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Matching in Power Graphs of Finite Groups

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Abstract. The power graph $P(G)$ of a finite group G is the undirected simple graph with vertex set G, where two elements are adjacent if one is a power of the other. In this paper, the matching numbers of power graphs of finite groups are investigated. We give upper and lower bounds, and conditions for the power graph of a group to possess a perfect matching. We give a formula for the matching number for any finite nilpotent group. In addition, using some elementary number theory, we show that the matching number of the enhanced power graph $P_e(G)$ of G (in which two elements are adjacent if both are powers of a common element) is equal to that of the power graph of G.

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1. Introduction

Associating graphs to algebraic structures is an interesting research topic. Cayley graphs, intersection graphs, zero divisor graphs, commuting graphs and power graphs are some examples of graphs constructed from semigroups and groups.

The directed power graph was first proposed in 2002 by Kelarev and Quinn [11]. For a group G, the directed power graph $\overrightarrow{P}(G)$ is a graph with vertex set G with an arc from the vertex x to the vertex y if and only if $y = x^n$ for some natural number $n \in \mathbf{N}$. Motivated by this, Chakrabarty et al. [7] defined the undirected power graph $P(G)$, whose vertex set is G and the edge set consists of pairs of distinct vertices x and y if $y = x^j$ or $x = y^k$ for some $j, k \in \mathbb{N}$. (These definitions were first given for semigroups, but we only consider groups here.)

Many fascinating results on directed and undirected power graphs were established by several authors. The authors of [7] proved that the necessary

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and sufficient condition for $P(G)$ to be complete is that G is a finite cyclic group having order 1 or p^i, for some prime number p and positive integer i.

Curtin and Pourgholi [8,9] proved that power graphs of cyclic groups have the largest clique and maximum number of edges, among all finite groups of a given order.

In [6], the first author and Shamik Ghosh proved that finite abelian groups having isomorphic power graphs are isomorphic. They also showed that the Klein 4-group is the only finite group whose automorphism group is isomorphic to that of its power graph. In [4], the first author proved that finite groups having isomorphic power graphs have isomorphic directed power graphs.

The chromatic number of power graphs of finite groups is investigated in [12,15] and some results on the independence number of the same is proved in [14].

Ma et al. [13] found, for certain graphs Γ, the smallest group G for which Γ can be embedded in $P(G)$.

We denote graphs by Γ: our graphs are simple graphs with vertex set V and edge set E. A matching or independent edge set M in a graph Γ is a set of edges in which no two of them share a common vertex. A vertex is said to be matched (or saturated) if it is incident to one of the edges in the matching. Otherwise the vertex is unmatched. A matching M of a graph Γ is maximal if it is not a subset of any other matching in Γ. A matching that contains the largest possible number of edges is called a maximum matching. The size (number of edges) in a maximum matching in a graph Γ is known as its matching number, which is denoted by $\mu(\Gamma)$. A perfect matching is a matching which saturates all vertices of the graph: it has size $|V|/2$.

Let G be a finite group. Together with the power graph, the enhanced power graph and the commuting graph are some of the examples of graphs whose vertex set is G and whose edges reflect the group structure in some way. In the *enhanced power graph* of G, denoted by $P_e(G)$, two vertices x and y are adjacent if and only if $\langle x, y\rangle$ is cyclic, and in the *commuting graph* $\mathrm{Com}(G)$ of G, two vertices are adjacent if they commute.

We denote the order of a group G by $|G|$, while for $a \in G$, the order of the element a is denoted by $o(a)$.

In this paper, we concentrate on finding matchings in the power graphs. We investigate several class of groups to obtain groups whose power graphs have a perfect matching. In particular, we find the size of a maximum matching in the power graph of any abelian group. We also include some results we obtained on the enhanced power graph and commuting graph. In particular, the power graph and enhanced power graph have the same matching number.

The context for this result is the classification of finite groups for which the power graph and enhanced power graph are equal: these are the groups in which every element has prime power order, determined by Brandl [2] in 1981; see [5] for a discussion of these groups and their connection with graphs. These form an interesting class of groups, and we thought to investigate wider classes as follows: choose a monotone graph parameter π, and determine the groups G for which π takes the same value on the power graph and enhanced power graph of G. To our surprise, we found that, for matching number, this is the class of all groups. (For further comments see the end of the paper.)

2. A Preliminary Result

We begin this section by noting that a finite group G of odd order has matching number $(|G|-1)/2$; that is, a maximum matching leaves just one vertex unmatched. To see this, note that, for any $x \in G$, if $x \neq 1$, then $x \neq x^{-1}$ and $\{x, x^{-1}\}$ is an edge of $P(G)$; these edges form a matching of the required size.

For groups of even order, we begin with the following observation.

Theorem 1. *Let G be a group of even order. Let $T = \{g \in G : g^2 = 1\}$ be the set consisting of the identity and the involutions in G. Let Γ be a graph with vertex set G with the property that every element of $G \backslash T$ is joined to its inverse. Then there is a maximum-size matching in Γ for which the set of unmatched vertices is contained in T.*

Proof. Clearly $|T| > 1$.

Take any matching M of Γ. We describe a transformation to another matching M' such that either $|M'| > |M|$, or $|M'| = |M|$ and the number of unmatched vertices not in T is smaller in M' than in M.

Suppose that g is an unmatched vertex which is not in T. If g^{-1} is also unmatched then we can match g to g^{-1}, increasing the size of the matching. So suppose g^{-1} is matched.

Put $g = g_0$. Let g_1 be the vertex matched to g_0^{-1}; let g_2 be the vertex matched to g_1^{-1}; and so on, as long as possible. The process terminates when either g_m^{-1} is unmatched, or $g_m \in T$.

In the first case, we replace the edges $\{g_0^{-1}, g_1\}$, $\{g_1^{-1}, g_2\}, \ldots, \{g_{m-1}^{-1}, g_m\}$ with the edges $\{g_0, g_0^{-1}\}$, $\{g_1, g_1^{-1}\}, \ldots, \{g_m, g_m^{-1}\}$, and the size of the matching is increased by one.

In the second case, we replace the edges $\{g_0^{-1}, g_1\}$, $\{g_1^{-1}, g_2\}, \ldots, \{g_{m-1}^{-1}, g_m\}$ with the edges $\{g_0, g_0^{-1}\}$, $\{g_1, g_1^{-1}\}, \ldots, \{g_{m-1}, g_{m-1}^{-1}\}$. The resulting matching has the same size, but we have replaced the unmatched vertex $g_0 \notin T$ by $g_m \in T$, so we have decreased by one the number of unmatched vertices not in T.

If we begin the process with a maximum matching, then each step must reduce the number of unmatched vertices not in T, and the process concludes when this number reaches zero. □

Now the power graph $P(G)$ has the property of Γ in this theorem, so we can take a maximum matching in $P(G)$ with all unmatched vertices lying in T. If $|T| > 1$ (so that $|G|$ is even) and the identity is unmatched, then we may add the edge $\{1, t\}$ to the matching, where t is an involution. So the following holds.

Corollary 1. (a) *Let G be a group of even order. Then the size of a maximum matching in the power graph or enhanced power graph of G is at least*

$1 + (|G| - |T|)/2$, *where* T *consists of the identity and the involutions in* G.

(b) *If* G *is a group with a unique involution, then* $P(G)$ *has a perfect matching.*

Remark 1. Groups with a unique involution are known; see [1].

3. Upper and Lower Bounds

In this section, we describe upper and lower bounds for the matching number of the power graph of a group of even order. First, an upper bound.

Theorem 2. *Let* G *be a finite group of even order. Let* $I(G)$ *be the set of involutions in* G, *and* $O(G)$ *the set of elements of odd order. Then*

(a) *any matching of* $P(G)$ *leaves at least* $|I(G)| - |O(G)|$ *vertices unmatched*;

(b) *if* G *has a perfect matching, then* $|I(G)| \leq |O(G)|$.

Proof. We use Tutte's 1-factor theorem [17], which asserts that a graph $\Gamma = (V, E)$ has a perfect matching if and only if for every subset U of V, the induced subgraph $\Gamma - U$ on $V \backslash U$ has at most $|U|$ connected components with an odd number of vertices. The Tutte–Berge formula (the "deficit form" of the theorem) asserts that the number of unmatched vertices in a maximum matching of a graph Γ is equal to the maximum, over all subsets U of V, of the number of odd components of $\Gamma - U$ minus $|U|$.

Let Γ' be the induced subgraph of $P(G)$ on $G \backslash O(G)$ (the set of elements of even order in G). For $t \in I(G)$, let

$$C_t = \{x \in G : t \in \langle x \rangle\}.$$

Note that elements of C_t have even order, and no element of G can lie in more than one of these sets, since a cyclic group contains at most one involution.

We will show that the sets C_t for $t \in I(G)$ are connected components of Γ', and that they all have odd cardinality. It follows from Tutte's 1-factor theorem that, if $P(G)$ has a perfect matching, then $|I(G)|$ (the number of odd components of Γ') does not exceed $|O(G)|$. Moreover, the deficit form of the theorem shows that, if $|I(G)| > |O(G)|$, there are at least $|I(G)| - |O(G)|$ vertices uncovered in any matching of $P(G)$.

Note that any element of C_t is joined to t in the power graph, so any two elements of C_t have distance at most 2; thus C_t is contained in a connected component. Take an edge $\{x, y\}$ of the power graph contained in $G \backslash O(G)$. Without loss of generality, x is a power of y. Suppose that t is the involution in $\langle x \rangle$, so that $x \in C_t$. Then $t \in \langle x \rangle \leq \langle y \rangle$, so also $y \in C_t$. This shows that C_t is a connected component of Γ.

Now all elements of $C_t \backslash \{t\}$ have order greater than 2; so they can be paired with their inverses, leaving only t unpaired. So $|C_t|$ is odd, as required. □

Now we give a lower bound.

Theorem 3. *Let G be a finite group of even order. Let $S = I(G)$ be the set of involutions in G, and $O(C_G(S))$ the set of elements of odd order which commute with all involutions.*

(a) *There is a matching leaving at most* $\max\{0, |I(G)| - |O(C_G(S))|\}$ *vertices unmatched.*
(b) *If $|I(G)| \leq |O(C_G(S))|$, then $P(G)$ has a perfect matching.*

Proof. Let $n = |I(G)|$ and $m = |O(C_G(S))|$. We start as usual with the matching M on G in which each element of order greater than 2 is matched to its inverse, leaving the identity and the involutions unmatched. In addition, we match the identity to one of the involutions. This leaves $n-1$ unmatched involutions, which we partition into $(n-1)/2$ pairs in any manner, and $m-1$ elements of odd order commuting with them, falling into $(m-1)/2$ inverse pairs.

Assume that both n and m are greater than 1. Let u, v be involutions, and x, x^{-1} a pair of elements of odd order commuting with u and v. In the given matching, we have edges $\{x, x^{-1}\}$, $\{ux, ux^{-1}\}$, and $\{vx, vx^{-1}\}$. We delete these and include instead the edges $\{u, ux\}$, $\{v, vx^{-1}\}$, $\{ux^{-1}, x^{-1}\}$ and $\{vx, x\}$. Now all previously matched elements are still matched, and in addition u and v are matched.

Repeating this process, if $m \geq n$ we match all the involutions, while if $m < n$ we match $(m-1)/2$ pairs of involutions with elements of odd order, leaving $n - m$ involutions unmatched. □

With these results we can calculate the matching number of the power graph of a nilpotent group.

Theorem 4. *Let G be nilpotent; let $I(G)$ and $O(G)$ be the sets of involutions and elements of odd order, respectively.*

(a) *If $|I(G)| < |O(G)|$, then $P(G)$ has a perfect matching.*
(b) *Otherwise, a maximum matching of $P(G)$ leaves $|I(G)| - |O(G)|$ vertices unmatched.*

Proof. If G is nilpotent, then the elements of odd order form a normal subgroup $O(G)$, and $G \cong H \times O(G)$ where H is a Sylow 2-subgroup. So all involutions commute with all elements of odd order. So the result follows from Theorems 2 and 3. □

4. Related Results

In this section, we give some miscellaneous related results.

4.1. Groups Whose Power Graph Has Small Matching Number

Theorem 5. *For every positive integer m, there are only finitely many finite groups G with $\mu(G) = m$, apart from elementary abelian 2-groups (with $\mu(G) = 1$); such groups satisfy $|G| < 8m + 4$.*

Proof. If $|G|$ is odd, then $m = \mu(G) = (|G|-1)/2$, so $|G| = 2m+1$. So suppose that $|G|$ is even. Then $|O(C_G(S))| \geq 1$, and the number of vertices uncovered in a maximum matching is $|G| - 2m$. So

$$|I(G)| - 1 \geq |G| - 2m,$$

whence $|I(G)| \geq |G| - 2m + 1$. However, if G is not elementary abelian, then $|I(G)| < \frac{3}{4}|G|$. (This known result has been described in the literature as an "easy exercise"). So $|G| < 8m + 4$. □

Using this, we can give the determination of groups whose power graph has matching number 1 or 2. The *dihedral group* D_n is the group of order $2n$ which is the symmetry group of a regular n-gon, for $n \geq 3$.

Theorem 6. *Let G be a finite group.*

(a) *If $\mu(P(G)) = 1$, then G is either an elementary abelian 2-group or C_3.*
(b) *If $\mu(P(G)) = 2$, then G is one of the following groups: C_4, C_5, D_3 or D_4.*

Proof. Part (a) follows immediately from the preceding theorem. For part (b), we know that such a group has order at most 11, and there are only a small number of groups to analyse. □

4.2. Groups with Few Involutions

We have seen that, if G has a unique involution, then $P(G)$ has a perfect matching. We now extend this result.

Theorem 7. *Let G be a group with exactly three involutions, not all pairs of which commute. Then either $G \cong S_3$, or $P(G)$ has a perfect matching.*

Proof. Let s, t, u be the involutions. If s and t do not commute, then $\langle s, t\rangle$ is a dihedral group of order $2n$ containing n involutions, with $n \geq 3$; so we must have $n = 3$, and s, t, u are the involutions in a normal subgroup of G isomorphic to S_3. Now S_3 is a complete group: this means that its centre and its outer automorphism group are both trivial. Hence every extension of S_3 splits: that is, if S_3 is a normal subgroup of G, then $G \cong S_3 \times H$. See [16, Section 13.5]. Now H contains no involutions, so has odd order. If $|H| = 1$, then $G \cong S_3$; otherwise $H = O(C_G(S))$, and the result follows from Theorem 3. □

For a group G, since $E(P(G)) \subseteq E(P_e(G)) \subseteq E(\mathrm{Com}(G))$, the possibility to have a perfect matching in the commuting graph is greater as compared to the power graph. The following theorem shows that, if the order of a group is much bigger than the number of involutions in it, then its commuting graph has a perfect matching.

Proposition 1. *There is a function F such that, if G is a group of even order which has exactly n involutions, and $|G| \geq F(n)$, then the commuting graph of G has a perfect matching.*

Proof. We take $F(n) = 2n{\cdot}n!$. So let G be a group with even order greater than $2n \cdot n!$ and suppose that G contains n involutions. We begin with a matching M as follows: elements of order greater than 2 are matched to their inverses;

the identity is mapped to one involution. If $n = 1$ we are finished, so suppose not.

The group G acts by conjugation on the set S of involutions. The kernel of this action, which is $C_G(S)$, has index at most $n!$ in G, and so has order at least $2n$; so, putting $X = C_G(S) \setminus (\{1\} \cup S)$, we have $|X| \geq n - 1$. Moreover, elements of X have order greater than 2, and so are matched with their inverses in M; and X is inverse-closed.

Pick $(n-1)/2$ inverse pairs in X, say $\{x_1, x_2\}, \ldots, \{x_{n-2}, x_{n-1}\}$. Let $t_1, \ldots, t_{n-1}$ be the unmatched involutions in S. Now delete the edges $\{x_{2i-1}, x_{2i}\}$ from M for $i = 1, \ldots, (n-1)/2$, and add the edges $\{x_1, t_1\}$, $\{x_2, t_2\}, \ldots$, $\{x_{n-1}, t_{n-1}\}$ instead. (These are edges since $t_i \in S$ and $x_i \in C_G(S)$.) The result is a perfect matching M'. □

The hypothesis in the above theorem is not enough in the case of power graphs, since, the power graph of $C_{2^n} \times C_{2^m}$ has no perfect matching even if we take n and m very large: the group has three involutions and one element of odd order.

Proposition 2. *There is a function F on the natural numbers with the following property: Let G be a finite group of even order, and S the set of involutions in G. Suppose that for every involution $u \in S$, there is an involution v in S which does not commute with u. If $|G| \geq F(|S|)$, then the power graph of G has a perfect matching.*

Proof. Take $F(n) = n.n!$. Again, G acts by conjugation on S, so $|G : C_G(S)| \leq n!$. Thus, $|C_G(S)| \geq n$. Now by hypothesis, no involution belongs to $C_G(S)$, so $C_G(S)$ is a group of odd order. Thus the assumptions of Theorem 3 are satisfied. □

4.3. Embedding in Groups Whose Power Graph Has a Perfect Matching

Theorem 8. *Let G be a finite group of even order, and suppose that the number of elements of G not matched in a matching of maximum size in $P(G)$ is s. If p is an odd prime greater than s, then $G \times C_p$ has a perfect matching.*

Proof. In the following, we use the internal direct product, so that G and C_p are subgroups of $G \times C_p$.

Let $t_1, \ldots, t_s$ be the elements unmatched in some matching of maximum size in $P(G)$. By Theorem 1 we can assume that $t_1, \ldots, t_s$ are involutions. (The set of unmatched vertices can be taken to be a subset of $I(G)$, and the identity can be matched to any other vertex.) Note that s is even.

Take $p > s$, and let x be a generator of C_p in the group $G \times C_p$. Let $A_0 = \langle x \rangle \setminus \{1\}$, and for $1 \leq i \leq s$ let $A_i = A_0 t_i$. Each set A_i for $0 \leq i \leq s$ induces a complete graph in $P(G \times C_p)$, and we have all possible edges between A_0 and A_i for $i > 0$. Moreover, t_i is joined to every vertex in A_i. Also, $|A_i| = p - 1$ for all i.

Choose an edge from t_i to a vertex in A_i for each i and add to the matching on G. There remain $p - 2$ unmatched vertices in A_i; choose one, and match it to a vertex in A_0, using distinct vertices for different i. This leaves

$p-3$ unmatched vertices in A_i, an even number, and $p-1-s$ unmatched vertices in A_0, also an even number since s is even. So we can extend the matching by pairing up the unmatched vertices in A_i for all i.

Finally, the vertices not yet matched come in inverse pairs, since they lie outside the union of the subgroups G and $\langle xt_i \rangle$; so we can match each remaining vertex with its inverse. □

As a companion piece we have the following:

Theorem 9. *Let G be a finite group of odd order. Then $P(G\times C_2)$ has a perfect matching.*

Proof. $G \times C_2$ has a unique involution. □

4.4. 2-Groups

The following theorem characterises the 2-groups having perfect matchings in their power graphs.

Theorem 10. *Let G be a finite group with $|G| = 2^n$. Then $P(G)$ has a perfect matching if and only if G is cyclic or generalized quaternion.*

Proof. We have $|O(G)| = |O(G_G(S))| = 1$, so Theorems 2 and 3 show that G has a perfect matching if and only if it has a unique involution. The 2-groups with unique involution are the cyclic and generalized quaternion groups. □

5. A Number-Theoretic Result

We now head towards a proof that the matching numbers of $P(G)$ and $P_e(G)$ are equal for any group G. First we require a little number theory.

The functions $\tau(n)$ (the number of divisors of n) and $\phi(n)$ (Euler's totient function) are two of the best-studied in number theory. The result we require about them is elementary, but we have not found a proof in the literature.

Theorem 11. *Let n be a positive integer. If $n \geq 30$, then $\tau(n) < \phi(n)$.*

The proof depends on the formulae for these functions: if $n = \prod_{i=1}^{r} p_i^{a_i}$, where $p_1, \ldots, p_r$ are distinct primes and $a_1, \ldots, a_r$ are positive integers, then

(a) $\tau(n) = \prod_{i=1}^{r}(a_i + 1)$,

(b) $\phi(n) = \prod_{i=1}^{r} p_i^{a_i-1}(p_i - 1)$.

We use the following technical lemma:

Lemma 1. *Let p be a prime, and a a positive integer.*

(a) *If $(p, a) \notin \{(2, 1), (2, 2)\}$, then $p^{a-1}(p-1) \geq a+1$, with equality only if $(p, a) \in \{(2, 3), (3, 1)\}$.*

(b) *If $p \neq 2$ and $(p, a) \neq (3, 1)$, then $p^{a-1}(p-1) \geq 2(a+1)$, with equality only if $(p, a) = (5, 1)$.*

Proof. The function $f(x) = p^{x-1}(p-1) - (x+1)$ has derivative $f'(x) = p^{x-1}(p-1)\log p - 1$, which is positive for $x \geq 1$ if $p \neq 2$, and for $x \geq 2$ if $p = 2$. So for each p we only have to check the smallest values of x. □

Proof of the Theorem. To prove the theorem, we see that if n is odd or divisible by 8, then $\phi(n) \geq \tau(n)$, with strict inequality if the factorization includes 2^4, 3^2, or a prime larger than 3. If n is exactly divisible by 2^a with $a = 1$ or $a = 2$, then $2^{a-1}(2-1) \geq \frac{1}{2}(a+1)$, and so as long as we have a factor 3^3, 5^2 or a prime greater than 5 the strict inequality holds. The cases $n = 20$ and $n = 36$ satisfy the conclusion. Thus, the only cases for which it fails are $1, 2, 3, 4, 6, 8, 10, 12, 18, 24, 30$.

The result we actually require is the following corollary of this theorem. Recall that the *independence number* $\alpha(\Gamma)$ of a graph Γ is the size of the largest set of vertices containing no edges.

Corollary 2. *Let n be a positive integer. If $n \notin \{2, 6\}$, then the independence number of the power graph of the cyclic group C_n is strictly less than $\phi(n)$.*

Proof. In a cyclic group C_n, if two elements have the same order, then each is a power of the other, so they are joined in the power graph. So an independent set in the power graph has at most one element of each possible order, and its cardinality is at most $\tau(n)$. By Theorem 11, the conclusion holds if $n > 30$; it is easily checked directly for smaller values of n. □

Remark 2. In fact, it is easy to see that the independence number of $P(C_n)$ is the size of the largest antichain in the lattice of divisors of n. If n is a product of m primes (not necessarily distinct), then an antichain of maximum size is obtained by taking all distinct products of $\lfloor m/2 \rfloor$ primes, or all distinct products of $\lceil m/2 \rceil$ primes. (This extension of the celebrated Sperner lemma was proved by de Bruijn et al. [3].) This fact can be used to simplify the calculations in the Corollary.

6. The Matching Number of the Enhanced Power Graph

Recall that the *enhanced power graph* $P_e(G)$ of a finite group G is the graph with vertex set G in which two vertices x and y are joined if there exists z such that both x and y are powers of z (in other words, if $\langle x, y \rangle$ is cyclic). So the enhanced power graph contains the power graph as a spanning subgraph, and its matching number is at least as great as that of the power graph.

From our earlier work, there are several cases where equality holds:

(a) If $|G|$ is odd, then the power graph has a matching covering all but one vertex; therefore the same is true of the enhanced power graph.
(b) If the power graph of G has a perfect matching, then so does the enhanced power graph.
(c) Examining the proof of the formula for the matching number of the power graph of a nilpotent group (Theorem 4), we see that the same formula holds for the enhanced power graph.

We are going to prove that the matching numbers are always equal, even in cases where we cannot compute them from familiar group parameters.

Theorem 12. *Let G be a finite group. Then the matching numbers of the power graph and the enhanced power graph of G are equal.*

Proof. Let G be any finite group. Choose a matching M of maximum size in the enhanced power graph. If all its edges belong to the power graph, there is nothing to prove. Otherwise, we are going to change M to M' so that M' is a matching of the same size and has one fewer edge which does not belong to the power graph.

So let $\{g, h\}$ be an edge of the matching M which belongs to the enhanced power graph but not to the power graph. Choose this edge so that $\mathrm{lcm}(o(g), o(h))$ is as large as possible. Let ℓ be this lcm. Then $\langle g, h\rangle = C$ is a cyclic group of order ℓ. Let $x_1, \ldots, x_{\phi(\ell)}$ be the generators of C. They are joined to all vertices in C in the power graph.

Assume first that at least one of $x_1, \ldots, x_{\phi(\ell)}$, say x_i, is not covered by the edges of M. Then we can replace the edge $\{g, h\}$ by the edge $\{g, x_i\}$, which is an edge of the power graph. Similarly, if $\{x_i, x_j\}$ is an edge of M, replace it and $\{g, h\}$ by $\{g, x_i\}$ and $\{h, x_j\}$.

So we can assume that all of $x_1, \ldots, x_{\phi(\ell)}$ are covered by edges in M. Let $\{x_i, y_i\}$ be an edge of M for $i = 1, \ldots, \phi(\ell)$, with $y_i \notin \{x_1, \ldots, x_{\phi(\ell)}\}$.

For each i, there are three cases:

(a) x_i is a power of y_i;
(b) y_i is a power of x_i;
(c) neither of the above.

In case (a), g and h are powers of x_i, and hence also powers of y_i. So we can replace the edges $\{g, h\}$ and $\{x_i, y_i\}$ by $\{g, x_i\}$ and $\{h, y_i\}$, both of which are edges of the power graph.

In case (c), $\{x_i, y_i\}$ is an edge of the enhanced power graph but not of the power graph, and $\mathrm{lcm}(o(x_i), o(y_i)) > \ell$ (since x_i and y_i are both powers of some $z_i \notin C$), contradicting the choice of the edge $\{g, h\}$.

So we must be in case (b) for all i. This means that all of $y_1, \ldots, y_{\phi(\ell)}$ belong to C.

Now suppose that $\ell \notin \{2, 6\}$. Then the independence number of the power graph of C is strictly smaller than $\phi(\ell)$; so the set $\{y_1, \ldots, y_{\phi(\ell)}\}$ is not an independent set in the power graph, and so it contains at least one edge, say $\{y_i, y_j\}$. In this case, we replace the three edges $\{g, h\}$, $\{x_i, y_i\}$, $\{x_j, y_j\}$ by $\{g, x_i\}$, $\{h, x_j\}$, $\{y_i, y_j\}$, all edges of the power graph.

Finally, the case $\ell = 2$ is clearly impossible. If $\ell = 6$, let $C = \langle z\rangle$ be the cyclic group of order 6. There are just two nonedges of the power graph, namely $\{z^3, z^2\}$ and $\{z^3, z^4\}$; without loss of generality, $\{g, h\} = \{z^2, z^3\}$. We have $\{x_1, x_2\} = \{z, z^5\}$. Hence, necessarily $\{y_1, y_2\} = \{1, z^4\}$. But this is an edge of the power graph, so the argument in the preceding paragraph applies. □

7. Conclusion and Open Problems

The most important problem we have been unable to solve is the following.

Problem 1. (a) Find a formula for the matching number of $P(G)$ for any finite group G, in terms of group-theoretic parameters of G.
(b) Find a necessary and sufficient condition on a group G for $P(G)$ to have a perfect matching.

As noted in Introduction, groups for which the power graph and enhanced power graph coincide are the so-called *EPPO groups* or *CP-groups*, those for which all elements have prime power order. (These are also the finite groups whose *Gruenberg–Kegel graph* is null, see [5].) Their classification has a long and interesting history. A natural extension of this problem runs as follows:

Problem 2. Let π be a monotone graph parameter (that is, if Γ is a spanning subgraph of Δ then $\pi(\Gamma) \leq \pi(\Delta)$). Determine the finite groups for which $\pi(P(G)) = \pi(P_e(G))$.

Theorem 12 shows that, if π is the matching number, then the solution is "all finite groups". Also, it is easy to show that, if π is the clique number ω, then the solution is "all groups where the largest order of an element is a prime power". The same is true if π is the chromatic number χ: for, if G is a group with $\chi(P(G)) = \chi(P_e(G))$, then

$$\omega(P_e(G)) \geq \omega(P(G)) = \chi(P(G)) = \chi(P_e(G)) \geq \omega(P_e(G)),$$

the second and third terms being equal because $P(G)$ is a perfect graph (see [10]); so equality holds throughout.

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