_{2d,2h}$ such that $\widetilde{u}_j \in V_b$ for all $j \in [d]$.*

Proof. The number of children of the vertex $\widetilde{v}$ in $\widetilde{T}_{2d,2h}$ is $2d$. Out of these, the number of children w such that $(w, \widetilde{v}) \in \vec{G}$ is at most $d-1$. Furthermore, each edge $(\widetilde{v}, w) \in \vec{G}$ with $w \in V_r$ is colored blue in ψ, by definition and since $\widetilde{v} \in V_b$. That implies that none of these edges can be part of $\widetilde{T}_{2d,2h}$. It follows that at least $d+1$ neighbors of $\widetilde{v}$ in $\widetilde{T}_{2d,2h}$ are elements of V_b. At most one of these vertices is the parent of $\widetilde{v}$, and the claim follows. ◻

Recall that $\widetilde{r}$ is the root of $\widetilde{T}_{2d,2h}$.

CLAIM 4.3. *For every vertex $\widetilde{v} \in V(\widetilde{T}_{2d,2h})$ at distance at most h from $\widetilde{r}$ in $\widetilde{T}_{2d,2h}$ we have that $\widetilde{v} \in V_r$.*

Proof. Assume that $\widetilde{v} \in V_b$ and has distance at most h in $\widetilde{T}_{2d,2h}$ from $\widetilde{r}$. Apply Claim 4.2 iteratively to $\widetilde{v}$ and all of its descendants $\widetilde{u}$ that lie in V_b. In h iterations (before reaching the leaves of $\widetilde{T}_{2d,2h}$), we construct a red copy of $T_{d,h}$ whose vertices all lie in V_b, that is, a red copy of $T_{d,h}$. This contradicts the property of ψ'. ◻

It follows that all vertices in $\widetilde{T}_{2d,2h}$ at distance at most h from $\widetilde{r}$ must lie in V_r, forming a red copy of $T_{d,h}$ in $G[V_r]$, which again contradicts the property of ψ'. ◻

After the first preprint of this paper was finished we learned [41] that Maximilian Geißer, Jonathan Rollin, and Peter Stumpf independently obtained a proof of Theorem 1.3. This proof is unpublished, yet short and nice, so we include their argument here.

Second proof of Theorem 4.1. Let G be a d-degenerate graph. We show that G is not Ramsey for $T_{d+1,d+1}$. Assume without loss of generality that the vertex set of G is $[n]$ for some n and that every $u \in V(G)$ has at most d neighbors v with $v < u$. Let $\phi : V(G) \to [d+1]$ denote a proper coloring of the vertices of G using at most $d+1$ colors. Define an edge coloring ψ by coloring an edge uv with $u < v$ red if $\phi(u) < \phi(v)$ and blue otherwise. A path $v_1 \ldots v_n$ in G is called monotone if its vertices are ordered $v_1 < \cdots < v_n$. Each monochromatic monotone path in ψ has at most d vertices, since the colors of its vertices are either increasing or decreasing under ϕ. On the other hand each copy of $T_{d+1,d+1}$ in G contains a monotone path on d vertices, since each inner vertex u has a child v with $u < v$ due to the d-degeneracy of G. Hence there are no monochromatic copies of $T_{d+1,d+1}$ in G. ◻

5. Concluding remarks. We have shown that for a graph H of bounded maximum degree and treewidth, there is a graph G with $O(|V(H)|)$ edges that is Ramsey for H. It is now natural to ask whether the size Ramsey number of a planar graph H of bounded degree is linear in $|V(H)|$. A first candidate to consider is the grid graph. Recently Clemens et al. [10] have shown that the size Ramsey number of the grid graph on $n \times n$ vertices is bounded from above by $n^{3+o(1)}$. There are no nontrivial lower bounds known.

QUESTION 5.1. *Is the size Ramsey number of the grid graph on $n \times n$ vertices $O(n^2)$?*

Furthermore, we propose a multicolor extension of our result.

QUESTION 5.2. *Given positive integers $w, d, n, s \geqslant 3$ and an n-vertex graph H of maximum degree d and treewidth w, do there exist $C = C(w, d, s)$ and a graph G with Cn edges such that every s-coloring of the edges of G contains a monochromatic copy of H?*

When H is a bounded-degree tree, a positive answer (and even a stronger density analog result) follows from the work of Friedman and Pippinger [25]. Han et al. [26] have recently shown that the above extension holds for graphs of bounded bandwidth (or, equivalently, for any fixed power of a path).

Our second result is that the edges of every d-degenerate graph can be 2-colored without a monochromatic copy of a fixed tree $T = T(d)$. The maximum degree of T in the proof of Theorem 4.1 is $2d+1$. It follows from [35, Lemma 5] that T cannot be replaced by a tree whose maximum degree is bounded by an absolute constant which is independent of d.

Ding et al. [15] also showed that for every tree T, there is a graph G of treewidth two such that every red/blue-coloring of the edges of G contains a red copy of T or a blue copy of a subdivision of T. We wonder whether the following generalization is true.

QUESTION 5.3. *Is there a function $f(k)$ with the following property: for every graph H of treewidth k, there is a graph G of treewidth $f(k)$ such that every red/blue-coloring of the edges of G contains a red copy of H or a blue copy of a subdivision of H?*

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