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Sub-elliptic pseudo-differential calculus on SU(2) and related results

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Sub-elliptic pseudo-differential calculus on $SU(2)$ and related results

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Declaration of authorship

I am the author of this thesis, and the work described therein was carried out by myself personally.

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Abstract

This thesis aims to develop a sub-elliptic pseudo-differential calculus on any compact Lie group G. We build an operator class Ψ which forms an algebra of operators.

We consider a Hörmander system on G and its associated sub-Laplacian $\mathcal{L}$. The Sobolev spaces that arise naturally from the sub-elliptic operator $\mathcal{L}$ are well known, and we check some important properties.

Our symbolic calculus is then developed, we define our symbol classes S^m on G and their associated operator classes Ψ^m, for $m \in \mathbb{R}$. A particular example of these symbol classes, $S^m(Q_0)$, is considered and we show that $S^m(Q_0)$ is contained in any S^m.

The core results of this thesis are then proved. We show that if $T_1 \in \Psi^{m_1}(Q_0)$ and $T_2 \in \Psi^{m_2}(Q_0)$, then the composition operator $T_1 \circ T_2$ satisfies

$$T_1 \circ T_2 \in \Psi^{m_1+m_2}(Q_0).$$

Contents

Chapter 1

Introduction

The research presented in this thesis aims to provide an in-depth study of a sub-elliptic pseudo-differential calculus on compact Lie groups. Our objective is to define a class of operators which forms a symbolic calculus, and to establish results analogous to the already well-understood Euclidean setting. We do not provide an overview of these well-known results in this thesis, but the reader is referred to Stein [47] for an introduction in this subject.

One of the main tools used in our research is Fourier multipliers. These have been extensively studied throughout the history of mathematics, with the first important results appearing in Marcinkiewicz [32], where conditions for L^p Fourier multipliers on the torus were given. In 1956, this result was extended to $\mathbb{R}^n$ in Mihlin [34]. It was proved that if the bounded function $\sigma : \mathbb{R}^n \longrightarrow \mathbb{C}$ satisfies

$$|\partial_\xi^\alpha \sigma(\xi)| \leq C_\alpha |\xi|^{-|\alpha|}, \quad \xi \in \mathbb{R}^n, \tag{1.0.1}$$

for all multi-indices $|\alpha| \leq \lceil n/2 \rceil + 1$, then σ is an L^p Fourier multiplier for $1 < p < +\infty$. In 1960, this result was expanded further in Hörmander [27]. The condition given by (1.0.1) has a natural connection to the definition of the Euclidean symbol class $S^m_{1,0}(\mathbb{R}^n)$ $(m \in \mathbb{R})$, which is given by all functions σ on $\mathbb{R}^n \times \mathbb{R}^n$ such that

$$|\partial_x^\beta \partial_\xi^\alpha \sigma(x,\xi)| \leq C_{\alpha,\beta}(1 + |\xi|^2)^{\frac{1}{2}(m-|\alpha|)},$$

for all multi-indices α, β. In the setting of Lie groups, the reader is referred, for example, to Coifman and Weiss [8], which serves as an introduction to Fourier multipliers on the compact Lie groups $SU(2)$.

Some recent results related to the research presented in this thesis include

Ruzhansky, Turunen and Wirth [44], which proposes a global characterisation of a Hörmander class of pseudo-differential operators on compact Lie groups. Furthermore, Fischer [17] expanded on the results by Ruzhansky et al., providing a complete intrinsic description of a pseudo-differential calculus of compact Lie groups. In the literature, one can also find results concerning a pseudo-differential calculus on non-compact Lie groups. See, for example, Taylor [52] or Hajer Bahouri, Fermanian-Kammerer and Gallagher [3], which focus on the Heisenberg group $\mathbb{H}$, and also the monograph Fischer and Ruzhansky [18], which provides a pseudo-differential calculus for nilpotent Lie groups.

A common theme among the results in the compact case is the use of elliptic operators in the calculus, such as the Laplace operator. However, in our analysis we explore a sub-elliptic setting instead, choosing an appropriate sub-Laplacian to suit our objectives. This change has some important implications in the functional analysis. For instance, in this case the Carnot-Carathéodory metric is a more appropriate tool than a Riemannian metric, which is often useful in the elliptic case. A study of sub-Riemannian metrics in the Euclidean case can be found, for example, in Nagel, Stein and Wainger [36], Fefferman and Phong [15], or Parmeggiani [37]. Another consequence of this choice is reflected on the Sobolev spaces we work with. As we shall see in Section 3.2, our Sobolev spaces L^2_s will be defined in terms of the sub-Laplacian we chose, and we will need to check that the fundamental properties of Sobolev spaces are satisfied; such as the interpolation theorem (see Theorem 3.3.1) or a Sobolev inequality (see Theorem 3.4.1). In this context, sub-elliptic operators are usually a natural consideration in the case of stratified nilpotent Lie groups, and a study of Sobolev spaces in this setting can be found, for example, in Folland [19] or Fisher and Ruzhansky [18].

One of the most important tools we use in our symbol classes S^m $(m \in \mathbb{R})$ are difference operators. These are defined in order to replace differentiation in the frequency variable. In the literature, they were used with the purpose of studying a pseudo-differential calculus of compact Lie groups in Fischer [17] and Ruzhansky et al. [44], and of nilpotent Lie groups in [18]. A first fundamental example of difference operators can be found in Ruzhansky and Turunen [43] for the case of the torus. If κ is a function on $\mathbb{T}^d$, for some $d \in \mathbb{N}$, then for any $j \in \{1, 2, \dots, d\}$ the difference operator Δ_j acts on κ by

$$\Delta_j \widehat{\kappa}(\xi) = \widehat{\kappa}(\xi + e_j) - \widehat{\kappa}(\xi), \quad \xi \in \mathbb{Z}^d, \tag{1.0.2}$$

where e_j denotes the unit vector in the j-th direction. It can then be shown that

$$\Delta_j \widehat{\kappa}(\xi) = \widehat{q_j \kappa}(\xi), \quad \xi \in \mathbb{Z}^d,$$

where $q_j : \mathbb{T}^d \to \mathbb{C}$ is the function given by

$$q_j(x) = e^{ijx} - 1, \quad x \in \mathbb{T}^d.$$

The concept of difference operators discussed in this thesis generalises this to the setting of any compact Lie group. Suppose q is a smooth, real-valued function on G. If κ is a distribution on G, we define the difference operator associated to q, Δ_q, by the relation

$$\Delta_q \widehat{\kappa} = \widehat{q\kappa}.$$

Higher order derivatives applications of difference operators can be explained in the following way. Suppose $Q = \{q_1, q_2, \ldots, q_\ell\}$ is a family of smooth, real-valued functions on G, for $\ell \in \mathbb{N}$. Then, for any $\alpha \in \mathbb{N}_0^\ell$, we let Δ_Q^α be the difference operator given by

$$\Delta_Q^\alpha \widehat{\kappa} = \widehat{\widetilde{q}_\alpha \kappa},$$

for a distribution κ, where

$$\widetilde{q}_\alpha(z) := q(z^{-1})^{\alpha_1} q(z^{-1})^{\alpha_2} \cdots q(z^{-1})^{\alpha_\ell}, \qquad z \in G.$$

A difficulty found in difference operators is that they do not satisfy Leibniz's rule, in general, which is usually an exploitable property of differential operators. However, we are able to prove an analogous result (see Theorem 4.10.1) for a particular family of difference operators, associated to a class of smooth, real-valued functions Q_0 (see (4.2.12)). In practice, this result plays a similar role to Leibniz's rule, and allows us to prove our main theorems. This choice of Q_0 is, in fact, not aleatory, with one of its main properties being that the functions in Q_0 appear in the Taylor expansion of any smooth function (see Theorem 4.3.3). Additionally, we are also able to show kernel estimates of Calderón-Zygmund type (in the sense of Coifman and Weiss [7]) for the symbols belonging to $S^m(Q_0)$, the family of symbols of class m, associated to Q_0.

This thesis is organised as follows. The preliminary chapter (Chapter 2) focuses on introducing the fundamental tools needed throughout, including Lie groups and Lie algebras, Plancherel's Theorem, the Schwartz kernel Theorem, Haar integration and the exponential map, just to name a few. Chapter 3 then focuses on establishing a foundation for our work, introducing our Sobolev spaces and confirming some expected results. In this chapter we also study the Fourier multipliers of our sub-Laplacian, and we show our first important result in Lemma 3.8.1. Although this chapter does not have any groundbreaking mathematics, we include it in the thesis to keep this exposition self-contained. Chapter 4 is the main chapter of this thesis and is dedicated to developing our sub-elliptic pseudo-differential calculus for any compact Lie group.

Summarising our results presented in Chapter 4, we begin with the introduction of our difference operators. In Section 4.4.2 is where we first discuss the classes of symbols S^m and their associated operator classes Ψ^m. We then show that $S^m(Q_0) \subset S^m(Q)$, for any family Q of smooth, real-valued functions, which satisfies a condition we call 'comparability to the Carnot-Carathéodory metric'. In particular, this shows that our calculus will be valid for

$$\bigcap S^m(Q),$$

where the intersection is taken over all $m \in \mathbb{R}$ and any Q comparable to the C-C metric. Our next major result appears in Section 4.10, where we prove the analogous result to Leibniz's rule for difference operators. We end the chapter with the analysis of the composition of two pseudo-differential operators $T_1 \circ T_2$, where $T_1 \in S^{m_1}(Q_0)$ and $T_2 \in S^{m_2}(Q_0)$, proving that $T_1 \circ T_2 \in S^{m_1+m_2}(Q_0)$.

We end this thesis with a conclusion, where we give a technical summary of the main results. Furthermore, we will provide a discussion about potential future work, indicating some of the directions that could be taken to expand on the results presented here.

Chapter 2

Preliminaries

The aim of this chapter is to introduce the fundamental definitions and techniques that we shall be using throughout the thesis. These preliminaries are based on introductory material appearing on several textbooks in the subject. Some of the material used in this chapter include Faraut [14], Folland [21] and Stein [46], which serve as an introduction to compact Lie groups, their representation theory and the Peter-Weyl Theorem. Moreover, the textbooks Lee [31] and Helgason [26] provide an extensive study of smooth manifolds. Furthermore, for a study of the representation theory of the Heisenberg group, the reader is referred to Folland [20]. Other references used throughout this chapter include Fischer and Ruzhansky [18], Folland [19], Folland and Stein [22], Hall [25], Knapp [30], Ricci [39], Stein [47], and Treves [53]. We shall state without proof a number of well known results, and redirect the reader to the relevant source when necessary.

First we discuss the theory in a general setting, for any Lie group G, but when necessary, we shall provide results for the case in which G is compact. Furthermore, we will also give an overview of the 3-dimensional Heisenberg group $\mathbb{H}$ and the compact Lie group $SU(2)$, including their representation theory.

2.1 Lie groups and representation theory

We begin this section with some fundamental definitions.

Definition 2.1.1 (Topological group)**.** A topological group is a space G, which is also a group, endowed with the continuous mappings

$$\begin{cases} G \times G & \longrightarrow G \\ (x,y) & \longmapsto xy \end{cases}, \qquad \begin{cases} G & \longrightarrow G \\ x & \longmapsto x^{-1} \end{cases}.$$

Definition 2.1.2 (Lie group)**.** A Lie group is a smooth manifold G, which when endowed with the smooth mappings

$$\begin{cases} G \times G & \longrightarrow G \\ (x,y) & \longmapsto xy, \end{cases} \qquad \begin{cases} G & \longrightarrow G \\ x & \longmapsto x^{-1}, \end{cases}$$

forms a group.

If for every point x, in a topological space G, there exists a neighbourhood $V \subset G$ of x which is compact, then we say that G is locally compact. It is important to remark that every Lie group is locally compact. Moreover, a Lie group G is said to be compact if it is compact as a topological space.

For two Hilbert spaces $\mathscr{H}_1, \mathscr{H}_2$, we let $\mathscr{L}(\mathscr{H}_1, \mathscr{H}_2)$ denote the space of linear bounded operators mapping $\mathscr{H}_1$ into $\mathscr{H}_2$. We equip $\mathscr{L}(\mathscr{H}_1, \mathscr{H}_2)$ with the topology given by the norm

$$||T||_{\mathscr{L}(\mathscr{H}_1,\mathscr{H}_2)} = \sup_{\substack{v_1 \in \mathscr{H}_1 \\ ||v_1||_{\mathscr{H}_1} \leq 1}} ||Tv_1||_{\mathscr{H}_2}, \quad T \in \mathscr{L}(\mathscr{H}_1, \mathscr{H}_2).$$

If $\mathscr{H} = \mathscr{H}_1 = \mathscr{H}_2$, we write $\mathscr{L}(\mathscr{H}) = \mathscr{L}(\mathscr{H}, \mathscr{H})$.

For the remainder of this section, we assume G is any Lie group, unless stated otherwise.

Definition 2.1.3 (Representations)**.** A representation π of G on a Hilbert space $\mathscr{H}_\pi$ is a mapping

$$\pi : G \longrightarrow \mathscr{L}(\mathscr{H}_\pi),$$

such that

(i) $\pi(g_1 g_2) = \pi(g_1)\pi(g_2)$, for all $g_1, g_2 \in G$,

(ii) for every $g \in G$, the mapping

$$\pi(g) : \mathscr{H}_\pi \longrightarrow \mathscr{H}_\pi,$$

is continuous and has a bounded inverse.

Usually we will write π instead of $(\pi, \mathscr{H}_\pi)$ whenever the context is clear. Moreover, for a representation $(\pi, \mathscr{H}_\pi)$ of G, we let d_π denote the dimension of π, which is defined to be the dimension of $\mathscr{H}_\pi$.

Suppose that $(\pi, \mathscr{H}_\pi)$ is a representation of G and consider an inner product $\langle \cdot, \cdot \rangle_{\mathscr{H}_\pi}$ on $\mathscr{H}_\pi$. For each $x \in G$, $\pi(x)$ is a bounded linear map $\mathscr{H}_\pi \to \mathscr{H}_\pi$, so we may consider its formal adjoint $\pi(x)^*$, which is given by the relation

$$\langle \pi(x) f, g \rangle_{\mathscr{H}_\pi} = \langle f, \pi(x)^* g \rangle_{\mathscr{H}_\pi} ,$$

for every $f, g \in \mathscr{H}_\pi$.

Definition 2.1.4 (Unitary representation)**.** A representation $(\pi, \mathscr{H}_\pi)$ of G is said to be unitary if, for every $x \in G$, the linear mapping $\pi(x) : \mathscr{H}_\pi \to \mathscr{H}_\pi$ is unitary; that is,

$$\pi(x)^{-1} = \pi(x)^*, \quad \forall\, x \in G.$$

If G is a compact Lie group, it is a routine argument to show that, for each representation $(\pi, \mathscr{H}_\pi)$ of G, there exists an inner product $\langle \cdot, \cdot \rangle_{\mathscr{H}_\pi}$ such that π is unitary with respect to $\langle \cdot, \cdot \rangle_{\mathscr{H}_\pi}$.

We let $||\cdot||_{\mathscr{H}_\pi}$ denote the norm associated to the inner product $\langle \cdot, \cdot \rangle_{\mathscr{H}_\pi}$. Then, if $(\pi, \mathscr{H}_\pi)$ is unitary, we have

$$||\pi(x)v||_{\mathscr{H}_\pi} = ||v||_{\mathscr{H}_\pi}, \quad \forall\, x \in G,\ v \in \mathscr{H}_\pi.$$

Thus, it follows that

$$||\pi(x)||_{\mathscr{L}(\mathscr{H}_\pi)} = 1, \quad \forall\, x \in G. \tag{2.1.1}$$

If $(\pi_1, \mathscr{H}_1)$, $(\pi_2, \mathscr{H}_2)$ are two finite dimensional representations of G, we say that π_1 and π_2 are equivalent if there exists an isomorphism $A : \mathscr{H}_1 \to \mathscr{H}_2$ such that

$$A\,\pi_1(g) = \pi_2(g) A, \quad \forall\, g \in G.$$

In such case, we call A an intertwining operator between π_1 and π_2. Moreover, equivalence between representations of G forms an equivalence relation, which we shall denote by $\sim$.

Definition 2.1.5 (Irreducible representation)**.** A representation $(\pi, \mathscr{H}_\pi)$ of G is said to be irreducible if whenever W is a closed subspace of $\mathscr{H}_\pi$, we have $\pi(x)W \subset W$ for every $x \in G$ if and only if $W = \{0\}$ or $W = \mathscr{H}_\pi$.

Definition 2.1.6. Let G be a Lie group. We define $\widehat{G}$ to be the set of equivalence classes of irreducible, unitary representations of a Lie group G.

Throughout the thesis, each equivalence class $[\pi]_\sim \in \widehat{G}$ shall be identified by a representation π equivalent to all other representations in $[\pi]_\sim$. If G is compact, then by the theory of compact Lie groups, we know that the set $\widehat{G}$ is discrete.

Example 2.1.7. If $G = \mathbb{T}^n$ is the torus, then it is well known that all the irreducible representations of G are one-dimensional. Moreover, they are given by

$$\chi_k : x \longmapsto e^{ik\cdot x}, \quad x \in \mathbb{T}^n,\ k \in \mathbb{Z}^n.$$

Thus,

$$\widehat{G} = \{\chi_k : k \in \mathbb{Z}^n\}.$$

2.2 Integration over a group

In this section we shall introduce the concept of integration over a locally compact group. We present the Peter-Weyl Theorem, and discuss Plancherel's Theorem for compact Lie groups. For a detailed discussion on these topics, see for instance Faraut [14], Folland [21], or Stein [46].

2.2.1 Haar measure

The following theorem is a fundamental fact of Lie theory.

Theorem 2.2.1. *Let G be a locally compact group. Then, there exists a non-zero Radon measure μ on G satisfying the following property:*

$$\mu(xB) = \mu(B), \quad \forall\, x \in G, \text{ and every Borel set } B \subset G. \tag{2.2.1}$$

This measure is unique up to a positive constant.

A measure μ satisfying Theorem 2.2.1, for a locally compact group G, is called a Haar measure. Moreover, a measure μ satisfying (2.2.1) is said to be a left-invariant measure. Throughout this text we shall write $|B|$ for the measure of the Borel set $B \subset G$, and we will denote this measure dx, dy or dz, depending on the variable of integration. Observe that statement (2.2.1) is equivalent to

$$\int_G f(xz)\,\mathrm{d}z = \int_G f(z)\,\mathrm{d}z, \quad \forall\, x \in G,$$

and all integrable functions f.

If G is a compact or nilpotent Lie group, then it can be shown that the Haar measure on G is also right-invariant. A group that is both left-invariant and right-invariant is called unimodular. In this thesis we shall only consider unimodular groups.

2.2.2 L^p spaces

Suppose G is a locally compact group. For $p \in [1, +\infty]$, we let $L^p(G)$ be the usual Lebesgue space with respect to the Haar measure on G, with the norm $||\cdot||_{L^p(G)}$. For $p \in [1, +\infty)$, the norm $||\cdot||_{L^p(G)}$ is defined by

$$||f||_{L^p(G)} = \left(\int_G |f(z)|^p \, \mathrm{d}z \right)^{1/p}, \quad \forall\, f \in L^p(G),$$

and moreover, for $p = +\infty$, we let

$$||f||_{L^\infty(G)} = \operatorname{ess\,sup} |f| = \inf\{a \in \mathbb{R} : ||f|^{-1}(a, +\infty)| = 0\}, \quad f \in L^\infty(G).$$

In general, when we write $\sup_{x \in G}$ it shall be assumed that this refers to the essential supremum.

Example 2.2.2. For a locally compact group G, we let π_L denote the left regular representation of G on $L^2(G)$; for each $g \in G$, $\pi_L(g) : L^2(G) \to L^2(G)$ is defined by

$$(\pi_L(g)f)(x) = f(g^{-1}x), \quad f \in L^2(G),\ x \in G. \tag{2.2.2}$$

Similarly, we let π_R denote the right regular representation of G on $L^2(G)$; for each $g \in G$, $\pi_R(g) : L^2(G) \to L^2(G)$ is defined by

$$(\pi_R(g)f)(x) = f(xg), \quad f \in L^2(G),\ x \in G. \tag{2.2.3}$$

The representations π_L and π_R are unitary and continuous on $L^2(G)$.

2.2.3 Peter-Weyl Theorem

Now, for a representation $(\pi, \mathscr{H}_\pi)$ of G (of possibly infinite dimension), we let the entry functions of π be the mappings of the form

$$x \longmapsto \langle \pi_1(x)\varphi, \psi \rangle_{\mathscr{H}_\pi}, \quad x \in G,\ \varphi, \psi \in \mathscr{H}_\pi.$$

Furthermore, suppose that the set

$$\mathcal{B}_\pi := \left\{\varphi_1^{(\pi)}, \varphi_2^{(\pi)}, \ldots, \varphi_{d_\pi}^{(\pi)}\right\} \subset \mathscr{H}_\pi$$

forms an orthonormal basis of $\mathscr{H}_\pi$, where it is understood that if π is an infinite dimensional representation, then $\mathcal{B}_\pi$ consists of infinitely many elements. We then define the matrix entries of π to be the entry functions

$$x \longmapsto \pi(x)^{(j,k)} := \left\langle \pi(x)\varphi_k^{(\pi)}, \varphi_j^{(\pi)} \right\rangle_{\mathscr{H}_\pi}, \quad x \in G,\ j,k = 1,2,\ldots,d_\pi. \tag{2.2.4}$$

We now suppose G is a compact group. For $[\pi]_\sim \in \widehat{G}$, we let M_π be the subspace of $L^2(G)$ spanned by the entry functions of the representations in the equivalence class $[\pi]_\sim$; that is

$$M_\pi = \operatorname{Span}\left\{ \langle \pi_1(\cdot)\varphi, \psi\rangle_{\mathscr{H}_{\pi_1}} : \varphi,\psi \in \mathscr{H}_{\pi_1},\ \pi_1 \in [\pi]_\sim \right\}. \tag{2.2.5}$$

The space M_π is independent of the choice of representative π, and is of dimension d_π^2. Moreover, let M be the space spanned by all entry functions of representations in $\widehat{G}$;

$$M = \operatorname{Span}\left\{ \langle \pi_1(\cdot)\varphi, \psi\rangle_{\mathscr{H}_{\pi_1}} : \varphi,\psi \in \mathscr{H}_{\pi_1},\ \pi_1 \in [\pi]_\sim,\ \pi \in \widehat{G} \right\}. \tag{2.2.6}$$

We are now in a position to state the Peter-Weyl Theorem, whose proof can be found, for example, in Faraut [14] or Stein [46].

Theorem 2.2.3 (Peter-Weyl Theorem)**.** *Let G be a compact Lie group. Then, the following assertions hold:*

(I) Every irreducible unitary representation of G is finite dimensional.

(II) The left regular representations π_L can be decomposed into an orthogonal direct sum of finite dimensional irreducible representations. In particular, when restricted to the space M_π, for $\pi \in \widehat{G}$, the representations π_L is equivalent to the decomposition

$$\pi \oplus \pi \oplus \cdots \oplus \pi = d_\pi \pi.$$

The right regular representation π_R satisfies the same property.

(III) The space M is dense in $L^2(G)$.

(IV) For each $(\pi, \mathscr{H}_\pi) \in \widehat{G}$, pick an orthonormal basis of $\mathscr{H}_\pi$:

$$\left\{\varphi_1^{(\pi)}, \varphi_2^{(\pi)}, \ldots, \varphi_{d_\pi}^{(\pi)}\right\}. \tag{2.2.7}$$

Consider the matrix entries of π, with respect to the basis given by (2.2.7),

$$x \longmapsto \pi(x)^{(j,k)} = \left\langle \pi(x)\, \varphi_k^{(\pi)}, \varphi_j^{(\pi)} \right\rangle_{\mathscr{H}_\pi}, \qquad 1 \le j, k \le d_\pi, \ \ \textit{for} \ \ x \in G.$$

Then, the set

$$\left\{\sqrt{d_\pi}\, \pi(\cdot)^{j,k} \ : \ 1 \le i, j \le d_\pi,\ \pi \in \widehat{G}\right\}$$

forms an orthonormal basis of $L^2(G)$.

2.2.4 Fourier transform

Suppose G is a locally compact group. If f is an integrable function on G, with respect to the Haar measure, and $(\pi, \mathscr{H}_\pi)$ is a representation of G, we define the Fourier transform of f at π by

$$\widehat{f}(\pi) = \mathcal{F}_G f(\pi) = \int_G f(x)\, \pi(x)^* \, \mathrm{d}x.$$

Observe that $\widehat{f}(\pi)$ defines a bounded linear operator on the Hilbert space $\mathscr{H}_\pi$.

The Fourier transform $\widehat{f}(\pi)$ depends on the choice of representative from the equivalence class $[\pi]_\sim$. In particular, if $(\pi_1, \mathscr{H}_{\pi_1}) \in [\pi]_\sim$, then there exists an isomorphism $A : \mathscr{H}_{\pi_1} \to \mathscr{H}_\pi$ such that

$$\pi_1(x) = A^{-1}\pi(x)A, \quad \forall\, x \in G.$$

By the linearity of $\widehat{f}(\pi)$, we then have

$$\widehat{f}(\pi_1) = A^{-1}\widehat{f}(\pi)A.$$

This means that we must consider the Fourier transforms at π modulo conjugations. Throughout the thesis, it will be assumed that this is understood.

We now have the following result, which follows from the definition of unitarity of representations (see (2.1.1)).

Lemma 2.2.4. *Let f be an integrable function on a compact Lie group G and suppose that $(\pi, \mathscr{H}_\pi)$ is a unitary representation of G. Then*

$$||\widehat{f}(\pi)||_{\mathscr{L}(\mathscr{H}_\pi)} \leq ||f||_{L^1(G)}.$$

Proof. Recall that

$$\widehat{f}(\pi) = \int_G f(x)\, \pi(x)^*\, \mathrm{d}x.$$

So,

$$\begin{aligned} ||\widehat{f}(\pi)||_{\mathscr{L}(\mathscr{H}_\pi)} &\leq \int_G ||f(x)\, \pi(x)^*||_{\mathscr{L}(\mathscr{H}_\pi)}\, \mathrm{d}x \\ &\leq \int_G |f(x)|\, ||\pi(x)^*||_{\mathscr{L}(\mathscr{H}_\pi)}\, \mathrm{d}x. \end{aligned}$$

Since π is unitary, then, by (2.1.1), we have

$$||\widehat{f}(\pi)||_{\mathscr{L}(\mathscr{H}_\pi)} \leq \int_G |f(x)|\, \mathrm{d}x = ||f||_{L^1(G)},$$

as claimed. □

2.2.5 Plancherel's Theorem on compact Lie groups

We now discuss Plancherel's Theorem in the case that G is a compact Lie group.

Let $\mathscr{H}$ be a Hilbert space. For an operator $A \in \mathscr{L}(\mathscr{H})$, we define the Hilbert-Schmidt norm of A by

$$||A||_{HS}^2 = \mathrm{Tr}\,(A\, A^*)\,, \tag{2.2.8}$$

where Tr denotes the trace on the Hilbert space $\mathscr{H}$.

Furthermore, we let $L^2(\widehat{G})$ denote the space of sequences of operators $T = (T_\pi)_{\pi \in \widehat{G}}$, with $T_\pi \in \mathscr{L}(\mathscr{H}_\pi)$, which satisfy

$$||T||_{L^2(\widehat{G})}^2 := \sum_{\pi \in \widehat{G}} d_\pi \big|\big|T_\pi\big|\big|_{HS}^2 < +\infty.$$

The space $L^2(\widehat{G})$ is a Hilbert space with the inner product

$$\langle T, S\rangle_{L^2(\widehat{G})} := \sum_{\pi \in \widehat{G}} d_\pi \mathrm{Tr}\,(T_\pi S_\pi^*)\,,$$

for sequences of operators $T = (T_\pi)_{\pi \in \widehat{G}}$, $S = (S_\pi)_{\pi \in \widehat{G}} \in L^2(\widehat{G})$.

Example 2.2.5. If $G = \mathbb{T}^n$ is the torus, then $\widehat{G} = \{\chi_k : k \in \mathbb{Z}^n\}$, as we saw in Example 2.1.7, and hence $L^2(\widehat{G})$ is the space given by

$$L^2(\widehat{G}) = \left\{ (a_k)_{k \in \mathbb{Z}^n} : \sum_{k \in \mathbb{Z}^n} |a_k|^2 < +\infty \right\},$$

which is the usual sequence space $\ell^2(\mathbb{Z}^n)$.

We can also define the following space of operators.

Definition 2.2.6. Let $L^\infty(\widehat{G})$ denote the space of operators

$$\sigma = \{\sigma(\pi) : \pi \in \widehat{G}\}$$

satisfying

$$\sup_{\pi \in \widehat{G}} ||\sigma(\pi)||_{\mathscr{L}(\mathscr{H}_\pi)} < +\infty.$$

We endow $L^\infty(\widehat{G})$ with the essential supremum norm

$$||\sigma||_{L^\infty(\widehat{G})} := \sup_{\pi \in \widehat{G}} ||\sigma(\pi)||_{\mathscr{L}(\mathscr{H}_\pi)}.$$

We can now state Plancherel's Theorem, which is a consequence of the Peter-Weyl Theorem, and its proof can be found, for example in [14].

Theorem 2.2.7 (Plancherel's Theorem)**.** *Suppose G is a compact Lie group and let $f \in L^2(G)$. Then, the following assertions hold:*

(i) The function f is equal to its Fourier series

$$f(x) = \sum_{\pi \in \widehat{G}} d_\pi \operatorname{Tr}\left(\widehat{f}(\pi)\,\pi(x)\right),$$

in the L^2 sense. This is also known as the Fourier inversion formula.

(ii) We have

$$||f||^2_{L^2(G)} = ||\widehat{f}||^2_{L^2(\widehat{G})} = \sum_{\pi \in \widehat{G}} d_\pi \big|\big|\widehat{f}(\pi)\big|\big|^2_{HS},$$

and if $f_1, f_2 \in L^2(G)$,

$$\langle f_1, f_2 \rangle_{L^2(G)} = \int_G f_1(x)\, \overline{f_2(x)}\, dx = \sum_{\pi \in \widehat{G}} d_\pi \operatorname{Tr}\left(\widehat{f}_1(\pi)\, \widehat{f}_2(\pi)^* \right).$$

(iii) The map $f \mapsto \widehat{f}$ is a unitary isomorphism from $L^2(G)$ onto $L^2(\widehat{G})$.

2.3 Lie algebras and vector fields

In this section we summarise the relevant aspects of the theory of Lie algebras. For a detailed discussion on the subject, see Lee [31] or Helgason [26], for example.

2.3.1 Vector fields

Suppose $\mathcal{M}$ is an n-dimensional smooth manifold. Recall that the space $\mathcal{C}^\infty(\mathcal{M})$ consists of functions $f : \mathcal{M} \to \mathbb{R}$ which are smooth, in the sense that for every smooth chart (φ, U) on $\mathcal{M}$, the composite function $f \circ \varphi^{-1}$ is smooth on the open subset $\varphi(U) \subset \mathbb{R}^n$. A function belonging to $\mathcal{C}^\infty(\mathcal{M})$ is said to be smooth. For $d \in \mathbb{N}$, the space $\mathcal{C}^d(\mathcal{M})$ is similarly defined, requiring instead that $f \circ \varphi^{-1}$ belongs to $\mathcal{C}^d(\mathbb{R}^n)$.

The tangent space at a point $x \in \mathcal{M}$, $T_x(\mathcal{M})$, is the n-dimensional vector space consisting of all linear functionals $V : \mathcal{C}^\infty(\mathcal{M}) \to \mathbb{R}$ which satisfy

$$V(fg) = V(f)g(x) + f(x)V(g), \quad \forall\ f, g \in \mathcal{C}^\infty(\mathcal{M}).$$

An element of $T_x(\mathcal{M})$ is called a tangent vector at x. Recall also that the tangent bundle of $\mathcal{M}$ is the disjoint union of the tangent spaces at all points of $\mathcal{M}$:

$$T(\mathcal{M}) = \bigsqcup_{x \in \mathcal{M}} T_x(\mathcal{M}).$$

Now, if $\mathcal{N}$ is another smooth manifold and $F : \mathcal{M} \to \mathcal{N}$ is a smooth map, then for each $x \in \mathcal{M}$, the push-forward associated with F is the mapping $F_* : T_x(\mathcal{M}) \to T_{F(x)}(\mathcal{N})$ given by

$$(F_* V)(f) = V(f \circ F), \quad \text{for } V \in T_x(\mathcal{M}),\ f \in \mathcal{C}^\infty(\mathcal{N}).$$

Definition 2.3.1 (Vector field)**.** A vector field on $\mathcal{M}$ is a continuous map

$$\begin{cases} X : \mathcal{M} & \longrightarrow\ T(\mathcal{M}) \\ \qquad x & \longmapsto\ X_x \end{cases},$$

where for each $x \in \mathcal{M}$, $X_x \in T_x(\mathcal{M})$. Throughout this thesis, a vector field on $\mathcal{M}$ will always be assumed to be smooth, unless stated otherwise.

If X is a vector field on $\mathcal{M}$, not necessarily smooth, and $f \in \mathcal{C}^\infty(M)$, then the action of X on f is given by

$$(Xf)(x) = X_x(f), \quad x \in \mathcal{M}.$$

If X, Y are smooth vector fields, then define Lie bracket of X and Y to be the operator

$$\begin{cases} [X,Y] : \mathcal{M} & \longrightarrow \; T(\mathcal{M}) \\ \qquad\qquad x & \longmapsto \; [X,Y]_x \end{cases},$$

with

$$[X,Y]_x(f) = X_x(Yf) - Y_x(Xf), \quad \text{for } x \in \mathcal{M},\ f \in \mathcal{C}^\infty(\mathcal{M}).$$

It is well-known that the map $[X,Y]$ is a smooth vector field (see, for example, [31]).

We shall now also define the push-forward of a vector field by a function F. Let $F : \mathcal{M} \to \mathcal{N}$ be a smooth map between the smooth manifolds $\mathcal{M}$ and $\mathcal{N}$, and suppose X is a vector field on $\mathcal{M}$. Tentatively we might define the push-forward of X to be the mapping given by

$$F_* X : x \longmapsto F_* X_x,$$

since $F_* X_x \in T_{F(x)}(\mathcal{N})$. However, $F_* X$ might not necessarily be a vector field on $\mathcal{N}$, so we need to impose some additional conditions on F. Namely, it is sufficient to have the following condition: if X is a smooth vector field on $\mathcal{M}$, then there exists a unique smooth vector field Z on $\mathcal{N}$ such that

$$\forall\, x \in \mathcal{M} \quad F_* X_x = Z_{F(x)}; \tag{2.3.1}$$

that is, for any smooth function f defined on an open subset of $\mathcal{N}$ we have

$$X_x(f \circ F) = (Zf)(F(x)), \quad \forall\, x \in \mathcal{M}. \tag{2.3.2}$$

One can then show that if F is a diffeomorphism, then this condition is satisfied. A proof of this can be found in [31] (see Chapter 8 therein). Hence, we can define the push-forward of the smooth vector field X by F to be the unique smooth vector field Z on $\mathcal{N}$ satisfying (2.3.1) (or equivalently (2.3.2)).

2.3.2 Basis of vector fields on a smooth manifold

Suppose $\mathcal{M}$ is a smooth manifold of dimension n. In this section we aim to define what it means for a family of vector fields to form a basis on $\mathcal{M}$.

In Lee [31] it is shown that the space of all vector fields on $\mathcal{M}$ is a module over the ring $\mathcal{C}^\infty(\mathcal{M})$. This means that, if $f \in \mathcal{C}^\infty(\mathcal{M})$ and X is a vector field on $\mathcal{M}$, then the mapping

$$\begin{aligned}(fX) : \mathcal{M} &\longrightarrow T(\mathcal{M}) \\ x &\longmapsto f(x)X_x\end{aligned}$$

defines a smooth vector field. We can now define a basis of vector fields on $\mathcal{M}$.

Definition 2.3.2. A family of vector fields

$$\{V_j\}_{j=1}^n$$

on $\mathcal{M}$ is said to be a basis of vector fields on $\mathcal{M}$ if, for each $x \in \mathcal{M}$, the set of tangent vectors

$$\{V_{j,x} \,:\, j = 1, 2, \ldots, n\} \subset T_x(\mathcal{M})$$

forms a basis of the tangent space $T_x(\mathcal{M})$.

Remark 2.3.3. A basis of vector on $\mathcal{M}$ is also known as a (smooth global) frame for $\mathcal{M}$ (see Chapter 8 in Lee [31]).

By definition, if $\{V_j \,:\, j = 1, 2 \ldots, n\}$ is basis of vector fields on $\mathcal{M}$ and W is any vector field on $\mathcal{M}$, then there exists a family of functions $\{c_j\}_{j=1}^n \subset \mathcal{C}^\infty(\mathcal{M})$ such that, for any $x \in \mathcal{M}$ we have

$$W_x = \sum_{j=1}^n c_j(x)V_{j,x}.$$

This observation can be summarised as follows.

Lemma 2.3.4. *Suppose $\{V_j \,:\, j = 1, 2 \ldots, n\}$ is basis of vector fields on $\mathcal{M}$. If W is any vector field on $\mathcal{M}$, then there exists a family of functions $\{c_j\}_{j=1}^n \subset \mathcal{C}^\infty(\mathcal{M})$ such that*

$$W = \sum_{j=1}^{n} c_j V_j.$$

Now, for $\ell \in \mathbb{N}$, let $\mathcal{I}(\ell)$ denote the set of multi-indices taking values in $\{1, 2, \ldots, \ell\}$, of arbitrary length. That is, $\mathcal{I}(\ell)$ is the disjoint union

$$\mathcal{I}(\ell) := \bigsqcup_{a \in \mathbb{N}} \{1, 2, \ldots, \ell\}^a. \tag{2.3.3}$$

Suppose that $\boldsymbol{V} = \{V_j\}_{j=1}^{\ell}$ is a family of vector fields on $\mathcal{M}$, for some $\ell \in \mathbb{N}$. For $\beta = (i_1, i_2, \ldots, i_b) \in \mathcal{I}(\ell)$, we let V_β denote the differential operator

$$V_\beta = V_{i_1} V_{i_2} \ldots V_{i_b}.$$

Corollary 2.3.5. *Suppose $\boldsymbol{V} = \{V_j\}_{j=1}^{n}$ and $\boldsymbol{W} = \{W_j\}_{j=1}^{n}$ are two bases of vector fields on $\mathcal{M}$, and let $\beta \in \mathcal{I}(n)$. Then, for any $\beta' \in \mathcal{I}(n)$, with $|\beta'| \leq |\beta|$, there exists a function $c_{\boldsymbol{V},\boldsymbol{W}}^{\beta'} \in \mathcal{C}^\infty(\mathcal{M})$, which depends on β' and the bases of vector fields $\boldsymbol{V}$ and $\boldsymbol{W}$, such that,*

$$V_\beta = \sum_{\substack{\beta' \in \mathcal{I}(n) \\ |\beta'| \leq |\beta|}} c_{\boldsymbol{V},\boldsymbol{W}}^{\beta'} W_{\beta'}. \tag{2.3.4}$$

Remark 2.3.6. Assume $\mathcal{M}$ is compact and suppose we have the same hypothesis as in Corollary 2.3.5. Then, for any $\beta' \in \mathcal{I}(n)$, with $|\beta'| \leq |\beta|$, the functions $c_{\boldsymbol{V},\boldsymbol{W}}^{\beta'} \in \mathcal{C}^\infty(\mathcal{M})$ which appear in (2.3.4) have compact support. Thus, there exists a constant $C^\beta > 0$, depending on β, such that

$$\big|\big|V_\beta f\big|\big|_{L^\infty(\mathcal{M})} \leq C^\beta \sup_{\substack{\beta' \in \mathcal{I}(n) \\ |\beta'| \leq |\beta|}} ||c_{\boldsymbol{V},\boldsymbol{W}}^{\beta'}||_{L^\infty(\mathcal{M})} \, ||W_{\beta'} f||_{L^\infty(\mathcal{M})}.$$

In particular, the quantity

$$C_{\boldsymbol{V},\boldsymbol{W}}^{\beta} := C^\beta \sup_{\substack{\beta' \in \mathcal{I}(n) \\ |\beta'| \leq |\beta|}} ||c_{\boldsymbol{V},\boldsymbol{W}}^{\beta'}||_{L^\infty(\mathcal{M})} < +\infty.$$

Therefore,

$$\big|\big|V_\beta f\big|\big|_{L^\infty(\mathcal{M})} \leq C_{\boldsymbol{V},\boldsymbol{W}}^{\beta} \sup_{\substack{\beta' \in \mathcal{I}(n) \\ |\beta'| \leq |\beta|}} ||W_{\beta'} f||_{L^\infty(\mathcal{M})}.$$

2.3.3 Lie algebras

Definition 2.3.7 (Lie algebra)**.** A Lie algebra is a real vector space $\mathfrak{g}$ endowed with a bilinear operation $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, called the Lie bracket of $\mathfrak{g}$, satisfying

(i) $[X, X] = 0$ for all $X \in \mathfrak{g}$, and

(ii) Jacobi's identity: $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$ for all $X, Y, Z \in \mathfrak{g}$.

Now suppose that G is a Lie group. It is well known that each $g \in G$ defines the diffeomorphism L_g given by left multiplication by g:

$$\begin{cases} L_g : G & \longrightarrow & G \\ \quad\; x & \longmapsto & gx \end{cases},$$

and similarly, each $g \in G$ also gives rise to the diffeomorphism R_g given by right multiplication by g:

$$\begin{cases} R_g : G & \longrightarrow & G \\ \quad\; x & \longmapsto & xg \end{cases}.$$

We say that a smooth vector field X is left-invariant if

$$X(f \circ L_g) \;=\; (Xf) \circ L_g, \quad \forall\, f \in \mathcal{C}^\infty(G),\, g \in G.$$

This can be written as

$$X(f(g\,\cdot\,))(x) \;=\; (Xf)(gx), \quad \forall\, \mathcal{C}^\infty(G),\, x, g \in G.$$

Similarly, we say that X is right-invariant if

$$X(f \circ R_g) \;=\; (Xf) \circ R_g, \quad \forall\, f \in \mathcal{C}^\infty(G),\, g \in G,$$

which can also be expressed as

$$X(f(\,\cdot\, g))(x) \;=\; (Xf)(xg), \quad \forall\, f \in \mathcal{C}^\infty(G),\, x, g \in G.$$

Definition 2.3.8 (Lie algebra of a Lie group)**.** Let G be a Lie group. The set $\mathfrak{g}$ of all left-invariant vector fields on G is called the Lie algebra of G.

Now let G be a Lie group and $\mathfrak{g}$ be its Lie algebra. If $X, Y \in \mathfrak{g}$, consider the Lie bracket of X and Y:

$$\begin{cases} [X,Y] : G & \longrightarrow T(G) \\ \qquad x & \longmapsto [X,Y]_x \end{cases}.$$

One checks easily that this mapping defines a smooth left-invariant vector field $[X,Y]$ on G. Indeed, suppose that $f \in \mathcal{C}^\infty(G)$ and $g \in G$. Then, for every $x \in G$, we have

$$\begin{aligned} [X,Y](f(g\cdot))(x) &= [X,Y]_x(f(g\cdot)) = X_x(Yf(g\cdot)) - Y_x(Xf(g\cdot)) \\ &= X(Yf(g\cdot))(x) - Y(Xf(g\cdot))(x) \\ &= X(Yf)(gx) - Y(Xf)(gx), \end{aligned}$$

by the left-invariance of X and Y. Since

$$\begin{aligned} X(Yf)(gx) - Y(Xf)(gx) &= X_{gx}(Yf) - Y_{gx}(Xf) \\ &= [X,Y]_{gx} f \\ &= ([X,Y]f)(gx), \end{aligned}$$

then we conclude that

$$[X,Y](f(g\cdot))(x) = ([X,Y]f)(gx),$$

which means that $[X,Y]$ is a left-invariant vector field, as claimed. This means that $\mathfrak{g}$ is closed under the bracket operation $[\cdot,\cdot]$, and in particular, one can show that $\mathfrak{g}$, equipped with the Lie bracket $[\cdot,\cdot]$, forms a Lie algebra in the sense of Definition 2.3.7 (see [31]).

It is a fundamental result that $\mathfrak{g}$ is of the same dimension as G. Furthermore, the evaluation map

$$\begin{cases} \varepsilon : \mathfrak{g} & \longrightarrow \ T_{e_G}(G) \\ \quad X & \longmapsto \ X_{e_G} \end{cases}$$

is a vector space isomorphism (see [31]). More precisely, we have the following relation. If X is a left-invariant vector field, then it is uniquely determined at $e_G \in G$. On the other hand, if V is a tangent vector at e_G (that is, $V \in T_{e_G}(G)$), then, as proved, for instance, in Hall [25] or Lee [31], there exists a unique left-invariant vector field X^V, with $X^V_{e_G} = V$, which can be constructed by

$$X^V_g \;=\; (L_g)_* V, \quad g \,\in\, G.$$

Or equivalently,

$$(X^V f)(g) = (L_g)_* V f(e_G) = V(f(g\cdot))(e_G), \quad \forall\, f \,\in\, \mathcal{C}^\infty(G),\, g \,\in\, G. \tag{2.3.5}$$

Observe that

$$X^V_{e_G} f = V(f(e_G\cdot))(e_G) \;=\; Vf, \quad \forall\, f \,\in\, \mathcal{C}^\infty(G),$$

and so $X^V_{e_G} = V$. In fact, one can show that the mapping

$$\begin{cases} T_{e_G}(G) & \longrightarrow \; \mathfrak{g} \\ \quad V & \longmapsto \; X^V \end{cases} \tag{2.3.6}$$

is the inverse of the evaluation map ε (see [31]). Hence, $\mathfrak{g}$ can be identified with the tangent space at the identity element e_G of G.

In particular, we have shown that if V is a tangent vector at the identity e_G, then left-translation by $g \in G$ yields a uniquely determined tangent vector at the point g. This implies that there is a one-to-one correspondence between $T_{e_G}(G)$ and $T_g(G)$, given by this relation. Hence, via the evaluation map, we obtain a one-to-one correspondence between $\mathfrak{g}$, the space of all left-invariant vector fields on G, and the tangent space $T_g(G)$, for any $g \in G$.

Throughout the thesis we will assume the following convention; for any tangent vector $V \in T_{e_G}(G)$, we shall identify the unique left-invariant vector field X^V, associated to V via the map (2.3.6), with V. Similarly, if a left-invariant vector field X is given, we shall identify X with its evaluation at the identity X_{e_G}.

2.3.4 Exponential map

In this section we provide an introduction to the exponential map in the context of Lie groups and Lie algebras. For a deeper study of this subject, the reader is redirected to Lee [31] (Chapter 20) or Helgason [26] (Chapter 1, Section 6). We shall also state some fundamental results linking Lie groups and Lie algebras, which can be found in Hall [25] (see Chapter 3) or Helgason [26] (see Chapter II).

Let G be a Lie group and suppose that $\mathfrak{g}$ denotes its Lie algebra. Now, assume that γ is a one-parameter subgroup of G; that is, suppose $\gamma : \mathbb{R} \to G$ is a Lie group homomorphism, and let

$$X := \gamma_* \left(\frac{d}{dt} \right).$$

Viewing $\mathbb{R}$ as a Lie group equipped with addition, we see that X is a left-invariant vector field on G. Then, as shown in [31] (see Theorem 20.1 therein), one can prove that

$$\gamma'(t_0) = X_{\gamma(t_0)}, \quad \forall\, t_0 \in \mathbb{R}.$$

On the other hand, if X is any vector field on G, it can be shown (see [31], Chapter 9) that there exists a unique mapping $\gamma : \mathbb{R} \to G$ satisfying

$$\gamma'(t) = X_{\gamma(t)}, \quad \forall\, t \in \mathbb{R}. \tag{2.3.7}$$

In fact, as demonstrated in [31] (see Theorem 20.1), we have that γ is a one-parameter subgroup. In particular, we see that there is a one-to-one correspondence between $\mathfrak{g}$, the set of all left-invariant vector fields on G, and the one-parameter subgroups of G.

Hence, we now define the exponential map $\exp : \mathfrak{g} \to G$ as follows: for each left-invariant vector field X let

$$e^X = \exp(X) := \gamma(1),$$

where γ is the unique one-parameter subgroup of G satisfying (2.3.7).

The exponential map has the following fundamental properties, the proof of which can be found in [31] (see Proposition 20.8):

Proposition 2.3.9. *Let G be a Lie group and suppose that $\mathfrak{g}$ denotes its Lie algebra. Then, the following assertions hold:*

(i) The exponential map $\exp : \mathfrak{g} \to G$ is smooth.

(ii) If $X \in \mathfrak{g}$, then

$$e^{(s+t)X} = e^{sX}\, e^{tX}, \quad \forall\, s, t \in \mathbb{R}.$$

(iii) If $X \in \mathfrak{g}$, then

$$\left(e^X\right)^{-1} = e^{-X}.$$

(iv) If $X \in \mathfrak{g}$, then

$$\left(e^X\right)^n = e^{nX}, \quad \forall\, n \in \mathbb{Z}.$$

(v) There exist a neighbourhood U of 0 in $\mathfrak{g}$ and a neighbourhood V of e_G in G, such that $\exp$ maps U diffeomorphically into V.

(vi) If $X \in \mathfrak{g}$, then the action of X, viewed as a left-invariant vector field, on a function $f \in \mathcal{C}^\infty(G)$ is given by

$$Xf(x) = \frac{d}{dt} f\left(xe^{tX}\right)\Big|_{t=0}. \tag{2.3.8}$$

(vii) Any left-invariant vector field $X \in \mathfrak{g}$ defines a right-invariant differential operator, which we denote by $\widetilde{X}$ and is given by

$$\widetilde{X}f(x) = \frac{d}{dt} f\left(e^{tX}x\right)\Big|_{t=0}. \tag{2.3.9}$$

Part (vi) of this result tells us that we can view $\mathfrak{g}$ as the vector space of first order left-invariant differential operators on G.

Now recall that if $n \in \mathbb{N}$, the general linear group, which we denote by $\mathrm{GL}(n, \mathbb{C})$, is defined to be the space consisting of all $n \times n$ invertible matrices with complex entries. We say that a Lie group G is a matrix Lie group if it is a closed subgroup of $\mathrm{GL}(n, \mathbb{C})$, for some $n \in \mathbb{N}$.

If G is a matrix Lie group, then its Lie algebra $\mathfrak{g}$ is a matrix Lie algebra, in the sense that it is a subalgebra of $\mathfrak{gl}(n, \mathbb{C})$, the Lie algebra consisting of all $n \times n$ matrices with complex entries. In this case, we can consider the elements of $\mathfrak{g}$ to be matrices, instead of left-invariant vector fields. As it turns out, the exponential map of a matrix is easy to compute. We have the following result, a proof of which can be found, for example, in [25] or [31].

Proposition 2.3.10. *Suppose G is a matrix Lie group and $\mathfrak{g}$ is its Lie algebra. Then, for every $X \in \mathfrak{g}$, we have*

$$e^X := \exp(X) = \sum_{k=0}^{\infty} \frac{1}{k!} X^k,$$

where X is considered as a matrix.

Furthermore, we can use this result to compute the Lie algebra of a matrix Lie group explicitly. We have the following fundamental result, a proof of which can be found in [26] (see Proposition 2.7 in Chapter II, Section 2).

Proposition 2.3.11. *Suppose G is a matrix Lie group. Then its Lie algebra $\mathfrak{g}$ is given by*

$$\mathfrak{g} = \{X \in \mathfrak{gl}(n,\mathbb{C}) : e^{tX} \in G, \forall\, t \in \mathbb{R}\}. \tag{2.3.10}$$

2.3.5 Hörmander system

Suppose G is a connected Lie group and let $\mathfrak{g}$ be its Lie algebra. Further suppose $\mathfrak{g}_\infty$ denotes the Lie algebra of all smooth real vector fields on G and consider a family of smooth real vector fields $\boldsymbol{X} = \{X_1, X_2, \ldots, X_k\}$. Let $\mathfrak{g}_{\boldsymbol{X}}$ be the vector subspace of $\mathfrak{g}_\infty$ generated by the vectors

$$\big[X_{i_1}, \big[X_{i_2}, \ldots, \big[X_{i_{a-1}}, X_{i_a}\big]\ldots\big]\big], \quad 1 \le i_1, i_2, \ldots, i_a \le k.$$

One checks easily that $\mathfrak{g}_{\boldsymbol{X}}$ is a Lie subalgebra of $\mathfrak{g}_\infty$. Furthermore, for each $x \in G$ let $\mathfrak{g}_{\boldsymbol{X}}(x)$ denote the linear subspace of $T_x(G)$ given by

$$\mathfrak{g}_{\boldsymbol{X}}(x) := \{X_x : X \in \mathfrak{g}_{\boldsymbol{X}}\}.$$

Definition 2.3.12 (Hörmander system of vector fields)**.** Let G be a connected Lie group of dimension n and consider a family of vector fields

$$\boldsymbol{X} = \{X_1, X_2, \ldots, X_k\}.$$

Suppose further that $\mathfrak{g}$ denotes the Lie algebra of G. We say that $\boldsymbol{X}$ forms a Hörmander system of vector fields if, for every $x \in G$,

$$\mathfrak{g}_{\boldsymbol{X}}(x) = T_x(G).$$

Suppose G is a connected Lie group and let $\boldsymbol{X}$ be a Hörmander system of left-invariant vector fields on G. Then, the sub-Laplacian associated with $\boldsymbol{X}$ is denoted by

$$\mathcal{L} := -(X_1^2 + X_2^2 + \cdots + X_k^2).$$

Now, recall that $\mathcal{I}(k)$ denote the set of multi-indices taking values in $\{1, 2, \dots, k\}$, of arbitrary length (see (2.3.3)). For $\alpha = (i_1, i_2, \dots, i_a) \in \mathcal{I}(k)$, we write

$$X_\alpha = X_{i_1} X_{i_2} \dots X_{i_a}, \tag{2.3.11}$$

and if $\widetilde{X}_{i_1}, \widetilde{X}_{i_2}, \dots, \widetilde{X}_{i_a}$ denote the right-invariant vector fields associated to $X_{i_1}, X_{i_2}, \dots, X_{i_a}$, respectively (see (2.3.9)), then we denote

$$\widetilde{X}_\alpha = \widetilde{X}_{i_1} \widetilde{X}_{i_2} \dots \widetilde{X}_{i_a}. \tag{2.3.12}$$

Definition 2.3.13. Suppose T is a differential operator on G of the form

$$T = \sum_{\alpha \in \mathcal{I}(k)} c_\alpha X_\alpha.$$

We define the transpose operator T^t to be the differential operator given by

$$T^t = \sum_{\alpha \in \mathcal{I}(k)} (-1)^{|\alpha|} c_\alpha X_{i_a} X_{i_{a-1}} \dots X_{i_1}. \tag{2.3.13}$$

2.4 Comparability of the Carnot-Carathéodory metric to the Euclidean distance

Suppose G is a compact Lie group of dimension n and let $\mathfrak{g}$ be the Lie algebra of G. Further suppose that, for some $k \in \mathbb{N}$, the set $\boldsymbol{X} = \{X_1, X_2, \dots, X_k\}$ forms a Hörmander system of left-invariant vector fields on G. The objective of this section is to show that the Carnot-Carathéodory metric is comparable to the Euclidean distance. In the case that $G = \mathbb{R}^n$, this result is well known; see, for example, Chapter 1 in Nagel et al [36].

2.4.1 An adapted basis of $\mathfrak{g}$

Let us first construct a basis of the Lie algebra $\mathfrak{g}$ of G. For a multi-index $I = (i_1, i_2, \dots, i_a) \in \mathcal{I}(k)$, we define the vector field $X_{[I]}$ by

$$X_{[I]} = \big[X_{i_1}, \big[X_{i_2}, \dots, \big[X_{i_{a-1}}, X_{i_a}\big] \dots\big]\big].$$

Let V_1 be the subspace of $\mathfrak{g}$ consisting of linear combinations of the vector fields $X_1, X_2, \dots, X_k$; that is,

$$V_1 := \operatorname{Span}\{X_j \,:\, j = 1, 2, \dots, k\}.$$

We now recursively define

$$V_s = V_{s-1} + \big[V_1, V_{s-1}\big], \quad \text{for } s \in \mathbb{N},$$

where we use the convention $V_0 := \{0\}$. Observe that V_s is spanned by the set of vector fields

$$\big\{X_{[I]} : \mathcal{I}(k) \ |I| \leq s\big\}.$$

Since $\boldsymbol{X} = \{X_1, X_2, \ldots, X_k\}$ is a Hörmander system, then there exists an integer $r > 0$ such that

$$V_r = \mathfrak{g}.$$

In fact, we have the increasing sequence of subspaces

$$\{0\} = V_0 \subset V_1 \subset \cdots \subset V_r = \mathfrak{g}.$$

We then denote

$$n_s = \dim(V_s), \quad s = 0, 1, 2, \ldots, r.$$

We have

$$0 = n_0 < n_1 < n_2 < \cdots < n_r = n.$$

We now construct a basis of $\mathfrak{g}$. Choose vector fields $Y_1, Y_2, \ldots, Y_{n_1}$, from our Hörmander system $\{X_1, X_2, \ldots, X_k\}$, such that the set $\{Y_1, Y_2, \ldots, Y_{n_1}\}$ forms a basis of V_1. Then, for $s = 2, 3, \ldots, r$ and for each $j = n_{s-1}+1, n_{s-1}+2, \ldots, n_s$, we let

$$Y_j = X_{[I_j^{(s)}]} = \big[X_{i_1}, \big[X_{i_2}, \ldots, \big[X_{i_{s-1}}, X_{i_s}\big] \ldots\big]\big], \tag{2.4.1}$$

for some multi-index $I_j^{(s)} \in \mathcal{I}(k)$, such that $\big|I_j^{(s)}\big| = s$ and the set of vector fields

$$\{Y_1, Y_2, \ldots, Y_{n_{s-1}}, Y_{n_{s-1}+1}, \ldots Y_{n_s}\}$$

forms a basis of V_s. Hence, we have constructed a set of vector fields

$$\boldsymbol{Y} := \big\{Y_1, Y_2, \ldots, Y_{n_r}\big\} = \{Y_1, Y_2, \ldots, Y_n\}, \tag{2.4.2}$$

which forms a basis of $\mathfrak{g}$. Observe that this basis of $\mathfrak{g}$ may not be orthonormal.

Furthermore, for each $j = 1, 2, \ldots, n$, we let

$$d_j := |I_j|, \tag{2.4.3}$$

and define

$$\delta = \max\{d_j : j = 1, 2, \ldots, n\}. \tag{2.4.4}$$

Example 2.4.1. Suppose $G = SU(2)$. In this case, we consider the Hörmander system of left-invariant vector fields on $SU(2)$ given by

$$\boldsymbol{X} = \{X_1, X_2\},$$

where

$$X_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Observe that

$$[X_1, X_2] = \begin{pmatrix} 2i & 0 \\ 0 & -2i \end{pmatrix},$$

and recall that the set

$$\left\{ \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \right\}$$

forms a (orthonormal) basis of $\mathfrak{su}(2)$ (see Section 2.8.2). Thus, the set

$$\{X_1, X_2, [X_1, X_2]\}$$

is a basis of $\mathfrak{su}(2)$. Letting Y_j $(j = 1, 2, 3)$ be the basis elements of $\mathfrak{g}$ given by (2.4.1), one readily checks that we can take

$$Y_1 = X_1, \quad Y_2 = X_2, \quad Y_3 = [X_1, X_2].$$

Hence, we see that $d_1 = d_2 = 1$ and $d_3 = 2$ in this case.

2.4.2 An important neighbourhood of x in G and the ball-box theorem

We continue with the setting of Section 2.4.1. Let $x \in G$. We know that there exist a neighbourhood V of x in G and a neighbourhood N of 0 in $\mathbb{R}^n$ such that the mapping $\phi : N \to V$, which is given by

$$\phi((z_1, z_2, \ldots, z_n)) := e^{z_1 Y_1} e^{z_2 Y_2} \ldots e^{z_n Y_n}(x), \tag{2.4.5}$$

is a diffeomorphism (see Proposition 2.3.9 (v)). For $z \in V$ we then let

$$(z_1, z_2, \ldots, z_n) \in N \subset \mathbb{R}^n$$

denote the coordinates of z given by the coordinate chart (ϕ^{-1}, V); that is, z and $(z_1, z_2, \ldots, z_n)$ satisfy (2.4.5).

To obtain our desired aim, we shall make use of a result known as the ball-box theorem. The statement of this theorem can be found, for example, in Section 2.4 in Montgomery [35] or in Section 0.5.A in Gromov [24]. In our case, the ball-box theorem implies that there exist constants ε_0, C, $C' > 0$ such that

$$C'\phi\left(\mathrm{Box}(\varepsilon)\right) \subset B_\varepsilon(x) \subset C\phi\left(\mathrm{Box}(\varepsilon)\right), \tag{2.4.6}$$

for all $\varepsilon \leq \varepsilon_0$, where for each $\varepsilon > 0$, we let

$$\mathrm{Box}(\varepsilon) := \left\{x \in \mathbb{R}^n : |x_i| \leq \varepsilon^{d_i},\ \forall\, i = 1, 2, \ldots, n\right\}. \tag{2.4.7}$$

It is important to note here that we can apply the ball-box theorem due to the construction of the basis of $\mathfrak{g}$ given in Section 2.4.1. A proof of this result can be found in several references; see, for example, Sections 2.4 and 2.6 in Montgomery [35] or Section 4 in Nagel et al [36].

Now, let N be a neighbourhood of 0 in $\mathbb{R}^n$ and V be a neighbourhood of x in G small enough such that the following properties are satisfied:

(a) $V \subset B_{\varepsilon_0}(x)$; that is, V satisfies (2.4.6).

(b) The mapping $\phi : N \to V$ given by (2.4.5) is a diffeomorphism.

(c) Any $(z_1, z_2, \ldots, z_n) \in N$ satisfies

$$||(z_1, z_2, \ldots, z_n)||_{\mathbb{R}^n} \leq 1.$$

For any $z \in V$ we can then re-write (2.4.6) as

$$C'\left(|z_1|^{1/d_1} + |z_2|^{1/d_2} + \cdots + |z_n|^{1/d_n}\right) \le d(x,z) \\ \le C\left(|z_1|^{1/d_1} + |z_2|^{1/d_2} + \cdots + |z_n|^{1/d_n}\right). \quad (2.4.8)$$

2.4.3 Comparing the Carnot-Carathéodory metric to the Euclidean distance

We continue with the setting of Sections 2.4.1 and 2.4.2. We then have the following result.

Proposition 2.4.2. *There exist constants* $C_1, C_2 > 0$ *such that*

$$C_1\, d_E(x,z) \le d(x,z) \le C_2\, d_E(x,z)^{1/\delta}, \quad \forall\, z \in V, \quad (2.4.9)$$

where $d_E(\cdot,\cdot)$ *denotes the Euclidean distance on* $\mathbb{R}^n$ *induced by the chart* (ϕ^{-1}, V).

Proof. Recall that the neighbourhood V of e_G in G and the neighbourhood N of 0 in $\mathbb{R}^n$ satisfy properties (a), (b) and (c) from Section 2.4.2. By property (c), for every $(z_1, z_2, \ldots, z_n) \in N$ we have $|z_j| \le 1$ for all $j = 1, 2, \ldots, n$. Moreover, $d_j \ge 1$ for all $j = 1, 2, \ldots, n$. So, for every $(z_1, z_2, \ldots, z_n) \in N$, we have

$$|z_j| \le |z_j|^{1/d_j} \le |z_j|^{1/\delta}, \qquad \forall\, j = 1, 2, \ldots, n,$$

where δ is the integer given by (2.4.4). Hence, by (2.4.8), for every $z \in V$ we have

$$C'\left(|z_1| + |z_2| + \cdots + |z_n|\right) \le d(x,z) \\ \le C\left(|z_1|^{1/\delta} + |z_2|^{1/\delta} + \cdots + |z_n|^{1/\delta}\right). \quad (2.4.10)$$

Furthermore, the equivalence of norms in $\mathbb{R}^n$ implies that there exist constants $c_1, c_2, > 0$ such that

$$c_1\left(|z_1|^2 + |z_2|^2 + \cdots + |z_n|^2\right)^{\frac{1}{2}} \le \big(|z_1| + |z_2| + \cdots + |z_n|\big), \quad (2.4.11)$$

and

$$\left(|z_1|^{1/\delta} + |z_2|^{1/\delta} + \cdots + |z_n|^{1/\delta}\right) \le c_2 \left(|z_1|^2 + |z_2|^2 + \cdots + |z_n|^2\right)^{\frac{1}{2\delta}}, \quad (2.4.12)$$

for all $(z_1, z_2, \ldots, z_n) \in N$. Hence, applying (2.4.11) and (2.4.12) to (2.4.10), we obtain that there exist constants $C_1, C_2 > 0$ such that for all $z \in V$ we have

$$C_1\left(|z_1|^2 + |z_2|^2 + \cdots + |z_n|^2\right)^{\frac{1}{2}} \le d(x,z) \le C_2 \left(|z_1|^2 + |z_2|^2 + \cdots + |z_n|^2\right)^{\frac{1}{2\delta}}.$$

This is equivalent to (2.4.9).

□

Example 2.4.3. Consider the 3-dimensional, connected and compact Lie group $SU(2)$, and let $\{X_1, X_2, X_3\}$ be the basis of $\mathfrak{su}(2)$ given by (2.8.8). We further consider the Hörmander system of vector fields $\boldsymbol{X} = \{X_1, X_2\}$. Let $x = I$ be the identity element of $SU(2)$.

In this case, let $I_1 = 1$, $I_2 = 2$ and $I_3 = (1, 2)$. Then, the set

$$\{X_{[I_1]}, X_{[I_2]}, X_{[I_3]}\} = \{X_1, X_2, [X_1, X_2]\}$$

forms a basis of $\mathfrak{su}(2)$. We have $d_1 = 1$, $d_2 = 1$ and $d_3 = 2$ (see (2.4.3)).

By the ball-box theorem, in particular (2.4.8), there exists a neighbourhood V of I in $SU(2)$ and constants $C_1, C_2 > 0$ such that

$$C_1(|z_1| + |z_2| + |z_3|^{1/2}) \le |z| \le C_2(|z_1| + |z_2| + |z_3|^{1/2}), \quad \forall\, z \in V. \quad (2.4.13)$$

2.5 Schwartz kernel theorem

Suppose $\mathcal{M}$ is a smooth manifold. The objective of this section is to introduce the Schwartz kernel theorem. Its purpose is to describe certain operators acting on $\mathcal{D}(\mathcal{M})$, the space of compactly supported smooth functions on $\mathcal{M}$, in terms of an integral kernel. For a detailed exposition of the subject, see Treves [53].

We now assume $\mathcal{M} = G$. Throughout the thesis we use a particular characterisation of the space of compactly supported smooth functions, $\mathcal{D}(G)$, which is relevant to our purposes. Let K_i be an increasing family of compact sets, such that $\bigcup_i K_i = G$. We have the countable increasing union

$$\mathcal{D}(G) = \bigcup_i \mathcal{D}(K_i).$$

We then define the topology on $\mathcal{D}(G)$ via the family of semi-norms which are given by

$$||f||_{\mathcal{D}(K_i),N} := \sup_{\substack{\alpha \in \mathcal{I}(k) \\ |\alpha| \leq N \\ x \in K_i}} |X_\alpha f(x)|, \quad f \in \mathcal{D}(K_i), \tag{2.5.1}$$

for each $N \in \mathbb{N}_0$ and each i. This topology is independent of the choice of the K_i.

We let $\mathcal{D}'(G)$ be the space of distributions on $\mathcal{D}(G)$; that is, the space of continuous linear functionals on $\mathcal{D}(G)$. For $u \in \mathcal{D}'(G)$ and $\phi \in \mathcal{D}(G)$, we denote the action of u on ϕ by $\langle u, \phi \rangle$. For a given $u \in L^p(G)$, for $1 \leq p < +\infty$, we can define a corresponding distribution, $T_u \in \mathcal{D}'(G)$, by

$$\langle T_u, \phi \rangle = \int_G u(x)\, \phi(x)\, \mathrm{d}x, \quad \phi \in \mathcal{D}(G).$$

We will usually abuse the notation and identify T_u with u. The topology on $\mathcal{D}'(G)$ is then defined to be given by the family of semi-norms $\{|| \cdot ||_{\mathcal{D}'(G),N} : N \in \mathbb{N}_0\}$, where for each $N \in \mathbb{N}_0$,

$$||u||_{\mathcal{D}'(G),N} := \sup_{\substack{\phi \in \mathcal{D}(G) \\ ||\phi||_{\mathcal{D}(G),N} \leq 1}} |\langle u, \phi \rangle|, \quad u \in \mathcal{D}'(G). \tag{2.5.2}$$

If X is a left-invariant vector field, we then define Xu to be the distribution given by

$$\langle Xu, \phi \rangle = -\langle u, X\phi \rangle, \quad \forall\, \phi \in \mathcal{D}(G). \tag{2.5.3}$$

This readily implies that, for any collection $\{X_1, X_2, \ldots, X_k\}$ of left-invariant vector fields, if $\beta = (i_1, i_2, \ldots, i_b) \in \mathcal{I}(k)$, then the differential operator

$$X_\beta = X_{i_1} X_{i_2} \ldots X_{i_b}$$

satisfies

$$\langle X_\beta u, \phi \rangle = \left\langle u, X_\beta^t \phi \right\rangle, \quad \forall\, \phi \in \mathcal{D}(G), \tag{2.5.4}$$

where X_β^t is the differential operator defined by (2.3.13). That is,

$$\langle X_{i_1} X_{i_2} \dots X_{i_b} u, \phi \rangle = (-1)^{|\beta|} \left\langle u, X_{i_b} X_{i_{b-1}} \dots X_{i_1} \phi \right\rangle, \quad \forall\, \phi \in \mathcal{D}(G), \quad (2.5.5)$$

Let us also introduce the following useful notation.

Definition 2.5.1. We let $\mathscr{L}(L^2(G))^G$ denote the space of continuous linear operators

$$T : \mathcal{D}(G) \longrightarrow \mathcal{D}'(G),$$

which are left-invariant and bounded in the L^2 norm; that is,

$$||Tf||_{L^2(G)} \leq C\, ||f||_{L^2(G)}, \qquad \forall\, f \in \mathcal{D}(G),$$

for some $C > 0$.

We now state the Schwartz kernel theorem on manifolds. The reader is referred to [53] for a proof of this result.

Theorem 2.5.2 (Schwartz kernel Theorem)**.** *Let $\mathcal{M}$ be a smooth connected manifold and suppose $T : \mathcal{D}(\mathcal{M}) \to \mathcal{D}'(\mathcal{M})$ is a continuous linear operator. Then there exists a unique distribution $\kappa \in \mathcal{D}'(\mathcal{M} \times \mathcal{M})$ such that*

$$Tf(x) = \int_{\mathcal{M}} f(z)\, \kappa(x, z)\, dz, \quad \forall\, f \in \mathcal{D}(\mathcal{M}),\ x \in \mathcal{M},$$

in the sense of distributions; that is

$$\langle Tf, \phi \rangle = \left\langle \int_{\mathcal{M}} f(z) \kappa(\cdot, z)\, dz, \phi \right\rangle, \quad f, \phi \in \mathcal{D}(\mathcal{M}).$$

The converse also holds. Furthermore, the map

$$T \longmapsto \kappa$$

is an isomorphism of topological vector spaces from the space of continuous linear operators $T : \mathcal{D}(\mathcal{M}) \to \mathcal{D}'(\mathcal{M})$ onto $\mathcal{D}'(\mathcal{M} \times \mathcal{M})$.

2.5.1 Convolution on groups

Suppose G is a locally compact group. For two functions $f, g \in L^1(G)$, the convolution $f * g$ is defined by

$$(f * g)(x) \;=\; \int_G f(z)\, g(z^{-1}x)\, \mathrm{d}z.$$

This is well-defined and moreover, $f * g \in L^1(G)$. The following properties of convolution can be readily checked.

Proposition 2.5.3. *Suppose $f, g, h \in L^1(G)$, for a Lie group G. Then, the following assertions hold:*

(i) We have

$$||f * g||_{L^1(G)} \;\leq\; ||f||_{L^1(G)}\, ||g||_{L^1(G)}.$$

(ii) Convolution is associative; that is,

$$f * (g * h) = (f * g) * h.$$

(iii) Now, suppose that $f, g \in \mathcal{D}(G)$, and X is a left-invariant vector field on G. Then, we have

$$X(f * g) \;=\; f * (Xg) \quad \text{and} \quad \widetilde{X}(f * g) \;=\; (\widetilde{X} f) * g,$$

and additionally,

$$(Xf) * g \;=\; f * (\widetilde{X} g),$$

whenever these expressions make sense.

The following result about convolutions of L^2 functions is also well-known.

Proposition 2.5.4. *(1) If $f_1, f_2 \in L^2(G)$, then $f_1 * f_2$ is continuous on G, with*

$$||f_1 * f_2||_{L^\infty(G)} \;\leq\; ||f_1||_{L^2(G)}\, ||f_2||_{L^2(G)}. \tag{2.5.6}$$

(2) Consequently, the map

$$\begin{cases} L^2(G) \times L^2(G) & \longrightarrow \quad \mathcal{C}(G) \\ \quad (f_1, f_2) & \longmapsto \quad f_1 * f_2 \end{cases}$$

is bilinear and continuous.

Proof. Observe that for any $f_1, f_2 \in L^2(G)$,

$$f_1 * f_2(x) = \langle f_1, \pi_L(x) f_2^* \rangle_{L^2(G)}, \qquad \forall\, x \in G,$$

where π_L denotes the left regular representation on G (see (2.2.2)), and where for any $f \in L^2(G)$ we denote

$$f^*(x) = \overline{f(x^{-1})}, \qquad \forall\, x \in G.$$

As the complex-conjugate linear map

$$\begin{cases} L^2(G) & \longrightarrow & L^2(G) \\ f & \longmapsto & f^* \end{cases}$$

is an isometry on $L^2(G)$, then the continuity of $f_1 * f_2$ on G follows from the continuity of π_L (see Example 2.2.2). Furthermore, by the Cauchy-Schwarz inequality, as well as the unitarity of π_L, we have (2.5.6). □

Convolution with distributions

Suppose G is a Lie group. For a function f on G, we denote

$$\widetilde{f}(x) = f(x^{-1}), \quad x \in G.$$

Observe that, if $f, g \in L^1(G)$, then we have

$$\begin{aligned} f * g(x) &= \int_G f(z)\, g(z^{-1}x)\, \mathrm{d}z \\ &= \int_G f(z)\, \pi_L(x)\widetilde{g}(z)\, \mathrm{d}z, \end{aligned}$$

where π_L denotes the left regular representation on G (see (2.2.2)). Hence, the convolution of f and g can be written as

$$f * g(x) = \langle f, \pi_L(x)\widetilde{g} \rangle .$$

Since G is assumed to be unimodular, we also obtain

$$\begin{aligned} f * g(x) &= \int_G f(z)\, g(z^{-1}x)\, \mathrm{d}z \\ &= \int_G f(xy^{-1})\, g(y)\, \mathrm{d}y. \end{aligned}$$

Hence, we can also write the convolution of f and g as

$$f * g(x) = \left\langle g, \pi_R(x^{-1})\widetilde{f} \right\rangle,$$

where π_R denotes the right regular representation on G (see (2.2.3)). This suggests the following definition:

Definition 2.5.5. Suppose G is a Lie group. Let $u \in \mathcal{D}'(G)$ and $f \in \mathcal{D}(G)$. We then define

$$f * u(x) = \left\langle u, \pi_R(x^{-1})\widetilde{f} \right\rangle, \quad x \in G,$$

and

$$u * f(x) = \left\langle u, \pi_L(x)\widetilde{f} \right\rangle, \quad x \in G.$$

Example 2.5.6. Suppose G is a Lie group and let $f \in \mathcal{D}(G)$. We shall consider the convolution of f with the Dirac distribution δ_{e_G}. For every $x \in G$, we have

$$f * \delta_{e_G}(x) = \left\langle \delta_{e_G}, \pi_R(x^{-1})\widetilde{f} \right\rangle = \pi_R(x^{-1})\widetilde{f}(e_G) = f(x).$$

The following properties can be readily checked.

Proposition 2.5.7. *Suppose G is a Lie group. If $u \in \mathcal{D}'(G)$ and $f \in \mathcal{D}(G)$, then $u * f, f * u \in \mathcal{D}(G)$.*

Observe that, using Definition 2.5.5, we can readily check the following property:

$$\forall\, u, v, \varphi \in \mathcal{D}(G), \qquad \langle u * v, \varphi \rangle = \langle u, \varphi * \widetilde{v} \rangle. \tag{2.5.7}$$

If $u \in \mathcal{D}'(G)$, we define the distribution $\widetilde{u}$ by

$$\langle \widetilde{u}, \varphi \rangle = \langle u, \widetilde{\varphi} \rangle, \qquad \forall\, \varphi \in \mathcal{D}(G).$$

In particular, the following property holds

$$\forall\, u \in \mathcal{D}'(G),\ \varphi \in \mathcal{D}(G), \qquad \varphi * \widetilde{u} \in \mathcal{D}(G). \tag{2.5.8}$$

Expression (2.5.7) naturally leads to the definition of convolution of two distributions.

Definition 2.5.8. Let $u \in \mathcal{E}'(G)$ and $v \in \mathcal{D}'(G)$. Then, we define $u * v$ by the relation

$$\langle u * v, \varphi\rangle := \langle u, \varphi * \widetilde{v}\rangle\,. \tag{2.5.9}$$

The definition of the convolution of two distributions given by (2.5.9) is well-defined by (2.5.8).

2.5.2 Schwartz kernel theorem on Lie groups

We can also state a consequence of Theorem 2.5.2 in the case that $\mathcal{M} = G$ is a Lie group, for left-invariant operators. More precisely, we have the following result.

Corollary 2.5.9. *Let G be a connected Lie group. Let $T : \mathcal{D}(G) \to \mathcal{D}'(G)$ be a continuous linear operator, which is left-invariant; that is,*

$$(Tf)(zx) = T(f(zx)), \quad \forall\, f \in \mathcal{D}(G),\ x, z \in G.$$

Then, there exists unique $\kappa \in \mathcal{D}'(G)$ such that T is a convolution operator, with right convolution kernel κ; that is,

$$Tf = f * \kappa, \quad f \in \mathcal{D}(G),$$

in the sense of distributions.

The converse also holds. Furthermore, the map

$$T \longmapsto \kappa$$

is an isomorphism of topological vector spaces from the space of continuous linear operators $T : \mathcal{D}(G) \to \mathcal{D}'(G)$, which are left-invariant, onto $\mathcal{D}'(G)$.

The proof of the converse in Corollary 2.5.9 is, in fact, a routine exercise, which we now show. If G is a connected Lie group and $\kappa \in \mathcal{D}'(G)$, then the right-convolution operator

$$\begin{cases} T_\kappa : \mathcal{D}(G) & \longrightarrow \mathcal{D}'(G) \\ \qquad f & \longmapsto f * \kappa \end{cases}$$

is left-invariant. Indeed, for any $x, z \in G$, we have

$$\begin{aligned}(T_\kappa f)(zx) \;=\; (f * \kappa)(zx) \;&=\; \int_G f(y)\, \kappa(y^{-1}zx)\, \mathrm{d}x \\ &= \int_G f(zy)\, \kappa(y^{-1}x)\, \mathrm{d}x \\ &= T_\kappa(f(zx)),\end{aligned}$$

as claimed.

Definition 2.5.10. Let G be a connected Lie group. Let $T : \mathcal{D}(G) \to \mathcal{D}'(G)$ be a continuous linear operator, which is left-invariant. Then, there exists unique $\kappa \in \mathcal{D}'(G)$ such that

$$Tf \;=\; f * \kappa, \quad f \in \mathcal{D}(G).$$

In this case, we shall call κ the right-convolution kernel associated to the operator T. Moreover, we denote

$$T\delta_0 \;=\; \kappa.$$

Now, we suppose G is a connected Lie group. If X is a left-invariant vector field on G, we claim that the right convolution kernel associated to X is the distribution $X\delta_{e_G}$. Suppose $f \in \mathcal{D}(G)$, then we consider the convolution of f with the distribution $X\delta_{e_G}$. For every $x \in G$ we have

$$f * (X\delta_{e_G})(x) \;=\; \left\langle X\delta_{e_G}, \pi_R(x^{-1})\widetilde{f} \right\rangle.$$

By the definition of $X\delta_{e_G}$ (see (2.5.3)), we then obtain

$$\begin{aligned} f * (X\delta_{e_G})(x) \;&=\; -\left\langle \delta_{e_G}, X\big\{\pi_R(x^{-1})\widetilde{f}\,\big\} \right\rangle \;=\; -X\big\{\pi_R(x^{-1})\widetilde{f}\,\big\}(e_G) \\ &=\; -\,\partial_{t=0}\big\{\pi_R(x^{-1})\widetilde{f}\,\big\}(e^{tX}) \\ &=\; -\,\partial_{t=0}\,\widetilde{f}(e^{tX}x^{-1}) \\ &=\; -\widetilde{X}\widetilde{f}(x^{-1}),\end{aligned}$$

where we recall that for a function f on G, we denote

$$\widetilde{f}(x) = f(x^{-1}), \qquad \forall\, x \in G.$$

Now we compute

$$\begin{aligned} -\widetilde{X}\widetilde{f}(x^{-1}) &= \partial_{t=0}\, \widetilde{f}(e^{-tX}x^{-1}) = \partial_{t=0}\, \widetilde{f}\left((xe^{tX})^{-1}\right) \\ &= \partial_{t=0}\, f(xe^{tX}) \\ &= Xf(x). \end{aligned}$$

So, we have shown that

$$Xf(x) = f * (X\delta_{e_G})(x), \quad \forall\, x \in G,$$

which proves the claim.

Since $f = f * \delta_{e_G}$ (see Example 2.5.6), then we have also shown that

$$f * (X\delta_{e_G}) = X(f * \delta_{e_G}). \tag{2.5.10}$$

By applying this operation recursively, we obtain the following result, which follows from (2.5.4).

Proposition 2.5.11. *Let G be a connected Lie group. Suppose $\{X_1, X_2, \ldots, X_k\}$ is any collection of left-invariant vector fields on G. For $\beta = (i_1, i_2, \ldots, i_k) \in \mathcal{I}(k)$, consider the differential operator*

$$X_\beta = X_{i_1}X_{i_2}\ldots X_{i_b}.$$

Then, the right-convolution kernel of X_β is the distribution

$$X_\beta^t\, \delta_{e_G},$$

where the operator X_β^t is defined by (2.3.13). Additionally, the distribution $X_\beta^t\, \delta_{e_G}$ satisfies

$$\left\langle X_\beta^t\, \delta_{e_G}, \phi\right\rangle = \langle \delta_{e_G}, X_\beta\phi\rangle = X_\beta\, \phi(e_G), \quad \forall\, \phi \in \mathcal{D}(G).$$

Furthermore, consider the right-invariant differential operator

$$\widetilde{X}_\beta = \widetilde{X}_{i_1}\widetilde{X}_{i_2}\dots\widetilde{X}_{i_b},$$

where $\widetilde{X}$ *denotes the unique right-invariant vector field associated to* X *(see Proposition 2.3.9 (vii)). Then, the right-convolution kernel associated to* $\widetilde{X}_\beta$ *is given by*

$$(-1)^\ell \widetilde{X}^t_\beta \, \delta_{e_G}(x^{-1}),$$

where the distribution $\widetilde{X}^t_\beta$ *satisfies*

$$\left\langle \widetilde{X}^t_\beta \, \delta_{e_G}, \phi \right\rangle = \left\langle \delta_{e_G}, \widetilde{X}_\beta \phi \right\rangle = \widetilde{X}_\beta \, \phi(e_G), \quad \forall \, \phi \in \mathcal{D}(G)$$

Example 2.5.12. Suppose G is a connected Lie group and let the set

$$\{X_1, X_2, \dots, X_k\}$$

be a Hörmander system of left-invariant vector fields on G, for some $k \in \mathbb{N}$. Furthermore, let

$$\mathcal{L} := -(X_1^2 + X_2^2 + \cdots X_k^2)$$

denote its associated sub-Laplacian. Then, by Proposition 2.5.11, the right convolution kernel associated to $\mathcal{L}$ is the distribution $\mathcal{L}\delta_{e_G}$.

2.6 Infinitesimal representations

Let G be a Lie group and suppose that $\mathfrak{g}$ denotes its Lie algebra. For a representation $(\pi, \mathscr{H}_\pi)$ of G, we aim to introduce the infinitesimal representation $d\pi$ of $\mathfrak{g}$. For a discussion on the subject, one can see, for example, Knapp [30] or Fischer and Ruzhansky [18].

In order to do this, we need to consider the subset of $\mathscr{H}_\pi$ consisting of smooth vectors.

Definition 2.6.1. Let $(\pi, \mathscr{H}_\pi)$ be a representation of a Lie group G. We say that a vector $v \in \mathscr{H}_\pi$ is smooth if the mapping

$$x \longmapsto \pi(x)v, \quad x \in G,$$

is smooth. We let $\mathscr{H}_\pi^\infty$ denote the space of all smooth vectors in $\mathscr{H}_\pi$.

A sketch proof of the following result can be found in [18].

Proposition 2.6.2. *Let G be a Lie group and suppose that $\mathfrak{g}$ denotes its Lie algebra. Let $(\pi, \mathscr{H}_\pi)$ be a strongly continuous representation of G. Then, for any $X \in \mathfrak{g}$ and $v \in \mathscr{H}_\pi^\infty$, we have that the limit*

$$d\pi(X)v := \lim_{t \to 0} \frac{\pi\left(e^{tX}\right) v - v}{t} \tag{2.6.1}$$

exists and is finite. Moreover, $d\pi$ is a representation of $\mathfrak{g}$ on $\mathscr{H}_\pi^\infty$, satisfying

$$d\pi\left([X, Y]\right) = d\pi(X)\, d\pi(Y) - d\pi(Y)\, d\pi(X), \quad \forall\, X, Y \in \mathfrak{g}.$$

Definition 2.6.3. Let G be a Lie group and suppose $\mathfrak{g}$ denotes its Lie algebra. If $(\pi, \mathscr{H}_\pi)$ is a strongly continuous representation of G, then the representation $d\pi$ of $\mathfrak{g}$ defined by (2.6.1) is called the infinitesimal representation associated to π.

Moreover, we also have the following definition.

Definition 2.6.4. Let G be a Lie group and $\mathfrak{g}$ denote its Lie algebra. Suppose that $\boldsymbol{X} = \{X_1, X_2, \ldots, X_k\}$ is a Hörmander system of left-invariant vector fields on G. If T be a differential operator of the form

$$T = \sum_{\substack{\alpha \in \mathcal{I}(k) \\ |\alpha| \leq a}} c_\alpha X_\alpha,$$

for some $a \in \mathbb{N}$, then we define

$$\pi(T) = d\pi(T).$$

If the representation $(\pi, \mathscr{H}_\pi)$ of G is finite dimensional, then all of the vectors in $\mathscr{H}_\pi$ are smooth; that is,

$$\mathscr{H}_\pi^\infty = \mathscr{H}_\pi.$$

Moreover, on a compact Lie group G, every irreducible representation is finite dimensional, by the Peter-Weyl Theorem. Hence, every representation $(\pi, \mathscr{H}_\pi) \in \widehat{G}$ consists of smooth vectors.

A proof of the following property of infinitesimal representations can be found in [18].

Proposition 2.6.5. *Suppose G is a Lie group and let $\mathfrak{g}$ denote its Lie algebra. If $(\pi, \mathscr{H}_\pi)$ is a strongly continuous unitary representation of G and $\varphi \in \mathcal{D}(G)$, then for any left-invariant vector field $X \in \mathfrak{g}$ we have*

$$\pi(X\varphi)v \;=\; \pi(X)\pi(\varphi)v, \qquad \forall\, v \,\in\, \mathscr{H}_\pi.$$

If $\widetilde{X}$ is a right-invariant vector field, then

$$\pi(\widetilde{X}\varphi)v \;=\; \pi(\varphi)\pi(X)v, \qquad \forall\, v \,\in\, \mathscr{H}_\pi.$$

2.7 The Lie group $\mathbb{H}$ and its representations

We now summarise the relevant theory related to the Heisenberg group and its Lie algebra. For a detailed exposition of the work presented here, see for example Stein [47], Folland [20], or Folland and Stein [22].

We shall explain two different ways of characterising the Heisenberg group. First we may consider $\mathbb{H}$ to be the manifold $\mathbb{R}^3$, with the following group operation:

$$(x, y, t)\cdot(x', y', t') = (x + x',\, y + y',\, t + t' - (xy' - x'y))\,.$$

We also let $\mathfrak{h}$ denote the Lie algebra of $\mathbb{H}$, which by the definition, is the vector space of all left-invariant vector fields on $\mathbb{H}$. We equip $\mathfrak{h}$ with the Lie bracket $[\cdot,\cdot]$ given by

$$[X, Y] = XY - YX, \quad \text{for } X, Y \,\in\, \mathfrak{h},$$

which, as we discussed in Section 2.3.3, defines a smooth left-invariant vector field on $\mathbb{H}$. By identifying $\mathfrak{h}$ with the tangent space at the identity of $\mathbb{H}$, one can show that its basis is given by

$$X = \frac{\partial}{\partial x} + y\,\frac{\partial}{\partial t}, \quad Y = \frac{\partial}{\partial y} - x\,\frac{\partial}{\partial t}, \quad T' = -2\,\frac{\partial}{\partial t}.$$

This calculation is done in great detail in [47] (Chapter XII, Section 2.6). Now, observe that

$$
\begin{aligned}
[X,Y] &= \left(\frac{\partial}{\partial x} + y\,\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial y} - x\,\frac{\partial}{\partial t}\right) - \left(\frac{\partial}{\partial y} - x\,\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial x} + y\,\frac{\partial}{\partial t}\right) \\
&= \left(\frac{\partial^2}{\partial x\partial y} - \frac{\partial}{\partial x}\left\{x\frac{\partial}{\partial t}\right\} + y\frac{\partial^2}{\partial t\partial y} - xy\frac{\partial^2}{\partial t^2}\right) \\
&\qquad\qquad - \left(\frac{\partial^2}{\partial y\partial x} + \frac{\partial}{\partial y}\left\{y\frac{\partial}{\partial t}\right\} - x\frac{\partial^2}{\partial t\partial x} - xy\frac{\partial^2}{\partial t^2}\right) \\
&= \frac{\partial}{\partial x}\{x\}\frac{\partial}{\partial t} - \frac{\partial}{\partial y}\{y\}\frac{\partial}{\partial t} \\
&= -2\frac{\partial}{\partial t}.
\end{aligned}
$$

Hence,

$$[X,Y] = T'.$$

Now, we can also identify the Heisenberg group $\mathbb{H}$ with the manifold $\mathbb{C}\times\mathbb{R}$, equipped with the group operation

$$(\zeta,t)\cdot(\zeta',t') = \left(\zeta+\zeta',\, t+t'+\operatorname{Im}\left(\zeta\,\overline{\zeta'}\right)\right).$$

Consider the space

$$\mathcal{U} := \left\{z=(z_1,z_2)\in\mathbb{C}^2 \,:\, \operatorname{Im}(z_2) > |z_1|^2+|z_2|^2\right\},$$

and its boundary

$$\partial\mathcal{U} := \left\{z=(z_1,z_2)\in\mathbb{C}^2 \,:\, \operatorname{Im}(z_2) = |z_1|^2+|z_2|^2\right\}.$$

It is shown in Stein [47] (see Chapter XII) that the Heisenberg group may be identified with the boundary $\partial\mathcal{U}$ via the mapping

$$
\begin{cases}
\mathbb{H} & \longrightarrow \partial\mathcal{U} \\
(\zeta,t) & \longmapsto (\zeta, t+i|\zeta|^2)
\end{cases}.
$$

We now consider differential operators

$$\frac{\partial}{\partial\zeta} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\,\frac{\partial}{\partial y}\right), \qquad \frac{\partial}{\partial\overline{\zeta}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\,\frac{\partial}{\partial y}\right),$$

for $\zeta = x+iy$, and the left-invariant complex vector fields

$$\overline{Z} = \frac{\partial}{\partial \overline{\zeta}} - i\,\frac{\zeta}{2}\,\frac{\partial}{\partial t}, \qquad Z = \frac{\partial}{\partial \zeta} + i\,\frac{\overline{\zeta}}{2}\,\frac{\partial}{\partial t}.$$

The right-invariant complex vector fields corresponding to $\overline{Z}$ and Z are given by

$$\widetilde{\overline{Z}} = \frac{\partial}{\partial \overline{\zeta}} + i\,\frac{\zeta}{2}\,\frac{\partial}{\partial t}, \qquad \widetilde{Z} = \frac{\partial}{\partial \zeta} - i\,\frac{\overline{\zeta}}{2}\,\frac{\partial}{\partial t}.$$

Moreover, set

$$T = \frac{\partial}{\partial t}.$$

We have

$$[\overline{Z}, Z] \;=\; iT.$$

Observe further that

$$\overline{Z} \;=\; \frac{1}{2}(X + iY), \qquad Z \;=\; \frac{1}{2}(X - iY).$$

Thus, we may identify the complexification of $\mathfrak{h}$ with $T_0(\partial\mathcal{U})$, the space of tangent vectors to $\partial\mathcal{U}$ at 0. In [47] it is further proved (Section 2.6.3) that the vector fields $\overline{Z}$, Z, T form a basis of the tangent space $T_0(\partial\mathcal{U})$. Thus, we can assume that the left-invariant vector fields $\overline{Z}$, Z, T form a basis of the complexification of $\mathfrak{h}$.

Haar measure

On the Heisenberg group, we consider the Haar measure given by

$$\int_{\mathbb{H}} f(g)\,\mathrm{d}g = \int_0^{+\infty} \int_0^{2\pi} \int_{-\infty}^{+\infty} f(\rho\, e^{i\varphi},\, t)\,\frac{\rho}{2\pi^2}\,\mathrm{d}\rho\,\mathrm{d}\varphi\,\mathrm{d}t.$$

2.7.1 Representations of $\mathbb{H}$

In this section we aim to describe the infinite dimensional irreducible unitary representations of $\mathbb{H}$. For $\lambda > 0$ we define $\mathscr{F}^\lambda$ to be the Fock space consisting of the entire functions $F : \mathbb{C} \to \mathbb{C}$ satisfying

$$||F||^2_{\mathscr{F}^\lambda} = \frac{\lambda}{\pi} \int_{\mathbb{C}} |F(z)|^2\, e^{-\lambda |z|^2}\,\mathrm{d}z < +\infty,$$

where $\mathrm{d}z$ represents the Lebesgue measure on $\mathbb{C}$. We consider the inner product on $\mathscr{F}^\lambda$ given by

$$\langle F, G\rangle_{\mathscr{F}^\lambda} = \frac{\lambda}{\pi}\int_{\mathbb{C}} F(z)\,\overline{G(z)}\,e^{-\lambda|z|^2}\,\mathrm{d}z, \qquad \text{for} \quad F, G \in \mathscr{F}^\lambda. \tag{2.7.1}$$

We now define, for each $\lambda > 0$, the representations $\sigma_\lambda, \sigma_{-\lambda}$ of $\mathbb{H}$ acting on $\mathscr{F}^\lambda$ by

$$[\sigma_\lambda(\zeta,t)F](z) = e^{-\lambda\left(it+\overline{\zeta}\,z+\frac{1}{2}|\zeta|^2\right)}\,F(z+\zeta),$$
$$[\sigma_{-\lambda}(\zeta,t)F](z) = e^{-\lambda\left(-it-\overline{\zeta}\,z+\frac{1}{2}|\zeta|^2\right)}\,F(z-\overline{\zeta}).$$

An orthonormal basis of $\mathscr{F}^\lambda$ with respect to the inner product $\langle\cdot,\cdot\rangle$ is given by

$$\left\{\eta_j^{(\lambda)}(z) = \left(\frac{\lambda^j}{j\,!}\right)^{1/2} z^j \;:\; j \geq 0\right\}. \tag{2.7.2}$$

Since $\mathscr{F}^\lambda$ is an infinite dimensional space, then so are the representations $\sigma_\lambda, \sigma_{-\lambda}$. Moreover, the representations $\sigma_\lambda, \sigma_{-\lambda}$ are unitary with respect to the inner product on $\mathscr{F}^\lambda$ given by (2.7.1). It can then be shown that the representations $\sigma_\lambda, \sigma_{-\lambda}$ $(\lambda > 0)$ are the only non-equivalent infinite dimensional irreducible unitary representations of $\mathbb{H}$ on $\mathscr{F}^\lambda$ (see, for instance, Folland [20]).

We can calculate the infinitesimal representations of σ_λ.

Proposition 2.7.1. *For $\lambda > 0$ and $F \in \mathscr{F}^\lambda$:*

(i) $d\sigma_\lambda\left(\overline{Z}\right) F(z) = -\lambda\, z\, F(z)$,

(ii) $d\sigma_\lambda\left(Z\right) F(z) = \partial_z\, F(z)$,

(iii) $d\sigma_\lambda\left(T\right) F(z) = \lambda\, F(z)$.

Proof. The proofs are all similar, so we only exhibit the proof of (iii). Let $\lambda > 0$, then for $F \in \mathcal{F}^\lambda$,

$$\begin{aligned} d\sigma_\lambda(T)\,F(z) &= i\partial_t\, e^{-\lambda(it+\overline{\zeta}\,z+\frac{1}{2}|\zeta|^2)}\,F(z+\zeta)\Big|_{(\zeta,t)=(0,0)} \\ &= i(-i\lambda)\, e^{-\lambda(it+\overline{\zeta}\,z+\frac{1}{2}|\zeta|^2)}\,F(z+\zeta)\Big|_{(\zeta,t)=(0,0)} \\ &= \lambda\, F(z). \end{aligned}$$

□

Similarly, we compute the infinitesimal representation of $\sigma_{-\lambda}$:

Proposition 2.7.2. *For* $\lambda > 0$ *and* $F \in \mathscr{F}^{\lambda}$,

(i) $d\sigma_{-\lambda}\left(\overline{Z}\right) F(z) = -\partial_z F(z)$,

(ii) $d\sigma_{-\lambda}(Z)\, F(z) = \lambda z\, F(z)$,

(iii) $d\sigma_{-\lambda}(T)\, F(z) = -\lambda F(z)$.

2.7.2 The Plancherel formula on $\mathbb{H}$

Observe that, for $\lambda > 0$, the matrix entries of σ_λ and $\sigma_{-\lambda}$ are given by

$$\sigma_\lambda(\zeta,t)^{(j,k)} = \left\langle \sigma_\lambda(\zeta,t)\eta_k^{(\lambda)}, \eta_j^{(\lambda)} \right\rangle_{\mathscr{F}^\lambda}, \quad j,k \geq 0,\ (\zeta,t) \in \mathbb{H},$$

and

$$\sigma_{-\lambda}(\zeta,t)^{(j,k)} = \left\langle \sigma_{-\lambda}(\zeta,t)\eta_k^{(\lambda)}, \eta_j^{(\lambda)} \right\rangle_{\mathscr{F}^\lambda}, \quad j,k \geq 0,\ (\zeta,t) \in \mathbb{H}.$$

For an integrable function f on $\mathbb{H}$ and for $\lambda > 0$, the Fourier transform of f at the representation σ_λ is given by

$$\widehat{f}(\sigma_\lambda) = \int_{\mathbb{H}} f(x)\,\sigma_\lambda(x)^*\,\mathrm{d}x = \int_0^{+\infty}\int_0^{2\pi}\int_{-\infty}^{+\infty} f(\rho\, e^{i\varphi},\, t)\,\sigma_\lambda(\zeta,t)^*\,\frac{\rho}{2\pi^2}\,\mathrm{d}\rho\,\mathrm{d}\varphi\,\mathrm{d}t,$$

and a similar formula is obtained by taking the Fourier transform of f at $\sigma_{-\lambda}$. Since $\widehat{f}(\sigma_\lambda) \in \mathscr{L}(\mathscr{F}^\lambda)$, then the Fourier transform $\widehat{f}(\sigma_\lambda)$ can be thought of as the countably infinite matrix with entries

$$\widehat{f}(\sigma_\lambda)^{(j,k)} = \int_0^{+\infty}\int_0^{2\pi}\int_{-\infty}^{+\infty} f(\rho\, e^{i\varphi},\, t)\ [\sigma_\lambda(\zeta,t)^*]^{(j,k)}\,\frac{\rho}{2\pi^2}\,\mathrm{d}\rho\,\mathrm{d}\varphi\,\mathrm{d}t, \quad j,k \geq 0.$$

Now, for an integrable function f on $\mathbb{H}$, the Plancherel formula of f is given by

$$\int_{\mathbb{H}} |f(x)|^2\,\mathrm{d}x = \int_{\mathbb{R}\setminus\{0\}} \mathrm{Tr}\left(\widehat{f}(\sigma_\lambda)\widehat{f}(\sigma_\lambda)^*\right)|\lambda|\,\mathrm{d}\lambda = \int_{\mathbb{R}\setminus\{0\}} \sum_{j,k=0}^{+\infty} \left|\widehat{f}(\sigma_\lambda)^{(j,k)}\right|^2 |\lambda|\,\mathrm{d}\lambda.$$

For a detailed presentation on the Plancherel formula on the Heisenberg group, see for example [18].

2.8 The compact Lie group $SU(2)$ and its representations

This section is aimed at introducing the compact Lie group $SU(2)$. For a deep study in the subject, the reader is referred to Faraut [14], Hall [25], or Folland [20].

The compact Lie group $SU(2)$ is defined by:

$$SU(2) = \{g \in GL_2(\mathbb{C}) : g^* = g^{-1}, \det(g) = 1\}.$$

It is not difficult to show that

$$SU(2) = \left\{ \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} : \alpha, \beta \in \mathbb{C}, \ |\alpha|^2 + |\beta|^2 = 1 \right\}.$$

Alternatively, one can parametrise the Lie group $SU(2)$ to obtain an equivalent definition in terms of Euler angles. Fix an element of $SU(2)$

$$x = \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix},$$

with $|\alpha|^2 + |\beta|^2 = 1$. Then, writing the complex numbers α, β in polar coordinates, we see that there exist unique $r_\alpha \geq 0$ and $t^* \in (-\pi, \pi]$, such that

$$\alpha = r_\alpha e^{it^*},$$

and $r_\beta \geq 0$ and $\varphi^* \in (-3\pi/2, \pi/2]$, satisfying

$$\beta = r_\beta e^{i\varphi^*}.$$

We have,

$$|\alpha|^2 + |\beta|^2 = r_\alpha^2 + r_\beta^2 = 1. \tag{2.8.1}$$

We write $t = -t^*$ and we let $\varphi \in (0, 2\pi]$ be given by $\varphi = \varphi^* + 3\pi/2$. Since

$$\cos(\varphi^*) = -\sin(\varphi^* - \pi/2) = -\sin(\varphi)$$

and

$$\sin(\varphi^*) = \cos(\varphi^* - \pi/2) = \cos(\varphi),$$

then we have

$$\beta \;=\; r_\beta\big(-\sin(\varphi)\;+\;i\cos(\varphi)\big)\;=\;i\,r_\beta\,e^{i\varphi}.$$

Hence, we can write

$$\beta = i\,r_\beta\,e^{i\varphi}, \qquad \text{for some} \quad r_\beta \geq 0, \quad \varphi\,\in\,(0,2\pi],$$

and

$$\alpha = r_\alpha\,e^{-it}, \qquad \text{for some} \quad r_\alpha \geq 0, \quad t\,\in\,[-\pi,\pi).$$

Moreover, by (2.8.1), there exists unique $0 \leq \rho \leq \pi/2$ such that

$$r_\alpha = \cos(\rho), \qquad r_\beta = \sin(\rho).$$

So every element $x \in SU(2)$ can be written uniquely in the form

$$\begin{pmatrix} e^{-it}\,\cos(\rho) & i\,e^{i\varphi}\,\sin(\rho) \\ i\,e^{-i\varphi}\,\sin(\rho) & e^{it}\,\cos(\rho) \end{pmatrix}.$$

Haar measure on $SU(2)$

For the compact Lie group $SU(2)$ we may consider the Haar measure given by

$$\int_{SU(2)} f(g)\,\mathrm{d}g = \int_0^{\pi/2}\int_0^{2\pi}\int_{-\pi}^{\pi} f(\rho,\,\varphi,\,t)\,\frac{\sin(2\rho)}{4\pi^2}\,\mathrm{d}\rho\,\mathrm{d}\varphi\,\mathrm{d}t,$$

where $(\rho,\,\varphi,\,t)$ is the $SU(2)$ element given by

$$(\rho,\,\varphi,\,t) := \begin{pmatrix} e^{-it}\,\cos(\rho) & i\,e^{i\varphi}\,\sin(\rho) \\ i\,e^{-i\varphi}\,\sin(\rho) & e^{it}\,\cos(\rho) \end{pmatrix}.$$

2.8.1 Representations of $SU(2)$

For each integer $n \geq 0$ we let $\mathscr{P}_n$ determine the space consisting of polynomials in one complex variable, of degree at most n. We define the irreducible representation of $SU(2)$, π_n, which acts on $\mathscr{P}_n$, by

$$[\pi_n(g)\varphi]\,(z) = \big(-i\,\overline{\beta}\,z+\alpha\big)^n\;\varphi\left(\frac{\overline{\alpha}\,z - i\,\beta}{-i\,\overline{\beta}\,z+\alpha}\right), \tag{2.8.2}$$

for

$$g = \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} \in SU(2), \quad \varphi \in \mathscr{P}_n.$$

An inner product on the space $\mathscr{P}_n$ is given by

$$\langle \varphi, \psi \rangle_{\mathscr{P}_n} = \frac{n+1}{\pi} \int_{\mathbb{C}} \varphi(z) \, \overline{\psi(z)} \, (1+|z|^2)^{-n-2} \, \mathrm{d}z, \tag{2.8.3}$$

and its induced norm is

$$||\varphi||^2_{\mathscr{P}_n} = \frac{n+1}{\pi} \int_{\mathbb{C}} |\varphi(z)|^2 (1+|z|^2)^{-n-2} \, \mathrm{d}z.$$

An orthonormal basis of $\mathscr{P}_n$ is given by

$$\left\{ \varphi_j^{(n)}(z) = \binom{n}{j}^{1/2} z^j \; : \; 0 \leq j \leq n \right\}. \tag{2.8.4}$$

The matrix entries of π_n associated to the basis (2.8.4) (see (2.2.4)) are then given by

$$\pi_n(g)^{(i,j)} = \left\langle \pi_n(g) \, \varphi_j^{(n)}, \, \varphi_i^{(n)} \right\rangle_{\mathscr{P}_n}, \qquad 0 \leq i, j \leq n, \; g \in SU(2). \tag{2.8.5}$$

It is clear that, for each $n \in \mathbb{N}_0$, π_n is an $n+1$ dimensional representation. Moreover, one can also show that π_n is unitary with respect to the inner product given by (2.8.3).

We now have the following result, whose fundamental proof can be found in [14].

Proposition 2.8.1. *For each $n \in \mathbb{N}_0$, the representation $(\pi_n, \mathscr{P}_n)$ of $SU(2)$ is irreducible. Moreover, if π is an irreducible finite dimensional representation of $SU(2)$, then there exists $n \in \mathbb{N}_0$ such that $\pi \sim \pi_n$.*

This implies that

$$\widehat{SU(2)} \simeq \{[\pi_n]_\sim \; : \; n \in \mathbb{N}_0\}.$$

Let $M_n := M_{\pi_n}$ denote the finite dimensional complex vector space spanned by the matrix entries of π_n; that is,

$$M_n := \operatorname{Span} \left\{ \pi_n(g)^{(i,j)} \; : \; 0 \leq i, j \leq n \right\}. \tag{2.8.6}$$

We then define M_{finite} to be the space consisting of finite linear combinations of vectors in some M_n, for some $n \in \mathbb{N}_0$; that is,

$$M_{\text{finite}} := \bigoplus_{n \in \mathbb{N}_0} M_n. \tag{2.8.7}$$

The Peter-Weyl Theorem (see Theorem 2.2.3) tells us that M_{finite} is a dense subset of $L^2(SU(2))$.

We also introduce the conjugate representation $\overline{\pi}_n$, which acts on $\mathscr{P}_n$ as follows:

$$[\overline{\pi}_n(g)\,\varphi]\,(z) = [\pi_n(\overline{g})\,\varphi]\,(z) = (-i\,\beta\,z + \overline{\alpha})^n\,\varphi\left(\frac{\alpha\,z - i\,\overline{\beta}}{-i\,\beta\,z + \overline{\alpha}}\right),$$

for

$$g = \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} \in SU(2), \quad \varphi \in \mathscr{P}_n.$$

For each $n \geq 0$, the representation $\overline{\pi}_n$ is equivalent to π_n, and an intertwining operator is given in [39] (p. 222).

2.8.2 Lie algebra $\mathfrak{su}(2)$

One can show that the Lie algebra of $SU(2)$, which we denote by $\mathfrak{su}(2)$, consists of the 2×2 skew-Hermitian matrices, with complex entries, which have trace 0; that is

$$\mathfrak{su}(2) = \left\{ \begin{pmatrix} ia & z \\ -\overline{z} & -ia \end{pmatrix} : a \in \mathbb{R},\ z \in \mathbb{C} \right\}.$$

For a proof of this result, see Hall [25] (see Section 3.4 therein).

We now consider linearly independent vectors X_1, X_2, $X_3 \in \mathfrak{su}(2)$ given by

$$X_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}. \tag{2.8.8}$$

One easily checks that the set $\{X_1, X_2, X_3\}$ forms an orthonormal basis of $\mathfrak{su}(2)$ with respect to the inner product defined by

$$\langle X, Y \rangle_{\mathfrak{su}(2)} = \frac{1}{2}\mathrm{Tr}(XY^*) = -\frac{1}{2}\mathrm{Tr}(XY), \quad X, Y \in \mathfrak{su}(2).$$

Indeed,

$$\langle X_1, X_1\rangle_{\mathfrak{su}(2)} = -\frac{1}{2}\mathrm{Tr}\left[\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}^2\right] = -\frac{1}{2}\mathrm{Tr}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = 1,$$

$$\langle X_2, X_2\rangle_{\mathfrak{su}(2)} = -\frac{1}{2}\mathrm{Tr}\left[\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^2\right] = -\frac{1}{2}\mathrm{Tr}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = 1,$$

$$\langle X_3, X_3\rangle_{\mathfrak{su}(2)} = -\frac{1}{2}\mathrm{Tr}\left[\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}^2\right] = -\frac{1}{2}\mathrm{Tr}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = 1.$$

Also,

$$\langle X_1, X_2\rangle_{\mathfrak{su}(2)} = -\frac{1}{2}\mathrm{Tr}\left[\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right] = -\frac{1}{2}\mathrm{Tr}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = 0,$$

$$\langle X_1, X_3\rangle_{\mathfrak{su}(2)} = -\frac{1}{2}\mathrm{Tr}\left[\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}\right] = -\frac{1}{2}\mathrm{Tr}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = 0,$$

$$\langle X_2, X_3\rangle_{\mathfrak{su}(2)} = -\frac{1}{2}\mathrm{Tr}\left[\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}\right] = -\frac{1}{2}\mathrm{Tr}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = 0.$$

Further, observe that

$$\begin{aligned} [X_1, X_2] &= \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \\ &= \begin{pmatrix} -2i & 0 \\ 0 & 2i \end{pmatrix}, \end{aligned}$$

so $[X_1, X_2] = -2X_3$. By identifying the vector X_j, for each $j = 1, 2, 3$, with the unique left-invariant vector field associated to it, we then see that the set $\{X_1, X_2\}$ forms a Hörmander system on $SU(2)$, and thus, the operator

$$\mathcal{L} = -(X_1^2 + X_2^2)$$

is a sub-Laplacian on $SU(2)$.

We now compute the infinitesimal representations of π_n and $\overline{\pi}_n$ on the basis elements of $\mathfrak{su}(2)$, X_1, X_2, X_3.

Proposition 2.8.2. *Let $n \geq 0$ and $\varphi \in \mathscr{P}_n$. Then,*

(i) $\pi_n(X_1)\,\varphi(z) = (z(-n + z\partial_z) + \partial_z)\ \varphi(z)$,

(ii) $\pi_n(X_2)\,\varphi(z) = i\,(z(-n + z\partial_z) - \partial_z)\ \varphi(z)$,

(iii) $\pi_n(X_3)\,\varphi(z) = i(n - 2z\partial_z)\,\varphi(z)$.

Proof. First observe that

$$e^{tX_1} = \begin{pmatrix} \cos t & i\sin t \\ i\sin t & \cos t \end{pmatrix}, \quad e^{tX_2} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}, \quad e^{tX_3} = \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix}.$$

We shall calculate $\pi_n(X_3)$ only, since all of the calculations are similar. Suppose first that $\varphi(z) = z^j$, for some $0 \leq j \leq n$. It is not difficult to calculate that

$$\begin{aligned} \pi_n(X_3)\,\varphi(z) &= \frac{d}{dt}\left[\pi_n\left(e^{tX_3}\right)\varphi\right](z)\bigg|_{t=0} \\ &= \frac{d}{dt}\left[e^{int}\,\varphi(e^{-2it}z)\right]\bigg|_{t=0} \\ &= \frac{d}{dt}\left[e^{i(n-2j)t}z^j\right]\bigg|_{t=0} \\ &= i(n-2j)z^j, \end{aligned}$$

by (2.6.1). In particular, we have

$$\pi_n(X_3)\,\varphi(z) = i\left(n - 2z\frac{\partial}{\partial z}\right)\varphi(z), \tag{2.8.9}$$

and hence, by the linearity of the operator $i(n - 2z\partial z)$, we deduce that (2.8.9) holds for any $\varphi \in \mathscr{P}_n$. Hence,

$$\pi_n(X_3) = i(n - 2z\partial_z),$$

as claimed. □

Similarly, we obtain:

Proposition 2.8.3. *Let $n \geq 0$ and $\varphi \in \mathscr{P}^n$. Then,*

(i) $\overline{\pi}_n(X_1)\,\varphi(z) = -\left(z(-n + z\partial_z) + \partial_z\right)\,\varphi(z)$,

(ii) $\overline{\pi}_n(X_2)\,\varphi(z) = -i\left(z(-n + z\partial_z) - \partial_z\right)\,\varphi(z)$,

(iii) $\overline{\pi}_n(X_3)\,\varphi(z) = -i(n - 2z\partial_z)\,\varphi(z)$.

Chapter 3

Analysis of the sub-Laplacian and Sobolev spaces on compact Lie groups

In this chapter we study the sub-Laplacian associated to a Hörmander system of a compact Lie group G. We shall first study its spectral theory closely, and use the spectral theorem to write down the action of its self-adjoint extension on $L^2(G)$. We shall consider the compact Lie group $SU(2)$ as an example, providing some calculations to put the theory into practice. The heat semigroup is also discussed in this chapter, and we explain some relevant properties.

Furthermore, we shall also consider the Sobolev spaces arising naturally from the sub-Laplacian for the case $p = 2$. We state some important properties of these, and prove the interpolation theorem and a Sobolev inequality for these spaces. A local version of Taylor's theorem is also proved in this chapter.

Lastly, we shall prove some important inequalities regarding the Fourier multipliers of a sub-Laplacian, using Littlewood-Paley decompositions. We base these on the results obtained in Alexopoulos [2] in the setting of Lie groups of polynomial growth, which were later adapted by Furioli et al [23].

3.1 Sub-Laplacians on compact Lie groups

This section is dedicated to providing an overview of the spectral analysis of a sub-Laplacian on a compact Lie group. The references Varopoulos [54] and [55] provide an extensive study of a sub-Laplacian on a Lie group of polynomial growth. For further results in this setting, see Alexopoulos [2] and Furioli et al [23]. For a comprehensive study of spectral theory, see Akhiezer and Glazman

[1], Conway [9], Davies [11], Reed and Simon [38], or Rudin [42]. One can also find in Fischer and Ruzhansky [18] the spectral theory for Rockland operators on homogeneous groups, which is a generalisation of sub-Laplacians. Additionally, further results in the case of stratified nilpotent Lie groups can be found in Folland [19].

In this section we shall also discuss the heat semigroup and its associated heat kernels (see Section 3.1.5). In Hunt [29] one can find important results on this topic, wherein the heat kernels are interpreted as positive measures. For results in the setting of Riemannian and sub-Riemannian geometry, the reader is redirected to Strichartz [49] and [50].

Throughout this section, we let G be a compact Lie group, and suppose $\mathfrak{g}$ denotes its Lie algebra, unless stated otherwise. Furthermore, we consider a set $\{X_1, X_2, \ldots, X_k\}$, for some $k \in \mathbb{N}$, which forms a Hörmander system of left-invariant vector fields on G, and let $\mathcal{L}$ be its associated sub-Laplacian:

$$\mathcal{L} = -\left(X_1^2 + X_2^2 + \cdots + X_k^2\right).$$

3.1.1 Definitions and the self-adjoint extension of $\mathcal{L}$

We recall the following definitions.

Definition 3.1.1. Suppose T is a densely defined linear operator (not necessarily bounded) on a Hilbert space $\mathscr{H}$, with domain $\mathrm{Dom}(T)$. Let also $\mathrm{Dom}(T^*)$ be the set of elements $v \in \mathscr{H}$ such that there exists $w \in \mathscr{H}$ which satisfies

$$\langle Tu, v\rangle_{\mathscr{H}} = \langle u, w\rangle_{\mathscr{H}}, \quad \forall\, u \in \mathrm{Dom}(T).$$

(a) We then define T^*, the adjoint of T, to be the operator satisfying

$$\langle Tu, v\rangle_{\mathscr{H}} = \langle u, T^*v\rangle, \quad \forall\, u \in \mathrm{Dom}(T),\, v \in \mathrm{Dom}(T^*).$$

(b) We say T is a symmetric operator if $\mathrm{Dom}(T) \subset \mathrm{Dom}(T^*)$ and

$$Tu = T^*u, \quad \forall\, u \in \mathrm{Dom}(T).$$

Or equivalently,

$$\langle Tu, v\rangle_{\mathscr{H}} = \langle u, Tv\rangle, \quad \forall\, u, v \in \mathrm{Dom}(T).$$

(c) We say T is self-adjoint if it is symmetric and $\mathrm{Dom}(T) = \mathrm{Dom}(T^*)$.

By definition, $\mathcal{L}$ is a differential operator acting on $\mathcal{C}^\infty(G)$. So $\mathcal{L}$ is an unbounded operator which is densely defined on $L^2(G)$. Observe that, if X is a left-invariant vector field and $f, g \in \mathcal{C}^\infty(G)$, then

$$\begin{aligned} \langle Xf, g\rangle_{L^2(G)} &= \int_G \left(\partial_{t=0} f(xe^{tX})\right) \overline{g(x)} \, \mathrm{d}x \\ &= \int_G f(x) \left(\partial_{t=0} \overline{g(xe^{-tX})}\right) \mathrm{d}x \\ &= -\langle f, Xg\rangle_{L^2(G)} . \end{aligned}$$

Therefore,

$$\begin{aligned} \langle \mathcal{L}f, g\rangle_{L^2(G)} &= -\sum_{j=1}^k \left\langle X_j^2 f, g\right\rangle_{L^2(G)} = \sum_{j=1}^k \langle X_j f, X_j g\rangle_{L^2(G)} \\ &= -\sum_{j=1}^k \left\langle f, X_j^2 g\right\rangle_{L^2(G)} \\ &= \langle f, \mathcal{L}g\rangle_{L^2(G)} , \end{aligned}$$

which means that $\mathcal{L}$ is a symmetric operator. Moreover, we check that

$$\begin{aligned} \langle \mathcal{L}f, f\rangle_{L^2(G)} &= \sum_{j=1}^k \langle X_j f, X_j f\rangle_{L^2(G)} \\ &= \sum_{j=1}^k ||X_j f||^2_{L^2(G)} \\ &\geq 0, \end{aligned}$$

and so $\mathcal{L}$ is a non-negative operator. Therefore, $\mathcal{L}$ admits a self-adjoint extension to $L^2(G)$ (see, for example, Section 85 in Akhiezer and Glazman [1]). Furthermore, one can show that this extension is unique (see, for instance, Section 12 in Strichartz [50]). Throughout this thesis, we will keep the same notation for the differential operator $\mathcal{L}$ and its unique self-adjoint extension to $L^2(G)$. We let $\mathrm{Dom}(\mathcal{L})$ denote the domain of the self-adjoint operator $\mathcal{L}$.

3.1.2 Fourier analysis of $\mathcal{L}$

In this section we aim to write down the spectrum of the self-adjoint operator $\mathcal{L}$, which we denote by $\mathrm{Spec}(\mathcal{L})$. For each $(\pi, \mathscr{H}_\pi) \in \widehat{G}$ we consider an inner product $\langle \cdot, \cdot \rangle_{\mathscr{H}_\pi}$ on $\mathscr{H}_\pi$, and we let d_π be the dimension of π. Suppose further that $|| \cdot ||_{\mathscr{H}_\pi}$ denotes the norm associated to $\langle \cdot, \cdot \rangle_{\mathscr{H}_\pi}$. Observe that for each $\pi \in \widehat{G}$ and each $j = 1, 2, \ldots, k$, $\pi(X_j)$ is a skew-adjoint operator on $\mathscr{H}_\pi$; that is,

$$\langle \pi(X_j)u, v \rangle_{\mathscr{H}_\pi} = \langle u, -\pi(X_j)v \rangle_{\mathscr{H}_\pi}, \quad \forall\, u, v \in \mathscr{H}_\pi.$$

In particular, this implies that

$$\pi(X_j^2)^* = \pi(X_j^2), \quad \forall\, j = 1, 2, \ldots, k.$$

Hence,

$$\pi(\mathcal{L}) = -\sum_{j=1}^{k} \pi(X_j^2)$$

is a self-adjoint operator acting on the finite dimensional inner product space $\mathscr{H}_\pi$. Therefore it is diagonalisable and there exists an orthonormal basis

$$\left\{ \varphi_j^{(\pi)} : j = 1, 2, \ldots, d_\pi \right\} \tag{3.1.1}$$

of $\mathscr{H}_\pi$ which consists of eigenvectors of $\pi(\mathcal{L})$. Thus, whenever $1 \leq j \leq d_\pi$ we have

$$\pi(\mathcal{L})\varphi_j^{(\pi)} = \lambda_j^{(\pi)} \varphi_j^{(\pi)}, \tag{3.1.2}$$

where $\lambda_j^{(\pi)} \in \mathbb{C}$ denotes the eigenvalues of $\pi(\mathcal{L})$ associated to the eigenfunction $\varphi_j^{(\pi)}$, and hence

$$\pi(\mathcal{L})^{(i,j)} = \left\langle \pi(\mathcal{L})\varphi_j^{(\pi)}, \varphi_i^{(\pi)} \right\rangle_{\mathscr{H}_\pi} = \begin{cases} \lambda_j^{(\pi)}, & \text{if } i = j, \\ 0, & \text{if } i \neq j \end{cases}. \tag{3.1.3}$$

Moreover, since for any $\varphi \in \mathscr{H}_\pi$,

$$\langle \pi(\mathcal{L})\varphi, \varphi\rangle_{\mathscr{H}_\pi} = -\sum_{j=1}^{k} \left\langle \pi(X_j^2)\varphi, \varphi\right\rangle_{\mathscr{H}_\pi} = \sum_{j=1}^{k} \langle \pi(X_j)\varphi, \pi(X_j)\varphi\rangle_{\mathscr{H}_\pi}$$
$$= \sum_{j=1}^{k} ||\pi(X_j)\varphi||^2_{\mathscr{H}_\pi} \geq 0,$$

then the eigenvalues $\lambda_j^{(\pi)}$ $(1 \leq j \leq d_\pi)$ of $\pi(\mathcal{L})$ are non-negative real numbers.

Now, for each $\pi \in \widehat{G}$, we consider the matrix entries of π (see (2.2.4)) with respect to the orthonormal basis of $\mathscr{H}_\pi$ given by (3.1.1):

$$\pi(x)^{(i,j)} = \left\langle \pi(x)\varphi_j^{(\pi)}, \varphi_i^{(\pi)}\right\rangle_{\mathscr{H}_\pi}, \quad 1 \leq i, j \leq d_\pi,\ x \in G.$$

By the Peter-Weyl Theorem (see Theorem 2.2.3), we have that the set

$$\left\{\sqrt{d_\pi}\, \pi(\,\cdot\,)^{(i,j)} \,:\, \pi \in \widehat{G},\ 1 \leq i, j \leq d_\pi\right\}$$

forms an orthonormal basis of $L^2(G)$. Now, if X is a left-invariant vector field on G, then whenever $1 \leq i, j \leq d_\pi$, using Proposition 2.3.9 (vi), we calculate that

$$\begin{aligned}
X\pi(x)^{(i,j)} &= X\left\langle \pi(x)\varphi_j^{(\pi)}, \varphi_i^{(\pi)}\right\rangle_{\mathscr{H}_\pi} \\
&= \left\langle \frac{d}{dt}\pi\left(xe^{tX}\right)\Big|_{t=0} \varphi_j^{(\pi)}, \varphi_i^{(\pi)}\right\rangle_{\mathscr{H}_\pi} \\
&= \left\langle \frac{d}{dt}\pi(x)\,\pi\left(e^{tX}\right)\Big|_{t=0} \varphi_j^{(\pi)}, \varphi_i^{(\pi)}\right\rangle_{\mathscr{H}_\pi} \\
&= \left\langle \pi(x)\pi(X)\varphi_j^{(\pi)}, \varphi_i^{(\pi)}\right\rangle_{\mathscr{H}_\pi}.
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}
\mathcal{L}\pi(x)^{(i,j)} &= -\sum_{j=1}^{k} \left\langle X_j^2\, \pi(x)\, \varphi_j^{(\pi)}, \varphi_i^{(\pi)}\right\rangle_{\mathscr{H}_\pi} \\
&= -\sum_{j=1}^{k} \left\langle \pi(x)\, \pi(X_j)^2\, \varphi_j^{(\pi)}, \varphi_i^{(\pi)}\right\rangle_{\mathscr{H}_\pi} \\
&= \left\langle \pi(x)\, \pi(\mathcal{L})\, \varphi_j^{(\pi)}, \varphi_i^{(\pi)}\right\rangle_{\mathscr{H}_\pi}.
\end{aligned}$$

By (3.1.2), we then have

$$\begin{aligned}\mathcal{L}\pi(x)^{(i,j)} &= \left\langle \pi(x)\,\pi(\mathcal{L})\,\varphi_j^{(\pi)}, \varphi_i^{(\pi)} \right\rangle_{\mathscr{H}_\pi} \\ &= \left\langle \pi(x)\,\lambda_j^{(\pi)}\varphi_j^{(\pi)}, \varphi_i^{(\pi)} \right\rangle_{\mathscr{H}_\pi} \\ &= \lambda_j^{(\pi)}\pi(x)^{(i,j)}.\end{aligned}$$

Therefore, we have the spectral decomposition

$$\operatorname{Spec}(\mathcal{L}) = \left\{\lambda_j^{(\pi)} \,:\, \pi \in \widehat{G},\ 1 \le j \le d_\pi\right\}. \tag{3.1.4}$$

Remark 3.1.2. One can show that the spectrum of $\mathcal{L}$ is discrete. Consider the operator $e^{-\mathcal{L}}$. As we shall see in a later section, this operator is compact (see Remark 3.1.7). Now, by spectral theory, the spectrum of $e^{-\mathcal{L}}$ is given by

$$\operatorname{Spec}(e^{-\mathcal{L}}) = \left\{e^{-\lambda} \,:\, \lambda \in \operatorname{Spec}(\mathcal{L})\right\}.$$

This implies that we can write the spectrum of $\mathcal{L}$ as

$$\operatorname{Spec}(\mathcal{L}) = \left\{-\ln(\mu) \,:\, \mu \in \operatorname{Spec}(e^{-\mathcal{L}})\right\}.$$

In fact, this relation yields a one-to-one correspondence between $\operatorname{Spec}(\mathcal{L})$ and $\operatorname{Spec}(e^{-\mathcal{L}})$. Additionally, eigenvalues associated via this relation have the same geometric multiplicity.

By the theory of compact operators (see, for example, Section VII.7.1 in Conway [9]), we know that $\operatorname{Spec}(e^{-\mathcal{L}})$ has at most one accumulation point and that it can only be 0, and moreover all non-zero eigenvalues of $e^{-\mathcal{L}}$ have finite multiplicity. By functional analysis, this implies that $\operatorname{Spec}(\mathcal{L})$ is discrete.

We also make the following observation.

Remark 3.1.3. (1) Suppose that L is the Laplace-Beltrami operator of G. By the theory of compact Lie groups, we know that for every $\pi \in \widehat{G}$, there exists a unique scalar $\lambda_\pi \ge 0$ such that

$$\pi(L) = \lambda_\pi \operatorname{Id}_{\mathscr{H}_\pi}.$$

In the case of a sub-Laplacian $\mathcal{L}$ this statement does not hold. We shall provide an example for the case $G = SU(2)$ in Section 3.1.4.

(2) Now, as we will also demonstrate in the example of $SU(2)$ in Section 3.1.4, there exists $\lambda \in \text{Spec}(\mathcal{L})$ for which there exist $\pi, \pi' \in \widehat{G}$, $i, j \in \{1, 2, \ldots, d_\pi\}$ and $i', j' \in \{1, 2, \ldots, d_{\pi'}\}$, where π and π' are non-equivalent, such that

$$\mathcal{L}\pi(x)^{(i,j)} = \lambda\pi(x)^{(i,j)}$$

and

$$\mathcal{L}\pi'(x)^{(i',j')} = \lambda\pi'(x)^{(i',j')}.$$

In this case the geometric multiplicity of λ is greater than 1 and the eigenspace associated to λ is not a subspace of either M_π or $M_{\pi'}$. Recall that, for each $\pi \in \widehat{G}$, M_π is the space given by

$$M_\pi = \text{Span}\left\{\langle \pi_1(\cdot)\varphi, \psi\rangle_{\mathscr{H}_{\pi_1}} : \varphi, \psi \in \mathscr{H}_{\pi_1},\ \pi_1 \in [\pi]_\sim\right\}.$$

3.1.3 Spectral decomposition of $\mathcal{L}$

Let E_λ denote the orthogonal projection onto the eigenspace corresponding to the eigenvalue λ; that is, if $\lambda \in \text{Spec}(\mathcal{L})$, then

$$E_\lambda f := \sum_{\substack{\pi \in \widehat{G},\, 1 \le i,j \le d_\pi \\ \lambda_j^{(\pi)} = \lambda}} \left\langle f, \pi(\cdot)^{(i,j)}\right\rangle_{L^2(G)} \pi^{(i,j)}.$$

Since the spectrum of $\mathcal{L}$ is discrete (see Remark 3.1.2), then the spectral theory (see, for instance, Theorem VIII.5 in [38]) tells us that,

$$\mathcal{L} = \sum_{\lambda \in \text{Spec}(\mathcal{L})} \lambda\, E_\lambda,$$

with domain $\text{Dom}(\mathcal{L})$ consisting of functions $f \in L^2(G)$, such that

$$\sum_{\lambda \in \text{Spec}(\mathcal{L})} |\lambda|^2 \left|\left|E_\lambda f\right|\right|^2_{L^2(G)} < +\infty.$$

So, for a function $f \in \text{Dom}(\mathcal{L})$, we have

$$\mathcal{L}f \;=\; \sum_{\lambda\,\in\,\mathrm{Spec}(\mathcal{L})} \lambda \sum_{\substack{\pi\,\in\,\widehat{G},\,1\le i,j\le d_\pi \\ \lambda_j^{(\pi)}=\lambda}} d_\pi \,\left\langle f, \pi(\cdot)^{(i,j)}\right\rangle_{L^2(G)} \pi^{(i,j)}$$

$$= \sum_{\pi\,\in\,\widehat{G}} d_\pi \sum_{i,j=1}^{d_\pi} \lambda_j^{(\pi)} \left\langle f, \pi(\cdot)^{(i,j)}\right\rangle_{L^2(G)} \pi^{(i,j)} \tag{3.1.5}$$

where we notice that

$$\left\langle f\,,\, \pi(\cdot)^{(i,j)}\right\rangle_{L^2(G)} \;=\; \int_G f(y)\,\overline{\pi(y)^{(i,j)}}\;\mathrm{d}y \;=\; \widehat{f}(\pi)^{(j,i)}, \quad \forall\, 1\le i,j\le d_\pi.$$

Hence,

$$\mathcal{L}f \;=\; \sum_{\pi\,\in\,\widehat{G}} d_\pi \sum_{i,j=1}^{d_\pi} \lambda_j^{(\pi)}\,\widehat{f}(\pi)^{(j,i)}\pi^{(i,j)}. \tag{3.1.6}$$

Furthermore, we have

$$\sum_{\lambda\,\in\,\mathrm{Spec}(\mathcal{L})} |\lambda|^2 \left|\left|E_\lambda f\right|\right|^2_{L^2(G)} \;=\; \sum_{\pi\,\in\,\widehat{G}} d_\pi \sum_{i,j=1}^{d_\pi} \left|\lambda_j^{(\pi)}\right|^2 \left|\widehat{f}(\pi)^{(j,i)}\right|^2, \tag{3.1.7}$$

and hence the domain $\mathrm{Dom}(\mathcal{L})$ of $\mathcal{L}$ is given by

$$\left\{ f \,\in\, L^2(G) \,:\, \sum_{\pi\,\in\,\widehat{G}} d_\pi \sum_{i,j=1}^{d_\pi} \left|\lambda_j^{(\pi)}\right|^2 \left|\widehat{f}(\pi)^{(j,i)}\right|^2 < +\infty \right\}.$$

More generally, by the spectral theorem (see, for example, [38]), if m is a Borel function on $\mathbb{R}$, we have

$$m(\mathcal{L}) \;=\; \sum_{\lambda\,\in\,\mathrm{Spec}(\mathcal{L})} m(\lambda) E_\lambda. \tag{3.1.8}$$

Hence,

$$m(\mathcal{L})f \;=\; \sum_{\pi\,\in\,\widehat{G}} d_\pi \sum_{i,j=1}^{d_\pi} m\big(\lambda_j^{(\pi)}\big)\,\widehat{f}(\pi)^{(j,i)}\,\pi^{(i,j)}, \tag{3.1.9}$$

and the domain $\mathrm{Dom}(m(\mathcal{L}))$ of $m(\mathcal{L})$ consists of functions $f \in L^2(G)$ such that

$$\sum_{\pi \in \widehat{G}} d_\pi \sum_{i,j=1}^{d_\pi} \left|m\left(\lambda_j^{(\pi)}\right)\right|^2 \left|\widehat{f}(\pi)^{(j,i)}\right|^2 < +\infty.$$

Let us now consider some important examples.

Example 3.1.4.

(i) For $s \in \mathbb{R}$, the operator $(I+\mathcal{L})^s$ satisfies

$$(I+\mathcal{L})^s f \;=\; \sum_{\pi \in \widehat{G}} d_\pi \sum_{i,j=1}^{d_\pi} \left(1+\lambda_j^{(\pi)}\right)^s \widehat{f}(\pi)^{(j,i)}\, \pi^{(i,j)}, \tag{3.1.10}$$

for $f \in \mathrm{Dom}\left((I+\mathcal{L})^{s/2}\right)$, which is the set consisting of functions $f \in L^2(G)$ such that

$$\sum_{\pi \in \widehat{G}} d_\pi \sum_{i,j=1}^{d_\pi} \left|1+\lambda_j^{(\pi)}\right|^{2s} \left|\widehat{f}(\pi)^{(j,i)}\right|^2 < +\infty.$$

(ii) For each $t>0$, the operator $e^{-t\mathcal{L}}$ can be written as

$$e^{-t\mathcal{L}} f \;=\; \sum_{\lambda \in \mathrm{Spec}(\mathcal{L})} e^{-t\lambda} E_\lambda(f) \;=\; \sum_{\pi \in \widehat{G}} d_\pi \sum_{i,j=1}^{d_\pi} e^{-t\lambda_j^{(\pi)}}\, \widehat{f}(\pi)^{(j,i)}\, \pi^{(i,j)},$$

for $f \in \mathrm{Dom}(e^{-t\mathcal{L}})$, which is the set consisting of functions $f \in L^2(G)$ such that

$$\sum_{\pi \in \widehat{G}} d_\pi \sum_{i,j=1}^{d_\pi} \left|e^{-t\lambda}\right|^2 \left|\widehat{f}(\pi)^{(j,i)}\right|^2 < +\infty.$$

We end this section with the following immediate observation.

Remark 3.1.5. If $m \in \mathcal{D}(G)$ is such that

$$\mathrm{supp}(m) \cap [0,+\infty) \;\subset\; [0,\lambda_1),$$

where λ_1 is the smallest non-zero eigenvalue of $\mathcal{L}$, then

$$m(\mathcal{L}) \;=\; m(0)\, E_0.$$

3.1.4 Sub-Laplacian on $SU(2)$

In this section we study a sub-Laplacian on the compact Lie group $SU(2)$ (see Section 2.8), using the theory developed so far in Section 3.1.

Recall that, viewing the elements of $\mathfrak{su}(2)$ as left-invariant vector fields, we know that the set $\{X_1, X_2\}$ forms a Hörmander system on $SU(2)$, where X_1, X_2, X_3 denote the basis elements of $\mathfrak{su}(2)$ given by (2.8.8) (see Section 2.8.2). We consider the following sub-Laplacian on $SU(2)$:

$$\mathcal{L} = -(X_1^2 + X_2^2).$$

Recall that we use the same notation for the differential operator $\mathcal{L}$ and its self-adjoint extension to $L^2(SU(2))$.

Fix $n \in \mathbb{N}_0$. By a direct computation, using Proposition 2.8.2, we obtain

$$\begin{aligned}\pi_n(\mathcal{L}) &= -\left(\pi_n(X_1)^2 + \pi_n(X_2)^2\right) \\ &= -\left[(z(-n+z\partial_z)+\partial_z)^2 + (i(z(-n+z\partial_z)-\partial_z))^2\right]. \qquad (3.1.11)\end{aligned}$$

Observe that

$$\begin{aligned}&(z(-n+z\partial_z)+\partial_z)^2 \\ &\quad = z^2(-n+z\partial_z)^2 + z(-n+z\partial_z)\partial_z + \partial_z\left\{z(-n+z\partial_z)\right\} + \partial_z^2, \qquad (3.1.12)\end{aligned}$$

and similarly,

$$\begin{aligned}&(i\left(z(-n+z\partial_z)-\partial_z\right))^2 \\ &\quad = -z^2(-n+z\partial_z)^2 + z(-n+z\partial_z)\partial_z + \partial_z\left\{z(-n+z\partial_z)\right\} - \partial_z^2. \qquad (3.1.13)\end{aligned}$$

Combining (3.1.12) and (3.1.13) with (3.1.11), we obtain

$$\begin{aligned}\pi_n(\mathcal{L}) &= -2z(-n+z\partial_z)\partial_z - 2\partial_z\left\{z(-n+z\partial_z)\right\} \\ &= -2\left(-zn\partial_z + z^2\partial_z^2\right) - 2\left(\partial_z\{z\}(-n+z\partial_z) - nz\partial_z + (z\partial_z)^2\right) \\ &= 2n + 4nz\partial_z - 2z^2\partial_z^2 - 2z\partial_z - 2(z\partial_z)^2. \qquad (3.1.14)\end{aligned}$$

But,

$$(z\partial_z)^2 \;=\; z\partial_z + z^2\partial_z^2,$$

and hence (3.1.14) becomes

$$\pi_n(\mathcal{L}) \;=\; 2n + 4nz\partial_z - 4(z\partial_z)^2. \tag{3.1.15}$$

For $n \in \mathbb{N}_0$, recall that (see Section 2.8.1) an orthonormal basis of $\mathscr{P}_n$ is given by

$$\left\{\varphi_j^{(n)}(z) = \binom{n}{j}^{1/2} z^j \,:\, 0 \le j \le n\right\}.$$

If $0 \le j \le n$, we have

$$\begin{aligned}\pi_n(\mathcal{L})\varphi_j^{(n)}(z) \;&=\; 2\binom{n}{j}^{1/2}\left[n + 2nz\partial_z - 2(z\partial_z)^2\right] z^j \\ &=\; 2\left[n(1+2j) - 2j^2\right]\varphi_j^{(n)}(z).\end{aligned}$$

So, the eigenvalue of the operator $\pi_n(\mathcal{L})$ associated to the eigenfunction $\varphi_j^{(n)}$ is given by

$$\lambda_j^{(n)} \;:=\; 2\left[n(1+2j) - 2j^2\right].$$

In particular, the basis (2.8.4) of $\mathscr{P}_n$ forms a complete set of eigenfunctions of $\pi_n(\mathcal{L})$. Furthermore,

$$\pi_n(\mathcal{L})^{(i,j)} \;=\; \left\langle \pi_n(\mathcal{L})\varphi_j^{(n)}, \varphi_i^{(n)}\right\rangle_{\mathscr{P}_n} \;=\; \begin{cases}\lambda_j^{(n)} & \text{if } i = j \\ 0 & \text{otherwise}\end{cases}. \tag{3.1.16}$$

Therefore, the spectrum of the operator $\mathcal{L}$ is given by

$$\mathrm{Spec}(\mathcal{L}) \;=\; \left\{2n(1+2j) - 4j^2 \,:\, 0 \le j \le n,\; n \in \mathbb{N}_0\right\}.$$

Let us now write down some examples of eigenvalues $\lambda_j^{(n)}$ corresponding to the eigenfunctions $\pi_n^{(i,j)}$ of $\mathcal{L}$. The values of j are displayed along the top horizontal row, whilst the values of n are displayed on the left-most vertical column. We consider the values of n ranging from 0 to 5 and $j \le n$.

$n\backslash j$	0	1	2	3	4	5
0	0					
1	2	2				
2	4	8	4			
3	6	14	14	6		
4	8	20	24	20	8	
5	10	26	34	34	26	10

The table above shows that there exists $n \in \mathbb{N}_0$, and $j, j' \in \{0, 1, 2, \ldots, n\}$ such that

$$\lambda_j^{(n)} \neq \lambda_{j'}^{(n)}.$$

In particular, this means that there exists $n \in \mathbb{N}_0$ such that

$$\pi_n(\mathcal{L}) \neq \lambda\, I_{n+1},$$

for any $\lambda > 0$, where I_{n+1} denotes the $(n+1) \times (n+1)$ identity matrix. This contrasts with the case of the Laplace-Beltrami operator $L = -(X_1^2 + X_2^2 + X_3^2)$. Recall the classical result

$$\pi_n(L) = (n^2 + 2n) I_{n+1}, \quad \forall\, n \in \mathbb{N}_0.$$

This can also be checked directly with a computation, using Proposition 2.8.2. Moreover, we also notice that, for example we have

$$\lambda_1^{(2)} = \lambda_0^{(4)} = \lambda_4^{(4)} = 8.$$

This means that

$$\mathcal{L}\pi_2(x)^{(i,1)} = 8\, \pi_2(x)^{(i,1)}, \quad \mathcal{L}\pi_4(x)^{(i',0)} = 8\, \pi_4(x)^{(i',0)}, \quad \mathcal{L}\pi_4(x)^{(i',4)} = 8\, \pi_4(x)^{(i',4)},$$

whenever $0 \leq i \leq 1$ and $0 \leq i' \leq 4$. This illustrates what we explained earlier in Remark 3.1.3 (2); the eigenvalue $\lambda = 8$ has geometric multiplicity at least $12 > 1$ and its associated eigenspace, which is given by

$$\{f \in L^2(SU(2)) : \mathcal{L}f = 8f\},$$

is not a subspace of either M_2 or M_4 (see (2.8.6)).

On the other hand, in the case of the Laplace-Beltrami operator on $SU(2)$, if $\lambda = n^2 + 2n$, for some $n \in \mathbb{N}_0$, is given, then the eigenspace associated to λ is exactly M_n. Therefore, there is a one-to-one correspondence between the eigenvalues $\lambda_n = n^2 + 2n$ of L and the spaces M_n.

Now, by (3.1.6), we have

$$\mathcal{L}f = \sum_{n=0}^{\infty}(n+1)\sum_{i,j=0}^{n}\big(2n(1+2j) - 4j^2\big)\,\widehat{f}(\pi_n)^{(j,i)}\,\pi_n^{(i,j)}, \tag{3.1.17}$$

for any $f \in \mathrm{Dom}(\mathcal{L})$, where the domain $\mathrm{Dom}(\mathcal{L})$ is the space consisting of functions $f \in L^2(SU(2))$, such that

$$\sum_{n=0}^{\infty}(n+1)\sum_{i,j=0}^{n}\big|2n(1+2j) - 4j^2\big|^2\,\big|\widehat{f}(\pi_n)^{(j,i)}\big|^2 < +\infty.$$

To put it another way, $\mathrm{Dom}(\mathcal{L})$ is given by

$$\left\{f \in L^2(SU(2)) : \sum_{n=0}^{\infty}(n+1)\sum_{i,j=0}^{n}\big|2n(1+2j) - 4j^2\big|^2\,\big|\widehat{f}(\pi_n)^{(j,i)}\big|^2 < +\infty\right\}.$$

More generally, by (3.1.9), if m is a Borel function on $\mathbb{R}$, we have

$$m(\mathcal{L})f = \sum_{n=0}^{\infty}(n+1)\sum_{i,j=0}^{n} m\big(2n(1+2j) - 4j^2\big)\,\widehat{f}(\pi_n)^{(j,i)}\,\pi_n^{(i,j)}, \tag{3.1.18}$$

and the domain of $m(\mathcal{L})$ consists of functions $f \in L^2(SU(2))$ such that

$$\sum_{n=0}^{\infty}(n+1)\sum_{i,j=0}^{n}\big|m(2n(1+2j) - 4j^2)\big|^2\,\big|\widehat{f}(\pi_n)^{(j,i)}\big|^2 < +\infty.$$

3.1.5 Heat semigroup

In this section we aim to introduce the heat semigroup associated to a sub-Laplacian on a compact Lie group, and the corresponding heat kernels. Results in the setting of Lie groups can be found, for example, in Hunt [29], Varopoulos [54] or Saloff-Coste [45]. See also Folland [19] for results in the setting of stratified nilpotent Lie groups. Further results for Lie groups of polynomial growth can be found in Alexopoulos [2] or Furioli et al [23]. More general results on manifolds are also known; for instance, see Strichartz [49], where the Laplace-Beltrami operator is considered, or Strichartz [50] for related results on the sub-Riemannian setting. Furthermore, some standard references for semigroups in functional analysis include Davies [11], and Reed and Simon [38] (see Sections VIII.3 and VIII.4 in the latter).

Using functional analysis, we may construct the strongly continuous semigroup of operators on $L^2(G)$ associated to the self-adjoint operator $\mathcal{L}$:

$$\left\{e^{-t\mathcal{L}}\right\}_{t\geq 0},$$

where it is understood that $e^{0\mathcal{L}}$ is the identity operator on G. This is known as the heat semigroup. Since for all $t \geq 0$ the mapping $\lambda \mapsto e^{-t\lambda}$ is bounded and continuous on $[0, +\infty)$, then by functional analysis, the operator $e^{-t\mathcal{L}}$ is bounded on $L^2(G)$, with bound

$$||e^{-t\mathcal{L}}||_{\mathscr{L}(L^2(G))} \leq \sup_{\lambda\geq 0} |e^{-t\lambda}| \leq 1. \tag{3.1.19}$$

Hence, $\{e^{-t\mathcal{L}}\}_{t\geq 0}$ is a contraction semigroup.

Since $\mathcal{L}$ is left-invariant, then for each $t > 0$, the operator $e^{-t\mathcal{L}}$ is also left-invariant. Additionally, since for each $t > 0$ the operator $e^{-t\mathcal{L}}$ is bounded on $L^2(G)$, then in particular, it is a continuous operator on $L^2(G)$. So, for each $t > 0$, the operator $e^{-t\mathcal{L}}$ satisfies the hypothesis of Corollary 2.5.9, and hence it admits a right-convolution kernel $p_t \in \mathcal{D}'(G)$:

$$e^{-t\mathcal{L}}f = f * p_t, \quad \forall\, t > 0,\, f \in L^2(G). \tag{3.1.20}$$

The kernel p_t corresponding to the operator $e^{-t\mathcal{L}}$ is known as the heat kernel associated with $\mathcal{L}$.

In this thesis we will use the following properties of the heat kernels, which are classical and well-known. One can find, for example, in Hunt [29], a proof of properties (i)-(iii), where the heat kernels p_t are considered as finite positive measures on G. The regularity of the heat kernels follows from the fact that they satisfy the heat equation as a distribution and a famous theorem by Hörmander in [28]. Further discussions on these properties in the more general case of Lie groups of polynomial growth can be found in Varopoulos [54]. In Folland [19], one can also find a proof of these properties in the case of stratified nilpotent Lie groups.

Proposition 3.1.6. *The heat kernels satisfy the following conditions:*

(i) For each $t > 0$, the heat kernel p_t is smooth on G. Furthermore, $p_t(z) \geq 0$

for all $z \in G$*, and it is integrable, with*

$$\int_G p_t(z)\,dz \;=\; 1.$$

(ii) If $t_1, t_2 > 0$*, then*

$$p_{t_1} * p_{t_2} \;=\; p_{t_1+t_2}.$$

(iii) For every $t > 0$ *and* $z \in G$*,* $p_t(z) \;=\; p_t(z^{-1})$*.*

(iv) For every $t > 0$*, there exists* $C > 0$ *such that*

$$p_t(z) \;\le\; C\,V(\sqrt{t})^{-1}\,e^{-\frac{|z|^2}{Ct}}, \qquad \text{for } z \in G,\ t > 0, \tag{3.1.21}$$

where, for each $r > 0$*, we let* $V(r)$ *denote the volume of the ball* $B_r(e_G)$*, with respect to the Carnot-Carathéodory metric on* G *(see Definition A.1.2).*

Additionally, one can also find in Varopoulos [55] (Chapter VIII) an estimate for the volume of a ball with respect to the Carnot-Carathéodory distance:

$$V(r) \;\approx\; \begin{cases} r^l, & \text{for } r \in (0,1), \\ 1, & \text{for } 1 \le r \le R, \end{cases}$$

where l denotes the local dimension of G (see Definition A.2.1) and R denotes the radius of G:

$$R \;=\; \sup_{z \in G} |z| \;<\; +\infty.$$

Moreover, whenever $r > R$, we have

$$V(r) \;=\; \int_{B_r(e_G)} \mathrm{d}x \;=\; \int_{B_R(e_G)} \mathrm{d}x \;\approx\; 1.$$

Hence,

$$V(r) \;\approx\; 1, \qquad \forall\, r \in [1, +\infty).$$

So, we have

$$V(r) \;\approx\; \begin{cases} r^l, & \text{for } r \in (0,1), \\ 1, & \text{for } 1 \le r < +\infty. \end{cases} \tag{3.1.22}$$

Proposition 3.1.6 can be generalised to a sub-Riemannian manifold which does not necessarily have a group structure. See Strichartz [50] and [49] for the properties of the heat kernels in this case.

Remark 3.1.7. Since G is compact and for each $t > 0$ we have $p_t \in \mathcal{C}^\infty(G)$, then $p_t \in L^2(G)$. This implies that, for each $t > 0$, the operator $e^{-t\mathcal{L}}$ is compact on $L^2(G)$ (see Theorem 2.3.2 in Bump [5]).

3.1.6 The operator $(I+\mathcal{L})^{-s/2}$ on a compact Lie group

For a given $s \in \mathbb{R}$ we let $\mathcal{B}_s$ denote the right-convolution kernel associated to the operator $(I+\mathcal{L})^{-s/2}$. That is, $\mathcal{B}_s$ satisfies

$$(I+\mathcal{L})^{-s/2} f \;=\; f * \mathcal{B}_s, \qquad \forall\, f \in \operatorname{Dom}\big((I+\mathcal{L})^{-s/2}\big). \tag{3.1.23}$$

The aim of this section is to show that, for a given $s \in \mathbb{R}$, the distribution $\mathcal{B}_s$, whenever it exists, is square integrable subject to a certain condition on s.

As we saw in Example 3.1.4, for $s \in \mathbb{R}$, the domain of the operator

$$(I+\mathcal{L})^{-s/2},$$

which we denote by $\operatorname{Dom}\big((I+\mathcal{L})^{-s/2}\big)$, is given by

$$\left\{ f \in L^2(G) \,:\, \sum_{\pi \in \widehat{G}} d_\pi \sum_{i,j=1}^{d_\pi} \big|1+\lambda_j^{(\pi)}\big|^{-s} \, \big|\widehat{f}(\pi)^{(j,i)}\big|^2 < +\infty \right\},$$

and the action of $(I+\mathcal{L})^{-s/2}$ on a function f contained in $\operatorname{Dom}\big((I+\mathcal{L})^{-s/2}\big)$ is given by

$$(I+\mathcal{L})^{-s/2} f \;=\; \sum_{\pi \in \widehat{G}} d_\pi \sum_{i,j=1}^{d_\pi} \big(1+\lambda_j^{(\pi)}\big)^{-s/2} \, \widehat{f}(\pi)^{(j,i)} \, \pi^{(i,j)}.$$

First observe that, if $s \geq 0$, then the mapping

$$\lambda \longmapsto (1+\lambda)^{-s/2}$$

is a bounded measurable function in $[0,+\infty)$. Hence, by spectral analysis, the operator $(I+\mathcal{L})^{-s/2}$ is bounded on $L^2(G)$. Since it is also left-invariant, then by Corollary 2.5.9, it admits a right-convolution kernel, which we shall denote by $\mathcal{B}_s$.

To begin with, we shall consider the case $G = SU(2)$. We have the following result.

Proposition 3.1.8. *Suppose* $G = SU(2)$ *and let*

$$\mathcal{L} \;=\; -(X_1^2 + X_2^2)$$

be the sub-Laplacian on $SU(2)$ *we considered in Section 3.1.4. If* $s > 2$*, then the right convolution kernel* $\mathcal{B}_s$ *associated to the operator* $(I + \mathcal{L})^{-s/2}$ *is square integrable.*

Proof. By definition, the kernel $\mathcal{B}_s$ satisfies

$$(I + \mathcal{L})^{-s/2} f \;=\; f * \mathcal{B}_s, \quad \forall\, f \,\in\, L^2(SU(2)).$$

Furthermore, by the properties of convolution, taking the Fourier transform yields

$$\mathcal{F}\big\{(I + \mathcal{L})^{-s/2} f\big\}(\pi_n) \;=\; \mathcal{F}\big\{f * \mathcal{B}_s\big\}(\pi_n) \;=\; \widehat{\mathcal{B}_s}(\pi_n)\widehat{f}(\pi_n), \quad \forall\, n \,\in\, \mathbb{N}_0.$$

Hence, for each $n \in \mathbb{N}_0$, we have

$$\mathcal{F}\big\{(I + \mathcal{L})^{-s/2}\big\}(\pi_n) \;=\; \pi_n(I + \mathcal{L})^{-s/2} \;=\; \widehat{\mathcal{B}_s}(\pi_n).$$

Moreover, for each $n \in \mathbb{N}_0$,

$$\mathcal{F}\big\{(I + \mathcal{L})^{-s/2}\big\}(\pi_n)^{(i,j)} \;=\; \big(1 + 2n(1 + 2j) - 4j^2\big)^{-s/2}, \quad \forall\, 0 \leq i, j \leq n.$$

Now, in this proof we shall show that

$$\begin{aligned} &\sum_{n=0}^{\infty} (n+1)\,||\pi_n(I + \mathcal{L})^{-s/2}||_{\mathrm{HS}}^2 \\ &\qquad = \sum_{n=0}^{\infty} (n+1) \sum_{j=0}^{n} \left|\big(1 + 2n(1 + 2j) - 4j^2\big)^{-s/2}\right|^2 \end{aligned} \tag{3.1.24}$$

is finite provided that $s > 2$. By Plancherel's Theorem (see Theorem 2.2.7), we have

$$||\mathcal{B}_s||_{L^2(SU(2))}^2 \;=\; \sum_{n=0}^{\infty} (n+1)\,||\pi_n(I + \mathcal{L})^{-s/2}||_{\mathrm{HS}}^2. \tag{3.1.25}$$

Hence, proving that (3.1.24) is finite for $s > 2$ yields the result.

First, we claim that

$$\sum_{n=0}^{\infty}(n+1)\sum_{j=0}^{n}\left|\left(1+2n(1+2j)-4j^2\right)^{-s/2}\right|^2 \leq \int_0^{\infty}\int_0^{x}(x+1)\,\left|\left(1+2x(1+2y)-4y^2\right)\right|^{-s}\,\mathrm{d}y\,\mathrm{d}x.$$

In order to prove this, we study the function

$$\begin{aligned}[0,x] &\longrightarrow \mathbb{R}\\ y &\longmapsto \left|\left(1+2x(1+2y)-4y^2\right)\right|^{-s} \;=\; \left|(2y-x)^2-(x+1)^2\right|^{-s},\end{aligned} \tag{3.1.26}$$

for a fixed $x > 0$ and $s > 0$. The expression $(2y-x)^2-(x+1)^2$ is equal to 0 if and only if $y = \frac{1}{2}(2x+1) = x+\frac{1}{2} \notin [0,x]$, and moreover, $(2y-x)^2 \geq (x+1)^2$ if and only if

$$2y - x \geq x+1 \qquad \text{or} \qquad 2y-x \leq -(x+1).$$

This holds if and only if

$$y \geq \frac{1}{2}(2x+1) \qquad \text{or} \qquad y \leq -\frac{1}{2}.$$

Hence, it follows that

$$\left|(2y-x)^2-(x+1)^2\right| \;=\; (x+1)^2-(2y-x)^2, \qquad \text{whenever} \qquad 0 \leq y \leq x.$$

Furthermore, for $y \in [0,x]$, its derivative is given by

$$\frac{d}{dy}\left\{\left|(2y-x)^2-(x+1)^2\right|^{-s}\right\} \;=\; 4s\,(2y-x)\,\left((x+1)^2-(2y-x)^2\right)^{-s-1},$$

by the chain rule. So,

$$\frac{d}{dy}\left\{\left|(2y-x)^2-(x+1)^2\right|^{-s}\right\} \;=\; 0 \iff y \;=\; \frac{x}{2},$$

provided that $y \in [0,x]$. Thus, the only turning point of this function in the interval $[0,x]$ is at $y = x/2$. Additionally, we have

$$\left((x+1)^2 - (2y-x)^2\right)^{-s} \longrightarrow +\infty \quad \text{as} \quad y \longrightarrow \frac{1}{2}(2x+1)^-.$$

Putting all of this information together yields that the function given by (3.1.26) is a convex function with a turning point at $y = x/2$. This implies that

$$\int_0^x \left|(2y-x)^2 - (x+1)^2\right|^{-s} \geq \sum_{j=1}^{\lfloor x \rfloor_-} \left|1 + 2x(1+2j) - 4j^2\right|^{-s},$$

where $\lfloor x \rfloor_-$ denotes the highest integer n such that $n < x$. Furthermore, it is clear that we also have

$$\sum_{n=0}^{\infty} (n+1) \left|\left(1 + 2n(1+2j) - 4j^2\right)\right|^{-s} \leq \int_0^{\infty} (x+1) \left|\left(1 + 2x(1+2y) - 4y^2\right)\right|^{-s} \,\mathrm{d}x,$$

for a fixed $y \geq 0$, whenever $s > 1$. So, we have shown that for $s > 1$, the norm given by (3.1.24) is bounded by

$$\int_0^{\infty} \left[\int_0^x (x+1) \left|\left(1 + 2x(1+2y) - 4y^2\right)\right|^{-s} \,\mathrm{d}y + 2(x+1)|1+2x|^{-s}\right] \mathrm{d}x$$
$$= \int_0^{\infty} \int_0^x (x+1) \left|\left(1 + 2x(1+2y) - 4y^2\right)\right|^{-s} \,\mathrm{d}y\,\mathrm{d}x + \int_0^{\infty} 2(x+1)|1+2x|^{-s}\,\mathrm{d}x.$$

Now, let

$$I_1 := \int_0^{\infty} \int_0^x (x+1) \left|\left(1 + 2x(1+2y) - 4y^2\right)\right|^{-s} \,\mathrm{d}y\,\mathrm{d}x,$$

with

$$I_{1,x} := \int_0^x (x+1) \left|\left(1 + 2x(1+2y) - 4y^2\right)\right|^{-s} \,\mathrm{d}y,$$

and

$$I_2 := \int_0^{\infty} 2(x+1)|1+2x|^{-s}\,\mathrm{d}x.$$

It is clear that $I_2 < +\infty$ if $s > 2$. Let us now consider the integral $I_{1,x}$. Applying the substitution $u = 1 + 2x(1+2y) - 4y^2$ for a fixed $x \geq 0$, we first compute that

$$\frac{du}{dy} \;=\; -4(2y-x) \;=\; -4\big[(x+1)^2 - u\big]^{1/2}.$$

Also using the fact that our function is symmetrical about the line $y = x/2$, we have

$$\begin{aligned}\int_0^x (x+1)\,\big|\big(1+2x(1+2y)-4y^2\big)\big|^{-s}\,\mathrm{d}y \\ = -\frac{1}{2}\int_{1+2x}^{(x+1)^2} (x+1)\,u^{-s}\,\big[(x+1)^2-u\big]^{-1/2}\,\mathrm{d}u.\end{aligned}$$

This is finite, provided that $s > 1/2$, and consequently, $I_1 < +\infty$ provided that $s > 1/2$. Hence, we deduce that (3.1.24) is finite, provided that $s > 2$. So, by (3.1.25), the result is proved. □

We can give an alternate proof of Proposition 3.1.8 which is valid for any compact Lie group G. In this proof we shall make use of the heat semigroup $\{e^{-t\mathcal{L}}\}_{t>0}$ and the results given in Section 3.1.5.

Proposition 3.1.9. *Suppose that G is a compact Lie group of local dimension l (see Definition A.2.1). Suppose further that $\{X_1, X_2, \ldots, X_k\}$ forms a Hörmander system of left-invariant vector fields on G, for some $k \in \mathbb{N}$, and let*

$$\mathcal{L} \;:=\; -(X_1^2 + X_2^2 + \cdots X_k^2)$$

denote its associated sub-Laplacian. If $s > l/2$, then the right-convolution kernel associated to the operator $(I+\mathcal{L})^{-s/2}$, $\mathcal{B}_s$, is square integrable.

Proof. Recall that the Γ function is defined as the convergent integral

$$\Gamma(s) = \int_0^\infty t^{s-1}\,e^{-t}\,\mathrm{d}t, \qquad \text{for } s > 0. \tag{3.1.27}$$

Thus, if $\lambda, s > 0$, we have

$$\Gamma(s/2) \;=\; \lambda^{s/2}\int_0^\infty t^{\frac{s}{2}-1}\,e^{-\lambda t}\,\mathrm{d}t.$$

So, on rearranging we obtain

$$\lambda^{-s/2} \;=\; \frac{1}{\Gamma(s/2)}\int_0^\infty t^{\frac{s}{2}-1}\,e^{-\lambda t}\,\mathrm{d}t.$$

Hence,

$$(1+\lambda)^{-s/2} \;=\; \frac{1}{\Gamma(s/2)}\int_0^\infty t^{\frac{s}{2}-1}\, e^{-t}\, e^{-\lambda t}\; \mathrm{d}t.$$

Integrating with respect to the spectral measure yields the equality

$$(I+\mathcal{L})^{-s/2} \;=\; \frac{1}{\Gamma(s/2)}\int_0^\infty t^{\frac{s}{2}-1}\, e^{-t}\, e^{-\mathcal{L}t}\; \mathrm{d}t.$$

Hence, for any $f \in \mathrm{Dom}((I+\mathcal{L})^{-s/2})$, we obtain

$$\begin{aligned}
(I+\mathcal{L})^{-s/2}f \;&=\; \frac{1}{\Gamma(s/2)}\int_0^\infty t^{\frac{s}{2}-1}\, e^{-t}\, e^{-\mathcal{L}t} f\; \mathrm{d}t \\
&=\; \frac{1}{\Gamma(s/2)}\int_0^\infty t^{\frac{s}{2}-1}\, e^{-t}\; (f * p_t)\; \mathrm{d}t \\
&=\; f * \left\{ \frac{1}{\Gamma(s/2)}\int_0^\infty t^{\frac{s}{2}-1}\, e^{-t}\, p_t\; \mathrm{d}t \right\}.
\end{aligned}$$

This shows that the kernel $\mathcal{B}_s$ is formally given by

$$\mathcal{B}_s \;=\; \frac{1}{\Gamma(s/2)}\int_0^\infty t^{\frac{s}{2}-1}\, e^{-t}\, p_t\; \mathrm{d}t.$$

By Fubini's Theorem, we have

$$\begin{aligned}
\int_0^\infty \left| t^{\frac{s}{2}-1}\, e^{-t}\, p_t \right| \mathrm{d}t \;&\le\; \frac{1}{\Gamma(s/2)}\int_G \int_0^\infty \left| t^{\frac{s}{2}-1} e^{-t} p_t(z) \right| \mathrm{d}t\, \mathrm{d}z \\
&\le\; \frac{1}{\Gamma(s/2)}\int_0^\infty t^{\frac{s}{2}-1} e^{-t} \int_G |p_t(z)|\, \mathrm{d}z\, \mathrm{d}t \\
&\le\; 1,
\end{aligned}$$

where the last equality is obtained by Proposition 3.1.6 and (3.1.27). This shows that $\mathcal{B}_s \in L^1(G)$, and $||\mathcal{B}_s||_{L^1(G)} \le 1$. Now, the L^2 norm of $\mathcal{B}_s$ is given by

$$\begin{aligned}
||\mathcal{B}_s||^2_{L^2(G)} \;&=\; \int_G |\mathcal{B}_s(z)|^2\, \mathrm{d}z \\
&=\; \frac{1}{|\Gamma(s/2)|^2}\int_G \left| \int_0^\infty t^{\frac{s}{2}-1}\, e^{-t}\, p_t(z)\, \mathrm{d}t \right|^2 \mathrm{d}z.
\end{aligned}$$

But, for each $z \in G$, we compute

$$\left|\int_0^\infty t^{\frac{s}{2}-1}\, e^{-t}\, p_t(z)\,\mathrm{d}t\right|^2 = \int_0^\infty t_1^{\frac{s}{2}-1}\, e^{-t_1}\, p_{t_1}(z)\,\mathrm{d}t_1 \int_0^\infty \overline{t_2^{\frac{s}{2}-1}\, e^{-t_2}\, p_{t_2}(z)}\,\mathrm{d}t_2$$
$$= \int_0^\infty \int_0^\infty t_1^{\frac{s}{2}-1}\, t_2^{\frac{s}{2}-1}\, e^{-(t_1+t_2)}\, p_{t_1}(z)p_{t_2}(z)\,\mathrm{d}t_1\,\mathrm{d}t_2.$$

Hence, by Fubini's Theorem, we have

$$||\mathcal{B}_s||^2_{L^2(G)} = \frac{1}{|\Gamma(s/2)|^2}\int_0^\infty \int_0^\infty t_1^{\frac{s}{2}-1}\, t_2^{\frac{s}{2}-1}\, e^{-(t_1+t_2)} \int_0^\infty p_{t_1}(z)p_{t_2}(z)\,\mathrm{d}z\,\mathrm{d}t_1\,\mathrm{d}t_2.$$

By the properties of the heat kernels (see Proposition 3.1.6), for each $t_1, t_2 > 0$ we obtain

$$\int_G p_{t_1}(z)p_{t_2}(z)\,\mathrm{d}z = \int_G p_{t_1}(z)p_{t_2}(z^{-1})\,\mathrm{d}z = p_{t_1} * p_{t_2}(e_G) = p_{t_1+t_2}(e_G).$$

Hence,

$$||\mathcal{B}_s||^2_{L^2(G)} = \int_G \frac{1}{|\Gamma(s/2)|^2}\int_0^\infty \int_0^\infty t_1^{\frac{s}{2}-1}\, t_2^{\frac{s}{2}-1}\, e^{-(t_1+t_2)}\, p_{t_1+t_2}(e_G)\,\mathrm{d}t_1\,\mathrm{d}t_2\,\mathrm{d}z.$$

Now, we do the substitutions $t = t_1 + t_2$ and $u = t_2/t$. We have

$$\int_0^\infty \int_0^\infty (t_1t_2)^{\frac{s}{2}-1}\, e^{-(t_1+t_2)}\, p_{t_1+t_2}(e_G)\,\mathrm{d}t_1\,\mathrm{d}t_2$$
$$= \int_{t_2=0}^\infty \int_{t=t_2}^\infty \big(t_2(t-t_2)\big)^{\frac{s}{2}-1} e^{-t}\, p_t(e_G)\,\mathrm{d}t\,\mathrm{d}t_2$$
$$= \int_{u=0}^1 \int_{t=0}^\infty t^{2\left(\frac{s}{2}-1\right)}(u-u^2)^{\frac{s}{2}-1}\, e^{-t}\, p_t(e_G)\, t\,\mathrm{d}t\,\mathrm{d}u.$$

Thus, we have

$$\int_0^\infty \int_0^\infty (t_1t_2)^{\frac{s}{2}-1}\, e^{-(t_1+t_2)}\, p_{t_1+t_2}(e_G)\,\mathrm{d}t_1\,\mathrm{d}t_2$$
$$= \int_{u=0}^1 (u-u^2)^{\frac{s}{2}-1}\,\mathrm{d}u \int_{t=0}^\infty t^{2\left(\frac{s}{2}-1\right)+1} e^{-t}\, p_t(e_G)\,\mathrm{d}t$$
$$= \int_{u=0}^1 (u-u^2)^{\frac{s}{2}-1}\,\mathrm{d}u \int_{t=0}^\infty t^{s-1} e^{-t}\, p_t(e_G)\,\mathrm{d}t.$$

Clearly,

$$\int_0^1 (u-u^2)^{\frac{s}{2}-1}\, \mathrm{d}u \;<\; +\infty,$$

provided that $\frac{s}{2}-1>-1$; that is, as long as $s>0$. Now, observe that

$$\begin{aligned}\int_0^\infty & t^{s-1}\, e^{-t}\, p_t(e_G)\, \mathrm{d}t \\ &= \int_0^1 t^{s-1}\, e^{-t}\, p_t(e_G)\, \mathrm{d}t \;+\; \int_1^\infty t^{s-1}\, e^{-t}\, p_t(e_G)\, \mathrm{d}t. \end{aligned} \tag{3.1.28}$$

Proposition 3.1.6 (iv) tells us that there exists $C'>0$ such that

$$p_t(e_G) \;\le\; C'\, V(\sqrt{t})^{-1} e^{-\frac{|e_G|^2}{C't}} \;=\; C'\, V(\sqrt{t})^{-1}, \quad \forall\, t>0.$$

Furthermore, by (3.1.22), we know that the quantity $V(\sqrt{t})$ is bounded and it satisfies:

$$V(\sqrt{t})^{-1} \;\approx\; \begin{cases} t^{-l/2}, & t \in (0,1) \\ 1, & 1 \le t < +\infty. \end{cases} \tag{3.1.29}$$

This implies that there exists $C_1>0$ such that

$$p_t(e_G) \;\le\; C_1\, t^{-l/2}, \quad \forall\, t \in (0,1), \tag{3.1.30}$$

and $C_2>0$ such that

$$p_t(e_G) \;\le\; C_2, \qquad \forall\, 1 \le t < \infty. \tag{3.1.31}$$

Combining (3.1.30) and (3.1.31) with (3.1.28), we deduce that there exists $C>0$ such that

$$\begin{aligned}\int_0^\infty & t^{s-1}\, e^{-t}\, p_t(e_G)\, \mathrm{d}t \\ &\le C\left(\int_0^1 t^{s-\frac{l}{2}-1}\, e^{-t}\, \mathrm{d}t \;+\; \int_1^\infty t^{s-1}\, e^{-t}\, \mathrm{d}t\right). \end{aligned}$$

For any $s>0$ we have

$$\int_1^\infty t^{s-1}\, e^{-t}\, \mathrm{d}t \;<\; +\infty.$$

Moreover, the integral

$$\int_0^1 t^{s-\frac{l}{2}-1}\, e^{-t}\, \mathrm{d}t \;<\; +\infty,$$

provided that $s - \frac{l}{2} - 1 > -1$; that is, if $s > l/2$. Hence, a sufficient condition to have $||\mathcal{B}_s||^2_{L^2(G)} < +\infty$ is that $s > l/2$. □

Remark 3.1.10. In the case of stratified nilpotent Lie groups, one can find the proof of the analogous result to Proposition 3.1.9 in Folland [19].

The following result is a consequence of Proposition 3.1.9.

Corollary 3.1.11. *If* $s \in \mathbb{R}$, *such that* $s > l/2$, *then,*

$$\sum_{\pi \in \widehat{G}} d_\pi \operatorname{Tr} \left| \pi(I+\mathcal{L})^{-s} \right| \;<\; +\infty. \tag{3.1.32}$$

Proof. For each $\pi \in \widehat{G}$, we have

$$\operatorname{Tr} \left| \pi(I+\mathcal{L})^{-s} \right| \;=\; \operatorname{Tr} \left| \pi(I+\mathcal{L})^{-\frac{s}{2}} \left(\pi(I+\mathcal{L})^{-\frac{s}{2}} \right)^* \right| \;=\; \left|\left| \pi(I+\mathcal{L})^{-\frac{1}{2}s} \right|\right|^2_{HS}.$$

So, by Plancherel's Theorem (see Theorem 2.2.7), we obtain

$$\sum_{\pi \in \widehat{G}} d_\pi \operatorname{Tr} \left| \pi(I+\mathcal{L})^{-s} \right| \;=\; \sum_{\pi \in \widehat{G}} d_\pi \left|\left| \pi(I+\mathcal{L})^{-\frac{1}{2}s} \right|\right|^2_{HS} \;=\; ||\mathcal{B}_s||^2_{L^2(G)},$$

where $||\cdot||_{HS}$ denotes the Hilbert-Schmidt norm (see (2.2.8)), and $\mathcal{B}_s$ denotes the right-convolution kernel associated to the operator $(I+\mathcal{L})^{-s}$. By Proposition 3.1.9, if $s > l/2$, then $||\mathcal{B}_s||_{L^2(G)} < +\infty$. Thus, the result is proved. □

3.1.7 Complex powers of $(I+\mathcal{L})$

For $\alpha \in \mathbb{C}$ the mapping

$$\lambda \longmapsto (1+\lambda)^\alpha$$

is continuous on $[0, +\infty)$. Hence, by the spectral theory (see (3.1.9)), the operator $(I+\mathcal{L})^\alpha$ is given by

$$(I+\mathcal{L})^\alpha f \;=\; \sum_{\pi \in \widehat{G}} d_\pi \sum_{i,j=1}^{d_\pi} \left(1+\lambda_j^{(\pi)}\right)^\alpha \widehat{f}(\pi)^{(j,i)}\, \pi^{(i,j)}, \tag{3.1.33}$$

for $f \in \mathrm{Dom}((I+\mathcal{L})^{\alpha})$, which is the space given by

$$\left\{ f \in L^2(G) \,:\, \sum_{\pi \in \widehat{G}} d_\pi \sum_{i,j=1}^{d_\pi} \left|1+\lambda_j^{(\pi)}\right|^{2\alpha} \left|\widehat{f}(\pi)^{(j,i)}\right|^2 < +\infty \right\}.$$

The objective in this section is to prove the following proposition.

Proposition 3.1.12. *Suppose $\alpha \in \mathbb{C}$, with $a \leq Re(\alpha) \leq b \leq 0$, for some $a, b \leq 0$. Then, the operator $(I+\mathcal{L})^{\alpha}$ extends to a bounded operator on $L^2(G)$ and satisfies the bound*

$$||(I+\mathcal{L})^{\alpha}||_{\mathscr{L}(L^2(G))} \leq C \left|\Gamma\big(1-iIm(\alpha)\big)\right|^{-1},$$

for some $C > 0$ depending only on a and b.

In order to prove this, we shall follow the same strategy as in Folland [19], we shall require the following result, a proof of which can be found in Stein [46] (see Chapter IV, Section 6).

Proposition 3.1.13. *Suppose $m : (0,+\infty) \to \mathbb{R}$ is a function of the form*

$$m(\lambda) = \lambda \int_0^{\infty} e^{-\lambda t} M(t)\, dt, \quad \lambda \in (0,+\infty), \tag{3.1.34}$$

where M is a bounded function on $(0,+\infty)$. Then, the operator $m(\mathcal{L})$, which is given by

$$m(\mathcal{L})f = \sum_{\pi \in \widehat{G}} d_\pi \sum_{i,j=1}^{d_\pi} m\big(\lambda_j^{(\pi)}\big)\, \widehat{f}(\pi)^{(j,i)}\, \pi^{(i,j)},$$

is bounded on $L^2(G)$. Moreover, there exists a constant $C > 0$ independent of M such that

$$||m(\mathcal{L})||_{\mathscr{L}(L^2(G))} \leq C \sup_{t>0} |M(t)|.$$

Remark 3.1.14. Suppose a function m satisfies (3.1.34). Then, m is said to be of Laplace transform type. Moreover, observe that the function given by

$$\lambda \longmapsto \frac{1}{\lambda} m(\lambda)$$

is the Laplace transform of M.

Proof of Lemma 3.1.12. First observe that

$$\begin{aligned} ||(I+\mathcal{L})^{\alpha}||_{\mathscr{L}(L^2(G))} &= \big|\big|(I+\mathcal{L})^{\mathrm{Re}(\alpha)}(I+\mathcal{L})^{i\mathrm{Im}(\alpha)}\big|\big|_{\mathscr{L}(L^2(G))} \\ &\leq \big|\big|(I+\mathcal{L})^{\mathrm{Re}(\alpha)}\big|\big|_{\mathscr{L}(L^2(G))}\big|\big|(I+\mathcal{L})^{i\mathrm{Im}(\alpha)}\big|\big|_{\mathscr{L}(L^2(G))}. \end{aligned}$$

Thus, in order to prove our result, it suffices find a suitable bound for the L^2 operator norms of $(I+\mathcal{L})^{\mathrm{Re}(\alpha)}$ and $(I+\mathcal{L})^{i\mathrm{Im}(\alpha)}$.

To achieve the former, first observe that, since $\mathrm{Re}(\alpha)\leq 0$, then the mapping

$$\lambda \longmapsto (1+\lambda)^{\mathrm{Re}(\alpha)}$$

is bounded by 1 on $(0,+\infty)$. Moreover, by functional analysis,

$$\big|\big|(I+\mathcal{L})^{\mathrm{Re}(\alpha)}\big|\big|_{\mathscr{L}(L^2(G))} \leq \sup_{\lambda>0}(1+\lambda)^{\mathrm{Re}(\alpha)} \leq 1.$$

Hence, the operator $(I+\mathcal{L})^{\mathrm{Re}(\alpha)}$ is bounded on $L^2(G)$. We now aim to find a bound for the operator norm of $(I+\mathcal{L})^{i\mathrm{Im}(\alpha)}$. Observe that, since $\mathrm{Re}(i\mathrm{Im}(\alpha))=0$, then by functional analysis, we have

$$\big|\big|(I+\mathcal{L})^{i\mathrm{Im}(\alpha)}\big|\big|_{\mathscr{L}(L^2(G))} \leq \sup_{\lambda>0}|1+\lambda|^{i\mathrm{Im}(\alpha)} = 1. \tag{3.1.35}$$

Hence, the operator $(I+\mathcal{L})^{i\mathrm{Im}(\alpha)}$ is bounded on $L^2(G)$. However, we are interested in the dependence in α of the bound, so (3.1.35) does not provide a suitable bound for us. In order to investigate the dependence on α, first recall that, as we saw in the proof of Proposition 3.1.9, for any $s>0$, we have

$$(1+\lambda)^{-s} = \frac{1}{\Gamma(s)}\int_0^{\infty} t^{s-1}\,e^{-(1+\lambda)t}\,\mathrm{d}t.$$

A similar identity holds for any $s\in\mathbb{C}$, with $\mathrm{Re}(s)>0$. Since $\mathrm{Re}(1-i\mathrm{Im}(\alpha))=1>0$, then we have

$$(1+\lambda)^{i\mathrm{Im}(\alpha)-1} = \frac{1}{\Gamma(1-i\mathrm{Im}(\alpha))}\int_0^{\infty} t^{-i\mathrm{Im}(\alpha)}\,e^{-(1+\lambda)t}\,\mathrm{d}t.$$

Hence,

$$(1+\lambda)^{i\mathrm{Im}(\alpha)} = \frac{1+\lambda}{\Gamma(1-i\mathrm{Im}(\alpha))}\int_0^{\infty} t^{-i\mathrm{Im}(\alpha)}e^{-(1+\lambda)t}\,\mathrm{d}t. \tag{3.1.36}$$

Now, we define the function

$$m(\lambda) := (1+\lambda)^{i\mathrm{Im}(\alpha)}, \quad \lambda>0.$$

The expression given by (3.1.36) can be formally rewritten as

$$
\begin{aligned}
(1+\lambda)^{i\mathrm{Im}(\alpha)} &= \frac{\lambda}{\Gamma(1-i\mathrm{Im}(\alpha))}\left(\frac{\lambda+1}{\lambda}\right)\int_0^\infty e^{-(1+\lambda)t}\,t^{-i\mathrm{Im}(\alpha)}\,\mathrm{d}t \\
&= \frac{\lambda}{\Gamma(1-i\mathrm{Im}(\alpha))}\int_0^\infty e^{-(1+\lambda)t}\,t^{-i\mathrm{Im}(\alpha)}\,\mathrm{d}t \\
&\qquad + \frac{\lambda}{\Gamma(1-i\mathrm{Im}(\alpha))}\int_0^\infty \frac{1}{\lambda}\,e^{-\lambda t}\,e^{-t}\,t^{-i\mathrm{Im}(\alpha)}\,\mathrm{d}t.
\end{aligned}
$$

Now, observe that

$$
\frac{1}{\lambda}\,e^{-\lambda t} = \int_t^\infty e^{-\lambda u}\,\mathrm{d}u, \quad \forall\, t>0.
$$

So, using Fubini's theorem and changing the dummy variables as necessary, we obtain

$$
\int_0^\infty \frac{1}{\lambda}\,e^{-\lambda t}\,e^{-t}\,t^{-i\mathrm{Im}(\alpha)}\,\mathrm{d}t = \int_0^\infty e^{-\lambda t}\int_0^t e^{-u}\,u^{-i\mathrm{Im}(\alpha)}\,\mathrm{d}u\,\mathrm{d}t.
$$

Hence,

$$
(1+\lambda)^{i\mathrm{Im}(\alpha)} = \frac{\lambda}{\Gamma(1-i\mathrm{Im}(\alpha))}\int_0^\infty e^{-\lambda t}\left[e^{-t}\,t^{-i\mathrm{Im}(\alpha)} + \int_0^t e^{-u}\,u^{-i\mathrm{Im}(\alpha)}\,\mathrm{d}u\right]\mathrm{d}t.
$$

Now, consider the function given by

$$
M(t) = \frac{1}{\Gamma(1-i\mathrm{Im}(\alpha))}\left[e^{-t}\,t^{-i\mathrm{Im}(\alpha)} + \int_0^t e^{-u}\,u^{-i\mathrm{Im}(\alpha)}\,\mathrm{d}u\right], \qquad t>0.
$$

Observe that

$$
\sup_{t>0}\left|e^{-t}\,t^{-i\mathrm{Im}(\alpha)}\right| \le 1.
$$

Moreover, for any $t>0$ we have

$$
\int_0^t \left|e^{-u}\,u^{-i\mathrm{Im}(\alpha)}\right|\,\mathrm{d}u \le \int_0^\infty e^{-u}\,\mathrm{d}u = 1.
$$

Hence, we have shown that there exists a constant $C_0>0$, independent of α, such that

$$\sup_{t>0} |M(t)| \le C_0 |\Gamma(1 - i\mathrm{Im}(\alpha))|^{-1}. \tag{3.1.37}$$

This shows that M is bounded on $(0, +\infty)$. Hence, the function m satisfies the hypothesis of Proposition 3.1.13. Thus, the operator $m(\mathcal{L}) = (I + \mathcal{L})^{i\mathrm{Im}(\alpha)}$ is bounded on $L^2(G)$, and there exists $C' > 0$, independent of α, such that

$$||(I + \mathcal{L})^{i\mathrm{Im}(\alpha)}||_{\mathscr{L}(L^2(G))} \le C' \sup_{t>0} |M(t)|.$$

Therefore, by (3.1.37), we have shown that there exists $C_1 > 0$, independent of α, such that

$$\left|\left|(I + \mathcal{L})^{i\mathrm{Im}(\alpha)}\right|\right|_{\mathscr{L}(L^2(G))} \le C_1 |\Gamma(1 - i\mathrm{Im}(\alpha))|^{-1}. \tag{3.1.38}$$

Thus, combining this result with (3.1.7), yields that there exists $C > 0$, independent of α, such that

$$||(I + \mathcal{L})^{\alpha}||_{\mathscr{L}(L^2(G))} \le C |\Gamma(1 - i\mathrm{Im}(\alpha))|^{-1},$$

which proves the result. □

3.2 Sobolev spaces

In this section we shall introduce the Sobolev spaces $L^2_s(G)$ for the compact Lie group G, for $s \in \mathbb{R}$, which will be defined in terms of the sub-Laplacian $\mathcal{L}$. The properties of these spaces are well known in greater generality, see for example Furioli et. al [23], Coulhon et. al [10], and Dungey et. al [13] for Sobolev spaces on Lie groups of polynomial growth. See also, for instance, Folland [19] for Sobolev spaces on nilpotent Lie groups.

Recall that the spectrum of $\mathcal{L}$ is given by

$$\mathrm{Spec}(\mathcal{L}) = \left\{\lambda_j^{(\pi)} : \pi \in \widehat{G},\, 1 \le j \le d_\pi\right\},$$

where for each $\pi \in \widehat{G}$ and every $1 \le j \le d_\pi$, $\lambda_j^{(\pi)}$ is the non-negative real number satisfying

$$\mathcal{L}\pi(x)^{(i,j)} = \lambda_j^{(\pi)} \pi(x)^{(i,j)}, \quad \forall\, x \in G,\, 1 \le i \le d_\pi.$$

3.2.1 Definitions of Sobolev spaces and properties

As we saw in Example 3.1.4, for $s \in \mathbb{R}$, the domain of the operator $(I+\mathcal{L})^{s/2}$, which we denote by $\operatorname{Dom}\left((I+\mathcal{L})^{s/2}\right)$, is given by

$$\left\{ f \in L^2(G) : \sum_{\pi \in \widehat{G}} d_\pi \sum_{i,j=1}^{d_\pi} \left|1+\lambda_j^{(\pi)}\right|^s \left|\widehat{f}(\pi)^{(j,i)}\right|^2 < +\infty \right\},$$

and the action of $(I+\mathcal{L})^{s/2}$ on a function f contained in $\operatorname{Dom}\left((I+\mathcal{L})^{s/2}\right)$ is given by

$$(I+\mathcal{L})^{s/2} f = \sum_{\pi \in \widehat{G}} d_\pi \sum_{i,j=1}^{d_\pi} \left(1+\lambda_j^{(\pi)}\right)^{s/2} \widehat{f}(\pi)^{(j,i)}\, \pi^{(i,j)}. \tag{3.2.1}$$

To define our Sobolev spaces, first we need to check the following brief result.

Proposition 3.2.1. *Suppose G is a compact Lie group. For any $s \in \mathbb{R}$, we have $\mathcal{D}(G) \subset \operatorname{Dom}\left((I+\mathcal{L})^{s/2}\right)$.*

Proof. First suppose that $s \leq 0$. In this case, the mapping

$$\lambda \longmapsto (1+\lambda)^{s/2},$$

is a bounded measurable function in $[0,+\infty)$. By the spectral decomposition of the operator $(I+\mathcal{L})^{s/2}$ (see (3.2.1)), this implies that $(I+\mathcal{L})^{s/2}$ is bounded on $L^2(G)$ and hence $\operatorname{Dom}\left((I+\mathcal{L})^{s/2}\right) = L^2(G)$. Thus, in particular, we have $\mathcal{D}(G) \subset \operatorname{Dom}\left((I+\mathcal{L})^{s/2}\right)$.

On the other hand, suppose that $s > 0$. Then let $t \in 2\mathbb{N}$ be such that $t > s$. Then, for all $\lambda \geq 0$ we have

$$(1+\lambda)^{s/2} = (1+\lambda)^{s'/2}(I+\lambda)^{t/2},$$

where $s' = s-t < 0$. Now, consider the operator $(I+\mathcal{L})^{s'/2}(I+\mathcal{L})^{t/2}$ given by

$$(I+\mathcal{L})^{s'/2}(I+\mathcal{L})^{t/2} f = (I+\mathcal{L})^{s'/2}\left((I+\mathcal{L})^{t/2} f\right),$$

for f in the domain of the operator $(I+\mathcal{L})^{s'/2}(I+\mathcal{L})^{t/2}$, which is the space given by

$$\left\{ f \in \operatorname{Dom}\left((I+\mathcal{L})^{t/2}\right) : (I+\mathcal{L})^{t/2} f \in \operatorname{Dom}\left((I+\mathcal{L})^{s'/2}\right) \right\}.$$

Since t is an even integer, then $(I+\mathcal{L})^{t/2}$ is a differential operator, and hence

$\mathcal{D}(G) \subset \text{Dom}\left((I+\mathcal{L})^{t/2}\right)$. Moreover, $\mathcal{D}(G) \subset \text{Dom}\left((I+\mathcal{L})^{s'/2}\right)$, since $s' < 0$. So, by functional analysis,

$$\mathcal{D}(G) \subset \text{Dom}\left((I+\mathcal{L})^{s/2}\right).$$

□

Definition 3.2.2 (Sobolev spaces)**.** Let $s \in \mathbb{R}$. We define $L^2_s(G) \subset \mathcal{D}'(G)$ to be the closure of the space $\mathcal{D}(G)$ for the norm

$$||f||_{L^2_s(G)} := ||(I+\mathcal{L})^{s/2} f||_{L^2(G)}, \qquad f \in L^2_s(G). \tag{3.2.2}$$

We could similarly define the Sobolev spaces $L^p_s(G)$, for any $p > 1$, however only the case of $p = 2$ will be relevant for the results discussed in this thesis, so it will be the only case we consider.

Recall that $\mathcal{I}(k)$ is the set given by

$$\mathcal{I}(k) := \bigsqcup_{b \in \mathbb{N}} \{1, 2, \ldots, k\}^b.$$

For $\beta = (i_1, i_2, \ldots, i_b) \in \mathcal{I}(k)$, we write

$$X_\beta = X_{i_1} X_{i_2} \ldots X_{i_b}.$$

The following result consists of well-known properties of the Sobolev spaces $L^2_s(G)$.

Theorem 3.2.3. *(a) For any $s \in \mathbb{R}$, $L^2_s(G)$ is a Hilbert space for the norm $||\cdot||_{L^2_s(G)}$ given by* (3.2.2)*. Moreover, $\mathcal{D}(G)$ is a dense subspace of $L^2_s(G)$.*

(b) For $s = 0$, $L^2_s(G)$ coincides with $L^2(G)$, and furthermore,

$$||f||_{L^2_0(G)} = ||f||_{L^2(G)}, \quad \forall\, f \in L^2_0(G).$$

(c) If $s > 0$, then

$$L^2_s(G) = Dom\left((I+\mathcal{L})^{s/2}\right) = Dom\left(\mathcal{L}^{s/2}\right) \subset L^2(G).$$

Furthermore, the norm $|||\cdot|||_s$ given by

$$|||f|||_s := ||f||_{L^2(G)} + ||\mathcal{L}^{s/2} f||_{L^2(G)}, \quad f \in Dom\left(\mathcal{L}^{s/2}\right),$$

is equivalent to $||\cdot||_{L^2_s(G)}$.

(d) If $s<0$*, then*

$$L^2_s(G)\subset \mathcal{D}'(G).$$

(e) For $s_1,s_2\in\mathbb{R}$*, if* $s_1\le s_2$ *then we have continuous inclusion*

$$L^2_{s_2}(G)\subset L^2_{s_1}(G).$$

(f) For each $s\in\mathbb{R}$*, the dual space of* $L^2_s(G)$ *is isomorphic to* $L^2_{-s}(G)$ *via the extension of the distributional duality.*

(g) Let $\beta\in\mathcal{I}(k)$ *and* $s\in\mathbb{R}$*. Then, the mapping*

$$X_\beta : L^2_s(G)\to L^2_{s-|\beta|}(G)$$

is continuous. Equivalently, there exists $C>0$ *such that*

$$\big|\big|X_\beta f\big|\big|_{L^2_{s-|\beta|}(G)}\le C\,||f||_{L^2_s(G)},\quad \forall\, f\in L^2_s(G).$$

(h) If $s\in\mathbb{N}$*, then* $f\in L^2_s(G)$ *if and only if* $X_\beta f\in L^2(G)$ *for every* $\beta\in\mathcal{I}(k)$ *with* $|\beta|\le s$*. Furthermore, in this case the Hilbert space* $L^2_s(G)$ *admits the following norm, which is equivalent to* $||\cdot||_{L^2_s(G)}$*:*

$$f\longmapsto\left(\sum_{|\beta|\le s}||X_\beta f||^2_{L^2(G)}\right)^{1/2}. \tag{3.2.3}$$

Sketch proof of Theorem 3.2.3. The proofs for (a) and (b) are trivial; they follow from Definition 3.2.2.

Statements (c), (d), (e) and (f) follow classically from functional analysis.

Results (g) and (h) are deeper. For a proof of these results, see for example Coulhon et al. [10] (see Section 3), Dungey et al. [13] (see Proposition II.6.2) and Robinson [41] (see Theorem 5.14 in Chapter IV). □

Remark 3.2.4. Observe that if we had considered the self-adjoint extension of the right-invariant sub-Laplacian

$$\widetilde{\mathcal{L}}=-(\widetilde{X}_1^2+\widetilde{X}_2^2+\cdots+\widetilde{X}_k^2),$$

we would have obtained an analogous result to Theorem 3.2.3 in terms of Sobolev spaces based on powers of $I + \widetilde{\mathcal{L}}$. This is true for any Lie group of polynomial growth.

The following property of Sobolev spaces shall also prove to be important.

Lemma 3.2.5. *Let $\psi \in \mathcal{D}(G)$ and, for $s \in \mathbb{R}$, define the operator*

$$\begin{cases} T_s^{(\psi)} : L_s^2(G) & \longrightarrow & L_s^2(G) \\ \phi & \longmapsto & \psi\phi \end{cases}.$$

Then, the operator $T_s^{(\psi)}$ is continuous on $L_s^2(G)$. Moreover, we have

$$||T_s^{(\psi)}||_{\mathscr{L}(L_s^2(G))} \leq C \max_{\substack{\beta \in \mathcal{I}(k) \\ |\beta| \leq \lceil |s| \rceil}} ||X_\beta \psi||_{L^\infty(G)},$$

for some $C > 0$.

Proof. First, we suppose that $s \in \mathbb{N}_0$. In this case, by Theorem 3.2.3 (h), for any $\phi \in L_s^2(G)$ we have

$$||\psi\phi||^2_{L_s^2(G)} \lesssim \sum_{\substack{\beta \in \mathcal{I}(k) \\ |\beta| \leq s}} ||X_\beta(\psi\phi)||^2_{L^2(G)}.$$

Hence, by Leibniz's rule for vector fields, we obtain

$$\begin{aligned} ||\psi\phi||^2_{L_s^2(G)} &\lesssim \sum_{\substack{\beta_1, \beta_2 \in \mathcal{I}(k) \\ |\beta_1| + |\beta_2| \leq s}} ||(X_{\beta_1}\psi)(X_{\beta_2}\phi)||^2_{L^2(G)} \\ &\lesssim \max_{\substack{\beta_1 \in \mathcal{I}(k) \\ |\beta_1| \leq s}} \sup_{x_1 \in G} |X_{\beta_1, x_1}\psi(x_1)|^2 \sum_{\substack{\beta_2 \in \mathcal{I}(k) \\ |\beta_2| \leq s}} \int_G |X_{\beta_2, x}\phi(x)|^2 \, \mathrm{d}x. \end{aligned}$$

Then, we have shown that there exists $C > 0$, independent of ϕ, such that

$$||\psi\phi||^2_{L_s^2(G)} \leq C \max_{\substack{\beta \in \mathcal{I}(k) \\ |\beta| \leq s}} ||X_\beta\psi||_{L^\infty(G)} \, ||\phi||_{L_s^2(G)},$$

and so

$$T_s^{(\psi)} \in \mathscr{L}(L_s^2(G)). \tag{3.2.4}$$

By the Interpolation Theorem (see Theorem 3.3.1 below), it follows that (3.2.4) holds for any $s \geq 0$. Furthermore, from Theorem 3.2.3 (f) we know that, for each $s \in L^2_s(G)$, the dual space of $L^2_s(G)$ is isomorphic to $L^2_{-s}(G)$. Hence, the conclusion given by (3.2.4) may be extended to any $s \in \mathbb{R}$. □

3.3 Interpolation theorem for Sobolev spaces

In this section we aim to prove the following interpolation theorem for the Sobolev spaces on the compact Lie group G we defined in the previous section (Definition 3.2.2):

Theorem 3.3.1 (Interpolation Theorem). *Let $\alpha_0, \alpha_1, \beta_0, \beta_1$ be any real numbers. Moreover, suppose T is a bounded linear map:*

$$T : L^2_{\alpha_0}(G) \longrightarrow L^2_{\beta_0}(G),$$
$$T : L^2_{\alpha_1}(G) \longrightarrow L^2_{\beta_1}(G).$$

Then T extends uniquely to a bounded linear map

$$T : L^2_{\alpha_t}(G) \longrightarrow L^2_{\beta_t}(G), \qquad \forall\, 0 \le t \le 1,$$

where

$$(\alpha_t\,,\, \beta_t) = \big(\alpha_0 + t(\alpha_1 - \alpha_0)\,,\, \beta_0 + t(\beta_1 - \beta_0)\big), \qquad \forall\, 0 \le t \le 1.$$

Although we only prove this theorem for the Sobolev spaces $L^2_s(G)$, it is possible to prove it for any $L^p_s(G)$, with $p > 1$. The proof we provide here follows the strategy exhibited in Folland [19], which proves the result in the case of stratified nilpotent Lie groups.

Before we prove the Interpolation Theorem, let us consider the following consequence.

Lemma 3.3.2. *Let $\kappa \in \mathcal{D}'(G)$. Furthermore, suppose that $\{\kappa_\ell\}_{\ell \in \mathbb{N}} \subset \mathcal{D}'(G)$ be a sequence of distributions such that*

$$\kappa_\ell \longrightarrow \kappa \quad \text{as} \quad \ell \longrightarrow \infty, \quad \text{in} \quad \mathcal{D}'(G).$$

For any $\kappa' \in \mathcal{D}'(G)$, we let $T_{\kappa'}$ denote the operator

$$\begin{cases} T_{\kappa'} : \mathcal{D}(G) & \longrightarrow \quad \mathcal{D}'(G) \\ \qquad f & \longmapsto \quad f * \kappa' \end{cases}.$$

Then, for all $a,\, b \in \mathbb{R}$,

$$\liminf_{\ell\to\infty} \left|\left|T_{\kappa_\ell}\right|\right|_{\mathscr{L}(L^2_a(G)\to L^2_b(G))} \geq \left|\left|T_\kappa\right|\right|_{\mathscr{L}(L^2_a(G)\to L^2_b(G))}.$$

Proof. Let us first consider the case $a = b = 0$. By Fatou's Lemma, for any $f \in \mathcal{D}(G)$ we have

$$\begin{aligned}\liminf_{\ell\to\infty} \left|\left|T_{\kappa_\ell} f\right|\right|^2_{L^2(G)} &= \liminf_{\ell\to\infty} \int_G |f * \kappa_\ell(x)|^2 \,\mathrm{d}x \\ &\geq \int_G \liminf_{\ell\to\infty} |f * \kappa_\ell(x)|^2 \,\mathrm{d}x \\ &= \left|\left|T_\kappa f\right|\right|^2_{L^2(G)}.\end{aligned}$$

So, for any $f \in \mathcal{D}(G)$, with $||f||_{L^2(G)} = 1$, we have

$$\begin{aligned}\liminf_{\ell\to\infty} \left|\left|T_{\kappa_\ell}\right|\right|_{\mathscr{L}(L^2(G))} &\geq \liminf_{\ell\to\infty} \left|\left|T_{\kappa_\ell} f\right|\right|_{L^2(G)} \\ &\geq \left|\left|T_\kappa f\right|\right|_{L^2(G)}.\end{aligned}$$

Thus,

$$\liminf_{\ell\to\infty} \left|\left|T_{\kappa_\ell}\right|\right|_{\mathscr{L}(L^2(G))} \geq \left|\left|T_\kappa\right|\right|_{\mathscr{L}(L^2(G))}. \tag{3.3.1}$$

Observe that, for any $\kappa' \in \mathcal{D}(G)$ we have

$$\left|\left|T_{\kappa'}\right|\right|_{\mathscr{L}\left(L^2_a(G),\, L^2_b(G)\right)} = \left|\left|(I+\mathcal{L})^{\frac{b}{2}}\, T_{\kappa'}\, (I+\mathcal{L})^{-\frac{a}{2}}\right|\right|_{\mathscr{L}(L^2(G))}.$$

Let us now suppose that either $a \in -2\mathbb{N}_0$ or $a > 0$, and either $b \in 2\mathbb{N}_0$ or $b < 0$. For $a \in -2\mathbb{N}_0$ and $b \in 2\mathbb{N}_0$, $(I+\widetilde{\mathcal{L}})^{-\frac{a}{2}}$ and $(I+\mathcal{L})^{\frac{b}{2}}$ are differential operators. On the other hand, for $a > 0$ and $b < 0$, $(I+\mathcal{L})^{\frac{b}{2}}$ and $(I+\widetilde{\mathcal{L}})^{-\frac{a}{2}}$ are convolution operators. Hence, for our choice of a and b, the right convolution kernel associated to the operator

$$(I+\mathcal{L})^{\frac{b}{2}}\, T_{\kappa'}\, (I+\mathcal{L})^{-\frac{a}{2}}$$

is given by

$$(I+\mathcal{L})^{\frac{b}{2}}\, (I+\widetilde{\mathcal{L}})^{-\frac{a}{2}}\, \kappa'. \tag{3.3.2}$$

Therefore,

$$||T_{\kappa'}||_{\mathscr{L}\left(L^2_a(G),\,L^2_b(G)\right)} \;=\; \left|\left|T_{(I+\mathcal{L})^{\frac{b}{2}}\,(I+\widetilde{\mathcal{L}})^{-\frac{a}{2}}\,\kappa'}\right|\right|_{\mathscr{L}(L^2(G))}. \tag{3.3.3}$$

We then define the sequence of distributions

$$\widetilde{\kappa}_\ell \;:=\; (I+\mathcal{L})^{\frac{b}{2}}\,(I+\widetilde{\mathcal{L}})^{-\frac{a}{2}}\,\kappa_\ell, \qquad \ell \in \mathbb{N}.$$

We have

$$\widetilde{\kappa}_\ell \longrightarrow \widetilde{\kappa} \quad \text{as} \quad \ell \longrightarrow \infty, \quad \text{in} \quad \mathcal{D}'(G),$$

where

$$\widetilde{\kappa} \;:=\; (I+\mathcal{L})^{\frac{b}{2}}\,(I+\widetilde{\mathcal{L}})^{-\frac{a}{2}}\,\kappa.$$

Applying the case $a = b = 0$ (see (3.3.1)) to the sequence $\{\widetilde{\kappa}_\ell\}_{\ell\in\mathbb{N}}$ and $\widetilde{\kappa}$ we obtain

$$\liminf_{\ell\to\infty}\,||T_{\widetilde{\kappa}_\ell}||_{\mathscr{L}(L^2(G))} \;\geq\; ||T_{\widetilde{\kappa}}||_{\mathscr{L}(L^2(G))}.$$

By (3.3.3) we have shown that

$$\liminf_{\ell\to\infty}\,||T_{\kappa_\ell}||_{\mathscr{L}\left(L^2_a(G),\,L^2_b(G)\right)} \;\geq\; ||T_{\kappa}||_{\mathscr{L}\left(L^2_a(G),\,L^2_b(G)\right)},$$

for $a, b \in \mathbb{R}$ such that either $a \in -2\mathbb{N}_0$ or $a > 0$, and either $b \in 2\mathbb{N}_0$ or $b < 0$.

It remains to check the case $a, b \in \mathbb{R}$. Suppose that either $a \in -2\mathbb{N}_0$ or $a > 0$, and either $b \in 2\mathbb{N}_0$ or $b < 0$. Then, we have

$$\liminf_{\ell\to\infty}\,||T_{\kappa_\ell}||_{\mathscr{L}\left(L^2_{a_0}(G),\,L^2_{b_0}(G)\right)} \;\geq\; ||T_{\kappa}||_{\mathscr{L}\left(L^2_{a_0}(G),\,L^2_{b_0}(G)\right)},$$

and

$$\liminf_{\ell\to\infty}\,||T_{\kappa_\ell}||_{\mathscr{L}\left(L^2_{a_1}(G),\,L^2_{b_1}(G)\right)} \;\geq\; ||T_{\kappa}||_{\mathscr{L}\left(L^2_{a_1}(G),\,L^2_{b_1}(G)\right)}.$$

By the Interpolation Theorem (see Theorem 3.3.1), we then obtain

$$||T_{\kappa}||_{\mathscr{L}\left(L^2_{a_t}(G),\,L^2_{b_t}(G)\right)} \;\leq\; \liminf_{\ell\to\infty}\,||T_{\kappa_\ell}||_{\mathscr{L}\left(L^2_{a_t}(G),\,L^2_{b_t}(G)\right)}, \tag{3.3.4}$$

where

$$(a_t\,,\,b_t) \;=\; \big(a_0 + t(a_1-a_0)\,,\,b_0 + t(b_1-b_0)\big), \qquad \forall\, 0 \leq t \leq 1.$$

This shows that

$$\liminf_{\ell\to\infty} ||T_{\kappa_\ell}||_{\mathscr{L}\left(L^2_a(G),\, L^2_b(G)\right)} \geq ||T_\kappa||_{\mathscr{L}\left(L^2_a(G),\, L^2_b(G)\right)},$$

for all $a, b \in \mathbb{R}$, as required.

□

3.3.1 Tools for the proof

In this section we aim to introduce the tools necessary to prove Theorem 3.3.1.

One of the main tools we shall require in our proof will be the following result, which is a consequence of Proposition 3.1.12.

Lemma 3.3.3. *Suppose $\alpha \in \mathbb{C}$, with $a \leq \mathrm{Re}(\alpha) \leq b \leq 0$, for some $a, b \leq 0$. Then, the operator $(I + \mathcal{L})^\alpha$ extends to a bounded operator on $L^2_s(G)$, for all $s \in \mathbb{R}$, and satisfies the bound*

$$||(I + \mathcal{L})^\alpha||_{\mathscr{L}(L^2_s(G))} \leq C\left|\Gamma\big(1 - iIm(\alpha)\big)\right|^{-1},$$

for some $C > 0$ depending only on a, b and s.

Proof. By the commutativity of the operator $(I + \mathcal{L})$, for any $s \in \mathbb{R}$ we have

$$\begin{aligned} \left|\left|(I + \mathcal{L})^\alpha\right|\right|_{\mathscr{L}(L^2_s(G))} &= \left|\left|(I + \mathcal{L})^{s/2}\,(I + \mathcal{L})^\alpha\,(I + \mathcal{L})^{-s/2}\right|\right|_{\mathscr{L}(L^2(G))} \\ &= \left|\left|(I + \mathcal{L})^\alpha\right|\right|_{\mathscr{L}(L^2(G))}. \end{aligned}$$

Thus, the result follows immediately from Proposition 3.1.12. □

Let us now introduce approximate identities, which will play an important role in the proof.

Definition 3.3.4. Let $\mathcal{U}$ be a neighbourhood base at the identity I in G. For a neighbourhood $U \in \mathcal{U}$, let $\varphi_U \in \mathcal{D}(G)$ be such that $\mathrm{supp}(\varphi_U) \subset U$, satisfying the following properties:

(I) $\varphi_U \geq 0$,

(II) $\varphi_U(x^{-1}) = \varphi_U(x)$ for all $x \in G$ and,

(III)

$$\int_G \varphi_U(x)\,\mathrm{d}x = 1.$$

A sequence of functions $\{\varphi_U\}_{U\in\mathcal{U}}$ satisfying these properties is called an approximate identity.

In what follows, we shall consider approximate identities to be of the form $\{\varphi_\varepsilon\}_{\varepsilon>0}$. Here, for each $\varepsilon > 0$, we denote $\varphi_\varepsilon = \varphi_{B_\varepsilon(e_G)}$, where $B_\varepsilon(e_G)$ is the ball of radius ε, centred at e_G, with respect to the Carnot-Carathéodory metric. The following result, a proof of which can be found in Folland [21], is an important property of approximate identities.

Proposition 3.3.5. *Let $\{\varphi_\varepsilon\}_{\varepsilon>0}$ be an approximate identity. Then, for every $f \in L^2(G)$,*

$$||f * \varphi_\varepsilon - f||_{L^2(G)} \to 0 \quad \textit{as } \varepsilon \to 0,$$

and

$$||\varphi_\varepsilon * f - f||_{L^2(G)} \to 0 \quad \textit{as } \varepsilon \to 0.$$

The proof of the interpolation theorem in [19] relies on the dilation properties of a nilpotent Lie group. However, since G is compact, the usual mollifier would not be a well-defined function on G. Thus, we must introduce an approximate identity which can be used in the proof.

We let $\varphi_{\mathbb{R}^n} \in \mathcal{D}(\mathbb{R}^n)$ be such that

$$\int_{\mathbb{R}^n} \varphi_{\mathbb{R}^n}(x)\,\mathrm{d}x = 1,$$

with $\varphi_{\mathbb{R}^n}(x) = \varphi_{\mathbb{R}^n}(-x)$, for every $x \in \mathbb{R}^n$, and $\varphi_{\mathbb{R}^n} \geq 0$. Assume further that $\mathrm{supp}(\varphi_{\mathbb{R}^n}) \subset B_1^{\mathbb{R}^n}(0)$, where $B_1^{\mathbb{R}^n}(0)$ is the ball centred at 0 and with radius 1 in $\mathbb{R}^n$. For each $\varepsilon > 0$, define the function $\varphi_{\mathbb{R}^n,\varepsilon} : \mathbb{R}^n \to \mathbb{R}$ by

$$\varphi_{\mathbb{R}^n,\varepsilon}(x) = \varepsilon^{-n}\,\varphi_{\mathbb{R}^n}(x/\varepsilon), \qquad x \in \mathbb{R}^n.$$

Then the family of functions $\{\varphi_{\mathbb{R}^n,\varepsilon}\}_{\varepsilon>0}$ forms an approximate identity on $\mathbb{R}^n$, and furthermore,

$$\mathrm{supp}(\varphi_{\mathbb{R}^n,\varepsilon}) \subset B_\varepsilon(0) \subset \mathbb{R}^n.$$

Moreover, let us also consider the exponential map $\exp : \mathfrak{g} \to G$. We know that there exist a neighbourhood V of 0 in $\mathfrak{g}$ and a neighbourhood U of e_G in G such that the map $\exp : V \to U$ is a diffeomorphism (see Proposition 2.3.9 (v)). We can consider the inverse map $\ln : U \to V$. Furthermore, there exists a continuous function ψ which maps V diffeomorphically onto a neighbourhood W of the origin in $\mathbb{R}^n$. In particular, $\psi : V \to W$ is a chart, and we have

$$\exp \circ\, \psi^{-1}(B_\varepsilon^{\mathbb{R}^n}(0)) \,=\, U.$$

Therefore,

$$\operatorname{supp}(\psi \circ \ln) \,=\, U.$$

Then, for each $\varepsilon > 0$ we define the function $\widetilde{\varphi}_\varepsilon : U \to \mathbb{R}$ by

$$\widetilde{\varphi}_\varepsilon(x) \,=\, \begin{cases} \varphi_{\mathbb{R}^n,\varepsilon} \circ \psi \circ \ln(x) & \text{if } x \in U \\ 0 & \text{if } x \notin U. \end{cases}$$

Furthermore, for each $\varepsilon > 0$, we define

$$\varphi_\varepsilon(x) \,=\, \frac{\widetilde{\varphi}_\varepsilon(x)}{||\widetilde{\varphi}_\varepsilon||_{L^1(G)}}, \qquad x \in G. \tag{3.3.5}$$

Since $\varphi_{\mathbb{R}^n} \geq 0$, then it follows that $\widetilde{\varphi}_\varepsilon(x) \geq 0$, for all $x \in G$ and every $\varepsilon > 0$, and hence

$$\int_G \varphi_\varepsilon(x)\,\mathrm{d}x \,=\, \frac{1}{||\widetilde{\varphi}_\varepsilon||_{L^1(G)}} \int_G \widetilde{\varphi}_\varepsilon(x)\,\mathrm{d}x \,=\, 1, \quad \forall\, \varepsilon > 0.$$

This means that the sequence of functions $\{\varphi_\varepsilon\}_{\varepsilon>0}$ satisfies property (III) from Definition 3.3.4. Additionally, conditions (I) and (II) hold trivially, and therefore the sequence of functions $\{\varphi_\varepsilon\}_{\varepsilon>0}$ forms an approximate identity on G.

3.3.2 Proof of interpolation theorem

Proof of Theorem 3.3.1. Consider the space $B := L^1(G) \cap L^\infty(G)$. We let $\{\varphi_\varepsilon\}_{\varepsilon>0}$ be the approximate identity given by (3.3.5). For each $\varepsilon > 0$, we define the family of operators on B

$$\big\{T_z^\varepsilon \,:\, 0 \leq \operatorname{Re}(z) \leq 1\big\},$$

given by

$$T_z^\varepsilon f \,=\, (I + \mathcal{L})^{\widetilde{\beta}_z}\, T\, (I + \mathcal{L})^{-\widetilde{\alpha}_z}(f * \varphi_\varepsilon), \quad f \,\in\, B,$$

where we define

$$(\widetilde{\alpha}_z, \widetilde{\beta}_z) \,=\, \frac{1}{2}\big(\alpha_0 + z(\alpha_1 - \alpha_0)\,,\, \beta_0 + z(\beta_1 - \beta_0)\big).$$

By Lemma 3.3.3, T_z^ε is well-defined on B. We set

$$A(z) \,=\, \big|\Gamma\big(1 + i\,\mathrm{Im}(\widetilde{\alpha}_z)\big)\,\Gamma\big(1 - i\,\mathrm{Im}(\widetilde{\beta}_z)\big)\big|^{-1}.$$

Suppose that $\beta_1 \geq \beta_0$ without loss of generality. We now aim to show that for every $\varepsilon > 0$, $z \in \mathbb{C}$, with $\mathrm{Re}(z) \in (0,1)$, and any $f \in B$, there exists a constant $C > 0$, independent of f and z, such that

$$||T_z^\varepsilon f||_{L^2(G)} \,\leq\, C\,A(z)\,||f * \varphi_\varepsilon||_{L^2_{\alpha_1}(G)}. \tag{3.3.6}$$

For the rest of the proof, let us fix $\varepsilon > 0$. Since $\beta_1 \geq \beta_0$, then $\beta_0 \,\leq\, 2\mathrm{Re}(\widetilde{\beta}_z) \,\leq\, \beta_1$, and so

$$\frac{\beta_0}{2} - \frac{\beta_1}{2} \,\leq\, \mathrm{Re}(\widetilde{\beta}_z) - \frac{\beta_1}{2} \,\leq\, 0.$$

Therefore, for all $f \in B$, we have

$$\begin{aligned} &\big|\big|(I+\mathcal{L})^{\widetilde{\beta}_z}\,T\,(I+\mathcal{L})^{-\widetilde{\alpha}_z}(f * \varphi_\varepsilon)\big|\big|_{L^2(G)} \\ &\qquad = \big|\big|(I+\mathcal{L})^{\widetilde{\beta}_z}\,(I+\mathcal{L})^{-\beta_1/2}\,T\,(I+\mathcal{L})^{-\widetilde{\alpha}_z}(f * \varphi_\varepsilon)\big|\big|_{L^2_{\beta_1}(G)}. \end{aligned} \tag{3.3.7}$$

By Lemma 3.3.3, there exists $C' > 0$, depending only on β_0, β_1, such that

$$\begin{aligned} &\big|\big|(I+\mathcal{L})^{\widetilde{\beta}_z}\,(I+\mathcal{L})^{-\beta_1/2}\,T\,(I+\mathcal{L})^{-\widetilde{\alpha}_z}(f * \varphi_\varepsilon)\big|\big|_{L^2_{\beta_1}(G)} \\ &\qquad \leq\, C'\,\big|\Gamma\big(1 - i\,\mathrm{Im}\big(\widetilde{\beta}_z \,-\, \beta_1/2\big)\big)\big|^{-1}\;\big|\big|T\,(I+\mathcal{L})^{-\widetilde{\alpha}_z}\,(f * \varphi_\varepsilon)\big|\big|_{L^2_{\beta_1}(G)}. \end{aligned}$$

Hence, we have obtained

$$\begin{aligned} &\big|\big|(I+\mathcal{L})^{\widetilde{\beta}_z}\,T\,(I+\mathcal{L})^{-\widetilde{\alpha}_z}(f * \varphi_\varepsilon)\big|\big|_{L^2(G)} \\ &\qquad \leq C'\,\big|\Gamma\big(1 - i\,\mathrm{Im}(\widetilde{\beta}_z)\big)\big|^{-1}\,\big|\big|T\,(I+\mathcal{L})^{-\widetilde{\alpha}_z}(f * \varphi_\varepsilon)\big|\big|_{L^2_{\beta_1}(G)}. \end{aligned} \tag{3.3.8}$$

Now,

$$\big|\big|T\,(I+\mathcal{L})^{-\widetilde{\alpha}_z}(f*\varphi_\varepsilon)\big|\big|_{L^2_{\beta_1}(G)} \leq ||T||_{L^2_{\alpha_1}\to L^2_{\beta_1}}\,\big|\big|(I+\mathcal{L})^{-\widetilde{\alpha}_z}\,(f*\varphi_\varepsilon)\big|\big|_{L^2_{\alpha_1}(G)}. \quad (3.3.9)$$

and, by the hypothesis,

$$||T||_{L^2_{\alpha_1}\to L^2_{\beta_1}}$$

is a finite constant. Moreover, by Lemma 3.3.3, there exists a constant $C'' > 0$, depending only on α_0 and α_1, such that

$$\big|\big|(I+\mathcal{L})^{-\widetilde{\alpha}_z}\,(f*\varphi_\varepsilon)\big|\big|_{L^2_{\alpha_1}(G)} \leq C''\,|\Gamma\,(1+i\mathrm{Im}(\widetilde{\alpha}_z))\,|^{-1}\,||f*\varphi_\varepsilon||_{L^2_{\alpha_1}(G)}. \quad (3.3.10)$$

Hence, combining (3.3.8), (3.3.9) and (3.3.10) we have shown (3.3.6).

Now, for any $f \in B$ and $g \in L^2(G)$, the mapping

$$z \mapsto \int_G \big(T^\varepsilon_z\,f\big)(x)\,g(x)\,\mathrm{d}x$$

is analytic for $0 < \mathrm{Re}(z) < 1$ and continuous for $0 \leq \mathrm{Re}(z) \leq 1$. Using the Cauchy-Schwarz inequality, we then have

$$\left|\int_G \big(T^\varepsilon_z\,f\big)(x)\,g(x)\,\mathrm{d}x\right| \lesssim A(z)\,||f*\varphi_\varepsilon||_{L^2_{\alpha_1}(G)}\,||g||_{L^2(G)}.$$

Let us suppose that for any $s \in \mathbb{R}$ and all $f \in B$, there exist constants $C_0, C_1 > 0$, independent of f, s and ε such that

$$||T^\varepsilon_{is}\,f\,||_{L^2(G)} \leq C_0\,A(is)\,||f*\varphi_\varepsilon||_{L^2(G)} \leq C_0\,A(is)\,||f||_{L^2(G)}, \quad (3.3.11)$$

$$||T^\varepsilon_{1+is}\,f\,||_{L^2(G)} \leq C_1\,A(1+is)\,||f*\varphi_\varepsilon||_{L^2(G)} \leq C_1\,A(1+is)\,||f||_{L^2(G)}, \quad (3.3.12)$$

for some constants C_0, $C_1 > 0$, which do not depend on f, s or ε. Then the Riesz-Thorin Interpolation Theorem (see Theorem 4.1 in [48]) implies that, for all $f \in B$,

$$||T^\varepsilon_t f||_{L^2(G)} \leq C_t\,||f||_{L^2(G)}, \quad 0 \leq t \leq 1, \quad (3.3.13)$$

for some constant $C_t > 0$, which only depends on the function A, and the constants C_0, C_1 and t. We now demonstrate that (3.3.11) and (3.3.12) hold. To see that the inequality

$$||T^{\varepsilon}_{is}\, f\,||_{L^2(G)} \,\leq\, C_0\, A(is)\; ||f * \varphi_\varepsilon||_{L^2(G)} \tag{3.3.14}$$

holds, we first note that, if $\mathrm{Re}(z) = 0$, then

$$\mathrm{Re}\left(\widetilde{\beta}_z - \frac{1}{2}\beta_0\right) \,=\, 0.$$

So, by Lemma 3.3.3, we obtain

$$\begin{aligned}
\big|\big|T^{\varepsilon}_{is} f\big|\big|_{L^2(G)} \,&=\, \big|\big|(I+\mathcal{L})^{\widetilde{\beta}_{is}-\frac{1}{2}\beta_0}\, T\,(I+\mathcal{L})^{-\widetilde{\alpha}_{is}}(f*\varphi_\varepsilon)\big|\big|_{L^2_{\beta_0}(G)} \\
&\lesssim\, \big|\Gamma\big(1 - i\,\mathrm{Im}(\widetilde{\beta}_{is})\big)\big|^{-1}\, \big|\big|T\,(I+\mathcal{L})^{-\widetilde{\alpha}_{is}}(f*\varphi_\varepsilon)\big|\big|_{L^2_{\beta_0}(G)} \\
&\lesssim\, \big|\Gamma\big(1 - i\,\mathrm{Im}(\widetilde{\beta}_{is})\big)\big|^{-1}\, ||T||_{L^2_{\alpha_0\to\beta_0}}\, \big|\big|(I+\mathcal{L})^{-\widetilde{\alpha}_{is}}(f*\varphi_\varepsilon)\big|\big|_{L^2_{\alpha_0}(G)}.
\end{aligned}$$

Now, observe that, if $\mathrm{Re}(z) = 0$, then

$$\mathrm{Re}\left(-\widetilde{\alpha}_z + \frac{1}{2}\alpha_0\right) \,=\, 0.$$

So, by Lemma 3.3.3, we have

$$\begin{aligned}
\big|\big|(I+\mathcal{L})^{-\widetilde{\alpha}_{is}}(f*\varphi_\varepsilon)\big|\big|_{L^2_{\alpha_0}(G)} \,&=\, \big|\big|(I+\mathcal{L})^{\frac{1}{2}\alpha_0-\widetilde{\alpha}_{is}}(f*\varphi_\varepsilon)\big|\big|_{L^2(G)} \\
&\lesssim\, \big|\Gamma\big(1 + i\,\mathrm{Im}(\widetilde{\alpha}_{is})\big)\big|^{-1}\, ||f*\varphi_\varepsilon||_{L^2(G)}.
\end{aligned}$$

By the hypothesis, $||T||_{L^2_{\alpha_0}\to L^2_{\beta_0}} < +\infty$. Hence, we have shown that there exists $C' > 0$ such that

$$\big|\big|(I+\mathcal{L})^{\widetilde{\beta}_z}\, T\,(I+\mathcal{L})^{-\widetilde{\alpha}_z}(f*\varphi_\varepsilon)\big|\big|_{L^2(G)} \,\leq\, C'\, A(is)\, ||f*\varphi_\varepsilon||_{L^2(G)},$$

which is exactly (3.3.14). Finally, observe that, by the definition of an approximate identity (see Definition 3.3.4 (III)), we have $||\varphi_\varepsilon||_{L^1(G)} = 1$, and so by Young's convolution inequality, we obtain

$$||f*\varphi_\varepsilon||_{L^2(G)} \,\leq\, ||\varphi_\varepsilon||_{L^1(G)}\, ||f||_{L^2(G)} \,=\, ||f||_{L^2(G)}.$$

Thus, (3.3.11) is proved. Inequality (3.3.12) is similar; it follows by realising that if $\mathrm{Re}(z) = 1$, then we have

$$\mathrm{Re}\left(\widetilde{\beta}_z - \frac{1}{2}\beta_1\right) \,=\, \mathrm{Re}\left(-\widetilde{\alpha}_z + \frac{1}{2}\alpha_1\right) \,=\, 0.$$

Now, let us define the space of functions

$$\mathscr{V} = \left\{ f * \varphi_\varepsilon : f \in B, \, \varepsilon > 0, \, ||f||_{L^2(G)} \leq 2||f * \varphi_\varepsilon||_{L^2(G)} \right\}.$$

If $f \in B$, then $f * \varphi_\varepsilon \to f$ in $L^2(G)$. So, for $\varepsilon > 0$ sufficiently small, we have $f * \varphi_\varepsilon \in \mathscr{V}$. This means that the space $\mathscr{V}$ is dense in $L^2(G)$. Hence, (3.3.13) implies that if $g = f * \varphi_\varepsilon \in \mathscr{V}$, then for $0 \leq t \leq 1$,

$$\left|\left| (I + \mathcal{L})^{\beta_t} T (I + \mathcal{L})^{-\alpha_t} (f * \varphi_\varepsilon) \right|\right|_{L^2(G)} \leq C_t \, ||f||_{L^2(G)} \leq 2 \, C_t \, ||f * \varphi_\varepsilon||_{L^2(G)}.$$

Therefore, the operator $(I + \mathcal{L})^{\beta_t} T (I + \mathcal{L})^{-\alpha_t}$ extends uniquely to a bounded operator on $L^2(G)$. And hence it follows that T extends uniquely to a bounded operator from $L^2_{\alpha_t}(G)$ to $L^2_{\beta_t}(G)$. □

3.4 Sobolev embedding

In this section we prove a Sobolev inequality for the Sobolev spaces introduced in Definition 3.2.2.

Theorem 3.4.1 (Sobolev embedding). *If $s > l/2$, where l denotes the local dimension of G (see Definition A.2.1), then the following embedding holds*

$$L^2_s(G) \subset \big(\mathcal{C}(G) \cap L^\infty(G)\big).$$

Moreover, if $s > l/2$ and $f \in L^2_s(G)$, then f is continuous on G and there exists $C > 0$, independent of f, such that

$$||f||_{L^\infty(G)} \leq C \, ||f||_{L^2_s(G)}. \tag{3.4.1}$$

Proof. Fix $s > 0$. First we recall that, by Proposition 3.1.9, the right-convolution kernel $\mathcal{B}_s$ associated to the operator $(I + \mathcal{L})^{-s/2}$ satisfies

$$\mathcal{B}_s \in L^1(G) \cap L^2(G).$$

By Theorem 3.2.3 (c), we know that $\mathrm{Dom}((I+\mathcal{L})^{s/2}) = L^2_s(G)$, so for $f \in L^2_s(G)$, we can define

$$f_s := (I + \mathcal{L})^{s/2} f \in L^2(G).$$

Now, observe that

$$(I+\mathcal{L})^{-s/2} f_s \;=\; f \;=\; f_s * \mathcal{B}_s,$$

so Hölder's inequality then implies that

$$||f||_{L^\infty} \;\leq\; ||f_s||_{L^2(G)}\, ||\mathcal{B}_s||_{L^2(G)} \;=\; ||f||_{L^2_s(G)}\, ||\mathcal{B}_s||_{L^2(G)}.$$

Since $||\mathcal{B}_s||_{L^2(G)} < +\infty$, then (3.4.1) holds. It remains to show that f is a continuous function. For $x \in G$ we have

$$f(x) = \int_G f_s(y)\, \mathcal{B}_s(y^{-1}x)\, \mathrm{d}y = \int_G f_s(xz^{-1})\, \mathcal{B}_s(z)\, \mathrm{d}z.$$

If $x_1 \in G$ is also fixed, by the Cauchy-Schwarz inequality we have

$$\begin{aligned} |f(x) - f(x_1)| \;&\leq\; \int_G \left| \left[f_s(xz^{-1}) - f_s(x_1 z^{-1}) \right] \mathcal{B}_s(z) \right| \mathrm{d}z \\ &\leq\; \left(\int_G \left| \mathcal{B}_s(z) \right|^2 \mathrm{d}z \right)^{1/2} \left(\int_G \left| f_s(xz) - f_s(x_1 z) \right|^2 \mathrm{d}z \right)^{1/2}. \end{aligned}$$

Since $\mathcal{B}_s \in L^2(G)$, then there exists $C > 0$ such that

$$|f(x) - f(x_1)| \;\leq\; C \left|\left| f_s(x\cdot) - f_s(x_1\cdot) \right|\right|_{L^2(G)}. \tag{3.4.2}$$

Now,

$$f_s(xz) - f_s(x_1 z) = \pi_L(x^{-1}) f_s(z) - \pi_L(x_1^{-1}) f_s(z),$$

and hence

$$\begin{aligned} \left|\left| f_s(x\cdot) - f_s(x_1\cdot) \right|\right|^2_{L^2(G)} &= \int_G \left| \pi_L(x^{-1}) f_s(z) - \pi_L(x_1^{-1}) f_s(z) \right|^2 \mathrm{d}z \\ &\lesssim ||\pi_L(x^{-1}) - \pi_L(x_1^{-1})||_{\mathscr{L}(L^2(G))}\, ||f_s||_{L^2(G)}. \end{aligned}$$

Since the left regular representation π_L is continuous on $L^2(G)$ (see Example 2.2.2) and $f_s \in L^2(G)$, it follows that

$$\left|\left| f_s(x\cdot) - f_s(x_1\cdot) \right|\right|_{L^2(G)} \longrightarrow 0 \quad \text{as} \quad x_1 \longrightarrow x.$$

Thus, from (3.4.2) it follows that

$$|f(x) - f(x_1)| \longrightarrow 0 \quad \text{as} \quad x_1 \longrightarrow x,$$

which proves the result. $\square$

3.5 Taylor's theorem

The objective in this section is to prove a local version of Taylor's Theorem for a compact Lie group G. Suppose $\{Y_1, Y_2, \ldots, Y_n\}$ is a basis of the Lie algebra $\mathfrak{g}$ of G, and recall that we denote

$$Y^\alpha = Y_1^{\alpha_1} Y_2^{\alpha_2} \ldots Y_n^{\alpha_n},$$

for all multi-indices $\alpha \in \mathbb{N}_0^n$. Throughout, we let $|| \cdot ||_{\mathbb{R}^n}$ denote the usual Euclidean norm on $\mathbb{R}^n$.

Theorem 3.5.1. *Let $x \in G$ and suppose that f is a smooth function on G. Then, there exists a neighbourhood V of e_G in G such that for any $M \in \mathbb{N}$, the following Taylor expansion of f at x holds;*

$$f(xz) = \sum_{|\alpha|<M} \frac{1}{\alpha!} z_1^{\alpha_1} z_2^{\alpha_2} \ldots z_n^{\alpha_n} Y^\alpha f(x) + R_{x,M}^f(z), \qquad \forall z \in V, \qquad (3.5.1)$$

where the remainder $R_{x,M}^f$ satisfies

$$\left|R_{x,M}^f(z)\right| \leq C \, ||(z_1, z_2, \ldots, z_n)||_{\mathbb{R}^n}^M \max_{|\alpha| \leq M} \left|\left|Y^\alpha f\right|\right|_{L^\infty(G)}, \quad \forall z \in V. \qquad (3.5.2)$$

Proof. Let $x \in G$. First recall that if Y is a smooth vector field on G, then we denote by $\phi_*^{-1} Y$ the push-forward of Y by ϕ^{-1} (see Section 2.3.1 for a discussion on the push-forward of a vector field). The map $\phi_*^{-1} Y$ is a smooth vector field on $\mathbb{R}^n$ and is given by

$$(\phi_*^{-1} Y) u := Y(u \circ \phi^{-1}), \qquad \text{for } u \in \mathcal{C}^\infty(\mathbb{R}^n).$$

For each $j = 1, 2, \ldots, n$, we shall consider the smooth vector fields $\phi_*^{-1} Y_j$. By identifying the vector fields $\phi_*^{-1} Y_j$ with their corresponding tangent vectors of $\mathbb{R}^n$ at $\phi^{-1}(x)$, we see that the set

$$\{\phi_*^{-1} Y_j \,:\, j = 1, 2, \ldots, n\}$$

forms a complete orthogonal set of tangent vectors to $\mathbb{R}^n$ at the point $\phi^{-1}(x)$. We write:

$$\left.\frac{\partial}{\partial x_j}\right|_{\phi^{-1}(x)} = \phi_*^{-1} Y_j, \qquad \text{for } j = 1, 2, \ldots, n.$$

Let f be a smooth map on G and suppose N is a neighbourhood of 0 in $\mathbb{R}^n$ and V is a neighbourhood of e_G in G such that the mapping $\phi : N \to V$ given by

$$\phi((z_1, z_2, \ldots, z_n)) \,=\, e^{z_1 Y_1}\, e^{z_2 Y_2} \,\ldots\, e^{z_n Y_n}, \quad (z_1, z_2, \ldots, z_n) \,\in\, N, \tag{3.5.3}$$

is a diffeomorphism. Such mapping exists due to Proposition 2.3.9 (v). For a given $z \in V$, we shall let $(z_1, z_2, \ldots, z_n)$ denote the coordinates of z given by the coordinate chart (ϕ^{-1}, V) (in the sense that (3.5.3) is satisfied). We then define the smooth function u on $N \subset \mathbb{R}^n$ by

$$u((z_1, z_2, \ldots, z_n)) \,=\, f \circ \phi((z_1, z_2, \ldots, z_n)), \qquad (z_1, z_2, \ldots, z_n) \,\in\, N.$$

Thus, by Taylor's Theorem on $\mathbb{R}^n$, for every $M \in \mathbb{N}$, the Taylor expansion of u at $\phi^{-1}(x)$ is given by

$$\sum_{|\alpha|<M} \frac{1}{\alpha!}\, z_1^{\alpha_1}\, z_2^{\alpha_2} \,\ldots\, z_n^{\alpha_n}\, \partial_x^{\alpha}\big|_{\phi^{-1}(x)}\, u \,+\, R^u_{\phi^{-1}(x),M}((z_1, z_2, \ldots, z_n)),$$

for all $(z_1, z_2, \ldots, z_n) \in N$, where $R^u_{\phi^{-1}(x),M}((z_1, z_2, \ldots, z_n))$ denotes the Taylor remainder, and we write

$$\partial_x^{\alpha}\big|_{\phi^{-1}(x)} \,=\, \left(\left.\frac{\partial}{\partial x_1}\right|_{\phi^{-1}(x)}\right)^{\alpha_1} \left(\left.\frac{\partial}{\partial x_2}\right|_{\phi^{-1}(x)}\right)^{\alpha_2} \cdots \left(\left.\frac{\partial}{\partial x_n}\right|_{\phi^{-1}(x)}\right)^{\alpha_n},$$

for any multi-index $\alpha \in \mathbb{N}_0^n$. Moreover, the remainder satisfies the estimate

$$\left|R^u_{\phi^{-1}(x),M}(z_1, z_2, \ldots, z_n)\right| \,\leq\, C\, ||(z_1, z_2, \ldots, z_n)||^M_{\mathbb{R}^n} \max_{|\alpha|\leq M} \left|\left|\partial_x^{\alpha}\, u\right|\right|_{L^\infty(\mathbb{R}^n)}.$$

We now let $z = \phi((z_1, z_2, \ldots, z_n))$ and $R^f_{x,M} : V \to \mathbb{R}$ be the function given by

$$R^f_{x,M}(z) \;=\; R^u_{\phi^{-1}(x),N} \circ \phi^{-1}(z), \qquad \text{for } \; z \,\in\, V.$$

Observe that

$$\left.\frac{\partial}{\partial x_j}\right|_{\phi^{-1}(x)} u \;=\; Y_j(u \circ \phi^{-1})(x) \;=\; Y_j f(x), \qquad \forall\, j = 1, 2, \ldots, n.$$

Hence, we have

$$f(xz) \;=\; \sum_{|\alpha| \leq N} \frac{1}{\alpha!}\, z_1^{\alpha_1}\, z_2^{\alpha_2}\, \ldots\, z_n^{\alpha_n}\, Y^\alpha f(x) \;+\; R^f_{x,M}(z), \tag{3.5.4}$$

for any $z \,\in\, V$. Moreover, the remainder $R^f_{x,M}$ satisfies the estimate

$$\left|R^f_{x,M}(z)\right| \;\leq\; C\, ||(z_1, z_2, \ldots, z_n)||^M_{\mathbb{R}^n} \max_{|\alpha| \leq M} \left|\left|Y^\alpha f\right|\right|_{L^\infty(G)}, \quad \forall\, z \,\in\, V,$$

as claimed.

□

Remark 3.5.2. We now express some important observations from the proof of Theorem 3.5.1.

(a) It is clear that the neighbourhood V of e_G only needs to be small enough such that the mapping $\phi : N \to V$ given by (3.5.3) is a diffeomorphism, for some neighbourhood N of 0 in $\mathbb{R}^n$.

(b) We can also bound the remainder from Theorem 3.5.1 in terms of the Carnot-Carathéodory metric (see Definition A.1.2), which we denote by $d(\cdot, \cdot)$. We use the following notation

$$|z| \;:=\; d(e_G, z), \quad \forall\, z \,\in\, G.$$

By a result from the appendix (see Proposition 2.4.2), we know that there exists a neighbourhood V of e_G in G such that

$$|z|_E \;\leq\; C_1\, |z|, \quad \forall\, z \,\in\, V, \tag{3.5.5}$$

for some constant $C_1 > 0$, where $|\cdot|_E$ denotes the Euclidean norm induced by the chart ϕ^{-1} (see (3.5.3)). We can now choose V small enough and a neighbourhood N of 0 in $\mathbb{R}^n$ such that the mapping $\phi : N \to V$ given by (3.5.3) is a diffeomorphism. Let $f \in \mathcal{C}^\infty(G)$, $x \in G$ and $M \in \mathbb{N}_0$, and consider the difference

$$R^f_{x,M}(z) = f(xz) - \sum_{|\alpha|<M} \frac{1}{\alpha!} z_1^{\alpha_1} z_2^{\alpha_2} \dots z_n^{\alpha_n} Y^\alpha f(x), \qquad \forall\, z \in V. \quad (3.5.6)$$

Then, by Theorem 3.5.1 and part (a), there exists $C_2 > 0$ such that

$$\left|R^f_{x,M}(z)\right| \leq C_2 \,||(z_1, z_2, \dots, z_n)||^M_{\mathbb{R}^n} \max_{|\alpha|\leq M} \left|\left|Y^\alpha f\right|\right|_{L^\infty(G)}, \quad \forall\, z \in V.$$

Hence, by (3.5.5), there exists $C > 0$ such that

$$\left|R^f_{x,M}(z)\right| \leq C\,|z|^M \max_{|\alpha|\leq M} \left|\left|Y^\alpha f\right|\right|_{L^\infty(G)}, \quad \forall\, z \in V. \qquad (3.5.7)$$

It is important to note that, for $z \in V$, some of the terms of the sum

$$\sum_{|\alpha|<M} \frac{1}{\alpha!} z_1^{\alpha_1} z_2^{\alpha_2} \dots z_n^{\alpha_n} Y^\alpha f(x)$$

might also be bounded by $|z|^M$, as we have chosen a bigger bound. If this is the case, then (3.5.6) can not be considered as a Taylor remainder in the context of the Carnot-Carathéodory norm.

Remark 3.5.3.

3.6 Formal degree of a vector field

In this section we discuss the formal degree of a vector field, which appears, for example, in Nagel et al [36] (see the introduction therein). Throughout this thesis we are interested in the action of left-invariant vector fields in a Hörmander system on functions, and introducing this term gives us a way of quantifying the derivatives we take. The formal degree of a vector field functions as an analogous term to the 'order' of a vector field in the elliptic case.

Recall that, for $k \in \mathbb{N}$, $\mathcal{I}(k)$ denotes the set of multi-indices taking values in $\{1, 2, \dots, k\}$, of arbitrary length. That is, $\mathcal{I}(k)$ is the disjoint union

$$\mathcal{I}(k) := \bigsqcup_{a \in \mathbb{N}} \{1, 2, \ldots, k\}^a. \tag{3.6.1}$$

Suppose G is a compact Lie group of dimension n and let $\mathfrak{g}$ be the Lie algebra of G. Further suppose that, for some $k \in \mathbb{N}$, the set $\boldsymbol{X} = \{X_1, X_2, \ldots, X_k\}$ forms a Hörmander system of left-invariant vector fields on G. Let

$$\boldsymbol{Y} := \{Y_1, Y_2, \ldots, Y_n\}$$

be a basis of $\mathfrak{g}$.

It is well known that there exist a neighbourhood V of x in G and a neighbourhood N of 0 in $\mathbb{R}^n$ such that the mapping $\phi : N \to V$, which is given by

$$\phi((z_1, z_2, \ldots, z_n)) := e^{z_1 Y_1}\, e^{z_2 Y_2} \, \ldots \, e^{z_n Y_n}(x), \tag{3.6.2}$$

is a diffeomorphism.

By the definition of a Hörmander system, there exists a subset $J \subset \mathcal{I}(k)$ such that for each left-invariant vector field X of G, we can write

$$X = \sum_{I=(i_1, i_2, \ldots, i_a) \in J} c_I \left[X_{i_1}, \left[X_{i_2}, \ldots, \left[X_{i_{a-1}}, X_{i_a}\right] \ldots\right]\right], \tag{3.6.3}$$

for some constants $c_I \in \mathbb{R}$. Observe that for each $I = (i_1, i_2, \ldots, i_a) \in J$, $X_{i_1}, X_{i_2}, \ldots, X_{i_a}$ is some iteration of a subset of $\{X_1, X_2, \ldots, X_k\}$.

Now, for every non-negative integer a, let $\boldsymbol{X}^{(a)}$ denote the subset of $\mathfrak{g}$ consisting of all commutators of length a; that is,

$$\boldsymbol{X}^{(a)} := \big\{[X_{i_1}, [X_{i_2}, \ldots, [X_{i_{\alpha-1}}, X_{i_\alpha}] \ldots]] \, : \, |(i_1, i_2, \ldots, i_\alpha)| = a\big\}.$$

Definition 3.6.1. Suppose X is a left-invariant vector field on G. We say that X has formal degree a, and write $d(X) = a$, if $X \in \boldsymbol{X}^{(a)}$.

Observe that for each $j = 1, 2, \ldots, k$, X_j has formal degree 1.

Definition 3.6.2. For $\alpha \in \mathbb{N}_0^n$, we define the quantity

$$[\alpha]_{\boldsymbol{Y}} := \sum_{j=1}^{n} d(Y_j)\, \alpha_j, \tag{3.6.4}$$

where, for each $j = 1, 2, \ldots, n$, Y_j denotes the basis element of $\mathfrak{g}$.

Example 3.6.3. Suppose $\boldsymbol{Y} = \{Y_1, Y_2, \ldots, Y_n\}$ is the basis of $\mathfrak{g}$ constructed in Section 2.4.1. In this case, for each $j = 1, 2, \ldots, n$, we have $d(Y_j) = d_j$ (see (2.4.3)). So, we have

$$[\alpha]_{\boldsymbol{Y}} = \sum_{j=1}^{n} d_j\, \alpha_j, \qquad \forall\, \alpha \in \mathbb{N}_0^n. \tag{3.6.5}$$

Remark 3.6.4. Suppose $\boldsymbol{Y} = \{Y_1, Y_2, \ldots, Y_n\}$ is the basis of $\mathfrak{g}$ constructed in Section 2.4.1. Observe that, for any multi-index $\alpha \in \mathbb{N}_0^n$, the expression

$$Y_1^{\alpha_1}\, Y_2^{\alpha_2}\, \ldots\, Y_n^{\alpha_n}$$

can be expressed in the form

$$Y_1^{\alpha_1}\, Y_2^{\alpha_2}\, \ldots\, Y_n^{\alpha_n} = \sum_{j=1}^{[\alpha]_{\boldsymbol{Y}}} \sum_{i_j=1}^{k} c^{\alpha}_{i_1, i_2, \ldots, i_{[\alpha]_{\boldsymbol{Y}}}}\, X_{i_1}\, X_{i_2}\, \ldots\, X_{i_{[\alpha]_{\boldsymbol{Y}}}}, \tag{3.6.6}$$

for some structure constants $c^{\alpha}_{i_1, i_2, \ldots, i_{[\alpha]_{\boldsymbol{Y}}}}$. This can be simplified by the following notation;

$$Y_1^{\alpha_1}\, Y_2^{\alpha_2}\, \ldots\, Y_n^{\alpha_n} = \sum_{\substack{\beta \in \mathcal{I}(k) \\ |\beta| = [\alpha]_{\boldsymbol{Y}}}} c_\beta X_\beta, \tag{3.6.7}$$

for some constants $c_\beta \in \mathbb{R}$. This, in fact, holds for any basis of the Lie algebra $\mathfrak{g}$, not just the one we chose.

We now show that any differential operator involving the Y_j can be rewritten as a linear combination of differential operators in an order of our choice.

Lemma 3.6.5. *Suppose $\boldsymbol{Y} = \{Y_1, Y_2, \ldots, Y_n\}$ is the basis of $\mathfrak{g}$ constructed in Section 2.4.1, and let $N \in \mathbb{N}$. For $\beta = (i_1, i_2, \ldots, i_N) \in \mathcal{I}(n)$, we define the quantity*

$$[Y_\beta] := \sum_{\ell=1}^{N} d(Y_{i_\ell})\, \beta_{i_\ell}. \tag{3.6.8}$$

Then, we have

$$Y_\beta = Y_{\beta_1} Y_{\beta_2} \ldots Y_{\beta_N} = \sum_{\substack{\alpha \in \mathbb{N}_0^n \\ [\alpha]_{\boldsymbol{Y}} = [Y_\beta],\, |\alpha| \le |\beta|}} c_{\alpha,\beta}\, Y^\alpha, \tag{3.6.9}$$

for some constants $c_{\alpha,\beta} \in \mathbb{R}$, with the convention that $Y_0 = \mathrm{Id}$, the identity operator.

Proof. We prove this recursively on N. In the case $N = 1$ the result is immediate. Let us now check this in the case $N = 2$. If $(\beta_1, \beta_2) \in \mathcal{I}(n)$, with $\beta_1 \le \beta_2$, then the result follows readily. On the other hand, if $\beta_1 > \beta_2$, we have

$$Y_{\beta_1} Y_{\beta_2} = Y_{\beta_2} Y_{\beta_1} + [Y_{\beta_1}, Y_{\beta_2}].$$

By construction,

$$[Y_{\beta_1}, Y_{\beta_2}] = \sum_{\substack{\ell \in \mathbb{N}_0 \\ d(Y_\ell) \le d(Y_{\beta_1}) + d(Y_{\beta_2})}} c_\ell \, Y_\ell,$$

for some $c_\ell \in \mathbb{R}$. Hence, we have shown that

$$Y_{\beta_1} Y_{\beta_2} = \sum_{\substack{\alpha \in \mathbb{N}_0^n \\ [Y^\alpha] = [Y_{\beta_1}] + [Y_{\beta_2}],\, |\alpha| \le 2}} c_\alpha \, Y^\alpha,$$

which proves that (3.6.9) holds for $N = 2$.

Let us then assume that the statement of the Lemma holds for N, and consider the expression

$$Y_{\beta_0} \, Y_{\beta_1} \ldots Y_{\beta_N},$$

for $\beta = (\beta_1, \ldots, \beta_N) \in \mathcal{I}(n)$. By the assumption,

$$Y_{\beta_1} \ldots Y_{\beta_N} = \sum_{\substack{\alpha \in \mathbb{N}_0^n \\ [Y^\alpha] = [Y_\beta],\, |\alpha| \le N}} c_{\alpha,\beta} \, Y^\alpha.$$

Hence, we may assume that

$$Y_{\beta_1} \, Y_{\beta_2} \ldots Y_{\beta_N} = Y_1^{\alpha_1} \, Y_2^{\alpha_2} \ldots Y_n^{\alpha_n},$$

for some $\alpha \in \mathbb{N}_0^n$, such that $[Y^\alpha] = [Y_\beta]$ and $|\alpha| \le N$. Furthermore, suppose $j \in \{1, 2, \ldots, N\}$ such that it is the smallest index for which $\alpha_j \ne 0$. If $\beta_0 \le j$, then the result follows readily. On the other hand, suppose that $\beta_0 > j$. In this case we have

$$\begin{aligned} Y_{\beta_0} \, Y_j &= Y_{\beta_j} Y_{\beta_0} + [Y_{\beta_0}, Y_j] \\ &= Y_{\beta_j} Y_{\beta_0} + \sum_{\substack{\ell \in \mathbb{N}_0 \\ d(Y_\ell) \le d(Y_{\beta_0}) + d(Y_j)}} c_\ell \, Y_\ell. \end{aligned}$$

Hence, we have shown that

$$\begin{aligned} Y_{\beta_0} Y_{\beta_1} Y_{\beta_2} \dots Y_{\beta_N} &= Y_{\beta_0} Y_j^{\alpha_j} \dots Y_n^{\alpha_n} \\ &= Y_j Y_{\beta_0} Y_j^{\alpha_j - 1} \dots Y_n^{\alpha_n} + \sum_{\substack{\ell \in \mathbb{N}_0 \\ d(Y_\ell) \le d(Y_{\beta_0}) + d(Y_j)}} c_\ell Y_\ell Y_j^{\alpha_j - 1} \dots Y_n^{\alpha_n}. \end{aligned}$$

We now apply this process recursively, so that we obtain

$$Y_{\beta_0} Y_j^{\alpha_j} \dots Y_n^{\alpha_n} = Y_j^{\alpha_j} \dots Y_{\beta_0} Y_{j'}^{\alpha_{j'}} \dots Y_n^{\alpha_n} + \sum_{\substack{\gamma \in \mathcal{I}(n) \\ [Y_\gamma] = [Y^\alpha],\, |\gamma| \le N}} Y_\gamma,$$

where $j' \in \mathbb{N}$ is such that $j' \geq \beta_0$ and $j' - 1 < \beta_0$. By the recursion hypothesis, the result is then proved.

□

We have the following consequence.

Corollary 3.6.6. *Suppose $\boldsymbol{Y} = \{Y_1, Y_2, \dots, Y_n\}$ is the basis of $\mathfrak{g}$ constructed in Section 2.4.1. For any $\beta \in \mathcal{I}(k)$, we have*

$$X_\beta = \sum_{\substack{\alpha \in \mathbb{N}_0^n \\ [\alpha] = |\beta|}} c_{\alpha,\beta} Y^\alpha,$$

for some $c_{\alpha,\beta} \in \mathbb{R}$.

Proof. By the construction of $\boldsymbol{Y}$, there exists $n_1 \in \mathbb{N}$, with $n_1 \leq k$, such that $\{X_1, X_2, \dots, X_{n_1}\}$ is the largest subset of $\boldsymbol{X}$ consisting of linearly independent left-invariant vector fields of G, with

$$X_j = Y_j, \qquad \forall\, j = 1, 2, \dots, n_1.$$

Therefore,

$$X_\beta = \sum_{\substack{\beta' \in \mathcal{I}(n) \\ |\beta'| = |\beta|}} c_{\beta'} Y_{\beta'},$$

for some constants $c_{\beta'} \in \mathbb{R}$. By Lemma 3.6.5 the result is then proved. □

We end this section with the following result, which follows from Corollary 3.6.6 and applying Corollary 2.3.5 to $\{Y_j\}_{j=1}^n$ and $\{\widetilde{Y}_j\}_{j=1}^n$.

Proposition 3.6.7. *(I) Suppose T is a differential operator of the form*

$$T = \sum_{\substack{\alpha \in \mathcal{I}(k) \\ |\alpha| \leq d}} C_\alpha X_\alpha,$$

with $C_\alpha \in \mathbb{R}$, for some $d \in \mathbb{N}$. Then,

$$T = \sum_{\substack{\beta \in \mathcal{I}(k) \\ |\beta| \leq d}} \widetilde{c}_\beta \widetilde{X}_\beta,$$

for some $\widetilde{c}_\beta \in \mathcal{C}^\infty(G)$.

(II) Suppose $\widetilde{T}$ is a differential operator of the form

$$\widetilde{T} = \sum_{\substack{\alpha \in \mathcal{I}(k) \\ |\alpha| \leq d}} \widetilde{C}_\alpha \widetilde{X}_\alpha,$$

with $\widetilde{C}_\alpha \in \mathbb{R}$, for some $d \in \mathbb{N}$. Then,

$$\widetilde{T} = \sum_{\substack{\beta \in \mathcal{I}(k) \\ |\beta| \leq d}} c_\beta X_\beta,$$

for some $c_\beta \in \mathcal{C}^\infty(G)$.

3.7 Vanishing functions

Throughout this section we remain in the same setting as in Section 3.6, with $x = e_G$.

Definition 3.7.1. Let (X, d) be a metric space and consider $x_0 \in X$. Moreover suppose that V is a neighbourhood of x_0 in X. For $a \in \mathbb{N}_0$, we say that a function $q : X \to \mathbb{C}$ vanishes at x_0 up to order $a-1$ on V with respect to the metric d if there exists $C > 0$ such that

$$\forall\, x \in V \implies |q(x)| \leq C\, d(x, x_0)^a.$$

Notation 3.7.2. Let $a \in \mathbb{N}_0$ and V be a neighbourhood of e_G in G. Throughout the thesis we shall use the following convention. If $q \in \mathcal{D}(G)$ vanishes at e_G up to order $a-1$ on V with respect to the Carnot-Carathéodory metric, then we shall write: q CC-vanishes at e_G up to order $a-1$ on V, for short. Moreover, if $V = G$, then we shall usually omit any mention of G; that is, we write: q CC-vanishes at e_G up to order $a-1$.

The main objective of this section is to show the following result, and Section 3.7.1 shall be devoted to its proof.

Proposition 3.7.3. *Let $a \in \mathbb{N}$ be a positive integer and suppose that $q \in \mathcal{D}(G)$. Then, the following statements are equivalent:*

(i) For any $\beta \in \mathcal{I}(k)$, with $|\beta| \le a-1$, we have

$$X_\beta \, q(e_G) \,=\, 0. \tag{3.7.1}$$

(ii) The function q CC-vanishes at e_G up to order $a-1$.

Furthermore, if (i) and (ii) hold, we have

$$|q(z)| \,\le\, C\,|z|^a, \qquad \forall\, z \,\in\, G, \tag{3.7.2}$$

where the constant C can be chosen to be

$$C \,:=\, \frac{k}{a} \sup_{\substack{z \,\in\, G \\ i=1,2,\ldots,k}} |z|^{1-a}\,|X_i\, q(z)|.$$

Remark 3.7.4. In the Euclidean case, the analogous result to Proposition 3.7.3 holds by Taylor's Theorem. More precisely, it is a consequence of the estimate of the Taylor remainder and the uniqueness of the Taylor expansion. Namely, if p is a smooth function on an open set $O \subset \mathbb{R}^n$ containing 0, then the following statements are equivalent:

(i) For any $\beta \in \mathbb{N}_0^n$, with $|\beta| \le a-1$, we have

$$\frac{\partial^{\beta_1}}{\partial z_1^{\beta_1}} \frac{\partial^{\beta_2}}{\partial z_2^{\beta_2}} \cdots \frac{\partial^{\beta_n}}{\partial z_n^{\beta_n}}\, p(0) \,=\, 0.$$

(ii) For any

$$\beta \,=\, (i_1, i_2, \ldots, i_b) \,\in\, \mathcal{I}(n),$$

with $|\beta| \le a-1$, we have

$$\frac{\partial}{\partial z_{i_1}} \frac{\partial}{\partial z_{i_2}} \cdots \frac{\partial}{\partial z_{i_b}} p(0) = 0.$$

(iii) There exists $C > 0$ and a neighbourhood N of 0 in $O \subset \mathbb{R}^n$ such that

$$|p(z)| \leq C \, ||z||_{\mathbb{R}^n}^a, \qquad \forall \, z \in N.$$

Example 3.7.5. Suppose the set $\{q_1, q_2, \ldots, q_d\}$ is a family of functions such that, for every $j = 1, 2, \ldots, d$, the function q_j CC-vanishes at e_G up to order $a_j - 1$. Then, for every multi-index $\beta \in \mathbb{N}_0^d$, the function

$$\prod_{j=1}^{d} q^{\beta_j}$$

CC-vanishes at e_G up to order

$$\sum_{j=1}^{d} a_j \, \beta_j - 1.$$

Corollary 3.7.6. *Let $a \in \mathbb{N}$ and $q \in \mathcal{D}(G)$. If q is CC-vanishing at e_G up to order $a-1$. Then, for any $\beta \in \mathcal{I}(k)$, with $|\beta| < a$, the function $X_\beta \, q$ is CC-vanishing at e_G up to order $a - |\beta| - 1$. Furthermore, there exists a constant $C > 0$, depending on a, β and k, such that*

$$\sup_{z \in G} |z|^{-a+|\beta|} \, |X_\beta \, q(z)| \leq C \sup_{\substack{z \in G \\ |\beta'|=a}} |X_{\beta'} \, q(z)|.$$

Proof. Since q is CC-vanishing at e_G up to order $a-1$, then the function $X_\beta \, q$ is CC-vanishing at e_G up to order $a - |\beta| - 1$, by Proposition 3.7.3. Hence, applying (3.7.2) to $X_\beta \, q$ we obtain

$$\sup_{z \in G} |z|^{-a+|\beta|} \, |X_\beta \, q(z)| \leq \frac{k}{a - |\beta|} \sup_{\substack{z \in G \\ i_1=1,2,\ldots,k}} |z|^{1-a+|\beta|} \, |X_{i_1} X_\beta \, q(z)|.$$

Applying this argument to the function $X_{i_1} X_\beta \, q$, for each $i_1 = 1, 2, \ldots, k$, we now get

$$\sup_{z \in G} |z|^{1-a+|\beta|} \, |X_{i_1} X_\beta \, q(z)| \leq \frac{k}{a - |\beta| - 1} \sup_{\substack{z \in G \\ i_2=1,2,\ldots,k}} |z|^{2-a+|\beta|} \, |X_{i_2} X_{i_1} X_\beta \, q(z)|,$$

and thus,

$$\sup_{z\in G} |z|^{-a+|\beta|}\, |X_\beta\, q(z)| \\ \leq \frac{k^2}{(a-|\beta|)(a-|\beta|-1)} \sup_{\substack{z\in G\\ i_1,i_2=1,2,\ldots,k}} |z|^{2-a+|\beta|}\, |X_{i_2}X_{i_1}X_\beta\, q(z)|.$$

Applying this argument recursively $a-|\beta|$ shows that there exists $C>0$, depending on α, β and k, such that

$$\begin{aligned}\sup_{z\in G} |z|^{-a+|\beta|}\, |X_\beta\, q(z)| &\leq C \sup_{\substack{z\in G\\ \beta'\in\mathcal{I}(k),\,|\beta'|=a-|\beta|}} |z|^{|\beta'|-a+|\beta|}\, |X_{\beta'}X_\beta\, q(z)| \\ &= C \sup_{\substack{z\in G\\ \beta'\in\mathcal{I}(k),\,|\beta'|=a}} |X_{\beta'}\, q(z)|,\end{aligned}$$

as required.

□

The following Lemma studies the differentiability of a function of the form $\frac{f_1}{f_2}$, where f_1, f_2 are smooth functions.

Lemma 3.7.7. *Let $f_1,\, f_2 \in \mathcal{D}(G)$ and suppose that, for $M_1,\, M_2 \in \mathbb{N}$, there exist constants $C_1,\, C_2,\, C_2' > 0$ such that*

$$|f_1(z)| \leq C_1\, |z|^{M_1}, \qquad C_2'\, |z|^{M_2} \leq |f_2(z)| \leq C_2\, |z|^{M_2}, \qquad \forall\, z\in G. \quad (3.7.3)$$

If $M_2 < M_1$, then the following assertions hold:

(1) The function given by

$$\frac{f_1}{f_2} : z \longmapsto \frac{f_1(z)}{f_2(z)}, \qquad \forall\, z\in G,$$

extends to a continuous function on G. Moreover,

$$||f_1/f_2||_{L^\infty(G)} \leq \frac{C_1}{C_2'}\, R^{M_1-M_2}, \quad (3.7.4)$$

where R is the radius of the Lie group G:

$$R = \sup_{z \in G} |z|.$$

(2) For any $\beta \in \mathcal{I}(k)$, with $|\beta| < M_1 - M_2$, the function

$$X_\beta \left(\frac{f_1}{f_2}\right) : z \longmapsto X_{\beta,z}\left(\frac{f_1(z)}{f_2(z)}\right), \qquad \forall\, z \in G, \tag{3.7.5}$$

coincides with a continuous function on G. Moreover, there exists a constant $C > 0$, depending on β, f_2, $\boldsymbol{X}$, M_1 and M_2, such that

$$||X_\beta(f_1/f_2)||_{L^\infty(G)} \leq C \sup_{\substack{\beta' \in \mathcal{I}(k) \\ |\beta'| = M_1}} ||X_{\beta'} f_1||_{L^\infty(G)}. \tag{3.7.6}$$

The same result holds for the operator $\widetilde{X}_\beta$, for any $\beta \in \mathcal{I}(k)$.

Proof. For the proof of part (1), note that,

$$\left|\frac{f_1(z)}{f_2(z)}\right| \leq \frac{C_1}{C_2'}\,|z|^{M_1 - M_2}, \qquad \forall\, z \in G.$$

Since $M_2 < M_1$, then it follows that the function $\frac{f_1}{f_2}$ extends to a continuous function on G. Estimate (3.7.4) then follows.

We now show part (2). Let us first fix $\beta \in \mathcal{I}(k)$, with $|\beta| < M_1 - M_2$. Then, observe that, by Leibniz's rule for vector fields,

$$X_\beta\left(\frac{f_1}{f_2}\right) = \sum_{\substack{\beta_1,\beta_2 \in \mathcal{I}(k) \\ |\beta_1| + |\beta_2| = |\beta|}} c^\beta_{\beta_1,\beta_2} \left(X_{\beta_1} f_1\right)\left(X_{\beta_2}(1/f_2)\right),$$

for some constants $c^\beta_{\beta_1,\beta_2} \in \mathbb{R}$. Taking absolute values yields

$$\left| X_\beta \left(\frac{f_1}{f_2}\right)(z)\right| \leq C_\beta \sum_{\substack{\beta_1,\beta_2 \in \mathcal{I}(k) \\ |\beta_1| + |\beta_2| = |\beta|}} \left|X_{\beta_1} f_1(z)\right| \left|X_{\beta_2}(1/f_2(z))\right|, \tag{3.7.7}$$

for any $z \in G \backslash \{e_G\}$, for some constant $C_\beta > 0$. Now, we check that for any $\beta_2 \in \mathcal{I}(k)$, with $|\beta_2| < M_1 - M_2$, we have

$$X_{\beta_2}\left(\frac{1}{f_2}\right) = \sum_{\substack{\gamma \in \mathcal{I}(k)^{|\beta_2|} \\ |\gamma| = |\beta_2|}} c_{\beta_2,\gamma}\, \frac{1}{f_2} \prod_{\ell=1}^{|\beta_2|} \frac{X_{\gamma_\ell} f_2}{f_2}. \tag{3.7.8}$$

Indeed, the cases $|\beta_2| = 0$ and $|\beta_2| = 1$ are clear, since for any $i = 1, 2, \ldots, k$ we have

$$X_i \left(\frac{1}{f_2}\right) = -\frac{X_i f_2}{f_2^2}.$$

The general case can be proved recursively, by noticing that

$$\begin{aligned} X_i X_{\beta_2}\left(\frac{1}{f_2}\right) &= \sum_{\substack{\gamma \in \mathcal{I}(k)^{|\beta_2|} \\ |\gamma| = |\beta_2|}} c_{\beta_2,\gamma} \Bigg\{ -\left(\frac{X_i f_2}{f_2^2}\right) \prod_{\ell=1}^{|\beta_2|} \frac{X_{\gamma_\ell} f_2}{f_2} \\ &\qquad + \frac{1}{f_2} \prod_{\ell=1}^{|\beta_2|} \left[\frac{X_i X_{\gamma_\ell} f_2}{f_2} - \left(\frac{X_i f_2}{f_2}\right)\left(\frac{X_{\gamma_\ell} f_2}{f_2}\right)\right] \Bigg\} \\ &= \sum_{\substack{\gamma \in \mathcal{I}(k)^{|\beta_2|+1} \\ |\gamma| = |\beta_2|+1}} c_{i,\beta_2,\gamma} \frac{1}{f_2} \prod_{\ell=1}^{|\beta_2|+1} \frac{X_{\gamma_\ell} f_2}{f_2}, \end{aligned}$$

for some constants $c_{i,\beta_2,\gamma} \in \mathbb{R}$.

Now, by (3.7.8), for any $z \in G\backslash\{e_G\}$ and $\beta_2 \in \mathcal{I}(k)$, with $|\beta_2| \leq |\beta|$, we have

$$\begin{aligned} |X_{\beta_2}(1/f_2)(z)| &\lesssim_{\beta_2, f_2} \sum_{\substack{\gamma \in \mathcal{I}(k)^{|\beta_2|} \\ |\gamma| = |\beta_2|}} \frac{1}{|z|^{M_2}} \prod_{\ell=1}^{|\beta_2|} \frac{|z|^{M_2 - |\gamma_\ell|}}{|z|^{M_2}} \\ &\lesssim_{\beta_2, f_2} |z|^{-M_2 - |\beta_2|}. \end{aligned} \tag{3.7.9}$$

Hence, by (3.7.7) and (3.7.9), for any $z \in G\backslash\{e_G\}$ we obtain

$$\begin{aligned} \left| X_\beta\left(\frac{f_1}{f_2}\right)(z)\right| &\lesssim_\beta \sum_{\substack{\beta_1, \beta_2 \in \mathcal{I}(k) \\ |\beta_1| + |\beta_2| = |\beta|}} |X_{\beta_1} f_1(z)|\, |X_{\beta_2}(1/f_2)(z)| \\ &\lesssim_{\beta, f_2} \sum_{\substack{\beta_1, \beta_2 \in \mathcal{I}(k) \\ |\beta_1| + |\beta_2| = |\beta|}} |X_{\beta_1} f_1(z)|\, |z|^{-M_2 - |\beta_2|}. \end{aligned} \tag{3.7.10}$$

Observe that, for any $z \in G\backslash\{e_G\}$ and $\beta_1 \in \mathcal{I}(k)$, with $|\beta_1| \leq |\beta|$, we have

$$
\begin{aligned}
|X_{\beta_1} f_1(z)| \;&=\; \frac{|X_{\beta_1} f_1(z)|}{|z|^{M_1-|\beta_1|}}\,|z|^{M_1-|\beta_1|} \\
&\le\; \left(\sup_{z_0\,\in\, G} |z_0|^{-(M_1-|\beta_1|)}\,|X_{\beta_1} f_j(z_0)|\right)|z|^{M_1-|\beta_1|}, \qquad (3.7.11)
\end{aligned}
$$

and moreover, by Corollary 3.7.6, for any $\beta_1 \in \mathcal{I}(k)$, with $|\beta_1| < M_1$, we have

$$
\sup_{z_0\,\in\, G} |z_0|^{-(M_1-|\beta_1|)}\,|X_{\beta_1} f_1(z_0)| \;\lesssim_{M_1,\beta_1,k}\; \sup_{\substack{\beta'\,\in\,\mathcal{I}(k)\\ |\beta'|=M_1}} ||X_{\beta'} f_1||_{L^\infty(G)} \;<\; +\infty. \qquad (3.7.12)
$$

Then, applying this to (3.7.10), for any $z \in G\backslash\{e_G\}$ we obtain

$$
\begin{aligned}
\left|X_\beta\left(\frac{f_1}{f_2}\right)(z)\right| \;&\lesssim_{\beta,f_2}\; \sum_{\substack{\beta_1,\beta_2\,\in\,\mathcal{I}(k)\\ |\beta_1|+|\beta_2|=|\beta|}} |z|^{M_1-|\beta_1|}\,|z|^{-M_2-|\beta_2|} \sup_{\substack{\beta'\,\in\,\mathcal{I}(k)\\ |\beta'|=M_1}} ||X_{\beta'} f_1||_{L^\infty(G)} \\
&\lesssim_{\beta,f_2}\; |z|^{M_1-M_2-|\beta|} \sup_{\substack{\beta'\,\in\,\mathcal{I}(k)\\ |\beta'|=M_1}} ||X_{\beta'} f_1||_{L^\infty(G)}, \qquad (3.7.13)
\end{aligned}
$$

which is finite by the hypothesis $|\beta| < M_1 - M_2$. Taking the supremum of both sides of (3.7.13) over $z \in G$ yields (3.7.6), as required. □

Remark 3.7.8. Lemma 3.7.7 also holds when f_1 is a vector-valued function. More precisely, suppose $(V, ||\cdot||_V)$ is a normed vector space, and let $f_1 : G \to V$ and $f_2 \in \mathcal{D}(G)$ be smooth functions on G. Furthermore, suppose that, for $M_1,\, M_2 \in \mathbb{N}$, with $M_1 > M_2$, there exist constants $C_1,\, C_2,\, C_2' > 0$ such that

$$
||f_1(z)||_V \;\le\; C_1\,|z|^{M_1}, \qquad C_2'\,|z|^{M_2} \;\le\; |f_2(z)| \;\le\; C_2\,|z|^{M_2}, \qquad \forall\, z \in G.
$$

Then, the functions f_1/f_2 and $X_\beta(f_1/f_2)$ extend to a continuous function on G, for any $\beta \in \mathcal{I}(k)$, with $|\beta| < M_1 - M_2$. Moreover,

$$
\sup_{z\,\in\, G} \left|\left|\frac{f_1(z)}{f_2(z)}\right|\right|_V \;\le\; \frac{C_1}{C_2'}\,R^{M_1-M_2},
$$

and there exists a constant $C > 0$, depending on β, f_2, $\boldsymbol{X}$, M_1 and M_2,, such that

$$\left|\left|X_\beta\left(\frac{f_1(z)}{f_2(z)}\right)\right|\right|_V \le C \sup_{\substack{z\in G\\ \beta_0\in\mathcal{I}(k),\,|\beta_0|=M_1}} ||X_{\beta_0} f_1(z)||_V.$$

3.7.1 Proof of Proposition 3.7.3

The demonstration of Proposition 3.7.3 we provide in this section is adapted from the proofs given in Bellaiche [4], and Montgomery [35]. In these references the result is proved for any finite dimensional manifold, endowed with a bracket-generating distribution, which is analogous to a Hörmander system of left-invariant vector fields in our setting.

First, let

$$\boldsymbol{Y} := \{Y_1, Y_2, \ldots, Y_n\}$$

be the basis of $\mathfrak{g}$ constructed in Section 2.4.1.

The ball-box theorem (see Section 2.4 in Montgomery [35], Section 0.5.A in Gromov [24] and Section 2.4 in this thesis) tells us that there exist constants $\varepsilon_0, C, C' > 0$ such that

$$C'\phi\left(\mathrm{Box}(\varepsilon)\right) \subset B_\varepsilon(e_G) \subset C\phi\left(\mathrm{Box}(\varepsilon)\right), \tag{3.7.14}$$

for all $\varepsilon \le \varepsilon_0$, where for each $\varepsilon > 0$ we denote

$$\mathrm{Box}(\varepsilon) = \left\{x \in \mathbb{R}^n : |x_i| \le \varepsilon^{d_i},\ \forall\, i = 1, 2, \ldots, n\right\}. \tag{3.7.15}$$

Here, ϕ denotes the mapping given by

$$\phi\big((z_1, z_2, \ldots, z_n)\big) = e^{z_1 Y_1} e^{z_2 Y_2} \ldots e^{z_n Y_n}, \tag{3.7.16}$$

for $(z_1, z_2, \ldots, z_n) \in \mathbb{R}^n$, where for each $j = 1, 2, \ldots, n$, Y_j is the basis element of $\mathfrak{g}$.

Now, let N be a neighbourhood of 0 in $\mathbb{R}^n$ and V be a neighbourhood of e_G in G small enough such that the following properties are satisfied:

(a) $V \subset B_{\varepsilon_0}(e_G)$; that is, V satisfies the ball-box theorem (see (3.7.14)).

(b) The restricted mapping $\phi : N \to V$ given by (3.7.16) is a diffeomorphism.

(c) Any $(z_1, z_2, \ldots, z_n) \in N$ satisfies

$$||(z_1, z_2, \ldots, z_n)||_{\mathbb{R}^n} \le 1.$$

For any $z \in V$ we can then re-write (3.7.14) as

$$C' \left(|z_1|^{1/d_1} + |z_2|^{1/d_2} + \cdots + |z_n|^{1/d_n} \right) \leq d(e_G, z)$$
$$\leq C \left(|z_1|^{1/d_1} + |z_2|^{1/d_2} + \cdots + |z_n|^{1/d_n} \right), \quad (3.7.17)$$

where for each $j = 1, 2, \ldots, n$, $d_j = d(Y_j)$ denotes the formal degree of the vector field Y_j (see Definition 3.6.1).

In order to prove Proposition 3.7.3, we require the following result.

Lemma 3.7.9. *Let $q \in \mathcal{D}(G)$. If $\beta = (i_1, i_2, \ldots, i_b) \in \mathcal{I}(k)$, for some $b \in \mathbb{N}_0$, then*

$$\frac{\partial}{\partial z_1} \frac{\partial}{\partial z_2} \cdots \frac{\partial}{\partial z_b} q_\beta(0) = X_\beta\, q(e_G) = X_{i_1} X_{i_2} \cdots X_{i_b}\, q(e_G), \qquad (3.7.18)$$

where q_β denotes the mapping on $\mathbb{R}^b$:

$$q_\beta\left((z_1, z_2, \ldots, z_b)\right) = q\left(e^{z_1 X_{i_1}}\, e^{z_2 X_{i_2}} \ldots e^{z_b X_{i_b}}\right), \qquad (3.7.19)$$

for $(z_1, z_2, \ldots, z_b) \in \mathbb{R}^b$.

Proof. First observe that, applying the operator $\frac{\partial}{\partial z_b}\big|_{z_b=0}$ to the function q_β yields:

$$\frac{\partial}{\partial z_b} q_\beta\left((z_1, z_2, \ldots, z_b)\right)\Big|_{z_b=0} = \frac{\partial}{\partial z_b} q\left(e^{z_1 X_{i_1}}\, e^{z_2 X_{i_2}} \ldots e^{z_b X_{i_b}}\right)\Big|_{z_b=0}$$
$$= (X_{i_b}\, q)\left(e^{z_1 X_{i_1}}\, e^{z_2 X_{i_2}} \ldots e^{z_{b-1} X_{i_{b-1}}}\right).$$

Similarly, we now apply the operator $\frac{\partial}{\partial z_{b-1}}\big|_{z_{b-1}=0}$ to this function to obtain

$$\frac{\partial}{\partial z_b} q_\beta\left((z_1, z_2, \ldots, z_b)\right)\Big|_{z_b=0} = (X_{i_b}\, q)\left(e^{z_1 X_{i_1}}\, e^{z_2 X_{i_2}} \ldots e^{z_{b-1} X_{i_{b-1}}}\right)$$

to obtain

$$\frac{\partial}{\partial z_{b-1}} \frac{\partial}{\partial z_b} q_\beta \left((z_1, z_2, \ldots, z_b)\right)\Bigg|_{z_{b-1}=z_b=0}$$
$$= \frac{\partial}{\partial z_{b-1}} (X_{i_b}\, q) \left(e^{z_1 X_{i_1}}\, e^{z_2 X_{i_2}} \,\ldots\, e^{z_{b-1} X_{i_{b-1}}}\right)\Bigg|_{z_{b-1}=0}$$
$$= (X_{i_{b-1}}\, X_{i_b}\, q) \left(e^{z_1 X_{i_1}}\, e^{z_2 X_{i_2}} \,\ldots\, e^{z_{b-2} X_{i_{b-2}}}\right).$$

Continuing in this way, we obtain

$$\frac{\partial}{\partial z_1} \frac{\partial}{\partial z_2} \cdots \frac{\partial}{\partial z_b} q_\beta \left((z_1, z_2, \ldots, z_b)\right)\Bigg|_{z_1=z_2=\cdots=z_b=0}$$
$$= \frac{\partial}{\partial z_1} \left(X_{i_2}\, X_{i_3} \,\cdots\, X_{i_b}\, q\right) \left(e^{z_1 X_{i_1}}\right)\Bigg|_{z_1=0}$$
$$= (X_{i_1}\, X_{i_2} \,\cdots\, X_{i_b}\, q)\,(e_G),$$

which is the desired result. □

We are now in a position to prove Proposition 3.7.3.

Proof that (i) $\Longrightarrow$ (ii) in Proposition 3.7.3

Assume that for every $\beta \in \mathcal{I}(k)$, with $|\beta| \leq a-1$, we have

$$X_\beta\, q(e_G) \;=\; 0. \tag{3.7.20}$$

We first prove (ii) in the case $a = 1$. Consider the function

$$p\,(x) \;:=\; q\,(\phi(x))\,, \qquad \text{for } \; x \in \mathbb{R}^n,$$

where ϕ is the mapping given by (3.7.16). Then, the function p is smooth on N. Moreover, by the hypothesis, we have

$$p(0) \;=\; q(e_G) \;=\; 0. \tag{3.7.21}$$

So, by Remark 3.7.4, there exists $C_E > 0$ such that

$$|p(x)| \;\leq\; C_E\, ||x||_{\mathbb{R}^n}, \tag{3.7.22}$$

for all x in a neighbourhood $N' \subset N$ of 0 in $\mathbb{R}^n$, where we recall that $||\cdot||_{\mathbb{R}^n}$ denotes the usual Euclidean norm on $\mathbb{R}^n$. But, by Proposition 2.4.2, there exists

$C' > 0$ such that

$$||x||_{\mathbb{R}^n} \leq C' |\phi(x)|, \qquad \forall\, x \in N'.$$

Hence, we deduce that there exists $C > 0$ such that

$$|q(\phi(x))| \leq C_E ||x||_{\mathbb{R}^n} \leq C |\phi(x)|,$$

for all $x \in N'$. Moreover, we can choose N' and a neighbourhood $V' \subset V$ of e_G in G small enough such that ϕ maps N' diffeomorphically onto V'. Thus, we have

$$|q(z)| \leq C |z|, \qquad \forall\, z \in V'.$$

Since G is compact, then it follows that

$$|q(z)| \leq C |z|, \qquad \forall\, z \in G,$$

which finishes the proof for the case $a = 1$.

Now, for the case $a > 1$ we proceed by induction. Hence, assume that if for any $\beta \in \mathcal{I}(k)$, with $|\beta| \leq a - 2$, we have

$$X_\beta\, q(e_G) = 0,$$

then there exists $C > 0$ such that

$$|q(z)| \leq C |z|^{a-1}, \qquad \forall\, z \in G. \tag{3.7.23}$$

This is our induction hypothesis. Now, suppose that for any $\beta \in \mathcal{I}(k)$, with $|\beta| \leq a - 1$, we have

$$X_\beta\, q(e_G) = 0. \tag{3.7.24}$$

Thus, any $(i_1, i_2, \ldots, i_{b-1}) \in \mathcal{I}(k)$, with $b \leq a - 1$, satisfies

$$(X_{i_1} X_{i_2} \cdots X_{i_{b-1}})(X_i\, q)(e_G) = 0,$$

for any $i = 1, 2, \ldots, k$. So, applying the induction hypothesis to $X_i\, q$ yields

$$|(X_i\, q)(z)| \leq C |z|^{a-1}, \qquad \forall\, z \in G, \tag{3.7.25}$$

for some $C > 0$. Now, fix $z \in G$ and set $T = |z|$. Let $\gamma : [0, T] \to G$ be a geodesic joining e_G and z such that

$$\gamma'(t) \;=\; \sum_{i=1}^{k} c_i(t) X_i(\gamma(t)), \qquad \text{a.e.},$$

for some functions c_i $(i = 1, 2, \ldots, k)$ integrable on $[0, T]$, and with velocity 1 (see Section A.1, and in particular, Definition A.1.1). In particular, for a.a. $t \in [0, T]$, we have

$$||\gamma'(t)||^2 = \sum_{i=1}^{k} c_i(t)^2 \;=\; 1. \tag{3.7.26}$$

Moreover, since $|\gamma(0)| = |e_G| = 0$ and $|\gamma(T)| = |z| = T$, and the velocity of γ is constant, then we deduce that

$$|\gamma(t)| = d(e_G, \gamma(t)) = t, \qquad \text{a.e.} \tag{3.7.27}$$

Now, we have

$$\frac{d}{dt}\,(q \circ \gamma)(t) \;=\; \sum_{i=1}^{k} c_i(t)\,(X_i\, q)(\gamma(t)) \qquad \text{a.e.}$$

Hence, by the triangle inequality, we have

$$\left| \frac{d}{dt} q(\gamma(t)) \right| \;\leq\; \sum_{i=1}^{k} |c_i(t)|\, |(X_i q)(\gamma(t))|, \tag{3.7.28}$$

noting that here $|\cdot|$ simply denotes the usual Euclidean norm on $\mathbb{R}$. Now, by (3.7.26) we have

$$|c_i(t)| \leq 1, \qquad \text{a.e.}, \tag{3.7.29}$$

and by the induction hypothesis (see (3.7.25)), there exists $C > 0$ such that

$$|(X_i q)(\gamma(t))| \;\leq\; C\, |\gamma(t)|^{a-1}, \qquad \text{a.e.}$$

But, by (3.7.27), in fact we have

$$|(X_i q)(\gamma(t))| \;\leq\; C\, t^{a-1}, \qquad \text{a.e.} \tag{3.7.30}$$

Therefore, applying (3.7.29) and (3.7.30) to (3.7.28), we obtain

$$\left|\frac{d}{dt}q(\gamma(t))\right| \le \sum_{i=1}^{k} C\, t^{a-1} \;=\; kC\, t^{a-1}. \tag{3.7.31}$$

Now, we know that

$$q(\gamma(T)) \;=\; q(z),$$

and moreover,

$$q(\gamma(0)) \;=\; q(e_G) \;=\; 0.$$

So, integrating both sides of (3.7.31) with respect to t, between 0 and T, yields

$$\begin{aligned}
|q(z)| \;=\; |q(\gamma(T)) - q(\gamma(0))| \;&=\; \left|\int_0^T \frac{d}{dt}\, q(\gamma(t))\, \mathrm{d}t\right| \\
&\le \int_0^T \left|\frac{d}{dt}\, q(\gamma(t))\right| \mathrm{d}t \\
&\le kC \int_0^T t^{a-1}\, \mathrm{d}t \\
&= \frac{kC}{a}\, T^a.
\end{aligned}$$

Hence, we have shown that

$$|q(z)| \;\le\; \frac{kC}{a}\, |z|^a, \qquad \forall\, z \,\in\, G, \tag{3.7.32}$$

which proves (ii).

Furthermore, observe that the constant C introduced in (3.7.25) may be chosen to be

$$C \;:=\; \sup_{\substack{z \in G \\ i=1,2,\dots,k}} \; |z|^{1-a}\, |X_i q(z)|.$$

Moreover note that, if q satisfies the hypothesis of (i), then, by the proof we have just completed, for every $i = 1, 2, \dots, k$, the function $X_i q$ is CC-vanishing at e_G up to order $a-2$. Hence,

$$\sup_{\substack{z \in G \\ i=1,2,\dots,k}} \; |z|^{1-a}\, |X_i q(z)| \;\lesssim\; \frac{k}{a} \sup_{z \in G} \; |z|^{1-a}\, |z|^{a-1} \;=\; \frac{k}{a},$$

which shows that C is a finite constant. Hence, by (3.7.32), (3.7.2) is also proved.

□

Proof that (ii) $\Longrightarrow$ (i) in Proposition 3.7.3

Assume that there exists $C > 0$ such that

$$|q(z)| \,\leq\, C\,|z|^a, \qquad \forall\, z \,\in\, G.$$

Then, for $b \in \mathbb{N}$, with $b \leq a$, consider the point

$$z \,:=\, e^{z_1 X_{i_1}}\, e^{z_2 X_{i_2}}\, \ldots\, e^{z_b X_{i_b}} \,\in\, G,$$

for $z_1, z_2, \ldots, z_b \in \mathbb{R}$ and $\beta = (i_1, i_2, \ldots, i_b) \in \mathcal{I}(k)$. The Carnot-Carathéodory distance between e_G and z satisfies

$$\begin{aligned} |z| \,=\, d(e_G, z) \,&\leq\, d(e_G, e^{z_1 X_{i_1}}) + d(e_G, e^{z_2 X_{i_2}}) + \cdots + d(e_G, e^{z_b X_{i_b}}) \\ &\leq\, |z_1| + |z_2| + \cdots + |z_b|. \end{aligned}$$

By the hypothesis, we then have

$$|q(z)| \,\leq\, C\,|z|^a \,\leq\, C\,(|z_1| + |z_2| + \cdots + |z_b|)^a\,. \tag{3.7.33}$$

In particular, the function q_β (see (3.7.19)) satisfies

$$|q_\beta\,((z_1, z_2, \ldots, z_b))\,| \,\leq\, C\, (|z_1| + |z_2| + \cdots + |z_b|)^a\,. \tag{3.7.34}$$

This holds for any $z \in G$, and in particular, for every $(z_1, z_2, \ldots, z_b)$ in a neighbourhood of 0 in $\mathbb{R}^n$. Thus, by Remark 3.7.4, we have

$$\frac{\partial}{\partial z_1}\,\frac{\partial}{\partial z_2}\,\cdots\,\frac{\partial}{\partial z_b}\, q_\beta(0) \,=\, 0.$$

Hence, by Lemma 3.7.9, the result is proved.

□

3.8 Spectral multipliers of the sub-Laplacian on a compact Lie group

In this section we remain in the same setting as in previous sections. Our objective focuses on proving some important results about the spectral multipliers of the sub-Laplacian

$$\mathcal{L} := -(X_1^2 + X_2^2 + \cdots + X_k^2).$$

The foundations of the results we present here were first laid out by Alexopoulos in [2], and later adapted by Furioli et al in [23] to dyadic decompositions of $\mathcal{L}$ and Besov spaces. The main difference between the result exhibited in [23] and Lemma 3.8.1 below lies in the weight of the integrals involved; in the reference mentioned the authors do not allow the weight to vanish at the identity element of the group, whereas in our case it is one of the defining properties of the function q to vanish at e_G.

This idea was already explored in the elliptic case, and it can be found, for example, in Fischer [17]. In that case, the action of the left-invariant vector fields belonging to the basis of the Lie algebra $\mathfrak{g}$ of G was considered. On the other hand, as can be seen in the statement of Lemma 3.8.1 (part (II)) below, in our case we consider the action of the left-invariant vector fields belonging to a Hörmander system. This choice stems from having chosen a sub-elliptic setting, and is a natural consideration due to Remark 3.6.4.

One other important difference between Lemma 3.8.1 below and its analogous version in the elliptic setting is in the implementation of dimension within the proof. One example of this arises with integrals of the form

$$\int_G |z|^r \, \mathrm{d}z, \tag{3.8.1}$$

for some $r > 0$, which we consider in the proof. In our case, $|\cdot|$ denotes the Carnot-Carathéodory metric (see Definition A.1.2), whereas in the elliptic case a Riemannian metric is considered instead. As we prove in the appendix (see Lemma A.3.2), the finiteness of (3.8.1) in our case depends on the local dimension l of G. On the other hand, in the elliptic setting this is dependent on the topological dimension of G, instead. Another example arises in the heat kernel estimates, as can be seen below in (3.8.5).

3.8.1 Main result on spectral multipliers in $\mathcal{L}$

The main result of this section is the following lemma.

Lemma 3.8.1. *(I) Let $q \in \mathcal{D}(G)$ and $m \in \mathbb{R}$. Then there exists a constant $C = C_{q,m} > 0$ such that if $f \in \mathcal{C}([0,\infty))$, with supp$(f) \subset [0,2]$, we have*

$$\int_G \left|q(z) f(t\mathcal{L})\, \delta_{e_G}(z)\right| \, dz \, \leq \, C \, ||f||_\infty, \tag{3.8.2}$$

for any $t \geq 1$.

(II) *Let* $m \in \mathbb{R}$*,* $a \in \mathbb{N}_0$ *and* $\beta, \beta' \in \mathcal{I}(k)$*. Suppose that* $q \in \mathcal{D}(G)$ *CC-vanishes at* e_G *up to order* $a-1$ *(see Notation 3.7.2). Then, there exist* $C > 0$ *and* $d \in \mathbb{N}$ *such that for any function* $f \in \mathcal{C}^d([0,\infty))$ *with supp*$(f) \subset [0,2]$*, we have*

$$\int_G \left|q(x) X_\beta \, \widetilde{X}_{\beta'} \{f(t\mathcal{L})\delta_{e_G}\}(x)\right| \, dx \\ \leq \, C \, t^{\frac{1}{2}(a-|\beta|-|\beta'|)} \max_{0 \leq j \leq d} \left|\left|\partial^j f\right|\right|_\infty, \tag{3.8.3}$$

whenever $t \in (0,1)$.

An easy corollary of Lemma 3.8.1 is the following, which follows from Leibniz's rule for vector fields.

Corollary 3.8.2. *Let* $m \in \mathbb{R}$*,* $a \in \mathbb{N}_0$ *and* $\gamma, \beta, \beta' \in \mathcal{I}(k)$*. Suppose that* $q \in \mathcal{D}(G)$ *CC-vanishes at* e_G *up to order* $a-1$*. Then, there exists* $C > 0$ *and* $d \in \mathbb{N}$ *such that for any function* $f \in \mathcal{C}^d([0,\infty))$ *with supp*$(f) \subset [0,2]$*, we have*

$$\int_G \left|X_\gamma \{q(x) X_\beta \widetilde{X}_{\beta'} \{f(t\mathcal{L})\delta_{e_G}\}(x)\}\right| \, dx \, \leq \, C \, t^{\frac{1}{2}(a-|\beta|-|\beta'|-|\gamma|)} \max_{0 \leq j \leq d} \left|\left|\partial^j f\right|\right|_\infty,$$

for all $t \in (0,1)$.

If $\boldsymbol{Y} = \{Y_1, Y_2, \ldots, Y_n\}$ denotes a basis of the Lie algebra $\mathfrak{g}$, one can also state a version of Lemma 3.8.1 part (II) and Corollary 3.8.2 in terms of differential operators Y_j, $j = 1, 2, \ldots, n$, imitating the elliptic case. However, in that case, which we do not explore further in this thesis, the condition on q as well as the dependence of the bound on $|\beta|$ are affected. In fact, if we consider the differential operator Y^α, for $\alpha \in \mathbb{N}_0^n$, the formal degree $[\alpha]_{\boldsymbol{Y}}$ of Y^α is substituted for $|\beta|$ in the formulae above.

The following sections will be devoted to the proof of Lemma 3.8.1.

3.8.2 Proof of Lemma 3.8.1

Although the result of Lemma 3.8.1 part (I) is already well known, we include it here for the sake of completeness. On the other hand, with part (II) we are introducing the first major original result in this thesis, which has already been proved in the elliptic setting with respect to the Laplace-Beltrami operator (see, for example, Fischer [17]). Despite being in a different setting, our proof's ideas stem from Alexopoulos [2] and Furioli et al [23], which also appear in the elliptic case in [17].

One also notices that the first part of Proposition 6 in [23], which is analogous to part (I) in Lemma 3.8.1, is proved for all $t > 0$. However, in our case it is not possible to have the estimate (3.8.2) for $t \in (0, 1)$, and thus we only prove it for $t \geq 1$.

Step 0

This preliminary step aims to set-up the strategy of the proof. Fix a function $f : [0, \infty) \to \mathbb{C}$, with $\operatorname{supp}(f) \subset [0, 2]$, and assume $f \in \mathcal{C}^d([0, \infty))$, with d to be determined later.

We now mention some relevant results. Recall that, as we saw in Proposition 3.1.6, the heat kernels p_t $(t \geq 0)$ associated with $\mathcal{L}$ satisfy:

$$|p_t(z)| \leq C\, V(\sqrt{t})^{-1}\, e^{-\frac{|z|^2}{Ct}}, \qquad \text{for } z \in G,\ t > 0. \tag{3.8.4}$$

Moreover, as shown in Varopoulos et al. [55], for each $\beta, \beta' \in \mathcal{I}(k)$, we also have that,

$$|X_\beta \widetilde{X}_{\beta'} p_t(z)| \leq C\, t^{-\frac{1}{2}(l+|\beta|+|\beta'|)}\, e^{-\frac{|z|^2}{Ct}}, \qquad \text{for } z \in G,\ t \in (0,1), \tag{3.8.5}$$

where X_β and $\widetilde{X}_{\beta'}$ are the differential operators defined by (2.3.11) and (2.3.12), respectively.

For a given $t \in (0, 1)$ and for $\beta, \beta' \in \mathcal{I}(k)$, we split up the integral in part (II) as follows:

$$\int_G \left|q(z) X_\beta \widetilde{X}_{\beta'}\{f(t\mathcal{L})\delta_{e_G}\}(z)\right| \mathrm{d}z$$
$$\leq \int_{B_{\sqrt{t}}(G)} \left|q(z) X_\beta \widetilde{X}_{\beta'}\{f(t\mathcal{L})\delta_{e_G}\}(z)\right| \mathrm{d}z$$
$$+ \int_{B_{\sqrt{t}}(G)^c} \left|q(z) X_\beta \widetilde{X}_{\beta'}\{f(t\mathcal{L})\delta_{e_G}\}(z)\right| \mathrm{d}z. \quad (3.8.6)$$

The objective of this proof is to first prove part (I), and then bound the integrals above separately to prove part (II).

Step 1

This step is dedicated to proving the following result.

Proposition 3.8.3. *For $\beta, \beta' \in \mathcal{I}(k)$ there exists $C > 0$ such that*

$$\left|\left|X_\beta \widetilde{X}_{\beta'} \{f(t\mathcal{L})\delta_{e_G}\}\right|\right|_{L^2(G)} \lesssim ||f||_\infty \, ||X_\beta \widetilde{X}_{\beta'} p_t||_{L^2(G)}, \quad (3.8.7)$$

for any $t > 0$.

Proof. For $t > 0$, let $h_t : [0, \infty) \to \mathbb{C}$ be the function given by

$$h_t(\mu) = e^{t\mu^2} f(t\mu^2), \qquad \mu \in [0, \infty). \quad (3.8.8)$$

Since $\operatorname{supp}(f) \subset [0, 2]$, then

$$||h_t||_\infty = \sup_{\mu \geq 0} |h_t(\mu)| \leq e^2 \, ||f||_\infty, \quad (3.8.9)$$

and moreover, observe that for $t > 0$,

$$f(t\lambda) = h_t(\sqrt{\lambda}) \, e^{-t\lambda}, \qquad \forall \, \lambda \geq 0.$$

The spectral theory discussed in Section 3.1 then implies that for every $t > 0$,

$$f(t\mathcal{L})\delta_{e_G} = h_t\big(\sqrt{\mathcal{L}}\big) p_t, \quad (3.8.10)$$

and consequently, since X_j and $\widetilde{X}_{j'}$ commute with $h_t(\sqrt{\mathcal{L}})$, for every $j, j' = 1, 2, \ldots, k$, we have

$$\left|\left|X_\beta \widetilde{X}_{\beta'} \{f(t\mathcal{L})\delta_{e_G}\}\right|\right|_{L^2(G)} \leq ||h_t||_\infty \, ||X_\beta \widetilde{X}_{\beta'} \, p_t||_{L^2(G)}. \quad (3.8.11)$$

Then (3.8.9) implies that

$$\big|\big|X_\beta \widetilde{X}_{\beta'} f(t\mathcal{L})\delta_{e_G}\big|\big|_{L^2(G)} \lesssim ||f||_\infty \, ||X_\beta \widetilde{X}_{\beta'}\, p_t||_{L^2(G)},$$

for $t > 0$, which is the desired result.

□

Step 2

This step is dedicated to the proof of part (I) of Lemma 3.8.1.

Proof of Lemma 3.8.1 part (I): Fix $t \geq 1$. First observe that, by the Cauchy Schwarz inequality, we have

$$\int_G |q(z)\, f(t\mathcal{L})\delta_{e_G}(z)| \,\mathrm{d}z \leq \left(\int_G |q(z)|^2 \,\mathrm{d}z\right)^{1/2} \left(\int_G |f(t\mathcal{L})\delta_{e_G}|^2\right)^{1/2}. \quad (3.8.12)$$

Proposition 3.8.3, with $\beta = \beta' = 0$, implies that

$$\big|\big|f(t\mathcal{L})\delta_{e_G}\big|\big|_{L^2(G)} \lesssim ||f||_\infty \, ||p_t||_{L^2(G)}. \quad (3.8.13)$$

Moreover, by (3.8.4), we obtain the estimate

$$\int_G \big|p_t(z)\big|^2 \,\mathrm{d}z \leq C_1 \int_G V(\sqrt{t})^{-2}\, e^{-\frac{2|z|^2}{Ct}} \,\mathrm{d}z, \quad (3.8.14)$$

for some $C_1 > 0$. Furthermore, a result from [55] (see the proof of Lemma VIII.2.5 therein) tells us that

$$\int_G e^{-\frac{|z|^2}{Ct}} \,\mathrm{d}z \lesssim V(\sqrt{t}), \quad \text{for } t > 0, \quad (3.8.15)$$

which, by (3.8.14), implies that

$$\int_G \big|p_t(z)\big|^2 \,\mathrm{d}z \lesssim C_1\, V(\sqrt{t})^{-2}\, V(\sqrt{t}) \;=\; C_1\, V(\sqrt{t})^{-1},$$

and hence

$$||p_t||_{L^2(G)} \lesssim C_1^{1/2}\, V(\sqrt{t})^{-1/2}.$$

Substituting this into (3.8.13) yields

$$\big|\big|f(t\mathcal{L})\delta_{e_G}\big|\big|_{L^2(G)} \lesssim C_1^{1/2}\, V(\sqrt{t})^{-1/2}\, ||f||_\infty,$$

and hence, as $t \geq 1$, the estimates of the volume given by (3.1.22) imply that there exists $C_1' > 0$ such that

$$\big|\big|f(t\mathcal{L})\delta_{e_G}\big|\big|_{L^2(G)} \leq C_1' \,||f||_\infty. \tag{3.8.16}$$

Now, since q CC-vanishes at e_G up to order $a-1$, then there exists $C_2 > 0$ such that

$$||q||^2_{L^2(G)} \leq C_2 \int_G |z^a|^2 \,\mathrm{d}z \leq C_2 \,|G| \sup_{z\in G} |z|^{2a} = C_2 \,|G|\, R^{2a} < +\infty,$$

where $|G|$ denotes the volume of G:

$$|G| := \int_G \mathrm{d}z < +\infty,$$

and R denotes the radius of G:

$$R := \sup_{z\in G} |z| < +\infty.$$

Hence, we have shown that there exists $C_2' > 0$ such that

$$||q||_{L^2(G)} \leq C_2'. \tag{3.8.17}$$

Combining (3.8.16) and (3.8.17) with (3.8.12), we then obtain that there exists $C > 0$ such that

$$\int_G |q(z)\, f(t\mathcal{L})\delta_{e_G}(z)| \,\mathrm{d}z \leq C\,||f||_\infty,$$

which is the result required. □

Step 3

The objective of Step 3 is to prove the following result, which gives a bound for the first integral in (3.8.6).

Proposition 3.8.4. *There exists a constant $C_q > 0$, depending on q, such that*

$$\int_{B_{\sqrt{t}}(e_G)} \big|q(z)X_\beta \widetilde{X}_{\beta'}\big\{f(t\mathcal{L})\delta_{e_G}\big\}(z)\big| \,dz \leq C_q\, t^{\frac{1}{2}(a-|\beta|-|\beta'|)}\,||f||_\infty, \tag{3.8.18}$$

whenever $t \in (0,1)$ and for any $\beta, \beta' \in \mathcal{I}(k)$.

Proof. By Cauchy-Schwarz's inequality, we have

$$\int_{B_{\sqrt{t}}(e_G)} \left|q(z) X_\beta \widetilde{X}_{\beta'} \{f(t\mathcal{L})\delta_{e_G}\}(z)\right| \,\mathrm{d}z$$
$$\leq ||q||_{L^2(B_{\sqrt{t}}(e_G))} \left|\left|X_\beta \widetilde{X}_{\beta'} f(t\mathcal{L})\delta_{e_G}\right|\right|_{L^2(B_{\sqrt{t}}(e_G))}, \quad (3.8.19)$$

for any $t > 0$ and $\beta, \beta' \in \mathcal{I}(k)$. By Lemma 3.8.5, which we prove below, we see that there exist a constant $C_1 > 0$ such that

$$\left|\left|X_\beta \widetilde{X}_{\beta'} f(t\mathcal{L})\delta_{e_G}\right|\right|_{L^2(B_{\sqrt{t}}(e_G))} \leq C_1\, t^{-\frac{1}{2}(l+|\beta|+|\beta'|)}\, V(\sqrt{t})^{1/2}\, ||f||_\infty, \qquad (3.8.20)$$

for every $t \in (0,1)$, and a constant $C_{2,q} > 0$, depending on q, such that

$$||q||_{L^2(B_{\sqrt{t}}(e_G))} \leq C_{2,q}\, \sqrt{t}^{\,a+\frac{l}{2}}, \qquad (3.8.21)$$

for every $t \in (0,1)$. Hence, by (3.8.19), there exists a constant $C > 0$, depending on q, such that

$$\int_{B_{\sqrt{t}}(e_G)} \left|q(z) X_\beta \widetilde{X}_{\beta'} \{f(t\mathcal{L})\delta_{e_G}\}(z)\right| \,\mathrm{d}z \leq C\, t^{-\frac{1}{2}(l+|\beta|+|\beta'|)}\, V(\sqrt{t})^{1/2}\, ||f||_\infty\, \sqrt{t}^{\,a+\frac{l}{2}},$$

for every $t \in (0,1)$. By (3.1.22), for any $t \in (0,1)$, we have

$$t^{\frac{1}{2}\left(a+\frac{l}{2}\right)}\, t^{-\frac{1}{2}(l+|\beta|+|\beta'|)}\, V(\sqrt{t})^{1/2} \approx t^{\frac{1}{2}(a+l)}\, t^{-\frac{1}{2}(l+|\beta|+|\beta'|)} = t^{\frac{1}{2}(a-|\beta|-|\beta'|)}.$$

Hence, we have obtained

$$\int_{B_{\sqrt{t}}(e_G)} \left|q(z) X_\beta \widetilde{X}_{\beta'} \{f(t\mathcal{L})\delta_{e_G}\}(z)\right| \,\mathrm{d}z \lesssim_q t^{\frac{1}{2}(a-|\beta|-|\beta'|)}\, \left|\left|f\right|\right|_\infty,$$

whenever $t \in (0,1)$. So, the result is proved. □

Lemma 3.8.5. *The following assertions hold.*

(1) There exists a constant $C > 0$ such that

$$\left|\left|X_\beta \widetilde{X}_{\beta'} f(t\mathcal{L})\delta_{e_G}\right|\right|_{L^2(B_{\sqrt{t}}(e_G))} \leq C\, t^{-\frac{1}{2}(l+|\beta|+|\beta'|)}\, V(\sqrt{t})^{1/2}\, ||f||_\infty, \quad (3.8.22)$$

whenever $t \in (0,1)$ *and for any* $\beta, \beta' \in \mathcal{I}(k)$*.*

(2) There exists a constant $C_q > 0$*, depending on* q*, such that*

$$||q||_{L^2(B_r(e_G))} \leq C_q\, r^{a+\frac{l}{2}}, \tag{3.8.23}$$

whenever $0 < r < 1$*.*

Proof. (1): By (3.8.5) we know that

$$\left|X_\beta \widetilde{X}_{\beta'} p_t(z)\right| \leq C\, t^{-\frac{1}{2}(l+|\beta|+|\beta'|)}\, e^{-\frac{|z|^2}{Ct}}, \quad \forall\, z \in G,\ t \in (0,1).$$

Hence,

$$\int_G \left|X_\beta \widetilde{X}_{\beta'} p_t(z)\right|^2 \mathrm{d}z \lesssim \int_G t^{-(l+|\beta|+|\beta'|)}\, e^{-\frac{2|z|^2}{Ct}}\, \mathrm{d}z \lesssim t^{-(l+|\beta|+|\beta'|)}\, V(\sqrt{t}),$$

by (3.8.15), for any $t \in (0,1)$ and any $\beta, \beta' \in \mathcal{I}(k)$, and so

$$\left|\left|X_\beta \widetilde{X}_{\beta'} p_t\right|\right|_{L^2(G)} \lesssim t^{-\frac{1}{2}(l+|\beta|+|\beta'|)}\, V(\sqrt{t})^{1/2}, \tag{3.8.24}$$

for any $t \in (0,1)$ and any $\beta, \beta' \in \mathcal{I}(k)$. So, applying (3.8.24) to the inequality (3.8.7) yields

$$\begin{aligned}\left|\left|X_\beta \widetilde{X}_{\beta'} \{f(t\mathcal{L})\delta_{e_G}\}\right|\right|_{B_{\sqrt{t}}(e_G)} &\lesssim \left|\left|X_\beta \widetilde{X}_{\beta'} \{f(t\mathcal{L})\delta_{e_G}\}\right|\right|_{L^2(G)} \\ &\lesssim t^{-\frac{1}{2}(l+|\beta|+|\beta'|)}\, V(\sqrt{t})^{1/2}\, \left|\left|f\right|\right|_\infty,\end{aligned}$$

for any $t \in (0,1)$ and any $\beta, \beta' \in \mathcal{I}(k)$, which is the required result.

(2): Since q CC-vanishes at e_G up to order $a-1$ (see Definition 3.7.1), then there exists $C_q > 0$, depending on q, such that

$$||q||^2_{L^2(B_r(e_G))} \leq C_q^2 \int_{B_r(e_G)} |z|^{2a}\, \mathrm{d}z,$$

for any $r > 0$. Now, by Lemma A.3.1 we have

$$\int_{B_r(e_G)} |z|^{2a}\, \mathrm{d}z \approx \int_0^r \rho^{2a}\, \rho^{l-1}\, \mathrm{d}\rho = \int_0^r \rho^{2a+l-1}\, \mathrm{d}\rho = r^{2a+l}.$$

Therefore, we have

$$||q||_{L^2(B_r(e_G))} \leq C_q \, r^{a+\frac{l}{2}},$$

as claimed.

□

Step 4

It remains show that there exists a constant $C_q > 0$, depending on q, such that

$$\int_{B_{\sqrt{t}}(e_G)^c} \left|q(z) X_\beta \widetilde{X}_{\beta'}\{f(t\mathcal{L})\delta_{e_G}\}(z)\right| \mathrm{d}z \leq C_q \, t^{\frac{1}{2}(a-|\beta|-|\beta'|)} \, ||f||_{\mathcal{C}^d([0,2])}, \quad (3.8.25)$$

whenever $t \in (0,1)$ and any $\beta, \beta' \in \mathcal{I}(k)$. The first thing we shall do is employ the following decomposition:

Proposition 3.8.6. *For $t \in (0,1)$ and $\beta, \beta' \in \mathcal{I}(k)$, we have the following inequality:*

$$\int_{B_{\sqrt{t}}(e_G)^c} \left|q(z) \, X_\beta \widetilde{X}_{\beta'}\{f(t\mathcal{L})\delta_{e_G}\}(z)\right| \, dz \leq \sum_{j=0}^{\infty} \left\{ \int_{A_{t,j}} \left|q(z) \, M_{t,j}^{(1)}(z)\right| \, dz + \int_{A_{t,j}} \left|q(z) \, M_{t,j}^{(2)}(z)\right| \, dz \right\}. \quad (3.8.26)$$

where, for each $j \in \mathbb{N}_0$, $A_{t,j}$ denotes the annulus

$$A_{t,j} = B_{2^{j+1}\sqrt{t}}(e_G) \backslash B_{2^j\sqrt{t}}(e_G),$$

and moreover,

$$M_{t,j}^{(1)} := h_t(\sqrt{\mathcal{L}}) \left\{ \left(X_\beta \widetilde{X}_{\beta'} p_t\right) \chi_{B_{2^{j-1}\sqrt{t}}(e_G)} \right\},$$

and

$$M_{t,j}^{(2)} := h_t(\sqrt{\mathcal{L}}) \left\{ \left(X_\beta \widetilde{X}_{\beta'} p_t\right) \chi_{B_{2^{j-1}\sqrt{t}}(e_G)^c} \right\}.$$

Proof. Let $t \in (0,1)$ and $\beta, \beta' \in \mathcal{I}(k)$. We begin by making the following straightforward observation

$$\int_{B_{\sqrt{t}}(e_G)^c} \left| q(z)\, X_\beta \widetilde{X}_{\beta'} \big\{ f(t\mathcal{L})\delta_{e_G} \big\}(z) \right| \mathrm{d}z$$
$$= \int_G \left| q(z)\, X_\beta \widetilde{X}_{\beta'} \big\{ f(t\mathcal{L})\delta_{e_G} \big\}(z) \right| \chi_{B_{\sqrt{t}}(e_G)^c}(z)\, \mathrm{d}z,$$

where χ_B denotes the indicator function of a set B. Moreover, using (3.8.10), we have

$$\begin{aligned}
X_\beta\, \widetilde{X}_{\beta'} \big\{ f(t\mathcal{L})\delta_{e_G} \big\} &= X_\beta\, \widetilde{X}_{\beta'} \big\{ h_t(\sqrt{\mathcal{L}}) p_t \big\} \\
&= h_t(\sqrt{\mathcal{L}}) \big\{ X_\beta\, \widetilde{X}_{\beta'} p_t \big\} \\
&= h_t(\sqrt{\mathcal{L}}) \big\{ X_\beta\, \widetilde{X}_{\beta'} p_t \big\} \chi_{B_{2^{j-1}\sqrt{t}}(e_G)} \\
&\qquad + h_t(\sqrt{\mathcal{L}}) \big\{ X_\beta\, \widetilde{X}_{\beta'} p_t \big\} \chi_{B_{2^{j-1}\sqrt{t}}(e_G)^c},
\end{aligned}$$

for every $j \in \mathbb{N}_0$. Additionally, observe that

$$B_{\sqrt{t}}(e_G)^c = \bigcup_{j=0}^{\infty} A_{t,j},$$

where the sets $A_{t,j}$ are pairwise disjoint. Thus, it follows that

$$\begin{aligned}
&\int_{B_{\sqrt{t}}(e_G)^c} \left| q(z)\, X_\beta \widetilde{X}_{\beta'} \big\{ f(t\mathcal{L})\delta_{e_G} \big\}(z) \right| \mathrm{d}z \\
&\leq \sum_{j=0}^{\infty} \Bigg\{ \int_G \left| q(z)\, h_t(\sqrt{\mathcal{L}}) \left\{ \big(X_\beta \widetilde{X}_{\beta'} p_t \big) \chi_{B_{2^{j-1}\sqrt{t}}(e_G)} \right\} (z) \right| \chi_{A_{t,j}}(z)\, \mathrm{d}z \\
&\quad + \int_G \left| q(z)\, h_t(\sqrt{\mathcal{L}}) \left\{ \big(X_\beta \widetilde{X}_{\beta'} p_t \big) \chi_{B_{2^{j-1}\sqrt{t}}(e_G)^c} \right\} (z) \right| \chi_{A_{t,j}}(z)\, \mathrm{d}z \Bigg\}, \qquad (3.8.27)
\end{aligned}$$

which yields the required result. □

Note that for each $t \in (0,1)$ the sum in (3.8.27) is, in fact, finite. In particular, the number of non-zero terms is equal to the smallest positive integer L such that $2^{L+1}\sqrt{t} > R_0$. However, we shall keep the sum as infinite for reasons that will be become clear later.

For the rest of the proof, fix $t \in (0,1)$ and arbitrary multi-indices $\beta, \beta' \in \mathcal{I}(k)$. Now, for $i = 1, 2$ and any $j \in \mathbb{N}_0$, Cauchy-Schwarz's inequality implies that

$$\int_{A_{t,j}} \left| q(z)\, M_{t,j}^{(i)}(z) \right| \, \mathrm{d}z \;\leq\; ||q||_{L^2(A_{t,j})} \, \big|\big| M_{t,j}^{(i)} \big|\big|_{L^2(A_{t,j})}.$$

Moreover, by (3.8.23) it follows that

$$||q||_{L^2(A_{t,j})} \;\leq\; ||q||_{L^2(B_{2^{j+1}\sqrt{t}}(e_G))} \;\leq\; C_q \left(2^{j+1}\sqrt{t}\right)^{a+\frac{l}{2}},$$

for some constant $C_q > 0$ depending on q. Hence, we have shown that for each $j \in \mathbb{N}_0$ and for $i = 1, 2$,

$$\int_{A_{t,j}} \left| q(z)\, M_{t,j}^{(i)}(z) \right| \, \mathrm{d}z \;\leq\; C_q \left(2^{j+1}\sqrt{t}\right)^{a+\frac{l}{2}} \big|\big| M_{t,j}^{(i)} \big|\big|_{L^2(A_{t,j})}. \tag{3.8.28}$$

We now analyse the bounds for $\big|\big| M_{t,j}^{(2)} \big|\big|_{L^2(A_{t,j})}$ and $\big|\big| M_{t,j}^{(1)} \big|\big|_{L^2(A_{t,j})}$ separately, splitting up the rest of Step 4 into Step 4a and Step 4b.

Step 4a

This step is dedicated to finding a bound for $\big|\big| M_{t,j}^{(2)} \big|\big|_{L^2(A_{t,j})}$, for each $j \in \mathbb{N}_0$. In particular, we have the following result:

Proposition 3.8.7. *There exists a constant $C > 0$ such that*

$$\big|\big| M_{t,j}^{(2)} \big|\big|_{L^2(A_{t,j})} \lesssim ||f||_\infty \, \sqrt{t}^{\,-\frac{l}{2}-|\beta|-|\beta'|} \, e^{-\frac{2^{2(j-1)}}{C}}, \tag{3.8.29}$$

for every $j \in \mathbb{N}_0$.

Proof. Let us fix $j \in \mathbb{N}_0$. We first obtain the simple estimate:

$$\begin{aligned} \big|\big| M_{t,j}^{(2)} \big|\big|_{L^2(A_{t,j})} &\leq \big|\big| M_{t,j}^{(2)} \big|\big|_{L^2(G)} \\ &\leq \big|\big| h_t(\sqrt{\mathcal{L}}) \big|\big|_{\mathscr{L}(L^2(G))} \big|\big| \big(X_\beta \widetilde{X}_{\beta'} p_t\big) \chi_{B_{2^{j-1}\sqrt{t}}(e_G)^c} \big|\big|_{L^2(G)}. \end{aligned} \tag{3.8.30}$$

By functional analysis and (3.8.9), we have

$$\big|\big| h_t(\sqrt{\mathcal{L}}) \big|\big|_{\mathscr{L}(L^2(G))} \;\leq\; ||h_t||_\infty \;\leq\; e^2 \, ||f||_\infty. \tag{3.8.31}$$

Now observe that, for any $C > 0$ and $z \in G$ with $|z| \geq 2^{j-1}\sqrt{t}$, we have

$$e^{-\frac{|z|^2}{Ct}} \;\leq\; e^{-\frac{(2^{j-1}\sqrt{t})^2}{Ct}} \;=\; e^{-\frac{2^{2(j-1)}}{C}}. \tag{3.8.32}$$

Moreover, we have

$$\begin{aligned}
&\big|\big|\big(X_\beta \widetilde{X}_{\beta'} p_t\big)\chi_{B_{2^{j-1}\sqrt{t}}(e_G)^c}\big|\big|^2_{L^2(G)} \\
&\qquad = \int_G \Big|\big(X_\beta \widetilde{X}_{\beta'} p_t\big)(z)\, \chi_{B_{2^{j-1}\sqrt{t}}(e_G)^c}(z)\Big|^2 \,\mathrm{d}z \\
&\qquad \leq \sup_{z_1 \in G} \big|\big(X_\beta \widetilde{X}_{\beta'} p_t\big)(z_1)\, \chi_{B_{2^{j-1}\sqrt{t}}(e_G)^c}(z_1)\big| \left(\int_G \big|X_\beta \widetilde{X}_{\beta'} p_t(z)\big| \,\mathrm{d}z\right) \\
&\qquad = \sup_{|z_1|\geq 2^{j-1}\sqrt{t}} \big|X_\beta \widetilde{X}_{\beta'} p_t(z_1)\big| \left(\int_G \big|X_\beta \widetilde{X}_{\beta'} p_t(z)\big| \,\mathrm{d}z\right).
\end{aligned}$$

So, by (3.8.5) and (3.8.32), there exists $C > 0$ such that

$$\begin{aligned}
\big|\big|\big(X_\beta \widetilde{X}_{\beta'} p_t\big)\chi_{B_{2^{j-1}\sqrt{t}}(e_G)^c}\big|\big|^2_{L^2(G)} &\leq \sup_{|z_1|\geq 2^{j-1}\sqrt{t}} \big|X_\beta \widetilde{X}_{\beta'} p_t(z_1)\big| \left(\int_G \big|X_\beta \widetilde{X}_{\beta'} p_t(z)\big| \,\mathrm{d}z\right) \\
&\leq \sqrt{t}^{\,-l-|\beta|-|\beta'|}\, e^{-\frac{2^{2(j-1)}}{C}} \int_G \sqrt{t}^{\,-l-|\beta|-|\beta'|}\, e^{-\frac{|z|^2}{Ct}} \,\mathrm{d}z \\
&\leq \sqrt{t}^{\,-l-|\beta|-|\beta'|}\, e^{-\frac{2^{2(j-1)}}{C}}\, \sqrt{t}^{\,-l-|\beta|-|\beta'|}\, V(\sqrt{t}) \\
&= \sqrt{t}^{\,-2l-2|\beta|-2|\beta'|}\, V(\sqrt{t})\, e^{-\frac{2^{2(j-1)}}{C}},
\end{aligned}$$

by (3.8.15). Thus, by (3.1.22), we have obtained

$$\big|\big|\big(X_\beta \widetilde{X}_{\beta'} p_t\big)\chi_{B_{2^{j-1}\sqrt{t}}(e_G)^c}\big|\big|^2_{L^2(G)} \lesssim t^{-\frac{l}{2}-|\beta|-|\beta'|}\, e^{-\frac{2^{2(j-1)}}{C}},$$

and so

$$\big|\big|\big(X_\beta \widetilde{X}_{\beta'} p_t\big)\chi_{B_{2^{j-1}\sqrt{t}}(e_G)^c}\big|\big|_{L^2(G)} \lesssim \sqrt{t}^{\,-\frac{l}{2}-|\beta|-|\beta'|}\, e^{-\frac{2^{2(j-1)}}{2C}}. \tag{3.8.33}$$

Hence, combining (3.8.31) and (3.8.33) with (3.8.30), we obtain

$$\big|\big|M^{(2)}_{t,j}\big|\big|_{L^2(A_{t,j})} \lesssim ||f||_\infty\, \sqrt{t}^{\,-\frac{l}{2}-|\beta|-|\beta'|}\, e^{-\frac{2^{2(j-1)}}{2C}},$$

which is the desired result. □

Applying the estimate (3.8.29) to (3.8.28) means that so far, we have proved that there exists a constant $C > 0$ and a constant $C_q > 0$, which depends on q, such that

$$\int_{A_{t,j}} \left| q(z)\, M^{(2)}_{t,j}(z) \right| \,\mathrm{d}z \;\le\; C_q \left(2^{j+1}\sqrt{t}\right)^{a+\frac{l}{2}} ||f||_\infty \left(\sqrt{t}\right)^{-\frac{l}{2}-|\beta|-|\beta'|} e^{-\frac{2^{2(j-1)}}{C}}$$

$$\le\; C_q\, ||f||_\infty \sqrt{t}^{\,a-|\beta|-|\beta'|}\, 2^{(j+1)(a+l/2)}\, e^{-\frac{2^{2(j-1)}}{C}}. \qquad (3.8.34)$$

Using the ratio test, for example. it is not difficult to show that,

$$\sum_{j=0}^{\infty} 2^{(j+1)(a+l/2)}\, e^{-\frac{2^{2(j-1)}}{C}} \;<\; +\infty,$$

and thus,

$$\sum_{j=0}^{\infty} \int_{A_{t,j}} \left| q(z)\, M^{(2)}_{t,j}(z) \right| \,\mathrm{d}z \;\lesssim\; C_q \left|\left|f\right|\right|_\infty \sqrt{t}^{\,a-|\beta|-|\beta'|}. \qquad (3.8.35)$$

Step 4b

Recall that we have fixed $t \in (0,1)$. The final step of the proof is to find an estimate for $\left|\left|M^{(1)}_{t,j}\right|\right|_{L^2(A_{t,j})}$, for each $j \in \mathbb{N}_0$. Recall that $f \in \mathcal{C}^d([0,\infty))$, with $d \ge 2$ to be determined. Then, by construction (see the proof of Proposition 3.8.3), the function h_t given by (3.8.8) belongs to $\mathcal{C}^d([0,\infty))$ and its Fourier transform is well-defined. Therefore,

$$h_t(\mu) \;=\; \frac{1}{2\pi} \int_{\mathbb{R}} \cos(s\mu)\, \widehat{h}_t(s)\,\mathrm{d}s, \qquad \forall\, \mu \in \mathbb{R},$$

and the integral is finite for every $\mu \in \mathbb{R}$. The spectral theory then implies that

$$h_t(\sqrt{\mathcal{L}}) \;=\; \frac{1}{2\pi} \int_{\mathbb{R}} \cos(s\sqrt{\mathcal{L}})\, \widehat{h}_t(s)\,\mathrm{d}s,$$

and hence,

$$M^{(1)}_{t,j}(z) \;=\; \frac{1}{2\pi} \int_{\mathbb{R}} \cos(s\sqrt{\mathcal{L}}) \big\{ \big(X_\beta \widetilde{X}_{\beta'} p_t \big) \chi_{B_{2^{j-1}\sqrt{t}}(e_G)} \big\}(z)\, \widehat{h}_t(s)\,\mathrm{d}s. \qquad (3.8.36)$$

In Melrose [33] (see Section 3) it is shown that

$$\mathrm{supp}\big(\cos(s\sqrt{\mathcal{L}})\delta_{e_G}\big) \;\subset\; B_{|s|}(e_G), \qquad \forall\, s \in \mathbb{R}.$$

So, for $z \in A_{t,j}$ and $s \in \mathbb{R}$, with $|s| \le 2^{j-1}\sqrt{t}$, we have

$$\cos(s\sqrt{\mathcal{L}})\big\{\big(X_\beta \widetilde{X}_{\beta'} p_t\big)\chi_{B_{2j-1}\sqrt{t}(e_G)}\big\}(z) \;=\; 0.$$

Now, let $g \in \mathcal{S}(\mathbb{R})$ be an even function such that its Euclidean Fourier transform $\widehat{g} \in \mathcal{D}(\mathbb{R})$ and

$$\widehat{g}(\xi) \;=\; \begin{cases} 1, & \text{for } \xi \in [-1/2\,,\,1/2] \\ 0, & \text{for } \xi \in (-\infty, 1] \cup [1, \infty). \end{cases}$$

Furthermore, consider the function

$$g_\delta \;:=\; \delta^{-1} g(\delta^{-1}\cdot), \qquad \text{for } \delta > 0.$$

We now prove the following result regarding g:

Lemma 3.8.8. *Let $d \in \mathbb{N}$ and suppose that $h \in \mathcal{S}'(\mathbb{R})$ such that $h \in \mathcal{C}^d(\mathbb{R})$, with*

$$\big|\big|\partial^d h\big|\big|_\infty \;=\; \sup_{s_1 \in \mathbb{R}} \left| \left[\frac{\partial^d}{\partial s^d}\, h(s)\right]_{s=s_1} \right| \;<\; +\infty.$$

Then, we have

$$\big|\big|h - h * g_\delta\big|\big|_\infty \;\leq\; \frac{\delta^d}{d!}\, \big|\big|\partial^d h\big|\big|_\infty \int_{\mathbb{R}} |y|^d\, |g(y)|\; dy, \tag{3.8.37}$$

for every $\delta > 0$.

Proof. Observe that, by the construction of g, we have

$$\int_{\mathbb{R}} g(x)\;\mathrm{d}x \;=\; \widehat{g}(0) \;=\; 1,$$

and moreover, for every $j \in \mathbb{N}$,

$$\int_{\mathbb{R}} x^j\, g(x)\;\mathrm{d}x \;=\; 0.$$

Using Taylor's Theorem on h we obtain

$$\begin{aligned} h * g_\delta(x) \;&=\; \int_{\mathbb{R}} h(x + \delta y)\, g(y)\;\mathrm{d}y \\ &=\; \int_{\mathbb{R}} \left(\sum_{j=0}^{d-1} \frac{h^{(j)}(x)}{j!}\,(\delta y)^j \;+\; R_d(x, \delta y) \right) g(y)\;\mathrm{d}y. \end{aligned}$$

Since

$$\sum_{j=0}^{d-1} \int_{\mathbb{R}} \frac{h^{(j)}(x)}{j!}\,(\delta y)^j\, g(y)\,\mathrm{d}y \;=\; h(x)\int_{\mathbb{R}} g(y)\,\mathrm{d}y \,+\, \sum_{j=1}^{d-1} \delta^j\,\frac{h^{(j)}(x)}{j!}\int_{\mathbb{R}} y^j\, g(y)\,\mathrm{d}y,$$

then

$$h * g_\delta(x) \;=\; h(x) \,+\, \int_{\mathbb{R}} R_d(x, \delta y)\, g(y)\,\mathrm{d}y.$$

By Taylor's Theorem, the remainder satisfies the following estimate:

$$|R_d(x, \delta y)| \;\le\; \frac{|\delta y|^d}{d!}\,\big|\big|\partial^d h\big|\big|_\infty,$$

and so,

$$\begin{aligned} \big|\big|h * g_\delta - h\big|\big|_\infty &\le \sup_{x \in \mathbb{R}} \int_{\mathbb{R}} |R_d(x, \delta y)\, g(y)|\,\mathrm{d}y \\ &\le \big|\big|\partial^d h\big|\big|_\infty \int_{\mathbb{R}} \frac{|\delta y|^d}{d!}\,|g(y)|\,\mathrm{d}y, \end{aligned}$$

which yields the result. □

Furthermore, we also have the following result associated to $h_t \in \mathcal{C}^d([0,\infty))$ $(t > 0)$, the function given by (3.8.8).

Lemma 3.8.9. *For any* $t > 0$ *,*

$$\big|\big|\partial^d\, h_t\big|\big|_\infty \;=\; t^{d/2}\,\big|\big|\partial^d\, h_1\big|\big|_\infty. \tag{3.8.38}$$

Proof. First observe that for any $t > 0$,

$$\begin{aligned} \sup_{\mu \in \mathbb{R}} \left|\frac{\partial^d}{\partial \mu^d}\, h_t(\mu)\right| &= \sup_{\mu \in \mathbb{R}} \left|\frac{\partial^d}{\partial \mu^d}\left\{e^{t\mu^2}\, f(t\mu^2)\right\}\right| \\ &= \sup_{\mu \in \mathbb{R}} \left|\sum_{j=0}^{d} \binom{d}{j}\left\{\frac{\partial^j}{\partial \mu^j}\, e^{t\mu^2}\right\}\left\{\frac{\partial^{d-j}}{\partial \mu^{d-j}} f(t\mu^2)\right\}\right|. \end{aligned}$$

Since $\mathrm{supp}(f) \subset [0,2]$, then

$$\sup_{\mu\in\mathbb{R}}\left|\frac{\partial^d}{\partial\mu^d}\,h_t(\mu)\right| = \sup_{\mu\le\sqrt{2/t}}\left|\sum_{j=0}^{d}\binom{d}{j}\left\{\frac{\partial^j}{\partial\mu^j}\,e^{t\mu^2}\right\}\left\{\frac{\partial^{d-j}}{\partial\mu^{d-j}}f(t\mu^2)\right\}\right|$$

$$= \sup_{\mu\le\sqrt{2/t}}\left|\sum_{j=0}^{d}\binom{d}{j}\left\{(2t\mu)^j e^{t\mu^2}\right\}\left\{(2t\mu)^{d-j}\left(\frac{\partial^{d-j}}{\partial\mu^{d-j}}\,f\right)(t\mu^2)\right\}\right|$$

$$= (2\sqrt{2})^d\,t^{d/2}\sup_{\mu\le\sqrt{2/t}}\left|\sum_{j=0}^{d}\binom{d}{j}\,e^2\,\left(\frac{\partial^{d-j}}{\partial\mu^{d-j}}\,f\right)(t\mu^2)\right|.$$

Hence, we deduce that

$$\sup_{\mu\in\mathbb{R}}\left|\frac{\partial^d}{\partial\mu^d}\,h_t(\mu)\right| = (2\sqrt{2})^d\,t^{d/2}\sup_{\mu\le 2}\left|\sum_{j=0}^{d}\binom{d}{j}\,e^2\,\frac{\partial^{d-j}}{\partial\mu^{d-j}}\,f(\mu)\right|. \qquad (3.8.39)$$

Similarly, we compute

$$\sup_{\mu\in\mathbb{R}}\left|\frac{\partial^d}{\partial\mu^d}\,h_1(\mu)\right| = \sup_{\mu\le\sqrt{2}}\left|\sum_{j=0}^{d}\binom{d}{j}\left\{\frac{\partial^j}{\partial\mu^j}\,e^{\mu^2}\right\}\left\{\frac{\partial^{d-j}}{\partial\mu^{d-j}}f(\mu^2)\right\}\right|$$

$$= (2\sqrt{2})^d\sup_{\mu\le\sqrt{2}}\left|\sum_{j=0}^{d}\binom{d}{j}\,e^2\,\left(\frac{\partial^{d-j}}{\partial\mu^{d-j}}\,f\right)(\mu^2)\right|.$$

Hence, by (3.8.39), we have

$$\sup_{\mu\in\mathbb{R}}\left|\frac{\partial^d}{\partial\mu^d}\,h_t(\mu)\right| = t^{d/2}\sup_{\mu\in\mathbb{R}}\left|\frac{\partial^d}{\partial\mu^d}\,h_1(\mu)\right|,$$

as required.

□

We are now in a position to prove the following estimate for $||M^{(1)}_{t,j}||_{L^2(A_{t,j})}$, for each $j\in\mathbb{N}_0$.

Proposition 3.8.10. *There exists $C>0$ such that*

$$||M^{(1)}_{t,j}||_{L^2(A_{t,j})} \le C\left(2^{j-1}\sqrt{t}\right)^{-d} t^{d/2}\max_{0\le j\le d}||\partial^j f||_\infty\,\gamma_0^{j/2}\,\sqrt{t}^{\,-\frac{l}{2}-|\beta|-|\beta'|}, \quad (3.8.40)$$

for every $j\in\mathbb{N}_0$, where

$$\gamma_0 := \sup_{r>0} \frac{V(2r)}{V(r)} < +\infty,$$

and where $d \geq 2$ *denotes the non-negative integer to be determined such that* $f \in \mathcal{C}^d([0,\infty))$.

Proof. Since

$$\operatorname{supp}\Big(\widehat{g}_{(2^{j-1}\sqrt{t})^{-1}}\Big) \subset \big[-2^{j-1}\sqrt{t}\,,\, 2^{j-1}\sqrt{t}\big],$$

then for every $z \in A_{t,j}$ we have

$$\int_{\mathbb{R}} \cos(s\sqrt{\mathcal{L}})\big\{\big(X_\beta \widetilde{X}_{\beta'} p_t\big)\chi_{B_{2^{j-1}\sqrt{t}}}\big\}(z)\, \widehat{h}_t(s)\, \widehat{g}_{(2^{j-1}\sqrt{t})^{-1}}(s)\, \mathrm{d}s = 0.$$

Since g is an even function, then (3.8.36) and the Fourier inversion theorem imply that $M^{(1)}_{t,j}(z)$ is exactly equal to

$$\begin{aligned}\frac{1}{2\pi}\int_{\mathbb{R}} &\cos(s\sqrt{\mathcal{L}})\big\{\big(X_\beta \widetilde{X}_{\beta'} p_t\big)\chi_{B_{2^{j-1}\sqrt{t}}(e_G)}\big\}(z)\, \Big(\widehat{h}_t(s) - \widehat{h}_t(s)\, \widehat{g}_{(2^{j-1}\sqrt{t})^{-1}}(s)\Big)\, \mathrm{d}s \\ &= \big(h_t - h_t * g_{(2^{j-1}\sqrt{t})^{-1}}\big)(\sqrt{\mathcal{L}})\,\big\{\big(X_\beta \widetilde{X}_{\beta'} p_t\big)\chi_{B_{2^{j-1}\sqrt{t}}(e_G)}\big\}(z),\end{aligned}$$

for every $z \in G$. Applying L^2 norms yields the estimate:

$$\begin{aligned}\big|\big|M^{(1)}_{t,j}\big|\big|_{L^2(A_{t,j})} &\leq \big|\big|\big(h_t - h_t * g_{(2^{j-1}\sqrt{t})^{-1}}\big)(\sqrt{\mathcal{L}})\,\big\{\big(X_\beta \widetilde{X}_{\beta'} p_t\big)\chi_{B_{2^{j-1}\sqrt{t}}(e_G)}\big\}\big|\big|_{L^2(G)} \\ &\leq \big|\big|h_t - h_t * g_{(2^{j-1}\sqrt{t})^{-1}}\big|\big|_\infty \big|\big|\big(X_\beta \widetilde{X}_{\beta'} p_t\big)\chi_{B_{2^{j-1}\sqrt{t}}(e_G)}\big|\big|_{L^2(G)},\end{aligned} \tag{3.8.41}$$

by the spectral theory. Since $h_t \in \mathcal{C}^d([0,\infty))$, then by Lemma 3.8.8, we obtain the estimate:

$$\big|\big|h_t - h_t * g_{(2^{j-1}\sqrt{t})^{-1}}\big|\big|_\infty \lesssim \big(2^{j-1}\sqrt{t}\big)^{-d}\, \big|\big|\partial^d h_t\big|\big|_\infty.$$

Then, Lemma 3.8.9 implies that

$$\big|\big|h_t - h_t * g_{(2^{j-1}\sqrt{t})^{-1}}\big|\big|_\infty \lesssim \big(2^{j-1}\sqrt{t}\big)^{-d}\, t^{d/2} \max_{0\leq j\leq d} \big|\big|\partial^j f\big|\big|_\infty. \tag{3.8.42}$$

On the other hand, now observe that by (3.8.5), we obtain

$$
\begin{aligned}
\left|\left|\left(X_\beta \widetilde{X}_{\beta'} p_t\right)\chi_{B_{2^{j-1}\sqrt{t}}(e_G)}\right|\right|^2_{L^2(G)} &= \int_{B_{2^{j-1}\sqrt{t}}(e_G)} \left|X_\beta \widetilde{X}_{\beta'} p_t(z)\right|^2 \mathrm{d}z \\
&\leq \int_{B_{2^{j-1}\sqrt{t}}(e_G)} \left|\sqrt{t}^{\,-2(l+|\beta|+|\beta'|)}\, e^{-\frac{2|z|^2}{Ct}}\right| \mathrm{d}z \\
&\leq \sqrt{t}^{\,-2(l+|\beta|+|\beta'|)}\, V(2^{j-1}\sqrt{t}) \sup_{|z|\leq 2^{j-1}\sqrt{t}} \left(e^{-\frac{2|z|^2}{Ct}}\right),
\end{aligned}
$$

by the definition of the volume. Since

$$
\sup_{|z|\leq 2^{j-1}\sqrt{t}} \left(e^{-\frac{2|z|^2}{Ct}}\right) = 1, \quad \forall\, j \in \mathbb{N}_0,
$$

then we have shown that

$$
\left|\left|\left(X_\beta \widetilde{X}_{\beta'} p_t\right)\chi_{B_{2^{j-1}\sqrt{t}}(e_G)}\right|\right|_{L^2(G)} \leq \sqrt{t}^{\,-l-|\beta|-|\beta'|}\, V(2^{j-1}\sqrt{t})^{1/2}. \tag{3.8.43}
$$

Now, note that

$$
\begin{aligned}
\sqrt{t}^{\,-l}\, V(2^{j-1}\sqrt{t})^{1/2} &\approx \sqrt{t}^{\,-l} \left(2^{j-1}\sqrt{t}\right)^{l/2} = 2^{\frac{l}{2}(j-1)} = 2^{-\frac{l}{2}} \left(2\sqrt{t}\right)^{j\frac{l}{2}} \sqrt{t}^{\,-j\frac{l}{2}} \\
&\lesssim \left(\frac{V(2\sqrt{t})}{V(\sqrt{t})}\right)^{j/2}.
\end{aligned}
$$

Thus, we have shown that

$$
\left|\left|\left(X_\beta \widetilde{X}_{\beta'} p_t\right)\chi_{B_{2^{j-1}\sqrt{t}}(e_G)}\right|\right|_{L^2(G)} \lesssim \sqrt{t}^{\,-\frac{l}{2}-|\beta|-|\beta'|}\, \gamma_0^{j/2}, \tag{3.8.44}
$$

where

$$
\gamma_0 = \sup_{r>0} \frac{V(2r)}{V(r)} < +\infty,
$$

by (3.1.22). Hence, by substituting (3.8.42) and (3.8.44) into (3.8.41), we have the estimate required. □

Proposition 3.8.10 and (3.8.28) then imply that there exists a constant $C_q > 0$ depending on q such that

$$\int_{A_{t,j}} \left|q(z)\, M_{t,j}^{(1)}(z)\right| \mathrm{d}z$$
$$\lesssim C_q \left(2^{j+1}\sqrt{t}\right)^{a+\frac{l}{2}} \left(2^{j-1}\sqrt{t}\right)^{-d} t^{d/2} \max_{0\le j\le d} \left|\left|\partial^j f\right|\right|_\infty \gamma_0^{j/2} \sqrt{t}^{\,-\frac{l}{2}-|\beta|-|\beta'|}$$
$$\lesssim C_q \max_{0\le j\le d} \left|\left|\partial^j f\right|\right|_\infty 2^{j\left(a+\frac{l}{2}-d+\frac{1}{2}\log_2(\gamma_0)\right)} \sqrt{t}^{\,a-|\beta|-|\beta'|},$$

since

$$\left(2^{j+1}\sqrt{t}\right)^{a+\frac{l}{2}} \left(2^{j-1}\sqrt{t}\right)^{-d} t^{d/2} \gamma_0^{j/2} \sqrt{t}^{\,-\frac{l}{2}-|\beta|-|\beta'|}$$
$$= 2^{j(a+\frac{l}{2}-d)}\, 2^{a+\frac{l}{2}+d} \sqrt{t}^{\,a-|\beta|-|\beta'|} \gamma_0^{j/2}$$
$$= 2^{j(a+\frac{l}{2}-d)}\, 2^{a+\frac{l}{2}+d}\, 2^{\log_2\left(\gamma_0^{j/2}\right)} \sqrt{t}^{\,a-|\beta|-|\beta'|}$$
$$\lesssim 2^{j\left(a+\frac{l}{2}-d+\frac{1}{2}\log_2(\gamma_0)\right)} \sqrt{t}^{\,a-|\beta|-|\beta'|}.$$

Hence, we choose d to be the smallest positive integer such that

$$d > a + \frac{l}{2} + \frac{1}{2}\log_2(\gamma_0),$$

so that the sum

$$\sum_{j=0}^{\infty} 2^{j\left(a+\frac{l}{2}-d+\frac{1}{2}\log_2(\gamma_0)\right)} < +\infty.$$

Thus, we obtain

$$\sum_{j=0}^{\infty} \int_{A_{t,j}} \left|q(z)\, M_{t,j}^{(1)}(z)\right| \mathrm{d}z \lesssim C_q \max_{0\le j\le d} \left|\left|\partial^j f\right|\right|_\infty \sqrt{t}^{\,a-|\beta|-|\beta'|}. \tag{3.8.45}$$

Applying (3.8.35) and (3.8.45) to (3.8.26) yields

$$\int_{B_{\sqrt{t}}(e_G)^c} \left|q(z)\, X_\beta \widetilde{X}_{\beta'}\left\{f(t\mathcal{L})\delta_{e_G}\right\}(z)\right| \mathrm{d}z \lesssim C_q \left|\left|f\right|\right|_{\mathcal{C}^d([0,2])} \sqrt{t}^{\,a-|\beta|-|\beta'|}, \tag{3.8.46}$$

which is exactly (3.8.25), and thus the proof of Lemma 3.8.1 is finished.

□

Chapter 4

Pseudo-differential calculus on compact Lie groups

Pseudo-differential operators have been studied extensively in the literature and are generally well understood, especially in the Euclidean cases. From the point of view of harmonic analysis, one can find, for example, a study of the symbolic calculus in the case of $\mathbb{R}^n$ in Chapter VI of Stein [47]. For applications of the pseudo-differential theory to PDEs in the Euclidean case, see for example the monograph Taylor [51]. A more recent result can be found, for instance, in Fischer and Ruzhansky [18], wherein the authors analysed the pseudo-differential theory on nilpotent Lie groups. The case of compact Lie groups, which is the main focus of this thesis, has also been studied in the past, although until now, the elliptic setting has been the central focus of research. See, for example, Ruzhansky and Turunen [43] for the case of the torus, or Ruzhansky et al [44] as well as Fischer [17] for the general case of any compact Lie group.

It is then natural to ask whether it is possible to define classes of pseudo-differential operators in a sub-elliptic setting without losing the important properties that can be found in the elliptic case. The aim of this chapter is thus to define symbol classes S^m and corresponding operator classes Ψ^m, using a sub-Laplacian, such that the space

$$\Psi := \bigcup_{m \in \mathbb{R}} \Psi^m$$

forms an pseudo-differential calculus. This means that Ψ is stable under taking the composition and the adjoint. In this chapter, we will prove the following result: Let $m_1, m_2 \in \mathbb{R}$. If $T_1 \in \Psi^{m_1}$ and $T_2 \in \Psi^{m_2}$, then their composition

$$T_1 \circ T_2 \in \Psi^{m_1+m_2},$$

and the mapping $(T_1, T_2) \mapsto T_1 \circ T_2$ is a continuous function $\Psi^{m_1} \times \Psi^{m_2} \to \Psi^{m_1+m_2}$.

The foundation of the ideas used in this thesis for the proof of this result stem from the classical Euclidean case, which can be found in Chapter VI in Stein [47]. Other inspirations for the work presented here include Fischer [17], which provides an intrinsic pseudo-differential calculus on any compact Lie group, and Fischer and Ruzhansky [18], which presents an adaptation of Stein's work to the case of nilpotent Lie groups.

Throughout this chapter, suppose G is a compact Lie group of dimension n and let $\mathfrak{g}$ be the Lie algebra of G. Further suppose that, for some $k \in \mathbb{N}$, the set $\boldsymbol{X} = \{X_1, X_2, \ldots, X_k\}$ forms a Hörmander system of left-invariant vector fields on G, and consider its associated sub-Laplacian

$$\mathcal{L} := -(X_1^2 + X_2^2 + \cdots + X_k^2).$$

4.1 Functions comparable to the C-C metric and difference operators

In this section we aim to introduce a way of comparing a family of functions to the Carnot-Carathéodory norm.

4.1.1 Definitions, vocabulary and notation

Definition 4.1.1. Let $\ell \in \mathbb{N}_0$ and suppose that $Q = \{q_1, q_2, \ldots, q_\ell\}$ is a family of smooth real-valued functions on G. We say that Q is comparable to the Carnot-Carathéodory metric (C-C metric, for short) if there exist $\omega = (\omega_1, \omega_2, \ldots, \omega_\ell) \in \mathbb{N}_0^\ell$ and constants $C, C' > 0$ such that

$$C \sum_{j=1}^{\ell} |q_j(z)|^{1/\omega_j} \leq |z| \leq C' \sum_{j=1}^{\ell} |q_j(z)|^{1/\omega_j}, \tag{4.1.1}$$

for all $z \in G$. In this case, we say that Q has weight ω.

Remark 4.1.2. Observe that, if Q is a family of smooth real-valued functions on G comparable to the C-C metric, then it follows that e_G is the only point in G where the functions q_j $(j = 1, 2, \ldots, \ell)$ vanish simultaneously. That is,

$$\bigcap_{j=1}\{z \in G : q_j(z) = 0\} = \{e_G\}.$$

We also introduce the following notations, which will help us shorten our future calculations considerably.

Notation 4.1.3. If $Q = \{q_1, q_2, \ldots, q_\ell\}$ is a family of smooth real-valued functions on G with weight $\omega = (\omega_1, \omega_2, \ldots, \omega_\ell) \in \mathbb{N}_0^\ell$, we will sometimes simplify (4.1.1) by writing

$$|z| \approx \sum_{j=1}^{\ell} |q_j(z)|^{1/\omega_j}, \quad z \in G. \tag{4.1.2}$$

Notation 4.1.4. Suppose that $Q := \{q_1, q_2, \ldots, q_\ell\}$ is a family of smooth real-valued functions on G with weight $\omega = (\omega_1, \omega_2, \ldots, \omega_\ell)$. Then, for $\beta \in \mathbb{N}_0^\ell$, we denote

$$[\beta]_Q := \sum_{j=1}^{\ell} \beta_j\, \omega_j. \tag{4.1.3}$$

Notation 4.1.5. Let $\ell \in \mathbb{N}$ and consider the family $Q = \{q_1, q_2, \ldots, q_\ell\}$ of smooth functions on G. For any $\alpha \in \mathbb{N}_0^\ell$, we denote

$$\begin{aligned} q_\alpha &:= q_1^{\alpha_1}\, q_2^{\alpha_2} \, \ldots \, q_\ell^{\alpha_\ell}, \\ \widetilde{q}_\alpha &:= q_\alpha(\cdot^{-1}). \end{aligned}$$

4.1.2 First properties

The following lemma illustrates a simple application of our new notation.

Lemma 4.1.6. *Let $\ell \in \mathbb{N}$ and consider the set $Q = \{q_1, q_2, \ldots, q_\ell\}$ of smooth functions on G. Suppose that Q has weight $\omega = (\omega_1, \omega_2, \ldots, \omega_\ell) \in \mathbb{N}^\ell$. Then, for each $j = 1, 2, \ldots, \ell$, the function q_j CC-vanishes at e_G up to order $\omega_j - 1$ (see Definition 3.7.1 and Notation 3.7.2) and, for any $\alpha \in \mathbb{N}_0^\ell$, the functions q_α and $\widetilde{q}_\alpha$ CC-vanish at e_G up to order $[\alpha]_Q - 1$.*

Proof. For each $j = 1, 2, \ldots, \ell$, the function q_j satisfies

$$|q_j(z)| \le \frac{1}{C^{\omega_j}} |z|^{\omega_j}, \qquad \forall\, z \in G. \tag{4.1.4}$$

This means that q_j CC-vanishes at e_G up to order $\omega_j - 1$.

Now, let $\alpha \in \mathbb{N}_0^\ell$. Applying (4.1.4), for any $z \in G$ we have

$$\begin{aligned} |q_\alpha(z)| &= \prod_{j=1}^{\ell} |q_j(z)|^{\alpha_j} \\ &\leq \prod_{j=1}^{\ell} \left(\frac{1}{C^{\omega_j}} |z|^{\omega_j} \right)^{\alpha_j} \\ &= \frac{1}{C^{[\alpha]_Q}} |z|^{[\alpha]_Q}. \end{aligned}$$

This means that the function q_α CC-vanishes at e_G up to order $[\alpha]_Q - 1$. The proof for $\widetilde{q}_\alpha$ is similar. $\square$

Lemma 4.1.7. *Suppose $Q = \{q_1, q_2, \ldots, q_\ell\}$ is a family of functions in $\mathcal{D}(G)$, with weight $\omega = (\omega_1, \omega_2, \ldots, \omega_\ell) \in \mathbb{N}^\ell$. Furthermore, let ω_0 be the lowest common multiple of the numbers $\{\omega_1, \omega_2, \ldots, \omega_\ell\}$. Then, for any $N' \in \mathbb{N}_0$, we have*

$$|z|^{N'\omega_0} \approx \sum_{[\alpha]_Q = N'\omega_0} |q_\alpha(z)|, \quad \forall\, z \in G. \tag{4.1.5}$$

Proof. First observe that, by Lemma 4.1.6, if $N' \in \mathbb{N}_0$, then for every $\alpha \in \mathbb{N}_0^\ell$, with $[\alpha]_Q = N'\omega_0$, we have

$$|q_\alpha(z)| \lesssim |z|^{N'\omega_0}, \quad \forall\, z \in G.$$

Hence, it follows that

$$\sum_{[\alpha]_Q = N'\omega_0} |q_\alpha(z)| \lesssim |z|^{N'\omega_0}, \quad \forall\, z \in G. \tag{4.1.6}$$

Let us now show the reverse inequality (up to a constant). By the equivalence of norms on $\mathbb{R}^\ell$, for any $N' \in \mathbb{N}$, we obtain

$$|z|^{N'\omega_0} \approx \left(\sum_{j=1}^{\ell} |q_j(z)|^{1/\omega_j} \right)^{N'\omega_0} \approx \left(\sum_{j=1}^{\ell} |q_j(z)|^{\omega_0/\omega_j} \right)^{N'}, \quad \forall\, z \in G. \tag{4.1.7}$$

Thus, using a multinomial expansion, we obtain

$$\left(\sum_{j=1}^{\ell} |q_j(z)|^{\omega_0/\omega_j} \right)^{N'} \approx \sum_{|\alpha| = N'} \prod_{j=1}^{\ell} |q_j(z)|^{\alpha_j \frac{\omega_0}{\omega_j}}, \quad \forall\, z \in G. \tag{4.1.8}$$

Observe that, by the definition of ω_0, for each $j = 1, 2, \ldots, \ell$, $\omega_0/\omega_j \in \mathbb{N}$. So, we do the following change of variables; let $\beta = (\beta_1, \beta_2, \ldots, \beta_\ell) \in \mathbb{N}_0^\ell$ be given by

$$\beta_j := \alpha_j \frac{\omega_0}{\omega_j} \in \mathbb{N}, \quad \forall\, j = 1, 2, \ldots, \ell.$$

We then have

$$|\alpha| = N' \iff \beta_1\omega_1 + \beta_2\omega_2 + \cdots + \beta_\ell\omega_\ell = N'\omega_0$$

By the definition of $[\,\cdot\,]_Q$ (see (4.1.3)), this is equivalent to

$$|\alpha| = N' \iff [\beta]_Q = N'\omega_0.$$

However, this only holds for $\beta \in \mathbb{N}_0^\ell$ of the form

$$\beta_j = \alpha_j \frac{\omega_0}{\omega_j}, \quad j = 1, 2, \ldots, \ell,$$

for some $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell) \in \mathbb{N}_0^\ell$. This means that

$$\sum_{|\alpha|=N'} \prod_{j=1}^{\ell} |q_j(z)|^{\alpha_j \frac{\omega_0}{\omega_j}} = \sum_{\substack{\beta_j \in \frac{\omega_0}{\omega_j}\mathbb{N}_0,\ j=1,2,\ldots,\ell \\ [\beta]_Q = N'\omega_0}} \prod_{j=1}^{\ell} |q_j(z)|^{\beta_j}.$$

But,

$$\frac{\omega_0}{\omega_j}\mathbb{N}_0 \subset \mathbb{N}_0$$

for each $j = 1, 2, \ldots, \ell$, thus we have

$$\sum_{|\alpha|=N'} \prod_{j=1}^{\ell} |q_j(z)|^{\alpha_j \frac{\omega_0}{\omega_j}} \leq \sum_{\substack{\beta \in \mathbb{N}_0^\ell \\ [\beta]_Q = N'\omega_0}} \prod_{j=1}^{\ell} |q_j(z)|^{\beta_j}, \quad \forall\, z \in G.$$

Hence, by (4.1.7) and (4.1.8), we have shown that

$$|z|^{N'\omega_0} \lesssim \sum_{[\alpha]_Q = N'\omega_0} |q_\alpha(z)|, \quad \forall\, z \in G. \tag{4.1.9}$$

Thus, combining (4.1.6) and (4.1.9), the result is proved. $\square$

4.1.3 Definition of difference operators

Definition 4.1.8. If q is a smooth, real-valued function on G, we define the difference operator Δ_q associated to q to be the operator acting on the space $\mathcal{F}_G\big(\mathcal{D}'(G)\big)$ given by

$$\Delta_q \widehat{f} = \widehat{qf}, \qquad f \in \mathcal{D}'(G).$$

We also introduce some useful notation.

Definition 4.1.9. Let $\ell \in \mathbb{N}$ and suppose $Q = \{q_1, q_2, \ldots, q_\ell\}$ is a family of smooth, real-valued functions on G. Furthermore, consider the collection of difference operators $\Delta_Q = \{\Delta_{q_1}, \Delta_{q_2}, \ldots, \Delta_{q_\ell}\}$ associated to Q. For a given $\alpha \in \mathbb{N}_0^\ell$, we denote

$$\Delta_Q^\alpha := \Delta_{\widetilde{q}_\alpha}.$$

Definition 4.1.10. Let $\ell \in \mathbb{N}$ and suppose $Q = \{q_1, q_2, \ldots, q_\ell\}$ is a family of smooth, real-valued functions on G. Furthermore, let $\omega = (\omega_1, \omega_2, \ldots, \omega_\ell) \in \mathbb{N}^\ell$. We shall say that the collection of difference operators Δ_Q associated to Q has weight ω if Q has weight ω.

4.2 An example of a family of functions on G comparable to the C-C metric

Here we consider an example of a family of functions Q which, as shown below in Proposition 4.2.2, is comparable to the C-C metric.

4.2.1 An important neighbourhood of e_G in G

We let

$$\boldsymbol{Y} := \{Y_1, Y_2, \ldots, Y_n\}$$

be the basis of $\mathfrak{g}$ constructed in Section 2.4.1. Recall that, for each $j = 1, 2, \ldots, n$, Y_j is written in the form

$$Y_j = X_{[I_j^{(s)}]} = \big[X_{i_1}, \big[X_{i_2}, \ldots, \big[X_{i_{s-1}}, X_{i_s}\big] \ldots\big]\big], \tag{4.2.1}$$

for some $I_j^{(s)} \in \mathcal{I}(k)$ (see (2.4.1)). Furthermore, recall that, for each $j = 1, 2, \ldots, n$, we denote d_j (see (2.4.3)) by

$$d_j = \left|I_j^{(s)}\right|, \tag{4.2.2}$$

and δ (see (2.4.4)) is the constant

$$\delta = \max\{d_1, d_2, \ldots, d_n\}. \tag{4.2.3}$$

Observe that there exist a neighbourhood V of e_G in G and a neighbourhood N of 0 in $\mathbb{R}^n$ such that the mapping $\phi : N \to V$, which is given by

$$\phi((z_1, z_2, \ldots, z_n)) := e^{z_1 Y_1 + z_2 Y_2 + \cdots z_n Y_n}, \tag{4.2.4}$$

is a diffeomorphism (see Proposition 2.3.9 (v)). For $z \in V$ we then let

$$(z_1, z_2, \ldots, z_n) \in N \subset \mathbb{R}^n$$

denote the coordinates of z given by the coordinate chart (ϕ^{-1}, V). We also know, by the ball-box theorem, (see Section 2.4 in Montgomery [35], Section 0.5.A in Gromov [24] and Section 2.4.2 in this thesis) that there exist constants $\varepsilon_0, C, C' > 0$ such that

$$C\phi\left(\text{Box}(\varepsilon)\right) \subset B_\varepsilon(e_G) \subset C'\phi\left(\text{Box}(\varepsilon)\right), \tag{4.2.5}$$

for all $\varepsilon \leq \varepsilon_0$, where for each $\varepsilon > 0$ we denote

$$\text{Box}(\varepsilon) = \left\{x \in \mathbb{R}^n : |x_i| \leq \varepsilon^{d_i}, \ \forall\, i = 1, 2, \ldots, n\right\}. \tag{4.2.6}$$

In particular, we can choose V and N small enough such that the following properties are satisfied:

(a) $V \subset B_{\varepsilon_0}(e_G)$; that is, V satisfies (4.2.5).

(b) The mapping $\phi : N \to V$ given by (4.2.4) is a diffeomorphism.

(c) Any $(z_1, z_2, \ldots, z_n) \in N$ satisfies

$$||(z_1, z_2, \ldots, z_n)||_{\mathbb{R}^n} \leq 1.$$

By these properties, for any $z \in V$ we have

$$C\left(|z_1|^{1/d_1}+|z_2|^{1/d_2}+\cdots+|z_n|^{1/d_n}\right)\;\le\;|z|$$
$$\le\; C'\left(|z_1|^{1/d_1}+|z_2|^{1/d_2}+\cdots+|z_n|^{1/d_n}\right). \quad (4.2.7)$$

Next, observe that there exists $r\in(0,1]$ such that $B_r(e_G)$, the ball of radius r centred at e_G, with respect to the Carnot-Carathéodory metric, is strictly contained in V; that is,

$$B_r(e_G)\;\subsetneq\;V. \quad (4.2.8)$$

4.2.2 Construction of the example

We continue with the same setting as in Section 4.2.1. Our objective is to define a family of functions $Q_0=\{q_{0,j}\,:\,j=1,2,\ldots,n\}$ such that, for each $j=1,2,\ldots,n$, we have

$$\begin{cases} q_{0,j}(z)=z_j, & \text{near the identity}\\ q_{0,j}(z)=1, & \text{far away from the identity}\end{cases},$$

and moreover

$$\bigcap_{j=1}^{n}\left\{z\in G\,:\,q_{0,j}(z)=0\right\}\;=\;\{e_G\}.$$

In other words, the only point at which all functions in Q_0 vanish simultaneously is the identity, e_G. In order to achieve this, we proceed in the following way.

We first consider the case $n=\dim G=1$. In this case, G is isomorphic to the one dimensional torus, $\mathbb{T}$. So, we may assume that $G=\mathbb{T}=\mathbb{R}/\pi\mathbb{Z}$. Moreover, the torus may be identified with one of its fundamental domains; we choose the interval $\left[-\frac{\pi}{2},\frac{\pi}{2}\right)$. In this setting, the Carnot-Carathéodory metric is equivalent to the Euclidean metric and additionally, the map ϕ (see (4.2.4)) is the natural identification between elements of the torus and $\left[-\frac{\pi}{2},\frac{\pi}{2}\right)$. We then take $N=\left[-\frac{\pi}{2},\frac{\pi}{2}\right)\cong V$.

To fix the ideas, we now let

$$r\;=\;\frac{\pi}{4},$$

so that (4.2.8) is satisfied, and furthermore,

$$r_1 \;=\; \frac{r}{10},$$

and

$$r_2 \;=\; 4r_1.$$

For $j = 1, 2$, we then let $\chi_j, \psi_j \in \mathcal{D}(\mathbb{T})$, taking values in $[0, 1]$, be such that

$$\chi_j(t) \equiv 1 \quad \text{on} \quad (-r_j, r_j), \qquad\qquad \chi_j(t) \equiv 0 \quad \text{on} \quad \left[-\frac{\pi}{2}, -r\right] \cup \left[r, \frac{\pi}{2}\right),$$

and

$$\psi_j(t) \equiv 0 \quad \text{on} \quad \left(-\frac{r_j}{2}, \frac{r_j}{2}\right), \qquad\qquad \psi(t) \equiv 1 \quad \text{on} \quad \left[-\frac{\pi}{2}, -r_j\right] \cup \left[r_j, \frac{\pi}{2}\right),$$

Then, we define the functions $q_{0,1}, q_{0,2}$ by

$$q_{0,j}(t) = t\,\chi_j(z) + \psi_j(z) \qquad \text{for} \;\; j = 1, 2. \tag{4.2.9}$$

Hence, we define

$$Q_0 \;=\; \{q_{0,1}, q_{0,2}\}. \tag{4.2.10}$$

The following result follows from Proposition 4.2.2 below.

Proposition 4.2.1. *Suppose G is a compact Lie group of dimension 1. The set Q_0 given by (4.2.10) is comparable to the C-C metric with weight (d_1, d_2).*

Let us now consider the case $n = \dim G > 1$. We first let

$$r_1 \;:=\; \frac{r}{4^{n-1}},$$

and for each $j = 2, 3, \ldots, n$ we define

$$r_j \;=\; 4^{j-1}\, r_1.$$

Observe that

$$B_{r_1}(e_G) \;\subsetneq\; B_{r_2}(e_G) \;\subsetneq\; \cdots \;\subsetneq\; B_{r_n}(e_G) \;=\; B_r(e_G) \;\subsetneq\; V. \tag{4.2.11}$$

Furthermore, let $\chi_j, \psi_j \in \mathcal{D}(G)$, taking values in $[0, 1]$, be such that

$$\chi_j(z) \equiv 1 \quad \text{on} \quad B_{r_j}(e_G), \qquad\qquad \chi_j(z) \equiv 0 \quad \text{on} \quad V^c,$$

and

$$\psi_j(z) \equiv 0 \quad \text{on} \quad B_{r_j/2}(e_G), \qquad\qquad \psi_j(z) \equiv 1 \quad \text{on} \quad B_{r_j}(e_G)^c.$$

Then, we define

$$q_{0,j}(z) = z_j\, \chi_j(z) + \psi_j(z) \quad \text{ for } \; j = 1, 2, \ldots, n. \tag{4.2.12}$$

For each $j = 1, 2, \ldots, n$, we can also write $q_{0,j}$ in the following way:

$$q_{0,j}(z) \;=\; \begin{cases} z_j, & \text{if } \; z \,\in\, B_{r_j/2}(e_G) \\ z_j + \psi_j(z), & \text{if } \; z \,\in\, B_{r_j}(e_G) \backslash B_{r_j/2}(e_G) \\ z_j \chi_j(z) + 1, & \text{if } \; z \,\in\, V \backslash B_{r_j}(e_G) \\ 1, & \text{if } \; z \,\in\, V^c. \end{cases} \tag{4.2.13}$$

We now define the family of functions

$$Q_0 \;:=\; \{q_{0,1}, q_{0,2}, \ldots, q_{0,n}\}. \tag{4.2.14}$$

For any $\alpha \in \mathbb{N}_0^n$, we shall denote by $q_{0,\alpha}$ the mapping given by

$$q_{0,\alpha}(z) \;=\; q_{0,1}(z)^{\alpha_1} q_{0,2}(z)^{\alpha_2} \cdots q_{0,n}(z)^{\alpha_n}, \qquad \forall\; z \,\in\, G. \tag{4.2.15}$$

Additionally, we let $\widetilde{q}_{0,\alpha}$ be the function defined by

$$\widetilde{q}_{0,\alpha}(z) \;=\; q_{0,\alpha}(z^{-1}), \qquad \forall\; z \,\in\, G.$$

Next, we prove that the family functions Q_0 is comparable to the C-C metric with weight $(d_1, d_2, \ldots, d_n)$.

Proposition 4.2.2. *Suppose G is a compact Le group of dimension $n > 1$. The family Q_0 of smooth, real-valued functions on G given by* (4.2.14) *is comparable to the C-C metric with weight $(d_1, d_2, \ldots, d_n)$.*

Proof. First observe that, by (4.2.7), there exist $C_1, C_1' > 0$ such that

$$C_1 \sum_{j=1}^{n} |z_j|^{1/d_j} \;\leq\; |z| \;\leq\; C_1' \sum_{j=1}^{n} |z_j|^{1/d_j}, \quad \forall\; z \,\in\, V. \tag{4.2.16}$$

Moreover, from (4.2.13) and the inclusion given by (4.2.11), it is clear that for each $j = 1, 2, \ldots, n$ we have

$$q_{0,j}(z) = z_j, \quad \forall\, z \in B_{r_1/2}(e_G).$$

Hence, it follows that

$$C_1 \sum_{j=1}^{n} |q_{0,j}(z)|^{1/d_j} \le |z| \le C_1' \sum_{j=1}^{n} |q_{0,j}(z)|^{1/d_j}, \quad \forall\, z \in B_{r_1/2}(e_G).$$

We now consider the annulus $B_{r_2/2}(e_G) \backslash B_{r_1/2}(e_G)$. Observe that, for any $z \in B_{r_2/2}(e_G) \backslash B_{r_1/2}(e_G)$,

$$\begin{aligned} q_{0,1}(z) &= z_1\, \chi_1(z) + \psi_1(z), \\ q_{0,j}(z) &= z_j, \qquad\qquad \forall\, j = 2, 3, \ldots, n. \end{aligned}$$

By (4.2.16), it follows that there exist constants $C_{2,0}, C_{2,0}' > 0$ such that, for all $z \in B_{r_2/2}(e_G) \backslash B_{r_1/2}(e_G)$,

$$C_{2,0} \sum_{j=2}^{n} |q_{0,j}(z)|^{1/d_j} \le |z| \le C_{2,0}' \sum_{j=2}^{n} |q_{0,j}(z)|^{1/d_j}.$$

It remains to check $q_{0,1}$. There exists $c_2 > 0$, only depending on n and r, such that

$$\begin{aligned} c_2 \le \chi_1(z) \le 1, &\qquad \forall\, z \in B_{r_2/2}(e_G) \backslash B_{r_1/2}(e_G), \\ 0 \le \psi_1(z) \le 1, &\qquad \forall\, z \in B_{r_2/2}(e_G) \backslash B_{r_1/2}(e_G). \end{aligned}$$

So, we have

$$c_2\, |z_1| \le |q_{0,1}(z)| \le |z_1| + 1, \qquad \forall\, z \in B_{r_2/2}(e_G) \backslash B_{r_1/2}(e_G),$$

and in particular, there exist constants $C_{2,1}, C_{2,1}' > 0$ such that

$$C_{2,1}\, |q_{0,1}(z)|^{1/d_1} \le |z| \le C_{2,1}'\, |q_{0,1}(z)|^{1/d_1}, \qquad \forall\, z \in B_{r_2/2}(e_G) \backslash B_{r_1/2}(e_G).$$

Hence, there exist $C_2, C_2' > 0$ such that, for all $z \in B_{r_2/2}(e_G) \backslash B_{r_1/2}(e_G)$,

$$C_2 \sum_{j=1}^{n} |q_{0,j}(z)|^{1/d_j} \;\le\; |z| \;\le\; C_2' \sum_{j=1}^{n} |q_{0,j}(z)|^{1/d_j},$$

Continuing in this way successively, we deduce that there exist constants $C_r, C_r' > 0$, depending on r, such that, for all $z \in B_{r/2}(e_G)$,

$$C_r \sum_{j=1}^{n} |q_{0,j}(z)|^{1/d_j} \;\le\; |z| \;\le\; C_r' \sum_{j=1}^{n} |q_{0,j}(z)|^{1/d_j}. \tag{4.2.17}$$

Next, let us consider the space $V \backslash B_{r/2}(e_G)$. For each $j = 1, 2, \ldots, n$ we have

$$1 \;\le\; |q_{0,j}(z)| \;\le\; z_j + 1, \qquad \forall\, z \in V \backslash B_{r/2}(e_G),$$

which implies that there exist constants $C_{r,V}, C_{r,V}' > 0$, depending on the choices of r and V, such that

$$C_{r,V}\,|z| \;\le\; \sum_{j=1}^{n} |q_{0,j}(z)|^{1/d_j} \;\le\; C_{r,V}'\,|z|, \qquad \forall\, z \in V \backslash B_{r/2}(e_G). \tag{4.2.18}$$

The inequality given by (4.2.18) can be extended to $G \backslash B_{r/2}(e_G)$. Indeed, as we saw in (4.2.13), for each $j = 1, 2, \ldots, n$, we have

$$q_{0,j}(z) \;=\; 1, \quad \forall\, z \in G \backslash V.$$

So, we deduce that there exist constants $C_{r,G}, C_{r,G}' > 0$, depending on r and G, such that

$$C_{r,G} \;\le\; \sum_{j=1}^{n} |q_{0,j}(z)|^{1/d_j} \;\le\; C_{r,G}', \quad \forall\, z \in G \backslash B_{r/2}(e_G). \tag{4.2.19}$$

Observe also that

$$\frac{r}{2} \;\le\; |z| \;\le\; R, \quad \forall\, z \in G \backslash B_{r/2}(e_G), \tag{4.2.20}$$

where $R > 0$ is the radius of G:

$$R \;:=\; \sup_{z \in G} |z|.$$

Hence,

$$\frac{r}{2C'_{r,G}} \sum_{j=1}^{n} |q_{0,j}(z)|^{1/d_j} \;\le\; |z| \;\le\; \frac{R}{C_{r,G}} \sum_{j=1}^{n} |q_{0,j}(z)|^{1/d_j}, \tag{4.2.21}$$

for all $z \in G \backslash B_{r/2}(e_G)$.

Now, we take

$$C \;:=\; \min\left\{C_r,\; \frac{r}{2C'_{r,G}}\right\} \qquad \text{and} \qquad C' \;:=\; \max\left\{C'_r,\; \frac{R}{C_{r,G}}\right\}.$$

Thus, combining (4.2.17) and (4.2.21), we obtain

$$C\sum_{j=1}^{n} |q_{0,j}(z)|^{1/d_j} \;\le\; |z| \;\le\; C'\sum_{j=1}^{n} |q_{0,j}(z)|^{1/d_j}, \quad \forall\, z \in G,$$

which shows that Q_0 has weight $(d_1, d_2, \ldots, d_n)$. □

Remark 4.2.3. One could replace χ_j and ψ_j $(j = 1, 2, \ldots, n)$ with any other cut-off functions. The resulting smooth functions $q_{0,1}, q_{0,2}, \ldots, q_{0,n}$ would then also be comparable to the C-C metric.

Remark 4.2.4. If Q_0 is the family of smooth, real-valued functions on G given by (4.2.14), then by Proposition 4.2.2, we have that

$$[\alpha]_{Q_0} \;=\; \sum_{j=1}^{n} d_j\, \alpha_j, \qquad \forall\, \alpha \in \mathbb{N}_0^n,$$

where, for each $j = 1, 2, \ldots, n$, d_j denotes the positive integer given by (2.4.3). Furthermore, suppose that $\boldsymbol{Y}$ denotes the basis of $\mathfrak{g}$ constructed in Section 2.4.1. Then, by Example 3.6.3, we have

$$[\alpha]_{Q_0} \;=\; [\alpha]_{\boldsymbol{Y}}, \qquad \forall\, \alpha \in \mathbb{N}_0^n. \tag{4.2.22}$$

Example 4.2.5. We consider the case $G = SU(2)$. In this case, $n = 3$ and we consider the family of functions $Q_0 = \{q_{0,1}, q_{0,2}, q_{0,3}\}$, where for each $j = 1, 2, 3$, the function $q_{0,j}$ is given by

$$q_{0,j}(z) \;=\; z_j\chi(z) + \psi(z), \quad z \in SU(2).$$

By Example 2.4.1 and Proposition 4.2.2, Q_0 has weight $(1, 1, 2)$, and in particular, there exist $C, C' > 0$ such that

$$C(|q_{0,1}(z)| + |q_{0,2}(z)| + |q_{0,3}(z)|^{1/2}) \le |z| \le C'(|q_{0,1}(z)| + |q_{0,2}(z)| + |q_{0,3}(z)|^{1/2}),$$

for all $z \in SU(2)$.

4.3 Taylor's Theorem revisited

For each $j = 1, 2, \ldots, n$, recall that $q_{0,j}$ is the smooth function on G given by (4.2.12) and Q_0 is the family of smooth, real-valued functions on G given by (4.2.14). The following observation is then an immediate consequence of Theorem 3.5.1 and the construction of the $q_{0,j}$.

Remark 4.3.1. Suppose f is a smooth function on G and let $x \in G$. Then, by Theorem 3.5.1, there exists a neighbourhood U of e_G in G, independent of f, such that for every $M \in \mathbb{N}$ and $z \in U$,

$$\begin{aligned} f(xz) &= \sum_{|\alpha|<M} \frac{1}{\alpha!}\, z_1^{\alpha_1} z_2^{\alpha_2} \cdots z^{\alpha_n}\, Y^\alpha f(x) \,+\, R^f_{x,M}(z) \\ &= \sum_{|\alpha|<M} \frac{1}{\alpha!}\, q_{0,\alpha}(z)\, Y^\alpha f(x) \,+\, R^f_{x,M}(z), \end{aligned}$$

where $R^f_{x,M}(z)$ satisfies

$$\left|R^f_M(z)\right| \,\leq\, C\,||(z_1, z_2, \ldots, z_n)||^M_{\mathbb{R}^n} \max_{|\alpha|=M} \left|\left|Y^\alpha f\right|\right|_{L^\infty(G)}, \quad \forall\, z \in U. \tag{4.3.1}$$

In fact, by the construction of the functions $q_{0,j}$, $j = 1, 2, \ldots, n$ (see (4.2.12)), it follows that $U = B_{r/2}(e_G)$ is a suitable choice, where $r \in (0, 1]$ is the real number satisfying (4.2.8).

Now, as we saw in Remark 3.5.2 (b) (see also Proposition 2.4.2), there exists $C' > 0$ such that

$$||(z_1, z_2, \ldots, z_n)||_{\mathbb{R}^n} \,\leq\, C'|z|, \quad \forall\, z \in B_{r/2}(e_G). \tag{4.3.2}$$

Hence, we have

$$\left|R^f_M(z)\right| \,\leq\, C\,|z|^M \max_{|\alpha|=M} \left|\left|Y^\alpha f\right|\right|_{L^\infty(G)}, \quad \forall\, z \in U, \tag{4.3.3}$$

for some $C > 0$. However, as a consequence of (4.3.2), there might exist $\alpha \in \mathbb{N}_0^n$, with $|\alpha| < M$, such that

$$|q_{0,\alpha}(z)| \,\leq\, C\,|z|^M, \quad \forall\, z \in B_{r/2}(e_G), \tag{4.3.4}$$

for some $C > 0$ (see Example 4.3.2 below). This means that $R^f_{x,M}$ is not a true Taylor remainder, when estimated via the Carnot-Carathéodory metric, because it does not encompass all elements with $O(|z|^M)$. This is illustrated in Example 4.11.20 below. In order to fix this issue, we must remove the unwanted terms from the sum, which is the main objective of Theorem 4.3.3 below.

Example 4.3.2. Let $G = SU(2)$ and $M = 3$. Furthermore, suppose f is a smooth function on $SU(2)$ and let x be any element of $SU(2)$. Additionally, for each $j = 1, 2, 3$, we let $q_{0,j}$ be the smooth function on $SU(2)$ defined by (4.2.12). As we worked out in Example 4.2.5, the family of functions $Q := \{q_{0,1}, q_{0,2}, q_{0,3}\}$ has weight $(1, 1, 2)$ (see also Example 2.4.1). Now, by Remark 4.3.1, we have

$$f(xz) \;=\; \sum_{|\alpha|<3} \frac{1}{\alpha!} q_{0,\alpha}(z) Y^\alpha f(x) + R^f_{x,3}(z), \quad \forall\, z \,\in\, B_{r/2}(I),$$

where

$$|R^f_{x,3}(z)| \;\leq\; C\,|z|^3 \max_{|\alpha|=3} ||Y^\alpha f||_{L^\infty(SU(2))}, \quad \forall\, z \,\in\, B_{r/2}(I),$$

for some $C > 0$. Consider, for instance, the multi-index $\alpha_0 = (0, 0, 2) \in \mathbb{N}_0^3$. Clearly, $|\alpha_0| = 2 < 3$, so the expression

$$\frac{1}{\alpha_0!} q_{0,\alpha_0}(z) Y^{\alpha_0} f(x), \quad z \,\in\, B_{r/2}(I),$$

is included in the sum $\sum_{|\alpha|<3}$.

Now, since Q_0 has weight $(1, 1, 2)$, we have

$$[\alpha_0]_{Q_0} \;=\; 1 \cdot 0 + 1 \cdot 0 + 2 \cdot 2 \;=\; 4 \quad \text{(see (4.1.3))}.$$

Moreover, by Lemma 4.1.6, there exists $C' > 0$ such that

$$|q_{0,\alpha_0}(z)| \;\leq\; C'\,|z|^{[\alpha_0]_Q} \;=\; C'\,|z|^4 \;\leq\; C'\,|z|^3, \quad \forall\, z \,\in\, B_{r/2}(I),$$

since $r \in (0, 1]$. This is precisely the scenario described in Remark 4.3.1 (see (4.3.4)).

Theorem 4.3.3. *Suppose f is a smooth function on G. Then, there exists a neighbourhood U of e_G in G, independent of f, such that for any $x \in G$ and any $M \in \mathbb{N}$, we have the Taylor expansion*

$$f(xz) \;=\; \sum_{[\alpha]_{Q_0}<M} \frac{1}{\alpha!} q_{0,\alpha}(z) Y^\alpha f(x) + R^f_{x,M}(z), \quad \forall\, z \,\in\, U, \tag{4.3.5}$$

where the remainder $R^f_{x,M}$ *satisfies*

$$|R^f_{x,M}(z)| \le C\,|z|^M \max_{\substack{[\alpha]_{Q_0}\ge M \\ |\alpha|\le M}} ||Y^\alpha f||_{L^\infty(G)}, \quad \forall\, z \in U, \tag{4.3.6}$$

for some $C > 0$ *independent of* x*.*

Remark 4.3.4. Suppose $\boldsymbol{Y} = \{Y_1, Y_2, \ldots, Y_n\}$ denotes the basis of $\mathfrak{g}$ constructed in Section 2.4.1. Furthermore, let f be a smooth function on G and suppose $x = e_G$. If f is CC-vanishing at e_G up to order $a-1$, for some $a \in \mathbb{N}$, then, by Remark 4.2.4 (in particular, see (4.2.22)), we have

$$Y^\alpha f(e_G) = 0, \qquad \forall\, \alpha \in \mathbb{N}_0^n, \quad [\alpha]_{\boldsymbol{Y}} = [\alpha]_{Q_0} \le a-1.$$

Hence, by Theorem 4.3.3, there exists a neighbourhood U of e_G in G such that, if $M > a$, then

$$f(z) = \sum_{\substack{\alpha \in \mathbb{N}_0^n \\ a\le[\alpha]_{Q_0}<M}} \frac{1}{\alpha!} q_{0,\alpha}(z) Y^\alpha f(e_G) + R^f_{e_G,M}(z), \quad \forall\, z \in U, \tag{4.3.7}$$

where the remainder $R^f_{e_G,M}$ satisfies

$$|R^f_{e_G,M}(z)| \le C\,|z|^M \max_{\substack{[\alpha]_{Q_0}\ge M \\ |\alpha|\le M}} ||Y^\alpha f||_{L^\infty(G)}, \quad \forall\, z \in U. \tag{4.3.8}$$

Recall that, for each $j = 1, 2, \ldots, n$, d_j is the positive integer given by (4.2.2). Moreover, we know that the family of functions Q_0 is comparable to the C-C metric with weight $(d_1, d_2, \ldots, d_n)$ (see Proposition 4.2.2). Hence, by definition (see (4.1.3)), we have

$$[\alpha]_{Q_0} = d_1\alpha_1 + d_2\alpha_2 + \cdots + d_n\alpha_n, \quad \forall\, \alpha \in \mathbb{N}_0^n.$$

Proof of Theorem 4.3.3. Let $r \in (0,1]$ be the real number satisfying (4.2.8) and suppose M is any positive integer. By Remark 4.3.1 we then obtain the expansion

$$f(xz) = \sum_{|\alpha|<M} \frac{1}{\alpha!}\, q_{0,\alpha}(z)\, Y^\alpha f(x) \,+\, R^f_{x,M}(z), \quad \forall\, z \in B_{r/2}(e_G), \tag{4.3.9}$$

where

$$\left|R^f_{x,M}(z)\right| \leq C\,|z|^M \max_{|\alpha|=M} \left|\left|Y^\alpha f\right|\right|_{L^\infty(G)}, \quad \forall\, z \in B_{r/2}(e_G). \tag{4.3.10}$$

Now, we aim to show that the sum $\sum_{|\alpha|<M}$ encompasses all elements of the sum $\sum_{[\alpha]_{Q_0}<M}$. In order to prove this, it is sufficient to show that if $\alpha \in \mathbb{N}_0^n$, with $[\alpha]_{Q_0} < M$, then $|\alpha| < M$. First observe that, for any $\alpha \in \mathbb{N}_0^n$, we have

$$|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n \leq d_1\alpha_1 + d_2\alpha_2 + \cdots + d_n\alpha_n = [\alpha]_{Q_0}.$$

Thus, for any $\alpha \in \mathbb{N}_0^n$, with $[\alpha]_{Q_0} < M$, we have

$$|\alpha| \leq [\alpha]_{Q_0} < M,$$

which implies the desired result. In particular, this means that

$$\begin{aligned}&\sum_{|\alpha|<M} \frac{1}{\alpha!}\, q_{0,\alpha}(z)\, Y^\alpha f(x) \\ &\quad = \sum_{[\alpha]_{Q_0}<M} \frac{1}{\alpha!}\, q_{0,\alpha}(z)\, Y^\alpha f(x) + \sum_{\substack{[\alpha]_{Q_0}\geq M \\ |\alpha|<M}} \frac{1}{\alpha!}\, q_{0,\alpha}(z)\, Y^\alpha f(x), \quad \forall\, z \in B_{r/2}(e_G).\end{aligned}$$

So, expression (4.3.9) can be rewritten as

$$f(xz) = \sum_{[\alpha]_{Q_0}<M} \frac{1}{\alpha!} q_{0,\alpha}(z) Y^\alpha f(x) + R^f_{x,M}(z), \quad \forall\, z \in B_{r/2}(e_G), \tag{4.3.11}$$

where

$$R^f_{x,M}(z) := \sum_{\substack{[\alpha]_{Q_0}\geq M \\ |\alpha|<M}} \frac{1}{\alpha!} q_{0,\alpha}(z) Y^\alpha f(x) + R^f_{x,M}(z), \quad \forall\, z \in B_{r/2}(e_G). \tag{4.3.12}$$

It remains to show that the remainder $R^f_{x,M}$ satisfies the estimate (4.3.6). We shall first find an estimate for the sum

$$\sum_{\substack{[\alpha]_{Q_0}\geq M \\ |\alpha|<M}} \frac{1}{\alpha!} q_{0,\alpha}(z) Y^\alpha f(x), \quad z \in B_{r/2}(e_G).$$

By Lemma 4.1.6, we know that $q_{0,\alpha}$ CC-vanishes at e_G up to order $[\alpha]_{Q_0} - 1$, for every $\alpha \in \mathbb{N}_0^n$, so we have

$$\sum_{\substack{[\alpha]_{Q_0} \geq M \\ |\alpha| < M}} \frac{1}{\alpha!} |q_{0,\alpha}(z)| \, |Y^\alpha f(x)| \lesssim \sum_{\substack{[\alpha]_{Q_0} \geq M \\ |\alpha| < M}} \frac{1}{\alpha!} |z|^{[\alpha]_{Q_0}} \, |Y^\alpha f(x)|, \tag{4.3.13}$$

for all $z \in B_{r/2}(e_G)$. In fact, (4.3.13) holds for every $z \in G$, but in this proof we are only interested in the local behaviour of these sums. Furthermore, for every $\alpha \in \mathbb{N}_0^n$ with $[\alpha]_{Q_0} \geq M$, we have

$$|z|^{[\alpha]_{Q_0}} \leq |z|^M, \quad \forall \, z \in B_{r/2}(e_G),$$

since $r \in (0, 1]$. Therefore,

$$\begin{aligned} &\sum_{\substack{[\alpha]_{Q_0} \geq M \\ |\alpha| < M}} \frac{1}{\alpha!} |z|^{[\alpha]_{Q_0}} \, |Y^\alpha f(x)| \\ &\qquad \lesssim |z|^M \max_{\substack{[\alpha]_{Q_0} \geq M \\ |\alpha| < M}} ||Y^\alpha f||_{L^\infty(G)}, \quad \forall \, z \in B_{r/2}(e_G). \end{aligned} \tag{4.3.14}$$

Moreover,

$$|R^f_{x,M}(z)| \lesssim |z|^M \max_{|\alpha| = M} ||Y^\alpha f||_{L^\infty(G)}, \quad \forall \, z \in B_{r/2}(e_G). \tag{4.3.15}$$

Therefore, by (4.3.14) and (4.3.15), the remainder $R^f_{x,M}$ satisfies

$$\begin{aligned} \left| R^f_{x,M}(z) \right| &\lesssim |z|^M \max_{\substack{[\alpha]_{Q_0} \geq M \\ |\alpha| < M}} ||Y^\alpha f||_{L^\infty(G)} + |z|^M \max_{|\alpha| = M} ||Y^\alpha f||_{L^\infty(G)} \\ &\lesssim |z|^M \max_{\substack{[\alpha]_{Q_0} \geq M \\ |\alpha| \leq M}} ||Y^\alpha f||_{L^\infty(G)}, \end{aligned}$$

for all $z \in B_{r/2}(e_G)$, which proves the result.

□

Remark 4.3.5. Theorem 4.3.3 can readily be extended to functions which are valued in a normed vector space. More precisely, if $(V, ||\cdot||_V)$ is a normed vector space and $f : G \to V$ is a smooth function on G, then for any $M \in \mathbb{N}$ we have

$$f(xz) \;=\; \sum_{[\alpha]_{Q_0}<M} \frac{1}{\alpha!} q_{0,\alpha}(z) Y^{\alpha} f(x) + R^{f}_{x,M}(z), \quad \forall\, z \,\in\, U, \tag{4.3.16}$$

where the remainder $R^{f}_{x,M}$ satisfies

$$\left|\left|R^{f}_{x,M}(z)\right|\right|_{V} \;\leq\; C\,|z|^{M} \max_{\substack{[\alpha]_{Q_0}\geq M \\ |\alpha|\leq M}} ||Y^{\alpha} f||_{L^{\infty}(G)}, \quad \forall\, z \,\in\, U, \tag{4.3.17}$$

for some $C > 0$ independent of x.

Lemma 4.3.6. *Suppose f is a smooth function on G, and let U be a neighbourhood of e_G in G such that, for any $x \in G$ and any $M \in \mathbb{N}$,*

$$f(xz) \;=\; \sum_{[\alpha]_{Q_0}<M} \frac{1}{\alpha!} q_{0,\alpha}(z) Y^{\alpha} f(x) + R^{f}_{x,M}(z), \quad \forall\, z \,\in\, U,$$

where

$$|R^{f}_{x,M}(z)| \;\leq\; C\,|z|^{M} \max_{\substack{[\alpha]_{Q_0}\geq M \\ |\alpha|\leq M}} ||Y^{\alpha} f||_{L^{\infty}(G)}, \quad \forall\, z \,\in\, U.$$

Then,

$$\sup_{\substack{z\,\in\, G \\ \beta\,\in\,\mathcal{I}(k),\, |\beta|=M}} \left|X_{\beta,z}\, R^{f}_{x,M}(z)\right| \;\lesssim\; \sup_{\substack{\alpha\,\in\,\mathbb{N}_0^n \\ [\alpha]_{Q_0}\leq M}} \left|\left|Y^{\alpha} f\right|\right|_{L^{\infty}(G)}. \tag{4.3.18}$$

Furthermore, the same result holds for the right-invariant operators $\widetilde{X}_{\beta}$.

Proof. Let us first fix $x \in G$. Then, for any $z \in U$,

$$X_{\beta,z}\big\{R^{f}_{x,M}(z)\big\} \;=\; (X_{\beta} f)(xz) - \sum_{[\alpha]_{Q_0}<M} \frac{1}{\alpha!}\,(X_{\beta}\, q)(z)\, Y^{\alpha} f(x).$$

Taking the supremum over $z \in G$ and $\beta \in \mathcal{I}(k)$, with $|\beta| = M$, we obtain

$$\sup_{\substack{z\,\in\, G \\ \beta\,\in\,\mathcal{I}(k),\, |\beta|=M}} \left|X_{\beta,z}\, R^{f}_{x,M}(z)\right|_{L^{\infty}(G)}$$

$$\leq \sup_{|\beta|=M} \left|\left|X_{\beta} f\right|\right|_{L^{\infty}(G)} + \sum_{[\alpha]_{Q_0}<M} \frac{1}{\alpha!} \sup_{\substack{\beta\,\in\,\mathcal{I}(k) \\ |\beta|=M}} ||X_{\beta}\, q||_{L^{\infty}(G)} \left|\left|Y^{\alpha} f\right|\right|_{L^{\infty}(G)}.$$

Since q is a smooth function on the compact Lie group G, then

$$\sup_{\substack{\beta \in \mathcal{I}(k) \\ |\beta|=M}} ||X_\beta \, q||_{L^\infty(G)} < +\infty.$$

By Corollary 3.6.6, for any $\beta \in \mathcal{I}(k)$ we have

$$X_\beta = \sum_{\substack{\alpha \in \mathbb{N}_0^n \\ [\alpha]_{Q_0}=|\beta|}} c_\alpha \, Y^\alpha.$$

In particular, we obtain

$$\sup_{|\beta|=M} ||X_\beta f||_{L^\infty(G)} \lesssim \sup_{[\alpha]_{Q_0}=M} ||Y^\alpha f||_{L^\infty(G)}.$$

Hence, we have shown that

$$\sup_{\substack{z \in G \\ \beta \in \mathcal{I}(k), \, |\beta|=M}} \left| X_{\beta,z} \, R^f_{x,M}(z) \right|_{L^\infty(G)} \lesssim \sup_{[\alpha]_{Q_0} \leq M} ||Y^\alpha f||_{L^\infty(G)},$$

as claimed.

The fact that the result also holds for differential operators $\widetilde{X}_\beta$ follows from Proposition 3.6.7. □

Remark 4.3.7. Lemma 4.3.6 may be extended to functions f which are vector valued. More precisely, suppose $(V, || \cdot ||_V)$ is a normed vector space, and let $f : G \to V$ be a smooth function on G. If the hypothesis of Lemma 4.3.6 is satisfied, then

$$\sup_{\substack{z \in G \\ \beta \in \mathcal{I}(k), \, |\beta|=M}} \left|\left| X_{\beta,z} \, R^f_{x,M}(z) \right|\right|_V \lesssim \sup_{\substack{\alpha \in \mathbb{N}_0^n \\ [\alpha]_{Q_0} \leq M}} \left|\left| Y^\alpha f \right|\right|_{L^\infty(G)}.$$

4.4 Symbols on G and their associated operators

In this section we shall introduce pseudo-differential operators on the compact Lie group G, as well as their associated symbols. This topic has been studied in the elliptic case in the context of compact Lie groups, see Ruzhansky et al [44] or Fischer [17].

Symbols shall be initially defined to be the Fourier transform of a right-convolution kernel, as a form of introduction, and in a later section we will define symbols more generally.

4.4.1 Introduction to symbols

Let $T : \mathcal{D}(G) \to \mathcal{D}'(G)$ be a continuous linear operator and, as a first example, suppose T is left-invariant. Then, as a consequence of the Schwartz kernel theorem (see Corollary 2.5.9), it is a convolution operator; that is, there exists a unique distribution $\kappa \in \mathcal{D}'(G)$ such that

$$Tf = f * \kappa, \quad f \in \mathcal{D}(G), \tag{4.4.1}$$

in the sense of distributions. Taking the Fourier transform, for any function $f \in \mathcal{D}(G)$ we have

$$\widehat{Tf}(\pi) = \widehat{f * \kappa}(\pi) = \sigma(\pi)\,\widehat{f}(\pi), \quad \pi \in \widehat{G},$$

where $\sigma := \widehat{\kappa}$ is known as the symbol of T. In this case, T is a Fourier multiplier operator with multiplier $\widehat{\kappa}$.

Now, suppose T is not necessarily a left-invariant operator. Then, by the Schwartz kernel theorem (see Theorem 2.5.2), there exists a unique distribution $\kappa \in \mathcal{D}'(G \times G)$ such that

$$Tf(x) = f * \kappa_x(x) = \int_G f(z)\,\kappa_x(z^{-1}x)\,\mathrm{d}z, \quad \forall\, f \in \mathcal{D}(G),$$

in the sense of distributions, where

$$\kappa_x(z) := \kappa(x, z), \quad \forall\, x, z \in G.$$

If we take the Fourier transform, we obtain

$$\widehat{Tf}(\pi) = \sigma(x, \pi)\,\widehat{f}(\pi), \quad \forall\, \pi \in \widehat{G}, \tag{4.4.2}$$

where σ is the field of operators which is given on $G \times \widehat{G}$ by

$$\sigma(x, \pi) := \widehat{\kappa_x}(\pi), \quad \forall\, x \in G,\ \pi \in \widehat{G}. \tag{4.4.3}$$

In this case, σ is called the symbol of T, and moreover, (4.4.3) can be rewritten as

$$\kappa_x(z) = \mathcal{F}_G^{-1}\{\sigma(x, \cdot)\}(z), \quad \forall\, x, z \in G. \tag{4.4.4}$$

Remark 4.4.1. Let us now explain why this definition of symbols is independent of the choice of π from its equivalence class $[\pi]_\sim \in \widehat{G}$. If $(\pi_1, \mathscr{H}_{\pi_1})$ and $(\pi_2, \mathscr{H}_{\pi_2})$

are equivalent representations, then, by definition, there exists an isomorphism $A : \mathscr{H}_1 \to \mathscr{H}_2$ such that

$$\pi_2(x) = A^{-1}\pi_1(x)A, \quad \forall\, x \in G.$$

Then, for any function $f \in \mathcal{D}(G)$, we have

$$\widehat{f}(\pi_2) = A^{-1}\widehat{f}(\pi_1)A,$$

and therefore,

$$\sigma(x, \pi_2) = A^{-1}\sigma(x, \pi_1)A, \quad \forall\, x \in G.$$

However, as we shall see below, the quantization we develop will not depend on this choice.

By applying the Fourier inversion formula to (4.4.2) (see Theorem 2.2.7 (i)), we have

$$Tf(x) = \sum_{\pi \in \widehat{G}} d_\pi \operatorname{Tr}\left(\pi(x)\, \sigma(x, \pi)\, \widehat{f}(\pi)\right) \tag{4.4.5}$$

in the sense of $L^2(G)$. If (4.4.5) holds, then we denote $T = \operatorname{Op}(\sigma)$. Furthermore, observe that

$$\operatorname{Tr}\left(\pi_1(x)\, \sigma(x, \pi_1)\, \widehat{f}(\pi_1)\right) = \operatorname{Tr}\left(\pi_2(x)\, \sigma(x, \pi_2)\, \widehat{f}(\pi_2)\right),$$

whenever π_1 and π_2 are equivalent representations. Thus, the sum given in (4.4.5) does not depend on the choice of a representation from its equivalence class. Therefore, the symbol defined by (4.4.3) is well-defined.

Remark 4.4.2. It is important to note here that not all symbols arise in this form. In general, we will not define symbols as the Fourier transform of a distribution. Nonetheless, symbols belonging to the class S^m, for some $m \in \mathbb{R}$, which we define later (see Definition 4.5.3), always admit a right-convolution kernel, as we shall see later (see Section 4.5.1).

Furthermore, observe that, by definition, if $\kappa \in \mathcal{D}'(G)$, then for every $(\pi, \mathscr{H}_\pi) \in \widehat{G}$, the expression $\widehat{\kappa}(\pi)$ is a bounded linear map on $\mathscr{H}_\pi$. This leads to the definition of a symbol given below (see Definition 4.4.3).

4.4.2 First definitions

We now state the definition of a symbol on a compact Lie group G.

Definition 4.4.3 (Symbol)**.** A symbol on G is a collection

$$\sigma := \{\sigma(x,\pi) : x \in G,\ \pi \in \widehat{G}\},$$

where for each $x \in G$ and $\pi \in \widehat{G}$, $\sigma(x,\pi)$ is a linear map $\mathscr{H}_\pi \to \mathscr{H}_\pi$.

A symbol $\sigma = \{\sigma(\pi) : \pi \in \widehat{G}\}$ that does not depend on the G variable is said to be an invariant symbol.

For a symbol σ, the notions of continuity and differentiability can be defined.

Definition 4.4.4. Suppose that for each $\pi \in \widehat{G}$, a matrix realisation of π is fixed.

(i) A symbol $\sigma = \{\sigma(x,\pi) : x \in G,\ \pi \in \widehat{G}\}$ is said to be continuous in x if, for each $\pi \in \widehat{G}$, the entries of $\sigma(x,\pi)$ are continuous.

(ii) Similarly, a symbol $\sigma = \{\sigma(x,\pi) : x \in G,\ \pi \in \widehat{G}\}$ is said to be smooth if, for each $\pi \in \widehat{G}$, the entries of $\sigma(x,\pi)$ are smooth.

An immediate observation from these definitions is that any invariant symbol $\sigma \in \mathcal{F}_G(\mathcal{D}'(G))$ is smooth. We will usually assume that any symbol we work with is smooth, unless stated otherwise.

We also define what it means for a symbol to admit an associated kernel, based on the discussion from the previous section (see (4.4.4)).

Definition 4.4.5 (Associated kernel)**.** A symbol σ on G is said to admit an associated kernel if, for each $x \in G$, we have $\sigma(x,\cdot) \in \mathcal{F}_G(\mathcal{D}'(G))$. In this case, its associated kernel is given by

$$\kappa_x(z) := \mathcal{F}_G^{-1}\{\sigma(x,\cdot)\}(z), \quad \forall\, x, z \in G. \tag{4.4.6}$$

For any symbol σ on G we have the notion of associated operator $\mathrm{Op}(\sigma)$. Let us now explain how we can define this. First recall that, for each $\pi \in \widehat{G}$, we define M_π to be the subspace of $L^2(G)$ spanned by the entry functions of the representations in the equivalence class $[\pi]_\sim$ of π; that is,

$$M_\pi = \operatorname{Span}\left\{\langle \pi_1(\cdot)\varphi, \psi\rangle_{\mathscr{H}_{\pi_1}} : \varphi, \psi \in \mathscr{H}_{\pi_1},\ \pi_1 \in [\pi]_\sim\right\}.$$

Additionally, recall that we define the space M (originally defined in (2.2.6)) to be the subspace of $L^2(G)$ consisting of finite linear combinations of vectors in M_π, for some $\pi \in \widehat{G}$:

$$M = \bigoplus_{\pi \in \widehat{G}} M_\pi .$$

By the Peter-Weyl Theorem (see Theorem 2.2.3), M is dense in $L^2(G)$ and, in particular, the closure of M, which we denote by $\overline{M}$, satisfies

$$\overline{M} = L^2(G).$$

If $\sigma = \{\sigma(x,\pi) : x \in G, \pi \in \widehat{G}\}$ is a symbol, then its associated operator, which we denote by $\mathrm{Op}(\sigma)$, is given by

$$\mathrm{Op}(\sigma) f(x) = \sum_{\pi \in \widehat{G}} d_\pi \operatorname{Tr}\Big(\pi(x)\, \sigma(x,\pi)\, \widehat{f}(\pi)\Big), \quad f \in M,\ x \in G.$$

Observe that the operator $\mathrm{Op}(\sigma)$ is well defined on M, since every function $f \in M$ is a finite linear combination of entry functions of representations $\pi \in \widehat{G}$; that is, the mappings belonging to M are of the form

$$x \longmapsto \langle \pi(x) u, v\rangle_{\mathscr{H}_\pi}, \quad x \in G,\ u, v \in \mathscr{H}_\pi.$$

If the operator $\mathrm{Op}(\sigma)$ is bounded in the L^2 norm, in the sense that there exists $C > 0$ such that

$$\big|\big|\mathrm{Op}(\sigma) f\big|\big|_{L^2(G)} \leq C\, ||f||_{L^2(G)}, \quad \forall\, f \in M, \tag{4.4.7}$$

then it can be extended uniquely to $L^2(G)$. This unique extension will also be denoted by $\mathrm{Op}(\sigma)$. More generally, it is a routine exercise to check that $M \subset \mathcal{C}^\infty(G)$ and that it is dense in the Sobolev space $L^2_s(G)$, for any $s \in \mathbb{R}$. If, for some $s_1, s_2 \in \mathbb{R}$, there exists $C > 0$ such that

$$\big|\big|(I+\mathcal{L})^{\frac{s_2}{2}}\, \mathrm{Op}(\sigma)\, (I+\mathcal{L})^{-\frac{s_1}{2}} f\big|\big|_{L^2(G)} \leq C\, ||f||_{L^2(G)}, \quad \forall\, f \in M,$$

then the operator $(I+\mathcal{L})^{\frac{s_2}{2}}\, \mathrm{Op}(\sigma)\, (I+\mathcal{L})^{-\frac{s_1}{2}}$ extends uniquely to an operator

$$(I+\mathcal{L})^{\frac{s_2}{2}}\, \mathrm{Op}(\sigma)\, (I+\mathcal{L})^{-\frac{s_1}{2}} : L^2(G) \longrightarrow L^2(G).$$

In particular, as M is dense in $L^2_{s_1}(G)$, this implies that the operator $\mathrm{Op}(\sigma)$ extends uniquely to an operator

$$\mathrm{Op}(\sigma) : L^2_{s_1}(G) \longrightarrow L^2_{s_2}(G).$$

As we did in the case of $L^2(G)$, we will denote this extension by $\mathrm{Op}(\sigma)$.

Remark 4.4.6. Suppose σ is a symbol on G. If $c \in \mathcal{D}(G)$, then, the field of operators

$$c\,\sigma = \{c(x)\,\sigma(x,\pi) \,:\, x \in G,\, \pi \in \widehat{G}\}$$

is a symbol on G. Moreover, its associated operator is $c\,\mathrm{Op}(\sigma)$.

If τ is another symbol on G, then

$$\sigma + \tau = \{\sigma(x,\pi) + \tau(x,\pi) \,:\, x \in G,\, \pi \in \widehat{G}\}$$

is also a symbol on G. Additionally, its associated operator is $\mathrm{Op}(\sigma) + \mathrm{Op}(\tau)$.

4.4.3 First examples

Suppose $\boldsymbol{V} = \{V_1, V_2, \ldots, V_r\}$ is any family of left-invariant vector fields on G, for some $r \in \mathbb{N}$, and, for $\beta \in \mathcal{I}(r)$, consider the symbol given by the collection of operators

$$\big\{\pi(V_\beta) \,:\, \pi \in \widehat{G}\big\}. \tag{4.4.8}$$

This symbol will usually be denoted by $\pi(V_\beta)$. We will assume this notation as long as there is no ambiguity between the symbol $\pi(V_\beta)$ and the infinitesimal representation associated to a given $\pi \in \widehat{G}$ of the differential operator V_β, which is also expressed as $\pi(V_\beta)$.

More generally, if $a \in \mathbb{N}$ and T is a differential operator of the form

$$T = \sum_{\substack{\alpha \in \mathcal{I}(r) \\ |\alpha| \leq a}} c_\alpha V_\alpha,$$

for some constant coefficients $c_\alpha \in \mathbb{R}$, then we let $\pi(T)$ denote the invariant symbol given by

$$\big\{\pi(T) \,:\, \pi \in \widehat{G}\big\}.$$

Example 4.4.7. If $\beta = (i_1, i_2, \ldots, i_b) \in \mathcal{I}(r)$, then $\pi(V_\beta)$ is an invariant symbol. This follows from the fact that, for each $(\pi, \mathscr{H}_\pi) \in \widehat{G}$, $\pi(V_\beta) \in \mathscr{L}(\mathscr{H}_\pi)$.

We can also establish the operator associated to the symbol $\pi(V_\beta)$ $(\beta \in \mathcal{I}(r))$.

Lemma 4.4.8. *Suppose* $\boldsymbol{V} = \{V_1, V_2, \ldots, V_r\}$ *is any family of left-invariant vector fields on* G*, for some* $r \in \mathbb{N}$*. Then, for any* $\beta \in \mathcal{I}(r)$*, the symbol* $\pi(V_\beta)$ *has associated operator* V_β*; that is,*

$$V_\beta = \mathrm{Op}(\pi(V_\beta)).$$

Moreover, the kernel associated to the symbol $\pi(V_\beta)$ *is the distribution* $V^t_\beta \delta_{e_G}$*, which, by* (2.5.5)*, is given by*

$$\begin{aligned} \left\langle V^t_\beta \delta_{e_G}, \varphi \right\rangle &= (-1)^{|\beta|} \left\langle V_{i_b} V_{i_{b-1}} \ldots V_{i_1} \delta_{e_G}, \varphi \right\rangle \\ &= \langle \delta_{e_G}, V_\beta \, \varphi \rangle \\ &= V_\beta \, \varphi(e_G), \end{aligned} \tag{4.4.9}$$

for $\varphi \in \mathcal{D}(G)$*.*

Remark 4.4.9. Once it has been shown that the associated operator of the symbol $\pi(V_\beta)$ is V_β, we can readily obtain that the symbol $\pi(V_\beta)$ has associated kernel $V^t_\beta \delta_{e_G}$. Indeed, by Proposition 2.5.11, the right-convolution kernel of the operator V_β is $V^t_\beta \delta_{e_G}$. This means that, for each $f \in \mathcal{D}(G)$, we have

$$V_\beta f = f * (V^t_\beta \delta_{e_G}). \tag{4.4.10}$$

Taking the Fourier transform of $V_\beta f$ yields

$$\widehat{V_\beta f}(\pi) = \pi(V_\beta) \, \widehat{f}(\pi), \quad \pi \in \widehat{G}.$$

But, by (4.4.10), we also have

$$\widehat{V_\beta f}(\pi) = \mathcal{F}\left\{ V^t_\beta \delta_{e_G} \right\}(\pi) \widehat{f}(\pi), \quad \pi \in \widehat{G}. \tag{4.4.11}$$

and so this shows that

$$\pi(V_\beta) = \mathcal{F}\left\{ V^t_\beta \, \delta_{e_G} \right\}(\pi), \quad \pi \in \widehat{G}.$$

Hence, the associated kernel of $\pi(V_\beta)$ is the distribution $V^t_\beta \delta_{e_G}$, as claimed in Lemma 4.4.8.

Proof of Lemma 4.4.8: If V is any left-invariant vector field and $\pi \in \widehat{G}$, we have

$$
\begin{aligned}
\pi(V\varphi) &= \int_G (V\varphi)(x)\,\pi(x)^*\,\mathrm{d}x \\
&= \int_G \left(\frac{d}{dt}\,\varphi(xe^{tV})\bigg|_{t=0} \right) \pi(x)^*\,\mathrm{d}x \\
&= \frac{d}{dt}\int_G \varphi(xe^{tV})\,\pi(x)^*\,\mathrm{d}x\bigg|_{t=0}.
\end{aligned}
$$

We now apply the substitution

$$
y = xe^{tV}.
$$

Since G is a unimodular group, we have

$$
\begin{aligned}
\pi(V\varphi) &= \frac{d}{dt}\int_G \varphi(y)\,\pi(ye^{-tV})^*\,\mathrm{d}y\bigg|_{t=0} \\
&= \frac{d}{dt}\int_G \varphi(y)\,\pi(e^{tV})\pi(y)^*\,\mathrm{d}y\bigg|_{t=0} \\
&= \left(\frac{d}{dt}\pi(e^{tV})\bigg|_{t=0} \right) \int_G \varphi(y)\,\pi(y)^*\,\mathrm{d}y.
\end{aligned}
$$

Since

$$
\frac{d}{dt}\pi(e^{tV})\bigg|_{t=0} = \pi(V),
$$

then we have

$$
\pi(V\varphi) = \pi(V)\,\pi(\varphi). \tag{4.4.12}
$$

In particular, by applying this method recursively to $V_{i_1}, V_{i_2}, \ldots, V_{i_b}$, we obtain that

$$
\mathcal{F}\left\{V_\beta\varphi\right\}(\pi) = \pi(V_\beta\varphi) = \pi(V_\beta)\pi(\varphi).
$$

Therefore, the operator associated to the symbol $\pi(V_\beta)$ is V_β, and the proof is finished, by Remark 4.4.9.

□

Example 4.4.10. Let us now consider the symbol $\pi(T)$, where T is a differential operator of the form

$$T = \sum_{\substack{\alpha \in \mathcal{I}(r) \\ |\alpha| \leq a}} c_\alpha V_\alpha,$$

for some coefficients $c_\alpha \in \mathcal{D}(G)$ and some $a \in \mathbb{N}$. By Remark 4.4.6 and Lemma 4.4.8, it follows that the symbol $\pi(T)$ has associated operator

$$\mathrm{Op}(\pi(T)) = T.$$

A more general example is given by the Fourier transform of a distribution, as we shall see next.

Example 4.4.11. If $\kappa \in \mathcal{D}'(G)$, then its group Fourier transform

$$\widehat{\kappa} = \left\{ \widehat{\kappa}(\pi) : \pi \in \widehat{G} \right\}$$

is an invariant symbol, and in fact, as saw in Section 4.4.1 (see (4.4.1)), the operator associated to $\widehat{\kappa}$ is the right convolution operator given by

$$\mathrm{Op}(\widehat{\kappa}) f = f * \kappa, \quad \forall\, f \in \mathcal{D}(G).$$

This shows that all functions in the space $\mathcal{F}_G\left(\mathcal{D}'(G)\right)$ are invariant symbols.

The scenario proposed in Example 4.4.11 is, in fact, not a rare occurrence. That is, it will often be the case that if σ is a symbol, which satisfies certain conditions, then it admits an associated kernel. In the following section we discuss some important examples of these conditions.

4.4.4 Sufficient condition for a symbol to admit an associated kernel

Let us now discuss an important sufficient condition for a symbol to admit an associated kernel (see Definition 4.4.5). Suppose σ is an invariant symbol. If

$$\sup_{\pi \in \widehat{G}} ||\sigma(\pi)||_{\mathscr{L}(\mathscr{H}_\pi)} < +\infty,$$

then by the Peter-Weyl Theorem (see Theorem 2.2.3), the operator $\mathrm{Op}(\sigma)$ is bounded in the $L^2(G)$ norm, in the sense that (4.4.7) holds. Hence, as we discussed in Section 4.4.2, $\mathrm{Op}(\sigma)$ extends uniquely to an operator $\mathrm{Op}(\sigma) : L^2(G) \to L^2(G)$, and we have

$$\big|\big|\mathrm{Op}(\sigma)\big|\big|_{\mathscr{L}(L^2(G))} \;=\; \sup_{\pi\,\in\,\widehat{G}} ||\sigma(\pi)||_{\mathscr{L}(\mathscr{H}_\pi)} \;<\; +\infty. \tag{4.4.13}$$

Therefore, by the Schwartz kernel theorem (see Theorem 2.5.2), the operator $\mathrm{Op}(\sigma)$ is given by right convolution against a distribution. That is, there exists a unique $\kappa \in \mathcal{D}'(G)$ such that

$$\mathrm{Op}(\sigma)f \;=\; f * \kappa, \quad \forall\, f \,\in\, L^2(G),$$

in the sense of distributions. Taking the Fourier transform yields

$$\mathcal{F}\{\mathrm{Op}(\sigma)f\}(\pi) \;=\; \widehat{\kappa}(\pi)\widehat{f}(\pi) \;=\; \sigma(\pi)\widehat{f}(\pi),$$

by the definition of $\mathrm{Op}(\sigma)$. This implies that the symbol σ admits an associated kernel, κ, in the sense of Definition 4.4.5.

More generally, suppose that, for some $s_1, s_2 \in \mathbb{R}$, we have

$$\sup_{\pi\,\in\,\widehat{G}} \big|\big|\pi(I+\mathcal{L})^{\frac{s_2}{2}}\,\sigma(\pi)\,\pi(I+\mathcal{L})^{-\frac{s_1}{2}}\big|\big|_{\mathscr{L}(\mathscr{H}_\pi)} \;<\; +\infty.$$

Then, the operator $(I+\mathcal{L})^{\frac{s_2}{2}}\,\mathrm{Op}(\sigma)\,(I+\mathcal{L})^{-\frac{s_1}{2}}$ is bounded in the $L^2(G)$ norm, in the sense that there exists $C > 0$ such that

$$\big|\big|(I+\mathcal{L})^{\frac{s_2}{2}}\,\mathrm{Op}(\sigma)\,(I+\mathcal{L})^{-\frac{s_1}{2}}f\big|\big|_{L^2(G)} \;\leq\; C\,||f||_{L^2(G)}, \quad \forall\, f \,\in\, M.$$

As was explained is Section 4.4.2, this means that the operator $\mathrm{Op}(\sigma)$ extends uniquely to an operator

$$\mathrm{Op}(\sigma) : L^2_{s_1}(G) \;\longrightarrow\; L^2_{s_2}(G),$$

and moreover, we have

$$\begin{aligned}
\big|\big|\mathrm{Op}(\sigma)\big|\big|&_{\mathscr{L}\left(L^2_{s_1}(G)\,,\,L^2_{s_2}(G)\right)} \\
&= \big|\big|(I+\mathcal{L})^{\frac{s_2}{2}}\,\mathrm{Op}(\sigma)\,(I+\mathcal{L})^{-\frac{s_1}{2}}\big|\big|_{\mathscr{L}(L^2(G))} \\
&= \sup_{\pi\,\in\,\widehat{G}} \big|\big|\pi(I+\mathcal{L})^{\frac{s_2}{2}}\,\sigma(\pi)\,\pi(I+\mathcal{L})^{-\frac{s_1}{2}}\big|\big|_{\mathscr{L}(\mathscr{H}_\pi)} \;<\; +\infty.
\end{aligned} \tag{4.4.14}$$

Hence, by the Schwartz kernel theorem, the operator $\mathrm{Op}(\sigma)$ is given by right convolution against a distribution. This implies that the symbol σ admits an associated kernel.

Remark 4.4.12. Suppose σ is a symbol on G such that, for some $\nu \in \mathbb{R}$,

$$\sup_{\substack{x \in G \\ \pi \in \widehat{G}}} \left|\left|\sigma(x,\pi)\,\pi(I+\mathcal{L})^{\nu}\right|\right|_{\mathscr{L}(\mathscr{H}_\pi)} < +\infty;$$

that is, there exists a constant $C > 0$ such that

$$\sup_{\substack{x \in G \\ \pi \in \widehat{G}}} \left|\left|\sigma(x,\pi)\,\pi(I+\mathcal{L})^{\nu}\right|\right|_{\mathscr{L}(\mathscr{H}_\pi)} \leq C. \tag{4.4.15}$$

Then, the operator $\mathrm{Op}(\sigma)$ associated to σ, which is originally defined on M, can be extended to $\mathcal{D}(G)$. Let us now prove this.

Note that it suffices to show that, for any $f \in \mathcal{D}(G)$ and $x \in G$, the sum

$$\sum_{\pi \in \widehat{G}} d_\pi \operatorname{Tr}\left(\pi(x)\,\sigma(x,\pi)\,\widehat{f}(\pi)\right) \tag{4.4.16}$$

is absolutely convergent. First observe that

$$\sum_{\pi \in \widehat{G}} d_\pi \left|\operatorname{Tr}\left(\pi(x)\,\sigma(x,\pi)\,\widehat{f}(\pi)\right)\right| \leq \sum_{\pi \in \widehat{G}} d_\pi \operatorname{Tr}\left|\pi(x)\,\sigma(x,\pi)\,\widehat{f}(\pi)\right| \tag{4.4.17}$$

For any $\pi \in \widehat{G}$ we have

$$\begin{aligned}
&\operatorname{Tr}\left|\pi(x)\,\sigma(x,\pi)\,\widehat{f}(\pi)\right| \\
&\qquad = \operatorname{Tr}\left|\pi(x)\,\sigma(x,\pi)\,\pi(I+\mathcal{L})^{\nu}\,\pi(I+\mathcal{L})^{-\nu}\widehat{f}(\pi)\right| \\
&\qquad \leq \left|\left|\sigma(x,\pi)\,\pi(I+\mathcal{L})^{\nu}\right|\right|_{\mathscr{L}(\mathscr{H}_\pi)} \operatorname{Tr}\left|\pi(I+\mathcal{L})^{-\nu}\,\widehat{f}(\pi)\right|.
\end{aligned} \tag{4.4.18}$$

Now, let $N \in \mathbb{N}_0$ to be determined. We have

$$\begin{aligned}
\operatorname{Tr}\left|\pi(I+\mathcal{L})^{-\nu}\,\widehat{f}(\pi)\right| &= \operatorname{Tr}\left|\pi(I+\mathcal{L})^{-N-\nu}\,\pi(I+\mathcal{L})^{N}\,\widehat{f}(\pi)\right| \\
&\leq \left|\left|\pi(I+\mathcal{L})^{N}\,\widehat{f}(\pi)\right|\right|_{\mathscr{L}(\mathscr{H}_\pi)} \operatorname{Tr}\left|\pi(I+\mathcal{L})^{-N-\nu}\right|.
\end{aligned}$$

Applying this to (4.4.18) yields the following bound for $\operatorname{Tr}\left|\pi(x)\,\sigma(x,\pi)\,\widehat{f}(\pi)\right|$:

$$\big|\big|\sigma(x,\pi)\,\pi(I+\mathcal{L})^{\nu}\big|\big|_{\mathscr{L}(\mathscr{H}_\pi)}\,\big|\big|\pi(I+\mathcal{L})^{N}\,\widehat{f}(\pi)\big|\big|_{\mathscr{L}(\mathscr{H}_\pi)}\operatorname{Tr}\big|\pi(I+\mathcal{L})^{-N-\nu}\big|. \quad (4.4.19)$$

Now, by Lemma 2.2.4,

$$\begin{aligned}\big|\big|\pi(I+\mathcal{L})^{N}\,\widehat{f}(\pi)\big|\big|_{\mathscr{L}(\mathscr{H}_\pi)} &= \big|\big|\mathcal{F}\{(I+\mathcal{L})^{N}f\}(\pi)\big|\big|_{\mathscr{L}(\mathscr{H}_\pi)} \\ &\le \;||(I+\mathcal{L})^{N}f||_{L^1(G)}. \end{aligned} \quad (4.4.20)$$

Since $f\in\mathcal{D}(G)$, then $(I+\mathcal{L})^{N}f\in L^1(G)$. Then, let

$$C_N := C\,||(I+\mathcal{L})^{N}f||_{L^1(G)} < +\infty,$$

where $C>0$ is the constant given in (4.4.15). Thus, by (4.4.19), we obtain

$$\operatorname{Tr}\big|\pi(x)\,\sigma(x,\pi)\,\widehat{f}(\pi)\big| \le C_N\operatorname{Tr}\big|\pi(I+\mathcal{L})^{-N-\nu}\big|. \quad (4.4.21)$$

Therefore,

$$\sum_{\pi\in\widehat{G}} d_\pi\operatorname{Tr}\Big|\pi(x)\,\sigma(x,\pi)\,\widehat{f}(\pi)\Big| \le C_N\sum_{\pi\in\widehat{G}} d_\pi\operatorname{Tr}\big|\pi(I+\mathcal{L})^{-N-\nu}\big|. \quad (4.4.22)$$

But, by Corollary 3.1.11, if we choose N such that $N+\nu>l/2$, where l denotes the local dimension of G, then

$$\sum_{\pi\in\widehat{G}} d_\pi\operatorname{Tr}\big|\pi(I+\mathcal{L})^{-N-\nu}\big| < \infty,$$

which proves the result.

4.4.5 The associated kernel of a non-invariant symbol

Fix a basis of vector fields $\{V_j : j=1,2,\ldots,n\}$ on G and $m\in\mathbb{R}$, and let σ be a symbol on G. For $\beta\in\mathbb{N}_0^n$, we let $V^\beta\sigma$ be the symbol

$$V^\beta\sigma := \big\{V_x^\beta\sigma(x,\pi) : x\in G,\ \pi\in\widehat{G}\big\},$$

where V_x^β denotes the differential operator

$$V_x^\beta = V_1^{\beta_1}V_2^{\beta_2}\ldots V_n^{\beta_n},$$

acting on each operator $\sigma(x,\pi)$ $(x \in G,\ \pi \in \widehat{G})$ with respect to x. Furthermore, let us assume that, for all $\beta \in \mathbb{N}_0^n$, σ satisfies

$$\sup_{\substack{x \in G \\ \pi \in \widehat{G}}} \left|\left|\pi(I+\mathcal{L})^{-\frac{1}{2}m}\, V_x^{\beta}\sigma(x,\pi)\right|\right|_{\mathscr{L}(\mathscr{H}_\pi)} < +\infty. \tag{4.4.23}$$

In this section, we explain how condition (4.4.23) implies that σ admits an associated kernel, in the sense of Definition 4.4.5.

Let us first consider the case $\beta = 0$. Then, for any $x \in G$ we have

$$\sup_{\pi \in \widehat{G}} \left|\left|\pi(I+\mathcal{L})^{-\frac{1}{2}m}\, \sigma(x,\pi)\right|\right|_{\mathscr{L}(\mathscr{H}_\pi)} < +\infty. \tag{4.4.24}$$

As was discussed in Section 4.4.4 above, condition (4.4.24) implies that the operator $\mathrm{Op}(\sigma(x,\cdot))$ associated to $\sigma(x,\cdot)$ extends uniquely to a bounded operator from the Sobolev space $L^2_{-m}(G)$ to $L^2(G)$. Hence, by the Schwartz kernel theorem (see Corollary 2.5.9), the operator $\mathrm{Op}(\sigma(x,\cdot))$ admits a right-convolution kernel. That is, there exists a unique distribution $\kappa_x \in \mathcal{D}'(G)$, depending on x, such that

$$\mathrm{Op}(\sigma)f \;=\; f * \kappa_x, \quad \forall\, f \,\in\, L^2_{-m}(G),$$

in the sense of distributions. Taking the Fourier transform, we obtain

$$\mathcal{F}\{\mathrm{Op}(\sigma)f\}(\pi) \;=\; \mathcal{F}\{f*\kappa_x\}(\pi) \;=\; \widehat{\kappa_x}(\pi)\,\widehat{f}(\pi), \quad \pi \,\in\, \widehat{G}.$$

This means that the symbol $\sigma(x,\cdot)$ admits an associated kernel, in the sense of Definition 4.4.5, and we have

$$\sigma(x,\cdot) \;=\; \widehat{\kappa_x}. \tag{4.4.25}$$

It is then a routine exercise to show that $x \mapsto \kappa_x$ is a continuous mapping. Let us sketch, as a case in point, the proof in the special case $m = 0$.

Recall that, if $T : \mathcal{D}(G) \to \mathcal{D}'(G)$ is a continuous linear operator, which is left-invariant, then $T\delta_0 \in \mathcal{D}'(G)$ denotes its associated right-convolution kernel (see Definition 2.5.10). The Schwartz kernel theorem on Lie groups (see Corollary 2.5.9) implies that the mapping

$$T \;\longmapsto\; T\delta_0,$$

is an isomorphism from $\mathscr{L}(L^2(G))^G$ (see Definition 2.5.1) onto its image in

$\mathcal{D}'(G)$. As a consequence, for each $N \in \mathbb{N}_0$ there exists $C > 0$ such that, for any $\kappa \in \mathcal{D}'(G)$,

$$||\kappa||_{\mathcal{D}'(G),N} \leq C\,||f \mapsto f * \kappa||_{\mathscr{L}(L^2(G))},$$

where $||\cdot||_{\mathcal{D}'(G),N}$ denotes the semi-norm on $\mathcal{D}'(G)$ given by (2.5.2). So, for any $x, x_1 \in G$, we have

$$\begin{aligned}\big|\big|\kappa_x - \kappa_{x_1}\big|\big|_{\mathcal{D}'(G),N} &\leq C\,\big|\big|\mathrm{Op}(\widehat{\kappa_x}) - \mathrm{Op}(\widehat{\kappa_{x_1}})\big|\big|_{\mathscr{L}(L^2(G))} \\ &= C \sup_{\pi \in \widehat{G}} \big|\big|\sigma(x,\pi) - \sigma(x_1,\pi)\big|\big|_{\mathscr{L}(\mathscr{H}_\pi)}.\end{aligned}$$

Now, for each $\pi \in \widehat{G}$, let us apply Taylor's Theorem to $\sigma(x,\pi) - \sigma(x_1,\pi)$, with respect to the basis of vector fields $\{V_j : j = 1, 2, \ldots, n\}$ (see Theorem 3.5.1). This yields the estimate

$$\big|\big|\sigma(x,\pi) - \sigma(x_1,\pi)\big|\big|_{\mathscr{L}(\mathscr{H}_\pi)} \lesssim d_R(x,x_1) \sup_{y \in G} ||V_j\,\sigma(y,\pi)||_{\mathscr{L}(\mathscr{H}_\pi)},$$

where $d_R(\cdot,\cdot)$ denotes the Riemannian metric. By the hypothesis (see (4.4.23)), this implies that there exists $C > 0$ such that

$$\big|\big|\kappa_x - \kappa_{x_1}\big|\big|_{\mathcal{D}'(G),N} \leq C\,d(x,x_1).$$

Thus, we have shown that the mapping

$$x \longmapsto \kappa_x$$

is continuous in the case $m = 0$.

This idea can be pushed further. For each $j = 1, 2, \ldots, n$, one can use the condition

$$\sup_{\substack{x \in G \\ \pi \in \widehat{G}}} \big|\big|\pi(I + \mathcal{L})^{-\frac{1}{2}m}\,V_{j,x}\,\sigma(x,\pi)\big|\big|_{\mathscr{L}(\mathscr{H}_\pi)} < +\infty,$$

to show that the mapping $x \mapsto \kappa_x$ is differentiable, and that the kernel of $V_j\sigma$ is $V_{j,x}\,\kappa_x$. So, proceeding recursively we obtain that the mapping $x \mapsto \kappa_x$ is smooth, and that, for each $\beta \in \mathbb{N}_0^n$, the symbol $V^\beta\sigma$ has associated kernel

$$(x,z) \longmapsto V_x^{\beta} \kappa_x(z).$$

Hence, the symbol σ has an associated kernel, $\kappa \in \mathcal{D}'(G \times G)$, which is given by

$$\kappa(x,z) = \kappa_x(z), \quad x,z \in G.$$

4.5 Definitions of the symbol classes S^m and their associated operator classes Ψ^m, and first examples

Our objective in this section is to define the symbol classes S^m on G, with respect to a sub-Laplacian. The definition we provide here is inspired, partly, by the classical definition of symbols on $\mathbb{R}^n$ (see, for example, Chapter VI in Stein [47]), and the symbol classes in the elliptic case of compact Lie groups, which can be found, for instance, in Fischer [17] or Ruzhansky et al [44]. Moreover, the nilpotent case (see Fischer and Ruzhansky [18]) also influenced the work presented here.

We continue on the same setting as in the previous sections; recall that the set $\boldsymbol{X} = \{X_1, X_2, \ldots, X_k\}$ forms a Hörmander system of left-invariant vector fields on G, for some $k \in \mathbb{N}$, and its associated sub-Laplacian is given by

$$\mathcal{L} := -(X_1^2 + X_2^2 + \cdots + X_k^2).$$

4.5.1 Definition of symbol classes S^m

Before we state our definition of the symbol classes S^m, let us first establish the following convention. Suppose that $Q = \{q_1, q_2, \ldots, q_\ell\}$ is a family of functions comparable to the C-C metric. When the context is clear, we shall denote

$$[\alpha] = [\alpha]_Q, \quad \forall\, \alpha \in \mathbb{N}_0^\ell.$$

Let us now define what it means for a symbol to be of class m.

Definition 4.5.1. Suppose $m \in \mathbb{R}$. Let us now fix a basis of vector fields

$$\boldsymbol{V} := \{V_j : j = 1, 2, \dots, n\}$$

on G (see Definition 2.3.2) and a family

$$Q = \{q_1, q_2, \dots, q_\ell\}$$

of smooth, real-valued functions on G, which is comparable to the C-C metric (see Definition 4.1.1), for some $\ell \in \mathbb{N}_0$. Furthermore, let $\Delta = \Delta_Q$ be the family of difference operators on G associated to Q. We then say that a symbol

$$\sigma = \{\sigma(x, \pi) : x \in G, \ \pi \in \widehat{G}\},$$

on G is of class m with respect to $\mathcal{L}$, $\boldsymbol{V}$ and Q if it has smooth entries in x and for each $\alpha \in \mathbb{N}_0^\ell$, $\beta \in \mathbb{N}_0^n$ and every $\nu \in \mathbb{R}$, there exists $C > 0$ such that,

$$\sup_{\substack{x \in G \\ \pi \in \widehat{G}}} \left|\left|\pi\,(I + \mathcal{L})^{-\frac{1}{2}(m - [\alpha] + \nu)}\, V_x^\beta \Delta^\alpha \sigma(x, \pi)\, \pi\,(I + \mathcal{L})^{\frac{1}{2}\nu}\,\right|\right|_{\mathscr{L}(\mathscr{H}_\pi)} \leq C. \tag{4.5.1}$$

Remark 4.5.2. Observe that the basis of vector fields $\boldsymbol{V}$ chosen in Definition 4.5.1 does not necessarily consist of left-invariant vector fields.

Notation 4.5.3. Consider the same hypothesis as in Definition 4.5.3. We then let

$$S^m(G, \mathcal{L}, \boldsymbol{V}, Q)^{\text{sub}}$$

be the space of symbols of class m, with respect to the sub-Laplacian $\mathcal{L}$, the basis of vector fields $\boldsymbol{V}$, the family of difference operators Δ and the family Q of smooth, real-valued functions on G, comparable to the C-C metric.

We shall often omit any mention of $\mathcal{L}$, $\boldsymbol{V}$, Δ and Q, as long as the context is clear. In this case, we shall write S^m instead of $S^m(G, \mathcal{L}, \boldsymbol{V}, Q)^{\text{sub}}$.

As was already mentioned in Remark 4.4.2 above, not every symbol on G is the Fourier transform of a distribution. However, our definition of the difference operators Δ^α (see Definition 4.1.8) requires a symbols in S^m to admit an associated kernel, in the sense of Definition 4.4.5. This is indeed the case, since any symbol σ of class m satisfies

$$\sup_{\pi \in \widehat{G}} \left|\left|\pi(I + \mathcal{L})^{-\frac{1}{2}m}\, \sigma(x, \pi)\right|\right|_{\mathscr{L}(\mathscr{H}_\pi)} < +\infty,$$

for every $x \in G$, so, as discussed in Section 4.4.5, it admits an associated kernel

κ_x. In particular, for any $\alpha \in \mathbb{N}_0^\ell$, the expression

$$\Delta^\alpha \sigma(x,\pi)$$

is well-defined for each $x \in G$, as required. For every $x \in G$, we have

$$\Delta^\alpha \sigma(x,\pi) = \mathcal{F}\{\widetilde{q}_\alpha \kappa_x\}(\pi), \quad \pi \in \widehat{G}.$$

Remark 4.5.4. Let us assume we remain in the same setting as in Definition 4.5.1. Then, suppose σ is a symbol on G, and let κ denote its associated convolution kernel. Furthermore, let $\{\nu_{n_1}\}_{n_1 \in \mathbb{Z}} \subset \mathbb{R}$ be a sequence of real numbers such that

$$\nu_{n_1} \longrightarrow +\infty \quad \text{as} \quad n_1 \longrightarrow +\infty, \qquad \nu_{n_1} \longrightarrow -\infty \quad \text{as} \quad n_1 \longrightarrow -\infty,$$

and suppose that for any $\alpha \in \mathbb{N}_0^n$ we have

$$\left|\left|\pi(I+\mathcal{L})^{-\frac{1}{2}(m-[\alpha]+\nu_{n_1})}\, \Delta^\alpha \sigma(\pi)\, \pi(I+\mathcal{L})^{\frac{\nu_{n_1}}{2}}\right|\right|_{L^\infty(\widehat{G})} < +\infty, \qquad \forall\, n_1 \in \mathbb{Z}.$$

Then, in particular (see (4.4.13)), we have

$$\left|\left|(I+\mathcal{L})^{-\frac{1}{2}(m-[\alpha]+\nu_{n_1})}\, \mathrm{Op}(\widehat{\widetilde{q}_\alpha \kappa})\, (I+\mathcal{L})^{\frac{\nu_{n_1}}{2}}\right|\right|_{\mathscr{L}(L^2(G))} < +\infty, \qquad \forall\, n_1 \in \mathbb{Z}.$$

By the Interpolation Theorem for Sobolev spaces (see Theorem 3.3.1), we have

$$\left|\left|(I+\mathcal{L})^{-\frac{1}{2}(m-[\alpha]+\nu)}\, \mathrm{Op}(\widehat{\widetilde{q}_\alpha \kappa})\, (I+\mathcal{L})^{\frac{\nu}{2}}\right|\right|_{\mathscr{L}(L^2(G))} < +\infty, \qquad \forall\, \nu \in \mathbb{R}.$$

In particular, this implies that $\sigma \in S^m$. In particular this shows that, in general, to verify $\sigma \in S^m$ it suffices to prove (4.5.1) for a sequence $\{\nu_{n_1}\}_{n_1 \in \mathbb{Z}} \subset \mathbb{R}$ which converges to both $+\infty$ and $-\infty$.

We have the following result.

Proposition 4.5.5. *Suppose $m_1, m_2 \in \mathbb{R}$, with $m_1 \leq m_2$. Fix a basis of vector fields $\boldsymbol{V} := \{V_j : j = 1, 2, \ldots, n\}$ on G and a family $Q = \{q_1, q_2, \ldots, q_\ell\}$ of smooth, real-valued functions on G, which is comparable to the C-C metric, for some $\ell \in \mathbb{N}_0$. Furthermore, let S^{m_1}, S^{m_2} be the families of symbols of class m_1, m_2, with respect to $\boldsymbol{\mathcal{L}}$, $\boldsymbol{V}$ and Q, respectively. Then,*

$$S^{m_1} \subset S^{m_2}.$$

Proof. Let $\sigma \in S^{m_1}$ and suppose that $\alpha \in \mathbb{N}_0^{\ell}$, $\beta \in \mathbb{N}_0^n$ and $\nu \in \mathbb{R}$. First observe that

$$\begin{aligned}
&\sup_{\substack{x \in G \\ \pi \in \widehat{G}}} \left|\left|\pi\left(I+\mathcal{L}\right)^{-\frac{1}{2}(m_2-[\alpha]+\nu)} V_x^{\beta} \Delta^{\alpha}\sigma(x,\pi)\, \pi\left(I+\mathcal{L}\right)^{\frac{\nu}{2}}\right|\right|_{\mathscr{L}(\mathscr{H}_\pi)} \\
&\quad \leq ||\pi(I+\mathcal{L})^{-\frac{1}{2}(m_1-m_2)}||_{L^\infty(\widehat{G})} \\
&\qquad \sup_{\substack{x \in G \\ \pi \in \widehat{G}}} \left|\left|\pi\left(I+\mathcal{L}\right)^{-\frac{1}{2}(m_1-[\alpha]+\nu)} V_x^{\beta} \Delta^{\alpha}\sigma(x,\pi)\, \pi\left(I+\mathcal{L}\right)^{\frac{\nu}{2}}\right|\right|_{\mathscr{L}(\mathscr{H}_\pi)}.
\end{aligned}$$

Since $m_1 \leq m_2$, then by functional analysis,

$$||\pi(I+\mathcal{L})^{-\frac{1}{2}(m_1-m_2)}||_{L^\infty(\widehat{G})} \leq \sup_{\lambda \geq 0} (1+\lambda)^{\frac{1}{2}(m_1+m_2)} \leq 1.$$

Since $\sigma_1 \in S^{m_1}$, then it follows that there exists $C > 0$, independent of σ, such that

$$\sup_{\substack{x \in G \\ \pi \in \widehat{G}}} \left|\left|\pi\left(I+\mathcal{L}\right)^{-\frac{1}{2}(m_1-[\alpha]+\nu)} V_x^{\beta} \Delta^{\alpha}\sigma(x,\pi)\, \pi\left(I+\mathcal{L}\right)^{\frac{1}{2}\nu}\right|\right|_{\mathscr{L}(\mathscr{H}_\pi)} \leq C.$$

Thus, we have obtained that

$$\sup_{\substack{x \in G \\ \pi \in \widehat{G}}} \left|\left|\pi\left(I+\mathcal{L}\right)^{-\frac{1}{2}(m_2-[\alpha]+\nu)} V_x^{\beta} \Delta^{\alpha}\sigma(x,\pi)\, \pi\left(I+\mathcal{L}\right)^{\frac{1}{2}\nu}\right|\right|_{\mathscr{L}(\mathscr{H}_\pi)} \leq C,$$

which shows that $\sigma \in S^{m_2}$, as required.

□

Definition 4.5.6. We define the space

$$S^{-\infty} = \bigcap_{m \in \mathbb{R}} S^m.$$

A symbol in the class $S^{-\infty}$ is called a smoothing symbol.

Observe that, if Q and P are any two families of smooth, real-valued functions on G, which are comparable to the C-C metric, we have

$$S^{-\infty}(G, \mathcal{L}, \boldsymbol{V}, Q)^{\mathrm{sub}} \;=\; S^{-\infty}(G, \mathcal{L}, \boldsymbol{V}, P)^{\mathrm{sub}}.$$

Thus, $S^{-\infty}$ is independent on the choice of Q.

4.5.2 Definition of operator classes Ψ^m

Recall that, for any symbol σ on G, we define its associated operator, $\mathrm{Op}(\sigma)$, by

$$\mathrm{Op}(\sigma)f(x) \;=\; \sum_{\pi \in \widehat{G}} d_\pi \operatorname{Tr}\left(\pi(x)\,\sigma(x,\pi)\,\widehat{f}(\pi)\right), \quad f \in M,\; x \in G. \tag{4.5.2}$$

As was explained in Section 4.4.2, this is well-defined. If $\sigma \in S^m$, then by Remark 4.4.12, for any $f \in \mathcal{D}(G)$ and $x \in G$, the sum

$$\sum_{\pi \in \widehat{G}} d_\pi \operatorname{Tr}\left(\pi(x)\,\sigma(x,\pi)\,\widehat{f}(\pi)\right)$$

is absolutely convergent. Furthermore, for each $x \in G$,

$$\mathrm{Op}(\sigma)f(x) \;=\; (f * \kappa_x)(x), \tag{4.5.3}$$

where κ_x is the kernel associated to σ.

We now show that, for each $x \in G$, the mapping $x \mapsto (f * \kappa_x)(x)$ has a meaning and is, in fact, smooth. Observe that, as $x \mapsto \kappa_x$ is a smooth mapping $G \to \mathcal{D}'(G)$ (see Section 4.4.5), then κ_x is well-defined as a distribution, for each $x \in G$, and for any $f \in \mathcal{D}(G)$, the convolution given by $f * \kappa_x$ has a meaning (see Definition 2.5.5). Moreover, in this case, $f * \kappa_x \in \mathcal{D}(G)$ (see Proposition 2.5.7), and hence the sum in (4.5.2) converges pointwise to $(f * \kappa_x)(x)$, for each $x \in G$. So, the definition of the operator $\mathrm{Op}(\sigma)$ may be extended to $\mathcal{D}(G)$.

Definition 4.5.7. Let $m \in \mathbb{R}$. If $\sigma \in S^m$ then its associated operator, which is given by

$$\mathrm{Op}(\sigma)f(x) \;=\; \sum_{\pi \in \widehat{G}} d_\pi \operatorname{Tr}\left(\pi(x)\,\sigma(x,\pi)\,\widehat{f}(\pi)\right), \quad f \in \mathcal{D}(G),\; x \in G,$$

is said to be of class m.

We now also define the space of operators Ψ^m.

Definition 4.5.8. For each $m \in \mathbb{R}$, we let Ψ^m denote the space of operators of class m. That is,

$$\Psi^m = \mathrm{Op}(S^m).$$

Definition 4.5.9. We define the space

$$\Psi^{-\infty} = \bigcap_{m \in \mathbb{R}} \Psi^m.$$

An operator in the class $\Psi^{-\infty}$ is called a smoothing operator.

4.5.3 Example: The symbol $\pi(X_\beta)$

Recall that $\boldsymbol{X} = \{X_1, X_2, \dots, X_k\}$ forms a Hörmander system of left-invariant vector fields on G, and $\mathcal{L}$ is its corresponding sub-Laplacian. For a multi-index $\beta = (i_1, i_2, \dots, i_b) \in \mathcal{I}(k)$, consider the symbol $\pi(X_\beta)$, which is given by

$$\{\pi(X_\beta) : \pi \in \widehat{G}\}.$$

By Lemma 4.4.8), the operator associated to $\pi(X_\beta)$ is X_β.

Let us fix a basis of vector fields

$$\boldsymbol{V} := \{V_j : j = 1, 2, \dots, n\}$$

on G (see Definition 2.3.2). Furthermore, let Q_0 be the set of smooth,real-valued functions on G given by (4.2.14), and suppose Δ denotes the family of difference operators associated to Q_0. For $m \in \mathbb{R}$, we then let S^m be the space of symbols of class m, with respect to $\mathcal{L}$, $\boldsymbol{V}$ and the family of difference operators Δ.

The objective in this section is to prove the following result.

Proposition 4.5.10. *For any $\beta \in \mathcal{I}(k)$, the symbol $\pi(X_\beta)$ belongs to the symbol class $S^{|\beta|}$.*

Remark 4.5.11. Proposition 4.5.10 implies that if $a \in \mathbb{N}_0$, then any differential operator of the form

$$\sum_{\substack{\beta \in \mathcal{I}(k) \\ |\beta| \le a}} c_\beta(x) X_\beta,$$

where the coefficients $c_\beta \in \mathcal{D}(G)$, belongs to the operator class Ψ^a.

In order to prove Proposition 4.5.10, we first calculate $\Delta^\alpha \pi(X_\beta)$, for $\alpha \in \mathbb{N}_0^n$.

Lemma 4.5.12. *Let $\beta \in \mathcal{I}(k)$. Then, the symbol*

$$\pi(X_\beta) = \{\pi(X_\beta) : \pi \in \widehat{G}\}$$

satisfies

$$\Delta^\alpha \pi(X_\beta) = 0, \quad \text{for any } \alpha \in \mathbb{N}_0^n, \text{ with } [\alpha] > |\beta|. \tag{4.5.4}$$

On the other hand,

$$\Delta^\alpha \pi(X_\beta) = \sum_{\substack{\beta_2 \in \mathcal{I}(k) \\ [\alpha]+|\beta_2|=|\beta|}} c^{\alpha,\beta}_{\beta_2} \pi(X_{\beta_2}), \quad \text{for any } \alpha \in \mathbb{N}_0^n, \text{ with } [\alpha] \leq |\beta|, \tag{4.5.5}$$

for some constants $c^{\alpha,\beta}_{\beta_2} \in \mathbb{R}$, depending on α, β, and on the dummy variable β_2.

Proof. Let $\alpha \in \mathbb{N}_0^n$. We have already seen that the right convolution kernel associated to the symbol $\pi(X_\beta)$ is the distribution $X^t_\beta \delta_{e_G}$ (see Proposition 2.5.11). Thus, we have

$$\Delta^\alpha \pi(X_\beta) = \pi(\widetilde{q}_{0,\alpha} X^t_\beta \delta_{e_G}),$$

where we recall that, for any function f on G, $\widetilde{f}$ is given by

$$\widetilde{f}(z) = f(z^{-1}), \quad z \in G.$$

Now, using (4.4.9), we can see that the distribution $\widetilde{q}_{0,\alpha} X^t_\beta \delta_{e_G}$ is given by

$$\begin{aligned} \left\langle \widetilde{q}_{0,\alpha} X^t_\beta \delta_{e_G}, \varphi \right\rangle &= \left\langle X^t_\beta \delta_{e_G}, \widetilde{q}_{0,\alpha} \varphi \right\rangle = \left\langle \delta_{e_G}, X_\beta \{\widetilde{q}_{0,\alpha} \varphi\} \right\rangle \\ &= X_\beta \{\widetilde{q}_{0,\alpha} \varphi\}(e_G), \end{aligned}$$

for any $\varphi \in \mathcal{D}(G)$. By the Leibniz rule for vector fields, we have

$$X_\beta \{\widetilde{q}_{0,\alpha} \varphi\}(e_G) = \sum_{\substack{\beta_1,\beta_2 \in \mathcal{I}(k) \\ |\beta_1|+|\beta_2|=|\beta|}} c^\beta_{\beta_1,\beta_2} (X_{\beta_1} \widetilde{q}_{0,\alpha})(e_G)(X_{\beta_2} \varphi)(e_G), \tag{4.5.6}$$

for some constants $c^\beta_{\beta_1,\beta_2} \in \mathbb{R}$. Now, by Proposition 3.7.9 and the definition of the $q_{0,j}$ (see (4.2.12)), we have

$$X_{\beta_1}\widetilde{q}_{0,\alpha}(e_G) \;=\; \sum_{\substack{\alpha' \in \mathcal{I}(k) \\ [\alpha']=[\alpha]-|\beta_1|}} \widetilde{c}_{\alpha'}^{\,\beta_1,\alpha}\, \widetilde{q}_{0,\alpha'}(e_G),$$

for some constants $\widetilde{c}_{\alpha'}^{\,\beta_1,\alpha} \in \mathbb{R}$, where it is understood that, if $|\beta_1| > [\alpha]$, then

$$X_{\beta_1}\widetilde{q}_{0,\alpha}(e_G) \;=\; 0.$$

So, we have

$$\begin{aligned} X_\beta\{\widetilde{q}_{0,\alpha}\varphi\}(e_G) \;&=\; \sum_{|\beta_1|+|\beta_2|=|\beta|} c^{\beta}_{\beta_1,\beta_2}(X_{\beta_1}\widetilde{q}_{0,\alpha})(e_G)(X_{\beta_2}\varphi)(e_G) \\ &=\; \sum_{\substack{|\beta_1|+|\beta_2|=|\beta| \\ [\alpha']=[\alpha]-|\beta_1|}} c^{\beta}_{\beta_1,\beta_2}\widetilde{c}_{\alpha'}^{\,\beta_1,\alpha}(\widetilde{q}_{0,\alpha'}X_{\beta_2}\varphi)(e_G). \end{aligned} \tag{4.5.7}$$

Now, suppose $\beta_1 \in \mathcal{I}(k)$, with $|\beta_1| \le |\beta|$, and let $\alpha' \in \mathbb{N}_0^n$, with $[\alpha'] = [\alpha]-|\beta_1|$. By Lemma 4.1.6 the function $q_{0,\alpha'}$ CC-vanishes at e_G up to order $[\alpha'] - 1 = [\alpha] - |\beta_1| - 1$. This means that

$$|\widetilde{q}_{0,\alpha'}(z)| \;\lesssim\; |z|^{[\alpha']}, \qquad \forall\; z \in G,$$

and in particular, by Proposition 3.7.3,

$$\widetilde{q}_{0,\alpha'}(e_G) \;=\; 0, \quad \text{ whenever } \; [\alpha] > |\beta_1|. \tag{4.5.8}$$

Furthermore, $\widetilde{q}_{0,\alpha'}(e_G) = 1$ for $\alpha' = 0$, so, by (4.5.7), we have

$$X_\beta\{\widetilde{q}_{0,\alpha}\varphi\}(e_G) \;=\; \sum_{\substack{|\beta_1|+|\beta_2|=|\beta| \\ [\alpha]=|\beta_1|}} c^{\beta}_{\beta_1,\beta_2}\widetilde{c}_{\alpha'}^{\,\beta_1,\alpha}(X_{\beta_2}\varphi)(e_G).$$

Thus,

$$\widetilde{q}_{0,\alpha}X_\beta\delta_{e_G} \;=\; \sum_{\substack{\beta_2 \in \mathcal{I}(k) \\ [\alpha]+|\beta_2|=|\beta|}} c^{\alpha,\beta}_{\beta_2}X_{\beta_2}\delta_{e_G},$$

for some constants $c^{\alpha,\beta}_{\beta_2} \in \mathbb{R}$. Taking the Fourier transform, we obtain

$$\Delta^\alpha\pi(X_\beta) \;=\; \sum_{\substack{\beta_2 \in \mathcal{I}(k) \\ [\alpha]+|\beta_2|=|\beta|}} c^{\alpha,\beta}_{\beta_2}\pi(X_{\beta_2}),$$

which shows (4.5.5). Observe that (4.5.4) follows from the fact that, if $\alpha \in \mathcal{I}(k)$, with $[\alpha] > |\beta|$, then for any $\beta_1, \beta_2 \in \mathcal{I}(k)$ such that $|\beta_1| + |\beta_2| = |\beta|$, we have $|\beta_1| < [\alpha]$, and by (4.5.8).

□

We now show the main result of this section.

Proof of Proposition 4.5.10: Let $\alpha \in \mathbb{N}_0^n$ and $\beta \in \mathcal{I}(k)$. By Lemma 4.5.12, we have

$$\begin{aligned}
&\sup_{\pi \in \widehat{G}} \left|\left|\pi(I+\mathcal{L})^{-\frac{1}{2}(|\beta|-[\alpha]+\nu)}\Delta^{\alpha}\pi(X_\beta)\,\pi(I+\mathcal{L})^{\nu/2}\right|\right|_{\mathscr{L}(\mathscr{H}_\pi)} \\
&\qquad \lesssim \sum_{\substack{\beta_2 \in \mathcal{I}(k) \\ [\alpha]+|\beta_2|=|\beta|}} \sup_{\pi \in \widehat{G}} \left|\left|\pi(I+\mathcal{L})^{-\frac{1}{2}(|\beta|-[\alpha]+\nu)}\pi(X_{\beta_2})\,\pi(I+\mathcal{L})^{\nu/2}\right|\right|_{\mathscr{L}(\mathscr{H}_\pi)} \\
&\qquad \lesssim \sum_{\substack{\beta_2 \in \mathcal{I}(k) \\ |\beta_2|=|\beta|-[\alpha]}} \sup_{\pi \in \widehat{G}} \left|\left|\pi(I+\mathcal{L})^{-\frac{1}{2}(|\beta_2|+\nu)}\pi(X_{\beta_2})\,\pi(I+\mathcal{L})^{\nu/2}\right|\right|_{\mathscr{L}(\mathscr{H}_\pi)}.
\end{aligned}$$

Now, by Proposition 3.2.3 (g), for each $\beta_2 \in \mathcal{I}(k)$, the operator X_{β_2} maps $L^2_{-\nu}(G)$ continuously into $L^2_{-\nu-|\beta_2|}(G)$, so the operator

$$(I+\mathcal{L})^{-\frac{1}{2}(|\beta_2|+\nu)}\,X_{\beta_2}\,(I+\mathcal{L})^{\nu/2}$$

is bounded on $L^2(G)$. This implies that

$$\begin{aligned}
&\sup_{\pi \in \widehat{G}} \left|\left|\pi(I+\mathcal{L})^{-\frac{1}{2}(|\beta_2|+\nu)}\pi(X_{\beta_2})\,\pi(I+\mathcal{L})^{\nu/2}\right|\right|_{\mathscr{L}(\mathscr{H}_\pi)} \\
&\qquad\qquad = \left|\left|(I+\mathcal{L})^{-\frac{1}{2}(|\beta_2|+\nu)}\,X_{\beta_2}\,(I+\mathcal{L})^{\nu/2}\right|\right|_{\mathscr{L}(L^2(G))} \\
&\qquad\qquad < +\infty.
\end{aligned}$$

Since the sum over all $\beta_2 \in \mathcal{I}(k)$, with $|\beta_2| = |\beta| - [\alpha]$, is finite, it follows that

$$\sup_{\pi \in \widehat{G}} \left|\left|\pi(I+\mathcal{L})^{-\frac{1}{2}(|\beta|-[\alpha]+\nu)}\Delta^{\alpha}\pi(X_\beta)\,\pi(I+\mathcal{L})^{\nu/2}\right|\right|_{\mathscr{L}(\mathscr{H}_\pi)} < +\infty,$$

which shows that $\pi(X_\beta) \in S^{|\beta|}$, as required.

□

Remark 4.5.13. As a consequence of Proposition 4.5.10 and Lemma 4.4.8, for any $\beta \in \mathcal{I}(k)$, the operator X_β is of class $|\beta|$.

4.6 First properties of symbol classes S^m and their associated operator classes Ψ^m

4.6.1 Independence of S^m on the choice of basis of vector fields

In Definition 4.5.1 we fixed a basis of vector fields $\{V_1, V_2, \ldots, V_n\}$ on G to define our symbols classes S^m. However, one might ask whether our definition of S^m is dependent on this choice. We shall now explain why, for any $m \in \mathbb{R}$, the definition of these symbol classes is, in fact, independent of the choice of basis of vector fields $\boldsymbol{V}$.

Let $\boldsymbol{V} = \{V_1, V_2, \ldots, V_n\}$ be a basis of vector fields on G and suppose Q is a family of smooth, real-valued functions on G, which is comparable to the C-C metric. Furthermore, suppose Δ denotes the family of difference operators associated to Q. Consider a symbol $\sigma \in S^m(G, \mathcal{L}, \boldsymbol{V}, Q)^{\mathrm{sub}}$, for $m \in \mathbb{R}$. Then, for $\alpha \in \mathbb{N}_0^\ell$ and $\beta \in \mathbb{N}_0^n$, we have

$$\sup_{\substack{x \in G \\ \pi \in \widehat{G}}} \left|\left|\pi\left(I + \mathcal{L}\right)^{-\frac{1}{2}(m - [\alpha] + \nu)} V_x^\beta \Delta^\alpha \sigma(x, \pi)\, \pi\left(I + \mathcal{L}\right)^{\frac{1}{2}\nu}\right|\right|_{\mathscr{L}(\mathscr{H}_\pi)} < +\infty.$$

Suppose now that $\boldsymbol{W} = \{W_1, W_2, \ldots, W_n\}$ is another basis of vector fields on G. Then, by Corollary 2.3.5 and Remark 2.3.6, for every $\alpha \in \mathbb{N}_0^\ell$ and $\beta \in \mathbb{N}_0^n$, there exists a constant $C^\beta_{\boldsymbol{W},\boldsymbol{V}} > 0$, depending on β and the families of vector fields $\boldsymbol{W}$ and $\boldsymbol{V}$, such that

$$\begin{aligned}
&\sup_{\substack{x \in G \\ \pi \in \widehat{G}}} \left|\left|\pi\left(I + \mathcal{L}\right)^{-\frac{1}{2}(m - [\alpha] + \nu)} W_x^\beta \Delta^\alpha \sigma(x, \pi)\, \pi\left(I + \mathcal{L}\right)^{\frac{1}{2}\nu}\right|\right|_{\mathscr{L}(\mathscr{H}_\pi)} \\
&\qquad \leq C^\beta_{\boldsymbol{W},\boldsymbol{V}} \sup_{\substack{|\beta'| \leq |\beta| \\ x \in G \\ \pi \in \widehat{G}}} \left|\left|\pi\left(I + \mathcal{L}\right)^{-\frac{1}{2}(m - [\alpha] + \nu)} V_x^{\beta'} \Delta^\alpha \sigma(x, \pi)\, \pi\left(I + \mathcal{L}\right)^{\frac{1}{2}\nu}\right|\right|_{\mathscr{L}(\mathscr{H}_\pi)} \\
&\qquad < +\infty.
\end{aligned}$$

Hence $\sigma \in S^m(G, \mathcal{L}, \boldsymbol{W}, Q)^{\mathrm{sub}}$. The converse also holds, since $\boldsymbol{V}$ and $\boldsymbol{W}$ play a symmetric role.

4.6.2 S^m as a Fréchet space

Let $m \in \mathbb{R}$. In view of the condition for a symbol to belong to the space S^m (see (4.5.1)), for each $a, b \in \mathbb{N}_0$ and $c \geq 0$, we define the quantity

$$||\sigma||_{S^m,a,b,c} := \sup_{\substack{[\alpha]\leq a,\, [\beta]\leq b \\ x\in G \\ \pi\in\widehat{G},\, |\nu|\leq c}} \left|\left|\pi\,(I+\mathcal{L})^{-\frac{1}{2}(m-[\alpha]+\nu)}\, V_x^\beta\, \Delta^\alpha\sigma(x,\pi_n)\, \pi\,(I+\mathcal{L})^{\frac{1}{2}\nu}\right|\right|_{\mathscr{L}(\mathscr{H}_\pi)},$$

for $\sigma \in S^m$. We also define

$$||\sigma||^R_{S^m,a,b} := \max_{[\alpha]\leq a,\, [\beta]\leq b} \sup_{\substack{x\in G \\ \pi\in\widehat{G}}} \left|\left|V_x^\beta\, \Delta^\alpha\sigma(x,\pi)\, \pi\,(I+\mathcal{L})^{-\frac{1}{2}(m-[\alpha])}\right|\right|_{\mathscr{L}(\mathscr{H}_\pi)},$$

for $\sigma \in S^m$. Note that $\sigma \in S^m$ if and only if

$$||\sigma||_{S^m,\,a,b,c} < +\infty,$$

for all $a, b \in \mathbb{N}_0$ and all $c \geq 0$. It is not difficult to show that, for any $a, b \in \mathbb{N}_0$ and $c \geq 0$, the functions $||\cdot||_{S^m,\,a,b,c}$ and $||\cdot||^R_{S^m,\,a,b}$ are semi-norms on S^m. Additionally, S^m becomes a Fréchet space when equipped with the semi-norm given by $||\cdot||_{S^m,\,a,b,c}$, for every $a, b \in \mathbb{N}_0$ and $c \geq 0$.

Moreover, for each $m \in \mathbb{R}$, the space Ψ^m admits a Fréchet topology given by the family of semi-norms $\{||\cdot||_{\Psi^m,\,a,b,c} : a, b \in \mathbb{N}_0,\ c \geq 0\}$, which are defined by

$$||\mathrm{Op}(\sigma)||_{\Psi^m,a,b,c} := ||\sigma||_{S^m,\,a,b,c}, \quad \sigma \in S^m.$$

4.6.3 Continuity of operators in Ψ^m on $\mathcal{D}(G)$

The objective in this section is to prove that, for any $m \in \mathbb{R}$, any operator $T \in \Psi^m$ maps $\mathcal{D}(G)$ continuously into itself. We first need the following result.

Proposition 4.6.1. *Let $\sigma \in S^m$, for some $m \in \mathbb{R}$, and suppose that κ_x denotes its associated kernel. If $m < -\frac{l}{2}$ then for any $x \in G$, the following estimates hold:*

$$\begin{aligned}||\kappa_x||_{L^2(G)} &\le C \sup_{\pi \in \widehat{G}} \left|\left|\pi(I+\mathcal{L})^{-\frac{m}{2}}\, \sigma(x,\pi)\right|\right|_{\mathscr{L}(\mathscr{H}_\pi)},\\ ||\kappa_x||_{L^2(G)} &\le C \sup_{\pi \in \widehat{G}} \left|\left|\sigma(x,\pi)\, \pi(I+\mathcal{L})^{-\frac{m}{2}}\right|\right|_{\mathscr{L}(\mathscr{H}_\pi)},\end{aligned}$$

for some $C > 0$*, which does not depend on* σ *or* x*.*

Proof. By Plancherel's Theorem we hobtain

$$||\kappa_x||^2_{L^2(G)} = ||\widehat{\kappa_x}||^2_{L^2(\widehat{G})} = \sum_{\pi \in \widehat{G}} d_\pi\, ||\sigma(x,\pi)||^2_{\mathrm{HS}}.$$

Now, we observe that, for any $\pi \in \widehat{G}$, we have

$$\begin{aligned}||\sigma(x,\pi)||_{\mathrm{HS}} &= \left|\left|\pi(I+\mathcal{L})^{\frac{m}{2}}\, \pi(I+\mathcal{L})^{-\frac{m}{2}}\, \sigma(x,\pi)\right|\right|_{\mathrm{HS}}\\ &\le \left|\left|\pi(I+\mathcal{L})^{\frac{m}{2}}\right|\right|_{\mathrm{HS}} \left|\left|\pi(I+\mathcal{L})^{-\frac{m}{2}}\, \sigma(x,\pi)\right|\right|_{\mathscr{L}(\mathscr{H}_\pi)}.\end{aligned}$$

Hence, we have

$$||\sigma(x,\pi)||_{\mathrm{HS}} \le \sup_{\pi_1 \in \widehat{G}} \left|\left|\pi_1(I+\mathcal{L})^{-\frac{m}{2}}\, \sigma(x,\pi_1)\right|\right|_{\mathscr{L}(\mathscr{H}_\pi)} \left|\left|\pi(I+\mathcal{L})^{\frac{m}{2}}\right|\right|_{\mathrm{HS}}. \quad (4.6.1)$$

So,

$$\begin{aligned}&\sum_{\pi \in \widehat{G}} d_\pi \left|\left|\sigma(x,\pi)\right|\right|^2_{\mathrm{HS}}\\ &\quad\le \sup_{\pi_1 \in \widehat{G}} \left|\left|\pi_1(I+\mathcal{L})^{-\frac{1}{2}m}\, \sigma(x,\pi_1)\right|\right|_{\mathscr{L}(\mathscr{H}_\pi)} \left(\sum_{\pi \in \widehat{G}} d_\pi \left|\left|\pi(I+\mathcal{L})^{\frac{1}{2}m}\right|\right|^2_{\mathrm{HS}}\right).\end{aligned}$$

Applying the Plancherel Theorem to $\pi(I+\mathcal{L})^{\frac{m}{2}}$, we obtain

$$\sum_{\pi \in \widehat{G}} d_\pi \left|\left|\pi(I+\mathcal{L})^{\frac{1}{2}m}\right|\right|^2_{\mathrm{HS}} = ||\mathcal{B}_{-m}||^2_{L^2(G)},$$

where $\mathcal{B}_{-m}$ is the right-convolution kernel associated to the operator $(I+\mathcal{L})^{\frac{m}{2}}$. By Proposition 3.1.8,

$$||\mathcal{B}_{-m}||^2_{L^2(G)} < +\infty,$$

if $m < -\frac{l}{2}$, so the first inequality is proved. The other inequality is similar. □

We now consider the space $\mathcal{D}(G)$ of smooth compactly supported functions on G. We endow this space with the family of semi-norms

$$||f||_{\mathcal{D}(G),N} := \sup_{\substack{\beta \in \mathcal{I}(k) \\ |\beta| < N \\ x \in K_i}} |X_\beta f(x)|.$$

Theorem 4.6.2. *Let $m \in \mathbb{R}$. Then any pseudo-differential operator $T \in \Psi^m$ maps $\mathcal{D}(G)$ continuously into itself. That is, for any $N' \in \mathbb{N}_0$, there exist a constant $C > 0$ and $N \in \mathbb{N}_0$ such that*

$$||Tf||_{\mathcal{D}(G),N'} \leq C\,||f||_{\mathcal{D}(G),N},$$

for every $f \in \mathcal{D}(G)$. In particular, for any $\beta \in \mathcal{I}(k)$, if

$$N > \frac{1}{2}\left(m + |\beta| + \frac{l}{2}\right),$$

then there exists a constant $C_1 > 0$, depending on G, X, β and N, but is independent of σ, such that

$$\left|\left|X_\beta \operatorname{Op}(\sigma) f\right|\right|_{L^2(G)} \leq C_1\,||\sigma||_{S^m,0,|\beta|,N}\,\left|\left|(I+\mathcal{L})^N f\right|\right|_{L^2(G)}, \tag{4.6.2}$$

Proof. Let $T \in \Psi^m$ and suppose that $f \in \mathcal{D}(G)$. We know that if $\kappa : (x,z) \mapsto \kappa_x(z)$ is the right convolution kernel associated to T, then

$$Tf(x) = \int_G f(z)\,\kappa_x(z^{-1}x)\,\mathrm{d}z, \qquad x \in G.$$

For $\beta \in \mathcal{I}(k)$, the Leibniz formula for vector fields implies that

$$X_\beta Tf(x) = \sum_{|\beta_1|+|\beta_2|=|\beta|} c^\beta_{\beta_1,\beta_2} \int_G f(z)\,X_{\beta_1,x_1=x}\,X_{\beta_2,x_2=z^{-1}x}\,\kappa_{x_1}(x_2)\,\mathrm{d}z,$$

for some constants $c^\beta_{\beta_1,\beta_2}$. We write

$$f(z) = (I+\mathcal{L})^{-N}\,(I+\mathcal{L})^N\,f(z),$$

for some $N \in \mathbb{N}_0$ to be determined later. Then, we have

$$
\begin{aligned}
&X_\beta Tf(x)\\
&= \sum_{|\beta_1|+|\beta_2|=|\beta|} c^{\beta}_{\beta_1,\beta_2} \int_G \big[(I+\mathcal{L})^{-N}(I+\mathcal{L})^N f(z)\big] X_{\beta_1,x_1=x} X_{\beta_2,x_2=z^{-1}x} \kappa_{x_1}(x_2)\,\mathrm{d}z\\
&= \sum_{|\beta_1|+|\beta_2|=|\beta|} c^{\beta}_{\beta_1,\beta_2} \int_G \big[(I+\mathcal{L})^{N} f(z)\big] \big[(I+\widetilde{\mathcal{L}})^{-N} X_{\beta_1,x_1=x} X_{\beta_2,x_2=z^{-1}x} \kappa_{x_1}\big](x_2)\,\mathrm{d}z,
\end{aligned}
$$

by employing integration by parts. By the Cauchy-Schwartz inequality, we obtain the estimate

$$
|X_\beta Tf(x)| \leq C' \sum_{|\beta_1|+|\beta_2|=|\beta|} \big|\big|(I+\mathcal{L})^N f\big|\big|_{L^2(G)} \big|\big|(I+\widetilde{\mathcal{L}})^{-N} X_{\beta_1,x} X_{\beta_2,z} \kappa_x\big|\big|_{L^2(G)},
$$

for some $C' > 0$ depending on β. Now, since for every $\beta_1, \beta_2 \in \mathcal{I}(k)$ the symbol given by

$$
\mathcal{F}\big\{(I+\widetilde{\mathcal{L}})^{-N} X_{\beta_1,x} X_{\beta_2,z}\, \kappa_x\big\}(\pi) \;=\; \pi(X_{\beta_2})\, X_{\beta_1,x}\; \sigma(x,\pi)\, \pi(I+\mathcal{L})^{-N}
$$

is of class $m + |\beta_2| - 2N$, then Proposition 4.6.1 implies that

$$
\begin{aligned}
&\big|\big|\, (I+\widetilde{\mathcal{L}})^{-N} X_{\beta_1,x}\, X_{\beta_2,z}\, \kappa_x \,\big|\big|_{L^2(G)}\\
&\quad \leq\; C \sup_{\pi \in \widehat{G}} \big|\big|\, \pi(I+\mathcal{L})^{-\frac{1}{2}(m+|\beta_2|-2N)}\, \pi(X_{\beta_2})\, X_{\beta_1,x}\; \sigma(x,\pi)\, \pi(I+\mathcal{L})^{-N} \,\big|\big|_{\mathscr{L}(\mathscr{H}_\pi)}\\
&\quad \leq\; C\, ||\sigma||_{S^m,0,|\beta_1|,N}, \qquad (4.6.3)
\end{aligned}
$$

whenever $m + |\beta_2| - 2N < -2$. Since $|\beta_2| \leq |\beta|$, then the condition

$$
m + |\beta| - 2N < -\frac{l}{2}
$$

is sufficient. Additionally, since $|\beta_1| \leq |\beta|$, (4.6.3) implies that

$$
\big|\big|\, (I+\widetilde{\mathcal{L}})^{N} X_{\beta_1,x}\, X_{\beta_2,z}\, \kappa_x \,\big|\big|_{L^2(G)} \;\leq\; C\, ||\sigma||_{S^m,0,|\beta|,N}, \qquad (4.6.4)
$$

whenever $m + |\beta| - 2N < -\frac{l}{2}$. Let us then fix $N > \frac{1}{2}\left(m + |\beta| + \frac{l}{2}\right)$. Thus, we obtain the estimate

$$|X_\beta Tf(x)| \lesssim \left|\left|(I+\mathcal{L})^N f\right|\right|_{L^2(G)} ||\sigma||_{S^m,0,|\beta|,N}. \tag{4.6.5}$$

Since $f \in \mathcal{D}(G)$, then it follows that there exists $N' \in \mathbb{N}_0$, depending on N, such that

$$\left|\left|(I+\mathcal{L})^N f\right|\right|_{L^2(G)} \leq ||f||_{\mathcal{D}(G),N'},$$

which proves the result.

More precisely, by (4.6.5), we have shown that there exists $C_1 > 0$ such that for any $\sigma \in S^m$ and every $\beta \in \mathcal{I}(k)$ we have

$$\left|\left|X_\beta \operatorname{Op}(\sigma) f\right|\right|_{L^2(G)} \leq C_1 ||\sigma||_{S^m,0,|\beta|,N} \left|\left|(I+\mathcal{L})^N f\right|\right|_{L^2(G)},$$

where N denotes the smallest non-negative integer satisfying

$$N > \frac{1}{2}\left(m + |\beta| + \frac{l}{2}\right).$$

□

4.7 Kernel estimates

As has been the case in previous sections, G denotes a compact Lie group of dimension n and local dimension l (see Definition A.2.1). Furthermore, we shall suppose that $Q = \{q_1, q_2, \ldots, q_\ell\}$ is a family of smooth, real-valued functions on G, which is comparable to the C-C metric, with weight $(\omega_1, \omega_2, \ldots, \omega_\ell)$. For any $\alpha \in \mathbb{N}_0^\ell$, we shall denote

$$[\alpha] := [\alpha]_Q,$$

and

$$\Delta^\alpha := \Delta_Q^\alpha.$$

Furthermore, we let ω_0 denote the lowest common multiple of the numbers $\omega_1, \omega_2, \ldots, \omega_\ell$, and for a given $m \in \mathbb{R}$, we define

$$N_m = \left\lceil \frac{m+2l}{2\omega_0} \right\rceil. \tag{4.7.1}$$

In this section we aim to prove the following result.

Theorem 4.7.1. *Let $\sigma \in S^m$, for $m \in \mathbb{R}$, and suppose that κ_x denotes its associated kernel. Then, for each $x \in G$, the mapping $z \mapsto \kappa_x(z)$ is smooth on $G \backslash \{e_G\}$, and furthermore there exist $C > 0$, $a, b \in \mathbb{N}_0$ and $c > 0$ such that*

$$|\kappa_x(z)| \leq C \, ||\sigma||_{S^m,a,b,c} \, |z|^{N_m}, \qquad \forall \, x \in G, \; z \in G \backslash \{e_G\}. \tag{4.7.2}$$

The rest of this section is devoted to the proof of Theorem 4.7.1. We start Lemma 4.7.2.

Lemma 4.7.2. *For $m \in \mathbb{R}$ let $\sigma \in S^m$ and suppose that κ_x is its associated convolution kernel.*

(1) If $\beta \in \mathcal{I}(k)$, $\alpha \in \mathbb{N}_0^\ell$ and $\beta_1, \beta_2 \in \mathcal{I}(k)$ are such that

$$m - [\alpha] + |\beta_1| + |\beta_2| < -\frac{l}{2},$$

then the distribution $X_{\beta_1,z} \widetilde{X}_{\beta_2,z} \left(X_{\beta,x} \, \widetilde{q}_\alpha(z) \, \kappa_x(z) \right)$ is square integrable and for every $x \in G$, there exist $C > 0$ and a, $b \in \mathbb{N}_0$ and $c > 0$, such that

$$\int_G \left| X_{\beta_1,z} \widetilde{X}_{\beta_2,z} \left(X_{\beta,x} \, \widetilde{q}_\alpha(z) \, \kappa_x(z) \right) \right|^2 dz \leq C \, ||\sigma||^2_{S^m,a,b,c} \, .$$

(2) If $\beta \in \mathcal{I}(k)$, $\alpha \in \mathbb{N}_0^\ell$ and $\beta_1, \beta_2 \in \mathcal{I}(k)$ are such that

$$m - [\alpha] + |\beta_1| + |\beta_2| < -l,$$

then the distribution $X_{\beta_1,z} \widetilde{X}_{\beta_2,z} \left(X_{\beta,x} \, \widetilde{q}_\alpha(z) \kappa_x(z) \right)$ is continuous on G for every $x \in G$ and there exist $C > 0$ and a, $b \in \mathbb{N}_0$ and $c > 0$ such that

$$\sup_{z \in G} \left| X_{\beta_1,z} \widetilde{X}_{\beta_2,z} \left(X_{\beta,x} \, \widetilde{q}_\alpha(z) \, \kappa_x(z) \right) \right| \leq C \, ||\sigma||_{S^m,a,b,c}. \tag{4.7.3}$$

Proof. Observe that to prove this result it suffices to assume that σ is an invariant symbol; that is,

$$\sigma(x, \pi) = \sigma(\pi), \qquad \forall \, x \in G, \pi \in \widehat{G}.$$

To prove (1), observe that, by Plancherel's Theorem,

$$\left|\left| X_{\beta_1} \widetilde{X}_{\beta_2} (\widetilde{q}_\alpha \kappa) \right|\right|_{L^2(G)} = \left|\left| \pi(X_{\beta_1}) \, \Delta^\alpha \sigma(\pi) \, \pi(X_{\beta_2}) \right|\right|_{L^2(\widehat{G})}.$$

Now,

$$\begin{aligned}
&\big|\big|\pi(X_{\beta_1})\,\Delta^\alpha\sigma(\pi)\,\pi(X_{\beta_2})\big|\big|_{L^2(\widehat{G})}\\
&\quad\leq \big|\big|\pi(X_{\beta_1})\,\pi(I+\mathcal{L})^{\frac{1}{2}(m-[\alpha]+|\beta_2|)}\big|\big|_{L^2(\widehat{G})}\\
&\qquad\qquad \big|\big|\pi(I+\mathcal{L})^{-\frac{1}{2}(m-[\alpha]+|\beta_2|)}\Delta^\alpha\sigma(\pi)\,\pi(I+\mathcal{L})^{\frac{|\beta_2|}{2}}\big|\big|_{L^\infty(\widehat{G})}\\
&\qquad\qquad\qquad \big|\big|\pi(I+\mathcal{L})^{-\frac{|\beta_2|}{2}}\pi(X_{\beta_2})\big|\big|_{L^\infty(\widehat{G})}.
\end{aligned}$$

Since $\pi(X_{\beta_2}) \in S^{|\beta_2|}$, by Proposition 4.5.10, then

$$\big|\big|\pi(I+\mathcal{L})^{-\frac{|\beta_2|}{2}}\pi(X_{\beta_2})\big|\big|_{L^\infty(\widehat{G})} < +\infty.$$

We also have

$$\big|\big|\pi(I+\mathcal{L})^{-\frac{1}{2}(m-[\alpha]+|\beta_2|)}\Delta^\alpha\sigma(\pi)\,\pi(I+\mathcal{L})^{\frac{|\beta_2|}{2}}\big|\big|_{L^\infty(\widehat{G})} \leq ||\sigma||_{S^m,[\alpha],0,|\beta_2|},$$

as $\sigma \in S^m$. Moreover, by Plancherel's Theorem (see Theorem 2.2.7), we obtain

$$\big|\big|\pi(X_{\beta_1})\,\pi(I+\mathcal{L})^{\frac{1}{2}(m-[\alpha]+|\beta_2|)}\big|\big|_{L^2(\widehat{G})} = ||X_{\beta_1}\,\mathcal{B}_{-(m-[\alpha]+|\beta_2|)}||_{L^2(G)},$$

where $\mathcal{B}_{-(m-[\alpha]+|\beta_2|)}$ denotes the right convolution kernel associated with the operator $(I+\mathcal{L})^{\frac{1}{2}(m-[\alpha]+|\beta_2|)}$. Moreover, by Theorem 3.2.3 (h), we obtain

$$\begin{aligned}
||X_{\beta_1}\,\mathcal{B}_{-(m-[\alpha]+|\beta_2|)}||_{L^2(G)} &\leq \big|\big|(I+\mathcal{L})^{\frac{|\beta_1|}{2}}\mathcal{B}_{-(m-[\alpha]+|\beta_2|)}\big|\big|_{L^2(G)}\\
&= \big|\big|\mathcal{B}_{-(m-[\alpha]+|\beta_1|+|\beta_2|)}\big|\big|_{L^2(G)}.
\end{aligned}$$

By Proposition 3.1.9, we then have that this is finite if

$$m-[\alpha]+|\beta_1|+|\beta_2| < -\frac{l}{2}. \tag{4.7.4}$$

In particular, we have shown that if (4.7.4) holds, then there exist constants a, b, $c \in \mathbb{N}_0$ and $C>0$, depending only on α, β_1, β_2, such that

$$||X_{\beta_1}\widetilde{X}_{\beta_2}(\widetilde{q}_\alpha\kappa)||_{L^2(G)} \leq C\,||\sigma||_{S^m,a,b,c},$$

which proves part (1).

In order to prove part (2), first let $s \in \mathbb{R}$ to be determined. By the Sobolev embedding (see Theorem 3.4.1), if $s > l/2$ and $X_{\beta_1}\widetilde{X}_{\beta_2}(\widetilde{q}_\alpha\kappa) \in L^2_s(G)$, then $X_{\beta_1}\widetilde{X}_{\beta_2}(\widetilde{q}_\alpha\kappa)$ is continuous on G and there exists $C > 0$ such that

$$\sup_{z\in G}\left|X_{\beta_1}\widetilde{X}_{\beta_2}\left(\widetilde{q}_\alpha\kappa\right)(z)\right| \,\leq\, C\left|\left|X_{\beta_1}\widetilde{X}_{\beta_2}\left(\widetilde{q}_\alpha\kappa\right)\right|\right|_{L^2_s(G)}. \tag{4.7.5}$$

Let us then fix $s > l/2$. By Theorem 3.2.3 (h), there exists $C > 0$ such that

$$\begin{aligned}\left|\left|X_{\beta_1}\widetilde{X}_{\beta_2}\left(\widetilde{q}_\alpha\kappa\right)\right|\right|_{L^2_s(G)} &\leq\, C\left|\left|(I+\mathcal{L})^{\frac{|\beta_1|}{2}}(I+\widetilde{\mathcal{L}})^{\frac{|\beta_2|}{2}}\left(\widetilde{q}_\alpha\kappa\right)\right|\right|_{L^2_s(G)}\\ &=\, C\left|\left|(I+\mathcal{L})^{\frac{s+|\beta_1|}{2}}(I+\widetilde{\mathcal{L}})^{\frac{|\beta_2|}{2}}\left(\widetilde{q}_\alpha\kappa\right)\right|\right|_{L^2(G)}.\end{aligned}$$

By Plancherel's Theorem (see Theorem 2.2.7), we then have

$$\begin{aligned}&\left|\left|(I+\mathcal{L})^{\frac{s+|\beta_1|}{2}}(I+\widetilde{\mathcal{L}})^{\frac{|\beta_2|}{2}}\left(\widetilde{q}_\alpha\kappa\right)\right|\right|_{L^2(G)}\\ &\quad=\,\left|\left|\pi(I+\mathcal{L})^{\frac{s+|\beta_1|}{2}}\,\Delta^\alpha\sigma(\pi)\,\pi(I+\mathcal{L})^{\frac{|\beta_2|}{2}}\right|\right|_{L^2(\widehat{G})}\\ &\quad\leq\,\left|\left|\pi(I+\mathcal{L})^{\frac{1}{2}(s+|\beta_1|)}\,\Delta^\alpha\sigma(\pi)\,\pi(I+\mathcal{L})^{-\frac{1}{2}(m-[\alpha]+s+|\beta_1|)}\right|\right|_{L^\infty(\widehat{G})}\\ &\qquad\qquad\left|\left|\pi(I+\mathcal{L})^{\frac{1}{2}(m-[\alpha]+s+|\beta_1|+|\beta_2|)}\right|\right|_{L^2(\widehat{G})}.\end{aligned}$$

Now, observe that

$$\begin{aligned}&\left|\left|\pi(I+\mathcal{L})^{\frac{1}{2}(s+|\beta_1|)}\,\Delta^\alpha\sigma(\pi)\,\pi(I+\mathcal{L})^{-\frac{1}{2}(m-[\alpha]+s+|\beta_1|)}\right|\right|_{L^\infty(\widehat{G})}\\ &\quad=\,\left|\left|\pi(I+\mathcal{L})^{-\frac{1}{2}(m-[\alpha]+\nu_1)}\,\Delta^\alpha\sigma(\pi)\,\pi(I+\mathcal{L})^{\frac{\nu_1}{2}}\right|\right|_{L^\infty(\widehat{G})}\\ &\quad\leq\,||\sigma||_{S^m,[\alpha],0,|\nu_1|},\end{aligned}$$

where

$$\nu_1 \,:=\, -(m-[\alpha]+s+|\beta_1|).$$

Moreover,

$$\left|\left|\pi(I+\mathcal{L})^{\frac{1}{2}(m-[\alpha]+s+|\beta_1|+|\beta_2|)}\right|\right|_{L^2(\widehat{G})} \,<\, +\infty,$$

provided that

$$m-[\alpha]+s+|\beta_1|+|\beta_2| \,<\, -\frac{l}{2},$$

by Proposition 3.1.9. Since $s > l/2$, then the condition becomes

$$m - [\alpha] + |\beta_1| + |\beta_2| \, < \, -l,$$

proving part (2).

□

We can now show Theorem 4.7.1.

Proof of Theorem 4.7.1. The Leibniz property for vector fields implies that it suffices to prove the result for invariant symbols. For $N \in \mathbb{N}_0$, we now define the function

$$f_N \, = \, \sum_{j=1}^{\ell} q_{N,j},$$

where, for each $j = 1, 2, \ldots, n$, the function $q_{N,j}$ is given by

$$q_{N,j} \, := \, q_j^{\frac{2\omega_0 N}{\omega_j}},$$

where ω_0 is the highest common divisor of $\omega_1,\ \omega_2,\ \ldots,\ \omega_\ell$. Observe that, for any $z \in G$,

$$|f_N(z)| \, \approx \, \left(|q_1(z)|^{\frac{1}{\omega_1}} + |q_2(z)|^{\frac{1}{\omega_2}} + \cdots + |q_n(z)|^{\frac{1}{\omega_n}}\right)^{2\omega_0 N} \, \approx \, |z|^{2\omega_0 N}, \tag{4.7.6}$$

since Q has weight $(\omega_1, \omega_2, \ldots, \omega_\ell)$. We now define the multi-index

$$\alpha_{N,j} \, = \, \left(0,\, 0,\, \ldots,\, 0,\, \frac{2\omega_0 N}{\omega_j},\, 0,\, \ldots,\, 0\right), \qquad j = 1, 2, \ldots, \ell,$$

with the non-zero value $\frac{2\omega_0 N}{\omega_j}$ in the j-th position. Then,

$$f_N(z) \, = \, \sum_{j=1}^{\ell} q_{N,j}(z) \, = \, \sum_{j=1}^{\ell} q_{\alpha_{N,j}}(z).$$

Observe that, for every $j = 1, 2, \ldots, \ell$,

$$\left[\alpha_{N,j}\right] \, = \, 2\omega_0 N.$$

Hence, by Lemma 4.7.2 (2), the mapping $f_N \kappa$ is continuous provided that

$$m - 2\omega_0 N \, < \, -l, \tag{4.7.7}$$

and there exist $C > 0$ and $a,\, b,\, c \in \mathbb{N}_0$ such that

$$||f_N \kappa||_{L^\infty(G)} \leq C\, ||\sigma||_{S^m,a,b,c}.$$

By (4.7.6), for any $z \in G$, we then obtain

$$|z|^{2\omega_0 N} |\kappa(z)| \lesssim ||f_N \kappa||_{L^\infty(G)} \lesssim ||\sigma||_{S^m,a,b,c}.$$

If we choose

$$N := N_m = \left\lceil \frac{m + 2l}{2\omega_0} \right\rceil,$$

then, in particular, (4.7.7) is satisfied. Hence, (4.7.2) is obtained, and the result is proved.

□

4.8 Smoothing symbols

We shall continue with the same setting as in previous sections. Suppose that $Q = \{q_1, q_2, \ldots, q_\ell\}$ is a family of smooth, real-valued functions on G, which is comparable to the C-C metric, with weight $(\omega_1, \omega_2, \ldots, \omega_\ell)$. Additionally, suppose that

$$\boldsymbol{Y} = \{Y_1, Y_2, \ldots, Y_n\}$$

is the basis of the Lie algebra $\mathfrak{g}$ of G constructed in Section 2.4.1. Throughout this section, for any $m \in \mathbb{R}$ we let S^m denote the space of symbols of class m, with respect to $\mathcal{L}$, $\boldsymbol{Y}$ and Q.

4.8.1 Main result

Recall that, for a given function $\kappa \in \mathcal{D}(G \times G)$, with

$$\kappa : (x, z) \longmapsto \kappa_x(z),$$

its associated symbol is the collection

$$\sigma = \{\widehat{\kappa_x}(\pi) : x \in G,\, \pi \in \widehat{G}\},$$

and we often write $\sigma = \widehat{\kappa_x}$ to denote this relationship.

The objective in this section is to prove the following theorem.

Theorem 4.8.1. *The mapping*

$$\begin{cases} \mathcal{D}(G \times G) & \longrightarrow & S^{-\infty} \\ \kappa & \longmapsto & \sigma = \widehat{\kappa_x} \end{cases} \tag{4.8.1}$$

is an isomorphism of topological vector spaces. Hence, a symbol is smoothing if and only if its associated convolution kernel is smooth.

Recall that a symbol σ on G is said to be smoothing if

$$\sigma \in S^{-\infty} := \bigcap_{m \in \mathbb{R}} S^m.$$

This is a notion we introduced in Definition 4.5.6.

Furthermore, the space $S^{-\infty}$ is equipped with the projective limit induced by $\bigcap_{m \in \mathbb{R}} S^m$; that is, the topology for which the inclusion $S^{-\infty} \subset S^m$ is continuous, for any $m \in \mathbb{R}$.

Proof of Theorem 4.8.1. We first show that, if $\sigma \in S^{-\infty}$, then its associated convolution, $\kappa : (x, z) \mapsto \kappa_x(z)$, is smooth; that is,

$$\sup_{x,z \in G} \sup_{\substack{\alpha,\beta \in \mathcal{I}(k) \\ |\alpha| < N_1 \\ |\beta| < N_2}} \left| X_{\alpha,z} X_{\beta,x} \kappa_x(z) \right| < +\infty,$$

for every $N_1, N_2 \in \mathbb{N}_0$. Since $\sigma \in S^{-\infty}$, it follows that for every $N_1, N_2 \in \mathbb{N}_0$, there exists $m \in \mathbb{R}$ such that

$$m + N_1 < -l,$$

and $\sigma \in S^m$. In particular, whenever $\alpha \in \mathcal{I}(k)$, with $|\alpha| < N_1$, we have

$$m + |\alpha| < -l.$$

Hence, by Lemma 4.7.2 part (2), it follows that there exist $C > 0$, $a, b \in \mathbb{N}_0$ and $c > 0$ such that

$$\sup_{x,z \in G} \sup_{\substack{\alpha \in \mathcal{I}(k) \\ |\alpha| < N_1 \\ |\beta| < N_2}} \left| X_{\alpha,z} X_{\beta,x} \kappa_x(z) \right| \le C \, ||\sigma||_{S^m,a,b,c} < +\infty,$$

as required. In particular, this shows that the map

$$\begin{cases} S^{-\infty} & \longrightarrow \quad \mathcal{D}(G\times G) \\ \sigma = \widehat{\kappa_x} & \longmapsto \quad \kappa \end{cases}$$

is continuous, which means that the map given by (4.8.1) has a continuous inverse. Furthermore, this also shows that the map given by (4.8.1) is surjective.

We now show that the map given by (4.8.1) is continuous. Let $\kappa : (x,z) \mapsto \kappa_x(z)$ be a smooth function on $G\times G$; that is, suppose that

$$\sup_{x,z\in G} \sup_{\substack{\alpha,\beta\in\mathcal{I}(k)\\ |\alpha|\le N_1 \\ |\beta|\le N_2}} \left| X_{\alpha,z}\, X_{\beta,x}\, \kappa_x(z)\right| \;<\; +\infty, \tag{4.8.2}$$

for every $N_1, N_2 \in \mathbb{N}_0$. Then, consider the symbol given by

$$\sigma \;=\; \widehat{\kappa_x}.$$

Furthermore, let $\gamma_1,\gamma_2 \in \mathbb{R}$ and suppose that $N_1, N_2 \in \mathbb{N}_0$ such that $\gamma_j \le N_j$, for each $j = 1,2$. Then, for $x \in G$ and $\beta_0 \in \mathbb{N}_0^n$ we have

$$\begin{aligned} &\big|\big|\pi(I+\mathcal{L})^{\gamma_1} Y^{\beta_0}\Delta^{\alpha}\sigma(x,\pi)\pi(I+\mathcal{L})^{\gamma_2}\big|\big|_{L^\infty(\widehat{G})} \\ &\qquad\le \big|\big|\pi(I+\mathcal{L})^{N_1} Y^{\beta_0}\Delta^{\alpha}\sigma(x,\pi)\pi(I+\mathcal{L})^{N_2}\big|\big|_{L^\infty(\widehat{G})} \\ &\qquad= \big|\big|\mathcal{F}\big\{(I+\mathcal{L})^{N_1}(I+\widetilde{\mathcal{L}})^{N_2}\, Y_x^{\beta_0} q_\alpha\, \kappa_x\big\}(\pi)\big|\big|_{L^\infty(\widehat{G})}. \end{aligned}$$

By Lemma 2.2.4, we then have

$$\begin{aligned} &\big|\big|\mathcal{F}\big\{(I+\mathcal{L})^{N_1}(I+\widetilde{\mathcal{L}})^{N_2}\, Y_x^{\beta_0} q_\alpha\, \kappa_x\big\}(\pi)\big|\big|_{L^\infty(\widehat{G})} \\ &\qquad\qquad \le \big|\big|(I+\mathcal{L})^{N_1}(I+\widetilde{\mathcal{L}})^{N_2}\, Y_x^{\beta_0} q_\alpha\, \kappa_x\big|\big|_{L^1(G)}. \end{aligned}$$

Moreover, there exist constants $c_\beta \in \mathbb{R}$ such that

$$Y^{\beta_0} \;=\; \sum_{\substack{\beta\in\mathcal{I}(k)\\ |\beta|\le|\beta_0|}} c_\beta X_\beta.$$

Similarly,

$$(I+\mathcal{L})^{N_1} \;=\; \sum_{\substack{\beta_1\in\mathcal{I}(k)\\ |\beta_1|\le 2N_1}} c_{\beta_1} X_{\beta_1},$$

for some constants $c_{\beta_1} \in \mathbb{R}$ and

$$(I + \widetilde{\mathcal{L}})^{N_2} = \sum_{\substack{\beta_2 \in \mathcal{I}(k) \\ |\beta_2| \leq 2N_2}} \widetilde{c}_{\beta_2} \widetilde{X}_{\beta_2},$$

for some $c_{\beta_2} \in \mathbb{R}$. Hence, it follows that

$$\begin{aligned} &\big|\big|(I + \mathcal{L})^{N_1} (I + \widetilde{\mathcal{L}})^{N_2}\, Y_x^{\beta_0} q_\alpha\, \kappa_x\big|\big|_{L^1(G)} \\ &\qquad \lesssim \sum_{\substack{\beta, \beta_1, \beta_2 \in \mathcal{I}(k) \\ |\beta_1| \leq 2N_1,\, |\beta_2| \leq 2N_2 \\ |\beta| \leq |\beta_0|}} \big|\big|X_{\beta_1} \widetilde{X}_{\beta_2} X_{\beta,x}\, q_\alpha\, \kappa_x\big|\big|_{L^1(G)} \\ &\qquad \lesssim \sup_{x,z \in G} \sup_{\substack{\beta, \beta' \in \mathcal{I}(k) \\ |\beta| \leq |\beta_0| \\ |\beta'| \leq 2N_1 + 2N_2}} \big|X_{\beta',z}\, X_{\beta,x}\, \kappa_x(z)\big|, \end{aligned}$$

which is finite, by the hypothesis (see (4.8.2)). Hence, we have shown that the map given by (4.8.1) is continuous.

Additionally, we know that this map is linear and one-to-one. Therefore, we conclude that it is an isomorphism of topological vector spaces.

□

4.8.2 Consequence

Many properties we study throughout this thesis associated with symbols will hold for smoothing symbols. So, the argument we provide below shows that, given a symbol $\sigma \in S^m$, we may assume its associated convolution kernel is supported in a neighbourhood of e_G, whenever the context is appropriate.

Suppose that, for $m \in \mathbb{R}$, $\sigma \in S^m$ is a symbol on G and let $\kappa : (x, z) \mapsto \kappa_x(z)$ denote its associated convolution kernel. Furthermore, let U, V be neighbourhoods of e_G in G satisfying

$$\{e_G\} \subset U \subset V,$$

and let $\chi \in \mathcal{D}(G)$ be a cut-off function, taking values in $[0, 1]$, such that

$$\chi(z) \equiv 1 \text{ on } U, \qquad \chi(z) \equiv 0 \text{ on } V^c,$$

Then, we may write

$$\kappa_x(z) \,=\, \kappa_x(z)\chi(z) + \kappa_x(z)(1-\chi(z)), \qquad \forall\,(x,z) \,\in\, G\times G.$$

We now let $\kappa_{1,x}, \kappa'_{1,x}$ be given by

$$\kappa_{1,x}(z) \,:=\, \kappa_x(z)\chi(z), \qquad \kappa'_{1,x}(z) \,=\, \kappa_x(z)(1-\chi(z)), \qquad (x,z) \,\in\, G\times G,$$

and suppose $\tau_1,\ \tau'_1$ denote their associated symbols, respectively:

$$\tau_1(x,\pi) \,=\, \mathcal{F}\{\kappa_{1,x}\}(\pi), \qquad \tau'_1(x,\pi) \,=\, \mathcal{F}\{\kappa'_{1,x}\}(\pi), \qquad x \,\in\, G,\ \pi \,\in\, \widehat{G}.$$

Observe that, as $\mathrm{supp}(\kappa'_{1,x}) \subset V^c$, then $\kappa'_{1,x}$ is smooth on G, by Proposition 4.7.1. So, by Theorem 4.8.1, the symbol τ'_1 is smoothing; that is, for any $m' \,\in\, \mathbb{R}$ the following assertion holds:

$$\forall\, a,b \,\in\, \mathbb{N}_0,\ c>0, \qquad ||\tau'_1||_{S^{m'},a,b,c} \,<\, +\infty. \tag{4.8.3}$$

Now, for any $\alpha \,\in\, \mathbb{N}_0^\ell,\ \beta \,\in\, \mathbb{N}_0^n$ and $\nu \,\in\, \mathbb{R}$ we have

$$\begin{aligned}
&\big|\big|\pi(I+\mathcal{L})^{-\frac{1}{2}(m-[\alpha]+\nu)}\, Y^\beta \Delta^\alpha\, \sigma(x,\pi)\, \pi(I+\mathcal{L})^{\frac{\nu}{2}}\big|\big|_{L^\infty(\widehat{G})}\\
&\qquad \le\, \big|\big|\pi(I+\mathcal{L})^{-\frac{1}{2}(m-[\alpha]+\nu)}\, Y^\beta \Delta^\alpha\, \tau_1(x,\pi)\, \pi(I+\mathcal{L})^{\frac{\nu}{2}}\big|\big|_{L^\infty(\widehat{G})}\\
&\qquad\qquad +\, \big|\big|\pi(I+\mathcal{L})^{-\frac{1}{2}(m-[\alpha]+\nu)}\, Y^\beta \Delta^\alpha\, \tau'_1(x,\pi)\, \pi(I+\mathcal{L})^{\frac{\nu}{2}}\big|\big|_{L^\infty(\widehat{G})}.
\end{aligned}$$

Moreover, by (4.8.3),

$$\big|\big|\pi(I+\mathcal{L})^{-\frac{1}{2}(m-[\alpha])}\, Y^\beta \Delta^\alpha\, \tau'_1(x,\pi)\, \pi(I+\mathcal{L})^{\frac{\nu}{2}}\big|\big|_{L^\infty(\widehat{G})} \,\le\, ||\tau'_1||_{S^m,[\alpha],[\beta],|\nu|} \,<\, +\infty,$$

and in particular, there exists $C>0$ such that

$$\begin{aligned}
&\big|\big|\pi(I+\mathcal{L})^{-\frac{1}{2}(m-[\alpha]+\nu)}\, Y^\beta \Delta^\alpha\, \sigma(x,\pi)\, \pi(I+\mathcal{L})^{\frac{\nu}{2}}\big|\big|_{L^\infty(\widehat{G})}\\
&\qquad \le\, \big|\big|\pi(I+\mathcal{L})^{-\frac{1}{2}(m-[\alpha]+\nu)}\, Y^\beta \Delta^\alpha\, \tau_1(x,\pi)\, \pi(I+\mathcal{L})^{\frac{\nu}{2}}\big|\big|_{L^\infty(\widehat{G})} \,+\, C.
\end{aligned}$$

Here, τ_1 is a symbol on G of class m, whose associated kernel is supported in V. This shows we may assume the convolution kernel associated to σ is supported in a neighbourhood of e_G.

4.9 The continuous inclusion $S^m(Q_0) \subset S^m(Q)$

Recall that, for any family Q of smooth real-valued functions on G, which is comparable to the C-C metric, any basis of vector fields $\boldsymbol{V}$ on G, and any $m \in \mathbb{R}$,

$$S^m(G, \mathcal{L}, \boldsymbol{V}, Q)^{\text{sub}}$$

denotes the space of symbols of class m with respect to $\mathcal{L}$, $\boldsymbol{V}$ and Q (see Definition 4.5.1). By the work done in Section 4.6.1, S^m is independent of the choice of basis of vector fields $\boldsymbol{V}$, so it shall be convenient for us to fix one throughout the rest of the section. To this aim, we let

$$\boldsymbol{Y} := \{Y_1, Y_2, \ldots, Y_n\}$$

denote the basis of left-invariant vector fields constructed in Section 2.4.1. For any family Q of smooth, real-valued functions on G, which is comparable to the C-C metric, and $m \in \mathbb{R}$, we shall then write

$$S^m(Q)$$

for the space of symbols of class m with respect to $\mathcal{L}$, $\boldsymbol{Y}$ and Q, omitting any mention of $\mathcal{L}$ and $\boldsymbol{Y}$. Moreover, recall that

$$Q_0 := \{q_{0,1}, q_{0,2}, \ldots, q_{0,n}\} \tag{4.9.1}$$

denotes the family of smooth, real-valued functions on G given by (4.2.12). Throughout this section we shall consider the family of symbols of class m, with respect to Q_0,

$$S^m(Q_0).$$

The objective in this section is to show that, if Q is any family of smooth, real-valued functions comparable to the C-C metric, then for any $m \in \mathbb{R}$, the space $S^m(Q_0)$ is contained in $S^m(Q)$. The following proposition, which we shall prove later in Section 4.9.2, summarises this result.

Proposition 4.9.1. *For some $\ell \in \mathbb{N}$, let $Q = \{q_1, q_2, \ldots, q_\ell\}$ be any family* *of smooth, real-valued functions on G, which is comparable to the C-C metric. Then,*

$$S^m(Q_0) \subset S^m(Q).$$

Furthermore, the inclusion is continuous.

The proof of Proposition 4.9.1 will require an important lemma, which allows us to compare symbol classes semi-norms for different difference operators. We study this lemma in the following section.

4.9.1 An important lemma

Lemma 4.9.2. *Let $q, q' \in \mathcal{D}(G)$ be such that the function*

$$\frac{q'}{q} : z \longmapsto \frac{q'(z)}{q(z)}, \qquad z \in G$$

extends to a smooth function on G. Let $s_1 \in \mathbb{N}_0$ and $s_2 \in \mathbb{R}$, and suppose σ is an invariant symbol such that

$$\left|\left|\pi(I+\mathcal{L})^{\frac{s_1}{2}}\Delta_q\sigma(\pi)\pi(I+\mathcal{L})^{\frac{s_2}{2}}\right|\right|_{L^\infty(\widehat{G})} < +\infty.$$

Then, there exists a constant $C > 0$, depending on G, $\mathcal{L}$, s_1, s_2, q and q', such that

$$\begin{aligned}\left|\left|\pi(I+\mathcal{L})^{\frac{s_1}{2}}\Delta_{q'}\sigma(\pi)\,\pi(I+\mathcal{L})^{\frac{s_2}{2}}\right|\right|_{L^\infty(\widehat{G})}& \\ \leq C\left|\left|\pi(I+\mathcal{L})^{\frac{s_1}{2}}\Delta_{q}\sigma(\pi)\,\pi(I+\mathcal{L})^{\frac{s_2}{2}}\right|\right|_{L^\infty(\widehat{G})}&. \qquad (4.9.2)\end{aligned}$$

Proof. Let $\kappa \in \mathcal{D}'(G)$ be the convolution kernel associated to σ. Moreover, we also let

$$\sigma_1 := \mathcal{F}\{q\kappa\}, \qquad \sigma_1' := \mathcal{F}\{q'\kappa\}.$$

Observe that,

$$\begin{aligned}\left|\left|\pi(I+\mathcal{L})^{\frac{s_1}{2}}\Delta_q\,\sigma(\pi)\,\pi(I+\mathcal{L})^{\frac{s_2}{2}}\right|\right|_{L^\infty(\widehat{G})}& \\ = \left|\left|(I+\mathcal{L})^{\frac{s_1}{2}}\,\mathrm{Op}(\sigma_1)\,(I+\mathcal{L})^{\frac{s_2}{2}}\right|\right|_{\mathscr{L}(L^2(G))}&.\end{aligned}$$

The same equality holds if we substitute q' for q and σ_1' for σ_1. Thus, it suffices to show that there exists $C > 0$ such that

$$\begin{aligned}\big|\big|(I+\mathcal{L})^{\frac{s_1}{2}}\,\mathrm{Op}(\sigma_1')\,(I+\mathcal{L})^{\frac{s_2}{2}}\big|\big|_{\mathscr{L}(L^2(G))} \\ \leq C\,\big|\big|(I+\mathcal{L})^{\frac{s_1}{2}}\,\mathrm{Op}(\sigma_1)\,(I+\mathcal{L})^{\frac{s_2}{2}}\big|\big|_{\mathscr{L}(L^2(G))}.\end{aligned} \tag{4.9.3}$$

Or equivalently,

$$\big|\big|\mathrm{Op}(\sigma_1')\big|\big|_{\mathscr{L}\left(L^2_{-s_2}(G),L^2_{s_1}(G)\right)} \leq C_0\,\big|\big|\mathrm{Op}(\sigma_1)\big|\big|_{\mathscr{L}\left(L^2_{-s_2}(G),L^2_{s_1}(G)\right)}.$$

Let $\phi \in \mathcal{D}(G)$. For any $x \in G$, we have

$$\mathrm{Op}(\sigma_1')\,\phi(x) = \phi * (q'\kappa)(x) = \int_G \phi(y)\,(q'\kappa)(y^{-1}x)\,\mathrm{d}y.$$

For a fixed $x \in G$, we now define the mapping

$$\psi_x(y) := \frac{q'}{q}(y^{-1}x), \quad y \in G.$$

By our hypothesis, this map extends to a smooth function both in x and y. Then, we have

$$\mathrm{Op}(\sigma_1')\,\phi(x) = \int_G \phi(y)\,\psi_x(y)\,(q\kappa)(y^{-1}x)\,\mathrm{d}y.$$

Since $s_1 \in \mathbb{N}_0$, by Theorem 3.2.3 (h) we then have

$$\big|\big|\mathrm{Op}(\sigma_1')\,\phi\big|\big|^2_{L^2_{s_1}(G)} \approx \int_G \sum_{\substack{\beta\in\mathcal{I}(k)\\|\beta|\leq s_1}} \left|X_{\beta,x}\int_G \phi(y)\,\psi_x(y)\,(q\kappa)(y^{-1}x)\,\mathrm{d}y\right|^2 \mathrm{d}x.$$

Using Leibniz's rule for vector fields, we have

$$\begin{aligned}&\big|\big|\mathrm{Op}(\sigma_1')\,\phi\big|\big|^2_{L^2_{s_1}(G)} \\ &\quad\lesssim \int_G \sum_{\substack{\beta_1,\beta_2\in\mathcal{I}(k)\\|\beta_1|+|\beta_2|\leq s_1}} \left|\int_G \phi(y)\,X_{\beta_1,x_1=x}\psi_{x_1}(y)\,X_{\beta_2,x_2=x}(q\kappa)(y^{-1}x_2)\,\mathrm{d}y\right|^2 \mathrm{d}x.\end{aligned}$$

We now take the supremum over $x_1 \in G$ of $X_{\beta_1,x_1}\psi_{x_1}(y)$ outside of the integral over y, to obtain

$$\big|\big|\mathrm{Op}(\sigma_1')\,\phi\big|\big|^2_{L^2_{s_1}(G)}$$
$$\lesssim \int_G \sup_{x_1\in G} \sum_{\substack{\beta_1,\beta_2\in\mathcal{I}(k)\\ |\beta_1|+|\beta_2|\le s_1}} \left|\int_G \phi(y) X_{\beta_1,x_1}\psi_{x_1}(y) X_{\beta_2,x_2=x}(q\kappa)(y^{-1}x_2)\,\mathrm{d}y\right|^2 \mathrm{d}x.$$

Moreover, by the Sobolev embedding (see Theorem 3.4.1), there exists a constant $C_1>0$ such that, for any $x\in G$ and every $\beta_1,\beta_2\in\mathcal{I}(k)$, with $|\beta_1|+|\beta_2|\le s_1$, we have

$$\sup_{x_1\in G}\left|\int_G \phi(y) X_{\beta_1,x_1}\psi_{x_1}(y) X_{\beta_2,x_2=x}(q\kappa)(y^{-1}x_2)\,\mathrm{d}y\right|^2$$
$$\le C_1 \int_G \left|(I+\mathcal{L}_{x_1})^{\frac{s'}{2}}\, X_{\beta_1,x_1}\int_G \phi(y)\,\psi_{x_1}(y)\, X_{\beta_2,x_2=x}(q\kappa)(y^{-1}x_2)\,\mathrm{d}y\right|^2 \mathrm{d}x_1,$$

whenever $s'>l/2$, where l denotes the local dimension of G (see Definition A.2.1). For convenience we may choose $s'=\lceil\frac{l}{2}\rceil+1$. Moreover, observe that for each $x_1\in G$ we have

$$\int_G \phi(y)\,\psi_{x_1}(y)\,X_{\beta_2,x}(q\kappa)(y^{-1}x)\,\mathrm{d}y \;=\; X_{\beta_2}\mathrm{Op}(\sigma_1)(\phi\psi_{x_1})(x), \qquad x\in G.$$

Hence, by Theorem 3.2.3 (h), we obtain

$$\sup_{x_1\in G}\left|\int_G \phi(y) X_{\beta_1,x_1}\psi_{x_1}(y) X_{\beta_2,x_2=x}(q\kappa)(y^{-1}x_2)\,\mathrm{d}y\right|^2$$
$$\le C_1 \int_G \left|(I+\mathcal{L}_{x_1})^{\frac{s'}{2}}\, X_{\beta_1,x_1}X_{\beta_2,x}\mathrm{Op}(\sigma_1)(\phi\psi_{x_1})(x)\right|^2 \mathrm{d}x_1.$$

By our choice of s', we have

$$\sup_{x_1\in G}\left|\int_G \phi(y) X_{\beta_1,x_1}\psi_{x_1}(y) X_{\beta_2,x_2=x}(q\kappa)(y^{-1}x_2)\,\mathrm{d}y\right|^2$$
$$\lesssim \sum_{\substack{\alpha\in\mathcal{I}(k)\\ |\alpha|\le s'}} \int_G \left|X_{\alpha+\beta_1,x_1}X_{\beta_2,x}\mathrm{Op}(\sigma_1)(\phi\psi_{x_1})(x)\right|^2\,\mathrm{d}x_1.$$

Hence, we have obtained

$$\big|\big|\mathrm{Op}(\sigma_1')\phi\big|\big|^2_{L^2_{s_1}(G)} \lesssim \sum_{\substack{\beta_1,\beta_2\,\in\,\mathcal{I}(k)\\ |\beta_1|+|\beta_2|\leq s_1}} \sum_{\substack{\alpha\,\in\,\mathcal{I}(k)\\ |\alpha|\leq s'}} \int_G \big|\big|X_{\beta_2}\mathrm{Op}(\sigma_1)\big(\phi\, X_{\alpha+\beta_1,x_1}\,\psi_{x_1}\big)\big|\big|^2_{L^2(G)}\,\mathrm{d}x_1$$
$$\lesssim \big|\big|\mathrm{Op}(\sigma_1)\big|\big|^2_{\mathscr{L}\left(L^2_{-s_2}(G),L^2_{s_1}(G)\right)}$$
$$\sum_{\substack{\beta_1\,\in\,\mathcal{I}(k)\\ |\beta_1|\leq s_1}} \sum_{\substack{\alpha\,\in\,\mathcal{I}(k)\\ |\alpha|\leq s'}} \int_G \big|\big|\phi\, X_{\alpha+\beta_1,x_1}\,\psi_{x_1}\big|\big|^2_{L^2_{-s_2}(G)}\mathrm{d}x_1, \tag{4.9.4}$$

since, for every $x_1 \in G$ and each $\alpha,\beta_2 \in \mathcal{I}(k)$, with $|\beta_2| \leq s_1$ and $|\alpha| \leq s'$, we have

$$\sum_{\substack{\beta_1\,\in\,\mathcal{I}(k)\\ |\beta_1|+|\beta_2|\leq s_1}} \big|\big|X_{\beta_2}\mathrm{Op}(\sigma_1)\big(\phi\, X_{\alpha+\beta_1,x_1}\,\psi_{x_1}\big)\big|\big|^2_{L^2(G)}$$
$$\lesssim \big|\big|(I+\mathcal{L})^{\frac{s_1}{2}}\mathrm{Op}(\sigma_1)(I+\mathcal{L})^{\frac{s_2}{2}}(I+\mathcal{L})^{-\frac{s_2}{2}}\left(\phi\, X_{\alpha+\beta_1,x_1}\,\psi_{x_1}\right)\big|\big|^2_{L^2(G)}$$
$$\leq \big|\big|\mathrm{Op}(\sigma_1)\big|\big|^2_{\mathscr{L}\left(L^2_{-s_2}(G),L^2_{s_1}(G)\right)}\big|\big|(I+\mathcal{L})^{-\frac{s_2}{2}}\left(\phi\, X_{\alpha+\beta_1,x_1}\,\psi_{x_1}\right)\big|\big|^2_{L^2(G)}.$$

Moreover, by Lemma 3.2.5, there exists $C > 0$, independent of ϕ, such that

$$\big|\big|\phi\, X_{\alpha+\beta_1,x_1}\,\psi_{x_1}\big|\big|_{L^2_{-s_2}(G)} \leq C\,||\phi||_{L^2_{-s_2}(G)}.$$

Hence, by (4.9.4), we obtain

$$\big|\big|\mathrm{Op}(\sigma_1')\,\phi\big|\big|^2_{L^2_{s_1}(G)} \lesssim \big|\big|\mathrm{Op}(\sigma_1)\big|\big|^2_{\mathscr{L}\left(L^2_{-s_2}(G),L^2_{s_1}(G)\right)}\,||\phi||_{L^2_{-s_2}(G)},$$

which yields (4.9.3).

□

The following result is a consequence of the proof of Lemma 4.9.2.

Corollary 4.9.3. *Let* $s_1 \in \mathbb{N}_0$, $s_2 \in \mathbb{R}$ *and fix*

$$s_1' := s_1 + \left\lceil \frac{l}{2} \right\rceil + 1.$$

Moreover, suppose $q, q' \in \mathcal{D}(G)$ *are such that, for every* $\beta \in \mathcal{I}(k)$*, with* $|\beta| \leq s_1'$*, the function*

$$X_\beta\left(\frac{q'}{q}\right) : z \longmapsto X_{\beta,z}\left(\frac{q'(z)}{q(z)}\right), \qquad z \in G,$$

extends to a continuous function on G*. Furthermore, suppose* σ *is an invariant symbol such that*

$$\big|\big|\pi(I+\mathcal{L})^{\frac{s_1}{2}}\,\Delta_q\sigma(\pi)\,\pi(I+\mathcal{L})^{\frac{s_2}{2}}\big|\big|_{L^\infty(\widehat{G})} < +\infty.$$

Then, there exists a constant $C>0$ *such that*

$$\begin{aligned}\big|\big|\pi(I+\mathcal{L})^{\frac{s_1}{2}}\,\Delta_{q'}\sigma(\pi)\,\pi(I+\mathcal{L})^{\frac{s_2}{2}}\big|\big|_{L^\infty(\widehat{G})}&\\ \leq C\,C_{s_1'}\,\big|\big|\pi(I+\mathcal{L})^{\frac{s_1}{2}}\,\Delta_q\sigma(\pi)\,\pi(I+\mathcal{L})^{\frac{s_2}{2}}\big|\big|_{L^\infty(\widehat{G})},&\end{aligned}$$

where

$$C_{s_1'} := \sup_{\substack{z\in G\\ \beta_1\in\mathcal{I}(k),\,|\beta_1|\leq s_1'}} \left|X_{\beta_1}\left(\frac{q'}{q}\right)(z)\right|.$$

For $N\in\mathbb{N}_0$, we now define the function

$$f_N = \sum_{j=1}^{n} q_{N,j}, \tag{4.9.5}$$

where, for each $j=1,2,\ldots,n$, the function $q_{N,j}$ is given by

$$q_{N,j} := q_{0,j}^{\frac{2N_0N}{d_j}},$$

where N_0 is the highest common divisor of $d_1,\ d_2,\ \ldots,\ d_n$. Observe that, for any $z\in G$,

$$|f_N(z)| \approx \left(|q_{0,1}(z)|^{\frac{1}{d_1}}+|q_{0,2}(z)|^{\frac{1}{d_2}}+\cdots+|q_{0,n}(z)|^{\frac{1}{d_n}}\right)^{2N_0N} \approx |z|^{2N_0N}, \tag{4.9.6}$$

since Q_0 has weight $(d_1,d_2,\ldots,d_n)$.

4.9.2 Proof of Proposition 4.9.1

In order to prove this result, observe that it suffices to consider the case of invariant symbols. Then, let $\sigma\in S^m(Q_0)$ be an invariant symbol. We want to show that, for any $\beta \in \mathbb{N}_0^\ell$ and $\nu \in \mathbb{R}$, there exists $C > 0$, $a, b \in \mathbb{N}_0$ and $c > 0$ such that

$$\left|\left|\pi(I+\mathcal{L})^{-\frac{1}{2}(m-[\beta]_Q+\nu)}\,\Delta_Q^\beta\sigma(\pi)\,\pi(I+\mathcal{L})^{\frac{\nu}{2}}\right|\right|_{L^\infty(\widehat{G})} \le C\,||\sigma||_{S^m(Q_0),a,b,c}. \quad (4.9.7)$$

Let $\kappa \in \mathcal{D}'(G)$ denotes the convolution kernel associated to σ. By the work done in Section 4.8.2, we may assume that $\mathrm{supp}(\kappa) \subset B_{r/2}(e_G)$.

Step 1

In the first step of the proof, we find a decomposition for q_β, for any $\beta \in \mathbb{N}_0^\ell$. So, let us fix $\beta \in \mathbb{N}_0^\ell$. Then, by Theorem 4.3.3, for any $M \in \mathbb{N}$ we have

$$q_\beta(z) = \sum_{\substack{\alpha\in\mathbb{N}_0^n\\ [\alpha]_{Q_0}<M}} \frac{1}{\alpha!} q_{0,\alpha}(z)\,Y^\alpha q_\beta(e_G) + R^{q_\beta}_{e_G,M}(z), \qquad \forall\, z \in B_{r/2}(e_G),$$

where $r \in (0,1]$ is the real number satisfying (4.2.8), and

$$|R^{q_\beta}_{e_G,M}(z)| \le C\,|z|^M \max_{\substack{[\alpha]_{Q_0}\ge M\\ |\alpha|\le M}} ||Y^\alpha f||_{L^\infty(G)}, \quad \forall\, z \in B_{r/2}(e_G), \qquad (4.9.8)$$

for some $C > 0$. By Remark 4.3.4, since q_β is CC-vanishing at e_G up to order $[\beta]_Q - 1$, then, assuming $M > [\beta]_P$, we have

$$q_\beta(z) = \sum_{\substack{\alpha\in\mathbb{N}_0^n\\ [\beta]_Q\le[\alpha]_{Q_0}<M}} c_\alpha\, q_{0,\alpha}(z) + R^{q_\beta}_{e_G,M}(z), \qquad \forall\, z \in B_{r/2}(e_G),$$

where for some constants $c_\alpha \in \mathbb{R}$. For simplicity, we shall write

$$\rho_M := R^{q_\beta}_{e_G,M}.$$

Then, for $\nu \in \mathbb{R}$,

$$\big|\big|\pi(I+\mathcal{L})^{-\frac{1}{2}(m-[\beta]_Q+\nu)}\,\Delta_{q_\beta}\sigma(\pi)\,\pi(I+\mathcal{L})^{\frac{\nu}{2}}\big|\big|_{L^\infty(\widehat{G})}$$
$$\lesssim \sum_{\substack{\alpha\in\mathbb{N}_0^n\\ [\beta]_Q\le[\alpha]_{Q_0}<M}} \big|\big|\pi(I+\mathcal{L})^{-\frac{1}{2}(m-[\beta]_Q+\nu)}\,\Delta_{q_{0,\alpha}}\sigma(\pi)\,\pi(I+\mathcal{L})^{\frac{\nu}{2}}\big|\big|_{L^\infty(\widehat{G})}$$
$$+\big|\big|\pi(I+\mathcal{L})^{-\frac{1}{2}(m-[\beta]_Q+\nu)}\,\Delta_{\rho_M}\sigma(\pi)\,\pi(I+\mathcal{L})^{\frac{\nu}{2}}\big|\big|_{L^\infty(\widehat{G})}.$$

Step 2

In this step we examine the sum over α. For each $\alpha\in\mathbb{N}_0^n$, with $[\beta]_Q\le[\alpha]_{Q_0}<M$, we write

$$-\frac{1}{2}(m-[\beta]_Q+\nu)\,=\,\frac{1}{2}([\beta]_Q-[\alpha]_{Q_0})-\frac{1}{2}(m-[\alpha]_{Q_0}+\nu),$$

so that

$$\big|\big|\pi(I+\mathcal{L})^{-\frac{1}{2}(m-[\beta]_Q+\nu)}\,\Delta_{q_{0,\alpha}}\sigma(\pi)\,\pi(I+\mathcal{L})^{\frac{\nu}{2}}\big|\big|_{L^\infty(\widehat{G})}$$
$$\lesssim\big|\big|\pi(I+\mathcal{L})^{\frac{1}{2}([\beta]_Q-[\alpha]_{Q_0})}\big|\big|_{L^\infty(\widehat{G})}$$
$$\big|\big|\pi(I+\mathcal{L})^{-\frac{1}{2}(m-[\alpha]_{Q_0}+\nu)}\,\Delta_{q_{0,\alpha}}\sigma(\pi)\,\pi(I+\mathcal{L})^{\frac{\nu}{2}}\big|\big|_{L^\infty(\widehat{G})}.$$

In this case, by functional analysis,

$$\big|\big|\pi(I+\mathcal{L})^{\frac{1}{2}([\beta]_Q-[\alpha]_{Q_0})}\big|\big|_{L^\infty(\widehat{G})}\,\le\,\sup_{\lambda>0}\,(1+\lambda)^{\frac{1}{2}([\beta]_Q-[\alpha]_{Q_0})}\,<\,+\infty,$$

as $[\beta]_Q-[\alpha]_{Q_0}\le 0$, and since $\sigma\in S^m(Q_0)$, then

$$\big|\big|\pi(I+\mathcal{L})^{-\frac{1}{2}(m-[\alpha]_{Q_0}+\nu)}\,\Delta_{q_{0,\alpha}}\sigma(\pi)\,\pi(I+\mathcal{L})^{\frac{\nu}{2}}\big|\big|_{L^\infty(\widehat{G})}\,\le\,||\sigma||_{S^m(Q_0),[\alpha]_{Q_0},0,|\nu|}.$$

So, we have shown that there exists $C>0$, independent of σ, such that

$$\big|\big|\pi(I+\mathcal{L})^{-\frac{1}{2}(m-[\beta]_Q+\nu)}\,\Delta_{q_{0,\alpha}}\sigma(\pi)\,\pi(I+\mathcal{L})^{\frac{\nu}{2}}\big|\big|_{L^\infty(\widehat{G})}\,\le\,C\,||\sigma||_{S^m(Q_0),[\alpha]_{Q_0},0,|\nu|}.$$

Step 3

We now consider the remainder term:

$$\big|\big|\pi(I+\mathcal{L})^{-\frac{1}{2}(m-[\beta]_Q+\nu)}\,\Delta_{\rho_M}\sigma(\pi)\,\pi(I+\mathcal{L})^{\frac{\nu}{2}}\big|\big|_{L^\infty(\widehat{G})}.$$

First note that we have

$$\begin{aligned}\big|\big|\pi(I+\mathcal{L})^{-\frac{1}{2}(m-[\beta]_Q+\nu)}\,\Delta_{\rho_M}\sigma(\pi)\,\pi(I+\mathcal{L})^{\frac{\nu}{2}}\big|\big|_{L^\infty(\widehat{G})} \\ \leq \big|\big|\pi(I+\mathcal{L})^{\frac{1}{2}([\beta]_Q-2N_0M_1)}\big|\big|_{L^\infty(\widehat{G})} \\ \big|\big|\pi(I+\mathcal{L})^{-\frac{1}{2}(m-2N_0M_1+\nu)}\,\Delta_{\rho_M}\sigma(\pi)\,\pi(I+\mathcal{L})^{\frac{\nu}{2}}\big|\big|_{L^\infty(\widehat{G})}.\end{aligned}$$

Now, consider the function f_{M_1} defined by (4.9.5), for some $M_1\in\mathbb{N}_0$ to be determined. By Lemma 3.7.7, using (4.9.6) and (4.9.8), we have that for any $\beta'\in\mathcal{I}(k)$, with $|\beta'|<M-2N_0M_1$, the function

$$X_{\beta'}\left(\frac{\rho_M}{f_{M_1}}\right)$$

extends to a continuous function on G. Let us now choose $M,M_1\in\mathbb{N}$ such that

$$M-s_1'\,>\,2N_0M_1\,>\,\max\{m+\nu,[\beta]_Q\},$$

where

$$s_1'\,:=\,-(m-[\beta]_Q+\nu)+\left\lceil\frac{l}{2}\right\rceil+1.$$

In this case, $2N_0M_1>[\beta]_Q$, so by functional analysis we have

$$\big|\big|\pi(I+\mathcal{L})^{\frac{1}{2}([\beta]_Q-2N_0M_1)}\big|\big|_{L^\infty(\widehat{G})}\,<\,+\infty.$$

Moreover, $M-2N_0M_1>s_1'$, and if $-(m-2N_0M_1+\nu)\in\mathbb{N}$, then the mapping

$$X_{\beta'}\left(\frac{\rho_M}{f_{M_1}}\right)$$

extends to a continuous function on G, for any $\beta'\in\mathcal{I}(k)$, with $|\beta'|\leq s_1'$. So, by Corollary 4.9.3, we have

$$\begin{aligned}\big|\big|\pi(I+\mathcal{L})^{-\frac{1}{2}(m-2N_0M_1+\nu)}\,\Delta_{\rho_M}\sigma(\pi)\,\pi(I+\mathcal{L})^{\frac{\nu}{2}}\big|\big|_{L^\infty(\widehat{G})} \\ \leq\,C_{s_1'}\,\big|\big|\pi(I+\mathcal{L})^{-\frac{1}{2}(m-2N_0M_1+\nu)}\,\Delta_{f_{M_1}}\sigma(\pi)\,\pi(I+\mathcal{L})^{\frac{\nu}{2}}\big|\big|_{L^\infty(\widehat{G})},\end{aligned}$$

where

$$C_{s_1'} \approx \sup_{\substack{z \in G \\ \beta_1 \in \mathcal{I}(k),\, |\beta_1| \le s_1'}} \left| X_{\beta_1} \left(\frac{\rho_M}{f_{M_1}} \right)(z) \right|.$$

We now define the multi-index

$$\beta_{M_1,j} = \left(0,\, 0,\, \ldots,\, 0,\, \frac{2N_0 M_1}{d_j},\, 0,\, \ldots,\, 0\right), \qquad j = 1, 2, \ldots, n,$$

with the non-zero value $\frac{2N_0M_1}{d_j}$ in the j-th position. Hence,

$$\begin{aligned} &\left|\left|\pi(I+\mathcal{L})^{-\frac{1}{2}(m-2N_0M_1+\nu)}\, \Delta_{f_{M_1}} \sigma(\pi)\, \pi(I+\mathcal{L})^{\nu/2}\right|\right|_{L^\infty(\widehat{G})} \\ &\qquad \lesssim \sum_{j=1}^{n} \left|\left|\pi(I+\mathcal{L})^{-\frac{1}{2}(m-2N_0M_1+\nu)}\, \Delta_{Q_0}^{\beta_{M_1,j}} \sigma(\pi)\, \pi(I+\mathcal{L})^{\nu/2}\right|\right|_{L^\infty(\widehat{G})}. \end{aligned} \tag{4.9.9}$$

Since $[\beta_{M_1,j}]_{Q_0} = 2N_0M_1$ for every $j = 1, 2, \ldots, n$, then it follows that

$$\left|\left|\pi(I+\mathcal{L})^{-\frac{1}{2}(m-2N_0M_1+\nu)}\, \Delta_{f_{M_1}} \sigma(\pi)\, \pi(I+\mathcal{L})^{\nu/2}\right|\right|_{L^\infty(\widehat{G})} \lesssim ||\sigma||_{S^m(Q_0), 2N_0M_1, 0, |\nu|}.$$

Hence, we have shown that if $M, M_1 \in \mathbb{N}_0$ are such that

$$M - s_1' > 2N_0M_1 > \max\{m+\nu, [\beta]_Q\},$$

then there exists $C > 0$, independent of σ, such that

$$\left|\left|\pi(I+\mathcal{L})^{-\frac{1}{2}(m-[\beta]_Q+\nu)}\, \Delta_{\rho_M} \sigma(\pi)\, \pi(I+\mathcal{L})^{\frac{\nu}{2}}\right|\right|_{L^\infty(\widehat{G})} \le C\, ||\sigma||_{S^m(Q_0), 2N_0M_1, 0, |\nu|}.$$

Step 4

We have obtained that (4.9.7) holds for any $\nu \in -m + \mathbb{Z}$, and therefore for any $\nu \in \mathbb{R}$, by Remark 4.5.4. Hence, the result is proved.

□

4.10 Product of symbols

Throughout this section, for any $m \in \mathbb{R}$, the symbol class S^m shall be assumed to be defined with respect to Q_0 (see (4.2.14)) and the basis $\boldsymbol{Y}$ constructed in

Section 2.4.1 (see (2.4.2) and (2.4.1) therein).

4.10.1 Main result

The objective in this section is to prove the following result.

Theorem 4.10.1. *Let* $m_1, m_2 \in \mathbb{R} \cup \{-\infty\}$. *Then, the mapping*

$$\begin{cases} S^{m_1} \times S^{m_2} & \longrightarrow \quad S^{m_1+m_2} \\ (\sigma_1, \sigma_2) & \longmapsto \quad \sigma_1 \sigma_2 \end{cases} \tag{4.10.1}$$

is a morphism of topological vector spaces.

This may be viewed as a generalised Leibniz property for symbols.

For convenience, throughout this section we shall let

$$m := m_1 + m_2, \tag{4.10.2}$$

with the following convention:

$$m' + (-\infty) := -\infty, \qquad \forall\, m' \in \mathbb{R} \cup \{-\infty\}.$$

We now prove the result in the following cases, separately:

(I) $m_1 \in \mathbb{R}$ and $m_2 = -\infty$,

(II) $m_1 = -\infty$ and $m_2 \in \mathbb{R}$,

(III) $m_1 = \infty$ and $m_2 = -\infty$,

(IV) $m_1, m_2 \in \mathbb{R}$.

4.10.2 Proof of cases I, II, III

In this section we aim to show Theorem 4.10.1 for the cases I, II and III. We shall prove this result only for invariant symbols (see Definition 4.4.3), and the general case follows by the Leibniz property for vector fields. We first consider the case that $m_1 \in \mathbb{R}$ and $m_2 = -\infty$, and consider the map

$$(\sigma_1, \sigma_2) \longmapsto \sigma_1 \sigma_2, \qquad \forall\, \sigma_1 \in S^{m_1},\, \sigma_2 \in S^{-\infty} \text{ invariant.} \tag{4.10.3}$$

Let us then fix invariant symbols $\sigma_1 \in S^{m_1}$, $\sigma_2 \in S^{-\infty}$, and furthermore, suppose that κ_1 and κ_2 denote their associated convolution kernels, respectively. Since σ_2 is smoothing, by Theorem 4.8.1 it follows that $\kappa_2 \in \mathcal{D}(G)$. Hence, by Proposition 2.5.7, we have $\kappa_2 * \kappa_1 \in \mathcal{D}(G)$. In particular, this means that the symbol $\sigma_1\sigma_2$ is smoothing, by Theorem 4.8.1. Hence, for any $a \in \mathbb{N}_0$ and $m' \in \mathbb{R}$ we have

$$||\sigma_1\sigma_2||_{S^{m'},a,0,0} = \sup_{\substack{\alpha \in \mathbb{N}_0 \\ [\alpha]\leq a}} \left|\left|\pi(I+\mathcal{L})^{-\frac{1}{2}(m'-[\alpha])}\,\Delta^\alpha(\sigma_1\sigma_2)\right|\right|_{L^\infty(\widehat{G})} < +\infty,$$

thus proving that the map given by (4.10.3) is continuous. We also know that it is linear, hence the result is proved in this case.

The case that $m_1 = -\infty$ and $m_2 \in \mathbb{R}$ is obtained in a similar way, and both of these cases readily imply the case $m_1 = -\infty$ and $m_2 = -\infty$.

4.10.3 Proof of case IV

In this section we shall prove case IV. It suffices to prove this result only for invariant symbols, and the general case follows by the Leibniz property for vector fields.

Fix $m_1, m_2 \in \mathbb{R}$, and let $\sigma_1 \in S^{m_1}$ and $\sigma_2 \in S^{m_2}$ be invariant symbols. We also let $\kappa_1, \kappa_2 \in \mathcal{D}'(G)$ denote their associated convolution kernels, respectively, and fix $m := m_1 + m_2$.

Furthermore, let $\chi \in \mathcal{D}(G)$ be a cut-off function, taking values in $[0,1]$, such that

$$\chi(z) \equiv 1 \;\text{ on }\; B_{r/2}(e_G), \qquad \chi(z) \equiv 0 \;\text{ on }\; B_r(e_G)^c,$$

where r is the real number satisfying (4.2.8). We define the symbols

$$\tau_j(\pi) = \mathcal{F}\{\kappa_j\chi\}(\pi), \qquad \tau_j'(\pi) = \mathcal{F}\{\kappa_j(1-\chi)\}(\pi),$$

for $j = 1, 2$. Observe that, by Theorem 4.8.1, it follows that $\tau_j' \in S^{-\infty}$, for $j = 1, 2$. Moreover, we have

$$\sigma_1\sigma_2 = \tau_1\tau_2 + \tau_1\tau_2' + \tau_1'\tau_2 + \tau_1'\tau_2'.$$

By the cases I, II, III already proved, $\tau_1\tau_2'$, $\tau_1'\tau_2$ and $\tau_1'\tau_2'$ are smoothing symbols.

Since the inclusion $S^{-\infty} \subset S^m$ is continuous, it remains to study the symbol $\tau_1\tau_2$. This implies that case IV follows from the following lemma.

Lemma 4.10.2. *Let $m_1, m_2 \in \mathbb{R}$ and fix $m := m_1 + m_2$. Suppose $\sigma_1 \in S^{m_1}$ and $\sigma_2 \in S^{m_2}$ are invariant symbols and let κ_1 and κ_2 denote their associated convolution kernels, respectively, and assume that $supp(\kappa_j) \subset B_{r/2}(e_G)$, for $j = 1, 2$. Furthermore, let q be a smooth, real-valued function on G, which is CC-vanishing at e_G up to order $a - 1$, for $a \in \mathbb{N}$. Then, for any $\nu \in \mathbb{R}$, there exist $C > 0$, $a_j, b_j \in \mathbb{N}_0$ and $c_j > 0$ $(j = 1, 2)$, independent of σ_1 and σ_2, such that*

$$\begin{aligned}\big|\big|\pi(I+\mathcal{L})^{-\frac{1}{2}(m-a+\nu)}\,\Delta_q(\sigma_1\sigma_2)(\pi)\,\pi(I+\mathcal{L})^{\frac{\nu}{2}}\big|\big|_{L^\infty(\widehat{G})}&\\ \leq C\,||\sigma_1||_{S^{m_1},a_1,b_1,c_1}\,||\sigma_2||_{S^{m_2},a_2,b_2,c_2}&.\end{aligned}$$

Moreover, we can choose $c_2 = |\nu|$.

The next section is devoted to the proof of this result.

4.10.4 Proof of Lemma 4.10.2

In this section we prove Lemma 4.10.2. We split up the proof in several steps.

Step 1

In this step we present a decomposition of the expression $\Delta_q(\sigma_1\sigma_2)$.

Fix $x \in G$ and let $M \in \mathbb{N}_0$, with $M > a$, to be determined. Applying Theorem 4.3.3 to the mapping

$$q(x\cdot) \longmapsto q(xy), \qquad \forall\, y \in G, \tag{4.10.4}$$

we obtain

$$q(xy) = \sum_{[\alpha_1]<M} \frac{1}{\alpha_1!}\, q_{0,\alpha_1}(y)\,(Y^{\alpha_1}q)(x) + R^q_{x,M}(y), \tag{4.10.5}$$

for all $y \in B_{r/2}(e_G)$, where we recall that for each $\alpha_1 \in \mathcal{I}(k)$, q_{0,α_1} is the function given by (4.2.15). Moreover, we have

$$|R^q_{x,M}(y)| \lesssim |y|^M, \qquad \forall\, y \in B_{r/2}(e_G). \tag{4.10.6}$$

Note that

$$\Delta_q(\sigma_1\sigma_2)(\pi) \;=\; \mathcal{F}\{\widetilde{q}(\kappa_2 * \kappa_1)\}(\pi), \qquad \forall\, \pi \,\in\, \widehat{G}.$$

We now have

$$\begin{aligned}(\widetilde{q}(\kappa_2 * \kappa_1))(x) \;&=\; \widetilde{q}(x) \int_G \kappa_2(y)\, \kappa_1(y^{-1}x)\, \mathrm{d}y \\ &=\; \int_G q(x^{-1}yy^{-1})\, \kappa_2(y)\, \kappa_1(y^{-1}x)\, \mathrm{d}y.\end{aligned}$$

By (4.10.5), for any $y \,\in\, B_{r/2}(e_G)$ we obtain

$$q(x^{-1}yy^{-1}) \;=\; \sum_{[\alpha_1]<M} \frac{1}{\alpha_1!}\, q_{0,\alpha_1}(y^{-1})\, (Y^{\alpha_1}q)(x^{-1}y) + R^q_{x^{-1}y,M}(y^{-1}). \tag{4.10.7}$$

Since $\mathrm{supp}(\kappa_1)$, $\mathrm{supp}(\kappa_2) \subset B_{r/2}(e_G)$, then by (4.10.7) we have

$$\begin{aligned}&\int_G q(x^{-1}yy^{-1})\, \kappa_2(y)\, \kappa_1(y^{-1}x)\, \mathrm{d}y \\ &\qquad = \sum_{[\alpha_1]<M} \frac{1}{\alpha_1!} \int_G (Y^{\alpha_1}q)(x^{-1}y)\, q_{0,\alpha_1}(y^{-1})\, \kappa_2(y)\, \kappa_1(y^{-1}x)\, \mathrm{d}y \\ &\qquad\qquad + \int_G R^q_{x^{-1}y,M}(y^{-1})\, \kappa_2(y)\, \kappa_1(y^{-1}x)\, \mathrm{d}y.\end{aligned} \tag{4.10.8}$$

Now observe that

$$\begin{aligned}&(\widetilde{q}(\kappa_2 * \kappa_1))(x) \\ &\qquad = \sum_{[\alpha_1]<M} \frac{1}{\alpha_1!} \big(\widetilde{q}_{0,\alpha_1}\, \kappa_2\big) * \big((\widetilde{Y^{\alpha_1}q})\kappa_1\big)(x) \\ &\qquad\qquad + \int_G R^q_{x^{-1}y,M}(y^{-1})\, \kappa_2(y)\, \kappa_1(y^{-1}x)\, \mathrm{d}y,\end{aligned} \tag{4.10.9}$$

We now consider the function η_x given by

$$\eta_x(y) \;:=\; R^q_{x^{-1}y,M}(y^{-1})\, \kappa_2(y), \qquad \forall\, y \,\in\, G, \tag{4.10.10}$$

and define the distribution

$$\begin{aligned}\rho &= \widetilde{q}(\kappa_2 * \kappa_1) - \sum_{[\alpha_1]<M} \frac{1}{\alpha_1!}\left(\widetilde{q}_{0,\alpha_1}\,\kappa_2\right) * \left((\widetilde{Y^{\alpha_1}q})\kappa_1\right) \\ &= \int_G \eta_x(y)\,\kappa_1(y^{-1}x)\,\mathrm{d}y. \end{aligned} \tag{4.10.11}$$

By (4.10.9), taking the Fourier transform of $(\widetilde{q}(\kappa_2 * \kappa_1))$ yields

$$\begin{aligned}\Delta_q(\sigma_1\sigma_2)(\pi) &= \sum_{[\alpha_1]<M} \frac{1}{\alpha_1!}\,\mathcal{F}\{(\widetilde{Y^{\alpha_1}q})\kappa_1\}(\pi)\,\mathcal{F}\{\widetilde{q}_{0,\alpha_1}\,\kappa_2\} + \pi(\rho) \\ &= \sum_{[\alpha_1]<M} \frac{1}{\alpha_1!}\,\Delta_{Y^{\alpha_1}q}\,\sigma_1(\pi)\,\Delta^{\alpha_1}\sigma_2(\pi) + \pi(\rho), \end{aligned} \tag{4.10.12}$$

for all $\pi \in \widehat{G}$, which gives us the decomposition we were seeking. In particular, we have

$$\begin{aligned}&\left|\left|\pi(I+\mathcal{L})^{-\frac{1}{2}(m-a+\nu)}\Delta_q(\sigma_1\sigma_2)(\pi)\,\pi(I+\mathcal{L})^{\frac{\nu}{2}}\right|\right|_{L^\infty(\widehat{G})} \\ &\quad\lesssim \sum_{\substack{\alpha_1\in\mathbb{N}_0^n \\ [\alpha_1]<M}} \left|\left|\pi(I+\mathcal{L})^{-\frac{1}{2}(m-a+\nu)}\Delta_{Y^{\alpha_1}q}\,\sigma_1(\pi)\,\Delta^{\alpha_1}\sigma_2(\pi)\,\pi(I+\mathcal{L})^{\frac{\nu}{2}}\right|\right|_{L^\infty(\widehat{G})} \\ &\qquad + \left|\left|\pi(I+\mathcal{L})^{-\frac{1}{2}(m-a+\nu)}\,\pi(\rho)\,\pi(I+\mathcal{L})^{\frac{\nu}{2}}\right|\right|_{L^\infty(\widehat{G})}. \end{aligned} \tag{4.10.13}$$

Step 2

In this step, we find an estimate for the terms in the sum given by (4.10.13). For each $\alpha_1 \in \mathbb{N}_0^n$, with $[\alpha_1] < M$, we have

$$\begin{aligned}&\left|\left|\pi(I+\mathcal{L})^{-\frac{1}{2}(m-a+\nu)}\Delta_{Y^{\alpha_1}q}\,\sigma_1(\pi)\,\Delta^{\alpha_1}\sigma_2(\pi)\,\pi(I+\mathcal{L})^{\frac{\nu}{2}}\right|\right|_{L^\infty(\widehat{G})} \\ &\quad\leq \left|\left|\pi(I+\mathcal{L})^{-\frac{1}{2}(m_1-(a-[\alpha_1])+\nu_1)}\,\Delta_{Y^{\alpha_1}q}\,\sigma_1(\pi)\,\pi(I+\mathcal{L})^{\frac{\nu_1}{2}}\right|\right|_{L^\infty(\widehat{G})} \\ &\qquad\qquad \left|\left|\pi(I+\mathcal{L})^{-\frac{1}{2}(m_2-[\alpha_1]+\nu)}\,\Delta^{\alpha_1}\sigma_2(\pi)\,\pi(I+\mathcal{L})^{\frac{\nu}{2}}\right|\right|_{L^\infty(\widehat{G})},\end{aligned}$$

where $\nu_1 := m_2-[\alpha_1]+\nu$. Since $Y^{\alpha_1}q$ is CC-vanishing at e_G up to order $a-[\alpha_1]$, then by Proposition 4.9.1, we have

$$\left|\left|\pi(I+\mathcal{L})^{-\frac{1}{2}(m_1-(a-[\alpha_1])+\nu_1)}\,\Delta_{Y^{\alpha_1}q}\,\sigma_1(\pi)\,\pi(I+\mathcal{L})^{\frac{\nu_1}{2}}\right|\right|_{L^\infty(\widehat{G})} \lesssim ||\sigma_1||_{S^{m_1,a_1,b_1,c_1}},$$

for some $a_1, b_1 \in \mathbb{N}_0$ and $c_1 > 0$, since $\sigma_1 \in S^{m_1}$. For $\sigma_2 \in S^{m_2}$ we see that

$$\big|\big|\pi(I+\mathcal{L})^{-\frac{1}{2}(m_2-[\alpha_1]+\nu)}\,\Delta^{\alpha_1}\sigma_2(\pi)\,\pi(I+\mathcal{L})^{\frac{\nu}{2}}\big|\big|_{L^\infty(\widehat{G})} \leq ||\sigma_2||_{S^{m_2},[\alpha_1],0,|\nu|}. \quad (4.10.14)$$

Then, by (4.10.13), we have

$$\begin{aligned} &\big|\big|\pi(I+\mathcal{L})^{-\frac{1}{2}(m-a+\nu)}\Delta_q(\sigma_1\sigma_2)(\pi)\,\pi(I+\mathcal{L})^{\frac{\nu}{2}}\big|\big|_{L^\infty(\widehat{G})} \\ &\qquad\lesssim \sum_{\substack{\alpha_1\in\mathbb{N}_0^n\\ [\alpha_1]<M}} ||\sigma_1||_{S^{m_1},a_1,b_1,c_1}\,||\sigma_2||_{S^{m_2},a_2,b_2,c_2} \\ &\qquad\qquad + \big|\big|\pi(I+\mathcal{L})^{-\frac{1}{2}(m-a+\nu)}\,\pi(\rho)\,\pi(I+\mathcal{L})^{\frac{\nu}{2}}\big|\big|_{L^\infty(\widehat{G})}. \quad (4.10.15)\end{aligned}$$

It remains to study the norm

$$\big|\big|\pi(I+\mathcal{L})^{-\frac{1}{2}(m-a+\nu)}\,\pi(\rho)\,\pi(I+\mathcal{L})^{\frac{\nu}{2}}\big|\big|_{L^\infty(\widehat{G})}.$$

Step 3

In this step we analyse the remainders $y \mapsto R^q_{x^{-1}y,M}$ and ρ (see (4.10.11)). First observe that, since the map

$$\begin{cases} G \longrightarrow \mathcal{D}(G) \\ x \longmapsto \big(y \mapsto R^q_{x^{-1}y,M}(y^{-1})\big) \end{cases},$$

is smooth, then the mapping

$$\begin{cases} G \longrightarrow \mathcal{D}'(G) \\ x \longmapsto \eta_x \end{cases}$$

is smooth. Hence, note that

$$\rho = \eta_x * \kappa_1(x), \quad (4.10.16)$$

for $x \in G$, in the sense of distributions.

Let us first study the derivatives in x_1 of the remainder $R^q_{x_1,M}(y^{-1})$, for a given $y \in B_{r/2}(e_G)$. By (4.10.5), for any $\beta \in \mathcal{I}(k)$ and $x_1 \in G$ we have

$$\begin{aligned}
&\widetilde{X}_{\beta,x_1} R^q_{x_1,M}(y^{-1}) \\
&\quad = \widetilde{X}_{\beta,x_1} q(x_1 y^{-1}) - \widetilde{X}_{\beta,x_1} \sum_{[\alpha_1]<M} \frac{1}{\alpha_1!}\, q_{0,\alpha_1}(y^{-1})\,(Y^{\alpha_1} q)(x_1) \\
&\quad = (\widetilde{X}_\beta\, q)(x_1 y^{-1}) - \sum_{[\alpha_1]<M} \frac{1}{\alpha_1!}\, q_{0,\alpha_1}(y^{-1})\,(Y^{\alpha_1} \widetilde{X}_\beta\, q)(x_1),
\end{aligned}$$

which means that

$$\widetilde{X}_{\beta,x_1} \widetilde{R}^q_{x_1,M} \;=\; \widetilde{R}^{\widetilde{X}_\beta q}_{x_1,M}. \tag{4.10.17}$$

Furthermore, for any $x \in G$ we have

$$X_{\beta,x} R^q_{x^{-1}y,M}(y^{-1}) \;=\; (-1)^{|\beta|} \widetilde{X}_{\beta,x_1=x^{-1}y} R^q_{x_1,M}(y^{-1}) \;=\; (-1)^{|\beta|} R^{\widetilde{X}_\beta q}_{x^{-1}y,M}(y^{-1}),$$

by (4.10.17). So, we have obtained

$$X_{\beta,x} R^q_{x^{-1}y,M}(y^{-1}) \;=\; (-1)^{|\beta|} R^{\widetilde{X}_\beta q}_{x^{-1}y,M}(y^{-1}), \qquad \forall\, y \in B_{r/2}(e_G). \tag{4.10.18}$$

Next, we study the distribution $X_{\beta_0}\rho$, for $\beta_0 \in \mathcal{I}(k)$. Observe that, by (4.10.16), for any $x \in G$ we have

$$\begin{aligned}
X_{\beta_0}\, \rho(x) \;&=\; \sum_{\substack{\beta_1,\beta_2 \in \mathcal{I}(k) \\ |\beta_1|+|\beta_2|=|\beta_0|}} c_{\beta_1,\beta_2}\, X_{\beta_1,x_1=x} X_{\beta_2,x_2=x}\big((\eta_{x_1} * \kappa_1)(x_2)\big) \\
&=\; \sum_{\substack{\beta_1,\beta_2 \in \mathcal{I}(k) \\ |\beta_1|+|\beta_2|=|\beta_0|}} c_{\beta_1,\beta_2} \big(X_{\beta_1,x_1=x}\, \eta_{x_1}\big) * \big(X_{\beta_2}\, \kappa_1\big)(x),
\end{aligned}$$

for some constants $c_{\beta_1,\beta_2} \in \mathbb{R}$. By the definition of η_x (see (4.10.10)), for any $\beta_1 \in \mathcal{I}(k)$, $|\beta_1| \le |\beta_0|$, and any $y \in G$ we have

$$\begin{aligned}
X_{\beta_1,x}\, \eta_x(y) \;&=\; X_{\beta_1,x}\big(\widetilde{R}^q_{x^{-1}y,M}\, \kappa_2\big)(y) \\
&=\; (-1)^{|\beta_1|}\big(\widetilde{R}^{\widetilde{X}_{\beta_1} q}_{x^{-1}y,M}\, \kappa_2\big)(y),
\end{aligned}$$

by (4.10.18). So,

$$X_{\beta_0}\,\rho(x) \;=\; \sum_{\substack{\beta_1,\beta_2\,\in\,\mathcal{I}(k)\\|\beta_1|+|\beta_2|=|\beta_0|}} c'_{\beta_1,\beta_2}\big(\widetilde{R}^{\widetilde{X}_{\beta_1}q}_{x^{-1}\cdot,M}\,\kappa_2\big)*\big(X_{\beta_2}\,\kappa_1\big)(x), \tag{4.10.19}$$

for some constants $c'_{\beta_1,\beta_2}\in\mathbb{R}$.

Step 4

Let us now show that ρ is continuous for a suitably chosen M. Observe that, for every $x\in G$ and any $N\in\mathbb{N}_0$, as the operator $(I+\mathcal{L})^N$ is self-adjoint, we formally have

$$\begin{aligned}\rho(x) \;=\; \eta_x*\kappa_1(x) \;&=\; \int_G \eta_x(y)\kappa_1(y^{-1}x)\,\mathrm{d}y\\ &=\; \int_G \big((I+\mathcal{L}_y)^N\eta_x(y)\big)\big((I+\mathcal{L}_y)^{-N}\kappa_1(y^{-1}x)\big)\,\mathrm{d}y\\ &=\; \big((I+\mathcal{L})^N\eta_x\big)*\big((I+\widetilde{\mathcal{L}})^{-N}\kappa_1\big)(x).\end{aligned}$$

Furthermore, by Proposition 2.5.4 (1) we have

$$\begin{aligned}&\big|\big|\big((I+\mathcal{L})^N\eta_x\big)*\big((I+\widetilde{\mathcal{L}})^{-N}\kappa_1\big)\big|\big|_{L^\infty(G)}\\ &\qquad\qquad\lesssim\;\big|\big|(I+\mathcal{L})^N\eta_x\big|\big|_{L^2(G)}\,\big|\big|(I+\widetilde{\mathcal{L}})^{-N}\kappa_1\big|\big|_{L^2(G)},\end{aligned}$$

provided that the distributions $(I+\widetilde{\mathcal{L}})^{-N}\kappa_1$ and $(I+\mathcal{L})^N\eta_x$ are square integrable, which we shall now prove.

Step 4a

We first study $(I+\widetilde{\mathcal{L}})^{-N}\kappa_1$. This has a meaning in the sense of distributions,

$$(I+\widetilde{\mathcal{L}})^{-N}\kappa_1 \;=\; \mathcal{B}_{2N}*\kappa_1,$$

where $\mathcal{B}_{2N}$ is the right convolution kernel associated to the operator $(I+\mathcal{L})^{-N}$ (see (3.1.23)).

Observe that, by Plancherel's Theorem,

$$\begin{aligned}\big\|(I+\widetilde{\mathcal{L}})^{-N}\kappa_1\big\|_{L^2(G)} &= \big\|\sigma_1(\pi)\,\pi(I+\mathcal{L})^{-N}\big\|_{L^2(\widehat{G})}\\ &\lesssim \big\|\sigma_1(\pi)\,\pi(I+\mathcal{L})^{-\frac{1}{2}m_1}\big\|_{L^\infty(\widehat{G})}\,\big\|\pi(I+\mathcal{L})^{\frac{1}{2}(m_1-2N)}\big\|_{L^2(\widehat{G})}\\ &\lesssim \|\sigma_1\|_{S^{m_1,0,0,c_1}}\,\|\mathcal{B}_{-(m_1-2N)}\|_{L^2(G)},\end{aligned}$$

for some $c_1 > 0$, and this is finite provided that $2N > m_1 + \frac{l}{2}$, by Proposition 3.1.9. Hence, for this choice of N, we have $(I+\widetilde{\mathcal{L}})^{-N}\kappa_1 \in L^2(G)$.

Step 4b

Now we look at $(I+\mathcal{L})^N\eta_x$. First recall that, for $M_1 \in \mathbb{N}_0$ to be determined, the function f_{M_1} (see also (4.9.5)) is defined by

$$f_{M_1} = \sum_{j=1}^{n} q_{M_1,j},$$

where N_0 is the lowest common multiple of $d_1,\ d_2,\ \dots,\ d_n$ (see (2.4.3)), and for each $j = 1,2,\dots,n$, the function $q_{M_1,j}$ is given by

$$q_{M_1,j} := q_{0,j}^{\frac{2N_0M_1}{d_j}}.$$

We have

$$|f_{M_1}(z)| \approx |z|^{2N_0M_1}, \qquad \forall\, z \in G. \tag{4.10.20}$$

We now write

$$\eta_x = \frac{\widetilde{R}^q_{x^{-1}y,M}}{f_{M_1}}\, f_{M_1}\,\kappa_2. \tag{4.10.21}$$

Let us fix $x \in G$. We first have

$$(I+\mathcal{L}_y)^N\eta_x(y) = \sum_{|\beta|\le 2N} c_\beta\, X_{\beta,y}\,\eta_x(y), \qquad \forall\, y \in G,$$

for some constants $c_\beta \in \mathbb{R}$. Hence, by (4.10.21), the definition of η_x (see (4.10.10)) and Leibniz's rule for vector fields, for any $y \in G$ we obtain

$$
\begin{aligned}
(I+\mathcal{L}_y)^N \eta_x(y) \\
&= \sum_{\substack{\beta\in\mathcal{I}(k)^3\\ |\beta|\le 2N}} c_\beta\, X_{\beta_1,y_1=y}\, X_{\beta_2,y_2=y}\, X_{\beta_3,y_3=y}\big(\widetilde{R}^q_{x^{-1}y_1,M}/f_{M_1}\big)(y_2)\big(f_{M_1}\,\kappa_2\big)(y_3)\\
&= \sum_{\substack{\beta\in\mathcal{I}(k)^3\\ |\beta|\le 2N}} c_\beta\, X_{\beta_1,x_1=x^{-1}y}\big(\widetilde{X}_{\beta_2}\big(\widetilde{R}^q_{x_1,M}/f_{M_1}\big)\big)(y^{-1})\big(X_{\beta_3} f_{M_1}\,\kappa_2\big)(y)
\end{aligned}
$$

for some $c_\beta \in \mathcal{C}^\infty(G)$. Furthermore, for any $\beta_1 \in \mathcal{I}(k)$, we have

$$
X_{\beta_1} = \sum_{\substack{\beta_1'\in\mathcal{I}(k)\\ |\beta_1'|\le|\beta_1|}} c_{\beta_1'}\, \widetilde{X}_{\beta_1'},
$$

for some $c_{\beta_1'} \in \mathcal{C}^\infty(G)$, by Proposition 3.6.7. Thus, for a fixed $y \in G$, and for every $\beta_1, \beta_2 \in \mathcal{I}(k)$ and any $x_1 \in G$, we have, by (4.10.17),

$$
\begin{aligned}
X_{\beta_1,x_1}\big(\widetilde{X}_{\beta_2}\big(\widetilde{R}^q_{x_1,M}/f_{M_1}\big)\big)(y^{-1}) &= \sum_{\substack{\beta_1'\in\mathcal{I}(k)\\ |\beta_1'|\le|\beta_1|}} c_{\beta_1'}(y)\, \widetilde{X}_{\beta_1',x_1}\big(\widetilde{X}_{\beta_2}\big(\widetilde{R}^q_{x_1,M}/f_{M_1}\big)\big)(y^{-1})\\
&= \sum_{\substack{\beta_1'\in\mathcal{I}(k)\\ |\beta_1'|\le|\beta_1|}} c_{\beta_1'}(y)\big(\widetilde{X}_{\beta_2}\big(\widetilde{R}^{\widetilde{X}_{\beta_1'}q}_{x_1,M}/f_{M_1}\big)\big)(y^{-1}).
\end{aligned}
$$

Hence,

$$
\big|\big|(I+\mathcal{L})^N\eta_x\big|\big|_{L^2(G)} \lesssim \sum_{\substack{\beta\in\mathcal{I}(k)^3\\ |\beta|\le 2N}} \big|\big|\widetilde{X}_{\beta_2}\big(\widetilde{R}^{\widetilde{X}_{\beta_1}q}_{x^{-1}\cdot,M}/f_{M_1}\big)\big|\big|_{L^\infty(G)}\,\big|\big|X_{\beta_3}(f_{M_1}\,\kappa_2)\big|\big|_{L^2(G)}.
$$

Now, we have

$$
\big|\big|\widetilde{X}_{\beta_2}\big(\widetilde{R}^{\widetilde{X}_{\beta_1}q}_{x^{-1}\cdot,M}/f_{M_1}\big)\big|\big|_{L^\infty(G)} < +\infty, \tag{4.10.22}
$$

by Lemma 3.7.7 (2), whenever $|\beta_2| < M - 2N_0M_1$ (see (4.10.6) and (4.10.20)). Thus, it is sufficient to have

$$
2N < M - 2N_0M_1. \tag{4.10.23}
$$

Moreover, by Theorem 3.2.3 and Plancherel's Theorem, for any $\beta_3 \in \mathcal{I}(k)$, with

$|\beta_3| \leq 2N$, we obtain

$$\begin{aligned}\left|\left|X_{\beta_3}(f_{M_1}\,\kappa_2)\right|\right|_{L^2(G)} &\lesssim \left|\left|(I+\mathcal{L})^{\frac{|\beta_3|}{2}}(f_{M_1}\kappa_2)\right|\right|_{L^2(G)} \\ &= \left|\left|\pi(I+\mathcal{L})^{\frac{|\beta_3|}{2}}\Delta_{f_{M_1}}\sigma_2\right|\right|_{L^2(\widehat{G})}.\end{aligned}$$

Additionally,

$$\begin{aligned}&\left|\left|\pi(I+\mathcal{L})^{\frac{|\beta_3|}{2}}\Delta_{f_{M_1}}\sigma_2\right|\right|_{L^2(\widehat{G})} \\ &\quad\lesssim \left|\left|\pi(I+\mathcal{L})^{\frac{1}{2}(|\beta_3|+m_2-2N_0M_1)}\right|\right|_{L^2(\widehat{G})}\left|\left|\pi(I+\mathcal{L})^{-\frac{1}{2}(m_2-2N_0M_1)}\Delta_{f_{M_1}}\sigma_2\right|\right|_{L^\infty(\widehat{G})} \\ &\quad\lesssim \left|\left|\pi(I+\mathcal{L})^{\frac{1}{2}(|\beta_3|+m_2-2N_0M_1)}\right|\right|_{L^2(\widehat{G})}||\sigma_2||_{S^{m_2},2N_0M_1,0,0},\end{aligned}$$

and this is finite, provided that

$$2N+m_2-2N_0M_1 \,<\, -\frac{l}{2}, \tag{4.10.24}$$

by Proposition 3.1.9. Let us then choose $M, M_1 \in \mathbb{N}_0$ such that

$$2N+m_2+\frac{l}{2} \,<\, 2N_0M_1 \,<\, M-2N. \tag{4.10.25}$$

In this case, both (4.10.23) and (4.10.24) are satisfied. Thus, we have shown that for any $x \in G$ and any $N \in \mathbb{N}_0$ there exists $M_N \in \mathbb{N}_0$ such that, for all $M > M_N$, $(I+\mathcal{L})^N\eta_x \in L^2(G)$, with

$$\left|\left|(I+\mathcal{L})^N\eta_x\right|\right|_{L^2(G)} \,\leq\, C\,||\sigma_2||_{S^{m_2},a_2,0,0},$$

for some $C>0$, independent of x and σ_1, and some $a_2 \in \mathbb{N}_0$. In particular, we may choose

$$M_N \,:=\, 2N_0M_1+2N,$$

where $M_1 \in \mathbb{N}_0$ is chosen such that

$$2N_0M_1 \,>\, 2N+m_2+\frac{l}{2}.$$

Step 4c

In conclusion, by Proposition 2.5.4 (1), for each fixed $x \in G$ we have that for every $N, M \in \mathbb{N}_0$ satisfying

$$N > m_1 + \frac{l}{2}, \qquad M > M_N,$$

the mapping

$$\rho_{1,x} : y \longmapsto \big((I+\mathcal{L})^N \eta_x\big) * \big((I+\widetilde{\mathcal{L}})^{-N}\kappa_1\big)(y)$$

is continuous on G. Moreover,

$$\begin{aligned} \|\rho_{1,x}\|_{L^\infty(G)} &= \big\|\big((I+\mathcal{L})^N \eta_x\big) * \big((I+\widetilde{\mathcal{L}})^{-N}\kappa_1\big)\big\|_{L^\infty(G)} \\ &\leq \big\|(I+\mathcal{L})^N \eta_x\big\|_{L^2(G)} \big\|(I+\widetilde{\mathcal{L}})^{-N}\kappa_1\big\|_{L^2(G)} \\ &\leq C\, \|\sigma_1\|_{S^{m_1},0,0,c_1}\, \|\sigma_2\|_{S^{m_2},a_2,0,0}, \end{aligned}$$

for some constant $C > 0$, independent of $\sigma_1,\ \sigma_2$ and of x, some $c_1 > 0$ and $a_2 \in \mathbb{N}_0$. Furthermore, Proposition 2.5.4 (2) implies that the mapping $x \longmapsto \rho_{1,x}$ is continuous $G \to \mathcal{C}(G)$. Hence, by composition, $x \longmapsto \rho(x)$ is a continuous function on G, with

$$\|\rho\|_{L^\infty(G)} \leq \sup_{x\in G} \|\rho_{1,x}\|_{L^\infty(G)} \leq C\, \|\sigma_1\|_{S^{m_1},0,0,c_1}\, \|\sigma_2\|_{S^{m_2},a_2,0,0}, \tag{4.10.26}$$

provided that

$$N > m_1 + \frac{l}{2}, \qquad M > M_N.$$

Hence we have obtained that for all $m_1,\ m_2 \in \mathbb{R}$, there exists $M_{m_1,m_2} \in \mathbb{N}$ such that, for all $M > M_{m_1,m_2}$ there exist $C > 0$ and $c_1,\ a_2 \in \mathbb{N}$ satisfying

$$\|\rho\|_{L^\infty(G)} \leq C\, \|\widehat{\kappa_1}\|_{S^{m_1},0,0,c_1}\, \|\widehat{\kappa_2}\|_{S^{m_2},a_2,0,0},$$

where $\kappa_1,\ \kappa_2 \in \mathcal{C}^\infty(G)$ are such that

$$\rho(x) := \Big(R^q_{x^{-1}\cdot,M}(\cdot^{-1})\,\kappa_2(\cdot)\Big) * \kappa_1(x) = \rho_{q,\kappa_2,\kappa_1,M}(x). \tag{4.10.27}$$

Step 4c'

Let us suppose that

$$g = X_{\beta_0}\kappa_1,$$

for $\beta_0 \in \mathcal{I}(k)$. In this case, for any $N \in \mathbb{N}$ we have

$$\begin{aligned}\big|\big|(I+\widetilde{\mathcal{L}})^{-N} g\big|\big|_{L^2(G)} &= \big|\big|(I+\widetilde{\mathcal{L}})^{-N} X_{\beta_0}\, \kappa_1\big|\big|_{L^2(G)} \\ &= \big|\big|\pi(X_{\beta_0})\, \widehat{\kappa_1}(\pi)\, \pi(I+\mathcal{L})^{-N}\big|\big|_{L^2(\widehat{G})},\end{aligned}$$

by Plancherel's Theorem. Furthermore, by Lemma 4.5.10, for any $N' \in \mathbb{N}_0$ we have

$$\begin{aligned}&\big|\big|\pi(X_{\beta_0})\, \widehat{\kappa_1}(\pi)\, \pi(I+\mathcal{L})^{-N}\big|\big|_{L^2(\widehat{G})} \\ &\quad\leq \big|\big|\pi(I+\mathcal{L})^{\frac{1}{2}(|\beta_0|-N')}\big|\big|_{L^\infty(\widehat{G})} \big|\big|\pi(I+\mathcal{L})^{-\frac{1}{2}(|\beta_0|-N')}\, \pi(X_{\beta_0})\, \pi(I+\mathcal{L})^{-\frac{1}{2}N'}\big|\big|_{L^\infty(\widehat{G})} \\ &\qquad \big|\big|\pi(I+\mathcal{L})^{-\frac{1}{2}(m_1-(m_1+N'))}\, \widehat{\kappa_1}(\pi)\, \pi(I+\mathcal{L})^{-\frac{1}{2}(m_1+N')}\big|\big|_{L^\infty(\widehat{G})} \\ &\qquad\qquad \big|\big|\pi(I+\mathcal{L})^{\frac{1}{2}(m_1+N'-2N)}\big|\big|_{L^2(\widehat{G})} \\ &\quad\lesssim ||\sigma_1||_{S^{m_1},0,0,c'},\end{aligned}$$

for some $c' > 0$, provided that $N' > |\beta_0|$ and $2N > m_1 + N' + \frac{l}{2}$, by Proposition 3.1.9.

Therefore, we have obtained that for all $m_1,\, m_2 \in \mathbb{R}$ and $\beta_0 \in \mathcal{I}(k)$, there exists $M_{m_1,m_2,\beta_0} \in \mathbb{N}$ such that, for all $M > M_{m_1,m_2,\beta_0}$ there exist $C > 0$ and $c_1,\, a_2 \in \mathbb{N}$ satisfying

$$||\rho_{q,\kappa_2,X_{\beta_0}\kappa_1,M}||_{L^\infty(G)} \leq C\, ||\widehat{\kappa_1}||_{S^{m_1},0,0,c_1}\, ||\widehat{\kappa_2}||_{S^{m_2},a_2,0,0}.$$

Step 5

We now consider the distribution $X_{\beta_0}\rho$, for $\beta_0 \in \mathcal{I}(k)$. Using (4.10.27), we may rewrite (4.10.19) as

$$X_{\beta_0}\rho = \sum_{\substack{\beta_1,\beta_2 \in \mathcal{I}(k) \\ |\beta_1|+|\beta_2|=|\beta_0|}} c'_{\beta_1,\beta_2}\, \rho_{\widetilde{X}_{\beta_1} q,\, \kappa_2,\, X_{\beta_2}\kappa_1,\, M}.$$

Now, for each $\beta_1, \beta_2 \in \mathcal{I}(k)$, with $|\beta_1| + |\beta_2| = |\beta_0|$, we apply Step 4 (from Step 4a through to Step 4c') to $\rho_{\widetilde{X}_{\beta_1} q,\, \kappa_2,\, X_{\beta_2}\kappa_1,\, M}$. There exist $N \in \mathbb{N}$ and M_N are such that, if $M > M_N$, then we have

$$||X_{\beta_0}\rho||_{L^\infty(G)} \lesssim \sum_{\substack{\beta_1,\beta_2\in\mathcal{I}(k)\\ |\beta_1|+|\beta_2|=|\beta_0|}} ||\widehat{\kappa_1}||_{S^{m_1},0,0,c_1(m_1,\beta_2)}\,||\widehat{\kappa_2}||_{S^{m_2},a_2,0,0}, \tag{4.10.28}$$

for some constants $c_1(m_1,\beta_2)>0$, depending on m_1 and β_2, and $a_2\in\mathbb{N}_0$.

Thus, we have obtained that for all $\beta_0\in\mathcal{I}(k)$ and all m_1, $m_2\in\mathbb{R}$ there exists $M_{m_1,m_2,\beta_0}\in\mathbb{N}$ such that, for any $M>M_{m_1,m_2,\beta_0}$, there exist constants $C>0$ and c_1', $a_2'\in\mathbb{N}$ satisfying

$$||X_{\beta_0}\rho||_{L^\infty(G)} \leq C\,||\widehat{\kappa_1}||_{S^{m_1},0,0,c_1'}\,||\widehat{\kappa_2}||_{S^{m_2},a_2',0,0},$$

for any κ_1, $\kappa_2\in\mathcal{C}^\infty(G)$ such that (4.10.27) holds.

Step 6

Next, we study the distribution $\widetilde{X}_{\widetilde{\beta}_0}X_{\beta_0}\rho$, for $\widetilde{\beta}_0$, $\beta_0\in\mathcal{I}(k)$. Observe that, by Proposition 3.6.7, the right-invariant operator $\widetilde{X}_{\widetilde{\beta}_0}$ can be written as

$$\widetilde{X}_{\widetilde{\beta}_0} = \sum_{\substack{\beta_0'\in\mathcal{I}(k)\\ |\beta_0'|\leq|\widetilde{\beta}_0|}} c_{\beta_0'}X_{\beta_0'},$$

for some $c_{\beta_0'}\in\mathcal{C}^\infty(G)$. Hence, proceeding as in Step 5 we obtain that for all $\widetilde{\beta}_0,\beta_0\in\mathcal{I}(k)$ and all m_1, $m_2\in\mathbb{R}$ there exists $M_{m_1,m_2,\widetilde{\beta}_0,\beta_0}\in\mathbb{N}$ such that, for any $M>M_{m_1,m_2,\widetilde{\beta}_0,\beta_0}$, there exist constants $C>0$ and $\widetilde{c}_1$, $\widetilde{a}_2\in\mathbb{N}_0$ satisfying

$$||\widetilde{X}_{\widetilde{\beta}_0}X_{\beta_0}\rho||_{L^\infty(G)} \leq C\,||\widehat{\kappa_1}||_{S^{m_1},0,0,\widetilde{c}_1}\,||\widehat{\kappa_2}||_{S^{m_2},\widetilde{a}_2,0,0},$$

for any κ_1, $\kappa_2\in\mathcal{C}^\infty(G)$ such that (4.10.27) holds.

Step 7: Conclusion

Let $\nu\in\mathbb{R}$. By Lemma 2.2.4 we have

$$\begin{aligned}\big|\big|\pi(I+\mathcal{L})^{-\frac{1}{2}(m-a+\nu)}\,\pi(\rho)\,\pi(I+\mathcal{L})^{\frac{\nu}{2}}\big|\big|_{L^\infty(\widehat{G})}&\\ \leq \big|\big|(I+\mathcal{L})^{-\frac{1}{2}(m-a+\nu)}(I+\widetilde{\mathcal{L}})^{\frac{\nu}{2}}\rho\big|\big|_{L^1(G)}&\\ \lesssim \big|\big|(I+\mathcal{L})^{-\frac{1}{2}(m-a+\nu)}(I+\widetilde{\mathcal{L}})^{\frac{\nu}{2}}\rho\big|\big|_{L^2(G)}.&\end{aligned}$$

We now define

$$s_1 \;=\; \max(-(m-a+\nu),0), \qquad s_2 \;=\; \max(\lceil \nu \rceil + 1, 0).$$

By Theorem 3.2.3, we get

$$\begin{aligned} \big|\big|(I+\mathcal{L})^{-\frac{1}{2}(m-a+\nu)}(I+\widetilde{\mathcal{L}})^{\frac{\nu}{2}}\rho\big|\big|_{L^2(G)} &\lesssim \big|\big|(I+\mathcal{L})^{\frac{s_1}{2}}(I+\widetilde{\mathcal{L}})^{\frac{s_2}{2}}\rho\big|\big|_{L^2(G)} \\ &\lesssim \sum_{\substack{\beta_0,\widetilde{\beta}_0\in\mathcal{I}(k)\\ |\beta_0|\le s_1,\,|\widetilde{\beta}_0|\le s_2}} ||\widetilde{X}_{\widetilde{\beta}_0}X_{\beta_0}\rho||_{L^2(G)}. \end{aligned}$$

Hence, by Step 6, we have obtained that there exists $M_{s_1,s_2} \in \mathbb{N}$ such that, for any $M > M_{s_1,s_2}$, there exist constants $C > 0$ and $\widetilde{c}_1,\, \widetilde{a}_2 \in \mathbb{N}_0$ satisfying

$$\big|\big|\pi(I+\mathcal{L})^{-\frac{1}{2}(m-a+\nu)}\,\pi(\rho)\,\pi(I+\mathcal{L})^{\frac{\nu}{2}}\big|\big|_{L^\infty(\widehat{G})} \le C\,||\sigma_1||_{S^{m_1},0,0,\widetilde{c}_1}\,||\sigma_2||_{S^{m_2},\widetilde{a}_2,0,0}.$$

By Remark 4.5.4, Lemma 4.10.2 is thus proved.

□

The following result is an immediate consequence of the proof of this result (see (4.10.12)).

Corollary 4.10.3. *Let $\sigma_1 \in S^{m_1}$ and $\sigma_2 \in S^{m_2}$, for $m_1, m_2 \in \mathbb{R}$, and set $m := m_1 + m_2$. Furthermore, suppose q is a smooth, real-valued function on G, which is CC-vanishing at e_G up to order $a-1$, for $a \in \mathbb{N}$. For $M \in \mathbb{N}$, let*

$$\tau_M \;:=\; \Delta_q(\sigma_1\sigma_2) \;-\; \sum_{[\alpha_1]<M} \frac{1}{\alpha_1!}\,\big(\Delta_{Y^{\alpha_1}q}\,\sigma_1\big)\big(\Delta^{\alpha_1}\sigma_2\big).$$

Then,

$$\Delta_q(\sigma_1\sigma_2) \;\sim\; \sum_{[\alpha_1]<M} \frac{1}{\alpha_1!}\,\big(\Delta_{Y^{\alpha_1}q}\,\sigma_1\big)\big(\Delta^{\alpha_1}\sigma_2\big),$$

in the sense that, for all $M \in \mathbb{N}$ and any $\nu \in \mathbb{R}$, there exist $C > 0$, $a_1,\, a_2,\, b_1,\, b_2 \in \mathbb{N}_0$ and $c_1,\, c_2 > 0$ such that

$$\sup_{\substack{x\in G\\ \pi\in\widehat{G}}} \big|\big|\pi(I+\mathcal{L})^{-\frac{1}{2}(m-[\alpha]+\nu)}\,\tau_M(x,\pi)\,\pi(I+\mathcal{L})^{\frac{\nu}{2}}\big|\big|_{\mathscr{L}(\mathscr{H}_\pi)}$$

$$\le C\,||\sigma_1||_{S^{m_1},a_1,b_1,c_1}\,||\sigma_2||_{S^{m_2},a_2,b_2,c_2}.$$

4.11 Spectral multipliers of $\mathcal{L}$

We continue on the same setting as in previous sections. Recall that

$$Q_0 := \{q_{0,1}, q_{0,2}, \ldots, q_{0,n}\},$$

where, for each $j = 1, 2, \ldots, n$, $q_{0,j}$ denotes the smooth, real-valued function on G given by (4.2.12). We know that Q_0 has weight $(d_1, d_2, \ldots, d_n)$ (see (2.4.3)). Throughout this section, for any $m \in \mathbb{R}$, the symbol class S^m shall be assumed to be defined with respect to Q_0.

4.11.1 Definition of $\mathcal{M}_m$ and main result

Throughout this section we shall consider the following class of functions:

Definition 4.11.1. For $m \in \mathbb{R}$, let $\mathcal{M}_m$ be the space consisting of smooth functions on $(0, +\infty)$ such that the quantities

$$||f||_{\mathcal{M}_{m,d}} := \sup_{\substack{\lambda > 0 \\ 0 \le j \le d}} (1+\lambda)^{-m+j} \left|\partial_\lambda^j f(\lambda)\right|$$

are finite for every $d \in \mathbb{N}_0$.

The objective of Section 4.11 is to prove the following result.

Theorem 4.11.2. *Let $m \in \mathbb{R}$. If $f \in \mathcal{M}_{\frac{m}{2}}$, then $f(\mathcal{L}) \in \Psi^m$. Furthermore, for all $a, b \in \mathbb{N}_0$ and $c > 0$, there exists $d \in \mathbb{N}$ and $C > 0$, independent of f, such that its corresponding symbol, which is given by*

$$\{f(\pi(\mathcal{L})) : \pi \in \widehat{G}\},$$

satisfies

$$\left|\left|f(\pi(\mathcal{L}))\right|\right|_{S^m,a,b,c} \le C\,||f||_{\mathcal{M}_{\frac{m}{2}},d}. \tag{4.11.1}$$

This theorem is a consequence of Proposition 4.11.3 and Corollary 4.11.6 below, which are more precise results.

Proposition 4.11.3. *Let $m \in \mathbb{R}$ and $a \in \mathbb{N}_0$. Suppose that $q \in \mathcal{D}(G)$ CC-vanishes at e_G up to order $a-1$ (see Definition 3.7.1). Then, there exists $d \in \mathbb{N}_0$ satisfying the following statements:*

(A) Let $f \in \mathcal{M}_{\frac{m}{2}}$*, such that* $\operatorname{supp}(f) \subset [r_1, +\infty)$*, for some* $r_1 > 0$*. Then, there exists* $C > 0$ *such that for all* $\nu \in \mathbb{R}$ *we have*

$$\begin{aligned} \big|\big|\pi(I+\mathcal{L})^{-\frac{1}{2}(m-a+\nu)}\Delta_q\{f(t\,\pi(\mathcal{L}))\}\,\pi(I+\mathcal{L})^{\frac{\nu}{2}}\big|\big|_{L^\infty(\widehat{G})} \\ \leq C\,t^{\frac{m}{2}}||f||_{\mathcal{M}_{\frac{m}{2}},d}, \end{aligned} \tag{4.11.2}$$

for every $t \in (0,1)$*.*

(B) Let $f \in \mathcal{D}(\mathbb{R})$ *and* $\nu \in \mathbb{R}$*. If* m *and* ν *satisfy* $-m+a-\nu \geq 0$ *and* $\nu \geq 0$*, then there exists* $C > 0$*, depending on* q *and* m*, such that*

$$\begin{aligned} \big|\big|\pi(I+\mathcal{L})^{-\frac{1}{2}(m-a+\nu)}\Delta_q\{f(t\,\pi(\mathcal{L}))\}\,\pi(I+\mathcal{L})^{\frac{\nu}{2}}\big|\big|_{L^\infty(\widehat{G})} \\ \leq C\,t^{\frac{m}{2}}\max_{0\leq j\leq d}\big|\big|\partial^j f\big|\big|_\infty, \end{aligned} \tag{4.11.3}$$

for every $t \in (0,1)$*.*

Remark 4.11.4. In Proposition 4.11.3 (A) and (B), the condition $t \in (0,1)$ may be changed to $t \in (0,t_0)$, for any $t_0 > 0$. In this thesis we use $t_0 = 1$.

Remark 4.11.5. The hypothesis on the support of f in Proposition 4.11.3 (A) does not affect our analysis. Let us expand on this. Let λ_1 be the smallest non-zero eigenvalue of $\mathcal{L}$ and suppose that $f \in \mathcal{D}(\mathbb{R})$, with

$$\operatorname{supp}(f) \cap [0,+\infty) \subset [0,\lambda_1).$$

Then, by Remark 3.1.5,

$$f(\pi(\mathcal{L})) = \begin{cases} 0, & \text{if} \quad \pi \neq 1_{\widehat{G}} \\ f(0), & \text{if} \quad \pi = 1_{\widehat{G}} \end{cases},$$

where $1_{\widehat{G}}$ denotes the trivial representation of G. Moreover, in this case,

$$\operatorname{Op}\big(f(\pi(\mathcal{L}))\big) = f(0)E_0,$$

where E_0 denotes the spectral projection onto the 0-eigenspace (see Section 3.1.3). Furthermore, the right convolution kernel associated to the operator E_0 is $E_0\delta_0 = 1$, the constant function 1 on G. Hence, the operator E_0 is smoothing, and so the operator $\operatorname{Op}\big(f(\pi(\mathcal{L}))\big)$ is also smoothing.

Proposition 4.11.3 has the following consequence.

Corollary 4.11.6. *Suppose* $m_0 \in \mathbb{R}$, $a_0, b_0 \in \mathbb{N}_0$ *and* $c_0 \geq 0$, *such that* $m_0 \leq -c_0$. *Then, there exist* $C > 0$ *and* $d \in \mathbb{N}_0$ *such that, for all* $f \in \mathcal{D}(\mathbb{R})$, *we have*

$$\big|\big|f(t\,\pi(\mathcal{L}))\big|\big|_{S^{m_0}_{a_0,b_0,c_0}} \leq C\,t^{\frac{m_0}{2}} \max_{0\leq j\leq d} \big|\big|\partial^j f\big|\big|_{\infty}.$$

In the following sections we prove Proposition 4.11.3, following the strategy presented in Fischer [17] (see Appendix A therein). However let us first show Theorem 4.11.2.

Proof of Theorem 4.11.2. As we saw in Section 3.1.2 (see (3.1.4)), the spectrum of $\mathcal{L}$ is given by

$$\mathrm{Spec}(\mathcal{L}) = \big\{\lambda_j^{(\pi)} \,:\, \pi \in \widehat{G},\ 1 \leq j \leq d_\pi\big\}.$$

Since the spectrum of $\mathcal{L}$ is discrete (see Remark 3.1.2), then the eigenvalues of $\mathcal{L}$ may be ordered. We then let λ_1 be the smallest positive eigenvalue of $\mathcal{L}$.

We now let $\chi \in \mathcal{D}(\mathbb{R})$, taking values in $[0,1]$, be such that

$$\chi \equiv 1 \quad \text{on} \quad \left(-\frac{\lambda_1}{4}, \frac{\lambda_1}{4}\right), \qquad \chi \equiv 0 \quad \text{on} \quad \left[\frac{\lambda_1}{2}, +\infty\right).$$

Then,

$$f \equiv f\chi + f_1$$

on $(0,+\infty)$, where we write $f_1 = f(1-\chi)$. Now, by the spectral decomposition of $\mathcal{L}$ (see (3.1.8)), we have

$$f(\mathcal{L}) = \sum_{\lambda \in \mathrm{Spec}(\mathcal{L})} f(\lambda) E_\lambda,$$

where E_λ denotes the orthogonal projection onto the eigenspace corresponding to the eigenvalue λ. Thus, by the construction of f_1, we have

$$
\begin{aligned}
f(\mathcal{L}) &= \sum_{\lambda \in \operatorname{Spec}(\mathcal{L})} f(\lambda) E_\lambda \\
&= f(0)E_0 + \sum_{\lambda \in \operatorname{Spec}(\mathcal{L})\setminus\{0\}} f(\lambda) E_\lambda \\
&= f(0)E_0 + f_1(\mathcal{L}).
\end{aligned}
$$

Now, for any $\varphi \in \mathcal{D}(G)$ and $z \in G$ we have

$$
E_0\, \varphi(z) \;=\; \int_G \varphi(x) \cdot 1 \, \mathrm{d}x.
$$

Therefore, the right-convolution kernel associated to the operator E_0 is the constant function 1, which is a smooth function. Hence, by Theorem 4.8.1, this means that E_0 is a smoothing operator in the sense of Definition 4.5.9. Since $\operatorname{supp}(f_1) \subset \big[\frac{\lambda_1}{2}, +\infty\big)$, then we can apply Proposition 4.11.3 (A) to f_1, which yields the result.

□

4.11.2 Proof of Proposition 4.11.3 (B)

Fix $t \in (0,1)$. We first let κ_t be the distribution

$$
\kappa_t \;:=\; qf(t\mathcal{L})\delta_{e_G}. \tag{4.11.4}
$$

Observe that the field of operators

$$
\pi \longmapsto \pi(I+\mathcal{L})^{-\frac{1}{2}(m-a+\nu)}\, \widehat{\kappa_t}(\pi)\, \pi(I+\mathcal{L})^{\frac{\nu}{2}}, \qquad \pi \in \widehat{G},
$$

is exactly the Fourier transform of the distribution

$$
(I+\mathcal{L})^{-\frac{1}{2}(m-a+\nu)}\, (I+\widetilde{\mathcal{L}})^{\frac{\nu}{2}}\, \kappa_t. \tag{4.11.5}
$$

Moreover, its associated operator is given by

$$
(I+\mathcal{L})^{-\frac{1}{2}(m-a+\nu)}\, T_{\kappa_t}\, (I+\mathcal{L})^{\frac{\nu}{2}},
$$

where T_{κ_t} denotes the operator associated to the right-convolution kernel $\kappa_t = qf(t\mathcal{L})\delta_{e_G}$. By Lemma 2.2.4 it then follows that

$$\big|\big|\pi(I+\mathcal{L})^{-\frac{1}{2}(m-a+\nu)}\,\widehat{\kappa_t}(\pi)\,\pi(I+\mathcal{L})^{\nu/2}\big|\big|_{L^\infty(\widehat{G})}$$
$$\leq \int_G \big|(I+\mathcal{L})^{-\frac{1}{2}(m-a+\nu)}\,(I+\widetilde{\mathcal{L}})^{\nu/2}\,\kappa_t(z)\big|\,\mathrm{d}z.$$

Now, by Lemma 3.8.1 and the Leibniz formula for vector fields, it follows that if $-m+a-\nu \in 2\mathbb{N}$ and $\nu \in 2\mathbb{N}$, then

$$\int_G \big|(I+\mathcal{L})^{-\frac{1}{2}(m-a+\nu)}\,(I+\widetilde{\mathcal{L}})^{\frac{\nu}{2}}\,\kappa_t(z)\big|\,\mathrm{d}z$$
$$\leq \sum_{\substack{|\beta|\leq -m+a-\nu \\ |\widetilde{\beta}|\leq \nu}} \int_G \big|X_\beta\,\widetilde{X}_{\widetilde{\beta}}\{qf(t\mathcal{L})\delta_{e_G}\}(z)\big|\,\mathrm{d}z$$
$$\leq \sum_{\substack{|\beta|\leq -m+a-\nu \\ |\widetilde{\beta}|\leq \nu}} t^{\frac{1}{2}(a-|\beta|-|\widetilde{\beta}|)} \max_{0\leq j\leq d} \big|\big|\partial^j f\big|\big|_\infty.$$

Observe that for any $\beta, \widetilde{\beta} \in \mathcal{I}(k)$, with $|\beta| \leq -m+a-\nu$ and $|\widetilde{\beta}| \leq \nu$, we have

$$t^{\frac{1}{2}(a-|\beta|-|\widetilde{\beta}|)} \leq t^{\frac{1}{2}(a+m-a+\nu-\nu)} = t^{\frac{m}{2}},$$

as $t \in (0,1)$. Hence, we have shown that if $-m+a-\nu \in 2\mathbb{N}$ and $\nu \in 2\mathbb{N}$, then there exists a constant $C>0$, which depends on m, q and a, such that

$$\big|\big|\pi(I+\mathcal{L})^{-\frac{1}{2}(m-a+\nu)}\Delta_q\{f(t\,\pi(\mathcal{L}))\}\,\pi(I+\mathcal{L})^{\frac{\nu}{2}}\big|\big|_{L^\infty(\widehat{G})}$$
$$\leq C\,t^{\frac{m}{2}} \max_{0\leq j\leq d} \big|\big|\partial^j f\big|\big|_\infty. \quad (4.11.6)$$

Moreover, we also obtain

$$\big|\big|(I+\mathcal{L})^{-\frac{1}{2}(m-a+\nu)}\,T_{\kappa_t}\,(I+\mathcal{L})^{\frac{\nu}{2}}\big|\big|_{\mathscr{L}(L^2(G))} \leq C\,t^{\frac{m}{2}} \max_{0\leq j\leq d} \big|\big|\partial^j f\big|\big|_\infty. \quad (4.11.7)$$

So, T_{κ_t} is a bounded operator:

$$T_{\kappa_t} : L^2_{-\nu}(G) \longrightarrow L^2_{-m+a-\nu}(G), \quad (4.11.8)$$

with bound

$$||T_{\kappa_t}||_{\mathscr{L}(L^2_{-\nu}(G),\, L^2_{-m+a-\nu}(G))} \lesssim t^{\frac{m}{2}} \max_{0\leq j\leq d} \big|\big|\partial^j f\big|\big|_\infty.$$

Now, by the Interpolation Theorem (see Theorem 3.3.1), the operator T_{κ_t} extends uniquely to a bounded linear map $T_{\kappa_t} : L^2_{-\nu}(G) \longrightarrow L^2_{-m+a-\nu}(G)$, whenever $-m+a-\nu$ and ν are non-negative real numbers.

By Theorem 3.2.3 (f), the dual to T_{κ_t} extends uniquely to the operator

$$T^*_{\kappa_t} : L^2_{m-a+\nu}(G) \longrightarrow L^2_{\nu}(G),$$

whenever $-m+a-\nu \geq 0$ and $\nu \geq 0$. Moreover, it satisfies $T^*_{\kappa_t} = T_{\kappa^*_t}$, where κ^*_t is the distribution given by

$$\kappa^*_t = \widetilde{q} f(t\mathcal{L})\delta_{e_G} = q(\cdot^{-1}) f(t\mathcal{L})\delta_{e_G}.$$

We now let

$$\alpha_0 = -\nu_0, \qquad \beta_0 = -m_0 + a - \nu_0,$$

with $-m_0 + a - \nu_0 \geq 0$ and $\nu_0 \geq 0$, and

$$\alpha_1 = m_1 - a + \nu_1, \qquad \beta_1 = \nu_1,$$

with $-m_1 + a - \nu_1 \geq 0$ and $\nu_1 \geq 0$, such that, without loss of generality, $m_0 \leq m_1$ and $\nu_0 \leq \nu_1$. Then, we have that T_{κ_t} is a bounded operator

$$\begin{aligned} T_{\kappa_t} &: L^2_{\alpha_0}(G) \longrightarrow L^2_{\beta_0}(G), \\ T_{\kappa_t} &: L^2_{\alpha_1}(G) \longrightarrow L^2_{\beta_1}(G). \end{aligned}$$

Hence, by the Interpolation Theorem (see Theorem 3.3.1), T_{κ_t} extends uniquely to a bounded operator

$$T_{\kappa_t} : L^2_{\alpha_s}(G) \longrightarrow L^2_{\beta_s}(G),$$

where, for each $s \in [0,1]$,

$$\begin{aligned} \alpha_s &:= -\nu_0 + s(m_1 - a + \nu_1 + \nu_0), \\ \beta_s &:= (-m_0 + a - \nu_0) + s(\nu_1 + m_0 - a + \nu_0), \end{aligned}$$

For each $s \in [0,1]$, we now define m_s, $\nu_s \in \mathbb{R}$ via the relationship

$$\begin{cases} \alpha_s = -\nu_s, \\ \beta_s = -m_s + a - \nu_s \end{cases}.$$

Hence, one easily checks that

$$\begin{aligned} \nu_s &= (1-s)\nu_0 + s(-m_1 + a - \nu_1), \\ m_s &= (1-s)m_0 + s\, m_1. \end{aligned}$$

This implies that $m_s \in [m_0, m_1]$. So, we have shown that, for any m, $\nu \in \mathbb{R}$, such that $-m + a - \nu \geq 0$ and $\nu \geq 0$, we have

$$\begin{aligned} &\left|\left|\pi(I+\mathcal{L})^{-\frac{1}{2}(m-a+\nu)} \Delta_q\{f(t\,\pi(\mathcal{L}))\}\,\pi(I+\mathcal{L})^{\frac{\nu}{2}}\right|\right|_{L^\infty(\widehat{G})} \\ &\qquad\qquad\qquad\qquad \leq C\, t^{\frac{m}{2}} \max_{0\leq j\leq d} \left|\left|\partial^j f\right|\right|_\infty, \end{aligned}$$

as claimed.

4.11.3 Reduction of the proof of Proposition 4.11.3 (A)

We claim that it suffices to prove Proposition 4.11.3 (A) for $m < 0$. Let us then suppose the result holds for any $m' < 0$. Now, fix $m \geq 0$ and consider a function $f \in \mathcal{D}(\mathbb{R})$, such that

$$\sup_{\substack{\lambda\geq 0 \\ 0\leq j\leq d}} (1+\lambda)^{-\frac{1}{2}(m+2j)} \, |\partial_\lambda^j f(\lambda)| < \infty.$$

Observe that, by the hypothesis on f, we may assume that $\text{supp}(f) \subset [1, +\infty)$. Thus, we have

$$\sup_{\substack{\lambda\geq 1 \\ 0\leq j\leq d}} \lambda^{-\frac{1}{2}(m+2j)} \, |\partial_\lambda^j f(\lambda)| < \infty.$$

Then, define the function f_1 by

$$f_1(\lambda) = \lambda^{-N} f(\lambda), \qquad \lambda \in [0,\infty),$$

for some $N \in \mathbb{N}$ to be determined. As $\text{supp}(f) \subset [1,\infty)$, f_1 is well-defined, also supported in $[1,\infty)$, and $f_1 \in \mathcal{C}^d([0,\infty))$. We choose $N > m/2$ and let

$$m_1 := m - 2N.$$

Then, it follows that

$$\sup_{\substack{\lambda \geq 1 \\ 0 \leq j \leq d}} \lambda^{-\frac{1}{2}(m_1+2j)} \, |\partial_\lambda^j f_1(\lambda)| \leq \sup_{\substack{\lambda \geq 1 \\ 0 \leq j \leq d}} \lambda^{-\frac{1}{2}(m+2j)} \, |\partial_\lambda^j f(\lambda)| < \infty.$$

In particular, $f_1 \in \mathcal{M}_{\frac{m_1}{2},d}$. Next, fix $t \in (0,1)$ and observe that $f(\lambda) = \lambda^N f_1(\lambda)$ for all $\lambda \geq 0$. Then, the spectral theory developed in Section 3.1.4 implies that for any $\phi \in L^2(G)$ for which $||f(t\mathcal{L})\,\phi||_{L^2(G)} < +\infty$, we have

$$f(t\mathcal{L})\,\phi = (t\mathcal{L})^N f_1(t\,\mathcal{L})\phi.$$

So, we have

$$f(t\,\pi(\mathcal{L})) = t^N \pi(\mathcal{L}^N) f_1(t\,\pi(\mathcal{L})), \qquad \forall\, \pi \in \widehat{G}. \tag{4.11.9}$$

Now, observe that $\pi(\mathcal{L}^N)$ can be written as

$$\pi(\mathcal{L}^N) = \sum_{\substack{\beta' \in \mathcal{I}(k) \\ |\beta'| \leq 2N}} c_{\beta'} \pi(X_{\beta'}),$$

for some constants $c_{\beta'} \in \mathbb{R}$. By Proposition 4.5.10, for each $\beta' \in \mathcal{I}(k)$, with $|\beta'| = 2N$, the symbol $\pi(X_{\beta'})$ is of class $2N$. Furthermore, as $m_1 = m - 2N < 0$ and $f_1 \in \mathcal{M}_{\frac{m_1}{2}}$, then we can apply Proposition 4.11.3 (A) to f_1 and m_1. Indeed, if q_1 is a smooth, real-valued function on G, which is CC-vanishing at e_G up to order $a_1 - 1$, for $a_1 \in \mathbb{N}$, then there exists $C_1 > 0$, independent of t, such that for any $\nu \in \mathbb{R}$ we have

$$\begin{aligned} \big|\big|\pi(I+\mathcal{L})^{-\frac{1}{2}(m_1-a_1+\nu)}\big\{\Delta_{q_1} f_1(t\pi(\mathcal{L}))\big\}\pi(I+\mathcal{L})^{\frac{\nu}{2}}\big|\big|_{L^\infty(\widehat{G})} \\ \leq C_1\, t^{\frac{m_1}{2}} ||f_1||_{\mathcal{M}_{\frac{m_1}{2},d}}. \end{aligned} \tag{4.11.10}$$

This also tells us that the symbol $f_1(t\,\pi(\mathcal{L}))$ is of class m_1.

Therefore, by Proposition 4.10.2, for any $\nu \in \mathbb{R}$ there exist semi-norms $||\cdot||_{S^{m_1},a_1,b_1,c_1}$ and $||\cdot||_{S^{2N},a_2,b_2,c_2}$ such that

$$\big\|\pi(I+\mathcal{L})^{-\frac{1}{2}(m-a+\nu)}\Delta_q\big\{f(t\,\pi(\mathcal{L}))\big\}\,\pi(I+\mathcal{L})^{\nu/2}\big\|_{L^\infty(\widehat{G})}$$
$$\lesssim \sum_{\substack{\beta_1'\in\mathcal{I}(k)\\|\beta_1'|=2N}} t^N\,\big\|f_1(t\pi(\mathcal{L}))\big\|_{S^{m_1},a_1,b_2,c_2}\,\big\|\pi(X_{\beta_1'})\big\|_{S^{2N},a_2,b_2,c_2}.$$

By (4.11.10) we readily deduce that there exists $C>0$, independent of t, such that

$$\begin{aligned}\big\|\pi(I+\mathcal{L})^{-\frac{1}{2}(m-a+\nu)}\Delta_q\big\{f(t\,\pi(\mathcal{L}))\big\}\,\pi(I+\mathcal{L})^{\nu/2}\big\|_{L^\infty(\widehat{G})} &\le C\,t^N\,t^{\frac{m_1}{2}}\,\|f_1\|_{\mathcal{M}_{\frac{m_1}{2}},d}\\ &\le C\,t^{\frac{m}{2}}\,\|f\|_{\mathcal{M}_{\frac{m}{2}},d},\end{aligned}$$

which is the desired result.

4.11.4 Proof of Proposition 4.11.3 (A)

In this section we show Proposition 4.11.3 (A). We shall split up the proof in several steps, starting by laying out the strategy.

Strategy

Let $q\in\mathcal{D}(G)$ and fix $t\in(0,1)$. By the work done in Section 4.11.3, we may assume $m<0$ and $f\in\mathcal{C}^d([0,\infty))$ such that $\mathrm{supp}(f)\subset[1,+\infty)$. The properties of Sobolev spaces imply that it suffices to show that

$$\big\|\mathcal{L}^{\frac{b}{2}}\,T_{\kappa_t}\,\mathcal{L}^{\frac{b'}{2}}\big\|_{\mathscr{L}(L^2(G))} \le C\,t^{\frac{m}{2}}\sup_{\substack{\lambda>0\\0\le j\le d}}(1+\lambda)^{-\frac{1}{2}(m-2j)}\,\big|\partial_\lambda^j f(\lambda)\big|, \qquad (4.11.11)$$

for $b=b'=0$, and for $b=-m+a-\nu$ and $b'=\nu$.

In order to prove this we shall first construct a dyadic decomposition. This allows us to study the case $b,b'\in 2\mathbb{N}_0$, and provides us with a bound for the L^1-norm of $\mathcal{L}^{\frac{b}{2}}\mathcal{L}^{\frac{b'}{2}}\,\kappa_t$.

Furthermore, to extend the result to any $b,b'\in\mathbb{R}$, we shall use the almost orthogonality of the operator $\mathcal{L}^{\frac{b}{2}}\,T_{\kappa_t}\,\mathcal{L}^{\frac{b'}{2}}$, via the Cotlar-Stein Lemma (see Theorem B.0.1).

Dyadic decomposition

We now construct the following dyadic decomposition. Let $\chi \in \mathcal{D}(\mathbb{R})$ be such that

$$0 \le \chi \le 1, \quad \chi\Big|_{[\frac{3}{4},\frac{3}{2}]} = 1, \quad \operatorname{supp}(\chi) \subset \left[\frac{1}{2}, 2\right].$$

Next, we define

$$\chi_\ell(\lambda) \;=\; \chi(2^{-\ell}\lambda), \quad \ell \in \mathbb{N},$$

such that

$$\sum_{\ell=1}^{\infty} \chi_\ell(\lambda) = 1, \qquad \forall\, \lambda \ge 1.$$

For each $\ell \in \mathbb{N}$ and $\lambda \ge 0$ we set

$$f_\ell(\lambda) := 2^{-\ell\frac{m}{2}}\, f(2^\ell\lambda)\,\chi(\lambda). \tag{4.11.12}$$

Then, for any $\ell \in \mathbb{N}$, the function f_ℓ is smooth and $\operatorname{supp}(f_\ell) \subset \left[\frac{1}{2}, 2\right]$. Moreover, for any $d \in \mathbb{N}_0$, it satisfies

$$\begin{aligned}
\sup_{\frac{1}{2}\le\lambda\le 2} \left|\partial_\lambda^d f_\ell(\lambda)\right| \;&\le\; \sup_{\substack{\frac{1}{2}\le\lambda\le 2\\ 0\le j\le d}} 2^{-\ell\frac{m}{2}} \left|\partial_\lambda^j f\left(2^\ell\lambda\right)\right| |\chi(\lambda)| \\
&\le\; ||\chi||_{L^\infty(\mathbb{R})} \sup_{\substack{\frac{1}{2}\le\lambda\le 2\\ 0\le j\le d}} 2^{-\ell\frac{m}{2}+j}\left|\left(\partial_\lambda^j f\right)\left(2^\ell\lambda\right)\right|.
\end{aligned}$$

By substituting λ for $2^{-\ell}\lambda$ on the right hand side, we obtain the estimate

$$\sup_{\frac{1}{2}\le\lambda\le 2} \left|\partial_\lambda^d f_\ell(\lambda)\right| \;\le\; ||\chi||_{L^\infty(\mathbb{R})} \sup_{\substack{2^{\ell-1}\le\lambda\le 2^{\ell+1}\\ 0\le j\le d}} 2^{-\ell\frac{m}{2}+j}\left|\partial_\lambda^j f(\lambda)\right|$$

Since the supremum is taken over $\lambda \in [2^{\ell-1}, 2^{\ell+1}]$ and $||\chi||_{L^\infty(\mathbb{R})} \le 1$, then it follows that there exists $C > 0$ such that

$$\sup_{\frac{1}{2}\le\lambda\le 2} \left|f_\ell(\lambda)\right| \;\le\; C \sup_{\substack{2^{\ell-1}\le\lambda\le 2^{\ell+1}\\ 0\le j\le d}} \lambda^{-\frac{m}{2}+j}\left|\partial_\lambda^j f(\lambda)\right|.$$

Thus, we have shown that there exists $C > 0$ such that

$$\left|\left|\partial_\lambda^d f_\ell\right|\right|_{L^\infty(\mathbb{R})} \le C \sup_{\substack{\lambda>0 \\ 0\le j\le d}} (1+\lambda)^{-\frac{1}{2}(m-2j)} \left|\partial_\lambda^j f(\lambda)\right|, \qquad \forall\, \ell \in \mathbb{N}_0. \quad (4.11.13)$$

Observe that

$$f(\lambda) = \sum_{\ell=1}^{\infty} 2^{\ell \frac{m}{2}} f_\ell(2^{-\ell}\lambda),$$

and furthermore, since $m < 0$, the sum

$$\sum_{\ell=1}^{\infty} 2^{\ell \frac{m}{2}} < +\infty.$$

So, we obtain

$$\begin{aligned}\left|\left|f(t\mathcal{L})\right|\right|_{\mathscr{L}(L^2(G))} &\le \sum_{\ell=1}^{\infty} 2^{\ell\frac{m}{2}} \left|\left|f_\ell(2^{-\ell}t\mathcal{L})\right|\right|_{\mathscr{L}(L^2(G))} \\ &\le C \sup_{\lambda\ge 1} \lambda^{-\frac{m}{2}} \left|f(\lambda)\right| \\ &< +\infty,\end{aligned}$$

by (4.11.13). This means that

$$f(t\mathcal{L}) = \sum_{\ell=1}^{\infty} 2^{\ell\frac{m}{2}} f_\ell(2^{-\ell}t\mathcal{L}) \quad \text{in} \quad \mathscr{L}(L^2(G)), \qquad (4.11.14)$$

and also,

$$f(t\mathcal{L})\delta_{e_G} = \sum_{\ell=1}^{\infty} 2^{\ell\frac{m}{2}} f_\ell(2^{-\ell}t\mathcal{L})\delta_{e_G} \quad \text{in} \quad \mathcal{D}'(G). \qquad (4.11.15)$$

Estimates for the dyadic pieces in the case $b = b' = 0$

Let us show that (4.11.11) holds in the case that $b = b' = 0$. First observe that

$$\begin{aligned}||T_{\kappa_t}||_{\mathscr{L}(L^2(G))} &= \left|\left|\Delta_q\left\{f(t\pi(\mathcal{L}))\right\}\right|\right|_{L^\infty(\widehat{G})} \\ &\le \int_G \left|\{q f(t\mathcal{L})\delta_{e_G}\}(z)\right| \mathrm{d}z,\end{aligned}$$

by Lemma 2.2.4. Now, by (4.11.15) we have

$$||qf(t\mathcal{L})\delta_{e_G}||_{L^1(G)} \leq \sum_{\ell=1}^{\infty} 2^{\ell\frac{m}{2}} ||qf_\ell(2^{-\ell}t\mathcal{L})\delta_{e_G}||_{L^1(G)}. \tag{4.11.16}$$

By Lemma 3.8.1 (II), there exists $d \in \mathbb{N}$ such that, for each $\ell \in \mathbb{N}$ there exists $C > 0$, depending on q, such that

$$\begin{aligned} ||qf_\ell(2^{-\ell}t\mathcal{L})\delta_{e_G}||_{L^1(G)} &\leq C\left(2^{-\ell}t\right)^{\frac{a}{2}} \max_{0\leq j\leq d} \left|\left|\partial_\lambda^j f_\ell\right|\right|_{L^\infty(\mathbb{R})} \\ &\leq C\left(2^{-\ell}t\right)^{\frac{a}{2}} \sup_{\substack{\lambda\geq 1\\ 0\leq j\leq d}} \lambda^{-\frac{1}{2}(m-2j)} \left|\partial_\lambda^j f(\lambda)\right|, \end{aligned} \tag{4.11.17}$$

by (4.11.13). By (4.11.16), this yields the estimate

$$\begin{aligned} ||qf(t\mathcal{L})\delta_{e_G}||_{L^1(G)} &\lesssim \sum_{\ell=1}^{\infty} 2^{\ell\frac{m}{2}} \left(2^{-\ell}t\right)^{\frac{a}{2}} \sup_{\substack{\lambda\geq 1\\ 0\leq j\leq d}} \lambda^{-\frac{1}{2}(m-2j)} \left|\partial_\lambda^j f(\lambda)\right| \\ &\lesssim t^{\frac{a}{2}} \sup_{\substack{\lambda\geq 1\\ 0\leq j\leq d}} \lambda^{-\frac{1}{2}(m-2j)} \left|\partial_\lambda^j f(\lambda)\right| \\ &\lesssim t^{\frac{m}{2}} \sup_{\substack{\lambda\geq 1\\ 0\leq j\leq d}} \lambda^{-\frac{1}{2}(m-2j)} \left|\partial_\lambda^j f(\lambda)\right| \end{aligned}$$

as $m - a < 0$. This implies (4.11.11) for the case $b = b' = 0$.

Estimates for the dyadic pieces in the case $b, b' \in 2\mathbb{N}_0$

We first let

$$\kappa_{t,\ell} := qf_\ell(2^{-\ell}t\mathcal{L})\delta_{e_G}, \qquad \ell \in \mathbb{N}. \tag{4.11.18}$$

The aim of this section is to show that for every $\ell \in \mathbb{N}$ and any $b, b' \in 2\mathbb{N}_0$, we have

$$\left|\left|\mathcal{L}^{\frac{b}{2}}\, T_{\kappa_{t,\ell}}\, \mathcal{L}^{\frac{b'}{2}}\right|\right|_{\mathscr{L}(L^2(G))} \lesssim \left(2^{-\ell}t\right)^{\frac{a-b-b'}{2}} \sup_{\substack{\lambda\geq 1\\ 0\leq j\leq d}} \lambda^{-\frac{1}{2}(m-2j)} \left|\partial_\lambda^j f(\lambda)\right|, \tag{4.11.19}$$

where the function f_ℓ is defined by (4.11.12) and $T_{\kappa_{t,\ell}}$ is the right-convolution operator associated to the distribution $\kappa_{t,\ell}$.

Observe that, if $b, b' \in 2\mathbb{N}_0$, then

$$\begin{aligned}\big\|\mathcal{L}^{\frac{b}{2}}\,T_{\kappa_{t,\ell}}\,\mathcal{L}^{\frac{b'}{2}}\big\|_{\mathscr{L}(L^2(G))} &= \big\|\pi(\mathcal{L})^{\frac{b}{2}}\Delta_q\big\{f_\ell(2^{-\ell}t\,\pi(\mathcal{L}))\big\}\pi(\mathcal{L})^{\frac{b'}{2}}\big\|_{L^\infty(\widehat{G})}\\ &\leq \int_G \big|\mathcal{L}^{\frac{b}{2}}\widetilde{\mathcal{L}}^{\frac{b'}{2}}\big\{qf_\ell(2^{-\ell}t\mathcal{L})\delta_{e_G}\big\}(z)\big|\,\mathrm{d}z,\end{aligned}$$

by Lemma 2.2.4. Moreover,

$$\begin{aligned}\int_G \big|\mathcal{L}^{\frac{b}{2}}\widetilde{\mathcal{L}}^{\frac{b'}{2}}\big\{qf_\ell(2^{-\ell}t\mathcal{L})\delta_{e_G}\big\}(z)\big|\,\mathrm{d}z &\leq \sum_{\substack{\beta,\widetilde{\beta}\in\mathcal{I}(k)\\|\beta|=b,|\widetilde{\beta}|=b'}}\int_G\big|X_\beta\widetilde{X}_{\widetilde{\beta}}\big\{qf_\ell(2^{-\ell}t\mathcal{L})\delta_{e_G}\big\}(z)\big|\,\mathrm{d}z\\ &= \sum_{\substack{\beta,\widetilde{\beta}\in\mathcal{I}(k)\\|\beta|=b,|\widetilde{\beta}|=b'}}\big\|X_\beta\widetilde{X}_{\widetilde{\beta}}\big\{qf_\ell(2^{-\ell}t\mathcal{L})\delta_{e_G}\big\}\big\|_{L_1(G)},\end{aligned}$$

So, we have obtained

$$\big\|\mathcal{L}^{\frac{b}{2}}\,T_{\kappa_{t,\ell}}\,\mathcal{L}^{\frac{b'}{2}}\big\|_{\mathscr{L}(L^2(G))} \leq \sum_{\substack{\beta,\widetilde{\beta}\in\mathcal{I}(k)\\|\beta|=b,|\widetilde{\beta}|=b'}}\big\|X_\beta\widetilde{X}_{\widetilde{\beta}}\big\{qf_\ell(2^{-\ell}t\mathcal{L})\delta_{e_G}\big\}\big\|_{L_1(G)}. \tag{4.11.20}$$

Now, by Lemma 3.8.1, for very $\beta,\ \widetilde{\beta}\ \in\mathcal{I}(k)$ and each $\ell\in\mathbb{N}$ there exists $C>0$, depending on q, β, $\widetilde{\beta}$ and m such that

$$\begin{aligned}&\big\|X_\beta\widetilde{X}_{\widetilde{\beta}}\{qf_\ell(2^{-\ell}t\mathcal{L})\delta_{e_G}\}\big\|_{L^1(G)}\\ &\qquad\leq C\,\big(2^{-\ell}t\big)^{\frac{a-|\beta|-|\widetilde{\beta}|}{2}}\max_{0\leq j\leq d}\big\|\partial_\lambda^j f_\ell\big\|_{L^\infty(\mathbb{R})}\\ &\qquad\leq C\,\big(2^{-\ell}t\big)^{\frac{a-|\beta|-|\widetilde{\beta}|}{2}}\sup_{\substack{\lambda\geq1\\0\leq j\leq d}}\lambda^{-\frac{1}{2}(m-2j)}\big|\partial_\lambda^j f(\lambda)\big|,\end{aligned}\tag{4.11.21}$$

by (4.11.13). We can then apply (4.11.21) to (4.11.20), giving us (4.11.19).

Estimate for the dyadic pieces for the case $b,b'\in\mathbb{R}$

We now generalise the result obtained in the previous step for $b,b'\in\mathbb{R}$. First observe that, by duality, the result obtained in (4.11.19) is extended to $b,b'\in2\mathbb{Z}$. The case for $b,b'\in\mathbb{R}$ follows from an argument of convexity summarised in the following lemma.

Lemma 4.11.7. *Let $\kappa \in \mathcal{D}'(G)$ and suppose that T_κ denotes the right-convolution operator associated to κ. For $b, b' \in \mathbb{R}$, let us fix*

$$\theta := \frac{b}{2} - \left\lfloor \frac{b}{2} \right\rfloor, \qquad \theta' := \frac{b'}{2} - \left\lfloor \frac{b'}{2} \right\rfloor. \tag{4.11.22}$$

Then, we have

$$\left|\left|\mathcal{L}^{\frac{b}{2}}\, T_\kappa\, \mathcal{L}^{\frac{b'}{2}}\right|\right|_{\mathscr{L}(L^2(G))} \lesssim \max_{\substack{b_1 = \lfloor b/2 \rfloor,\, \lceil b/2 \rceil \\ b_1' = \lfloor b'/2 \rfloor,\, \lceil b'/2 \rceil \\ \theta_1 = \theta,\, 1-\theta \\ \theta_1' = \theta',\, 1-\theta'}} \left|\left|\mathcal{L}^{b_1}\, T_\kappa\, \mathcal{L}^{b_1'}\right|\right|^{\theta_1 \theta_1'}_{\mathscr{L}(L^2(G))}, \tag{4.11.23}$$

in the sense that, if the right-hand side is finite, then the left-hand side is also finite and the inequality holds.

Proof. By the spectral decomposition of $\mathcal{L}$ (see Section 3.1.3), for any $\phi \in \mathrm{Dom}\big(\mathcal{L}^{b/2}\big)$, we have

$$\left|\left|\mathcal{L}^{\frac{b}{2}} \phi\right|\right|^2_{L^2(G)} = \sum_{\lambda \in \mathrm{Spec}(\mathcal{L})} |\lambda|^b \left|\left|E_\lambda \phi\right|\right|^2, \tag{4.11.24}$$

where $E_\lambda \phi$ denotes the orthogonal projection onto the eigenfunction ϕ. Now, there exists a real number $\theta \in [0, 1]$ such that $b/2 = \lceil b/2 \rceil\, \theta + \lfloor b/2 \rfloor\, (1 - \theta)$. In particular, $\theta = b/2 - \lfloor b/2 \rfloor$. Hence

$$\begin{aligned}
\left|\left|\mathcal{L}^{\frac{b}{2}} \phi\right|\right|^2_{L^2(G)} &= \sum_{\lambda \in \mathrm{Spec}(\mathcal{L})} |\lambda|^{2(\lceil b/2 \rceil\, \theta + \lfloor b/2 \rfloor\, (1-\theta))} \left|\left|E_\lambda \phi\right|\right|^2 \\
&= \sum_{\lambda \in \mathrm{Spec}(\mathcal{L})} \left|\lambda^{\frac{2\lceil b/2 \rceil}{b} \theta}\right|^b \left|\lambda^{\frac{2\lfloor b/2 \rfloor}{b} (1-\theta)}\right|^b \left|\left|E_\lambda \phi\right|\right|^2.
\end{aligned} \tag{4.11.25}$$

Furthermore, as a consequence of Hölder's inequality, we obtain

$$\begin{aligned}
&\sum_{\lambda \in \mathrm{Spec}(\mathcal{L})} \left|\lambda^{\frac{2\lceil b/2 \rceil}{b} \theta}\right|^b \left|\lambda^{\frac{2\lfloor b/2 \rfloor}{b} (1-\theta)}\right|^b \left|\left|E_\lambda \phi\right|\right|^2 \\
&\qquad \leq \left(\sum_{\lambda \in \mathrm{Spec}(\mathcal{L})} \left|\lambda^{2\lceil b/2 \rceil}\right| \left|\left|E_\lambda \phi\right|\right|^2 \right)^\theta \left(\sum_{\lambda \in \mathrm{Spec}(\mathcal{L})} \left|\lambda^{2\lfloor b/2 \rfloor}\right| \left|\left|E_\lambda \phi\right|\right|^2 \right)^{1-\theta}.
\end{aligned}$$

Combining this with (4.11.25), we deduce that

$$\big|\big|\mathcal{L}^{\frac{b}{2}}\phi\big|\big|_{L^2(G)} \leq \big|\big|\mathcal{L}^{\lceil b/2\rceil}\phi\big|\big|^{\theta}_{L^2(G)}\,\big|\big|\mathcal{L}^{\lfloor b/2\rfloor}\phi\big|\big|^{1-\theta}_{L^2(G)}.$$

Similarly, if $\phi \in \operatorname{Dom}\left(\mathcal{L}^{b'/2}\right)$, then there exists $\theta' \in [0,1]$ such that

$$\big|\big|\mathcal{L}^{\frac{b'}{2}}\phi\big|\big|_{L^2(G)} \leq \big|\big|\mathcal{L}^{\lceil b'/2\rceil}\phi\big|\big|^{\theta'}_{L^2(G)}\,\big|\big|\mathcal{L}^{\lfloor b'/2\rfloor}\phi\big|\big|^{1-\theta'}_{L^2(G)}.$$

Therefore, for $\ell \in \mathbb{N}$, if $\phi \in L^2(G)$ satisfies

$$\big|\big|\mathcal{L}^{\frac{b}{2}}\,T_\kappa\,\mathcal{L}^{\frac{b'}{2}}\phi\big|\big|_{L^2(G)} < +\infty,$$

then we have

$$\big|\big|\mathcal{L}^{\frac{b}{2}}\,T_\kappa\,\mathcal{L}^{\frac{b'}{2}}\phi\big|\big|_{L^2(G)} \lesssim \max_{\substack{b_1=\lfloor b/2\rfloor,\,\lceil b/2\rceil\\ b_1'=\lfloor b'/2\rfloor,\,\lceil b'/2\rceil\\ \theta_1=\theta,\,1-\theta\\ \theta_1'=\theta',\,1-\theta'}} \big|\big|\mathcal{L}^{b_1}\,T_\kappa\,\mathcal{L}^{b_1'}\phi\big|\big|^{\theta_1\theta_1'}_{L^2(G)}.$$

This means that

$$\big|\big|\mathcal{L}^{\frac{b}{2}}\,T_\kappa\,\mathcal{L}^{\frac{b'}{2}}\big|\big|_{\mathscr{L}(L^2(G))} \lesssim \max_{\substack{b_1=\lfloor b/2\rfloor,\,\lceil b/2\rceil\\ b_1'=\lfloor b'/2\rfloor,\,\lceil b'/2\rceil\\ \theta_1=\theta,\,1-\theta\\ \theta_1'=\theta',\,1-\theta'}} \big|\big|\mathcal{L}^{b_1}\,T_\kappa\,\mathcal{L}^{b_1'}\big|\big|^{\theta_1\theta_1'}_{\mathscr{L}(L^2(G))},$$

as required.

$\square$

Fix $b, b' \in \mathbb{R}$ and let θ, θ' be as in (4.11.22). By (4.11.19), for any $\ell \in \mathbb{N}$ and every $b_1 = \lfloor b/2\rfloor$, $\lceil b/2\rceil$, $b_1' = \lfloor b'/2\rfloor$, $\lceil b'/2\rceil$, $\theta_1 = \theta$, $1-\theta$ and $\theta_1' = \theta'$, $1-\theta'$, we have

$$\big|\big|\mathcal{L}^{b_1}\,T_{\kappa_{t,\ell}}\,\mathcal{L}^{b_1'}\big|\big|^{\theta_1\theta_1'}_{\mathscr{L}(L^2(G))} \lesssim \left(2^{-\ell}t\right)^{\frac{a-2b_1-2b_1'}{2}(\theta_1\theta_1')} \left(\sup_{\substack{\lambda\geq 1\\ 0\leq j\leq d}} \lambda^{-\frac{1}{2}(m-2j)}\left|\partial_\lambda^j f(\lambda)\right|\right)^{\theta_1\theta_1'}.$$

Hence, by Lemma 4.11.7,

$$\big|\big|\mathcal{L}^{\frac{b}{2}}\,T_{\kappa_{t,\ell}}\,\mathcal{L}^{\frac{b'}{2}}\big|\big|_{\mathscr{L}(L^2(G))} \lesssim \left(2^{-\ell}t\right)^{\frac{a-b-b'}{2}} \sup_{\substack{\lambda\geq 1\\ 0\leq j\leq d}} \lambda^{-\frac{1}{2}(m-2j)}\left|\partial_\lambda^j f(\lambda)\right|, \qquad (4.11.26)$$

as required.

Application of the Cotlar-Stein Lemma

Next, we do the final step of the proof, where we finally show (4.11.11) for $b = -m + a - \nu$ and $b' = \nu$. We define the operators

$$T_\ell \;=\; 2^{\ell\frac{m}{2}}\mathcal{L}^{\frac{b}{2}}\,T_{\kappa_{t,\ell}}\,\mathcal{L}^{\frac{b'}{2}}, \qquad \ell \in \mathbb{N},$$

where $T_{\kappa_{t,\ell}}$ denotes the right-convolution operator associated to the distribution $\kappa_{t,\ell}$ (see (4.11.18)). So, by (4.11.14), we have, in the strong operator topology,

$$T \;:=\; \mathcal{L}^{\frac{b}{2}}\,T_{\kappa_t}\,\mathcal{L}^{\frac{b'}{2}} \;=\; \sum_{\ell=1}^{\infty} T_\ell \quad \text{in} \quad \mathscr{L}(L^2(G)). \tag{4.11.27}$$

We cannot immediately conclude that the operator T defined by (4.11.27) is bounded on $L^2(G)$. Thus, we must rely on the almost orthogonality of the operator T in order to prove this.

We now aim to find a bound for operators $T_{\ell_1}T_{\ell_2}^*$, for $\ell_1, \ell_2 \in \mathbb{N}$, to show that the hypothesis of the Cotlar-Stein Lemma holds. Observe that the kernel associated to $T_{\ell_1}T_{\ell_2}^*$ is

$$\kappa_{\ell_1,\ell_2} := 2^{(\ell_1+\ell_2)\frac{m}{2}}\big\{\mathcal{L}^{b/2}\,\widetilde{\mathcal{L}}^{b'/2}\kappa_{t,\ell_1}\big\} * \big\{\mathcal{L}^{b/2}\,\widetilde{\mathcal{L}}^{b'/2}\kappa_{t,\ell_2}\big\}.$$

Let $c \in \mathbb{R}$ to be determined. We now write

$$\kappa_{\ell_1,\ell_2} = 2^{(\ell_1+\ell_2)\frac{m}{2}}\big\{\mathcal{L}^{(b+c)/2}\,\widetilde{\mathcal{L}}^{b'/2}\kappa_{t,\ell_1}\big\} * \big\{\mathcal{L}^{b/2}\,\widetilde{\mathcal{L}}^{(b'-c)/2}\kappa_{t,\ell_2}\big\}.$$

We have

$$\begin{aligned}||T_{\ell_1}T_{\ell_2}^*||_{\mathscr{L}(L^2(G))} \;\le\; 2^{(\ell_1+\ell_2)\frac{m}{2}}\, &\big|\big|\mathcal{L}^{(b+c)/2}\,T_{\kappa_{t,\ell_1}}\,\mathcal{L}^{b'/2}\big|\big|_{\mathscr{L}(L^2(G))}\\ &\times \big|\big|\mathcal{L}^{b/2}\,T_{\kappa_{t,\ell_2}}\,\mathcal{L}^{(b'-c)/2}\big|\big|_{\mathscr{L}(L^2(G))}.\end{aligned}$$

Then, by (4.11.26), we obtain

$$\begin{aligned}&||T_{\ell_1}T_{\ell_2}^*||_{\mathscr{L}(L^2(G))}\\ &\quad\le\; 2^{(\ell_1+\ell_2)\frac{m}{2}}\big(2^{-\ell_1}t\big)^{\frac{a-(b+c)-b'}{2}}\big(2^{-\ell_2}t\big)^{\frac{a-b-(b'-c)}{2}}\left(\sup_{\substack{\lambda\ge 1\\ 0\le j\le d}}\lambda^{-\frac{1}{2}(m-2j)}\,\big|\partial_\lambda^j f(\lambda)\big|\right)^2.\end{aligned}$$

For $b = -m + a - \nu$ and $b' = \nu$, we have

$$\begin{aligned} 2^{(\ell_1+\ell_2)\frac{m}{2}} \left(2^{-\ell_1} t\right)^{\frac{a-(b+c)-b'}{2}} \left(2^{-\ell_2} t\right)^{\frac{a-b-(b'-c)}{2}} &= 2^{(\ell_1+\ell_2)\frac{m}{2}}\, 2^{-\frac{\ell_1}{2}(m-c)}\, 2^{-\frac{\ell_2}{2}(m+c)}\, t^m \\ &= 2^{(\ell_1-\ell_2)\frac{c}{2}}\, t^m, \end{aligned}$$

and thus, we obtain the estimate

$$||T_{\ell_1} T_{\ell_2}^*||_{\mathscr{L}(L^2(G))} \le 2^{(\ell_1-\ell_2)\frac{c}{2}}\, t^m \left(\sup_{\substack{\lambda \ge 1 \\ 0 \le j \le d}} \lambda^{-\frac{1}{2}(m-2j)} \left|\partial_\lambda^j f(\lambda)\right| \right)^2 .$$

We now let $c = c(\ell_1) = -4\,\mathrm{sgn}(\ell_1)$. Then, we define

$$\gamma(\ell_1) = 2^{\ell_1 \frac{c(\ell_1)}{4}}\, t^{\frac{m}{2}} \sup_{\substack{\lambda \ge 1 \\ 0 \le j \le d}} \lambda^{-\frac{1}{2}(m-2j)} \left|\partial_\lambda^j f(\lambda)\right|, \qquad \ell_1 \in \mathbb{N}.$$

Hence, we have

$$A := \sum_{\ell_1=-\infty}^{\infty} \gamma(\ell_1) = 3\, t^{\frac{m}{2}} \sup_{\substack{\lambda \ge 1 \\ 0 \le j \le d}} \lambda^{-\frac{1}{2}(m-2j)} \left|\partial_\lambda^j f(\lambda)\right| < +\infty.$$

By a consequence of the Cotlar-Stein Lemma (see Corollary B.0.2) we then conclude that

$$||T||_{\mathscr{L}(L^2(G))} \le C\, t^{\frac{m}{2}} \sup_{\substack{\lambda \ge 1 \\ 0 \le j \le d}} \lambda^{-\frac{1}{2}(m-2j)} \left|\partial_\lambda^j f(\lambda)\right|,$$

for a constant $C > 0$, which is exactly (4.11.11). This proves Proposition 4.11.3.
□

4.12 Density of $S^{-\infty}$ and $\Psi^{-\infty}$

We continue in the same setting as in previous sections. Recall that

$$Q_0 = \{q_{0,1}, q_{0,2}, \ldots, q_{0,n}\}$$

is the set of smooth, real-valued functions on G, which is comparable to the C-C metric (see Definition 4.1.1), given by (4.2.14). Furthermore, let $\Delta = \Delta_{Q_0}$ be the family of difference operators on G associated to Q_0. For $m \in \mathbb{R}$, we then let S^m be the space of symbols of class m, with respect to Q_0 and $\boldsymbol{Y}$, where $\boldsymbol{Y}$ is the basis of $\mathfrak{g}$ constructed in Section 2.4.1.

The aim of this section is to show that, for any $m \in \mathbb{R}$, the spaces of smoothing symbols $S^{-\infty}$ and their associated operators $\Psi^{-\infty}$ are weakly dense in S^m and Ψ^m, respectively, in the sense explained in Lemma 4.12.1 below. Recall that $S^{-\infty}$ is the space of smoothing symbols introduced in Definition 4.5.6.

Lemma 4.12.1. *Let $m \in \mathbb{R}$ and suppose that $\sigma \in S^m$. Then there exists a family of symbols $\{\sigma_\varepsilon\}_{\varepsilon>0} \subset S^{-\infty}$ such that the following properties are satisfied:*

(i) For any $m_1 \in \mathbb{R}$ and any $a, b, c \in \mathbb{N}_0$, such that $m_1 \leq -c$, there exist $C > 0$, a_1, $b_1 \in \mathbb{N}_0$ and $c_1 > 0$ such that

$$||\sigma_\varepsilon||_{S^{m_1},a,b,c} \leq C\, \varepsilon^{\frac{1}{2}(m_1-m)}\, ||\sigma||_{S^m,a_1,b_1,c_1}, \qquad \forall\, \varepsilon \in (0,1),$$

and whenever $m_1 \geq m$, there exist $C' > 0$, a', $b' \in \mathbb{N}_0$ and $c' > 0$ such that

$$||\sigma_\varepsilon - \sigma||_{S^{m_1},a,b,c} \leq C'\, \varepsilon^{\frac{1}{2}(m_1-m)}\, ||\sigma||_{S^m,a',b',c'}, \qquad \forall\, \varepsilon \in (0,1). \quad (4.12.1)$$

(ii) If $f \in \mathcal{D}(G)$, then

$$\mathrm{Op}(\sigma_\varepsilon) f \longrightarrow \mathrm{Op}(\sigma) f \quad \text{as} \quad \varepsilon \to 0,$$

in $\mathcal{D}(G)$. More precisely, there exist $C > 0$, $\alpha > 0$, a seminorm $||\cdot||_{S^m,a,b,c}$ and $M > 0$ such that, for all $\beta \in \mathcal{I}(k)$ and $\sigma \in S^m$, we have

$$\big|\big|X_\beta\, \mathrm{Op}(\sigma - \sigma_\varepsilon) f\big|\big|_{L^2(G)} \leq C\, \varepsilon^{\alpha}\, ||\sigma||_{S^m,a,b,c}\, \big|\big|(I+\mathcal{L})^M f\big|\big|_{L^2(G)}, \quad (4.12.2)$$

for all $\varepsilon \in (0,1)$.

(iii) For any a, $b \in \mathbb{N}_0$ and $c > 0$, we have

$$\liminf_{\varepsilon \to 0}\, ||\sigma_\varepsilon||_{S^m,a,b,c} \geq ||\sigma||_{S^m,a,b,c}. \quad (4.12.3)$$

Moreover, there exist $C' \geq 0$, a', $b' \in \mathbb{N}_0$ and $c' > 0$ such that

$$||\sigma_\varepsilon||_{S^m,a,b,c} \leq C'\, ||\sigma||_{S^m,a',b',c'}, \quad (4.12.4)$$

for all $\varepsilon \in (0,1)$.

Proof. Let $\sigma \in S^m$. Consider a cut-off function $\eta \in \mathcal{D}(\mathbb{R})$, satisfying

$$\eta \equiv 1 \quad \text{on} \quad [0,1], \quad \operatorname{supp}(\eta) \subset [0,+\infty).$$

Then, for each $\varepsilon > 0$, define the symbol

$$\sigma_\varepsilon(x,\pi) = \sigma(x,\pi)\,\eta(\varepsilon\pi(\mathcal{L})), \quad x \in G,\ \pi \in \widehat{G}.$$

Observe that, by the definition of η, the symbol $\eta(\varepsilon\pi(\mathcal{L}))$ is smoothing for every $\varepsilon > 0$. So, by Lemma 4.10.2 and Corollary 4.11.6, whenever $m_1 \le -c$ there exist $a_1,\ a_2,\ b_1,\ b_2 \in \mathbb{N}_0$ and $c_1 > 0$ such that

$$\begin{aligned}
||\sigma_\varepsilon||_{S^{m_1},a,b,c} &= ||\sigma(x,\pi)\,\eta(\varepsilon\,\pi(\mathcal{L}))||_{S^{m_1},a,b,c}\\
&\lesssim ||\sigma||_{S^m,a_1,b_1,c_1}\,||\eta(\varepsilon\pi(\mathcal{L}))||_{S^{m_1-m},a_2,b_2,c}\\
&\lesssim \varepsilon^{\frac{1}{2}(m_1-m)}\,||\sigma||_{S^m,a_1,b_1,c_1}.
\end{aligned}$$

Hence, the first part of (i) is proved.

We now show the second part of (i). First observe that, by Lemma 4.10.2, there exist $C' > 0$, and $a_1',\ a_2',\ b_1',\ b_2' \in \mathbb{N}_0$ and $c_1' > 0$, such that, for every $\varepsilon \in (0,1)$ we have

$$\begin{aligned}
||\sigma - \sigma_\varepsilon||_{S^{m_1},a,b,c} &= ||(I - \eta(\varepsilon\,\pi(\mathcal{L})))\sigma||_{S^{(m_1-m)+m},a,b,c}\\
&\le C'\,||I - \eta(\varepsilon\,\pi(\mathcal{L}))||_{S^{m_1-m},a_1',b_1',c_1'}\,||\sigma||_{S^m,a_2',b_2',c}.
\end{aligned}$$

Furthermore, $\operatorname{supp}(1-\eta) \subset [1,+\infty)$ and, since $m_1 \ge m$, $1-\eta \in \mathcal{M}_{\frac{1}{2}(m_1-m)}$. Hence, by Proposition 4.11.3 (A), for any $m_1 \in \mathbb{R}$ there exists $d \in \mathbb{N}_0$ such that

$$||I - \eta(\varepsilon\,\pi(\mathcal{L}))||_{S^{m_1-m},a_1',b_1',c_1'} \le C_1'\,\varepsilon^{\frac{1}{2}(m_1-m)}||1-\eta||_{\mathcal{M}_{\frac{1}{2}(m_1-m)},d},$$

for some $C_1' > 0$, for every $\varepsilon \in (0,1)$. So, we have shown that there exist $C' > 0$, $a',\ b' \in \mathbb{N}_0$ and $c' > 0$ such that

$$||\sigma_\varepsilon - \sigma||_{S^{m_1},a,b,c} \le C'\,\varepsilon^{\frac{1}{2}(m_1-m)}\,||\sigma||_{S^m,a',b',c'}, \qquad \forall\,\varepsilon \in (0,1),$$

as required.

We now show part (ii). For $\varepsilon \in (0,1)$ and $\beta \in \mathcal{I}(k)$ we have

$$\left|\left|X_\beta \operatorname{Op}(\sigma - \sigma_\varepsilon) f\right|\right|_{L^2(G)} = \left|\left|X_\beta \operatorname{Op}(\sigma - \sigma_\varepsilon)(I+\mathcal{L})^{-N_1}(I+\mathcal{L})^{N_1} f\right|\right|_{L^2(G)},$$

for $N_1 \in \mathbb{N}$. Observe that

$$\operatorname{Op}(\sigma - \sigma_\varepsilon)(I+\mathcal{L})^{-N_1} = \operatorname{Op}\left((\sigma - \sigma_\varepsilon)\,\pi(I+\mathcal{L})^{-N_1}\right).$$

Hence, we have

$$\begin{aligned} &\left|\left|X_\beta \operatorname{Op}(\sigma - \sigma_\varepsilon) f\right|\right|_{L^2(G)} \\ &\quad = \left|\left|X_\beta \operatorname{Op}\left((\sigma - \sigma_\varepsilon)\,\pi(I+\mathcal{L})^{-N_1}\right)(I+\mathcal{L})^{N_1} f\right|\right|_{L^2(G)} \\ &\quad \leq \left|\left|(\sigma - \sigma_\varepsilon)\,\pi(I+\mathcal{L})^{-N_1}\right|\right|_{S^{m_1},0,|\beta|,N} \left|\left|(I+\mathcal{L})^{N+N_1} f\right|\right|_{L^2(G)}, \end{aligned} \tag{4.12.5}$$

where

$$N > \frac{1}{2}\left(m_1 + |\beta| + \frac{l}{2}\right).$$

by Theorem 4.6.2 (in particular, see (4.6.2)). Moreover, by Lemma 4.10.2, there exist $C' > 0$, $a_1', a_2', b_1', b_2' \in \mathbb{N}_0$ and $c_1', c_2' > 0$ such that

$$\begin{aligned} &\left|\left|(\sigma - \sigma_\varepsilon)\,\pi(I+\mathcal{L})^{-N_1}\right|\right|_{S^{m_1},0,|\beta|,N} \\ &\qquad \leq \left|\left|\sigma - \sigma_\varepsilon\right|\right|_{S^{m+2N_1},a_1',b_1',c_1'} \left|\left|\pi(I+\mathcal{L})^{-N_1}\right|\right|_{S^{m_1-(m+2N_1)},a_2',b_2',N}. \end{aligned}$$

Observe that $\pi(I+\mathcal{L})^{-N_1} \in S^{-2N_1}$, and since $m_1 \geq m$, then by Proposition 4.5.5 we have $\pi(I+\mathcal{L})^{-N_1} \in S^{m_1-(m+2N_1)}$. Additionally, by part (i) of the Lemma (see (4.12.1)), there exist $C' > 0$, $a, b \in \mathbb{N}_0$ and $c > 0$ such that

$$\left|\left|\sigma - \sigma_\varepsilon\right|\right|_{S^{m+2N_1},a_1',b_1',c_1'} \leq C'\,\varepsilon^{2N_1} \left|\left|\sigma\right|\right|_{S^m,a,b,c}.$$

By (4.12.5) we have then proved that there exists $C > 0$ such that

$$\left|\left|X_\beta \operatorname{Op}(\sigma - \sigma_\varepsilon) f\right|\right|_{L^2(G)} \leq C\,\varepsilon^{2N_1} \left|\left|(I+\mathcal{L})^{N+N_1} f\right|\right|_{L^2(G)} \left|\left|\sigma\right|\right|_{S^m,a,b,c},$$

as required.

It remains to show part (iii). Observe that, if $\sigma' \in S^m$ is an invariant symbol, with associated right-convolution kernel $\kappa' \in \mathcal{D}'(G)$, then for every $\alpha \in \mathbb{N}_0^n$ and $\nu \in \mathbb{R}$ we have

$$\begin{aligned} &\big|\big|\pi(I+\mathcal{L})^{-\frac{1}{2}(m-a+\nu)}\,\Delta^\alpha\sigma'(\pi)\,\pi(I+\mathcal{L})^{\frac{\nu}{2}}\big|\big|_{L^\infty(\widehat{G})} \\ &\qquad = \big|\big|(I+\mathcal{L})^{-\frac{1}{2}(m-a+\nu)}\,\mathrm{Op}\big(\mathcal{F}\{\widetilde{q}_\alpha\,\kappa'\}\big)\,(I+\mathcal{L})^{\frac{\nu}{2}}\big|\big|_{\mathscr{L}(L^2(G))} \\ &\qquad = \big|\big|\mathrm{Op}\big(\mathcal{F}\{\widetilde{q}_\alpha\,\kappa'\}\big)\big|\big|_{\mathscr{L}\left(L^2_{-\nu}(G),\,L^2_{-(m-a+\nu)}(G)\right)}. \end{aligned} \tag{4.12.6}$$

Hence, part (iii) follows from applying (4.12.6) to σ and σ_ε, for every $\varepsilon \in (0,1)$, and Lemma 3.3.2.

□

Remark 4.12.2. Suppose $\sigma \in S^m$. Lemma 4.12.1 allows us to apply a density argument when we want to prove a quantitative result about σ. More precisely, we may assume that $\sigma = \sigma_\varepsilon \in S^{-\infty}$, for some $\varepsilon \in (0,1)$, and then use parts (i) and (iii) of the Lemma to take the limit as $\varepsilon \to 0$. By Theorem 4.8.1, this means that the convolution kernel associated to σ is smooth. An instance of the usefulness of this Lemma is in the proof of Theorem 4.15.1 below.

4.13 Improved kernel estimates for $S^m(Q_0)$

As in previous sections, G denotes a compact Lie group of dimension n and local dimension l. Here,

$$\boldsymbol{Y} = \{Y_1, Y_2, \ldots, Y_n\}$$

denotes the basis of the Lie algebra $\mathfrak{g}$ of G constructed in Section 2.4.1. Furthermore, recall also that

$$Q_0 = \{q_{0,1}, q_{0,2}, \ldots, q_{0,n}\}$$

is the set of smooth, real-valued functions on G, which is comparable to the C-C metric (see Definition 4.1.1), given by (4.2.14). Recall that Q_0 has weight $(d_1, d_2, \ldots, d_n)$, where, for each $j = 1, 2, \ldots, n$, d_j is the positive integers given by (2.4.3). Throughout this section we then let

$$N_0 := \operatorname{lcm}(d_1, d_2, \ldots, d_n) \tag{4.13.1}$$

be the lowest common multiple of $d_1, d_2, \ldots, d_n$.

Furthermore, let $\Delta = \Delta_{Q_0}$ be the family of difference operators on G associated to Q_0. For $m \in \mathbb{R}$, we then let S^m be the space of symbols of class m, with respect to Q_0 and $\boldsymbol{Y}$.

For a distribution $\kappa' \in \mathcal{D}'(G)$ and smooth function $\varphi \in \mathcal{D}(G)$, recall that $\langle \kappa', \varphi \rangle$ denotes the action of κ' on φ.

We aim to show the following result.

Proposition 4.13.1. *Let $\sigma \in S^m$ and suppose that κ_x denotes its associated kernel. Then for any $\alpha, \beta_0 \in \mathbb{N}_0^n$, $\beta_1, \beta_2 \in \mathcal{I}(k)$ satisfying*

$$l + m - [\alpha] + |\beta_1| + |\beta_2| > 0,$$

there exist $C > 0$ and non-negative integers a, b and $c > 0$, which do not depend on σ, such that for all $(x, z) \in G \times \big(G \backslash \{e_G\}\big)$ we have

$$\left| X_{\beta_1, z}\, \widetilde{X}_{\beta_2, z} \left\{ Y_x^{\beta_0}\, \widetilde{q}_\alpha(z) \big(\kappa_x(z) - \langle \kappa_x, 1 \rangle\big) \right\} \right| \leq C\, |z|^{-(l+m-[\alpha]+|\beta_1|+|\beta_2|)}\, ||\sigma||_{S^m, a, b, c},$$

where $\langle \kappa_x, 1 \rangle$ denotes the action of κ_x on the smooth constant function $1 \in \mathcal{D}(G)$.

Remark 4.13.2. Observe that, for any $x \in G$

$$\langle \kappa_x, 1 \rangle = \int_G \kappa_x(z)\, \mathrm{d}z = \sigma(x, 1_{\widehat{G}}),$$

where $1_{\widehat{G}}$ is the trivial representation of G, and

$$\sigma(x, 1_{\widehat{G}}) = \widehat{\kappa_x}(1_{\widehat{G}}).$$

Furthermore, $\sigma(x, 1_{\widehat{G}})$ is a smooth function in x, constant in z.

This result will be proved in the following sections. Observe that, in the case $m = 0$, the kernel is Calderón-Zygmund in the sense of Coifman and Weiss (see Chapter III in [7]). For the Euclidean case, the reader is referred to Stein [47].

4.13.1 Tools for the proof of Proposition 4.13.1

In this section we prove two important lemmata regarding a dyadic decomposition of symbols belonging to the class S^m.

Lemma 4.13.3. *Let $\sigma \in S^m$ and $c_0 > 0$. Furthermore, consider a function $\eta \in \mathcal{D}(\mathbb{R})$, with $\operatorname{supp}(\eta) \subset [r_1, +\infty)$, for some $r_1 > 0$. For $\ell \in \mathbb{N}_0$, we define the symbols*

$$\sigma_{R,\ell}(x,\pi) = \sigma(x,\pi)\, \eta\big(2^{-\ell c_0}\, \pi(\mathcal{L})\big),$$

and

$$\sigma_{L,\ell}(x,\pi) = \eta\big(2^{-\ell c_0}\, \pi(\mathcal{L})\big)\, \sigma(x,\pi).$$

Then $\sigma_{R,\ell}$, $\sigma_{L,\ell} \in S^{-\infty}$ and satisfy the following property: For any $a, b \in \mathbb{N}_0$ and $c > 0$, and $m_1 \in \mathbb{R}$. there exist $C > 0$, a_1, a_2, b_1, $b_2 \in \mathbb{N}_0$ and c_1, $c_2 > 0$, which do not depend on σ, such that for any $\ell \in \mathbb{N}_0$ we have

$$||\sigma_{R,\ell}||_{S^{m_1},a,b,c} \leq C\, 2^{\ell \frac{c_0}{2}(m-m_1)}\, ||\sigma||_{S^m,a_1,b_1,c_1},$$

and

$$||\sigma_{L,\ell}||_{S^{m_1},a,b,c} \leq C\, 2^{\ell \frac{c_0}{2}(m-m_1)}\, ||\sigma||_{S^m,a_2,b_2,c_2}.$$

Proof. Fix $\beta_0 \in \mathbb{N}_0^n$, $\alpha_0 \in \mathbb{N}_0^\ell$ and $\nu \in \mathbb{R}$, with $[\alpha_0]_{Q_0} \leq a$, $[\beta_0]_{\boldsymbol{Y}} \leq b$ and $|\nu| \leq c$. We study the quantity

$$\big|\big|\pi(I+\mathcal{L})^{\frac{1}{2}([\alpha_0]-m_1+\nu)}\, Y_x^{\beta_0}\, \Delta^{\alpha_0} \sigma_{R,\ell}(x,\pi)\, \pi(I+\mathcal{L})^{-\frac{\nu}{2}}\big|\big|_{\mathscr{L}(\mathscr{H}_\pi)}.$$

By Lemma 4.10.2, there exist $a_1, a_2, b_1, b_2 \in \mathbb{N}_0$ and $c_1 > 0$ such that

$$\begin{aligned} \big|\big|\pi(I+\mathcal{L})^{\frac{1}{2}([\alpha_0]-m_1+\nu)}\, Y_x^{\beta_0}\, \Delta^{\alpha_0} \sigma_{R,\ell}(x,\pi)\, \pi(I+\mathcal{L})^{-\frac{\nu}{2}}\big|\big|_{\mathscr{L}(\mathscr{H}_\pi)} \\ \lesssim ||\sigma||_{S^m,a_1,b_1,c_1}\, \big|\big|\eta\big(2^{-\ell c_0}\, \pi(\mathcal{L})\big)\big|\big|_{S^{m_1-m},a_2,b_2,c}. \end{aligned} \tag{4.13.2}$$

Since $\eta \in \mathcal{M}_{\frac{m_1-m}{2}}$, then by Proposition 4.11.3 part (A) we have

$$\big|\big|\eta\big(2^{-\ell c_0}\, \pi(\mathcal{L})\big)\big|\big|_{S^{m_1-m},a_2,b_2,c} \lesssim 2^{\ell c_0 \frac{m-m_1}{2}} \sup_{\substack{\lambda>0\\ 0\leq j\leq d}} (1+\lambda)^{-\frac{1}{2}(m_1-m-2j)}\, \big|\partial_\lambda^j\, \eta(\lambda)\big|.$$

for some $d \in \mathbb{N}_0$. Therefore, there exists a constant $C' > 0$, depending on d, m_1, m, such that

$$\big|\big|\eta\big(2^{-\ell c_0}\, \pi(\mathcal{L})\big)\big|\big|_{S^{m_1-m},a_2,b_2,c} \leq C'\, 2^{\ell c_0 \frac{m-m_1}{2}}$$

In particular, by (4.13.2), we have shown that there exists $C > 0$, independent of σ, such that

$$\big|\big|\pi(I+\mathcal{L})^{\frac{1}{2}([\alpha_0]-m_1+\nu)}\, Y_x^{\beta_0}\, \Delta^{\alpha_0}\, \sigma_{R,\ell}(x,\pi)\, \pi(I+\mathcal{L})^{-\frac{\nu}{2}}\big|\big|_{\mathscr{L}(\mathscr{H}_\pi)} \leq C\, 2^{\ell c_0 \frac{(m-m_1)}{2}}\, ||\sigma||_{S^m, a_1, b_1, c_1}.$$

which yields the desired estimate for $\sigma_{R,\ell}$ on taking suprema. The estimate for $\sigma_{L,\ell}$ is similar.

□

Now, we let λ_1 be the smallest non-zero eigenvalue of $\mathcal{L}$ and consider functions $\eta_0, \eta_1 \in \mathcal{D}((0,\infty))$, taking values in $[0,1]$, such that

$$\operatorname{supp}(\eta_0) \cap [0,+\infty) \subset [0,\lambda_1), \qquad \eta_0(0) = 1,$$

and

$$\operatorname{supp}(\eta_1) \subset \left[\frac{\lambda_1}{2}, \lambda_1\right).$$

We also assume that

$$\sum_{\ell=0}^{\infty} \eta_\ell(\lambda) = 1, \qquad \lambda > 0,$$

where we define η_ℓ by

$$\eta_\ell(\lambda) = \eta_1\big(2^{-(\ell-1)}\,\lambda\big), \quad \ell \in \mathbb{N}, \quad \lambda > 0.$$

Proposition 4.13.4. *For $m \in \mathbb{R}$, let $\sigma \in S^m$ and suppose that κ_x denotes its associated kernel.*

(a) Then, for each $\ell \in \mathbb{N}$, the symbol given by

$$\sigma_\ell(x,\pi) := \sigma(x,\pi)\, \eta_\ell\big(\pi(\mathcal{L})\big) = \sigma(x,\pi)\, \eta_1\big(2^{-(\ell-1)}\, \pi(\mathcal{L})\big), \qquad (4.13.3)$$

belongs to $S^{-\infty}$, and furthermore, its associated kernel of $\sigma_\ell(x,\pi)$ is given by

$$\kappa_\ell(x,z) = \kappa_{\ell,x}(z) = \big(\eta_\ell(\mathcal{L})\delta_0\big) * \kappa_x(z), \qquad (x,z) \in G \times G. \qquad (4.13.4)$$

Moreover, the sum

$$\sum_{\ell=0}^{N} \kappa_{\ell,x}$$

converges to κ_x*, as* $N \to +\infty$*, in the sense of distributions.*

(b) For $\ell = 0$*, the symbol given by*

$$\sigma_0(x,\pi) \;=\; \begin{cases} 0, & \text{if} \quad \pi \neq 1_{\widehat{G}}, \\ \sigma(x, 1_{\widehat{G}}), & \text{if} \quad \pi = 1_{\widehat{G}} \end{cases}. \tag{4.13.5}$$

belongs to $S^{-\infty}$*, and furthermore, its associated kernel is given by*

$$\kappa_{0,x}(z) \;=\; \left\langle \kappa_x, 1_{\widehat{G}} \right\rangle. \tag{4.13.6}$$

(c) We have

$$\left|\kappa_x(z) - \kappa_{0,x}(z)\right| \;\le\; \sum_{\ell=1}^{\infty} |\kappa_{\ell,x}(z)|, \quad \forall\, z \,\in\, G\backslash\{e_G\}. \tag{4.13.7}$$

Proof. We begin with the proof of (a). For each $\ell \in \mathbb{N}$, the symbol $\eta_\ell(\pi(\mathcal{L}))$ belongs to $S^{-\infty}$, by Theorem 4.11.2, so $\sigma_\ell \in S^{-\infty}$. Additionally, its associated right-convolution kernel, which is given by

$$\eta_\ell(\mathcal{L})\delta_0,$$

is in $\mathcal{D}(G)$. Furthermore, the operator given by

$$\sum_{\ell=0}^{N} \eta_\ell(\mathcal{L})$$

converges in the strong operator topology of $\mathscr{L}(L^2(G))$ to the identity operator I, as $N \to \infty$. That is,

$$\left\| \sum_{\ell=0}^{N} \eta_\ell(\mathcal{L})\, f - f \right\|_{L^2(G)} \longrightarrow 0 \quad \text{as} \;\; N \longrightarrow \infty, \tag{4.13.8}$$

for every $f \in L^2(G)$. Thus, the sum of smooth functions

$$\sum_{\ell=0}^{N} \eta_\ell(\mathcal{L})\delta_0$$

converges to δ_0 as $N \to \infty$ in $\mathcal{D}'(G)$. Convolving on the right with κ_x yields the convergence

$$\sum_{\ell=0}^{N} \kappa_{\ell,x} \longrightarrow \kappa_x \quad \text{as} \quad N \longrightarrow \infty,$$

in $\mathcal{D}'(G)$, which shows part (a).

Observe that (b) is a consequence of Remark 4.13.2. Hence, it remains to prove (c). By Lemma 4.13.3, we have that if $m_1 \in \mathbb{R}$, then there exists $C > 0$ such that

$$||\sigma_\ell||_{S^{m_1},a,b,c} \le C\, 2^{\ell(m-m_1)}\, ||\sigma||_{S^m,a',b',c'}, \qquad \forall\, \ell \in \mathbb{N},$$

and thus

$$\sum_{\ell=1}^{\infty} ||\sigma_\ell||_{S^{m_1},a,b,c} < \infty,$$

whenever $m_1 > m$. This implies that the sum

$$\sum_{\ell=1}^{N} \sigma_\ell$$

converges to $\sigma - \sigma_0$ in S^{m_1}, for $m_1 > m$; that is,

$$\left|\left|\sigma - \sigma_0 - \sum_{\ell=1}^{N} \sigma_\ell\right|\right|_{S^{m_1},a,b,c} \longrightarrow 0, \quad \text{as} \quad N \longrightarrow +\infty. \tag{4.13.9}$$

Since $\sigma_\ell \in S^{-\infty}$, for each $\ell \in \mathbb{N}$, then the kernels κ_ℓ are smooth on $G \times G$, by Theorem 4.8.1. Now, by Theorem 4.7.1, observe that there exist $C > 0$, a, $b \in \mathbb{N}_0$ and $c > 0$ such that, for all $x \in G$ and $z \in G\backslash\{e_G\}$, we have

$$\left|\kappa_x(z) - \kappa_{0,x}(z) - \sum_{\ell=1}^{N} \kappa_{\ell,x}(z)\right| \le C\,|z|^{N_{m_1}} \left|\left|\sigma - \sigma_0 - \sum_{\ell=1}^{N} \sigma_\ell\right|\right|_{S^{m_1},a,b,c},$$

where

$$N_{m_1} = \left\lceil \frac{m_1 + 2l}{2N_0} \right\rceil.$$

By (4.13.9), we then have

$$\sum_{\ell=1}^{N} \kappa_{\ell,x}(z) \longrightarrow \kappa_x(z) - \kappa_{0,x}(z) \quad \text{as} \quad N \longrightarrow \infty, \tag{4.13.10}$$

for every $z \in G\backslash\{e_G\}$. So,

$$\left|\kappa_x(z) - \kappa_{0,x}\right| = \left|\sum_{\ell=1}^{\infty} \kappa_{\ell,x}(z)\right| \le \sum_{\ell=1}^{\infty} |\kappa_{\ell,x}(z)|, \qquad \forall\, z \in G\backslash\{e_G\},$$

which proves the result.

□

4.13.2 Proof of Proposition 4.13.1

We now prove the main result of this section.

Step 1: Set up

Let

$$\sigma_1 = \mathcal{F}\big\{X_{\beta_1,z}\, \widetilde{X}_{\beta_2,z}\, \big\{Y_x^{\beta_0}\, \widetilde{q}_\alpha(z)\big(\kappa_x(z) - \langle \kappa_x, 1\rangle\,\big)\big\}\big\}.$$

Routine arguments show this symbol belongs to the class $S^{m-[\alpha]+|\beta_1|+|\beta_2|}$, so it suffices to prove the result for $\alpha = \beta_0 = \beta_1 = \beta_2 = 0$. Our hypothesis then becomes

$$l + m > 0,$$

which we assume. By the work done in Section 4.12, we may assume that the kernel $\kappa : (x,z) \mapsto \kappa_x(z)$ is in $\mathcal{D}(G \times G)$. Recall that, for $\ell \in \mathbb{N}_0$, σ_ℓ is the symbol on G given by (4.13.3) (with the case $\ell = 0$ given by (4.13.5)), and its associated kernel $\kappa_{\ell,x}$ is given by (4.13.4) (with the case $\ell = 0$ given by (4.13.6)). Now, whenever $\ell \in \mathbb{N}$ and $r \in N_0\mathbb{N}$, by Lemma 4.1.7 we have

$$|z|^r\, |\kappa_{\ell,x}(z)| \le C \sum_{[\alpha]=r} |q_\alpha(z)|\, |\kappa_{\ell,x}(z)|, \qquad \forall\, (x,z) \in G \times G,$$

for some $C > 0$. If $m_1 \in \mathbb{R}$ is such that $m_1 - r < -l$, then by Lemma 4.7.2 (2) there exist a constant $C' > 0$, $a,\, b \in \mathbb{N}_0$ and $c > 0$ such that, for each $\ell \in \mathbb{N}_0$,

$$\sum_{[\alpha]=r} |q_\alpha(z)|\,|\kappa_{\ell,x}(z)| \,\leq\, C'\,||\sigma_\ell||_{S^{m_1},a,b,c}, \qquad \forall\,(x,z)\,\in\,G\times G.$$

Hence, for all $(x,z)\,\in\,G\times G$ we have

$$\begin{aligned}|z|^r\,|\kappa_{\ell,x}(z)| &\leq C\,||\sigma_\ell||_{S^{m_1},a,b,c}\\ &\leq C\,2^{\frac{\ell-1}{2}(m-m_1)}\,||\sigma||_{S^m,a,b,c},\end{aligned} \tag{4.13.11}$$

by Lemma 4.13.3. Recall that R denotes the radius of G:

$$R \,:=\, \sup_{z\in G}|z|.$$

For a fixed $z\,\in\,G\backslash\{e_G\}$, we now let ℓ_0 be the unique integer such that

$$2^{-\ell_0} < \frac{|z|}{R+1} \leq 2^{-(\ell_0-1)}. \tag{4.13.12}$$

By Proposition 4.13.4, we have

$$\begin{aligned}|\kappa_x(z)-\kappa_{0,x}(z)| \,&\leq\, \sum_{\ell=1}^{\infty}\left|\kappa_{\ell,x}(z)\right|\\ &= \sum_{1\leq\ell<\ell_0}|\kappa_{\ell,x}(z)| + \sum_{\ell\geq\ell_0}|\kappa_{\ell,x}(z)|.\end{aligned} \tag{4.13.13}$$

We then study the sums in (4.13.13) separately.

Step 2: Sum $\sum_{1\leq\ell<\ell_0}$

Let us first consider the sum

$$\sum_{1\leq\ell<\ell_0}|\kappa_{\ell,x}(z)|,$$

for $(x,z)\,\in\,G\times\big(G\backslash\{e_G\}\big)$. We use (4.13.11) with $m_1\,\in\,\mathbb{R}$ and $r\,\in\,N_0\mathbb{N}$ such that

$$\frac{m-m_1}{2} \,=\, m+l-r \,>\, 0 \qquad \text{and} \qquad m_1-r \,<\, -l.$$

To this aim we choose $r=0$ and

$$m_1 \,:=\, -m-2l \,=\, -m-l-l \,<\, -l.$$

By (4.13.11), for any $\ell \in \mathbb{N}_0$, with $1 \leq \ell < \ell_0$, we then obtain

$$|\kappa_{\ell,x}(z)| \,\leq\, C\, 2^{\frac{\ell-1}{2}(m-m_1)}\, ||\sigma||_{S^m,a,b,c},$$

as $m_1 < -l$. So, we have the bound

$$\begin{aligned}\sum_{\ell=1}^{\ell_0-1} |\kappa_{\ell,x}(z)| &\lesssim \sum_{\ell=1}^{\ell_0-1} 2^{\frac{\ell-1}{2}(m-m_1)}\, ||\sigma||_{S^m,a,b,c} \\ &\lesssim 2^{\frac{\ell_0-1}{2}(m-m_1)}\, ||\sigma||_{S^m,a,b,c}.\end{aligned}$$

Since $\frac{1}{2}(m-m_1) = \frac{1}{2}(2m+2l) = m+l$, then we have

$$2^{\frac{\ell_0}{2}(m-m_1)} \,=\, 2^{(\ell_0-1)(m+l)} \,\leq\, (R+1)\,|z|^{-(m+l)},$$

by our choice of ℓ_0 (see (4.13.12)). Thus, we have shown that

$$\sum_{\ell=1}^{\ell_0-1} |\kappa_{\ell,x}(z)| \,\lesssim\, |z|^{-(m+l)}\, ||\sigma||_{S^m,a,b,c}, \qquad \forall\, z \in G\backslash\{e_G\}. \tag{4.13.14}$$

Step 3: The sum $\sum_{\ell\geq\ell_0}$

It remains to bound the sum $\sum_{\ell=\ell_0}^{\infty} |\kappa_{\ell,x}(z)|$. In this step we use (4.13.11) with $r \in N_0\mathbb{N}$ and $m_1 \in \mathbb{R}$ such that

$$\frac{m-m_1}{2} = m+l-r \qquad \text{and} \qquad \frac{1}{2}(m-m_1) < 0.$$

We set

$$r \,=\, N_0 \left\lceil \frac{m+l}{N_0} \right\rceil, \qquad m_1 \,=\, -m-2l+2r.$$

For this choice of r and m_1 we have

$$\frac{1}{2}(m-m_1) \,=\, m+l-r \,<\, 0.$$

Thus, there exist $a,\, b \in \mathbb{N}_0$ and $c > 0$ such that, for any $z \in G$,

$$\sum_{\ell=\ell_0}^{\infty} |z|^r \, |\kappa_{\ell,x}(z)| \leq C \, ||\sigma||_{S^m,a,b,c} \sum_{\ell=\ell_0}^{\infty} 2^{\frac{(\ell-1)}{2}(m-m_1)}$$

$$\leq C \, 2^{\frac{\ell_0-1}{2}(m-m_1)} \, ||\sigma||_{S^m,a,b,c}.$$

Therefore,

$$\sum_{\ell=\ell_0}^{\infty} |\kappa_{\ell,x}(z)| \leq C \, 2^{\frac{(\ell_0-1)}{2}(m-m_1)} \, |z|^{-r} \, ||\sigma||_{S^m,a,b,c}$$

$$\leq C \, |z|^{-r-\frac{1}{2}(m-m_1)} \, ||\sigma||_{S^m,a,b,c},$$

by our choice of ℓ_0. Since $-r - \frac{1}{2}(m-m_1) = -(m+l)$, then we have shown that

$$\sum_{\ell=\ell_0}^{\infty} |\kappa_{\ell,x}(z)| \leq C \, |z|^{-(m+l)} \, ||\sigma||_{S^m,a,b,c}, \qquad \forall \, z \, \in \, G \backslash \{e_G\}.$$

In combination with (4.13.14) and (4.13.13), this shows the result.

□

4.13.3 A consequence of Proposition 4.13.1

The following result is a consequence of Proposition 4.13.1.

Corollary 4.13.5. *Let $\sigma \in S^m$ and suppose that κ_x denotes its associated kernel. For any $\gamma \in \mathbb{R}$, if*

$$\gamma + l \; > \; \max\{m + l, \, 0\},$$

then there exists $C > 0$ and $a, \, b, \, c \in \mathbb{N}_0$ such that

$$\int_G |z|^\gamma \, |\kappa_x(z)| \; dz \; \leq \; C \, ||\sigma||_{S^m,a,b,c}.$$

Proof. We first write

$$\int_G |z|^\gamma \, |\kappa_x(z)| \, \mathrm{d}z = \int_{|z|>1} |z|^\gamma \, |\kappa_x(z)| \, \mathrm{d}z + \int_{|z|\leq 1} |z|^\gamma \, |\kappa_x(z)| \, \mathrm{d}z.$$

Since G is compact, it follows that

$$\int_{|z|>1} |z|^\gamma \, |\kappa_x(z)| \, \mathrm{d}z \; < \; +\infty.$$

Let us now suppose that $|z| \leq 1$. By Proposition 4.13.1, if $m+l > 0$, then there exist $C > 0$, a, $b \in \mathbb{N}_0$ and $c > 0$ such that, for all $x \in G$ and $z \in G \backslash \{e_G\}$, we have

$$|\kappa_x(z)| \leq C\,|z|^{-(m+l)}\,||\sigma||_{S^m,a,b,c}.$$

Therefore, we have

$$\int_{|z|\leq 1} |z|^{\gamma}\,|\kappa_x(z)|\,\mathrm{d}z \lesssim \int_{|z|\leq 1} |z|^{\gamma-(m+l)}\,\mathrm{d}z,$$

and this is finite if $\gamma + l > m + l$, by Lemma A.3.2.

On the other hand, if $m + l < 0$, then there exists $C' > 0$ such that, for any $x \in G$, $z \in G$ we have

$$|\kappa_x(z)| \leq C'\,||\sigma||_{S^m,a,b,c},$$

by Lemma 4.7.2 (2). So,

$$\int_{|z|\leq 1} |z|^{\gamma}\,|\kappa_x(z)|\,\mathrm{d}z \lesssim \int_{|z|\leq 1} |z|^{\gamma}\,\mathrm{d}z,$$

which is finite if $\gamma + l > 0$.

Finally, suppose that $m + l = 0$. In this case,

$$\int_{|z|\leq 1} |z|^{\gamma}\,\mathrm{d}z < +\infty,$$

provided that $\gamma + l > 0$. Therefore, the condition $\gamma + l > \max\{m + l, 0\}$ is sufficient to have the desired bound. □

4.14 Boundedness on $L^2(G)$

Let us fix a basis of vector fields

$$\boldsymbol{V} := \{V_j : j = 1, 2, \ldots, n\}$$

on G (see Definition 2.3.2). Furthermore, let us also fix a family

$$Q = \{q_1, q_2, \ldots, q_\ell\}$$

of functions on G comparable to the C-C metric (see Definition 4.1.1), for some $\ell \in \mathbb{N}_0$, and let $\Delta = \Delta_Q$ be the family of difference operators on G associated to Q. For $m \in \mathbb{R}$, we then let S^m be the space of symbols of class m, with respect to the family of difference operators Q.

In this section we shall prove the boundedness of any pseudo-differential operator $\mathrm{Op}(\sigma)$, for $\sigma \in S^0$, on $L^2(G)$.

Theorem 4.14.1. *Suppose that $\sigma \in S^0$. Then, the pseudo-differential operator $\mathrm{Op}(\sigma)$ extends to a bounded operator $L^2(G) \to L^2(G)$, with*

$$\big|\big|\mathrm{Op}(\sigma)\big|\big|_{\mathscr{L}(L^2(G))} \le C\,||\sigma||_{S^0,0,l,0}, \tag{4.14.1}$$

for some $C > 0$ independent of σ, where l denotes the local dimension of G.

Proof. Let $\eta_0 \in \mathcal{D}((0,\infty))$, taking values in $[0,1]$, be such that

$$\mathrm{supp}(\eta_0) \cap [0,+\infty) \subset [0,\lambda_1), \qquad \eta_0(0) = 1,$$

where λ_1 denotes the smallest non-zero eigenvalue of $\mathcal{L}$. We now write

$$\sigma = \sigma - \sigma\eta_0(\pi(\mathcal{L})) + \sigma\eta_0(\pi(\mathcal{L})).$$

By Remarks 3.1.5, 4.11.5 and 4.13.2, the operator associated to the symbol $\sigma\eta_0(\pi(\mathcal{L}))$ is given by

$$\mathrm{Op}\big(\sigma\eta_0(\pi(\mathcal{L}))\big) = \sigma(\cdot, 1_{\widehat{G}})\,E_0,$$

where $1_{\widehat{G}}$ denotes the trivial representation of G and E_0 denotes the orthogonal projection onto the 0-eigenspace of $\mathcal{L}$. Since $\sigma(\cdot,1_{\widehat{G}})$ is a smooth function on G, then for any $f \in L^2(G)$,

$$||\sigma(\cdot,1_{\widehat{G}})\,E_0 f||_{L^2(G)} \le ||\sigma(\cdot,1_{\widehat{G}})||_{L^\infty(G)}\,||E_0 f||_{L^2(G)}.$$

And since E_0 is an orthogonal projection of $L^2(G)$, then

$$||E_0 f||_{L^2(G)} \le ||f||_{L^2(G)}.$$

Hence, the operator $\sigma(\cdot,1_{\widehat{G}})\,E_0$ is bounded in $L^2(G)$. Thus, it suffices to show the L^2 boundedness of $\mathrm{Op}(\sigma - \sigma\eta_0)$. Additionally, Proposition 4.13.1 is applicable in this case and we may assume that $\sigma(\cdot,1_{\widehat{G}}) \equiv 0$.

Furthermore, by Lemma 4.12.1 (see also Remark 4.12.2) and Theorem 4.8.1, we may assume that the mapping

$$x \longmapsto \kappa_x, \quad x \in G,$$

is smooth.

If $f \in \mathcal{D}(G)$, then

$$\begin{aligned}
||\mathrm{Op}(\sigma)f||^2_{L^2(G)} &= \int_G \left| \mathrm{Op}(\sigma)f(x) \right|^2 \mathrm{d}x = \int_G \left| f * \kappa_x(x) \right|^2 \mathrm{d}x \\
&\leq \int_G \left| \sup_{x_1 \in G} (f * \kappa_{x_1})(x) \right|^2 \mathrm{d}x.
\end{aligned}$$

By the Sobolev inequality (3.4.1) we have

$$\sup_{x_1 \in G} (f * \kappa_{x_1})(x) \lesssim \left[\int_G \sum_{\substack{\alpha \in \mathcal{I}(k) \\ |\alpha| \leq \lceil s \rceil}} \left| X_{\alpha, x_1} (f * \kappa_{x_1})(x) \right|^2 \mathrm{d}x_1 \right]^{1/2},$$

for any $x \in G$ and $s > l/2$, where l denotes the local dimension of G. We may assume that $\lceil s \rceil = l$; thus we have obtained the estimate

$$\begin{aligned}
||\mathrm{Op}(\sigma)f||^2_{L^2(G)} &\leq \int_G \int_G \sum_{\substack{\alpha \in \mathcal{I}(k) \\ |\alpha| \leq \lceil s \rceil}} \left| X_{\alpha, x_1} (f * \kappa_{x_1})(x) \right|^2 \mathrm{d}x_1 \, \mathrm{d}x \\
&= \sum_{\substack{\alpha \in \mathcal{I}(k) \\ |\alpha| \leq \lceil s \rceil}} \int_G \int_G \left| X_{\alpha, x_1} (f * \kappa_{x_1})(x) \right|^2 \mathrm{d}x \, \mathrm{d}x_1,
\end{aligned}$$

by Fubini's Theorem. Now, $X_{\alpha,x_1}(f * \kappa_{x_1}) = f * (X_{\alpha,x_1} \kappa_{x_1})$. Hence, by Plancherel's Theorem, for every $\alpha \in \mathcal{I}(k)$, with $|\alpha| \leq l$, we have

$$\begin{aligned}
\int_G \left| X_{\alpha, x_1} (f * \kappa_{x_1})(x) \right|^2 \mathrm{d}x &= ||f * (X_{\alpha,x_1} \kappa_{x_1})||^2_{L^2(G)} \\
&= \left|\left| \mathcal{F} \left\{ f * (X_{\alpha,x_1} \kappa_{x_1}) \right\} \right|\right|^2_{L^2(\widehat{G})} \\
&= \left|\left| \mathcal{F} \left\{ X_{\alpha,x_1} \kappa_{x_1} \right\} \widehat{f} \right|\right|^2_{L^2(\widehat{G})}.
\end{aligned}$$

Moreover,

$$\begin{aligned}
\left|\left| \mathcal{F} \left\{ X_{\alpha,x_1} \kappa_{x_1} \right\} \widehat{f} \right|\right|^2_{L^2(\widehat{G})} &= \sum_{\pi \in \widehat{G}} d_\pi \left|\left| \mathcal{F} \left\{ X_{\alpha,x_1} \kappa_{x_1} \right\} (\pi) \, \widehat{f}(\pi) \right|\right|^2_{HS} \\
&\leq \left|\left| \mathcal{F} \left\{ X_{\alpha,x_1} \kappa_{x_1} \right\} \right|\right|^2_{L^\infty(\widehat{G})} \left|\left| \widehat{f} \right|\right|^2_{L^2(\widehat{G})},
\end{aligned}$$

where

$$\left|\left|\mathcal{F}\left\{X_{\alpha,x_1}\kappa_{x_1}\right\}\right|\right|_{L^\infty(\widehat{G})} = \sup_{\pi\in\widehat{G}} \left|\left|\mathcal{F}\left\{X_{\alpha,x_1}\kappa_{x_1}\right\}(\pi)\right|\right|_{\mathscr{L}(\mathscr{H}_\pi)}.$$

Applying Plancherel's Theorem once again, we obtain

$$\int_G \left|X_{\alpha,x_1}(f*\kappa_{x_1})(x)\right|^2 \mathrm{d}x \leq C\left|\left|\mathcal{F}\left\{X_{\alpha,x_1}\kappa_{x_1}\right\}\right|\right|^2_{L^\infty(\widehat{G})} ||f||^2_{L^2(G)}.$$

Since $\mathcal{F}\big\{X_{\alpha,x_1}\,\kappa_{x_1}\big\} = X_{\alpha,x_1}\,\sigma(x_1,\cdot)$ belongs to the symbol class S^0, then

$$\left|\left|\mathcal{F}\big\{X_{\alpha,x_1}\,\kappa_{x_1}\big\}\right|\right|_{L^\infty(\widehat{G})} = ||X_{\alpha,x_1}\,\sigma(x_1,\cdot)||_{L^\infty(\widehat{G})} < +\infty,$$

and thus,

$$\begin{aligned} ||\mathrm{Op}(\sigma)f||^2_{L^2(G)} &\lesssim \sum_{|\alpha|\leq l}\int_G ||X_{\alpha,x_1}\,\sigma(x_1,\cdot)||^2_{L^\infty(\widehat{G})}\,||f||^2_{L^2(G)}\,\mathrm{d}x_1 \\ &\lesssim \max_{|\alpha|\leq l}\,\sup_{x_1\in G}\,||X_{\alpha,x_1}\,\sigma(x_1,\cdot)||^2_{L^\infty(\widehat{G})}\,||f||^2_{L^2(G)}. \end{aligned}$$

Hence, there exists $C>0$ such that

$$||\mathrm{Op}(\sigma)||_{\mathscr{L}(L^2(G))} \leq C\,||\sigma||_{S^m,0,l,0},$$

as required.

□

4.15 Composition of pseudo-differential operators

Throughout this section, the set

$$\boldsymbol{Y} = \{Y_1, Y_2, \ldots, Y_n\}$$

denotes the basis of the Lie algebra $\mathfrak{g}$ of G constructed in Section 2.4.1. Recall also that

$$Q_0 = \{q_{0,1}, q_{0,2}, \ldots, q_{0,n}\}$$

is the set of smooth, real-valued functions on G, which is comparable to the C-C metric (see Definition 4.1.1), given by (4.2.14). Furthermore, we let $\Delta = \Delta_{Q_0}$ be the family of difference operators on G associated to Q_0. We shall assume that, for $m \in \mathbb{R}$, S^m is the space of symbols of class m, with respect to Q_0 and $\boldsymbol{Y}$. We shall also assume that Ψ^m denotes the family of operators of class m, corresponding to S^m.

4.15.1 Main result

The main objective of this section is to prove the following result:

Theorem 4.15.1. *Let $m_1, m_2 \in \mathbb{R}$. If $T_1 \in \Psi^{m_1}$ and $T_2 \in \Psi^{m_2}$, then their composition*

$$T_1 \circ T_2 \in \Psi^{m_1+m_2},$$

and the mapping $(T_1, T_2) \mapsto T_1 \circ T_2$ is continuous $\Psi^{m_1} \times \Psi^{m_2} \to \Psi^{m_1+m_2}$.

If $T_1 = \mathrm{Op}(\sigma_1)$ and $T_2 = \mathrm{Op}(\sigma_2)$, with $\sigma_1 \in S^{m_1}$ and $\sigma_2 \in S^{m_2}$, then we must show that the symbol $\sigma_1 \circ \sigma_2$, associated to the operator $T_1 \circ T_2$, exists and satisfies

$$\sigma_1 \circ \sigma_2 \in S^{m_1+m_2}.$$

Furthermore, for any a, $b \in \mathbb{N}_0$ and $c > 0$, there exist $C > 0$, a_1, a_2, b_1, $b_2 \in \mathbb{N}_0$ and c_1, $c_2 > 0$, independent of σ_1, σ_2, such that

$$||\sigma_1 \circ \sigma_2||_{S^{m_1+m_2},a,b,c} \leq C \, ||\sigma_1||_{S^{m_1},a_1,b_1,c_1} \, ||\sigma_2||_{S^{m_2},a_2,b_2,c_2}. \tag{4.15.1}$$

Throughout this section, we let $\kappa_{1,x}$ and $\kappa_{2,x}$ denote the right-convolution kernels associated to σ_1 and σ_2, respectively.

Observe that, by Lemma 4.12.1 (see also Remark 4.12.2) and Theorem 4.8.1, we may assume both $\kappa_{1,x}$ and $\kappa_{2,x}$ are smooth on G.

4.15.2 The composition symbol $\sigma_1 \circ \sigma_2$

Lemma 4.15.2. *Let $\sigma_1, \sigma_2 \in S^{-\infty}$ and suppose that κ_1 and κ_2 denote their associated convolution kernels, respectively. Set*

$$\kappa_x(z) = \int_G \kappa_{2,xy^{-1}}(zy^{-1}) \, \kappa_{1,x}(y) \, dy,$$

for $x, z \in G$. Then the Fourier transform

$$\sigma(x,\pi) := \widehat{\kappa_x}(\pi), \quad x \in G, \quad \pi \in \widehat{G},$$

defines a smooth symbol. Furthermore, it satisfies

$$\mathrm{Op}(\sigma) = \mathrm{Op}(\sigma_1) \circ \mathrm{Op}(\sigma_2), \tag{4.15.2}$$

and

$$\sigma(x,\pi) = \int_G \kappa_{1,x}(y)\, \pi(y)^* \, \sigma_2(xy^{-1},\, \pi)\; dy. \tag{4.15.3}$$

We will write $\sigma = \sigma_1 \circ \sigma_2$.

Proof. The kernel $\kappa : (x,z) \mapsto \kappa_x(z)$ is smooth on $G \times G$ and compactly supported in x. Furthermore,

$$\begin{aligned}
\int_G |\kappa_x(z)|\, \mathrm{d}z &\le \int_G \int_G \left|\kappa_{2,xy^{-1}}(zy^{-1})\, \kappa_{1,x}(y)\right| \, \mathrm{d}y\, \mathrm{d}z \\
&\le \int_G |\kappa_{2,xy^{-1}}(w)| \, \mathrm{d}w \int_G |\kappa_{1,x}(y)| \, \mathrm{d}y \\
&\le \sup_{x' \in G} \left(\int_G |\kappa_{2,x'}(w)| \, \mathrm{d}w \right) \int_G |\kappa_{1,x}(y)| \, \mathrm{d}y,
\end{aligned}$$

where in the second inequality we have applied the substitution $w = zy^{-1}$. Thus, κ_x is integrable, for every $x \in G$. Now, using Leibniz's rule for vector fields, for every $\beta_0 \in \mathcal{I}(k)$, we obtain

$$\begin{aligned}
\widetilde{X}_{\beta_0,x}\, \kappa_x(z) &= \int_G \widetilde{X}_{\beta_0,x} \left\{ \kappa_2(xy^{-1}, zy^{-1})\, \kappa_1(x,y) \right\} \, \mathrm{d}y \\
&= \sum_{|\beta_1|+|\beta_2|=|\beta_0|} c^{\beta_0}_{\beta_1,\beta_2} \int_G \widetilde{X}_{\beta_2,x_2=xy^{-1}}\, \kappa_{2,x_2}(zy^{-1})\, \widetilde{X}_{\beta_1,x_1=x}\, \kappa_{1,x_1}(y)\, \mathrm{d}y.
\end{aligned}$$

Hence, proceeding as before, we have

$$\begin{aligned}
&\int_G \left| \widetilde{X}_{\beta_0,x}\, \kappa_x(z) \right| \, \mathrm{d}z \\
&\lesssim \sum_{|\beta_1|+|\beta_2|=|\beta_0|} \sup_{x' \in G} \left(\int_G \left| \widetilde{X}_{\beta_2,x_2=x'}\, \kappa_{2,x_2}(w) \right| \, \mathrm{d}w \right) \int_G \left| \widetilde{X}_{\beta_1,x_1=x}\, \kappa_{1,x_1}(y) \right| \, \mathrm{d}y.
\end{aligned}$$

This implies that σ is a smooth symbol.

Next, we must prove that (4.15.2) and (4.15.3) hold. To this aim, first observe that formally, for $f \in \mathcal{D}'(G)$ and $x \in G$, we have

$$\begin{aligned}
\mathrm{Op}(\sigma_1) \circ \mathrm{Op}(\sigma_2)\, f(x) &= \int_G \left[\mathrm{Op}(\sigma_2)\, f\right](z)\, \kappa_{1,x}(z^{-1}x)\, \mathrm{d}z \\
&= \int_G \int_G f(y)\, \kappa_{2,z}(y^{-1}z)\, \kappa_{1,x}(z^{-1}x)\, \mathrm{d}y\, \mathrm{d}z \\
&= \int_G \int_G f(y)\, \kappa_{2,xw^{-1}}(y^{-1}xw^{-1})\, \kappa_{1,x}(w)\, \mathrm{d}y\, \mathrm{d}w,
\end{aligned}$$

using the substitution $w = z^{-1}x$. Since

$$\int_G \kappa_{2,xw^{-1}}(y^{-1}xw^{-1})\, \kappa_{1,x}(w)\, \mathrm{d}w = \kappa_x(y^{-1}x), \qquad \forall\, y \in G,$$

then applying Fubini's Theorem to swap the order of integration, we obtain

$$\mathrm{Op}(\sigma_1) \circ \mathrm{Op}(\sigma_2)\, f(x) = \int_G f(y)\, \kappa_x(y^{-1}x)\, \mathrm{d}y = f * \kappa_x(x).$$

It follows that

$$\mathrm{Op}(\sigma_1) \circ \mathrm{Op}(\sigma_2) = \mathrm{Op}(\sigma),$$

as required.

Finally, for $x \in G$ and $\pi \in \widehat{G}$,

$$\begin{aligned}
\sigma(x,\pi) = \widehat{\kappa_x}(\pi) &= \int_G \kappa_x(z)\, \pi(z)^*\, \mathrm{d}z \\
&= \int_G \int_G \kappa_{2,xy^{-1}}(zy^{-1})\, \kappa_{1,x}(y)\, \pi(z)^*\, \mathrm{d}y\, \mathrm{d}z.
\end{aligned}$$

Since κ_x is integrable, we can apply Fubini's Theorem to obtain that

$$\begin{aligned}
\sigma(x,\pi) &= \int_G \int_G \kappa_{2,xy^{-1}}(zy^{-1})\, \pi(z)^*\, \mathrm{d}z\ \kappa_{1,x}(y)\ \mathrm{d}y \\
&= \int_G \kappa_{1,x}(y)\, \pi(y)^* \left(\int_G \kappa_{2,xy^{-1}}(zy^{-1})\, \pi(zy^{-1})^*\, \mathrm{d}z \right) \mathrm{d}y,
\end{aligned}$$

as $\pi(z^{-1}) = \pi(y^{-1})\, \pi(yz^{-1}) = \pi(y)^*\, \pi(zy^{-1})^*$. Since

$$\int_G \kappa_{2,xy^{-1}}(zy^{-1})\,\pi(zy^{-1})^*\,\mathrm{d}z = \mathcal{F}\left\{\kappa_{2,xy^{-1}}\right\}(\pi) = \sigma_2(xy^{-1},\pi),$$

then we have

$$\sigma(x,\pi) \;=\; \int_G \kappa_{1,x}(y)\,\pi(y)^*\,\sigma_2(xy^{-1},\pi)\,\mathrm{d}y,$$

as required.

□

The following result is an immediate consequence of Lemma 4.15.2 and Theorem 4.10.1.

Corollary 4.15.3. *Let* $\sigma_1, \sigma_2 \in S^{-\infty}$*, and suppose that* σ_2 *is an invariant symbol. Then,*

$$\sigma_1 \circ \sigma_2 \;=\; \sigma_1\sigma_2.$$

Let us now consider the cut-off function $\eta_0 \in \mathcal{D}((0,+\infty))$, taking values in $[0,1]$, such that

$$\mathrm{supp}(\eta_0) \cap [0,+\infty) \;\subset\; [0,\lambda_1), \qquad \eta_0(0) \;=\; 1.$$

We write

$$\sigma_1 \;=\; \sigma_1 - \sigma_1\eta_0(\pi(\mathcal{L})) + \sigma_1\eta_0(\pi(\mathcal{L})), \qquad \sigma_2 \;=\; \sigma_2 - \sigma_2\eta_0(\pi(\mathcal{L})) + \sigma_2\eta_0(\pi(\mathcal{L})).$$

Hence, we have

$$\begin{aligned}\mathrm{Op}(\sigma_1)\circ\mathrm{Op}(\sigma_2) = \big(&\mathrm{Op}\big(\sigma_1 - \sigma_1\eta_0(\pi(\mathcal{L}))\big) + \mathrm{Op}\big(\sigma_1\eta_0(\pi(\mathcal{L}))\big)\big)\\ &\circ\big(\mathrm{Op}\big(\sigma_2 - \sigma_2\eta_0(\pi(\mathcal{L}))\big) + \mathrm{Op}\big(\sigma_2\eta_0(\pi(\mathcal{L}))\big)\big).\end{aligned}$$

Furthermore, by Proposition 4.11.3 (B), $\sigma_1\eta_0(\pi(\mathcal{L})),\ \sigma_2\eta_0(\pi(\mathcal{L})) \in S^{-\infty}$, so by Corollary 4.15.3 and Theorem 4.10.1, it suffices to show the result for

$$\mathrm{Op}\big(\sigma_1 - \sigma_1\eta_0(\pi(\mathcal{L}))\big)\circ\mathrm{Op}\big(\sigma_2 - \sigma_2\eta_0(\pi(\mathcal{L}))\big).$$

Therefore, Proposition 4.13.1 is applicable in this case, and we may assume that

$$\sigma_j(\cdot, 1_{\widehat{G}}) \;=\; \sigma_j\eta_0(\pi(\mathcal{L})) \;\equiv\; 0, \qquad j = 1,2.$$

4.15.3 First step of the proof of Theorem 4.15.1

We now start the proof of Theorem 4.15.1. First observe that, by Theorem 4.3.3 and Remark 4.3.5, for fixed $x \in G$ and $\pi \in \widehat{G}$ we have

$$\sigma_2(xy^{-1},\pi) \;=\; \sum_{[\alpha]<M} \frac{1}{\alpha!}\, q_{0,\alpha}(y^{-1})\, Y_x^{\alpha}\sigma_2(x,\pi) \;+\; R_{x,M}^{\sigma_2(\cdot,\pi)}(y^{-1}), \tag{4.15.4}$$

for any $y \in G$, where

$$\begin{aligned} &\big|\big|R_{x,M}^{\sigma_2(\cdot,\pi)}(y^{-1})\big|\big|_{\mathscr{L}(\mathscr{H}_\pi)} \\ &\qquad\qquad \le C\,|y|^M \sup_{\substack{z\in G\\ [\alpha]\ge M\\ |\alpha|\le M}} ||Y_z^{\alpha}\sigma_2(z,\pi)||_{\mathscr{L}(\mathscr{H}_\pi)}, \quad \forall\, y \in G, \end{aligned} \tag{4.15.5}$$

for some $C > 0$ independent of x. Consequently, for all $y \in G$, we have

$$\big|\big|R_{x,M}^{\sigma_2(\cdot,\pi)}(y^{-1})\big|\big|_{\mathscr{L}(\mathscr{H}_\pi)} \;\le\; C\,|y|^M \sup_{\substack{z\in G\\ |\alpha|\le M}} \big|\big|Y_z^{\alpha}\sigma_2(z,\pi)\big|\big|_{\mathscr{L}(\mathscr{H}_\pi)} \tag{4.15.6}$$

Then, by (4.15.3), for any $x \in G$ and $\pi \in \widehat{G}$ we have

$$\begin{aligned} \sigma(x,\pi) \;&=\; \int_G \kappa_{1,x}(y)\,\pi(y)^*\,\sigma_2(xy^{-1},\pi)\,\mathrm{d}y \\ &=\; \int_G \sum_{[\alpha]<M} \frac{1}{\alpha!}\, q_{0,\alpha}(y^{-1})\,\kappa_{1,x}(y)\,\pi(y)^*\,Y_x^{\alpha}\sigma_2(x,\pi)\,\mathrm{d}y \\ &\qquad\qquad\qquad\qquad + \int_G \kappa_{1,x}(y)\,\pi(y)^*\,R_{x,M}^{\sigma_2(\cdot,\pi)}(y^{-1})\,\mathrm{d}y \\ &=\; \sum_{[\alpha]<M} \frac{1}{\alpha!}\,\Delta^{\alpha}\sigma_1(x,\pi)\,Y_x^{\alpha}\sigma_2(x,\pi) \\ &\qquad\qquad\qquad\qquad + \int_G \kappa_{1,x}(y)\,\pi(y)^*\,R_{x,M}^{\sigma_2(\cdot,\pi)}(y^{-1})\,\mathrm{d}y. \end{aligned} \tag{4.15.7}$$

Note that, by Theorem 4.10.1, we have

$$(\Delta^{\alpha}\sigma_1)\cdot(Y^{\alpha}\sigma_2) \;\in\; S^{m_1+m_2-[\alpha]}, \qquad \forall\, \alpha \,\in\, \mathbb{N}_0^n,\; [\alpha] < M. \tag{4.15.8}$$

Hence,

$$\sum_{[\alpha]<M} \frac{1}{\alpha!} \Delta^\alpha \sigma_1(x,\pi_n)\, Y_x^\alpha \sigma_2(x,\pi_n) \in S^{m_1+m_2}. \tag{4.15.9}$$

For $M \in \mathbb{N}$, we now let $\rho_{M,\sigma_1,\sigma_2}$ be the symbol given by

$$\rho_{M,\sigma_1,\sigma_2}(x,\pi) := \sigma(x,\pi) - \sum_{[\alpha]<M} \frac{1}{\alpha!} \Delta^\alpha \sigma_1(x,\pi)\, Y_x^\alpha \sigma_2(x,\pi), \tag{4.15.10}$$

for $x \in G$, $\pi \in \widehat{G}$. By (4.15.7) we have

$$\rho_{M,\sigma_1,\sigma_2}(x,\pi) = \int_G \kappa_{1,x}(y)\, \pi(y)^*\, R_{x,M}^{\sigma_2(\cdot,\pi)}(y^{-1})\, \mathrm{d}y, \qquad x \in G,\ \pi \in \widehat{G}. \tag{4.15.11}$$

We then need to show that, for any a, $b \in \mathbb{N}_0$ and $c > 0$, there exist $M \in \mathbb{N}$, $C > 0$, a_1, a_2, b_1, $b_2 \in \mathbb{N}_0$ and c_1, $c_2 > 0$, independent of σ_1, σ_2, such that

$$||\rho_{M,\sigma_1,\sigma_2}||_{S^{m_1+m_2},a,b,c} \leq C\, ||\sigma_1||_{S^{m_1},a_1,b_1,c_1}\, ||\sigma_2||_{S^{m_2},a_2,b_2,c_2}. \tag{4.15.12}$$

We prove this result in the following sections.

4.15.4 Analysis of the remainder

Step 1: The symbol $\rho_{M,\sigma_1,\sigma_2}(x,\pi)$

In this section we study the symbol $\rho_{M,\sigma_1,\sigma_2}$, given by (4.15.10), and claim that there exists $M_0 \in \mathbb{N}_0$ such that, for any $M \geq M_0$, there exist a_1, b_1, $b_2 \in \mathbb{N}_0$ and $c_1 > 0$ such that, for all $x \in G$,

$$\big|\big|\rho_{M,\sigma_1,\sigma_2}(x,\pi)\big|\big|_{L^\infty(\widehat{G})} \lesssim ||\sigma_1||_{S^{m_1},a_1,b_1,c_1}\, ||\sigma_2||_{S^{m_2},0,b_2,0}.$$

Then, let $M \in \mathbb{N}$ to be determined and fix $\pi \in G$. For any $y \in G$, we now write

$$\pi(y)^* = \pi(y)^*\, \pi(I+\mathcal{L})^{M_1}\, \pi(I+\mathcal{L})^{-M_1},$$

for some $M_1 \in \mathbb{N}$ to be determined. Note that

$$(I+\mathcal{L})^{M_1} \;=\; \sum_{\substack{\gamma\,\in\,\mathcal{I}(k)\\ |\gamma|\le 2M_1}} c_\gamma\, X_\gamma, \tag{4.15.13}$$

for some constants $c_\gamma \in \mathbb{R}$. And similarly,

$$(I+\widetilde{\mathcal{L}})^{M_1} \;=\; \sum_{\substack{\widetilde{\gamma}\,\in\,\mathcal{I}(k)\\ |\widetilde{\gamma}|\le 2M_1}} \widetilde{c}_{\widetilde{\gamma}}\, \widetilde{X}_{\widetilde{\gamma}},$$

for some constants $\widetilde{c}_{\widetilde{\gamma}} \in \mathbb{R}$. So, $\pi(y)^*$ can be re-written as

$$\sum_{\substack{\gamma\,\in\,\mathcal{I}(k)\\ |\gamma|\le 2M_1}} c_\gamma\, \pi(y)^*\, \pi(X_\gamma)\, \pi(I+\mathcal{L})^{-M_1}. \tag{4.15.14}$$

Moreover, for any left-invariant vector field X, we have

$$\pi(y)^*\, \pi(X) = -\left(\pi(X)\, \pi(y)\right)^* = -\big(\widetilde{X}_y\, \pi(y)\big)^*. \tag{4.15.15}$$

Combining (4.15.14) and (4.15.15), we obtain

$$\begin{aligned}
&\int_G \kappa_{1,x}(y)\, \pi(y)^*\, R_{x,M}^{\sigma_2(\cdot,\pi)}(y^{-1})\, \mathrm{d}y \\
&\quad = \sum_{\substack{\gamma\,\in\,\mathcal{I}(k)\\ |\gamma|\le 2M_1}} (-1)^{|\gamma|}\, c_\gamma \int_G \kappa_{1,x}(y)\, \big(\widetilde{X}_{\gamma,y}\, \pi(y)\big)^*\, \pi(I+\mathcal{L})^{-M_1}\, R_{x,M}^{\sigma_2(\cdot,\pi)}(y^{-1})\, \mathrm{d}y \\
&\quad = \sum_{\substack{\gamma\,\in\,\mathcal{I}(k)\\ |\gamma|\le 2M_1}} (-1)^{|\gamma|}\, c_\gamma \int_G \kappa_{1,x}(y)\, \big(\widetilde{X}_{\gamma,y}\, \pi(y)\big)^*\, R_{x,M}^{\pi(I+\mathcal{L})^{-M_1}\sigma_2(\cdot,\pi)}(y^{-1})\, \mathrm{d}y.
\end{aligned}$$

Recall that, for $N \in \mathbb{N}$ to be determined, f_N is the function given by

$$f_N \;=\; \sum_{j=1}^{n} q_{0,j}^{\frac{2N_0N}{d_j}}, \tag{4.15.16}$$

where N_0 denotes the lowest common multiple of $(d_1, d_2, \ldots, d_n)$. We have

$$|f_N(z)| \;\approx\; |z|^{2N_0N}, \qquad \forall\, z \,\in\, G. \tag{4.15.17}$$

We then write

$$
\begin{aligned}
&\rho_{M,\sigma_1,\sigma_2}(x,\pi)\\
&\quad= \int_G \kappa_{1,x}(y)\,\pi(y)^*\,R_{x,M}^{\sigma_2(\cdot,\pi)}(y^{-1})\,\mathrm{d}y\\
&\quad= \sum_{\substack{\gamma\,\in\,\mathcal{I}(k)\\ |\gamma|\le 2M_1}} (-1)^{|\gamma|}c_\gamma \int_G (f_N\,\kappa_{1,x})(y)\,\big(\widetilde{X}_{\gamma,y}\,\pi(y)\big)^*\left(\frac{1}{f_N}\,\widetilde{R}_{x,M}^{\pi(I+\mathcal{L})^{-M_1}\sigma_2(\cdot,\pi)}\right)(y)\,\mathrm{d}y.
\end{aligned}
$$

Using integration by parts and Leibniz's rule for vector fields yields

$$
\begin{aligned}
&\rho_{M,\sigma_1,\sigma_2}(x,\pi)\\
&\qquad= \sum_{\substack{\gamma\,\in\,\mathcal{I}(k)^2\\ |\gamma|\le 2M_1}} c_\gamma \int_G \widetilde{X}_{\gamma_1,y_1=y}\big(f_N\,\kappa_{1,x}\big)(y_1)\,\pi(y)^*\\
&\qquad\qquad\qquad\qquad \widetilde{X}_{\gamma_2,y_2=y}\left(\frac{1}{f_N}\,\widetilde{R}_{x,M}^{\pi(I+\mathcal{L})^{-M_1}\sigma_2(\cdot,\pi)}\right)(y_2)\,\mathrm{d}y.
\end{aligned}
\tag{4.15.18}
$$

Taking norms, we obtain

$$
\begin{aligned}
&||\rho_{M,\sigma_1,\sigma_2}(x,\pi)||_{L^\infty(\widehat{G})}\\
&\qquad\lesssim \sum_{\substack{\gamma\,\in\,\mathcal{I}(2)^2\\ |\gamma|\le 2M_1}} \int_G \big|\widetilde{X}_{\gamma_1,y_1=y}\,(f_N\,\kappa_{1,x})(y_1)\big|\,\mathrm{d}y\\
&\qquad\qquad\qquad \sup_{y'\,\in\,G}\big|\big|\widetilde{X}_{\gamma_2,y_2=y'}\big(\widetilde{R}_{x,M}^{\pi(I+\mathcal{L})^{-M_1}\,\sigma_2(\cdot,\pi)}/f_N\big)(y_2)\big|\big|_{L^\infty(\widehat{G})}.
\end{aligned}
\tag{4.15.19}
$$

Observe that, by Lemma 3.7.7 (2) and Remark 3.7.8, for any $\gamma_2\in\mathcal{I}(k)$, with $|\gamma_2|\le 2M_1$, there exists $C>0$, depending on γ_2, f_N and k, such that

$$
\begin{aligned}
&\sup_{y'\,\in\,G}\big|\big|\widetilde{X}_{\gamma_2,y'}\big(\widetilde{R}_{x,M}^{\pi(I+\mathcal{L})^{-M_1}\,\sigma_2(\cdot,\pi)}/f_N\big)(y')\big|\big|_{L^\infty(\widehat{G})}\\
&\qquad\qquad\le C \sup_{\substack{y'\,\in\,G\\ |\gamma_{0,2}|=M}}\big|\big|\widetilde{X}_{\gamma_{0,2},y'}\,\widetilde{R}_{x,M}^{\pi(I+\mathcal{L})^{-M_1}\,\sigma_2(\cdot,\pi)}(y')\big|\big|_{L^\infty(\widehat{G})}
\end{aligned}
\tag{4.15.20}
$$

provided that $|\gamma_2|<M-2N_0N$, having used (4.15.17) and adapted (4.15.6). As $|\gamma_2|\le 2M_1$, it suffices to assume that

$$2M_1 \;<\; M - 2N_0N. \tag{4.15.21}$$

Moreover, by Lemma 4.3.6 and Remark 4.3.7, we obtain

$$\sup_{\substack{y' \in G \\ |\gamma_{0,2}|=M}} \big|\big|\widetilde{X}_{\gamma_{0,2},y'}\, \widetilde{R}_{x,M}^{\pi(I+\mathcal{L})^{-M_1}\,\sigma_2(\cdot,\pi)}(y')\big|\big|_{L^\infty(\widehat{G})}$$
$$\lesssim \sup_{\substack{y' \in G \\ [\beta_0]\leq M}} \big|\big|\pi(I+\mathcal{L})^{-M_1} Y_{y'}^{\beta_0}\sigma_2(y',\pi)\big|\big|_{L^\infty(\widehat{G})},$$

Furthermore,

$$\begin{aligned}\sup_{\substack{y' \in G \\ [\beta_0]\leq M}} &\big|\big|\pi(I+\mathcal{L})^{-M_1} Y_{y'}^{\beta_0}\sigma_2(y',\pi)\big|\big|_{L^\infty(\widehat{G})} \\ &\lesssim \sup_{\substack{y' \in G \\ [\beta_0]\leq M}} \big|\big|\pi(I+\mathcal{L})^{\frac{1}{2}(m_2-2M_1)}\big|\big|_{L^\infty(\widehat{G})} \big|\big|\pi(I+\mathcal{L})^{-\frac{m_2}{2}}\, Y_{y'}^{\beta_0}\sigma_2(y,\pi)\big|\big|_{L^\infty(\widehat{G})} \\ &\lesssim ||\sigma_2||_{S^{m_2},0,M,0},\end{aligned} \tag{4.15.22}$$

provided that

$$m_2 - 2M_1 \;\leq\; 0. \tag{4.15.23}$$

Thus, by (4.15.21) and (4.15.23), we have have shown that, if

$$m_2 \;\leq\; 2M_1 \;<\; M - 2N_0N, \tag{4.15.24}$$

then

$$\sup_{y \in G} \big|\big|\widetilde{X}_{\gamma_2,y}\big(\widetilde{R}_{x,M}^{\pi(I+\mathcal{L})^{-M_1}\,\sigma_2(\cdot,\pi)}/f_N\big)(y)\big|\big|_{L^\infty(\widehat{G})} \lesssim ||\sigma_2||_{S^{m_2},0,M,0}. \tag{4.15.25}$$

On the other hand, we have

$$\int_G \big|\widetilde{X}_{\gamma_1,y_1=y}\,(f_N\,\kappa_{1,x})(y_1)\big|\,\mathrm{d}y \lesssim \sum_{j=1}^{n} \int_G \big|\widetilde{X}_{\gamma_1,y_1=y}\,(q_{0,\alpha_{N,j}}\,\kappa_{1,x})(y_1)\big|\,\mathrm{d}y, \tag{4.15.26}$$

where $\alpha_{N,j} = \big(0,0,\dots,\frac{2N_0N}{d_j},\dots,0\big)$, with the only non-zero term of the multi-index being in the j-th position. Therefore, by Lemma 4.7.2 (2), for any $j = 1, 2, \ldots, n$, if

$$m - [\alpha_{N,j}] + |\gamma_1| \, < \, -l,$$

then there exist $C > 0$, a_1, $b_1 \in \mathbb{N}_0$ and $c_1 > 0$ such that

$$\sup_{y \in G} \left| \widetilde{X}_{\gamma_1, y} \, (q_{0,\alpha_{N,j}} \, \kappa_{1,x})(y) \right| \, \leq \, C \, ||\sigma_1||_{S^{m_1}, a_1, b_1, c_1}.$$

Since $[\alpha_{N,j}] = 2N_0 N$ and $|\gamma_1| \leq 2M_1$, then a sufficient condition is

$$m_1 - 2N_0 N + 2M_1 \, < \, -l. \tag{4.15.27}$$

Thus, we have shown that, if $m_1 - 2N_0 N + 2M_1 < -l$, then we have

$$\int_G \left| \widetilde{X}_{\gamma_1, y_1 = y} \, (f_N \, \kappa_{1,x})(y_1) \right| \mathrm{d}y \, \lesssim \, ||\sigma_1||_{S^{m_1}, a_1, b_1, c_1}. \tag{4.15.28}$$

Combining (4.15.25) and (4.15.28) with (4.15.19) (see also (4.15.24) and (4.15.27)), we obtain that if we choose M, M_1, $N \in \mathbb{N}$ such that

$$\begin{cases} m_2 \, \leq \, 2M_1 \, < \, M - 2N_0 N \\ m_1 - 2N_0 N + 2M_1 \, < \, -l \end{cases},$$

then, there exist a_1, b_1, $b_2 \in \mathbb{N}_0$ and $c_1 > 0$ such that

$$\left|\left| \rho_{M,\sigma_1,\sigma_2}(x, \pi) \right|\right|_{L^\infty(\widehat{G})} \, \lesssim \, ||\sigma_1||_{S^{m_1}, a_1, b_1, c_1} \, ||\sigma_2||_{S^{m_2}, 0, b_2, 0}, \tag{4.15.29}$$

which concludes the first step.

Step 2: $\widetilde{X}_{\beta_0, x} \, \rho_M(x, \pi)$

Let $M \in \mathbb{N}$ to be determined. In this section, we study the symbol $\widetilde{X}_{\beta_0, x} \, \rho_{M,\sigma_1,\sigma_2}(x, \pi)$, for a given $\beta_0 \in \mathcal{I}(k)$. By the definition of $\rho_{M,\sigma_1,\sigma_2}$ (see (4.15.10)), for any $x \in G$ and $\pi \in \widehat{G}$, we have

$$\begin{aligned} &\widetilde{X}_{\beta_0, x} \, \rho_{M,\sigma_1,\sigma_2}(x, \pi) \\ &\qquad = \widetilde{X}_{\beta_0, x} \, \sigma(x, \pi) \; - \sum_{[\alpha] < M} \frac{1}{\alpha!} \, \widetilde{X}_{\beta_0, x} \big(\Delta^\alpha \sigma_1(x, \pi) Y_x^\alpha \sigma_2(x, \pi) \big). \end{aligned} \tag{4.15.30}$$

Observe that, by Leibniz's rule for vector fields we have

$$
\begin{aligned}
&\widetilde{X}_{\beta_0,x}\,\sigma(x,\pi)\\
&\qquad = \widetilde{X}_{\beta_0,x}\int_G \kappa_{1,x}(y)\,\pi(y)^*\,\sigma_2(xy^{-1},\pi)\,\mathrm{d}y\\
&\qquad = \sum_{|\beta_1|+|\beta_2|=|\beta_0|} c^{\beta_0}_{\beta_1,\beta_2}\int_G \widetilde{X}_{\beta_1,x_1=x}\,\kappa_{1,x_1}(y)\,\pi(y)^*\,\widetilde{X}_{\beta_2,x_2=x}\,\sigma_2(x_2y^{-1},\pi)\,\mathrm{d}y\\
&\qquad = \sum_{|\beta_1|+|\beta_2|=|\beta_0|} c^{\beta_0}_{\beta_1,\beta_2}\int_G \big(\widetilde{X}_{\beta_1,x}\,\kappa_{1,x}\big)(y)\,\pi(y)^*\,\big(\widetilde{X}_{\beta_2,x_2=xy^{-1}}\,\sigma_2\big)(x_2,\pi)\,\mathrm{d}y,
\end{aligned}
$$

for some constants $c^{\beta_0}_{\beta_1,\beta_2}\in\mathbb{R}$.

Moreover, applying Leibniz's rule for vector fields once again, for any $\alpha\in\mathbb{N}_0^n$, with $[\alpha]<M$, we have

$$
\begin{aligned}
&\widetilde{X}_{\beta_0,x}\big(\Delta^\alpha\sigma_1(x,\pi)Y_x^\alpha\sigma_2(x,\pi)\big)\\
&\qquad\qquad = \sum_{|\beta_1|+|\beta_2|=|\beta_0|} c^{\beta_0}_{\beta_1,\beta_2}\,\Delta^\alpha\big(\widetilde{X}_{\beta_1,x}\,\sigma_1\big)(x,\pi)\,Y_x^\alpha\big(\widetilde{X}_{\beta_2,x}\,\sigma_2\big)(x,\pi)
\end{aligned}
$$

Observe that, for each $\beta_1,\,\beta_2\in\mathcal{I}(k)$, $\widetilde{X}_{\beta_1}\sigma_1=\widetilde{X}_{\beta_1,x}\,\sigma_1(x,\pi)$ belongs to the symbol class S^{m_1}, with associated kernel

$$
\widetilde{X}_{\beta_1}\kappa_1\,:\,(x,y)\longmapsto\big(\widetilde{X}_{\beta_1,x}\,\kappa_{1,x}\big)(y),
$$

and $\widetilde{X}_{\beta_2}\sigma_2=\widetilde{X}_{\beta_2,x}\,\sigma_2(x,\pi)$ belongs to the symbol class S^{m_2} with associated kernel

$$
\widetilde{X}_{\beta_2}\kappa_2\,:\,(x,y)\longmapsto\big(\widetilde{X}_{\beta_2,x}\,\kappa_{2,x}\big)(y).
$$

Thus, by (4.15.30) we have obtained

$$
\widetilde{X}_{\beta_0}\,\rho_{M,\sigma_1,\sigma_2}=\sum_{|\beta_1|+|\beta_2|=|\beta|} c^{\beta_0}_{\beta_1,\beta_2}\,\rho_{M,\widetilde{X}_{\beta_1}\sigma_1,\widetilde{X}_{\beta_2}\sigma_2}.
$$

Applying Step 1 to each $\rho_{M,\widetilde{X}_{\beta_1}\sigma_1,\widetilde{X}_{\beta_2}\sigma_2}$ we conclude that there exists $M_0\in\mathbb{N}_0$ such that, for any $M\geq M_0$, there exist $a_1,\,b_1,\,b_2\in\mathbb{N}_0$ and $c_1>0$ such that, for all $x\in G$,

$$
\big|\big|\widetilde{X}_{\beta_0,x}\,\rho_{M,\sigma_1,\sigma_2}(x,\pi)\big|\big|_{L^\infty(\widehat{G})}\lesssim||\sigma_1||_{S^{m_1},a_1,b_1,c_1}\,||\sigma_2||_{S^{m_2},0,b_2,0}.
$$

Step 3a: $\rho_{M,\sigma_1,\sigma_2}\,\pi(X_{\beta_0})$

In this step of the proof, we consider the symbol $\rho_{M,\sigma_1,\sigma_2}\,\pi(X_{\beta_0})$, for $\beta_0 \in \mathcal{I}(k)$. We let $M \in \mathbb{N}$ to be determined.

First observe that, by the definition of $\rho_{M,\sigma_1,\sigma_2}$ (see (4.15.10)), we have

$$\rho_{M,\sigma_1,\sigma_2}\,\pi(X_{\beta_0}) \;=\; \sigma\,\pi(X_{\beta_0}) - \sum_{[\alpha]<M} \frac{1}{\alpha!}\,(\Delta^\alpha\sigma_1)(Y^\alpha\sigma_2)\pi(X_{\beta_0}).$$

Now, note that, for any $x \in G$ and $\pi \in \widehat{G}$ we have

$$\sigma(x,\pi)\,\pi(X_{\beta_0}) \;=\; \mathcal{F}\big\{\widetilde{X}_{\beta_0}\,\kappa_x\big\}(\pi),$$

and moreover, for any $z \in G$,

$$\begin{aligned}
\widetilde{X}_{\beta_0,z}\,\kappa_x(z) \;&=\; \int_G \widetilde{X}_{\beta_0,z}\,\kappa_{2,xy^{-1}}(zy^{-1})\,\kappa_{1,x}(y)\,\mathrm{d}y \\
&=\; \int_G \big(\widetilde{X}_{\beta_0}\,\kappa_{2,xy^{-1}}\big)(zy^{-1})\,\kappa_{1,x}(y)\,\mathrm{d}y.
\end{aligned}$$

So,

$$\begin{aligned}
\mathcal{F}\big\{\widetilde{X}_{\beta_0,z}\,\kappa_x\big\}(\pi) \;&=\; \int_G \kappa_{1,x}(y)\,\pi(y)^*\,\mathcal{F}\{\widetilde{X}_{\beta_0}\,\kappa_{2,xy^{-1}}\}(\pi)\,\mathrm{d}y \\
&=\; \int_G \kappa_{1,x}(y)\,\pi(y)^*\,(\sigma_2\,\pi(X_{\beta_0}))(xy^{-1},\pi)\,\mathrm{d}y \\
&=\; \sigma_1 \circ (\sigma_2\,\pi(X_{\beta_0}))(x,\pi).
\end{aligned}$$

Therefore, we have shown that

$$\begin{aligned}
\rho_{M,\sigma_1,\sigma_2}\,\pi(X_{\beta_0}) \;&=\; \sigma_1 \circ (\sigma_2\,\pi(X_{\beta_0})) - \sum_{[\alpha]<M} \frac{1}{\alpha!}\,(\Delta^\alpha\sigma_1)(Y^\alpha\sigma_2\,\pi(X_{\beta_0})) \\
&=\; \rho_{M,\sigma_1,\sigma_2\pi(X_{\beta_0})}.
\end{aligned}$$

Moreover, by Theorem 4.10.1 and Proposition 4.5.10, $\sigma_2\,\pi(X_{\beta_0}) \in S^{m_2+|\beta_0|}$. Thus, applying Step 1 to $\rho_{M,\sigma_1,\sigma_2\pi(X_{\beta_0})}$, we conclude that there exists $M_0 \in \mathbb{N}_0$, depending on β_0, σ_1, σ_2, such that, for any $M \geq M_0$, there exist a_1, b_1, $\widetilde{b}_2 \in \mathbb{N}_0$ and $c_1 > 0$ such that, for all $x \in G$,

$$\big|\big|\rho_{M,\sigma_1,\sigma_2}(x,\pi)\,\pi(X_{\beta_0})\big|\big|_{L^\infty(\widehat{G})} \lesssim ||\sigma_1||_{S^{m_1},a_1,b_1,c_1}\,||\sigma_2\,\pi(X_{\beta_0})||_{S^{m_2+|\beta_0|},0,\widetilde{b}_2,0}.$$

By applying Lemma 4.10.2 to $\sigma_2\,\pi(X_{\beta_0})$, there exist $a_2,\,a_2',\,a_2,\,b_2' \in \mathbb{N}_0$ and $c_2,\,c_2' > 0$ such that

$$||\sigma_2\,\pi(X_{\beta_0})||_{S^{m_2+|\beta_0|},0,b_2,0} \lesssim ||\sigma_2||_{S^{m_2},a_2,b_2,c_2}\,||\pi(X_{\beta_0})||_{S^{|\beta_0|},a_2',b_2',c_2'}.$$

So, for every $x \in G$ and all $M \geq M_0$ we have

$$\big|\big|\rho_{M,\sigma_1,\sigma_2}(x,\pi)\,\pi(X_{\beta_0})\big|\big|_{L^\infty(\widehat{G})} \lesssim ||\sigma_1||_{S^{m_1},a_1,b_1,c_1}\,||\sigma_2||_{S^{m_2},a_2,b_2,c_2}. \qquad (4.15.31)$$

Step 3b: $\widetilde{X}_{\beta_0}\,\kappa_{M,\sigma_1,\sigma_2,x}$

Let $M \in \mathbb{N}_0$ to be chosen and $\beta_0 \in \mathcal{I}(k)$. Furthermore, suppose that the distribution

$$\kappa_{M,\sigma_1,\sigma_2} : (x,z) \longmapsto \kappa_{M,\sigma_1,\sigma_2,x}(z)$$

denotes the right convolution kernel associated to $\rho_{M,\sigma_1,\sigma_2}$. In this step we compute an estimate for the L^∞-norm of $\widetilde{X}_{\beta_0}\,\kappa_{M,\sigma_1,\sigma_2,x}$, for a fixed $x \in G$.

By Theorems 3.4.1 and 3.2.3, $\widetilde{X}_{\beta_0}\,\kappa_{M,\sigma_1,\sigma_2,x}$ is continuous on G and there exists $C > 0$, independent of $\widetilde{X}_{\beta_0}\,\kappa_{M,\sigma_1,\sigma_2,x}$, such that

$$\big|\big|\widetilde{X}_{\beta_0}\,\kappa_{M,\sigma_1,\sigma_2,x}\big|\big|_{L^\infty(G)} \leq C \sum_{\substack{\beta_0' \in \mathcal{I}(k) \\ |\beta_0'| \leq \lceil \frac{l}{2} \rceil + |\beta_0|}} \big|\big|\widetilde{X}_{\beta_0}\,\kappa_{M,\sigma_1,\sigma_2,x}\big|\big|_{L^2(G)}, \qquad (4.15.32)$$

provided that the right hand side of this inequality is finite. We now prove this. By Plancherel's Theorem (see Theorem 2.2.7), for any $\beta_0' \in \mathcal{I}(k)$, with $|\beta_0'| \leq |\beta_0|$, we have

$$\begin{aligned}\big|\big|\widetilde{X}_{\beta_0'}\,\kappa_{M,\sigma_1,\sigma_2,x}\big|\big|_{L^2(G)} &= \big|\big|\rho_{M,\sigma_1,\sigma_2}(x,\pi)\,\pi(X_{\beta_0'})\big|\big|_{L^2(\widehat{G})} \\ &\leq \big|\big|\rho_{M,\sigma_1,\sigma_2}(x,\pi)\,\pi(I+\mathcal{L})^{N_1}\big|\big|_{L^\infty(\widehat{G})} \\ &\qquad\qquad \big|\big|\pi(I+\mathcal{L})^{-N_1}\,\pi(X_{\beta_0'})\big|\big|_{L^2(\widehat{G})},\end{aligned}$$

for $N_1 \in \mathbb{N}_0$ to be determined.

Observe that, by the work done in Step 3 (see (4.15.31)), there exists $M_0 \in \mathbb{N}_0$ such that, for any $M \geq M_0$, there exist $\overline{a}_1, \overline{a}_2, \overline{b}_1, \overline{b}_2 \in \mathbb{N}_0$ and $\overline{c}_1, \overline{c}_2 > 0$ such that

$$\left|\left|\rho_{M,\sigma_1,\sigma_2}(x,\pi)\,\pi(I+\mathcal{L})^{N_1}\right|\right|_{L^\infty(\widehat{G})} \lesssim ||\sigma_1||_{S^{m_1},\overline{a}_1,\overline{b}_1,\overline{c}_1}\,||\sigma_2||_{S^{m_2},\overline{a}_2,\overline{b}_2,\overline{c}_2}. \quad (4.15.33)$$

Furthermore, we have

$$\begin{aligned}\left|\left|\pi(I+\mathcal{L})^{-N_1}\,\pi(X_{\beta_0'})\right|\right|_{L^2(\widehat{G})} &\lesssim \left|\left|\pi(I+\mathcal{L})^{-\frac{1}{2}(2N_1-|\beta_0'|)}\right|\right|_{L^2(\widehat{G})} \\ &= \left|\left|\mathcal{B}_{2N_1-|\beta_0'|}\right|\right|_{L^2(G)},\end{aligned}$$

where $\mathcal{B}_{2N_1-|\beta_0'|}$ denotes the right-convolution kernel associated to the operator $(I+\mathcal{L})^{-\frac{1}{2}(2N_1-|\beta_0'|)}$. This is finite, provided that $2N_1 - |\beta_0'| > \frac{l}{2}$, by Proposition 3.1.9. Thus, it suffices to assume that N_1 is such that

$$2N_1 - \left\lceil \frac{l}{2} \right\rceil - |\beta_0| > \frac{l}{2}.$$

Therefore, by (4.15.33), we have proved that, for each $\beta_0 \in \mathcal{I}(k)$, for any $M \geq M_0$, there exists $\overline{C} > 0$, independent of x, such that

$$\left|\left|\widetilde{X}_{\beta_0}\,\kappa_{M,\sigma_1,\sigma_2,x}\right|\right|_{L^\infty(G)} \leq \overline{C}\,||\sigma_1||_{S^{m_1},\overline{a}_1,\overline{b}_1,\overline{c}_1}\,||\sigma_2||_{S^{m_2},\overline{a}_2,\overline{b}_2,\overline{c}_2}.$$

Step 4: $\pi(X_{\beta_0})\,\rho_{M,\sigma_1,\sigma_2}$

In this step of the proof, we consider the symbol $\pi(X_{\beta_0})\,\rho_{M,\sigma_1,\sigma_2}$, for $\beta_0 \in \mathcal{I}(k)$. We let $M \in \mathbb{N}$ to be determined, and suppose $\kappa_{M,\sigma_1,\sigma_2} : (x,z) \mapsto \kappa_{M,\sigma_1,\sigma_2,x}(z)$ denotes the right-convolution kernel associated to $\rho_{M,\sigma_1,\sigma_2}$.

First observe that, by Lemma 2.2.4,

$$\begin{aligned}\left|\left|\pi(X_{\beta_0})\,\rho_{M,\sigma_1,\sigma_2}\right|\right|_{L^\infty(\widehat{G})} &\leq \left|\left|X_{\beta_0}\,\kappa_{M,\sigma_1,\sigma_2,x}\right|\right|_{L^1(G)} \\ &\lesssim \left|\left|X_{\beta_0}\,\kappa_{M,\sigma_1,\sigma_2,x}\right|\right|_{L^\infty(G)}, \quad (4.15.34)\end{aligned}$$

for $x \in G$. Now, recall that X_{β_0} can be written as

$$X_{\beta_0} \;=\; \sum_{|\widetilde{\beta}_0|\le|\beta_0|} c_{\widetilde{\beta}_0}\, \widetilde{X}_{\widetilde{\beta}_0},$$

for some $c_{\widetilde{\beta}_0} \in \mathcal{C}^\infty(G)$. Hence,

$$X_{\beta_0}\, \kappa_{M,\sigma_1,\sigma_2,x} \;=\; \sum_{|\widetilde{\beta}_0|\le|\beta_0|} c_{\widetilde{\beta}_0}\, \widetilde{X}_{\widetilde{\beta}_0}\, \kappa_{M,\sigma_1,\sigma_2,x}.$$

By applying in Step 3b to $\widetilde{X}_{\widetilde{\beta}_0}\, \kappa_{M,\sigma_1,\sigma_2,x}$, for each $\widetilde{\beta}_0 \in \mathcal{I}(k)$, with $|\widetilde{\beta}_0| \le |\beta_0|$, we then obtain that $X_{\beta_0}\, \kappa_{M,\sigma_1,\sigma_2,x}$ is continuous on G and satisfies the estimate

$$\big|\big|X_{\beta_0}\, \kappa_{M,\sigma_1,\sigma_2,x}\big|\big|_{L^\infty(G)} \lesssim ||\sigma_1||_{S^{m_1},a_1,b_1,c_1}\, ||\sigma_2||_{S^{m_2},a_2,b_2,c_2},$$

for some $a_1,\, a_2,\, b_1,\, b_2 \in \mathbb{N}_0$ and $c_1,\, c_2 > 0$, for M large enough. So, by (4.15.34), we have obtained that, for every $\beta_0 \in \mathcal{I}(k)$, there exists $M_0 \in \mathbb{N}_0$ such that, whenever $M \ge M_0$,

$$\big|\big|\pi(X_{\beta_0})\, \rho_{M,\sigma_1,\sigma_2}\big|\big|_{L^\infty(\widehat{G})} \lesssim ||\sigma_1||_{S^{m_1},a_1,b_1,c_1}\, ||\sigma_2||_{S^{m_2},a_2,b_2,c_2}.$$

Step 5: $\Delta^\alpha \rho_{M,\sigma_1,\sigma_2}$

Let q be a smooth, real-valued function on G, which is CC-vanishing at e_G up to order $a-1$, for $a \in \mathbb{N}$. In this step of the proof we consider the symbol $\Delta_q\, \rho_{M,\sigma_1,\sigma_2}$, for $M \in \mathbb{N}_0$ to be determined. First observe that

$$(\widetilde{q}\,\kappa_x)(z) \;=\; \int_G q(y^{-1}yz^{-1})\, \kappa_{2,xy^{-1}}(zy^{-1})\, \kappa_{1,x}(y)\, \mathrm{d}y. \tag{4.15.35}$$

By Theorem 4.3.3, for any $y, z \in G$ and $M \in \mathbb{N}_0$ we have

$$q(y^{-1}yz^{-1}) \;=\; \sum_{[\alpha]<M} \frac{1}{\alpha!}\, q_{0,\alpha}(yz^{-1})\, (Y^\alpha q)(y^{-1}) + R^q_{y^{-1},M}(yz^{-1}), \tag{4.15.36}$$

where

$$|R^q_{y^{-1},M}(z_1)| \;\le\; C\, |z_1|^M \max_{|\alpha|\le M} ||Y^\alpha q||_{L^\infty(G)}, \quad \forall\, z_1, y \in G, \tag{4.15.37}$$

for some $C > 0$ independent of y. Substituting (4.15.36) into (4.15.35) we obtain

$$
\begin{aligned}
(\widetilde{q}\,\kappa_x)(z) \;=\; & \sum_{[\alpha]<M} \frac{1}{\alpha!} \int_G q_{0,\alpha}(yz^{-1})\,(Y^\alpha q)(y^{-1})\,\kappa_{2,xy^{-1}}(zy^{-1})\,\kappa_{1,x}(y)\,\mathrm{d}y \\
& \qquad + \int_G R^q_{y^{-1},M}(yz^{-1})\,\kappa_{2,xy^{-1}}(zy^{-1})\,\kappa_{1,x}(y)\,\mathrm{d}y \\
=\; & \sum_{[\alpha]<M} \frac{1}{\alpha!} \int_G \big(\widetilde{q_{0,\alpha}}\,\kappa_{2,xy^{-1}}\big)(zy^{-1})((\widetilde{Y^\alpha q})\kappa_{1,x})(y)\,\mathrm{d}y \\
& \qquad + \int_G R^q_{y^{-1},M}(yz^{-1})\,\kappa_{2,xy^{-1}}(zy^{-1})\,\kappa_{1,x}(y)\,\mathrm{d}y. \qquad (4.15.38)
\end{aligned}
$$

Now note that, for each $\alpha \in \mathbb{N}_0^n$, with $[\alpha] < M$, the distribution $(x,z) \mapsto \big((\widetilde{Y^\alpha q})\kappa_{1,x}\big)(z)$ is the kernel associated to the symbol $\Delta_{Y^\alpha q}\,\sigma_1$. Hence,

$$
(x,z) \longmapsto \int_G \big(\widetilde{q_{0,\alpha}}\,\kappa_{2,xy^{-1}}\big)(zy^{-1})((\widetilde{Y^\alpha q})\kappa_{1,x})(y)\,\mathrm{d}y
$$

is the convolution kernel associated to the symbol

$$
(\Delta_{Y^\alpha q}\,\sigma_1) \circ (\Delta^\alpha \sigma_2),
$$

by Lemma 4.15.2. Hence, taking the Fourier transform of the expression given by (4.15.38), for any $x \in G$ and $\pi \in \widehat{G}$ we obtain

$$
\begin{aligned}
\Delta_q\,\sigma(x,\pi) \;=\; & \sum_{[\alpha]<M} \frac{1}{\alpha!}\,(\Delta_{Y^\alpha q}\,\sigma_1) \circ (\Delta^\alpha \sigma_2)(x,\pi) \\
& \qquad + \int_G \kappa_{1,x}(y)\,\pi(y)^*\,\mathcal{F}\big\{\widetilde{R}^q_{y^{-1},M}\,\kappa_{2,xy^{-1}}\big\}(\pi)\,\mathrm{d}y.
\end{aligned}
$$

Therefore, by the definition of $\rho_{M,\sigma_1,\sigma_2}$ (see (4.15.10)), we have

$$
\begin{aligned}
\Delta_q\,\rho_{M,\sigma_1,\sigma_2}(x,\pi) \;=\; & \sum_{[\alpha]<M} \frac{1}{\alpha!}\,(\Delta_{Y^\alpha q}\,\sigma_1) \circ (\Delta^\alpha \sigma_2)(x,\pi) \\
& \qquad + \int_G \kappa_{1,x}(y)\,\pi(y)^*\,\mathcal{F}\big\{\widetilde{R}^q_{y^{-1},M}\,\kappa_{2,xy^{-1}}\big\}(\pi)\,\mathrm{d}y \\
& \qquad - \sum_{[\alpha]<M} \frac{1}{\alpha!}\,\Delta_q\big(\Delta^\alpha\sigma_1(x,\pi)\,Y_x^\alpha\sigma_2(x,\pi)\big). \qquad (4.15.39)
\end{aligned}
$$

Step 5a: Simplification of (4.15.39)

We now aim to simplify (4.15.39). First we study the sum

$$\sum_{[\alpha]<M} \frac{1}{\alpha!} \Delta_q(\Delta^\alpha \sigma_1)(Y^\alpha \sigma_2).$$

Observe that, for each $\alpha \in \mathbb{N}_0^n$, with $[\alpha] < M$, applying Corollary 4.10.3 to $\Delta^\alpha \sigma_1$ and $Y^\alpha \sigma_2$, with $M' \in \mathbb{N}_0$ to be determined, we obtain

$$\Delta_q\big(\Delta^\alpha \sigma_1\big)\big(Y^\alpha \sigma_2\big) = \sum_{[\alpha_1]<M'} \frac{1}{\alpha_1!} \big(\Delta_{Y^{\alpha_1} q}\, \Delta^\alpha \sigma_1\big)\big(Y^\alpha \Delta^{\alpha_1} \sigma_2\big) + \tau_{M',\alpha},$$

where, for any $\widetilde{a}, \widetilde{b} \in \mathbb{N}_0$ and $\widetilde{c} > 0$, $\tau_{M',\alpha}$ satisfies

$$\begin{aligned} ||\tau_{M',\alpha}||_{S^{m_1-[\alpha]+m_2},\widetilde{a},\widetilde{b},\widetilde{c}} &\lesssim ||\Delta^\alpha \sigma_1||_{S^{m_1-[\alpha]},\widetilde{a}_1,\widetilde{b}_1,\widetilde{c}_1} ||Y^\alpha \sigma_2||_{S^{m_2},\widetilde{a}_2,\widetilde{b}_2,\widetilde{c}_2} \\ &\lesssim ||\sigma_1||_{S^{m_1},\widetilde{a}_1+[\alpha],\widetilde{b}_1,\widetilde{c}_1} ||\sigma_2||_{S^{m_2},\widetilde{a}_2,\widetilde{b}_2+[\alpha],\widetilde{c}_2}, \end{aligned} \tag{4.15.40}$$

for some $\widetilde{a}_1, \widetilde{a}_2, \widetilde{b}_1, \widetilde{b}_2 \in \mathbb{N}_0$, $\widetilde{c}_1, \widetilde{c}_2 > 0$. Hence, we have obtained

$$\begin{aligned} &\sum_{[\alpha]<M} \frac{1}{\alpha!} \Delta_q(\Delta^\alpha \sigma_1)(Y^\alpha \sigma_2) \\ &\qquad = \sum_{\substack{[\alpha]<M \\ [\alpha_1]<M'}} \frac{1}{\alpha_1!} \big(\Delta_{Y^{\alpha_1} q}\, \Delta^\alpha \sigma_1\big)\big(Y^\alpha \Delta^{\alpha_1} \sigma_2\big) + \sum_{[\alpha]<M} \tau_{M',\alpha}. \end{aligned} \tag{4.15.41}$$

Next, we study the sum

$$\sum_{[\alpha]<M} \frac{1}{\alpha!} (\Delta_{Y^\alpha q}\, \sigma_1) \circ (\Delta^\alpha \sigma_2).$$

Applying (4.15.7) to the symbols $\Delta_{Y^\alpha q}\, \sigma_1$ and $\Delta^\alpha \sigma_2$, with M', we obtain

$$(\Delta_{Y^\alpha q}\, \sigma_1) \circ (\Delta^\alpha \sigma_2) = \sum_{[\alpha_1]<M'} \frac{1}{\alpha_1!} \big(\Delta^{\alpha_1} \Delta_{Y^\alpha q}\, \sigma_1\big)\big(Y^{\alpha_1} \Delta^\alpha \sigma_2\big) + \rho_{M',\Delta_{Y^\alpha q}\sigma_1,\Delta^\alpha \sigma_2}.$$

Thus, we have shown that

$$\sum_{[\alpha]<M} \frac{1}{\alpha!} (\Delta_{Y^{\alpha}q}\, \sigma_1) \circ (\Delta^{\alpha}\sigma_2)$$

$$= \sum_{\substack{[\alpha]<M \\ [\alpha_1]<M'}} \frac{1}{\alpha_1!} \big(\Delta^{\alpha_1}\Delta_{Y^{\alpha}q}\, \sigma_1\big)\big(Y^{\alpha_1}\Delta^{\alpha}\sigma_2\big) + \sum_{[\alpha]<M} \rho_{M',\Delta_{Y^{\alpha}q}\sigma_1,\Delta^{\alpha}\sigma_2}. \quad (4.15.42)$$

Hence, combining (4.15.41) and (4.15.42) with (4.15.39), we obtain

$$\Delta_q\, \rho_{M,\sigma_1,\sigma_2}(x,\pi) \;=\; \sum_{[\alpha]<M} (\rho_{M',\Delta_{Y^{\alpha}q}\sigma_1,\Delta^{\alpha}\sigma_2} - \tau_{M',\alpha}) + \int_G \kappa_{1,x}(y)\, \pi(y)^*\, \mathcal{F}\big\{\widetilde{R}^q_{y^{-1},M}\, \kappa_{2,xy^{-1}}\big\}(\pi)\, \mathrm{d}y. \quad (4.15.43)$$

Next, we shall find an estimate for $\big|\big|\Delta_q\, \rho_{M,\sigma_1,\sigma_2}(x,\pi)\big|\big|_{L^\infty(\widehat{G})}$ using (4.15.43).

Step 5b

We first study the $L^\infty(\widehat{G})$-norm of the sum

$$\sum_{[\alpha]<M} (\rho_{M',\Delta_{Y^{\alpha}q}\sigma_1,\Delta^{\alpha}\sigma_2} - \tau_{M',\alpha}).$$

By the work done in Step 1, we readily obtain that there exists $M_0 \in \mathbb{N}_0$ such that, for any $M, M' \geq M_0$, whenever $[\alpha] < M$, there exist $\bar{a}_1, \bar{a}_2, \bar{b}_1, \bar{b}_2 \in \mathbb{N}_0$ and $\bar{c}_1, \bar{c}_2 > 0$ such that

$$\begin{aligned} ||\rho_{M',\Delta_{Y^{\alpha}q}\sigma_1,\Delta^{\alpha}\sigma_2}||_{L^\infty(\widehat{G})} &\lesssim ||\Delta_{Y^{\alpha}q}\sigma_1||_{S^{m_1-(a-[\alpha])},\bar{a}_1,\bar{b}_1,\bar{c}_1}\, ||\Delta^{\alpha}\sigma_2||_{S^{m_2-[\alpha]},\bar{a}_2,\bar{b}_2,\bar{c}_2} \\ &\lesssim ||\sigma_1||_{S^{m_1},\bar{a}_1+(a-[\alpha]),\bar{b}_1,\bar{c}_1}\, ||\sigma_2||_{S^{m_2},\bar{a}_2+[\alpha],\bar{b}_2,\bar{c}_2}. \end{aligned}$$

This, together with (4.15.40), shows that there exist $a'_1, a'_2, b'_1, b'_2 \in \mathbb{N}_0$ and $c'_1, c'_2 > 0$ such that

$$\sum_{[\alpha]<M} \big|\big|\rho_{M',\Delta_{Y^{\alpha}q}\sigma_1,\Delta^{\alpha}\sigma_2} - \tau_{M',\alpha}\big|\big|_{L^\infty(\widehat{G})} \lesssim ||\sigma_1||_{S^{m_1},a'_1,b'_1,c'_1}\, ||\sigma_2||_{S^{m_2},a'_2,b'_2,c'_2}. \quad (4.15.44)$$

Step 5c

It remains to check the $L^\infty(\widehat{G})$-norm of the integral

$$\int_G \kappa_{1,x}(y)\, \pi(y)^* \, \mathcal{F}\{\widetilde{R}^q_{y^{-1},M}\, \kappa_{2,xy^{-1}}\}(\pi)\, \mathrm{d}y.$$

Let $M_1 \in \mathbb{N}$ to be chosen. For $\pi \in \widehat{G}$ we have

$$\begin{aligned}
&\int_G \kappa_{1,x}(y)\, \pi(y)^* \, \mathcal{F}\{\widetilde{R}^q_{y^{-1},M}\, \kappa_{2,xy^{-1}}\}(\pi)\, \mathrm{d}y \\
&\qquad = \int_G \big((I+\widetilde{\mathcal{L}}_y)^{M_1}(I+\widetilde{\mathcal{L}}_y)^{-M_1}\kappa_{1,x}(y)\big)\, \pi(y)^* \, \mathcal{F}\{\widetilde{R}^q_{y^{-1},M}\, \kappa_{2,xy^{-1}}\}(\pi)\, \mathrm{d}y \\
&\qquad = \int_G \big((I+\widetilde{\mathcal{L}}_y)^{-M_1}\kappa_{1,x}\big)(y)\, (I+\widetilde{\mathcal{L}}_y)^{M_1}\big[\pi(y)^* \, \mathcal{F}\{\widetilde{R}^q_{y^{-1},M}\, \kappa_{2,xy^{-1}}\}(\pi)\big]\, \mathrm{d}y,
\end{aligned}$$

as $I+\widetilde{\mathcal{L}}$ is a symmetric operator on $L^2(G)$. Since

$$(I+\widetilde{\mathcal{L}})^{M_1} \;=\; \sum_{\substack{\beta_0 \in \mathcal{I}(k)\\ |\beta_0|\le 2M_1}} c_{\beta_0}\, \widetilde{X}_{\beta_0},$$

for some $c_{\beta_0} \in \mathbb{R}$, then

$$\begin{aligned}
&\int_G \big((I+\widetilde{\mathcal{L}}_y)^{-M_1}\kappa_{1,x}\big)(y)\, (I+\widetilde{\mathcal{L}}_y)^{M_1}\big[\pi(y)^* \, \mathcal{F}\{\widetilde{R}^q_{y^{-1},M}\, \kappa_{2,xy^{-1}}\}(\pi)\big]\, \mathrm{d}y \\
&\quad = \sum_{\substack{\beta_0 \in \mathcal{I}(k)\\ |\beta_0|\le 2M_1}} c_{\beta_0} \int_G \big((I+\widetilde{\mathcal{L}}_y)^{-M_1}\kappa_{1,x}\big)(y)\, \widetilde{X}_{\beta_0,y}\big[\pi(y)^* \, \mathcal{F}\{\widetilde{R}^q_{y^{-1},M}\, \kappa_{2,xy^{-1}}\}(\pi)\big]\, \mathrm{d}y \\
&\quad = \sum_{\substack{\beta \in \mathcal{I}(k)^3\\ |\beta|\le 2M_1}} c_{\beta} \int_G \big((I+\widetilde{\mathcal{L}}_y)^{-M_1}\kappa_{1,x}\big)(y)\, (\widetilde{X}_{\beta_1,y_1=y}\, \pi(y_1)^*) \\
&\qquad\qquad\qquad \mathcal{F}\big\{\big(\widetilde{X}_{\beta_2,y_2=y}\, \widetilde{R}^q_{y_2^{-1},M}\big)\, \big(\widetilde{X}_{\beta_3,y_3=y}\, \kappa_{2,xy_3^{-1}}\big)\big\}(\pi)\, \mathrm{d}y,
\end{aligned}$$

for some $c_\beta \in \mathbb{R}$, having used Leibniz's rule for vector fields on the last equality. Now, observe that, for $\beta \in \mathcal{I}(k)^3$ and $y \in G$ we have

$$\widetilde{X}_{\beta_1,y}\, \pi(y)^* \;=\; (-1)^{|\beta_1|}\, \pi(y)^*\pi(X_{\beta_1}).$$

Moreover, by (4.10.17),

$$\widetilde{X}_{\beta_2,y}\, \widetilde{R}^q_{y^{-1},M} \;=\; \widetilde{R}^{\widetilde{X}_{\beta_2} q}_{y^{-1},M}.$$

Additionally,

$$\widetilde{X}_{\beta_3,y}\,\kappa_{2,xy^{-1}} \;=\; (-1)^{|\beta_3|}\,X_{\beta_3,x_1=xy^{-1}}\,\kappa_{2,x_1}.$$

Thus, we have

$$\begin{aligned}
&\int_G \kappa_{1,x}(y)\,\pi(y)^*\,\mathcal{F}\big\{\widetilde{R}^q_{y^{-1},M}\,\kappa_{2,xy^{-1}}\big\}(\pi)\,\mathrm{d}y \\
&\qquad = \sum_{\substack{\beta\in\mathcal{I}(k)^3\\|\beta|\le 2M_1}} (-1)^{|\beta_1|+|\beta_3|}\,c_\beta \int_G \big((I+\widetilde{\mathcal{L}}_y)^{-M_1}\,\kappa_{1,x}\big)(y)\,\pi(y)^* \\
&\qquad\qquad\qquad \pi(X_{\beta_1})\,\mathcal{F}\big\{\widetilde{R}^{\widetilde{X}_{\beta_2}q}_{y^{-1},M}\,\big(X_{\beta_3,x_1=xy^{-1}}\,\kappa_{2,x_1}\big)\big\}(\pi)\,\mathrm{d}y.
\end{aligned}$$

Taking $L^\infty(\widehat{G})$-norm we obtain

$$\begin{aligned}
&\Big|\Big|\int_G \kappa_{1,x}(y)\,\pi(y)^*\,\mathcal{F}\big\{\widetilde{R}^q_{y^{-1},M}\,\kappa_{2,xy^{-1}}\big\}(\pi)\,\mathrm{d}y\Big|\Big|_{L^\infty(\widehat{G})} \\
&\quad\lesssim \sum_{\substack{\beta\in\mathcal{I}(k)^3\\|\beta|\le 2M_1}} \sup_{y\in G}\,\big|\big|\pi(X_{\beta_1})\,\mathcal{F}\big\{\widetilde{R}^{\widetilde{X}_{\beta_2}q}_{y^{-1},M}\,\big(X_{\beta_3,x_1=xy^{-1}}\,\kappa_{2,x_1}\big)\big\}(\pi)\big|\big|_{L^\infty(\widehat{G})} \\
&\qquad\qquad \times \big|\big|(I+\widetilde{\mathcal{L}})^{-M_1}\,\kappa_{1,x}\big|\big|_{L^1(G)}.
\end{aligned} \tag{4.15.45}$$

We first study the $L^1(G)$-norm on the right hand side of (4.15.45). Note that, by Plancherel's Theorem (see Theorem 2.2.7), for every $x\in G$ we have

$$\begin{aligned}
\big|\big|(I+\widetilde{\mathcal{L}})^{-M_1}\,\kappa_{1,x}\big|\big|_{L^1(G)} &\lesssim \big|\big|(I+\widetilde{\mathcal{L}})^{-M_1}\,\kappa_{1,x}\big|\big|_{L^2(G)} \\
&= \big|\big|\sigma_1(x,\pi)\,\pi(I+\mathcal{L})^{-M_1}\big|\big|_{L^2(\widehat{G})} \\
&\le \big|\big|\pi(I+\mathcal{L})^{-\frac{1}{2}(m_1-m_1)}\,\sigma_1(x,\pi)\,\pi(I+\mathcal{L})^{\frac{-m_1}{2}}\big|\big|_{L^\infty(\widehat{G})} \\
&\qquad\qquad \big|\big|\pi(I+\mathcal{L})^{-\frac{1}{2}(2M_1-m_1)}\big|\big|_{L^2(\widehat{G})}.
\end{aligned}$$

By Proposition 3.1.9,

$$\big|\big|\pi(I+\mathcal{L})^{-\frac{1}{2}(2M_1-m_1)}\big|\big|_{L^2(\widehat{G})} \;<\; +\infty,$$

provided that $2M_1 > m_1 + \frac{l}{2}$. Moreover, there exist $\widetilde{a}_1, \widetilde{b}_1 \in \mathbb{N}_0$ and $\widetilde{c}_1 > 0$ such that

$$\big|\big|\pi(I+\mathcal{L})^{-\frac{1}{2}(m_1-m_1)}\,\sigma_1(x,\pi)\,\pi(I+\mathcal{L})^{\frac{-m_1}{2}}\big|\big|_{L^\infty(\widehat{G})}\,\leq\,||\sigma_1||_{S^{m_1},\widetilde{a}_1,\widetilde{b}_1,\widetilde{c}_1}.$$

Hence, for $2M_1>m_1+\frac{l}{2}$,

$$\big|\big|(I+\widetilde{\mathcal{L}}_y)^{-M_1}\kappa_{1,x}\big|\big|_{L^1(G)}\,\lesssim\,||\sigma_1||_{S^{m_1},\widetilde{a}_1,\widetilde{b}_1,\widetilde{c}_1}. \tag{4.15.46}$$

Next we study the $L^\infty(\widehat{G})$-norm on the right hand side of (4.15.45). Now, observe that, by Lemma 2.2.4, for every $x,\,y\,\in\,G$ and $\beta\,\in\,\mathcal{I}(k)^3$, with $|\beta|\leq 2M_1$, we have

$$\begin{aligned}
&\big|\big|\pi(X_{\beta_1})\,\mathcal{F}\big\{\widetilde{R}^{\widetilde{X}_{\beta_2}q}_{y^{-1},M}\,\big(X_{\beta_3,x_1=xy^{-1}}\,\kappa_{2,x_1}\big)\big\}(\pi)\big|\big|_{L^\infty(\widehat{G})}\\
&\qquad\leq\,\big|\big|X_{\beta_1}\big\{\widetilde{R}^{\widetilde{X}_{\beta_2}q}_{y^{-1},M}\,\big(X_{\beta_3,y_3=y}\,\kappa_{2,xy_3^{-1}}\big)\big\}\big|\big|_{L^1(G)}\\
&\qquad\leq\sum_{\substack{\beta_{1,1},\beta_{1,2}\,\in\mathcal{I}(k)\\|\beta_{1,1}|+|\beta_{1,2}|=|\beta_1|}}\int_G\big|X_{\beta_{1,1},z_1=z}\,\widetilde{R}^{\widetilde{X}_{\beta_2}q}_{y^{-1},M}(z_1)\big|\\
&\qquad\qquad\qquad\qquad\big|X_{\beta_{1,2},z_2=z}\,X_{\beta_3,x_1=xy^{-1}}\,\kappa_{2,x_1}(z_2)\big|\,\mathrm{d}z,
\end{aligned}\tag{4.15.47}$$

by Leibniz's rule for vector fields. Furthermore, for any $\beta_{1,1},\,\beta_{1,2}\,\in\,\mathcal{I}(k)$, with $|\beta_{1,1}|+|\beta_{1,2}|=|\beta_1|$, we have

$$\big|X_{\beta_{1,1},z}\,\widetilde{R}^{\widetilde{X}_{\beta_2}q}_{y^{-1},M}(z)\big|\,\lesssim\,|z|^{M-|\beta_{1,1}|},\qquad\forall\,z\,\in\,G, \tag{4.15.48}$$

Hence, by (4.15.47) we obtain

$$\begin{aligned}
&\big|\big|\pi(X_{\beta_1})\,\mathcal{F}\big\{\widetilde{R}^{\widetilde{X}_{\beta_2}q}_{y^{-1},M}\,\big(X_{\beta_3,x_1=xy^{-1}}\,\kappa_{2,x_1}\big)\big\}(\pi)\big|\big|_{L^\infty(\widehat{G})}\\
&\quad\lesssim\sum_{\substack{\beta_{1,1},\beta_{1,2}\,\in\mathcal{I}(k)\\|\beta_{1,1}|+|\beta_{1,2}|=|\beta_1|}}\int_G|z|^{M-|\beta_{1,1}|}\,\big|X_{\beta_{1,2},z_2=z}\,X_{\beta_3,x_1=xy^{-1}}\,\kappa_{2,x_1}(z_2)\big|\,\mathrm{d}z.
\end{aligned}\tag{4.15.49}$$

Furthermore, for any $\beta_{1,1},\,\beta_{1,2}\,\in\,\mathcal{I}(k)$, with $|\beta_{1,1}|+|\beta_{1,2}|=|\beta_1|$, observe that the distribution

$$(x,z)\,\longmapsto\,X_{\beta_{1,2},z}\,X_{\beta_3,x}\,\kappa_{2,x}(z)$$

is the right-convolution kernel associated to the symbol

$$X_{\beta_3}\,\pi(X_{\beta_{1,2}})\,\sigma_2 \;=\; \big\{X_{\beta_3,x}\,\pi(X_{\beta_{1,2}})\,\sigma_2(x,\pi)\;:\;x\,\in\,G,\;\pi\,\in\,\widehat{G}\big\},$$

and additionally, by Proposition 4.5.10 and Theorem 4.10.1, $X_{\beta_3}\,\pi(X_{\beta_{1,2}})\,\sigma_2 \in S^{m_2+|\beta_{1,2}|}$. Therefore, if $M-|\beta_{1,1}|+l > \max(m_2+|\beta_{1,2}|+l,0)$, then by Corollary 4.13.5, there exist $a',\, b' \in \mathbb{N}_0$ and $c'>0$

$$\begin{aligned}\int_G |z|^{M-|\beta_{1,1}|}\,\big|X_{\beta_{1,2},z_2=z}\,X_{\beta_3,x_1=xy^{-1}}\,\kappa_{2,x_1}(z_2)\big|\,\mathrm{d}z&\\ \lesssim\;\big|\big|X_{\beta_3}\,\pi(X_{\beta_{1,2}})\,\sigma_2\big|\big|_{S^{m_2+|\beta_{1,2}|},a',b',c'}&\\ \lesssim\;\big|\big|\pi(X_{\beta_{1,2}})\big|\big|_{S^{|\beta_{1,2}|},a_1',b_1',c_1'}\,||\sigma_2||_{S^{m_2},a_2',b_2',c_2'},&\qquad(4.15.50)\end{aligned}$$

for some $a_1',\, a_2',\, b_1',\, b_2' \in \mathbb{N}_0$ and $c_1',\, c_2'>0$, by Lemma 4.10.2. It then suffices to choose $M\in\mathbb{N}$ such that

$$M \;>\; 2M_1+l+\max(m_2+2M_1+l,0). \qquad (4.15.51)$$

Combining (4.15.50) and (4.15.51) with (4.15.49) we then deduce that there exists $M_0\in\mathbb{N}_0$ such that, for any $M\geq M_0$, there exist $\widetilde{a}_2,\, \widetilde{b}_2 \in \mathbb{N}_0$ and $\widetilde{c}_2>0$ such that

$$\big|\big|\pi(X_{\beta_1})\,\mathcal{F}\big\{\widetilde{R}^{\widetilde{X}_{\beta_2}q}_{y^{-1},M}\,\big(X_{\beta_3,y_3=y}\,\kappa_{2,xy_3^{-1}}\big)\big\}(\pi)\big|\big|_{L^\infty(\widehat{G})} \lesssim ||\sigma_2||_{S^{m_2},\widetilde{a}_2,\widetilde{b}_2,\widetilde{c}_2}. \qquad (4.15.52)$$

Thus, by (4.15.45), (4.15.46) and (4.15.52), we conclude that there exists $M_0\in\mathbb{N}_0$ such that, for all $M\geq M_0$, there exist $\widetilde{a}_1,\, \widetilde{a}_2,\, \widetilde{b}_1,\, \widetilde{b}_2 \in \mathbb{N}_0$ and $\widetilde{c}_1,\, \widetilde{c}_2>0$ such that

$$\begin{aligned}\Big|\Big|\int_G \kappa_{1,x}(y)\,\pi(y)^*\,\mathcal{F}\big\{\widetilde{R}^q_{y^{-1},M}\,\kappa_{2,xy^{-1}}\big\}(\pi)\,\mathrm{d}y\Big|\Big|_{L^\infty(\widehat{G})}&\\ \lesssim\;||\sigma_1||_{S^{m_1},\widetilde{a}_1,\widetilde{b}_1,\widetilde{c}_1}\,||\sigma_2||_{S^{m_2},\widetilde{a}_2,\widetilde{b}_2,\widetilde{c}_2}.&\qquad(4.15.53)\end{aligned}$$

Step 5d: Conclusion of Step 5

By (4.15.43), combining (4.15.44) and (4.15.53), we conclude that there exist $a_1,\, a_2,\, b_1,\, b_2 \in \mathbb{N}_0$ and $c_1,\, c_2>0$ such that

$$\sup_{\substack{x\in G\\ \pi\in\widehat{G}}}\big|\big|\Delta_q\,\rho_{M,\sigma_1,\sigma_2}(x,\pi)\big|\big|_{\mathscr{L}(\mathscr{H}_\pi)} \lesssim ||\sigma_1||_{S^{m_1},a_1,b_1,c_1}\,||\sigma_2||_{S^{m_2},a_2,b_2,c_2},$$

as required.

4.15.5 End of the proof of Theorem 4.15.1

Performing the analysis done in Steps 1-5 simultaneously, we obtain that for all β_0, β_1, $\beta_2 \in \mathcal{I}(k)$ and $\alpha \in \mathbb{N}_0$, there exists $M_0 \in \mathbb{N}_0$ such that for all $M \geq M_0$ there exist $C > 0$, and a_1', a_2', b_1', $b_2' \in \mathbb{N}_0$ and c_1', $c_2' > 0$ such that

$$\begin{aligned} \big|\big|\pi(X_{\beta_1})\,\Delta^\alpha \widetilde{X}_{\beta_0}\,\rho_{M,\sigma_1,\sigma_2}(x,\pi)\,\pi(X_{\beta_2})\big|\big|_{L^\infty(\widehat{G})} \\ \leq C\,||\sigma_1||_{S^{m_1},a_1',b_1',c_1'}\,||\sigma_2||_{S^{m_2},a_2',b_2',c_2'}. \end{aligned} \tag{4.15.54}$$

Now, if $\alpha \in \mathbb{N}_0^n$, $\beta \in \mathcal{I}(k)$ and $\nu \in \mathbb{R}$, then for any $M \geq M_0$ we have

$$\begin{aligned} &\big|\big|\pi(I+\mathcal{L})^{-\frac{1}{2}(m-[\alpha]+\nu)}\,\Delta^\alpha \widetilde{X}_{\beta_0}\,\sigma(x,\pi)\,\pi(I+\mathcal{L})^{\frac{\nu}{2}}\big|\big|_{L^\infty(\widehat{G})} \\ &\lesssim \sum_{[\alpha_1]<M} \big|\big|\pi(I+\mathcal{L})^{-\frac{1}{2}(m-[\alpha]+\nu)}\,\Delta^\alpha \widetilde{X}_{\beta_0}(\Delta^{\alpha_1}\sigma_1)(Y^{\alpha_1}\sigma_2)(x,\pi)\,\pi(I+\mathcal{L})^{\frac{\nu}{2}}\big|\big|_{L^\infty(\widehat{G})} \\ &\quad + \big|\big|\pi(I+\mathcal{L})^{-\frac{1}{2}(m-[\alpha]+\nu)}\,\Delta^\alpha \widetilde{X}_{\beta_0}\,\rho_{M,\sigma_1,\sigma_2}(x,\pi)\,\pi(I+\mathcal{L})^{\frac{\nu}{2}}\big|\big|_{L^\infty(\widehat{G})}, \end{aligned} \tag{4.15.55}$$

by the definition of $\rho_{M,\sigma_1,\sigma_2}$ (see (4.15.10)).

First, we analyse the sum. Note that, by Theorem 4.10.1, for every $\alpha_1 \in \mathbb{N}_0^n$, with $[\alpha_1] < M$, there exist $\widetilde{a}_1$, $\widetilde{a}_2$, $\widetilde{b}_1$, $\widetilde{b}_2 \in \mathbb{N}_0^n$ and $\widetilde{c}_1$, $\widetilde{c}_2 > 0$ such that

$$\begin{aligned} &\big|\big|\pi(I+\mathcal{L})^{-\frac{1}{2}(m-[\alpha]+\nu)}\,\Delta^\alpha \widetilde{X}_{\beta_0}(\Delta^{\alpha_1}\sigma_1)(Y^{\alpha_1}\sigma_2)(x,\pi)\,\pi(I+\mathcal{L})^{\frac{\nu}{2}}\big|\big|_{L^\infty(\widehat{G})} \\ &\qquad \lesssim ||\Delta^{\alpha_1}\sigma_1||_{S^{m_1-[\alpha_1]},\widetilde{a}_1,\widetilde{b}_1,\widetilde{c}_1}\,||Y^{\alpha_1}\sigma_2||_{S^{m_2},\widetilde{a}_2,\widetilde{b}_2,\widetilde{c}_2} \\ &\qquad \lesssim ||\sigma_1||_{S^{m_1},\widetilde{a}_1+[\alpha_1],\widetilde{b}_1,\widetilde{c}_1}\,||\sigma_2||_{S^{m_2},\widetilde{a}_2,\widetilde{b}_2+[\alpha_1],\widetilde{c}_2}. \end{aligned} \tag{4.15.56}$$

Moreover, we let

$$\gamma_1 := \max\left(-\frac{1}{2}(m-[\alpha]+\nu),0\right), \qquad \gamma_2 := \max\left(\frac{\nu}{2},0\right).$$

By (4.15.54), there exist $\bar{a}_1$, $\bar{a}_2$, $\bar{b}_1$, $\bar{b}_2 \in \mathbb{N}_0$ and $\bar{c}_1$, $\bar{c}_2 > 0$ such that

$$\left\|\pi(I+\mathcal{L})^{-\frac{1}{2}(m-[\alpha]+\nu)}\,\Delta^{\alpha}\widetilde{X}_{\beta_0}\,\rho_{M,\sigma_1,\sigma_2}(x,\pi)\,\pi(I+\mathcal{L})^{\frac{\nu}{2}}\right\|_{L^{\infty}(\widehat{G})}$$
$$\leq \sum_{\substack{|\beta_1|\leq\gamma_1\\|\beta_2|\leq\gamma_2}} \left\|\pi(X_{\beta_1})\,\Delta^{\alpha}\widetilde{X}_{\beta_0}\,\rho_{M,\sigma_1,\sigma_2}(x,\pi)\,\pi(X_{\beta_2})\right\|_{L^{\infty}(\widehat{G})}$$
$$\lesssim \|\sigma_1\|_{S^{m_1},\bar{a}_1,\bar{b}_1,\bar{c}_1}\,\|\sigma_2\|_{S^{m_2},\bar{a}_2,\bar{b}_2,\bar{c}_2}. \tag{4.15.57}$$

Combining (4.15.56) and (4.15.57) with (4.15.55), we conclude that there exist $a_1,\, a_2,\, b_1,\, b_2 \in \mathbb{N}_0$ and $c_1,\, c_2 > 0$ such that

$$\left\|\pi(I+\mathcal{L})^{-\frac{1}{2}(m-[\alpha]+\nu)}\,\Delta^{\alpha}\widetilde{X}_{\beta_0,x}\,\sigma(x,\pi)\,\pi(I+\mathcal{L})^{\frac{\nu}{2}}\right\|_{L^{\infty}(\widehat{G})}$$
$$\lesssim \|\sigma_1\|_{S^{m_1},a_1,b_1,c_1}\,\|\sigma_2\|_{S^{m_2},a_2,b_2,c_2}.$$

Observe that, for any $\beta \in \mathbb{N}_0^n$, the differential operator Y^{β} can be written as

$$Y^{\beta} = \sum_{\substack{\beta_0\in\mathcal{I}(k)\\|\beta_0|\leq[\beta]}} c_{\beta_0}\,\widetilde{X}_{\beta_0},$$

for some $c_{\beta_0} \in \mathcal{C}^{\infty}(G)$. Hence, we have shown that, for any $\alpha,\, \beta \in \mathbb{N}_0^n$, there exist $a_1,\, a_2,\, b_1,\, b_2 \in \mathbb{N}_0$ and $c_1,\, c_2 > 0$ such that

$$\left\|\pi(I+\mathcal{L})^{-\frac{1}{2}(m-[\alpha]+\nu)}\,\Delta^{\alpha}Y_x^{\beta}\sigma(x,\pi)\,\pi(I+\mathcal{L})^{\frac{\nu}{2}}\right\|_{L^{\infty}(\widehat{G})}$$
$$\lesssim \|\sigma_1\|_{S^{m_1},a_1,b_1,c_1}\,\|\sigma_2\|_{S^{m_2},a_2,b_2,c_2}.$$

This finishes the proof of Theorem 4.15.1.

□

4.15.6 Asymptotics for composition

Theorem 4.15.1 can be improved via the following result.

Corollary 4.15.4. *Let $m_1,\, m_2 \in \mathbb{R}$, and set $m := m_1 + m_2$. Furthermore, let $\sigma_1 \in S^{m_1}$ and $\sigma_2 \in S^{m_2}$, denoting $\sigma := \sigma_1 \circ \sigma_2$, and recall that, for $M \in \mathbb{N}$,*

$$\rho_{M,\sigma_1,\sigma_2} := \sigma - \sum_{[\alpha]<M} \frac{1}{\alpha!}\,(\Delta^{\alpha}\sigma_1)\,(Y^{\alpha}\sigma_2)\,.$$

Then, there exists $M_0 \in \mathbb{N}_0$ such that, for all $M \geq M_0$, we have

$$\sigma \sim \sum_{[\alpha]<M} \frac{1}{\alpha!} \left(\Delta^{\alpha}\sigma_1\right)\left(Y^{\alpha}\sigma_2\right),$$

in the sense that, for all $a,\, b \in \mathbb{N}_0$ *and* $c > 0$*, there exist* $a_1,\, a_2,\, b_1,\, b_2 \in \mathbb{N}_0$ *and* $c_1,\, c_2 > 0$ *such that*

$$||\rho_{M,\sigma_1,\sigma_2}||_{S^{m-M},a,b,c} \lesssim ||\sigma_1||_{S^{m_1},a_1,b_1,c_1}\, ||\sigma_2||_{S^{m_2},a_2,b_2,c_2}.$$

Proof. Let $\alpha,\, \beta \in \mathbb{N}_0^n$ and $\nu \in \mathbb{R}$. Furthermore, let $M' > M$ to be determined. First observe that

$$\begin{aligned}
\rho_{M,\sigma_1,\sigma_2} &= \sigma - \sum_{[\alpha_1]<M} \frac{1}{\alpha_1!} \left(\Delta^{\alpha_1}\sigma_1\right)\left(Y^{\alpha_1}\sigma_2\right) \\
&= \sigma - \sum_{[\alpha_1]<M'} \frac{1}{\alpha_1!} \left(\Delta^{\alpha_1}\sigma_1\right)\left(Y^{\alpha_1}\sigma_2\right) + \sum_{[\alpha_1]<M'} \frac{1}{\alpha_1!} \left(\Delta^{\alpha_1}\sigma_1\right)\left(Y^{\alpha_1}\sigma_2\right) \\
&\qquad - \sum_{[\alpha_1]<M} \frac{1}{\alpha_1!} \left(\Delta^{\alpha_1}\sigma_1\right)\left(Y^{\alpha_1}\sigma_2\right) \\
&= \rho_{M',\sigma_1,\sigma_2} + \sum_{M\le[\alpha_1]<M'} \frac{1}{\alpha_1!} \left(\Delta^{\alpha_1}\sigma_1\right)\left(Y^{\alpha_1}\sigma_2\right).
\end{aligned}$$

Hence, for every $x \in G$,

$$\begin{aligned}
&\left|\left|\pi(I+\mathcal{L})^{-\frac{1}{2}(m-M-[\alpha]+\nu)}\,\Delta^{\alpha}Y_x^{\beta}\rho_{M,\sigma_1,\sigma_2}(x,\pi)\,\pi(I+\mathcal{L})^{\frac{\nu}{2}}\right|\right|_{L^\infty(\widehat{G})} \\
&\le \left|\left|\pi(I+\mathcal{L})^{-\frac{1}{2}(m-M-[\alpha]+\nu)}\,\Delta^{\alpha}Y_x^{\beta}\,\rho_{M',\sigma_1,\sigma_2}(x,\pi)\,\pi(I+\mathcal{L})^{\frac{\nu}{2}}\right|\right|_{L^\infty(\widehat{G})} \\
&+ \sum_{M\le[\alpha_1]<M'} \left|\left|\pi(I+\mathcal{L})^{-\frac{m-M-[\alpha]+\nu}{2}}\Delta^{\alpha}Y_x^{\beta}(\Delta^{\alpha_1}\sigma_1)(Y^{\alpha_1}\sigma_2)(x,\pi)\pi(I+\mathcal{L})^{\frac{\nu}{2}}\right|\right|_{L^\infty(\widehat{G})}.
\end{aligned}$$

We now do the same analysis as in Section 4.15.5. First note that, for any $\alpha_1 \in \mathbb{N}_0^n$, with $M \le [\alpha_1] < M$, we have

$$\begin{aligned}
&\left|\left|\pi(I+\mathcal{L})^{-\frac{m-M-[\alpha]+\nu}{2}}\Delta^{\alpha}Y_x^{\beta}(\Delta^{\alpha_1}\sigma_1)(Y^{\alpha_1}\sigma_2)(x,\pi)\pi(I+\mathcal{L})^{\frac{\nu}{2}}\right|\right|_{L^\infty(\widehat{G})} \\
&\qquad \le \left|\left|\pi(I+\mathcal{L})^{-\frac{1}{2}([\alpha_1]-M)}\right|\right|_{L^\infty(\widehat{G})} \\
&\qquad\qquad \left|\left|\pi(I+\mathcal{L})^{-\frac{m-[\alpha_1]+\nu}{2}}\Delta^{\alpha}Y_x^{\beta}(\Delta^{\alpha_1}\sigma_1)(Y^{\alpha_1}\sigma_2)(x,\pi)\pi(I+\mathcal{L})^{\frac{\nu}{2}}\right|\right|_{L^\infty(\widehat{G})},
\end{aligned}$$

and by functional analysis,

$$\big|\big|\pi(I+\mathcal{L})^{-\frac{1}{2}([\alpha_1]-M)}\big|\big|_{L^\infty(\widehat{G})} < +\infty.$$

Hence observe that, by Theorem 4.10.1 (see also (4.15.56)), there exist $\widetilde{a}_1,\, \widetilde{a}_2,\, \widetilde{b}_1,\, \widetilde{b}_2 \in \mathbb{N}_0$ and $\widetilde{c}_1,\, \widetilde{c}_2 > 0$ such that

$$\sum_{M\le[\alpha_1]<M'} \big|\big|\pi(I+\mathcal{L})^{-\frac{1}{2}(m-M-[\alpha]+\nu)}\Delta^\alpha Y_x^\beta(\Delta^{\alpha_1}\sigma_1)(Y_x^{\alpha_1}\sigma_2)(x,\pi)\pi(I+\mathcal{L})^{\frac{\nu}{2}}\big|\big|_{L^\infty(\widehat{G})}$$
$$\lesssim ||\sigma_1||_{S^{m_1},\widetilde{a}_1,\widetilde{b}_1,\widetilde{c}_1}\,||\sigma_2||_{S^{m_2},\widetilde{a}_2,\widetilde{b}_2,\widetilde{c}_2}.$$

Furthermore, by (4.15.54) and the work done in (4.15.57), we also deduce that there exist $\overline{a}_1,\, \overline{a}_2,\, \overline{b}_1,\, \overline{b}_2 \in \mathbb{N}_0$ and $\overline{c}_1,\, \overline{c}_2 > 0$ such that

$$\big|\big|\pi(I+\mathcal{L})^{-\frac{1}{2}(m-M-[\alpha]+\nu)}\,\Delta^\alpha Y_x^\beta\,\rho_{M',\sigma_1,\sigma_2}(x,\pi)\,\pi(I+\mathcal{L})^{\frac{\nu}{2}}\big|\big|_{L^\infty(\widehat{G})}$$
$$\lesssim ||\sigma_1||_{S^{m_1},\overline{a}_1,\overline{b}_1,\overline{c}_1}\,||\sigma_2||_{S^{m_2},\overline{a}_2,\overline{b}_2,\overline{c}_2}.$$

Hence, we have shown that there exist $a_1,\, a_2,\, b_1,\, b_2 \in \mathbb{N}_0$ and $c_1,\, c_2 > 0$ such that

$$\big|\big|\pi(I+\mathcal{L})^{-\frac{1}{2}(m-M-[\alpha]+\nu)}\,\Delta^\alpha Y_x^\beta\rho_{M,\sigma_1,\sigma_2}(x,\pi)\,\pi(I+\mathcal{L})^{\frac{\nu}{2}}\big|\big|_{L^\infty(\widehat{G})}$$
$$||\sigma_1||_{S^{m_1},a_1,b_1,c_1}\,||\sigma_2||_{S^{m_2},a_2,b_2,c_2}.$$

□

The following result is a consequence of Theorems 4.14.1 and 4.15.1, and the fact that $(I+\mathcal{L})^{\frac{s}{2}}$ is in the calculus, for any $s \in \mathbb{R}$.

Corollary 4.15.5. *Let $m \in \mathbb{R}$. If $\sigma \in S^m$, then $\mathrm{Op}(\sigma)$ extends to a bounded operator from $L^2_s(G)$ to $L^2_{s-m}(G)$, for all $s \in \mathbb{R}$. Moreover, there exist $C > 0$, $a,\, b \in \mathbb{N}_0$ and $c > 0$, independent of σ, such that*

$$\big|\big|\mathrm{Op}(\sigma)\big|\big|_{\mathscr{L}(L^2_s(G),L^2_{s-m}(G))} \le C\,||\sigma||_{S^m,a,b,c}.$$

Chapter 5

Conclusion and future work

5.1 Conclusion

In this section we summarise the main results of this thesis. The objective of this exposition was to define a class of operators Ψ which forms a symbolic pseudo-differential calculus on a compact Lie group G, in a sub-elliptic setting.

The chosen sub-elliptic operator was the sub-Laplacian $\mathcal{L}$ associated to a Hörmander system of left-invariant vector fields on G. The Sobolev spaces $L^2_s(G)$ that arise naturally from $\mathcal{L}$ have relatively well known properties, and we checked some of them, such as the Interpolation Theorem (see Theorem 3.3.1) or a Sobolev embedding (see Theorem 3.4.1). In this chapter we also introduced a notion of order for a smooth function q. We have that q is CC-vanishing at e_G up to order $a-1$, for $a \in \mathbb{N}$, if

$$|q(z)| \lesssim |z|^a, \qquad \forall\, z \in G,$$

where $|\cdot|$ denotes the Carnot-Carathéodory norm on G.

The core of the new results of this thesis appear in Chapter 4. We first introduced the notion of comparability to the C-C metric, and the concept of difference operators. An example of a family of functions comparable to C-C metric is Q_0, which we define in the following way. For a small neighbourhood V of e_G in G and $r \in (0,1]$, satisfying (4.2.8), we let $\chi, \psi \in \mathcal{D}(G)$, taking values in $[0,1]$, be such that

$$\chi(z) \equiv 1 \quad \text{on} \quad B_r(e_G), \qquad\qquad \chi(z) \equiv 0 \quad \text{on} \quad V^c,$$

and

$$\psi(z) \equiv 0 \quad \text{on} \quad B_{r/2}(e_G), \qquad\qquad \psi(z) \equiv 1 \quad \text{on} \quad B_r(e_G)^c.$$

We then define

$$q_{0,j}(z) = z_j\,\chi(z) + \psi(z) \quad \text{ for } \ j = 1, 2, \ldots, n, \tag{5.1.1}$$

and let

$$Q_0 \; := \; \{q_{0,1}, q_{0,2}, \ldots, q_{0,\ell}\}.$$

Our next objective was to develop a pseudo-differential calculus, which is meant in the following way: If for each $m \in \mathbb{R}$, Ψ^m is a class of operators, then the space

$$\Psi \; := \; \bigcup_{m \,\in\, \mathbb{R}} \Psi^m$$

is said to form a calculus if it satisfies the following properties:

(I) If $T_1 \in \Psi^{m_1}$ and $T_2 \in \Psi^{m_2}$, for $m_1,\, m_2 \in \mathbb{R}$, then

$$T_1 \circ T_2 \; \in \; \Psi^{m_1+m_2}.$$

Moreover, the composition is a continuous map $\Psi^{m_1} \times \Psi^{m_2} \to \Psi^{m_1+m_2}$.

(II) If $T \in \Psi^m$, for $m \in \mathbb{R}$, then its adjoint

$$T^* \; \in \; \Psi^m.$$

Moreover, the adjoint is a continuous map $\Psi^m \to \Psi^m$.

We first defined our symbol classes S^m on G, for $m \in \mathbb{R}$, with respect to our sub-Laplacian $\mathcal{L}$, any basis of vector fields $\boldsymbol{V}$, and any Q comparable to the C-C metric, as well as their associated operator classes Ψ^m. In the case that $S^m = S^m(Q_0)$ is defined in terms of Q_0 and the basis of vector fields $\boldsymbol{Y}$ (see Section 2.4.1), we have some important properties. First of all, for any Q comparable to the C-C metric we have

$$S^m(Q_0) \; \subset S^m(Q).$$

Secondly, on the classes of symbols $S^m(Q_0)$, difference operators satisfy a property analogous to Leibniz's rule. More precisely, if q is CC-vanishing at e_G up to order $a-1$, for $a \in \mathbb{N}$, and $\sigma_1 \in S^{m_1}$, $\sigma_2 \in S^{m_2}$, for $m_1, m_2 \in \mathbb{R}$, then for any $a', b' \in \mathbb{N}_0$ and $c' > 0$ we have

$$||\Delta_q(\sigma_1\sigma_2)||_{S^{m_1+m_2-a},a',b',c'} \lesssim ||\sigma_1||_{S^{m_1},a_1,b_1,c_1} \, ||\sigma_2||_{S^{m_2},a_2,b_2,c_2},$$

for some $a_1, a_2, b_1, b_2 \in \mathbb{N}_0$ and $c_1, c_2 > 0$.

This result on the product of symbols provided us with the tools necessary to prove that if $T_1 \in \Psi^{m_1}(Q_0)$ and $T_2 \in \Psi^{m_2}(Q_0)$, for $m_1, m_2 \in \mathbb{R}$, then

$$T_1 \circ T_2 \in \Psi^{m_1+m_2}(Q_0),$$

and that the composition is a continuous map $\Psi^{m_1} \times \Psi^{m_2} \to \Psi^{m_1+m_2}$. Hence, we have proved that the space

$$\Psi(Q_0) := \bigcup_{m \in \mathbb{R}} \Psi^m(Q_0)$$

satisfies property (I) above.

5.2 Future work

The author of this thesis believes one can prove stability of $\Psi(Q_0)$ under taking the adjoint. That is, if $T \in \Psi^m(Q_0)$, for $m \in \mathbb{R}$, then its adjoint $T^* \in \Psi^m(Q_0)$. The main ideas for the proof of this result appear in Section 4.15, where we proved the stability of $\Psi(Q_0)$ under composition. This will imply that the space

$$\bigcup_{m \in \mathbb{R}} \Psi^m(Q_0)$$

has all the natural properties of a symbolic pseudo-differential calculus; that is, it is an algebra of operators with a notion of order compatible with the action on functional spaces.

The next questions will require more work and new ideas. For example, understanding the appropriate conditions for a function to be comparable to the C-C metric, so that

$$S^m(P) = S^m(Q),$$

whenever P, Q are comparable to the C-C metric. Another example involves describing a sub-elliptic symbolic calculus for the symbol classes $S^m_{\rho,\delta}$. A priori, for $m \in \mathbb{R}$ and $0 \le \delta \le \rho \le 1$, the class $S^m_{\rho,\delta}$ is defined to be the space consisting of symbols σ such that, for any $\alpha, \beta \in \mathbb{N}_0^n$ and any $\nu \in \mathbb{R}$, we have

$$\sup_{\substack{x \in G \\ \pi \in \widehat{G}}} \left|\left|\pi\,(I+\mathcal{L})^{-\frac{1}{2}(m-\rho[\alpha]+\delta[\beta]+\nu)}\, Y_x^{\beta}\Delta^{\alpha}\sigma(x,\pi)\pi\,(I+\mathcal{L})^{\frac{1}{2}\nu}\right|\right|_{\mathscr{L}(\mathscr{H}_\pi)} \le C.$$

Observe that the case $\rho = 1$ and $\delta = 0$ is the one presented in this thesis. Moreover, the case $0 < \delta < \rho < 1$ will certainly be a direct generalisation of the methods presented here. But there are other cases, such as $\rho = \delta = 0$ (in particular, the Calderón-Vaillancourt Theorem), which will require new ideas.

More generally, we may consider studying a sub-elliptic pseudo-differential calculus for a Lie group G of polynomial growth. One of the main difficulties in this task is in proving the Calderón-Zygmund-type estimates for the convolution kernel associated to a symbol belonging to the class S^m (see Section 4.13). In the case that G is compact the convolution kernel has compact support, and hence the behaviour away from the identity is clear, but in the non-compact setting this has to be considered. The author of this thesis expects that finding a family of difference operators such that the estimate away from the identity is satisfied to not be difficult. However, obtaining the estimate near the identity would require more work.

As the pseudo-differential calculus of $\mathbb{H}$ was developed in Fisher and Ruzhansky [18] (see also Bahouri et al. [3]), it is natural to compare it with the contraction of our pseudo-differential calculus on $G = SU(2)$. The setting of these investigations can be extended to any contraction of a compact Lie group to its nilpotent counterpart intervening in the Iwasawa decomposition of a non-compact semisimple Lie group (see Dooley and Ricci [12]).

In conclusion, having obtained a sub-elliptic pseudo-differential calculus in the compact setting, the author believes this exposition presents the groundwork for more general results.

Appendix A

The Carnot-Carathéodory metric

A.1 Connecting paths

Let G be a connected Lie group of dimension n and suppose that

$$\boldsymbol{X} = \{X_1, X_2, \ldots, X_k\}$$

forms a Hörmander system of vector fields in G (see Definition 2.3.12).

Suppose $\gamma : J \to G$ is a continuous map, where $J \subset \mathbb{R}$ is an interval. The velocity of γ at $t_0 \in J$ is defined to be the vector

$$\gamma'(t_0) = \gamma_* \left(\frac{d}{dt}\bigg|_{t=t_0} \right), \tag{A.1.1}$$

which is the push-forward of $\frac{d}{dt}\big|_{t=t_0}$ by γ (see Section 2.3.1). Here $\frac{d}{dt}\big|_{t=t_0}$ denotes the usual derivative on $\mathbb{R}$, or equivalently, the canonical tangent vector to $\mathbb{R}$ at t_0. Moreover, note that $\gamma'(t_0)$ is a tangent vector to G at the point $\gamma(t_0)$. The action of $\gamma'(t_0)$ on a smooth function f on G is given by

$$\gamma'(t_0)f = (f \circ \gamma)'(t_0).$$

For $a, b \in \mathbb{R}$, with $a \leq b$, we define $\mathcal{C}_{\boldsymbol{X}}([a,b])$ to be the set consisting of absolutely continuous paths $\gamma : [a,b] \to G$ such that

$$\gamma'(t) = \sum_{i=1}^{k} c_i(t)\, X_i(\gamma(t)) \quad \text{a.e.}, \tag{A.1.2}$$

where $c_1, c_2, \ldots, c_k$ are some integrable functions over the interval $[a,b]$. For a path $\gamma \in \mathcal{C}_{\boldsymbol{X}}([a,b])$, we define the length of γ as

$$|\gamma| \;=\; \int_a^b \left(\sum_{i=1}^{k} a_i(t)^2 \right)^{1/2} \mathrm{d}t. \tag{A.1.3}$$

Definition A.1.1. Suppose $a, b \in \mathbb{R}$, with $a \leq b$. Let $\gamma : [a, b] \to G$ be an absolutely continuous path belonging to $\mathcal{C}_{\boldsymbol{X}}([a, b])$. Then, its velocity γ' satisfies

$$\gamma'(t) \;=\; \sum_{i=1}^{k} c_i(t)\, X_i(\gamma(t)) \qquad \text{a.e.},$$

where $c_1, c_2, \ldots, c_k$ are some integrable functions over the interval $[a, b]$. We say γ has constant velocity $\lambda \geq 0$ if for a.a. $t \in [a, b]$, we have

$$||\gamma'(t)|| \;:=\; \left(\sum_{i=1}^{k} c_i(t)^2 \right)^{1/2} \;=\; \lambda.$$

Definition A.1.2. For $x, y \in G$, if there exists an absolutely continuous path $\gamma \in \mathcal{C}_{\boldsymbol{X}}([0, 1])$, for some $a, b \in \mathbb{R}$, with $\gamma(0) = x$ and $\gamma(1) = y$, we define the Carnot-Carathéodory distance between x and y by

$$d(x, y) \;=\; \inf \big\{ \, |\gamma| \;:\; \gamma \in \mathcal{C}_{\boldsymbol{X}}([0, 1]), \; \gamma(0) = x, \; \gamma(1) = y \big\}.$$

For $z \in G$, we denote

$$|z| \;:=\; d(e_G, z). \tag{A.1.4}$$

It is proved in [55] (p. 39) that $d(\cdot, \cdot)$ is indeed a metric on G. Moreover, for any two points $x, y \in G$, the existence of a path connecting x and y is due to the following well-known result by Chow and Rashevskii (for a proof, see, for example, Chapter 2 in Montgomery [35]).

Theorem A.1.3 (Chow's Theorem)**.** *If x, y are any two points on G, then there exists an absolutely continuous path γ which connects x and y.*

One can also study the special case of absolutely continuous paths γ satisfying

$$d(x, y) \;=\; |\gamma|,$$

for some points $x, y \in G$. Such paths are called (minimising) geodesics. In the general setting of manifolds, a geodesic between any two points does not always exist. However, as proved in Bellaiche and Risler [4] (see Theorem 2.7 therein), one always exists in our case due to Hörmander's condition on $\boldsymbol{X}$.

Theorem A.1.4. *Any two points in G can be joined by a geodesic. That is, for any $x, y \in G$, there exists an absolutely continuous path $\gamma : [a, b] \to G$, such that $\gamma(a) = x$ and $\gamma(b) = y$, for some $a, b \in \mathbb{R}$, which satisfies*

$$|\gamma| = d(x, y).$$

Remark A.1.5. Let x, y be two distinct points in the compact Lie group G. Furthermore, suppose $\gamma : [a, b] \to G$ is an absolutely continuous path connecting x and y belonging to $\mathcal{C}_{\boldsymbol{X}}([a, b])$. Then, consider the path $\gamma_0 : [0, 1] \to G$ given by

$$\gamma_0(t) = \gamma\left((1 - t)a + tb\right), \quad t \in [0, 1].$$

Then, γ_0 is also an absolutely continuous path connecting x and y, and moreover,

$$|\gamma_0| = \frac{1}{b - a}|\gamma|.$$

Thus, $\gamma_0 \in \mathcal{C}_{\boldsymbol{X}}([0, 1])$, and in particular, the Carnot-Carathéodory distance between x and y, $d(x, y)$, is well defined.

Furthermore, let $T = d(x, y)$. We can also construct the path $\gamma_1 : [0, T] \to G$ given by

$$\gamma_1(t) = \gamma\left(\frac{(T - t)a + tb}{T}\right), \quad t \in [0, T].$$

The path $\gamma_1 \in \mathcal{C}_{\boldsymbol{X}}([0, T])$ is also an absolutely continuous path lying in G, and connects x and y. Moreover, we have

$$|\gamma_1| = \frac{T}{b - a}|\gamma|.$$

A.2 Local theory

As in the previous section, we let G be a connected Lie group of dimension n and consider a Hörmander system of vector fields $\boldsymbol{X} = \{X_1, X_2, \ldots, X_k\}$, for some $k \in \mathbb{N}$. Suppose further that $\mathfrak{g}$ denotes the Lie algebra of G.

Now, let $B_r(e_G)$ be the ball centred at the identity element e_G of radius r, with respect to the Carnot-Carathéodory metric. Furthermore, let $V(r)$ denote the volume of the ball; that is,

$$V(r) = \int_{B_r(e_G)} \mathrm{d}z,$$

where $\mathrm{d}z$ denotes the Haar measure on G. The objective is to give an estimate for $V(r)$. We start by letting V_j, $j \in \mathbb{N}$, be the linear subspace of $\mathfrak{g}$ spanned by the commutators of vector fields of length at most a. That is,

$$V_j := \mathrm{Span}\big\{[X_{i_1}, [X_{i_2}, \ldots, [X_{i_{\alpha-1}}, X_{i_\alpha}] \ldots]] \,:\, |(i_1, i_2, \ldots, i_\alpha)| \leq j\big\}, \qquad \text{(A.2.1)}$$

where the span is taken over all multi-indices $(i_1, i_2, \ldots, i_\alpha)$, with $i_1, i_2, \ldots, i_\alpha$ taking values in $\{1, 2, \ldots, k\}$. The definition of Hörmander systems implies that there exists $s \in \mathbb{N}$ such that

$$\{0\} = V_0 \subset V_1 \subset \cdots \subset V_s = \mathfrak{g}.$$

We denote $n_j = \dim V_j$, so $0 = n_0 < n_1 < \cdots < n_s = n$.

Definition A.2.1 (Local dimension)**.** Let G be a connected Lie group of dimension n and consider a Hörmander system of vector fields $\boldsymbol{X} = \{X_1, X_2, \ldots, X_k\}$, for some $k \in \mathbb{N}$. Let V_j be the linear subspace of the Lie algebra of G given by (A.2.1), and let n_j denote its dimension. Then, the expression

$$l := n_1 + 2(n_2 - n_1) + \cdots + s(n_s - n_{s-1})$$

is called the local dimension of G.

Example A.2.2. In the case of $SU(2)$ it is not difficult to see that

$$V_0 = \mathrm{Span}\{I\}, \qquad V_1 = \mathrm{Span}\{X_1, X_2\}, \qquad V_2 = \mathrm{Span}\{X_1, X_2, [X_1, X_2]\}.$$

Since $X_3 = 2[X_1, X_2]$, then $V_2 = \mathfrak{su}(2)$, and so it follows that the local dimension of $SU(2)$ is given by

$$l = \dim V_1 + 2(\dim V_2 - \dim V_1) = 2 + 2 \cdot (3 - 2) = 4.$$

In general, we have the following result, whose proof may be found in [55] (Chapter V, Theorem V.1.1).

Theorem A.2.3. *Let G be a connected Lie group of dimension n and consider a Hörmander system of left-invariant vector fields $\boldsymbol{X} = \{X_1, X_2, \ldots, X_k\}$. Suppose* *that $B_r(e_G)$ denotes the ball of radius $r > 0$ centred at the identity e_G, with respect to the Carnot-Carathéodory metric. and let $V(r)$ denote the volume of this ball. Suppose further that l denotes the local dimension of G. Then there exists $C > 0$ such that*

$$C^{-1}\, r^l \;\leq V(r) \;\leq\; C\, r^l, \qquad \textit{for } \; 0 < r < 1.$$

Example A.2.4. In the case of $SU(2)$ we summarise this result as follows. For any $r > 0$, there exists $C > 0$ such that

$$C^{-1}\, r^4 \;\leq V(r) \;\leq\; C\, r^4, \qquad \text{for } \; 0 < r < 1. \tag{A.2.2}$$

Remark A.2.5. Since $SU(2)$ is a unimodular Lie group, then for any $x \in SU(2)$, the volume of the ball $B_r(x)$ of radius $r > 0$ centred at x is equal to the volume of the ball $B_r(I)$.

A.3 Integration of powers of $|z|$

Let G be a compact Lie group of dimension n and local dimension l. Furthermore, suppose $\boldsymbol{Y} = \{Y_1, Y_2, \ldots, Y_n\}$ denotes a basis of the Lie algebra $\mathfrak{g}$ of G.

Now, for any given $r \in (0,1)$ there exists a neighbourhood N of 0 in $\mathbb{R}^n$ such that the mapping $\phi : N \to B_r(e_G)$ given by

$$\phi((z_1, z_2, \ldots, z_n)) \;=\; e^{z_1 Y_1}\, e^{z_2 Y_2}\, \ldots\, e^{z_n Y_n}, \quad (z_1, z_2, \ldots, z_n) \in N, \tag{A.3.1}$$

is a diffeomorphism. For a given $z \in B_r(e_G)$, we shall let $(z_1, z_2, \ldots, z_n)$ denote the coordinates of z in the sense that (A.3.1) is satisfied.

Lemma A.3.1. *For any $\gamma \in \mathbb{R}$, we have*

$$\int_{B_r(e_G)} |z|^\gamma \; dz \;\approx\; \int_0^r \rho^{\gamma + l - 1}\; d\rho.$$

Proof. We consider the change of coordinates map $\phi : B_r(e_G) \to N$ given by

$$\phi(z) := (z_1, z_2, \ldots, z_n), \qquad z \in B_r(e_G).$$

Thus, we have

$$\int_{B_r(e_G)} |z|^\gamma \, \mathrm{d}z = \int_N \big|\big|(z_1, z_2, \ldots, z_n)\big|\big|_{\mathbb{R}^n}^\gamma \, |J_{\phi^{-1}}(z_1, z_2, \ldots, z_n)| \, \mathrm{d}z_1 \, \mathrm{d}z_2 \, \ldots \, \mathrm{d}z_n,$$

where $|J_{\phi^{-1}}(z_1, z_2, \ldots, z_n)|$ denotes the determinant of the Jacobian of ϕ^{-1}. Since $\mathrm{Vol}(B_r(e_G)) \approx r^l$ (see Theorem A.2.3), then it follows that we must have

$$r^l \approx \int_{B_r^{\mathbb{R}^n}(0)} |J_{\phi^{-1}}(z_1, z_2, \ldots, z_n)| \, \mathrm{d}z_1 \, \mathrm{d}z_2 \, \ldots \, \mathrm{d}z_n = \int_{B_r(e_G)} \mathrm{d}z.$$

We now apply the substitution $\rho = \big|\big|(z_1, z_2, \ldots, z_n)\big|\big|_{\mathbb{R}^n}$. Hence, the volume element in polar coordinates satisfies

$$|J_{\phi^{-1}}(z_1, z_2, \ldots, z_n)| \, \mathrm{d}z_1 \, \mathrm{d}z_2 \, \ldots \, \mathrm{d}z_n \approx \rho^{l-1} \, \mathrm{d}\rho,$$

which yields the result. □

Proposition A.3.2. *Let $\gamma \in \mathbb{R}$. Then the integral*

$$\int_G |z|^\gamma \, dz < +\infty,$$

provided that $\gamma > -l$.

Proof. Observe that, if $r > 0$, then

$$\int_G |z|^\gamma \, \mathrm{d}z = \int_{B_r(e_G)} |z|^\gamma \, \mathrm{d}z + \int_{B_r(e_G)^c} |z|^\gamma \, \mathrm{d}z.$$

By the compactness of G,

$$\int_{B_r(e_G)^c} |z|^\gamma \, \mathrm{d}z < +\infty,$$

for any $\gamma \in \mathbb{R}$.

We now study the integral

$$\int_{B_r(e_G)} |z|^\gamma \, \mathrm{d}z, \tag{A.3.2}$$

and assume that $\gamma < 0$. The case $\gamma \geq 0$ is immediate, as the integral (A.3.2) is finite whenever $\gamma \geq 0$. By Proposition A.3.1, we then have

$$\int_{B_r(e_G)} |z|^\gamma \, \mathrm{d}z \approx \int_0^r \rho^{\gamma+l-1} \, \mathrm{d}\rho < +\infty,$$

provided that $\gamma > -l$. This finishes the proof.

□

Appendix B

Cotlar-Stein Lemma

This brief section is dedicated to the proof of the Cotlar-Stein Lemma for the case in which we have a collection of infinitely many operators. First, let us state the result for finitely many operators, whose proof can be found in Stein [47] (Chapter VII, Section 2).

Theorem B.0.1 (Cotlar-Stein Lemma). *Suppose that* $\{T_\ell\}_{\ell=1}^N$ *is a collection of bounded operators on a Hilbert space* $\mathscr{H}$ *and assume that we are given a sequence of constants* $\{\gamma(\ell)\}_{\ell\in\mathbb{Z}} \subset \mathbb{R}^+$*, such that*

$$A := \sum_{\ell\in\mathbb{Z}} \gamma(\ell) < +\infty.$$

Furthermore, suppose that for every $\ell, k = 1, 2, \ldots, N$*, we have*

$$\begin{aligned} ||T_\ell^* T_k||_{\mathscr{L}(\mathscr{H})} &\le \gamma(\ell - k)^2, \\ ||T_\ell T_k^*||_{\mathscr{L}(\mathscr{H})} &\le \gamma(\ell - k)^2. \end{aligned}$$

Then, the operator

$$T := \sum_{\ell=1}^N T_\ell$$

satisfies

$$||T||_{\mathscr{L}(\mathscr{H})} \le A.$$

In the context of our work, we will use the following consequence of the Cotlar-Stein Lemma.

Corollary B.0.2. *Suppose that* $\{T_\ell\}_{\ell\in\mathbb{Z}}$ *is a collection of bounded operators on* $L^2(G)$ *and assume that we are given a sequence of constants* $\{\gamma(\ell)\}_{\ell\in\mathbb{Z}} \subset \mathbb{R}^+$, *such that*

$$A := \sum_{\ell\in\mathbb{Z}} \gamma(\ell) < +\infty.$$

Suppose that for every $\ell, k \in \mathbb{N}$, *we have*

$$\begin{aligned} \|T_\ell^* T_k\|_{\mathscr{L}(L^2(G))} &\leq \gamma(\ell-k)^2, \\ \|T_\ell T_k^*\|_{\mathscr{L}(L^2(G))} &\leq \gamma(\ell-k)^2. \end{aligned}$$

Furthermore, let us assume that for any $f \in \mathcal{D}(G)$, *the sum*

$$\sum_{\ell\in\mathbb{Z}} T_\ell f$$

converges in the sense of distributions. We denote by $Tf \in \mathcal{D}'(G)$ *the limit of this sum. Then,* T *extends to a bounded operator on* $L^2(G)$, *with*

$$\|T\|_{\mathscr{L}(L^2(G))} \leq A.$$

Proof. Theorem B.0.1 tells us that for every $N \in \mathbb{N}$,

$$T^{(N)} := \sum_{\ell=-N}^{N} T_\ell$$

extends to a bounded operator on $L^2(G)$ and satisfies

$$\|T^{(N)}\|_{\mathscr{L}(L^2(G))} \leq A,$$

where A is independent of N. In particular, this means that for every $f \in L^2(G)$,

$$\left\| \sum_{\ell=1}^{N} T_\ell f \right\|_{L^2(G)} \leq A \|f\|_{L^2(G)}. \tag{B.0.1}$$

Now, by the assumption that the sum $\sum_{\ell\in\mathbb{Z}} T_\ell$ converges in the sense of distributions, for $f, g \in \mathcal{D}(G)$, we have

$$\langle Tf, g\rangle_{L^2(G)} = \lim_{N\to\infty} \left\langle T^{(N)} f, g\right\rangle_{L^2(G)}.$$

Since

$$\begin{aligned}\left|\left\langle T^{(N)}f,g\right\rangle_{L^2(G)}\right| &\leq \left|\left|T^{(N)}\right|\right|_{\mathscr{L}(L^2(G))}||f||_{L^2(G)}\,||g||_{L^2(G)} \\ &\leq A||f||_{L^2(G)}\,||g||_{L^2(G)},\end{aligned}$$

where A is independent of N, then we have

$$\left|\langle Tf,g\rangle_{L^2(G)}\right| \leq A\,||f||_{L^2(G)}\,||g||_{L^2(G)}.$$

The density of $\mathcal{D}(G)$ in $L^2(G)$ implies the result.

□

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