Noname manuscript No.
(will be inserted by the editor)

The rigidity of infinite graphs II

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Received: date / Accepted: date

Abstract Inductive constructions are established for countably infinite simple graphs which have minimally rigid locally generic placements in $\mathbb{R}^2$. This generalises a well-known result of Henneberg for generically rigid finite graphs. Inductive methods are also employed in the determination of the infinitesimal flexibility dimension of countably infinite graphs associated with infinitely faceted convex polytopes in $\mathbb{R}^3$. In particular, a generalisation of Cauchy's rigidity theorem is obtained.

Keywords Infinite graphs, infinitesimal rigidity, Cauchy's rigidity theorem, graph rigidity

Mathematics Subject Classification (2000) 52C25 · 05C63

1 Introduction

A finite simple graph G is said to be *rigid* for the Euclidean space $\mathbb{R}^d$ (or d-rigid) if its generic vertex realisations $p : V \to \mathbb{R}^d$ determine bar-joint frameworks that are infinitesimally rigid. Also, G is *minimally rigid* if the removal of any edge leads to a non-rigid graph. A celebrated characterisation proved independently by Pollaczek-Geiringer [19] and Laman [14] asserts that minimal rigidity for the plane is equivalent to the edge count $|E| = 2|V| - 3$ together with the uniform sparsity condition $|E(H)| \le 2|V(H)| - 3$ over subgraphs H

This work was supported by the Engineering and Physical Sciences Research Council [grant numbers EP/J008648/1, EP/P01108X/1].

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with at least one edge. Recently we have obtained variants of this theorem for non-Euclidean ℓ^q-spaces and for countably infinite simple graphs in Euclidean and non-Euclidean ℓ^q-spaces (Kitson and Power [12,13]). The countably infinite variants derive from a relative rigidity principle, valid for all dimensions, to the effect that an infinite graph is rigid if and only if there exists an edge-complete inclusion tower $G_1 \subset G_2 \subset \dots$ of finite subgraphs each of which is relatively rigid in its successor (for an illustration of a tower with this property see Example 15). The characterisations of minimal rigidity for infinite graphs in the Euclidean and non-Euclidean settings for $\mathbb{R}^2$ are then obtained through the identification of a derived tower that consists of the appropriate minimally rigid graphs.

Henneberg [8] proved that if a finite graph is minimally rigid for the Euclidean plane then it may be constructed by applying a finite sequence of graph moves, starting with a single edge, where each move is either a *vertex addition* or *edge splitting* move. With the Laman graph characterisation this leads to the well-known fact that a finite graph is rigid for the Euclidean plane if and only if it contains a spanning subgraph which can be constructed by applying a finite sequence of these moves. In contrast to the sufficiency of this subgraph condition it is straightforward to see that a countable sequence of the moves can give rise to a non-rigid infinite graph. An example of this phenomenon is given in Example 12. We show that nevertheless, the necessity of the condition does hold for infinite graphs. In particular, the minimally rigid infinite graphs are no more mysterious than those which can be constructed one step at a time by Henneberg moves.

In Theorem 10 we show that a countably infinite minimally rigid graph may be constructed from a base graph by a countable sequence of construction moves. In the Euclidean case the base graph is K_2 and each of the moves is one of the two usual Henneberg moves for the plane. In the non-Euclidean cases, with distance metric determined by any of the ℓ^q-norms, $1 < q < \infty$, $q \neq 2$, we require the rigidity matroid associated with the sparsity count $|E| = 2|V| - 2$ and additional construction moves. Here the base graph is K_1 and there are four move types required, namely, the Henneberg vertex and edge moves, the vertex-to-K_4 move and the vertex-to-4-cycle move. The existence of a Henneberg move construction chain for a finite minimally rigid graph requires the identification of a low-degree vertex which admits an inverse Henneberg move to a minimally rigid graph (see [7,8]). We remark that such descent arguments are not available in the countable case and indeed a minimally rigid graph may have no vertices of finite degree. In Examples 11, 12 and 14, we indicate various countable graphs determined by inclusion towers and construction chain limits. Also we indicate the notion of relative rigidity, which becomes significant for dimensions $d \geq 3$.

Turning to three dimensions, the celebrated rigidity theorem of Cauchy [2] asserts that if all the faces of a closed convex polyhedron are infinitesimally rigid in the Euclidean space $\mathbb{R}^3$ then the polyhedron itself is infinitesimally rigid. (Alexandrov [1], page 125.) In particular, the bar-joint framework associated with a convex polyhedron with triangular faces is infinitesimally rigid.

The generic form of this phenomenon, which no longer requires convexity, has been obtained more directly by Whiteley who introduced *vertex splitting* construction moves as a key tool in geometric rigidity theory. (Whiteley [21,22].) In Theorem 19 we use such inductive methods to determine variants of the generic Cauchy theorem. In particular, we see that a generic finite triangulated sphere in the non-Euclidean space $(\mathbb{R}^3, \|\cdot\|_q)$ is infinitesimally flexible with generic infinitesimal flex dimension 3. We then consider the rigidity of countable graphs G in Euclidean 3-space which correspond to triangulations of a finitely punctured 2-sphere. The generic infinitesimal flex dimension is determined in terms of the asymptotic minima of cycle lengths at the puncture points. As a consequence we see that for these graphs the three properties of minimal rigidity, rigidity and sequential rigidity are equivalent (Theorem 26).

Finally, we determine the generic flex dimension of countable graphs associated with some infinitely-faceted convex polytopes in $\mathbb{R}^3$.

2 Preliminaries

A *bar-joint framework* (G,p) in $\mathbb{R}^d$ consists of a simple graph G with vertex set $V(G)$ and edge set $E(G)$ together with a *placement* of the vertices $p : V(G) \to \mathbb{R}^d$, $v \mapsto p_v$, with the property that $p_v \neq p_w$ for each edge $vw \in E(G)$. If H is a subgraph of G then the bar-joint framework (H,p) obtained by restricting p to the vertices of H is called a *subframework* of (G,p). The complete graph on the vertices of G is denoted $K_{V(G)}$.

We consider here the collection of ℓ^q norms on $\mathbb{R}^d$ where $q \in (1,\infty)$. An *infinitesimal flex* of (G,p), for a given ℓ^q-norm, is a map $u : V(G) \to \mathbb{R}^d$, $v \mapsto u_v$, which satisfies

$$\|(p_v + tu_v) - (p_w + tu_w)\|_q - \|p_v - p_w\|_q = o(t), \quad \text{as } t \to 0,$$

for each edge $vw \in E(G)$. The real vector space of infinitesimal flexes of (G,p) is denoted by $\mathcal{F}_q(G,p)$ and the subspace of trivial infinitesimal flexes of (G,p) is denoted $\mathcal{T}_q(G,p)$. For $q \neq 2$ this is simply the d-dimensional space of infinitesimal translations. (See [12].) The *infinitesimal flex dimension* of (G,p), denoted as $\dim_{d,q}(G,p)$, is defined to be the dimension of the quotient space $\mathcal{F}_q(G,p)/\mathcal{T}_q(G,p)$. A bar-joint framework is *infinitesimally rigid* in $(\mathbb{R}^d, \|\cdot\|_q)$ if $\dim_{d,q}(G,p) = 0$.

If G is a finite graph then the space of infinitesimal flexes $\mathcal{F}_q(G,p)$ is linearly isomorphic to the kernel of an associated *rigidity matrix* $R_q(G,p)$ as given in [12]. The rank of this matrix may vary depending on the choice of placement p and (G,p) is said to be *regular* in $(\mathbb{R}^d, \|\cdot\|_q)$ if the rank at p is the maximum possible. The framework (G,p) is *generic* in $(\mathbb{R}^d, \|\cdot\|_q)$ if $p_v \neq p_w$ for all pairs $v,w \in V(G)$ and given any graph H with $V(H) \subseteq V(G)$, the bar-joint framework (H,p) is regular. The generic placements of G, regarded as points in $\prod_{v\in V(G)} \mathbb{R}^d$, form an open and dense subset of this product space (see [13, Lemma 2.7] for further details). In this sense, almost all placements of G are generic. The infinitesimal flex dimension of (G,p), regarded as a function of p, is constant on the set of generic placements of G.

A countable bar-joint framework (G,p) is *locally generic* in $(\mathbb{R}^d, \|\cdot\|_q)$ if every finite subframework of (G,p) is generic. It is shown in [13, Section 3] that every countable simple graph has such a placement and that the infinitesimal flex dimension of a locally generic countable bar-joint framework (G,p) is independent of the choice of locally generic placement.

The *infinitesimal flex dimension* of a finite (resp. countable) simple graph G, denoted as $\dim_{d,q}(G)$, is defined to be the dimension of the quotient space $\mathcal{F}_q(G,p)/\mathcal{T}_q(G,p)$ where p is any generic (resp. locally generic) placement of G. A finite or countable graph is *rigid* in $(\mathbb{R}^d, \|\cdot\|_q)$ if $\dim_{d,q}(G) = 0$, and *minimally rigid* if, in addition, G does not contain a proper spanning subgraph which is rigid.

A *tower* of finite graphs is an inclusion chain $G_1 \subset G_2 \subset G_3 \subset \cdots$ of finite graphs G_k. We say that it is an *edge-complete tower* in a countable graph G if the G_k are subgraphs of G, respecting the given inclusions, and every edge of G is an edge of some G_k. If G has an edge-complete tower in which each G_k is rigid for $(\mathbb{R}^d, \|\cdot\|_q)$ then G is said to be *sequentially rigid* for $(\mathbb{R}^d, \|\cdot\|_q)$. In [13, Section 3] it is shown that a sequentially rigid countable graph is necessarily rigid and, in two-dimensions, sequential rigidity is equivalent to rigidity. This equivalence does not hold in general for higher dimensional spaces as we note in Example 15.

Given two simple graphs H and G we will use the notation $H \xrightarrow{\mu} G$ to indicate that G is the result of a graph move μ applied to H. A *construction chain* is a finite or countable sequence of graphs $G_1, G_2, G_3, \dots$ together with graph moves, $G_1 \xrightarrow{\mu_1} G_2 \xrightarrow{\mu_2} G_3 \xrightarrow{\mu_3} \cdots$.

Finally, we recall that a finite graph G is said to be,

(a) $(2,3)$-*sparse* if $|E(H)| \leq 2|V(H)| - 3$ for each subgraph H of G which contains at least two vertices,
(b) $(3,6)$-*sparse* if $|E(H)| \leq 3|V(H)| - 6$ for each subgraph H of G which contains at least three vertices,
(c) (k,k)-*sparse*, where $k \in \mathbb{N}$, if $|E(H)| \leq k|V(H)| - k$ for each subgraph H of G.

In addition, G is said to be $(2,3)$-*tight* if it is $(2,3)$-sparse and $|E(G)| = 2|V(G)| - 3$, $(3,6)$-*tight* if it is $(3,6)$-sparse and $|E(G)| = 3|V(G)| - 6$, and (k,k)-*tight* if it is (k,k)-sparse and $|E(G)| = k|V(G)| - k$.

3 Construction chains for countable rigid graphs in $\mathbb{R}^2$

In this section we establish the existence of Henneberg-type construction chains for the countable simple graphs that are minimally rigid for $(\mathbb{R}^2, \|\cdot\|_q)$ for $1 < q < \infty$. To begin we prove a simple lemma on the existence of low-degree vertices in $(2,2)$-tight and $(2,3)$-tight graphs. The degree of a vertex v in G is denoted $\deg(v)$.

Lemma 1 *Let G be a $(2,l)$-sparse graph with $l \in \{2,3\}$. Then*

(i) $\deg(v) \leq 3$ *for some* $v \in V(G)$.
(ii) *If G is $(2,l)$-tight then* $\min\{\deg(v) : v \in V(G)\} \in \{2,3\}$.
(iii) *If G is $(2,l)$-tight and H is a $(2,l)$-tight subgraph of G which is not a spanning subgraph then there exists a vertex $v \in V(G)\backslash V(H)$ such that v is either 2-valent or 3-valent in G.*

Proof We leave (i) and (ii) to the reader. To show (iii), suppose $\deg(v) \geq 4$ for every vertex v in $V(G)\backslash V(H)$. Then we obtain $\sum_{v\in V(G)\backslash V(H)} \deg(v) \geq 4|V(G)\backslash V(H)|$. The vertex-induced subgraph K of G determined by the vertices in $V(G)\backslash V(H)$ is $(2,l)$-sparse and so by (i) there exists a vertex $v_0 \in V(G)\backslash V(H)$ which is at most 3-valent in K. It follows that v_0 must be incident with a vertex w of H and that $\deg(w)$ must be strictly greater than the degree of w in H. Since H is $(2,l)$-tight we have $\frac{1}{2}\sum_{v\in V(H)} \deg(v) > |E(H)| = 2|V(H)| - l$. Hence,

$$|E(G)| = \frac{1}{2} \sum_{v\in V(G)} \deg(v) > 2|V(G)\backslash V(H)| + 2|V(H)| - l = 2|V(G)| - l.$$

This contradicts the $(2,l)$-sparsity count for G and so there must exist a vertex $v \in V(G)\backslash V(H)$ with $\deg(v) \leq 3$. The result now follows from (ii). □

We now define four graph moves which will allow us to construct a given $(2,l)$-tight graph from another $(2,l)$-tight graph with fewer vertices. The first two moves, vertex addition and edge splitting, are the standard Henneberg moves for the Euclidean plane. The two additional moves are required for the non-Euclidean setting.

Definition 2 A simple graph G' is obtained from a simple finite graph G by applying a move $\mu : G \to G'$.

(i) μ is a *vertex addition move* if G' results from adjoining a vertex v_0 to $V(G)$ and two edges v_0v, v_0w to $E(G)$.
(ii) μ is an *edge splitting move* if it results from removing an edge vw from $E(G)$ and adjoining a vertex v_0 of degree 3 to $G\backslash vw$ where the three new edges include v_0v and v_0w. We refer to these new edges as the *replacement-derived* edges of the move.
(iii) μ is a *vertex-to-K_4 move* if a vertex w_0 is chosen in G and three vertices w_1, w_2, w_3 are adjoined to $V(G)$ together with all interconnecting edges $\{w_iw_j : i,j = 0,1,2,3,\ i \neq j\}$ and every edge vw_0 in G which is incident with w_0 is either left unchanged, or, is reassigned to an edge of the form vw_j for some $j \in \{1,2,3\}$. We refer to these new edges as being *reassignment-derived.*
(iv) μ is a *vertex to 4-cycle move* if a vertex v is chosen in G together with edges vv_1, vv_2 and a vertex v_0 is adjoined to $V(G)$ together with the edges v_0v_1, v_0v_2, and every edge of the form $vw \in E(G)$, $w \neq v_1, v_2$, is either left unchanged, or, is reassigned to the edge v_0w. We again refer to these new edges as being *reassignment-derived.*

If a $(2,l)$-tight graph G contains a 2-valent vertex then an inverse of a vertex addition move may be applied to G by simply removing the 2-valent vertex and its two incident edges. It is clear that the resulting graph is again $(2,l)$-tight. The following lemma shows that if v is a 3-valent vertex in G then an inverse edge splitting move can be applied. However, if G is $(2,2)$-tight, then a proviso is that v is not contained in a copy of the complete graph K_4.

The following lemma, showing the possibility of an inverse edge splitting move for a degree 3 vertex, is well-known for $(2,3)$-sparse graphs while for $(2,2)$-sparse graphs it is a special case of Lemma 3.1 in Nixon and Owen [15]. For completeness we give proofs.

Lemma 3 *Let G be a $(2,l)$-sparse simple graph containing a 3-valent vertex v with adjacent edges $vv_1, vv_2, vv_3 \in E(G)$. Suppose that,*

(a) $l = 3$, or,
(b) $l = 2$ and $v_iv_j \notin E(G)$ for some distinct pair $v_i, v_j \in \{v_1, v_2, v_3\}$.

Then,

- *(i) there exists $v_i, v_j \in \{v_1, v_2, v_3\}$ with $v_iv_j \notin E(G)$ such that no $(2,l)$-tight subgraph of $G\backslash\{v\}$ contains both v_i and v_j,*
- *(ii) there exists an edge splitting move $H \xrightarrow{\mu} G$ where H is a $(2,l)$-sparse graph obtained by removing the vertex v and adjoining an edge with vertices in $\{v_1, v_2, v_3\}$.*

Moreover, if G is $(2,l)$-tight then H is also $(2,l)$-tight.

Proof The assumption in (b) that $v_iv_j \notin E(G)$ for some v_i and v_j is automatically satisfied in case (a). Otherwise, the subgraph of G induced by the vertices $\{v, v_1, v_2, v_3\}$ would contradict the $(2,3)$-sparsity count for G.

(i) Suppose that every pair of distinct vertices $v_i, v_j \in \{v_1, v_2, v_3\}$ with $v_iv_j \notin E(G)$ is contained in a $(2,l)$-tight subgraph $H_{i,j}$ of $G\backslash\{v\}$. We may assume that each $H_{i,j}$ is maximal in the sense that it is not a proper subgraph of any other $(2,l)$-tight subgraph of $G\backslash\{v\}$ which contains both v_i and v_j. Let H be the union of the subgraphs $H_{i,j}$ together with the vertices v, v_1, v_2, v_3, the edges vv_1, vv_2, vv_3 and the edge v_iv_j whenever $v_iv_j \in E(G)$. Then it can be verified that H is a subgraph of G which violates the $(2,l)$-sparsity count.

(ii) By (i) there exists $v_i, v_j \in \{v_1, v_2, v_3\}$ with $v_iv_j \notin E(G)$ such that no $(2,l)$-tight subgraph of G contains both v_i and v_j. Let H be the graph with vertex set $V(H) = V(G)\backslash\{v\}$ and edge set $E(H) = (E(G)\backslash\{vv_1, vv_2, vv_3\}) \cup \{v_iv_j\}$. Then

$$|E(H)| = |E(G)| - 2 \leq 2|V(G)| - l - 2 = 2(|V(H)| + 1) - l - 2 = 2|V(H)| - l.$$

If H' is a subgraph of H and $v_iv_j \notin E(H')$ then H' is a subgraph of G and so the $(2,l)$-sparsity count holds for H'. If $v_iv_j \in E(H')$ then $H'\backslash\{v_iv_j\}$ is a subgraph of G which contains the vertices v_i and v_j and so $H'\backslash\{v_iv_j\}$ is $(2,l)$-sparse but not $(2,l)$-tight. Hence

$$|E(H')| = |E(H'\backslash\{v_iv_j\})| + 1 \leq (2|V(H'\backslash\{v_iv_j\})| - l - 1) + 1 = 2|V(H')| - l.$$

Thus H is $(2, l)$-sparse and there exists an edge splitting move $H \xrightarrow{\mu} G$. The final statement is clear. □

Definition 4 A construction chain is called

1. a *Henneberg construction chain* if each graph move in the construction chain is either a vertex addition move or an edge splitting move. These moves are also known as the Henneberg 1 and Henneberg 2 moves (as well as 0-extension and 1-extension moves), and
2. a *non-Euclidean Henneberg construction chain* if each graph move is either a vertex addition move, an edge splitting move, a vertex-to-K_4 move or a vertex to 4-cycle move.

The existence of a finite Henneberg construction chain from K_2 to G is equivalent to the statement that G is minimally rigid for the Euclidean plane. This result is due to L. Henneberg [8]. Pollaczek-Geiringer [19] and Laman [14] later proved that such a construction chain exists if and only if G is $(2, 3)$-tight. We require the following more general statement.

Proposition 5 *Let G be a $(2, 3)$-tight subgraph of a finite simple graph G'. Then the following statements are equivalent.*

(i) G' is $(2, 3)$-tight.
(ii) There exists a Henneberg construction chain from G to G'.

Proof $(i) \Rightarrow (ii)$ Suppose G' is $(2, 3)$-tight and there does not exist a Henneberg construction chain from G to G'. We may assume that G' is minimal in the sense that it is a smallest (in terms of vertices) (2,3)-tight graph that contains G as a subgraph but cannot be reached from G by a Henneberg construction chain. By Lemma 1 (iii) there exists $v \in V(G')\backslash V(G)$ such that v has degree 2 or degree 3 in G'.

Suppose $\deg(v) = 2$, in G', with $vv_1, vv_2 \in E(G')$. Let H be the vertex-induced subgraph of G' with $V(H) = V(G')\backslash\{v\}$. Then G is a subgraph of H and H is $(2, 3)$-tight. By the minimality of G' there must exist a Henneberg construction chain from G to H. Applying a vertex addition move to H based on the vertices v_1 and v_2 we obtain G'. This is a contradiction.

Suppose $\deg(v) = 3$, in G', with $vv_1, vv_2, vv_3 \in E(G')$. By Lemma 3, there exists a $(2, 3)$-tight graph H obtained by removing the vertex v from G' and adjoining an edge of the form v_iv_j. Moreover, there is an edge splitting move $H \xrightarrow{\mu} G'$. Now G is a subgraph of H and so by the minimality of G' there must exist a Henneberg construction chain from G to H. Applying the edge splitting move to H based on the vertices v_1, v_2, v_3 and the edge v_iv_j we obtain G'. This is a contradiction and so we conclude that there must exist a Henneberg construction chain from G to G'.

$(ii) \Rightarrow (i)$ It is clear that vertex addition and edge splitting moves preserve the class of $(2, 3)$-tight graphs and so, since G is $(2, 3)$-tight, G' must also be $(2, 3)$-tight. □

The next proposition presents the corresponding fact for $(2,2)$-tight graphs. This generalises the statement, proved in [17], that there exists a non-Euclidean Henneberg construction chain from K_1 to G if and only if G is $(2,2)$-tight.

Proposition 6 *Let G be a $(2,2)$-tight subgraph of a finite simple graph G'. Then the following statements are equivalent.*

(i) G' is $(2,2)$-tight.
(ii) There exists a non-Euclidean Henneberg construction chain from G to G'.

Proof $(i) \Rightarrow (ii)$ Suppose G' is $(2,2)$-tight and there does not exist a non-Euclidean Henneberg construction chain from G to G'. We may assume that G' is minimal in the sense that it is a smallest (in terms of vertices) (2,2)-tight graph that contains G as a subgraph but cannot be reached from G by a non-Euclidean Henneberg construction chain. By Lemma 1 there exists $v_0 \in V(G')\backslash V(G)$ such that v_0 has degree 2 or degree 3 in G'. If $\deg(v_0) = 2$ in G' then we can apply an inverse vertex addition move as in Proposition 5 to obtain a contradiction.

Suppose $\deg(v_0) = 3$ in G' with incident edges $v_0v_1, v_0v_2, v_0v_3 \in E(G')$. If $v_iv_j \notin E(G')$ for some distinct pair $v_i, v_j \in \{v_1, v_2, v_3\}$ then we can apply an inverse edge splitting move as in Proposition 5 to obtain a contradiction. If the complete graph K on the vertices $\{v_0, v_1, v_2, v_3\}$ is a subgraph of G' then every vertex $v \in V(G')\backslash V(K)$ is incident with at most two vertices in $\{v_1, v_2, v_3\}$. Otherwise, the vertex-induced subgraph on $\{v, v_0, v_1, v_2, v_3\}$ would contradict the sparsity count for G'. We consider two possible cases. Firstly, the case when there does not exist a vertex in $V(G')\backslash V(K)$ which is incident with two vertices in $\{v_1, v_2, v_3\}$. And secondly, the case when there does exist a vertex in $V(G')\backslash V(K)$ which is incident with two vertices in $\{v_1, v_2, v_3\}$.

In the first case, let H be the $(2,2)$-tight graph obtained from G' by contracting the complete graph on $\{v_0, v_1, v_2, v_3\}$ to any one of the vertices v_0, v_1, v_2, v_3. From the $(2,2)$-sparsity of G' it follows that $K \cap G$ consists of at most one vertex $v_i \in \{v_1, v_2, v_3\}$ and so G is a subgraph of H. By the minimality of G', there exists a non-Euclidean Henneberg construction chain from G to H. We can now obtain G' by applying a vertex-to-K_4 move to H. This is a contradiction.

In the second case, suppose $w_0 \in V(G')\backslash V(K)$ and w_0 is incident with the vertices v_1 and v_2. Let H be the $(2,2)$-tight graph obtained from G' by identifying v_0 with w_0. Thus the edges v_0v_1 and v_0v_2 are removed, the edge v_0v_3 is replaced with w_0v_3 and the vertex v_0 is removed. Since $v_0 \notin V(G)$, G is a subgraph of H. By the minimality of G' there must exist a non-Euclidean Henneberg construction chain from G to H. Now G' may be obtained by applying a vertex to 4-cycle move to H based on the edges w_0v_1 and w_0v_2. This is a contradiction and so the result follows.

$(ii) \Rightarrow (i)$ It is clear that the four types of graph move which may occur in a non-Euclidean Henneberg construction chain each preserve the class of $(2,2)$-tight graphs. Thus, since G is $(2,2)$-tight, G' is also $(2,2)$-tight. □

A (k,l)-tight edge-complete tower in G is an edge-complete tower $G_1 \subset G_2 \subset G_3 \subset \cdots$ in G with the additional property that each G_k is (k,l)-tight. The following result is proved in [13, Section 4].

Theorem 7 *Let G be a countable simple graph.*

(A) *The following statements are equivalent.*
(i) G is minimally rigid for $(\mathbb{R}^2, \|\cdot\|_2)$.
(ii) G contains a $(2,3)$-tight edge-complete tower.

(B) *If $q \in (1,2) \cup (2,\infty)$ then the following statements are equivalent.*
(i) G is minimally rigid for $(\mathbb{R}^2, \|\cdot\|_q)$.
(ii) G contains a $(2,2)$-tight edge-complete tower.

A consequence of Theorem 7 is that every countable minimally rigid graph for $(\mathbb{R}^2, \|\cdot\|_q)$ necessarily contains arbitrarily large finite rigid subgraphs.

Definition 8 Let $\{G_k : k \in \mathbb{N}\}$ be a sequence of finite simple graphs such that $V(G_k) \subseteq V(G_{k+1})$ for all $k \in \mathbb{N}$. The *countable graph limit* $\varinjlim G_k$ is the countable graph with vertex set $V(\varinjlim G_k) = \bigcup_{k\in\mathbb{N}} V(G_k)$ and edge set $E(\varinjlim G_k) = \{vw : \text{for some } n \in \mathbb{N}, vw \in E(G_k) \text{ for all } k \geq n\}$.

The graph moves we have considered have the property that each move $G_k \xrightarrow{\mu} G_{k+1}$ is associated with a vertex set inclusion $V(G_k) \subset V(G_{k+1})$. In particular, every countable (Euclidean or non-Euclidean) Henneberg construction chain $G_1 \xrightarrow{\mu_1} G_2 \xrightarrow{\mu_2} G_3 \xrightarrow{\mu_3} \cdots$ has an associated graph limit $G = \varinjlim G_k$. It may be that every edge in every finite graph in the sequence is subject to later removal, in which case the graph limit G will have no edges. Such extreme oddities can be avoided by restricting attention to construction chains which are edge stable in the sense of the following definition.

Definition 9 Let $G = \varinjlim G_k$ be the countable graph limit associated to a (Euclidean or non-Euclidean) Henneberg construction chain with vertex inclusion chain $V(G_1) \subset V(G_2) \subset \ldots$. Let v be a vertex of G_j. Then a sequence of edges $vw_j, vw_{j+1}, \ldots$, with $vw_k \in E(G_k)$, is a *derived edge sequence* if, for every $k \geq j$, the edge vw_{k+1} is either,

(i) equal to vw_k, or,
(ii) replacement-derived from vw_k by an edge splitting move, or,
(iii) reassignment-derived from vw_k by a vertex-to-K_4 move on w_k, or,
(iv) reassignment-derived from vw_k by a vertex-to-4-cycle move on w_k.

A derived edge sequence is *stable* if there exists $k_0 \in \mathbb{N}$ such that $vw_k = vw_{k_0}$ for all $k \geq k_0$. The construction chain is *edge stable* if every derived edge sequence is stable.

We now prove that every countably infinite graph which is minimally rigid for $(\mathbb{R}^2, \|\cdot\|_q)$ arises as the graph limit determined by an edge stable construction chain.

Theorem 10 *Let G be a countable simple graph which is minimally rigid for $(\mathbb{R}^2, \|\cdot\|_q)$ where $q \in (1,\infty)$.*

(A) *If $q = 2$ then there exists an edge stable countable Henneberg construction chain $K_2 = G_1 \xrightarrow{\mu_1} G_2 \xrightarrow{\mu_2} G_3 \xrightarrow{\mu_3} \cdots$ such that $G = \varinjlim G_k$.*
(B) *If $q \neq 2$ then there exists an edge stable countable non-Euclidean Henneberg construction chain $K_1 = G_1 \xrightarrow{\mu_1} G_2 \xrightarrow{\mu_2} G_3 \xrightarrow{\mu_3} \cdots$ such that $G = \varinjlim G_k$.*

Proof (A) If G is minimally rigid for $(\mathbb{R}^2, \|\cdot\|_2)$ then by part (A) of Theorem 7 there exists an edge-complete tower $\{G_k : k \in \mathbb{N}\}$ in G such that each G_k is $(2,3)$-tight. By Proposition 5, for each $k \in \mathbb{N}$ there exists a Henneberg construction chain $G_k = H_{k,1} \xrightarrow{\mu_{k,1}} H_{k,2} \xrightarrow{\mu_{k,2}} \cdots \xrightarrow{\mu_{k,n_k-1}} H_{k,n_k} = G_{k+1}$. Also by Proposition 5, there exists a Henneberg construction chain from K_2 to G_1. By concatenating these construction chains of finite length and relabelling we obtain a countable construction chain $K_2 = \tilde{H}_1 \xrightarrow{\mu_1} \tilde{H}_2 \xrightarrow{\mu_2} \tilde{H}_3 \xrightarrow{\mu_3} \cdots$. Note that the countable graph limit for the sequence $\{\tilde{H}_k : k \in \mathbb{N}\}$ has vertex set

$$V(\varinjlim \tilde{H}_k) = \bigcup_{k\in\mathbb{N}} V(\tilde{H}_k) = \bigcup_{k\in\mathbb{N}} V(G_k) = V(G).$$

Also, the edges of each G_k are edges of G_{k+1} and, moreover, because of the nature of the Euclidean construction moves, no non-edge v_iv_j in $\tilde{H}_k$ can become an edge in $\tilde{H}_{k+1}$. Suppose $vv_j, vv_{j+1}, \ldots$ is a derived edge sequence in this construction chain. There exist $k_0 \geq j$ and l such that $\tilde{H}_{k_0} = G_l$. Thus the edge $vv_{k_0} \in E(\tilde{H}_{k_0})$ is not removed by any subsequent graph move and so the derived edge sequence is stable. It follows that $E(\varinjlim \tilde{H}_k) = \bigcup_{k\in\mathbb{N}} E(G_k) = E(G)$. Hence $G = \varinjlim \tilde{H}_k$.

For the proof of (B) apply a similar argument using part (B) of Theorem 7 and Proposition 6. □

The following curious example shows that a minimally rigid countably infinite graph may have no vertices of finite degree.

Example 11 Let T be the tree with countably many vertices and countably many branches at every vertex. Let G be the cone graph over T obtained by adding one new vertex w_0 and all edges vw_0 to the vertices v of T. Then every vertex of G has infinite degree. The graph G may also be derived as the graph limit of an edge stable Henneberg construction chain by starting with a single edge vw_0 and applying vertex addition moves. This construction chain provides an edge-complete $(2,3)$-tight tower for G and so G is minimally rigid for $(\mathbb{R}^2, \|\cdot\|_2)$ by Theorem 7.

Example 12 It is possible that an edge stable Henneberg construction chain composed of minimally rigid finite graphs G_k for $(\mathbb{R}^2, \|\cdot\|_2)$ has a limit G which is not rigid. This becomes evident, for example, for limits of planar Laman graphs G_k where edge splitting moves operate on the boundaries of the G_k. For example the semi-infinite strip graph indicated in Figure 1(e) is flexible and is obtained in this way. One can also obtain in this manner the infinite grid graph with vertices $v_{(m,n)}$ indexed by $\mathbb{Z}^2$ and edges of the form $v_{(m,n)}v_{(m+1,n)}$ and $v_{(m,n)}v_{(m,n+1)}$ for all $(m,n) \in \mathbb{Z}^2$. This graph fails to be rigid for $(\mathbb{R}^2, \|\cdot\|_2)$ since, for example, it has no finite rigid subgraphs with more than two vertices.

Let us also remark that although Euclidean plane countable graph rigidity is equivalent to sequential rigidity, this equivalence is very much a generic property and does not hold for general bar-joint frameworks. This can be seen in Figure 2. The infinite bar-joint framework is infinitesimally rigid but fails to be sequentially infinitesimally rigid. Also note that this bar-joint framework has a unique maximal proper infinitesimally rigid subframework.

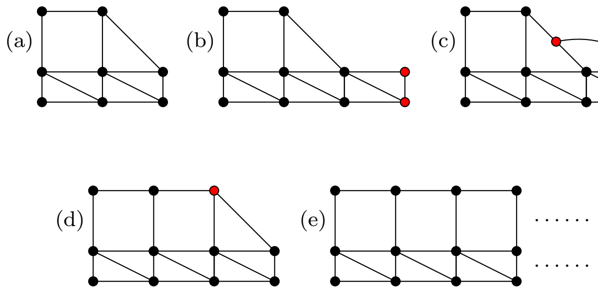

Fig. 1 A flexible graph (e) which is a limit of rigid graphs. The graph (c)=(d) is obtained from (a) by three Henneberg moves.

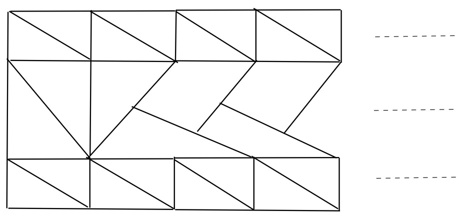

Fig. 2 An infinite bar-joint framework in the Euclidean plane that is infinitesimally rigid but not sequentially infinitesimally rigid.

Remark 13 In Nixon, Owen and Power [16] (see also [17]) a characterisation of finite $(2,2)$-tight graphs was established in order to obtain an analogue of Laman's theorem for graphs with respect to placements on a circular cylinder in $\mathbb{R}^3$. In this setting the placements are viewed as movably attached to the cylinder so that the admissible flexes (and velocity fields) are tangential to the cylinder. These considerations can be extended to countable simple graphs and it follows from the arguments here that graphs which are minimally rigid for the cylinder are constructible from K_1 by a non-Euclidean construction chain. A survey of inductive methods for bar-joint frameworks is given in [18].

Recall that the bipartite graph $K_{4,6}$ is rigid for $(\mathbb{R}^3, \|\cdot\|_2)$ and has no rigid subgraphs (with more than 2 vertices). We conjecture below that this double property is not possible for countably infinite graphs and dimensions $d \geq 3$.

Conjecture 14 Let $d \geq 3$ and let $q \in (1, \infty)$. Then every countably infinite graph G which is rigid for $(\mathbb{R}^d, \|\cdot\|_q)$ contains a finite rigid subgraph with at least 3 vertices.

The next example shows that countable graph rigidity is a more subtle consideration in higher dimensions.

Example 15 Figure 3 illustrates the first three graphs of an inclusion tower $\{G_n : n \in \mathbb{N}\}$. The first graph G_1 is the double banana graph, which is flexible for the Euclidean space $\mathbb{R}^3$. However, as a subgraph of G_2 it is *relatively rigid* in the sense that every infinitesimal flex of G_2 restricts to a rigid motion flex of G_1. Since every graph G_n is relatively rigid in its successor it follows that the countable graph G formed by their union is rigid. On the other hand, since the only rigid subgraphs of G are the the single banana graphs (copies of $K_5 \backslash e$) G is not sequentially rigid.

Substituting $K_{4,6}$ graphs for single banana graphs in this example gives another example of a minimally rigid countable graph, whose only rigid subgraphs with more than 2 vertices are copies of $K_{4,6}$.

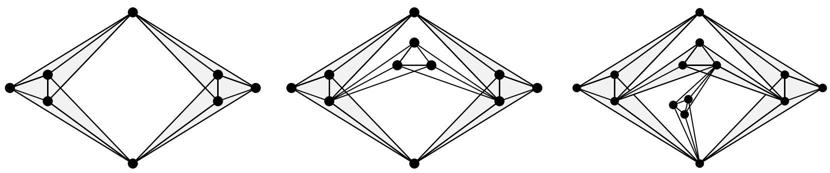

Fig. 3 The graphs G_1, G_2 and G_3 in Example 15.

4 Countable simplicial graphs and Cauchy's rigidity theorem

The generic form of Cauchy's rigidity theorem for finite Euclidean bar-joint frameworks in three dimensions may be stated as follows.

Theorem 16 (Cauchy, 1813) *Let G be the edge graph of a convex polyhedron with triangular faces. Then G is rigid for $(\mathbb{R}^3, \|\cdot\|_2)$.*

We first generalise this by determining the dimension of the space of infinitesimal flexes of a generic simplicial polytope which has κ nontriangular faces. To formalise this we define the following family of finite simple graphs.

Definition 17 A finite simple graph is said to be *simplicial with topological connectivity* κ if it has at least three edges and has a planar embedding in which exactly κ faces (including the unbounded face) are not triangular and where these faces have no vertices in common.

If $\kappa = 0$ then such a graph is the edge graph of a convex polyhedron with triangular faces (i.e. a convex simplicial polytope in $\mathbb{R}^3$).

The following vertex splitting construction move preserves the class of $(3,6)$-tight graphs and the class of $(3,3)$-tight graphs.

Definition 18 Let G and G' be finite simple graphs. Then G' is said to be obtained from G by applying a *vertex splitting move (for three dimensions)* if it results from,

(i) adding a vertex v_0 to $V(G)$,
(ii) adding an edge v_0v_1 for a vertex v_1 in $V(G)$ which has at least two incident edges v_1v_2 and v_1v_3,
(iii) adding the edges v_0v_2, v_0v_3,
(iv) replacing any number of the edges $wv_1, w \neq v_2, v_3$ by edges wv_0.

The preservation of 3-rigidity under the vertex splitting move is well-known. See Whiteley [22, Theorem 1.4.6]. This argument also has a straightforward ℓ^q-norm variant for $q \in (1,\infty)$. Note that a move which adds a 3-valent vertex to the face of a planar graph is a particular case of a vertex splitting move.

Vertex splitting plays a key role in the proof of the generic Cauchy theorem, which is the case $\kappa = 0$ in the following theorem. Since vertex splitting does not in general preserve infinitesimal flexibility dimension it is necessary to appeal to the linear independence of the rows of a generic rigidity matrix for a minimally rigid graph. We say that a finite graph G is $(3,q)$-*independent* if its rigidity matrix for the norm $\|\cdot\|_q$ is row independent. Also, we say that G is 3-independent if it is $(3,2)$-independent.

Theorem 19 *Let G be a finite simplicial graph with κ non-triangular faces which are bordered by cycles of length $\gamma_1, \ldots, \gamma_\kappa$.*

(i) For $\kappa > 0$

$$\dim_{3,2}(G) = \gamma_1 + \cdots + \gamma_\kappa - 3\kappa$$

while $\dim_{3,2}(G) = 0$ *for* $\kappa = 0$.

(ii) For $|V(G)| \geq 6$ and $1 < q < \infty$, $q \neq 2$,

$$\dim_{3,q}(G) = \dim_{3,2}(G) + 3$$

Proof Consider first the case $\kappa = 0$. Observe that any simplicial graph G of connectivity 0 can be constructed from K_3 by vertex splitting moves. To see this note that the inverse of a vertex splitting move (of planar type) is an edge contraction move (see Figure 4). Also, if G is not equal to K_3 then it can be shown (see [21, Corollary 7]) that there exist adjacent triangles, sharing an edge, such that edge contraction of this edge results in a simplicial graph of connectivity 0 which is an antecedent of G for vertex splitting. In view of this constructibility of G and the fact that K_3 is 3-rigid it follows that G is 3-rigid and $\dim_{3,2}(G) = 0$. Since $|E| = 3|V| - 6$ the graph G is minimally 3-rigid and in particular is 3-independent. The formula for $\dim_{3,2}(G)$ for $\kappa > 0$ now follows from the fact that the addition of $\gamma_1 + \cdots + \gamma_\kappa - 3\kappa$ edges to G creates a 3-independent simplicial graph with $\kappa = 0$.

For $q \neq 2$, note first that the $(3, 3)$-tight graph K_6 may be viewed as the graph of a regular octahedron together with three added internal edges. From the argument above, if $\kappa = 0$ then G is constructible from the regular octahedron graph by a sequence of vertex splitting moves. Observe that if the three internal edges of the regular octahedron are added then they are carried by these vertex splitting moves to produce a graph G^+ which consists of the simplicial graph G and three additional non-incident edges. Thus we have shown that G^+ is constructible from K_6 by a sequence of vertex splitting moves. The graph K_6 is known to be infinitesimally rigid for $(\mathbb{R}^3, \|\cdot\|)$ (see [5, Corollary 3.4] for a more general result) and so it follows that G^+ is infinitesimally rigid. On the other hand G^+ is $(3, 3)$-tight and so it is minimally infinitesimally rigid for $(\mathbb{R}^3, \|\cdot\|)$ and in particular is $(3, q)$-independent. The formula now follows, as for $q = 2$, by subtracting edges from G^+ to create G. □

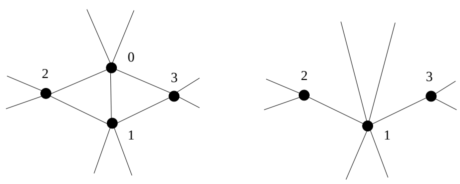

Fig. 4 Edge contraction as an inverse to a vertex splitting.

For non-Euclidean ℓ^q-norms we conjecture the following counterpart to the generic Cauchy theorem. Borrowing terminology from Whiteley [20] we refer to an internal bar that is added to a polytope framework as a *shaft*.

Conjecture 20 A generic simplicial polytope framework together with three non-incident shafts is minimally infinitesimally rigid in $(\mathbb{R}^3, \|\cdot\|_q)$ for $1 < q < \infty$, $q \neq 2$.

We remark that similar vertex splitting considerations hold for the analysis of the infinitesimal rigidity of block and hole frameworks in which, so to speak, simplicial polytope frameworks are adjusted by the deletion of a number of simplicial discs (holes) and the addition of extra edges to a number of other simplicial discs (creating blocks). See Finbow-Singh and Whiteley [6] and Cruickshank, Kitson and Power [3]. Vertex splitting has also been applied in recent work of Jordan and Tanigawa [10] on the generic global rigidity of simplicial graphs with added bracing edges in Euclidean 3-space.

4.1 Countable simplicial graphs.

We now consider the countable graphs that can be obtained from finite simplicial graphs of connectivity type κ by certain directed construction chains. Viewing such graphs as inscribed on the surface of a sphere this "directedness" corresponds, roughly speaking, to the κ nontriangular faces (the topological "holes") converging towards a set F on the sphere consisting of κ points.

The graphs can be thought of as simplicial triangulations of a finitely punctured sphere, $S^2\backslash F$, and we give the following direct definition in these terms, as a class of infinite planar graphs. This yields a somewhat larger class than that alluded to in the previous paragraph since it also allows for the infinite internal refinement of a finite number of triangular faces of the initial graph.

Recall that a countable graph has a *planar embedding*, or is a *planar graph*, if it may be realised by a set of distinct points in the plane and a family of non-crossing continuous paths. Let S^2 be the one-point compactification of the plane.

Definition 21 Let G be a locally finite countable graph and let ρ and κ be non-negative integers with $0 \leq \kappa \leq \rho$. Then G is said to be a *simplicial graph of type* (ρ, κ) if there is a planar embedding of G which is a triangulation of $S^2\backslash F$, where

(i) F is the set of accumulation points of the vertex points and $|F| = \rho$, and
(ii) there are exactly κ points p of F for which there exists a neighbourhood of p which contains no 3-cycles of represented edges around p.

We refer to ρ as the *refinement type* and κ as the *connectivity* of G. It follows that a countable simplicial graph of type (ρ, κ) has an edge-complete tower $G_1 \subset G_2 \subset G_3 \subset \cdots$ in which successive graphs are obtained by partially paving in the κ nontriangular faces and $\rho - \kappa$ triangular faces. In particular a simplicial graph with $(\rho, \kappa) = (1, 0)$ is obtained by joining a sequence of simplicial polyhedral graphs over common faces so that each polyhedron (except the first) has two neighbours. Similarly, a simplicial graph with $(\rho, \kappa) = (1, 1)$ is obtained by joining together a sequence of simplicial annuli (see the definition below) over identified end cycles.

We now formalise the general construction sequence for a simplicial graph of finite refinement type and introduce a limit form of the joint cycle index.

A *simplicial disc graph* of type $r \geq 3$ is a finite planar graph determined by the triangulation of an r-cycle. A *simplicial polytope graph* is a graph determined by the edges of a convex polyhedron with triangular faces. A *simplicial* *annulus graph* of type (r, s) is a graph which arises from a simplicial polytope graph following the removal of the interior edges and vertices of two face-disjoint simplicial disc subgraphs of type $r \geq 4$ and $s \geq 4$, respectively. A simplicial annulus graph is thus a planar graph with two faces that are not triangles. This includes the degenerate case of a cycle graph with two faces.

We define the *girth* $\gamma = \gamma(G)$ of a simplicial annulus graph G of type (r, s) as the minimum length of an edge cycle in G which winds once around the annulus. This is also the minimum length of cycles which provide a generator for the first simplicial homology group of the simplicial complex of G. We say that a simplicial annulus graph T is of type (r, s, γ) in this case, and if C is such a minimum length cycle subgraph then C is said to be a *girth cycle*.

The result of the next lemma appears as Theorem 5.1 of [6] with a proof based on r-connectivity and Menger's theorem. We give a direct proof based on graph reduction by division by girth cycles. In fact the lemma gives a direct proof that the block and hole r-towers of Finbow-Singh and Whiteley [6], for $r \geq 4$, are 3-rigid if their *girth* is not less than r. The necessity of this condition also holds. For more detail see [3], [6].

Lemma 22 *If T is a simplicial annulus graph of type (r, r, r) then there is a finite construction sequence $H_1 \to H_2 \to \cdots \to T$ where H_1 is an r-cycle and the moves are vertex splitting moves.*

Proof Let e be an edge which lies on no girth cycle of T. Then the contraction on this edge gives a simplicial annulus graph T' of the same type, (r, r, r). Since $T' \to T$ is a vertex splitting move we may assume that all edges lie on girth cycles. Suppose that T is a graph of this type with the fewest edges and suppose, by way of contradiction, that T is not an r-cycle. Then T has an edge e that does not lie on either of the two nontriangular faces of T, the outer and inner boundary cycles, C_a and C_b say. Let C be a girth cycle that includes e. Then the simplicial annuli, T_a and T_b say, with boundaries $\{C_a, C\}$ and $\{C_b, C\}$, respectively, also have girth r. Since T_a and T_b have fewer edges they are constructible from an r-cycle by vertex splitting. But in this case T is also constructible, a contradiction. □

Let G be a simplicial disc graph of type r and let T be a simplicial annulus graph of type (r, s). A graph obtained by identifying the boundary r-cycles of G and T is referred to as a *join*, denoted $G \sqcup T$. In a similar way we define the *join* $G \sqcup T$ when G is a finite simplicial graph of connectivity κ with an r-cycle face and where this r cycle is identified with a boundary r-cycle of T.

Lemma 23 *Let $G_1 \subseteq G_2$ where G_1 is a simplicial graph of type (ρ, κ) with a boundary cycle of length r_1 and where $G_2 = G_1 \sqcup T$ is a join with T a simplicial annulus graph of type (r_1, r_2, r).*

(i) If $r = r_1 = r_2$ then for generic placements p the natural restriction map,

$$\pi_{G_2,G_1} : \mathcal{F}_2(G_2, p) \to \mathcal{F}_2(G_1, p|_{G_1}),$$

is a linear isomorphism.

(ii) If $r = r_1 \leq r_2$ then the restriction map in (i) is a surjection.

Proof (i) The restriction map is clearly linear and so it is sufficient to show that it is injective. Consider a non-zero infinitesimal flex u in $\mathcal{F}_2(G_2, p)$ which lies in the kernel of the restriction map. This provides a non-zero infinitesimal flex for the framework for $K_r \sqcup T$ obtained by adding bars for all pairs of joints for the boundary r_1-cycle of T. However, K_r is 3-rigid. Also since vertex splitting preserves infinitesimal rigidity Lemma 22 implies that this framework is also infinitesimally rigid. This contradiction completes the proof.

(ii) By adjoining a copy of T to G_2 along the common boundary cycle of length r_2, we obtain $G_2' = G_1 \sqcup T'$ where $T' = T \sqcup T$ is a simplicial annulus graph of type (r_1, r_1, r). Let p be a generic placement of G_2' in $\mathbb{R}^3$. By (i), the natural restriction map $\pi_{G_2',G_1} : \mathcal{F}_2(G_2', p) \to \mathcal{F}_2(G_1, p|_{G_1})$ is a linear isomorphism. The result now follows since $\pi_{G_2',G_1}(\mathcal{F}_2(G_2', p))$ is a linear subspace of $\pi_{G_2,G_1}(\mathcal{F}_2(G_2, p))$. □

We remark that the isomorphism conclusion in (i) fails if T has type (r, r, s) with the girth s strictly less than r. This follows from the fact that a single block and single hole graph $K_r \sqcup T$ is not rigid for $(\mathbb{R}^3, \|\cdot\|_2)$ by the results in [3] and [6].

Definition 24 Let G be a countable simplicial graph of type (ρ, κ).

(i) If $\rho = 1$ then the *asymptotic girth* $\gamma(G)$, with value in the set $\{3, 4, \dots\} \cup \{\infty\}$, is the common value,

$$\gamma(G) := \lim_k \left(\lim_{l:l>k} \gamma(G_l \backslash G_k) \right),$$

associated with an inclusion chain $G_1 \subseteq G_2 \subseteq \dots$ for G of simplicial discs, where, for $l > k$, $G_l \backslash G_k$ is the simplicial annulus $T_{k,l}$ in the join representation $G_l = G_k \sqcup T_{k,l}$.

(ii) For $\rho \geq 2$ the *joint asymptotic girth* is the list $(\gamma_1(G), \dots, \gamma_\rho(G))$ of the asymptotic girths $\gamma_i(G)$ associated with the i^{th} refinement point in a spherical representation of G.

Note that $\gamma(G)$ in (i) is well-defined since $\gamma(G_l \backslash G_k)$ is a decreasing sequence in l, for fixed k, and any two inclusion chains for G have subchains which are also interlacing subchains of a single inclusion chain.

Theorem 25 *Let G be a countable simplicial graph of refinement type ρ and connectivity κ. Then the infinitesimal flex dimension of G is given in terms of the asymptotic girth of G by the formula,*

$$\dim_{3,2}(G) = \sum_{i=1}^{\rho} (\gamma_i(G) - 3).$$

Proof Suppose first that the joint asymptotic girth is a ρ-tuple of finite integers $(\gamma_1, \dots, \gamma_\rho)$. Then it follows that there exists an edge-complete tower $G_1 \subseteq G_2 \subseteq \dots$ where each G_k is a simplicial graph of connectivity κ with boundary cycles of lengths $\gamma_1, \dots, \gamma_\kappa$ obtained by the addition of κ simplicial annuli of type $(\gamma_i, \gamma_i, \gamma_i)$ to G_{k-1}. Let p be a generic placement of (G, p). By Lemma 23 the natural restriction maps provide linear isomorphisms between the infinitesimal flex spaces,

$$\mathcal{F}_2(G_1, p) \leftarrow \mathcal{F}_2(G_2, p) \leftarrow \mathcal{F}_2(G_3, p) \leftarrow \cdots .$$

Note that this sequence of vector spaces and linear maps is an inverse system and $\mathcal{F}_2(G, p)$, together with the natural restriction maps $\mathcal{F}_2(G, p) \to \mathcal{F}_2(G_k, p)$, is the inverse limit. It follows that $\dim \mathcal{F}_2(G, p) = \dim \mathcal{F}_2(G_k, p)$ for each k and so the dimension formula follows.

Suppose now that γ_i is infinite for some i. Once again we may assume that there is an edge complete tower for G as before. By Lemma 23 (ii), for a generic placement (G, p) we have a natural inverse system of surjections,

$$\mathcal{F}_2(G_1, p) \leftarrow \mathcal{F}_2(G_2, p) \leftarrow \mathcal{F}_2(G_3, p) \leftarrow \cdots .$$

Also, the infinitesimal flex space of (G, p) is the inverse limit of this system. By Theorem 19 this flex space has infinite dimension, as required. □

Theorem 26 *Let G be a countable simplicial graph of refinement type ρ and connectivity κ. Then the following conditions are equivalent.*

(i) *G is 3-rigid.*
(ii) *G is minimally 3-rigid.*
(iii) *G is sequentially 3-rigid.*
(iv) $\kappa = 0$.

Proof From the definition of κ it follows immediately that if $\kappa = 0$ then there is an edge-complete inclusion chain of finite graphs each of which is the edge graph of a simplicial polytope. From the generic Cauchy theorem (case $\kappa = 0$ of Theorem 19) it follows that G is sequentially 3-rigid, and hence 3-rigid. To see that G is also minimally 3-rigid we first note that if $G_1 \subseteq G_2$ is an inclusion of finite simplicial polytope graphs obtained by triangulation of several of the faces of G_1 and if e is an edge of G_1 then for any generic realisation the natural restriction map,

$$\mathcal{F}_2(G_2 \backslash e, p) \to \mathcal{F}_2(G_1 \backslash e, p)$$

is an isomorphism of real vector spaces of dimension 7. Indeed, not only are the spaces of dimension 7 as a consequence of minimal rigidity but every infinitesimal flex of $(G_1 \backslash e, p)$ extends to an infinitesimal flex of $(G_2 \backslash e, p)$ and so the map is a surjection. It follows that $\dim_{3,2}(G \backslash e, p) = 1$.

That (i) implies (iv) follows from the previous theorem and so the proof is complete. □

For a finite simple graph G and $i = 1, 2$ write $H_i(G)$ for the integral simplicial homology groups of the simplicial complex $\Delta(G)$ whose j-simplexes correspond to copies in G of the complete graph K_j. One may similarly define integral homology groups for a countable simple graph in terms of the chain groups which may now have countably many generators. It follows that $H_i(G)$ coincides with the direct limit $H_i(G) = \varinjlim_k H_i(G_k)$ associated with any edge-complete tower $G_1 \subseteq G_2 \subseteq \dots$ for G. These groups are therefore readily computable for infinite graphs given by construction chains.

For the countable graphs of Theorem 26 we have $H_1(G) = \mathbb{Z}^{\kappa-1}$ for $\kappa \geq 1$. Let us also remark that there are sequentially 3-rigid graphs of this type with $\kappa = 0$ and $H_2(G) = 0$. These extreme examples derive from the planar graph K_4 and a construction chain $K_4 \to G_2 \to G_3 \to \dots$ obtained by single vertex addition moves of planar type in the following sense. Each new vertex of degree 3 in G_{k+1} has its edges incident to the vertices of a face of G_k. In particular $H_2(G_k) = 0$ for all k. See Figure 6.

4.2 Infinite polytopes

We now give examples of countable simplicial graphs of type (ρ, κ) which arise from various infinitely faceted polytopes. We compute the infinitesimal flex dimensions by applying Theorem 25. Note first that countable simplicial graphs G of type (ρ, κ) can be constructed from the edge graph of a convex polyhedron $\mathcal{P}$ with triangular faces by means of the following general scheme.

(a) A finite set F of ρ vertices of $\mathcal{P}$ is chosen, κ of which have degree greater than 3.
(b) The faces of $\mathcal{P}$ which are incident to at least one of these vertices, v say, are countably triangulated towards v by means of sequences of added vertices on the edges that are incident to v.
(c) The countable graph G has vertex set equal to the union of the added vertices and the vertices of the complement $V(\mathcal{P})\backslash F$.

To be more explicit, we may take $F = V(\mathcal{P})$, for example, and triangulate each face of $\mathcal{P}$ in the manner of the template of Figure 5.

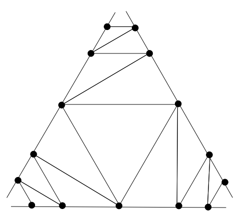

Fig. 5 Triangulation of a face of a convex polyhedron.

Example 27 Applying this refinement process to all vertices for the three platonic deltahedra one obtains

(i) tetrahedral countable simplicial graphs G with $(\rho, \kappa) = (4, 0)$, joint asymptotic girth $\gamma(G) = (3, 3, 3, 3)$ and $\dim_{3,2}(G) = 0$,
(ii) octahedral countable simplicial graphs with $(\rho, \kappa) = (6, 6)$, joint asymptotic girth $\gamma(G) = (4, 4, 4, 4, 4, 4)$ and $\dim_{3,2}(G) = 6$,
(iii) icosahedral countable simplicial graphs with $(\rho, \kappa) = (12, 12)$, a twelvefold asymptotic girth index $\gamma(G) = (5, \ldots, 5)$ and $\dim_{3,2}(G) = 24$.

Example 28 The countable simplicial graph G of type $(1, 0)$ indicated in Figure 6 is obtained by triangulating the faces of a 3-simplex towards a single vertex. The asymptotic girth is $\gamma(G) = 3$ and the infinitesimal flex dimension is $\dim_{3,2}(G) = 0$.

Example 29 Figure 7 illustrates a countable simplicial graph G of type $(1, 1)$ which is obtained by triangulating the faces of an octahedron. The asymptotic girth is $\gamma(G) = 4$ and the infinitesimal flex dimension is $\dim_{3,2}(G) = 1$.

The following example indicates a bar-joint framework which is associated with an infinitely faceted compact "diamond" polytope. (Its appearance has a passing resemblance to a cut diamond).

Example 30 Let $p_1, p_2, \ldots$ be a sequence of points on the unit sphere as indicated in Figure 8 and let $\mathcal{P}_{\mathrm{dia}}$ be the convex hull of this set of points together with the north polar point. The associated bar-joint framework (G, p) is determined by the edges of $\mathcal{P}_{\mathrm{dia}}$ and the underlying structure graph for the edges (omitting the north pole) is a countable simplicial graph of type $(\rho, \kappa) = (1, 1)$. In this example the number of vertices of G for successive latitudes doubles on passing to the next highest latitude and it follows that $\dim_{3,2}(G) = \infty$.

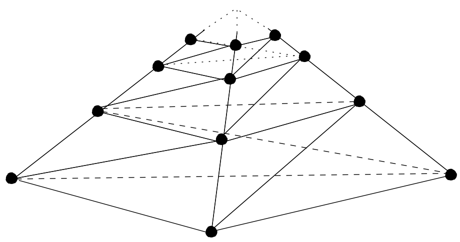

Fig. 6 A countable simplicial graph of type $(1, 0)$ obtained from the 3-simplex K_4.

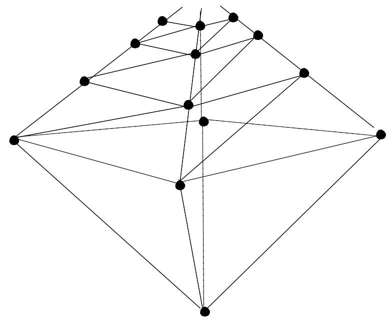

Fig. 7 A countable simplicial graph of type $(1,1)$ obtained from an octohedron.

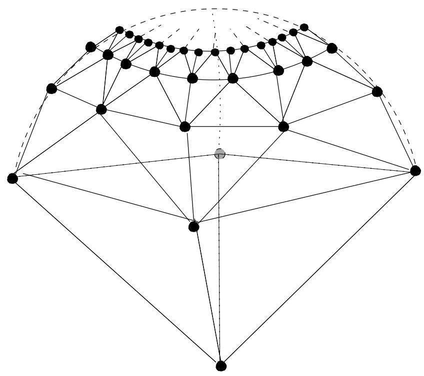

Fig. 8 A countable simplicial graph of type $(1,1)$, with infinite asymptotic girth, obtained from a diamond polytope.

4.3 Final remarks

We expect that the preceding arguments can be adapted to the determination of rigidity and flexibility dimension for bar-joint frameworks arising from various partial triangulations, both fine and infinite, of other surfaces. In particular this seems very likely for the projective plane (for all ρ, κ) and for the torus (for $\rho = 1$) since minimal infinitesimal rigidity, for finite partial triangulations, has been characterised in these cases. See [11] and [4]. The partially triangulated torus with $\rho = 2$ allows double banana phenomena and so there would be more to say in this case.

We remark that the countable simplicial graphs are also interesting from the point of view of the *continuous flexibility* and *continuous rigidity* of their placements in $\mathbb{R}^3$. For example it can be shown that the framework indicated in Figure 7 admits no nontrivial continuous flexes. On the other hand we do not know if the convex placement of the diamond polytope graph, indicated in Figure 8, is rigid in this sense. It could well be that the recent paper of Holmes-Cerfon, Theran and Gortler [9] on the almost-rigidity of finite frameworks could give quantitive techniques for determining when specific placements are continuously rigid.

Acknowledgements

The authors are grateful to the anonymous referees for their suggestions, which improved the presentation of this paper.

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