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Abstract

In this thesis we unify and find new, and recover all known, explicit local examples of extremal toric Kähler metrics and describe how to compactify them. To do so, we define explicitly a class of toric geometries of Sasaki type with toric Kähler quotients, both called *separable* geometries, using *factorization structures*. We conjecture factorization structures to be decomposable in which case we find their explicit description to be of Segre-Veronese type. A compatible factorization structure gives rise to *separable coordinates* on the image of the momentum map of a given separable geometry. In such coordinates the extremality equation for separable Kähler geometries becomes a functional system of ODEs which, in our case, is a system obtained from a generalisation of the method for separation of variables for PDEs. We derive necessary conditions for its solutions and find a complete set of solutions in the case of the product Segre-Veronese factorization structure with a decomposable Sasaki structure. We use generalised equipoised condition for extremal affine functions to geometrically characterise some compactifications of such extremal metrics.

Contents

Chapter 1

Introduction

This thesis concerns the construction of explicit toric extremal Kähler metrics on compact manifolds and orbifolds. Extremal metrics were originally defined by Calabi in [22, 23] as critical points of the L^2-norm of the scalar curvature as a functional on Kähler metrics corresponding to a fixed second cohomology class. The corresponding Euler-Lagrange equations, called the extremality equation, express that a metric g is extremal if and only if the symplectic gradient of its scalar curvature is a Killing vector field. This is a challenging-to-solve PDE for which no general methods are available. However, there is a formal picture due to Yau, Tian and Donaldson [52, 30, 29, 24, 47, 53, 17, 33, 46, 51, 25, 26] motivated by geometric invariant theory that suggests the existence of extremal Kähler metrics is equivalent to a stability condition, called K-stability. To formulate and test such a conjecture it is valuable to have explicit examples where the extremality and stability can be verified directly.

The known explicit examples of extremal Kähler metrics are either Calabi type or toric. The Calabi type examples were introduced by Calabi as examples on $\mathbb{CP}^1$-bundles and studied further in [13, 41, 15, 42, 43, 8, 7]. Toric geometries, geometries carrying an action of a torus of maximal dimension preserving all geometric structures, are a standard class of explicit examples in algebraic and differential geometry. We adopt differential-geometric approach and focus on sympelctic/Kähler and contact/CR/Sasaki geometries. A toric symplectic geometry is a compact connected symplectic geometry M (i.e. a manifold or orbifold) together with an effective hamiltonian action of a torus of dimension $\frac{1}{2}\dim(M)$. In momentum-angle coordinates, a compatible Kähler metric is given by the hessian of a function, called symplectic potential, on the image of the momentum map [37, 36, 38]. Despite toric symmetries the extremality equation remains a non-linear 4th order PDE in the symplectic potential [3, 2, 1].

The idea of separable toric geometry is to assume that the metric is given by unknown functions of one variable so that the extremality equation reduces into a functional system of ODEs. By this we mean a system obtained from a PDE system by an Ansatz in which the unknown functions are expressed in terms of functions of one variable. It is thus a generalization of the method of separation of variables for PDEs. A class of separable toric examples, called orthotoric, was found in [11, 6, 12, 10] which were motivated by the work on Bochner-Kähler metrics in [21] and by the previous work on weakly self-dual Kähler surfaces [5]. Another class of separable toric geometries, called ambitoric, was studied in [8, 7]. The second of these papers made two observations which are important for this thesis. First a notion of factorization structures was proposed to provide a common framework for ambitoric and orthotoric geometries. Secondly, different ambitoric geometries were observed to be quotients of a CR geometry with respect to different Sasaki structures.

Sasaki geometry is an odd-dimensional analogue of Kähler geometry. The notion of extremal metric can be transferred to Sasaki geometry and is studied in [19, 49, 27, 9, 14]. A Sasaki geometry N of dimension $2m+1$ is in particular a CR geometry, which is a contact geometry with a compatible complex structure on the contact distribution. It comes also equipped with a Sasaki-Reeb vector field which turns the corresponding quotient (if it is a manifold or orbifold) into a Kähler geometry. In the toric case, the image of contact momentum map was shown to be a convex polyhedral cone in $\mathfrak{h}^*$ (see [44]) which we regard as a projective polytope in $\mathbb{P}(\mathfrak{h}^*)$, where $\mathfrak{h}$ is the Lie algebra of the torus acting upon the geometry N and $\dim(\mathfrak{h}) = m+1$. Furthermore, the intersection of the projective polytope with the affine chart determined by a Sasaki-Reeb vector field X_β, $\beta \in \mathfrak{h}$, is the image of the momentum map of the quotient of the toric N by X_β (being a toric Kähler geometry). The idea of studying separable toric geometries as quotient Kähler metrics of CR geometries in [7] was further explored in [4] and this is the approach we will take in this thesis.

Since the momentum map of a toric contact geometry is $\mathbb{P}(\mathfrak{h}^*)$-valued, the idea of factorization structures is to introduce coordinates on an open subset U of $\mathbb{P}(\mathfrak{h}^*)$ such that a coordinate hypersurface (i.e. whenever one of the coordinates is fixed) is the intersection of U with a hyperplane in $\mathbb{P}(\mathfrak{h}^*)$. Motived by [8, 7, 4] these coordinates are introduced via a rational map $\mathbb{P}(V_1) \times \cdots \times \mathbb{P}(V_m) \dashrightarrow \mathbb{P}(\mathfrak{h}^*)$, $\dim(V_j) = 2$ for $j = 1, \dots, m$. For the coordinate hypersurfaces to be hyperplanes, the rational map is constructed by composing the Segre embedding,

$$\begin{aligned} S : \mathbb{P}(V_1) \times \cdots \times \mathbb{P}(V_m) &\to \mathbb{P}(V_1 \otimes \cdots \otimes V_m) \\ ([v_1], \dots, [v_m]) &\mapsto [v_1 \otimes \cdots \otimes v_m], \end{aligned}$$

with a projective map $\mathbb{P}(V_1 \otimes \cdots \otimes V_m) \dashrightarrow \mathbb{P}(\mathfrak{h}^*)$.

Having described the context of this thesis I will now detail the structure and results.

Chapter 2 surveys the background material used throughout this thesis on extremal Kähler metrics, Sasaki-Kähler correspondence and its toric counterparts. It also includes a remark on Schubert varieties which will be used in chapter 3.

The first main results appear in chapter 3 where factorization structures are studied in detail. They are defined via a linear injection $\varphi : \mathfrak{h} \to V_1^* \otimes \cdots \otimes V_m^*$, $\dim(\mathfrak{h}) = m+1$, such that $\mathbb{P}(\varphi^T)$ is the projective map above. A complete classification of factorization structures is at the moment out of reach, but we obtain detailed information on factorization structures and a partial classification which includes all previously known examples. A key ingredient is the observation that the coordinate hyperplane condition gives rise to maps $[\psi_j] : \mathbb{P}(V_j) \to \mathbb{P}(\mathfrak{h})$, $j = 1, \dots, m$, which we call factorization curves. These curves have a degree d_j, with $1 \le d_j \le m$. A factorization curve $[\psi_j]$ of degree d_j is said to be decomposable if the number of curves with the same image as $[\psi_j]$ is d_j. We conjecture all factorization curves are decomposable. We prove that if all factorization curves in a given factorization structure are decomposable, then, up to isomorphism, it is of Segre-Veronese type, i.e. there exists $k \ge 1$ and $d_1, \ldots, d_k$ positive integers such that for all $i \in \{1, \ldots, k\}$ there exists 1-dimensional subspace $\langle \Gamma_i \rangle \subset \bigotimes_{\substack{b=1 \\ b \neq i}}^{k} S^{d_b} W_b^*$ such that

$$\varphi(\mathfrak{h}) = \sum_{i=1}^{k} ins_i \left(S^{d_i} W_i^* \otimes \langle \Gamma_i \rangle \right), \tag{1.0.1}$$

where ins_i linearly inserts the first d_i slots of the tensor product into the missing slots of Γ_i, and $S^{d_i} W_i^*$ represents the d_ith symmetric power of the dual of a 2-dimensional vector space W_i, $i = 1, \ldots, k$. After establishing notions of a product, quotient and isomorphism of factorization structures we describe in detail the behaviour of degree in relation to quotient factorization structures. In particular, we show that any top degree factorization curve is decomposable and determines the underlying factorization structure to be of Veronese type, a special case of Segre-Veronese family, isomorphic with $\varphi : S^m W^* \to (W^*)^{\otimes m}$, $\dim(W^*) = 2$.

Chapter 4 demonstrates the effectiveness of factorization structures by deriving a functional system of ODEs equivalent to the extremality equation, giving necessary conditions for its solutions, and by finding new explicit examples generalising all previously known cases. We also explicitly describe their scalar curvature in terms of momentum coordinates.

A factorization structure determines a contact and CR geometry, and a family of Kähler geometries with separable coordinates as follows. A toric contact $(2m+1)$-geometry N with momentum map $[\mu] : N \to \mathbb{P}(\mathfrak{h}^*)$ is separable if there exist coordinates, called *separable*, $[x_j] : N \to \mathbb{P}(V_j)$, $\dim(V_j) = 2$, $j = 1, \ldots, m$, such that $[\mu] = \mathbb{P}(\varphi^T)[x_1 \otimes \cdots \otimes x_m]$, where $\varphi : \mathfrak{h} \to V_1^* \otimes \cdots \otimes V_m^*$ is a linear map. This is well-defined if and only if φ is a factorization structure. It is remarkable that the curves arising from the definition of factorization structure are precisely what is needed to define a compatible CR structure J on N using functions of one variable. It is given by

$$J d\tau \big|_{\mathcal{D}} = \sum_{j=1}^{m} -\frac{\psi_j(x_j)}{A_j(x_j)} dx_j \big|_{\mathcal{D}},$$

where $\tau : N \to \mathfrak{h}/2\pi\Lambda$ are the angle coordinates, $\mathcal{D}$ is the contact distribution given by the kernel of the one form $\langle \mu, d\tau \rangle$, and, in an affine chart on $\mathbb{P}(V_j)$, $j = 1, \ldots, m$, x_j is a separable coordinate, A_j is a function of one variable, ψ_j is an $\mathfrak{h}$-valued function associated with the factorization curve $[\psi_j]$, and μ is similarly related to $[\mu]$.

A family of (explicit) separable toric Kähler geometries corresponding to a fixed factorization structure is obtained as quotients of N by different Sasaki-Reeb vector fields. In particular, these Kähler metrics are given by unknown functions A_j of one variable. For example, when $m = 2$, the family corresponding to the Segre factorization structure,

$$V_1^* \otimes \langle \Gamma_1 \rangle + \langle \Gamma_2 \rangle \otimes V_2^* \hookrightarrow V_1^* \otimes V_2^*,$$

consists of toric Kähler products, toric Calabi geometries and negative orthotoric geometries, while regular ambitoric structures correspond to the Veronese factorization structure,

$$S^2 W^* \hookrightarrow W^* \otimes W^*.$$

In addition, these are the only factorization structures up to an isomorphism when $m = 2$ (see [7]). In general, Segre and Veronese factorization structures correspond to twisted toric product ansatz and twisted orthotoric geometries, respectively, both studied in [4]. These examples contain all known explicit extremal toric Kähler metrics.

In this chapter we mostly work with the decomposable Segre-Veronese factorization structure (i.e. the tensors Γ_i, $i = 1, \ldots, k$, from (1.0.1) are decomposable) whose special case is the product Segre-Veronese factorization structure, a common generalisation of factorization structures of Segre and Veronese type. It is called the product Segre-Veronese factorization structure because the simplest examples of the corresponding family of Kähler metrics are products of orthotoric geometries.

We exploit separable coordinates to the fullest by formulating the extremality equation for the class of separable toric Kähler geometries corresponding to a general Segre-Veronese factorization structure. To do so we calculate the Laplace and scalar curvature of such geometries which reveal the extremality equation to be a functional system of ODEs (4.3.24). Then we derive necessary conditions on its solutions A_i in terms of ODEs. In the case of the decomposable Segre-Veronese factorization structure, the ODEs are straightforward to solve yielding polynomial or rational functions as their solutions depending on the factorization structure. In particular, we use the generalised Vandermonde identities to show that these necessary conditions are also sufficient in the important case of the product Segre-Veronese factorization structure with Sasaki structure

$$\varphi(\beta) = ins_i \left((a, b)^{\otimes d_i} \otimes (1, 0) \otimes \cdots \otimes (1, 0) \right), \tag{1.0.2}$$

and we speculate that similar techniques can be used even in more generality. In this case, solutions are

$$A_{ir}(x_{ir}) = pol_i(x_{ir}) + (a + bx_{ir})^{m+1}(\nu_{ir}^0 + \nu_{ir}^1 x_{ir}), \quad r = 1, \ldots, d_i, \quad i = 1, \ldots, k$$

with the natural indexing related to the number k of factorization curves and degrees d_i, where $\nu_{ir}^1, \nu_{ir}^2 \in \mathbb{R}$ are arbitrary, and pol_i is an univariate r-independent polynomial with degree depending on $i, k, d_1, \ldots, d_k$ and b. As a by-product of these computations we express the scalar curvatures of these geometries in momentum coordinates and find that they form 4 families, which will be needed in chapter 5.

Chapter 5 outlines a geometrical characterisation of compactifications of Kähler metrics corresponding to the product Segre-Veronese factorization structure. We adopt an approach from [42], where compactifications are characterised as a class of Delzant polytopes whose extremal affine functions belong to a particular family, called equipoised.

To describe compactifications we specify their Delzant polytopes under the Delzant correspondence [28, 45] which associates a compact connected toric symplectic geometry to a Delzant polytope and vice versa. A Delzant polytope Δ in a vector space $\mathfrak{t}^*$ is defined via affine functions whose differentials are integral vectors in $\mathfrak{t}$ which represent normals of the hyperplanes bounding Δ, and if $\mathbb{T}$ is the torus acting on the geometry corresponding to Δ, then $\mathrm{Lie}(\mathbb{T}) = \mathfrak{t}$. We note the hyperplanes bounding Δ, or their normals, are defined by vectors rather than one-dimensional subspaces, and thus it makes sense to talk about their scales. In fact, finding compactifications boils down to finding suitable scales.

The condition for a Kähler metric to compactify was first computed by Abreu [3] in terms of a symplectic potential on the polytope Δ. This condition turns out to be equivalent to the first order boundary conditions on the inverse hessian of the symplectic potential [11]. In separable toric geometries this inverse hessian is determined by functions A_j of one variable which define the geometry. We compute these first order boundary conditions on A_j explicitly. It follows that the polytope Δ is compatible in the sense that it is the image of the m-cube in separable coordinates and has at most $2m$ facets. It remains to study the case in which the geometry is extremal, i.e. A_js solve the extremality equation. We do this in the case of geometries of the product Segre-Veronese type where we have an explicit description of A_js as polynomials. We derive that the boundary conditions form an over-determined system, call it (B), in coefficients of the polynomials A_{ir} with the right hand side consisting of inverse scales of normals of a Delzant polytope where it compactifies. This places conditions on scales and thus determines what Delzant polytopes can occur as compactifications.

Any Delzant polytope Δ has a unique extremal affine function which agrees with the scalar curvature of the corresponding geometry, if the metric is extremal (see [30]). In such case, the extremal affine function is also the L^2-projection of the scalar curvature into the space of affine functions on Δ, and thus satisfies a system (E) of linear equations expressing this fact. If boundary conditions (B) are satisfied on a Delzant polytope Δ for an extremal geometry given by A_{ir}, then A_{ir} determine the scalar curvature which is also the extremal affine function of Δ and thus satisfy (E).

In the case when $\varphi(\beta)$ is as in (1.0.2) and (B) and (E) have both full ranks, we show that compactifications can be described as Delzant polytopes with a particularly shaped extremal affine functions which generalises the equipoised condition from [42]. There are four families of such extremal affine functions depending on a particular choice of the product Segre-Veronese factorization structure and Sasaki structure: the number of factorization curves is either one or more than one, and either $b = 0$ or $b \neq 0$ in (1.0.2). Furthermore, we explicitly describe the system (B) in the case when the factorization structure has more than one factorization curve.

Chapter 2

Background

In this chapter we recapitulate background material used throughout this thesis. We start with recalling standard facts from Kähler, CR and Sasaki geometry on manifolds (see [18, 19]) and define extremal metrics (see [22, 23]). A generalisation of these constructs from manifolds to orbifolds, which shall be used later in this text, can be found in [19]. Then, we recall how the presence of a maximal torus action fits into this framework, e.g. momentum-angle coordinates, Delzant construction and compactification, and their relation to affine and projective geometry. At the end of this chapter we prepare a lemma for later use which shows that certain sets arising in the theory of factorization structures are Schubert varieties (see [31]).

2.1 (Toric) Kähler and (toric) Sasaki geometry

2.1.1 Kähler geometry. Kähler manifolds are smooth manifolds equipped with an integrable complex structure, symplectic form and Riemannian metric in a compatible way which makes their theory rich and interesting.

Definition 2.1.1. A *Kähler manifold* is a real manifold M equipped with an integrable almost complex structure J and a Hermitian metric g, i.e. a Riemannian metric g such that $g(JX, JY) = g(X, Y)$ for any vector fields X, Y on M, for which the associated *Kähler form* ω defined by $\omega(X, Y) = g(JX, Y)$ is closed.

Among almost direct consequences of this definition are properties such as $dim_{\mathbb{R}}(M)$ is even, say $2m$, ω is a symplectic form, and the Riemannian volume form is given by $\frac{\wedge^m \omega}{m!}$. Standard references on Kähler geometry and extremal metrics include [40, 34, 18, 35, 48].

As the Kähler form ω is a closed 2-form it defines the cohomology class $[\omega] \in H^2(M, \mathbb{R})$. For a fixed J, the set Ω of all J-compatible symplectic forms belonging to $[\omega]$ is called the *Kähler class*. By definition each $\omega \in \Omega$ has its associated metric g via J, and we can ask if there is a notion of preferred/canonical metric within Ω. One approach is via

Definition 2.1.2. Let M be a compact Kähler $2m$-manifold and Ω the corresponding

Kähler class. An *extremal metric* on M in Ω is a critical point of the functional

$$Cal(\omega) = \int_M \mathrm{Scal}(g)^2 \wedge^m \omega, \tag{2.1.1}$$

for $\omega \in \Omega$, where g is the Kähler metric associated to ω. This functional is called *the Calabi functional*.

This definition was introduced by Calabi in [22] and can be equivalently described via its Euler-Lagrange equation: a metric g is extremal if and only if the scalar curvature $\mathrm{Scal}(g)$ is a Killing potential, i.e. the vector field $J\mathrm{grad}_g\mathrm{Scal}(g)$ is a Killing vector field for g.

In order to calculate $\mathrm{Scal}(g)$ we shall use the Laplace operator. Thanks to Kähler identities the Laplace operator for functions on a Kähler manifold takes the form

$$\Delta f = -\langle \omega^\sharp, dJdf \rangle \tag{2.1.2}$$

where d is the exterior differential and J is the extension of the complex structure to one-forms defined by $(J\alpha)(X) = -\alpha(JX)$, and $\langle -, - \rangle$ is the natural contraction. Furthermore, for $\omega^\sharp$ we have

$$\omega^\sharp = -\omega^{-1} \tag{2.1.3}$$

Indeed, for dual frames (ϵ^a) and (e_a), $\langle \epsilon^a, e_b \rangle = \delta^a_b$, we calculate

$$\begin{aligned} \omega^\sharp &= \omega_{ab}(\epsilon^a)^\sharp \otimes (\epsilon^b)^\sharp = \omega_{ab}(g^{-1})^{ar} e_r \otimes (g^{-1})^{bs} e_s \\ &= J^t_a g_{tb}(g^{-1})^{ar}(g^{-1})^{bs} e_r \otimes e_s = J^s_a (g^{-1})^{ar} e_r \otimes e_s = -\omega^{-1}. \end{aligned} \tag{2.1.4}$$

Analogously to the definition of Kähler form we define the *Ricci form* ρ by

$$\rho(X, Y) = r(JX, Y), \tag{2.1.5}$$

where r is the Ricci curvature and X, Y are smooth vector fields on M. We infer

$$Scal(g) = \langle \omega^\sharp, \rho \rangle \tag{2.1.6}$$

Lemma 2.1.1. *A local expression for the Ricci form ρ is*

$$\rho = -\frac{1}{2} dJd \, \ln |l|^2, \tag{2.1.7}$$

where ln *stands for the natural logarithm, l is a section of the anti-canonical line bundle $\bigwedge^m(TM, J)$, and $|l|^2$ denotes the squared norm of l with respect to the induced Hermitian metric from the metric g on TM.*

The expression $-\frac{1}{2} ln|l|^2$ is called the Ricci potential. In a neighbourhood of a point on a Kähler manifold we have holomorphic coordinate system $z_1, \dots, z_m$ in which the corresponding holomorphic volume takes form $v_0 = \prod_{r=1}^m \frac{i}{2} dz_r \wedge d\bar{z}_r$ and relates to the

Riemannian volume form v_g by

$$v_g = 2^m |\partial_{z_1} \wedge \cdots \wedge \partial_{z_m}|^2 v_0, \tag{2.1.8}$$

where $\partial_{z_1} \wedge \cdots \wedge \partial_{z_m}$ is a nowhere vanishing holomorphic section of the anticanonical bundle. Thus

$$\rho = -\frac{1}{2} dJd\, ln \frac{v_g}{v_0}. \tag{2.1.9}$$

Finally, (2.1.9) and (2.1.2) give

$$Scal(g) = \frac{1}{2} \Delta\, ln \frac{v_g}{v_0}. \tag{2.1.10}$$

2.1.2 Sasaki geometry. Sasaki geometry is considered to be an odd-dimensional analogue of Kähler geometry. We shall use it to realise some Kähler manifolds as quotients of a given Sasaki manifold.

We start with a $(2m+1)$-manifold N equipped with a corank 1 distribution $\mathcal{D}$ and denote $\eta_{\mathcal{D}} : TN \to TN/\mathcal{D}$ the natural projection. We define the Levi form $L_{\mathcal{D}} : \mathcal{D} \times \mathcal{D} \to TN/\mathcal{D}$ by $L_{\mathcal{D}}(X, Y) = -\eta_{\mathcal{D}}([X, Y|])$. If $L_{\mathcal{D}}$ is non-degenerate, then $(N, \mathcal{D})$ is called a *contact geometry*. A *contact vector field* is a vector field X such that $\mathcal{L}_X(C^\infty(N, \mathcal{D})) \subset C^\infty(N, \mathcal{D})$. By [4] we have

Lemma 2.1.2. *The map $X \mapsto \eta_{\mathcal{D}}(X)$ from contact vector fields to sections of $TN/\mathcal{D}$ is a linear isomorphism, whose inverse $\xi \mapsto X_\xi$ is a first order linear differential operator.*

As contact vector fields form a Lie algebra we may use the isomorphism from this lemma to make sections of $TN/\mathcal{D}$ into a *contact Lie algebra* $\mathfrak{con}(N, \mathcal{D})$ with the Lie bracket defined by $[\xi, \chi] = \eta_{\mathcal{D}}([X_\xi, X_\chi])$.

The next ingredient in Sasaki geometry is a CR structure. A *CR structure* on $(N, \mathcal{D})$ is a complex structure J on $\mathcal{D}$ such that the subbundle $\{X \in \mathcal{D} \otimes \mathbb{C} \mid JX = iX\}$ is closed under Lie bracket and $L_{\mathcal{D}}$ is J-Hermitian. Contact vector fields which preserve a CR structure J form a Lie subalgebra

$$\mathfrak{cr}(N, \mathcal{D}, J) = \{\xi \in \mathfrak{con}(N, \mathcal{D}) \mid \mathcal{L}_{X_\xi} J = 0\} \tag{2.1.11}$$

of $\mathfrak{con}(N, \mathcal{D})$. If a CR structure J is *strictly pseudo-convex*, i.e. $L_{\mathcal{D}}(-, J-)$ is a definite fibre-wise metric on $\mathcal{D}$, then $TN/\mathcal{D}$ is orientable. We choose the orientation of $TN/\mathcal{D}$ so that positive sections χ are those for which $\chi^{-1} L_{\mathcal{D}}(-, J-)$ is positive definite. We let $\mathfrak{con}_+(N, \mathcal{D}) \subset \mathfrak{con}(N, \mathcal{D})$ be the open cone of positive sections χ of $TN/\mathcal{D}$.

Definition 2.1.3. Let $(N, \mathcal{D}, J)$ be a strictly pseudo-convex CR manifold. Then the *Sasaki cone* of $(N, \mathcal{D}, J)$ is $\mathfrak{cr}_+(N, \mathcal{D}, J) = \mathfrak{cr}(N, \mathcal{D}, J) \cap \mathfrak{con}_+(N, \mathcal{D})$. If $\mathfrak{cr}_+(N, \mathcal{D}, J)$ is nonempty then $(N, \mathcal{D}, J)$ is said to be of *Sasaki type*, an element $\chi \in \mathfrak{cr}_+(N, \mathcal{D}, J)$ is called a *Sasaki structure* on $(N, \mathcal{D}, J)$, with *Sasaki-Reeb vector field* X_χ, and $(N, \mathcal{D}, J, \chi)$ is called a *Sasaki manifold*. We say χ is quasi-regular if the flow of X_χ generated an $\mathbb{S}^1$ action on N, and moreover *regular* if this action is free.

The main use of Sasaki geometry appearing in this text is as follows. First observe $L_{\mathcal{D}} = d\eta_{\mathcal{D}}|_{\mathcal{D}}$. A choice of $\chi \in \mathfrak{con}_+(N, \mathcal{D})$ provides us with a contact form η_χ characterised by $\eta_\chi(X_\chi) = 1$, where $\eta_\chi = \chi^{-1}\eta_{\mathcal{D}}$. In a neighbourhood U of any point in N the leaf space M of the flow of X_χ is a Kähler manifold with the symplectic form ω given by $\pi^*\omega = d\eta_\chi$ and the complex structure J transferred via $\mathcal{D}|_U \cong \pi^*TM$, where $\pi : U \to M$. If χ is (quasi-)regular, then N is a principal $\mathbb{S}^1$-bundle (or orbibundle) $\pi : N \to M$ over a Kähler manifold (or orbifold) M.

This process can be inverted. For any Kähler manifold (M, ω, J, g) such that $[\omega/2\pi] \in H^2(M, \mathbb{Z})$ there exists $\mathbb{S}^1$-principal bundle $\pi : N \to M$ with a connection form η such that $d\eta = \pi^*\omega$. This bundle yields a Sasakian manifold $(N, \mathcal{D}, \hat{J}, \chi)$, where $\mathcal{D} := \mathrm{Ker}\,\eta$, $\hat{J}$ is the pullback of J to $\mathcal{D} \cong \pi^*TM$, and χ is the image in $TN/\mathcal{D}$ of the generator X_χ of the S^1-action.

2.1.3 Toric sympectric/Kähler geometry. Let (M, ω) be a symplectic manifold, i.e. ω is nondegenerate and closed 2-form, and let the torus $\mathbb{T}^k$ act effectively on (M, ω), i.e. the action preserves ω and the only element of $\mathbb{T}^k$ which acts as id_M is the identity of $\mathbb{T}^k$. Recall that for any vector a in the Lie algebra $\mathfrak{t}$ of $\mathbb{T}^k$ there is a *fundamental vector field* X_a on M given by

$$X_a(x) = \frac{d}{dt}\Big|_{t=0} (exp(ta) \cdot x)\,, \tag{2.1.12}$$

where $\cdot$ represents the action. We say that the action is *hamiltonian* if there is a $\mathbb{T}^k$-invariant smooth map $\mu_M : M \to \mathfrak{t}^*$, called a *momentum map* for the action, such that

$$d\langle \mu_M, a\rangle = -\iota_{X_a}\omega \qquad \forall\, a \in \mathfrak{t}. \tag{2.1.13}$$

Definition 2.1.4. A *toric* symplectic manifold is a compact connected symplectic manifold (M, ω) together with an effective Hamiltonian action ρ of the torus $\mathbb{T}^m$, where $2m = \mathrm{dim}M$.
Two toric symplectic manifolds $(M_i, \omega_i, \mathbb{T}_i, \rho_i)$, $i = 1, 2$, are isomorphic if there exist a Lie-group isomorphism $\phi : \mathbb{T}_1 \to \mathbb{T}_2$ and a diffeomorphism $\Phi : M_1 \to M_2$ with $\Phi^*\omega_2 = \omega_1$ such that

$$\begin{array}{ccc} M_1 & \xrightarrow{\rho_1(g)} & M_1 \\ {\scriptstyle\Phi}\downarrow & & \downarrow{\scriptstyle\Phi} \\ M_2 & \xrightarrow[\rho_2(g)]{} & M_2 \end{array}$$

commutes for every $g \in \mathbb{T}_1$.

The image of the momentum map of a toric symplectic geometry is a compact convex polytope, called the rational/integral Delzant polytope (see [38, 36, 37, 16]) as described in

Definition 2.1.5. Let $\mathfrak{t}$ be an m-dimensional real vector space. Then a *rational Delzant polytope* $(\Delta, \Lambda, L_1, \ldots, L_n)$ in $\mathfrak{t}^*$ is a compact convex polytope $\Delta \subset \mathfrak{t}^*$ equipped with normals u_j, $j = 1, \ldots, n$, belonging to a lattice $\Lambda \subset \mathfrak{t}$ such that

$$\Delta = \{x \in \mathfrak{t}^* \mid L_j(x) \geq 0, \;\; j = 1, \ldots, n\} \tag{2.1.14}$$
$$L_j(x) = \langle u_j, x \rangle + \lambda_j \tag{2.1.15}$$

for some $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$, and such that for any vertex $x \in \Delta$, the u_j with $L_j(x) = 0$ form a basis for $\mathfrak{t}$. If the normals form a basis for Λ at each vertex, then Δ is said to be *integral.*
The natural action of the affine group $\mathrm{Aff}(\mathfrak{t}^*)$ provides the notion of isomorphism of rational (integral) Delzant polytopes.

The rational Delzant theorem [28, 37] states that, up to an isomorphism, toric symplectic orbifolds are classified by rational Delzant polytopes (with manifolds corresponding to integral Delzant polytopes). Given such a polytope, (M, ω) is obtained as a symplectic quotient of $\mathbb{C}^n$ by an $(n-m)$-dimensional subgroup G of the standard n-torus $(S^1)^n = \mathbb{R}^n/2\pi\mathbb{Z}^n$: G is the kernel of the map $(S^1)^n \to \mathbb{T}^m = \mathfrak{t}/2\pi\Lambda$ induced by the map $(x_1, \ldots, x_n) \mapsto \sum_{j=1}^n x_j u_j$ from $\mathbb{R}^n$ to $\mathfrak{t}$, and the momentum level for the symplectic quotient is the image of $(\lambda_1, \ldots, \lambda_n) \in (\mathbb{R}^n)^*$ in $\mathrm{Lie}(G)^*$ under the transpose of the natural inclusion $\mathrm{Lie}(G) \hookrightarrow \mathbb{R}^n = \mathrm{Lie}((S^1)^n)$.

Complementary coordinates to the momentum map are angle coordinates which arise from the extension of the $\mathbb{T}^m$-action on a symplectic toric manifold to the holomorphic action of the algebraic torus $\mathbb{T} = (\mathbb{C}^\times)^m$ by choosing a compatible Kähler metric. Fixing a point $p_0 \in M^0$ we can identify M^0 with the orbit $\mathbb{T}(p_0) = (\mathbb{C}^\times)^m$ and using polar coordinates (r_i, t_i) on each $\mathbb{C}^\times$ we get the *angle coordinates*

$$t = (t_1, \ldots, t_m) : M^0 \to \mathbb{T}^m, \tag{2.1.16}$$

where $M^0 \subset M$ is the open dense subset where the $\mathbb{T}^m$-action is free (for transformation groups see [20]).

On $M^0 = \mu_M^{-1}(\Delta^0)$ (see [28]), Δ^0 being the interior of Δ, compatible Kähler metrics on toric symplectic orbifolds have an explicit description in momentum-angle coordinates due to [38, 36, 37]. In these coordinates we have

$$\omega = \langle d\mu_M \wedge dt \rangle. \tag{2.1.17}$$

Furthermore, they identify each tangent space with $\mathfrak{t}^* \oplus \mathfrak{t}$, and so any $\mathbb{T}^m$-invariant ω-compatible Kähler metric is given by

$$g = \langle d\mu_M, \mathbf{G}, d\mu_M \rangle + \langle dt, \mathbf{H}, dt \rangle, \tag{2.1.18}$$

where $\mathbf{G}$ is a positive definite $S^2\mathfrak{t}$-valued function on Δ^0, $\mathbf{H}$ is its inverse in $S^2\mathfrak{t}^*$, and $\langle -, -, - \rangle$ denotes the pointwise contraction $\mathfrak{t}^* \times S^2\mathfrak{t} \times \mathfrak{t}^* \to \mathbb{R}$ or the dual contraction. The corresponding complex structure is given by

$$Jdt = -\langle \mathbf{G}, d\mu_M \rangle \tag{2.1.19}$$

and it follows that J is integrable if and only if $\mathbf{G}$ is the hessian of a function on Δ^0.

In order to understand how a Kähler metric compactifies from M^0 to M we have

Theorem 2.1.3 ([11])**.** *Let (M, ω) be a compact toric symplectic $2m$-manifold or orbifold with momentum map $\mu_M : M \to \Delta \subset \mathfrak{t}^*$ and $\boldsymbol{H}$ be a positive definite $S^2\mathfrak{t}^*$-valued function on Δ^0. Then $\boldsymbol{H}$ comes from a $\mathbb{T}$-invariant, ω-compatible Kähler metric via* (2.1.18) *if and only if it satisfies the following conditions:*

- [smoothness] *$\boldsymbol{H}$ is the restriction to Δ^0 of a smooth $S^2\mathfrak{t}^*$-valued function on Δ;*
- [boundary values] *for any point y on the codimension one face $F_j \subset \Delta$ with inward normal u_j, we have*

$$\boldsymbol{H}_y(u_j, -) = 0 \qquad \text{and} \qquad (d\boldsymbol{H})_y(u_j, u_j) = 2u_j, \tag{2.1.20}$$

  where the differential $d\boldsymbol{H}$ is viewed as a smooth $S^2\mathfrak{t}^* \otimes \mathfrak{t}$-valued function on Δ;
- [positivity] *for any point y in interior of a face $F \subset \Delta$, $\boldsymbol{H}_y(-,-)$ is positive definite when viewed as a smooth function with values in $S^2(\mathfrak{t}/\mathfrak{t}_F)^*$, where $\mathfrak{t}_F \subset \mathfrak{t}$ is the vector subspace spanned by the normals $u_j \in \mathfrak{t}$ to all codimension one faces of Δ containing F.*

Recall that a metric g is extremal iff $J\mathrm{grad}_g\mathrm{Scal}(g)$ is a Killing vector field, and note $\mathrm{grad}_\omega\mathrm{Scal}(g) = J\mathrm{grad}_g\mathrm{Scal}(g)$. In the toric case, $\mathrm{Scal}(g)$ is torus invariant, and thus its symplectic gradient commutes with the Killing vector fields from the torus action. However the torus is maximal and hence the symplectic gradient is in their span. Note that the definition of the momentum map shows that these Killing vector fields are symplectic gradients of momentum coordinates. Thus the extremality equation for toric Kähler geometry says that $\mathrm{Scal}(g)$ is affine linear function of μ_M. This affine linear function is an important intrinsic characteristic of a Delzant polytope as explained in

Proposition 2.1.1. *Suppose Δ is a Delzant polytope. Then, there exists a unique affine linear function ζ_Δ on $\mathfrak{t}^*$, called the extremal affine function of Δ, such that for any affine linear function f*

$$2\int_{\partial\Delta} f d\sigma - \int_\Delta \zeta_\Delta f dv = 0, \tag{2.1.21}$$

where $dv = d\mu_1 \wedge \cdots \wedge d\mu_m$ is the standard Lebesgue measure on $\mathfrak{t}^*$, and $d\sigma$ is the measure induced on each facet $F_i \subset \partial\Delta$ by $dL_i \wedge d\sigma = -dv$. Furthermore, if g is extremal, i.e. satisfies

$$Scal(g) = s = \langle \mu_M, \zeta \rangle + \zeta_0 \tag{2.1.22}$$

then the affine linear function s must be equal to ζ_Δ.

Remark 2.1.1. *Observe that by introducing $\mu_0 = 1$, we have $\zeta_\Delta = \langle(\mu_0, \mu_M), (\zeta_0, \zeta)\rangle$.*

Equivalently, ζ_Δ is the $L^2(\Delta, dv)$-projection of $\mathrm{Scal}(g)$ (see [42]). Thus, it is the unique solution $\zeta = (\zeta_0, \dots, \zeta_m)$ of the linear system

$$W\zeta = Z \tag{2.1.23}$$

$$\text{with} \quad W_{ij} = \int_\Delta (\mu_M)_i(\mu_M)_j dv \quad \text{and} \quad Z_i = 2\int_{\partial\Delta} (\mu_M)_i d\sigma, \tag{2.1.24}$$

where we have $Z_i = \int_\Delta \mathrm{Scal}(g)(\mu_M)_i dv = 2\int_{\partial\Delta}(\mu_M)_i d\sigma$.

2.1.4 Toric contact geometry. Let $(N, \mathcal{D}, \mathfrak{h}/2\pi\Lambda)$ be a $(2m+1)$-dimensional contact manifold with the action of $(m+1)$-dimensional torus $\mathbb{T}^{m+1} = \mathfrak{h}/2\pi\Lambda$ which preserves the contact structure, i.e. N is *toric contact manifold* and each element of $\mathfrak{h}$ acts as a contact vector field. As before, we have angle coordinates $\tau : N^0 \to \mathbb{T}^{m+1}$ satisfying $(d\tau)(\tilde{a}) = a \in \mathfrak{h}$, where $N^0 \subset N$ is the open dense subset where the action is free.

We assume that the tautological bundle homomorphism

$$\begin{aligned} N \times \mathfrak{h} &\to TN/\mathcal{D} \\ (p, a) &\mapsto \xi_a(p) \end{aligned} \tag{2.1.25}$$

is surjective, and define the *momentum section* $\mu : N \to \mathfrak{h}^* \otimes (TN/\mathcal{D})$ by $\langle a, \mu(p)\rangle = \xi_a(p)$ for each $a \in \mathfrak{h}$ and $p \in N$. Using the transpose of (2.1.25) we define the *momentum map* $[\mu] : N \to \mathbb{P}(\mathfrak{h}^*)$ by $[\mu](p)$ being the image of $(TN/\mathcal{D})^*_p$ in $\mathfrak{h}^*$. Hence $[\mu]^*\mathcal{O}_{\mathfrak{h}^*}(-1) \cong (TN/\mathcal{D})^*$ and if $z : \mathbb{P}(\mathfrak{h}^*) \to \mathfrak{h}^* \otimes \mathcal{O}_{\mathfrak{h}^*}(1)$ denotes the tautological section, then under the isomorphism $[\mu]^*\mathcal{O}_{\mathfrak{h}^*}(1) \cong TN/\mathcal{D}$ we have $\mu = [\mu]^* z$, where $\mathcal{O}_{\mathfrak{h}^*}(-1)$ is the tautological line bundle over $\mathbb{P}(\mathfrak{h}^*)$ and $\mathcal{O}_{\mathfrak{h}^*}(1) := \mathcal{O}_{\mathfrak{h}^*}(-1)^*$.

Any non-zero $\beta \in \mathfrak{h}$ determines an affine chart

$$\mathcal{A} = \{v \in \mathfrak{h}^* \mid \langle \beta, v\rangle = 1\} \hookrightarrow \mathbb{P}(\mathfrak{h}^*) \tag{2.1.26}$$

and $\langle \beta, z\rangle$ restricts to a trivialisation of $\mathcal{O}_{\mathfrak{h}^*}(1)\big|_U$ on any open subset $U \subset \mathbb{P}(\mathfrak{h}^*)$ of the image of $\mathcal{A}$. Thus we have $\frac{z}{\langle\beta,z\rangle} : U \to \mathcal{A}$ and $\langle \beta, d\frac{z}{\langle\beta,z\rangle}\rangle = 0$, i.e.

$$d\frac{z}{\langle\beta, z\rangle} : TU \to U \otimes \beta^0, \qquad \text{where} \quad \beta^0 = \{v \in \mathfrak{h}^* \mid \langle\beta, v\rangle = 0\}, \tag{2.1.27}$$

is the trivialisation of TU in this affine chart. In this setting, $\mathfrak{h}$ can be naturally identified with the space of affine function on U: $v \in \mathfrak{h}$ defines the affine function $\langle v, \frac{z}{\langle\beta,z\rangle}\rangle$ with β corresponding to the constant function 1. The duality $\mathfrak{t}^* \cong \beta^0$ provided by the short exact sequence

$$0 \to \mathrm{span}\{\beta\} \to \mathfrak{h} \xrightarrow{\delta} \mathfrak{t} \to 0 \tag{2.1.28}$$

identifies $\mathfrak{t}$ with T^*_pU for any $p \in U$, while the quotient map δ sends an affine function to (the constant value of) its derivative.

The momentum map $[\mu] : N \to \mathbb{P}(\mathfrak{h}^*)$ takes values in the affine chart determined by

β if and only if $\xi_\beta \in \mathfrak{con}(N, \mathcal{D})$ is a non-vanishing section of $TN/\mathcal{D}$. From the definition of momentum section, for $a \in \mathfrak{h}$ we have

$$\eta_{\mathcal{D}}(X_a) = \xi_a = \langle a, \mu \rangle = \langle (d\tau)(X_a), \mu \rangle, \tag{2.1.29}$$

which implies

$$\eta_{\mathcal{D}} = \langle \mu, d\tau \rangle, \tag{2.1.30}$$

since complementary vector field satisfy (2.1.29) trivially. Hence, in the trivialisation provided by β we get

$$\eta_\beta = \left\langle \frac{\mu}{\langle \mu, \beta \rangle}, d\tau \right\rangle = \frac{\eta_{\mathcal{D}}}{\langle \mu, \beta \rangle} \tag{2.1.31}$$

where we write η_β instead of $\eta_{\langle \mu, \beta \rangle}$. Using Cartan's magic identity it is straightforward to observe that $d\eta_\beta$ is X_β-basic, and thus descends to the quotient M_β of N by the vector field X_β where it plays a role of symplectic form ω_β. We claim that the momentum map μ_β for M_β is given by

$$\mu_\beta = \frac{\mu}{\langle \mu, \beta \rangle} \tag{2.1.32}$$

Indeed for $a \in \mathfrak{t}$ we have

$$d\langle \mu_\beta, a \rangle = d\left\langle \frac{\mu}{\langle \mu, \beta \rangle}, a \right\rangle = -\iota_{X_a}\omega_\beta \tag{2.1.33}$$

We showed that (M_β, ω_β) is a toric symplectic geometry with the momentum map μ_β given by (2.1.32) which also shows that its image, being a Delzant polytope, is the affine slice of the image of μ, considered as a projectivisation of a convex polyhedral cone (see [44]), by the affine chart given by β. In the case of a toric geometry of Sasaki type we obtain toric Kähler metrics by the construction described below Definition 2.1.3 with relations between momentum maps and their images as in the contact-symplectic case above.

2.2 Schubert variety

In this section we show that sets $\mathcal{U}_d$ (2.2.4), which arise in the theory of factorization structures investigated in the next chapter, are Zariski-closed. A convenient way how to prove this is to show that these are Schubert varieties.

We denote $Gr(k, V)$ the Grassmannian of k-dimensional subspaces of an n-dimensional vector space V equipped with Zariski topology, and $\mathcal{V}$ a complete flag in V, i.e. a nested sequence of subspaces

$$0 \subset F_1 \subset \cdots \subset F_{n-1} \subset F_n = V \tag{2.2.1}$$

with $\dim F_i = i$. We define the *Schubert variety* $\Sigma_a(\mathcal{V})$ indexed by a sequence of integers

$a = (a_1, \ldots, a_k)$ with

$$n - k \geq a_1 \geq a_2 \cdots \geq a_k \geq 0 \tag{2.2.2}$$

to be the Zariski-closed set

$$\Sigma_a(\mathcal{V}) = \{\Lambda \in Gr(k, V) \mid \dim(F_{n-k+i-a_i} \cap \Lambda) \geq i \quad \text{for} \quad i = 1, \ldots, k\}. \tag{2.2.3}$$

Lemma 2.2.1. *For $n \geq 3$ suppose $1 < k < n$, $1 \leq s < n$, and $1 \leq d \leq min\{s, k\}$ are such that $n - k + d - s \geq 0$. If $S \subset V$ is an s-dimensional subspace then the set*

$$\mathcal{U}_d = \{\Lambda \in Gr(k, V) \mid dim(S \cap \Lambda) \geq d\} \tag{2.2.4}$$

is the Schubert variety $\Sigma_a(\mathcal{V})$, where $a_1 = \cdots = a_d = n - k + d - s$, $a_j = 0$ for $j = d + 1, \ldots, k$, and $\mathcal{V}$ is an arbitrary complete flag in V such that $F_{n-k+d-a_d} = S$. In particular, $\mathcal{U}_d$ is Zariski-closed.

Proof. First, note if $s = 1$, then $d = 1$ and the process described below becomes trivial up to a point where the completion $\mathcal{V}$ is chosen.

If (2.2.4) is to be a Schubert variety, then (2.2.3) forces $S = F_{n-k+d-a_d}$ which implies

$$s = \dim(S) = \dim(F_{n-k+d-a_d}) = n - k + d - a_d. \tag{2.2.5}$$

We extend S into a complete flag $\mathcal{V}$ in V so that $\mathcal{U}_d$ becomes a Schubert variety $\Sigma_a(\mathcal{V})$ for some a.

Observe that for every hyperplane H in S and every $\Lambda \in \mathcal{U}_d$ we have $\dim(H \cap \Lambda) \geq d - 1$. Thus, for any such a fixed hyperplane H we have that $\mathcal{U}_d$ equals to

$$\{\Lambda \in Gr(k, V) \mid \dim(S \cap \Lambda) \geq d \ \text{ and } \ \dim(H \cap \Lambda) \geq d - 1\}. \tag{2.2.6}$$

To proceed in defining the flag $\mathcal{V}$ we set $F_{n-k+d-1-a_{d-1}} = H$. By taking dimensions we see $n - k + d - 1 - a_{d-1} = s - 1$ which together with (2.2.5) yields $a_d = a_{d-1}$.

We observe once again that for any codimension 1 subspaces G in H the set $\mathcal{U}_d$ equals to

$$\{\Lambda \in Gr(k, V) \mid \dim(S \cap \Lambda) \geq d \ \text{ and } \ \dim(H \cap \Lambda) \geq d - 1 \ \text{ and } \ \dim(G \cap \Lambda) \geq d - 2\}. \tag{2.2.7}$$

We define $F_{n-k+d-2-a_{d-2}} = G$ which gives $a_d = a_{d-1} = a_{d-2}$ and proceed in the similar way until we obtain a partial flag

$$0 \subset F_1 \subset F_2 \subset \cdots \subset F_{s-2} = G \subset F_{s-1} = H \subset F_s = S \subset V \tag{2.2.8}$$

with $\dim F_i = i$ and $a_d = \cdots = a_1$. Furthermore, $\mathcal{U}_d$ coincides with

$$\mathcal{U}_d = \{\Lambda \in Gr(k, V) \mid \dim(F_{s-(d-i)} \cap \Lambda) \geq i \quad \text{for} \quad i = 1, \ldots, d\} \tag{2.2.9}$$

as before.

Now consider an arbitrary completion $\mathcal{V}$:

$$0 \subset F_1 \subset F_2 \subset \cdots \subset F_{s-1} = H \subset F_s = S \subset F_{s+1} \subset \cdots \subset F_n \subset V \tag{2.2.10}$$

of (2.2.8) into a complete flag. If $d < k$ we set $a_i = 0$ for $i = d+1, \ldots, k$, and then $\dim(F_{n-k+i} \cap \Lambda) \geq i$ are trivially satisfied. This shows $\mathcal{U}_d = \Sigma_a(\mathcal{V})$ for any complete flag $\mathcal{V}$ such that $S = F_{n-k+d-a_d}$. □

Chapter 3

Factorization structures: Structure theory

The main topic of this chapter are factorization structures. The only place of their occurrence in the literature is in [7] where were used in dimension 2 to establish what rational Delzant quadrilaterals arise as compactifications of ambitoric geometries. Here we analyse them in full generality. To study their structure we introduce quotients. However, it is not straightforward to show that quotients are factorization structures, so we define weak factorization structures which are easier to work with. Eventually we show that weak factorization structures are factorization structures. First indications of this phenomenon can be seen in examples of weak factorization structures we find. All of them are factorization structures and include a crucial class of model examples called Segre-Veronese factorization structures. The first examples of (weak) factorization structures occur in dimension 2. In fact, there are two isomorphism classes of these, called Segre and Veronese factorization structures, as it was found in [7]. Further exploration of examples motivates notions of quotient and product of weak factorization structures. The quotient construction together with an inductive argument with respect to the dimension of a weak factorization structure, starting from the dimension 2, is used as a main technique to show that weak factorization structures are factorization structures.

As described in Chapter 1 (Introduction) a factorization structure is, in particular, a map $\mathbb{P}(V_1) \times \cdots \times \mathbb{P}(V_m) \to \mathbb{P}(\mathfrak{h}^*)$ from a product of m projective lines to projective m-space such that a generic coordinate hypersurface is sent onto a hyperplane. Thus, for each $\ell_j \in \mathbb{P}(V_j)$ we get a hyperplane in $\mathbb{P}(\mathfrak{h}^*)$ which we represent by its normal $[\psi_j](\ell_j) \in \mathbb{P}(\mathfrak{h})$. This way we obtain factorization curves $[\psi_j] : \mathbb{P}(V_j) \to \mathbb{P}(\mathfrak{h})$, $j = 1, \ldots, m$. These are injective regular maps, have degrees and can be decomposable. In particular, factorization curves in factorization structures of Segre-Veronese type are rational normal curves in subspaces they span in the usual sense of algebraic geometry. Factorization curves carry substantial information on the underlying factorization structure. For example, once we describe the behaviour of degree in relation to quotient factorization structures, we prove that a factorization curve of degree m (top degree) is decomposable and determines the whole factorization structure to be of Veronese type. In addition, we show that a factorization structure has decomposable factorization curves if and only if it is of Segre-Veronese type. This underlines the importance of factorization curves in the study of factorization structures. We conjecture all factorization curves are decomposable, or equivalently, all factorization structures are of Segre-Veronese type. We believe the language of factorization curves provides the right context for this conjecture.

3.1 Definition, examples and some properties

In this section we define (weak) factorization structures, classify them in two dimensions, and explore some examples.

Let $m \geq 2$ be an integer and let $V_1, \ldots, V_m$ be real/complex vector spaces with dimension $|V_j| = 2$. For a fixed $j \in \{1, \ldots, m\}$ and any 1-dimensional subspace $\ell \subset V_j$ we define

$$\Sigma_{j,\ell} = V_1 \otimes \cdots \otimes V_{j-1} \otimes \ell \otimes V_{j+1} \otimes \cdots \otimes V_m \tag{3.1.1}$$

and denote its annihilator in $V^* := V_1^* \otimes \cdots \otimes V_m^*$ by $\Sigma^0_{j,\ell}$. We denote the product $V_1 \otimes \cdots V_{j-1} \otimes V_{j+1} \otimes \cdots \otimes V_m$ by $\hat{V}_j$. Furthermore, we shall use the linear operator $ins_j : V_j^* \otimes \hat{V}_j^* \to V^*$ defined on decomposable tensors by

$$v_j \otimes (v_1 \otimes \cdots \otimes v_{j-1} \otimes v_{j+1} \otimes \cdots \otimes v_m) \mapsto v_1 \otimes \cdots \otimes v_{j-1} \otimes v_j \otimes v_{j+1} \otimes \cdots \otimes v_m, \tag{3.1.2}$$

i.e. ins_j inserts the first slot into the missing slot of $\hat{V}_j^*$. For example, $\Sigma^0_{j,\ell_j} = ins_j(\ell_j^0 \otimes \hat{V}_j^*)$. We consider the projective space $\mathbb{P}(V_k)$ to be the set of 1-dimensional subspaces ℓ_k in the vector space V_k equipped with Zariski topology. We say a condition holds for a *generic* point or generically if there exists an open non-empty subset $U \subset \mathbb{P}(V_k)$ such that for each point in U the condition holds.

Definition 3.1.1. An inclusion (injective linear mapping) $\varphi : \mathfrak{h} \to V^*$ of a real/complex $(m+1)$-dimensional vector space $\mathfrak{h}$ into real/complex V^* is called a weak factorization structure of dimension m if

$$d_k(\ell) := |\varphi(\mathfrak{h}) \cap \Sigma^0_{k,\ell}| \geq 1 \tag{3.1.3}$$

holds for every $k \in \{1, \ldots, m\}$ and generic $\ell \in \mathbb{P}(V_k)$. If $d_k(\ell) = 1$ for every k and generic ℓ, then the weak factorization structure is called a factorization structure. An isomorphism between two weak factorization structures is the commutative diagram

$$\begin{array}{ccc} \mathfrak{h}_1 & \xrightarrow{\quad\Phi\quad} & \mathfrak{h}_2 \\ {\scriptstyle \varphi_1}\downarrow & & \downarrow{\scriptstyle \varphi_2} \\ V_1^* \otimes \cdots \otimes V_m^* & \xrightarrow{\phi_1 \otimes \cdots \otimes \phi_m} & W_1^* \otimes \cdots \otimes W_m^* \end{array}$$

where Φ and $\phi_j : V_j^* \to W_j^*$ are linear isomorphisms for all $j \in \{1, \ldots, m\}$.

A factorization structure gives rise to the rational map $\mathbb{P}(V_1)\times\cdots\times\mathbb{P}(V_m) \dashrightarrow \mathbb{P}(\mathfrak{h}^*)$ defined by the composition of the Segre embedding,

$$\begin{aligned} S : \mathbb{P}(V_1) \times \cdots \times \mathbb{P}(V_m) &\to \mathbb{P}(V) \\ ([v_1], \ldots, [v_m]) &\mapsto [v_1 \otimes \cdots \otimes v_m], \end{aligned} \tag{3.1.4}$$

and projectivised $\varphi^T : V \to \mathfrak{h}^*$. Clearly, for any $\ell \in \mathbb{P}(V_k)$ S maps the coordinate hyperplane

$$\mathbb{P}(V_1) \times \cdots \times \mathbb{P}(V_{k-1}) \times \{\ell\} \times \mathbb{P}(V_{k+1}) \times \cdots \times \mathbb{P}(V_m) \tag{3.1.5}$$

onto the projective subspace $\mathbb{P}(\Sigma_{k,\ell})$ which is further mapped onto the projective space $\mathbb{P}(\varphi^T\Sigma_{k,\ell})$. As outlined in the Introduction we want $\mathbb{P}(\varphi^T\Sigma_{k,\ell})$ to be a hyperplane so the rational map gives the desired coordinate system on an open subset of $\mathbb{P}(\mathfrak{h}^*)$. Suppose $\mathbb{P}(\varphi^T\Sigma_{k,\ell})$ is a hyperplane and, for now, denote $\mathbb{P}(\psi_k(\ell)) \in \mathbb{P}(\mathfrak{h})$ the projective normal corresponding to it. Then, for their natural contraction we have

$$0 = \langle \varphi^T\Sigma_{k,\ell}, \psi_k(\ell)\rangle = \langle \Sigma_{k,\ell}, \varphi \circ \psi_k(\ell)\rangle, \tag{3.1.6}$$

and thus $\varphi\circ\psi_k(\ell) \subset \Sigma^0_{k,\ell}\cap\varphi(\mathfrak{h})$. Now, if $\Sigma^0_{k,\ell}\cap\varphi(\mathfrak{h})$ would be more than 1-dimensional, then the first equality in (3.1.6) shows that $\varphi^T\Sigma_{k,\ell}$ is annihilated by more than 1-dimensional space and hence it could not be a hyperplane which contradicts assumptions. Thus, on an open subset we have $\varphi \circ \psi_k(\ell) = \Sigma^0_{k,\ell} \cap \varphi(\mathfrak{h})$. This is how the description of factorization structures from the Introduction fits Definition 3.1.1.

Remark 3.1.1. *All results of this chapter hold for real and complex (weak) factorization structures. Therefore, no distinction between these is made.*

Lemma 3.1.1. *For $m = 2$, $\varphi : \mathfrak{h} \to V_1^* \otimes V_2^*$ is a weak factorization iff it is a factorization structure.*

Proof. It follows

$$2 \geq |\varphi(\mathfrak{h}) \cap \ell_1^0 \otimes V_2^*| = |\varphi(\mathfrak{h}) \cap \Sigma^0_{1,\ell_1}| \geq 1$$

for generic $\ell_1 \in \mathbb{P}(V_1)$. If this intersection were 2-dimensional in two distinct points $\ell_1, \ell \in \mathbb{P}(V_1)$, then two 2-dimensional subspaces $\ell_1^0 \otimes V_2^*$ and $\ell^0 \otimes V_2^*$ with the trivial intersection must lie in the three dimensional space $\varphi(\mathfrak{h})$ which is a contradiction. Therefore the intersection is two dimensional at most at one point and hence is generically one dimensional.
The other intersection is similar. □

Example 3.1.1 ($m = 2$)**.** Two-dimensional factorization structures were classified in [7]. To recall this classification note that the image of a factorization structure $\mathfrak{h} \to V_1^* \otimes V_2^*$ is the annihilator of an element $\chi \in V_1 \otimes V_2$.

If $\langle\chi\rangle = \langle\gamma_1 \otimes \gamma_2\rangle$ is decomposable, the image of $\mathfrak{h}$ is $V_1^* \otimes \gamma_2^0 + \gamma_1^0 \otimes V_2^*$, where $\gamma_j^0 \subset V_j^*$ is the annihilator of the 1-dimensional subspace $\langle\gamma_j\rangle \subset V_j$. Clearly, this is a factorization structure as intersections

$$(V_1^* \otimes \gamma_2^0 + \gamma_1^0 \otimes V_2^*) \cap \ell_1^0 \otimes V_2^*$$

and

$$(V_1^* \otimes \gamma_2^0 + \gamma_1^0 \otimes V_2^*) \cap V_1^* \otimes \ell_2^0$$

are one dimensional for generic values $\ell_1 \in \mathbb{P}(V_1)$ and $\ell_2 \in \mathbb{P}(V_2)$, and $|V_1^* \otimes \gamma_2^0 + \gamma_1^0 \otimes V_2^*| = 3$.

To analyse the case when χ is indecomposable let e_1 and e_2 span V_1, and let f_1 and f_2 span V_2, and let $\{E^1, E^2\}$ and $\{F^1, F^2\}$ be the dual bases respectively. χ, viewed as a map from V_1^* to V_2, is invertible and hence, by fixing any area form $\omega = c(E^1 \otimes E^2 - E^2 \otimes E^1)$ on V_1, $c \in \mathbb{R}^\times$, defines an isomorphism $\omega \otimes \chi^{-1} : V_1 \otimes V_2 \to V_1^* \otimes V_1^*$. Under this isomorphism the image of $\mathfrak{h}$ is annihilated by $span\{c(E^1 \otimes E^1 - E^2 \otimes E^2)\}$ which can be further transformed by $T \otimes Id$ into $\bigwedge^2 V_1^*$, where $T = e_1 \otimes E_2 + e_2 \otimes E^1$ is considered as an automorphism of V_1^*. This means that the image of $\mathfrak{h}$ is isomorphic in the sense of (weak) factorization structures to $S^2 W^*$, where $|W| = 2$. Again, this is a factorization structure as $S^2W^* \cap \ell^0 \otimes W^*$ and $S^2W^* \cap W^* \otimes \ell^0$ are (generically) one dimensional, and $|S^2W^*| = 3$.

By generalising the first of the two examples from above we get

Example 3.1.2 (**Segre**). Let $\langle \Gamma_j \rangle \subset \hat{V}_j^*$, $j \in \{1, \ldots, m\}$, be such that the sum $\varphi(\mathfrak{h}) := \sum_{j=1}^m ins_j \left(V_j^* \otimes \langle \Gamma_j \rangle \right)$ has dimension $m+1$. The inclusion $\varphi : \mathfrak{h} \to V^*$ is a factorization structure as $\varphi(\mathfrak{h}) \cap \Sigma^0_{j,\ell_j} = ins_j \left(\ell_j^0 \otimes \langle \Gamma_j \rangle \right)$ is one dimensional for a generic ℓ_j. For example, for each $b \in \{1, \ldots, m\}$ we can choose a 1-dimensional subspace $\langle v_b \rangle \subset V_b^*$ and define $\langle \Gamma_j \rangle = \langle \otimes_{\substack{b=1 \\ b \neq j}}^m v_b \rangle$. When $V_1 = \cdots = V_m$ we can choose $\langle v_1 \rangle = \ldots = \langle v_m \rangle$.

Similarly, a generalisation of the second example in Example 3.1.1 is

Example 3.1.3 (**Veronese**). Let W be a 2-dimensional vector space. The natural inclusion $S^m W^* \to \otimes^m W^*$ is a factorization structure. Indeed, $|S^m W^*| = m+1$ and for $\ell \in \mathbb{P}(W)$ the dimension of $S^m W^* \cap \Sigma^0_{j,\ell} = \underbrace{\ell^0 \otimes \cdots \otimes \ell^0}_{m\text{-times}}$ is one for any $j \in \{1, \cdots, m\}$.

For easier description of a position of an element in a tensor product we establish the following conventions. The position of a term in the tensor product of m elements is referred to as a slot, e.g. terms a, b and c in $a \otimes b \otimes c$ are in the first, second and third slot, respectively. For partition of m, $m = d_1 + \cdots + d_k$, $d_j \geq 1$, slots decompose into k groups with jth group containing d_j slots, $j \in \{1, \ldots, k\}$. Slots belonging to the jth group are referred to as grouped j-slots. In fact, positions in the tensor product of m elements can be labelled by pairs (j, r), where $j \in \{1, \ldots, k\}$ and $r \in \{1, \ldots, d_j\}$. For a partition of m as above and a fixed $j \in \{1, \ldots, k\}$ we define the operator

$$Ins_j : (W_j^*)^{\otimes d_j} \otimes \bigotimes_{\substack{i=1 \\ i \neq j}}^k (W_i^*)^{\otimes d_i} \to \bigotimes_{i=1}^k (W_i^*)^{\otimes d_i}$$

on decomposable tensors by

$$w_j^1 \otimes \cdots \otimes w_j^{d_j} \otimes \bigotimes_{\substack{i=1\\ i\neq j}}^{k} \left(w_i^1 \otimes \cdots \otimes w_i^{d_i}\right) \mapsto \bigotimes_{i=1}^{k} \left(w_i^1 \otimes \cdots \otimes w_i^{d_i}\right)$$

and extend it linearly. In other words, Ins_j inserts the first d_j slots into the missing slots in the same order. Note that for partition $m = 1 + \cdots + 1$ operators Ins_j and ins_j agree (see (3.1.2)). We regard Ins_j as a generalisation of ins_j and from now on we shall denote both by ins_j.

Combining the two examples above we find

Example 3.1.4 (**Segre-Veronese**)**.** Let $d_1 + \cdots + d_k = m$, where d_j are positive natural numbers. Let $\langle \Gamma_j \rangle \subset \bigotimes_{r=1, r\neq j}^{k} (W_r^*)^{\otimes d_r}$, $j \in \{1, \ldots, k\}$, be such that

$$\sum_{j=1}^{k} ins_j \left(S^{d_j} W_j^* \otimes \langle \Gamma_j \rangle\right) \tag{3.1.7}$$

has dimension $m+1$. Clearly, the intersections are generically one dimensional, thus it is a factorization structure. Note that when $d_1 = \cdots = d_k = 1$ or $k = 1$ we recover Segre or Veronese factorization structures respectively. To give another special case, for each $b \in \{1, \ldots, k\}$ we can choose a 1-dimensional subspace $\langle v_b \rangle \subset W_b^*$ and define $\langle \Gamma_j \rangle = \langle \bigotimes_{\substack{b=1\\ b\neq j}}^{k} (v_b)^{\otimes d_b} \rangle$. The factorization structure corresponding to these particular Γ_js is called the product factorization structure because it corresponds to the product of orthotoric geometries as observed later in the text. More generally, we say that a Segre-Veronese factorization structure is decomposable if for each $j \in \{1, \ldots, m\}$ we can write $\langle \Gamma_j \rangle = \langle \bigotimes_{\substack{b=1\\ b\neq j}}^{k} (v_j^b)^{\otimes d_b} \rangle$ for some $v_j^b \in W_b^*$.

Example 3.1.5 (**Product of (weak) factorization structures**)**.** Let $\varphi : \mathfrak{h} \to V_1^* \otimes \cdots \otimes V_m^*$ and $\chi : \mathfrak{g} \to W_1^* \otimes \cdots \otimes W_n^*$ be (weak) factorization structures, and let $\langle S \rangle \subset \chi(\mathfrak{g})$ and $\langle T \rangle \subset \varphi(\mathfrak{h})$ be one-dimensional subspaces. An $(\langle S \rangle, \langle T \rangle)$-product of these (weak) factorization structures is defined to be the canonical inclusion

$$\varphi(\mathfrak{h}) \otimes \langle S \rangle + \langle T \rangle \otimes \chi(\mathfrak{g}) \hookrightarrow V_1^* \otimes \cdots \otimes V_m^* \otimes W_1^* \otimes \cdots \otimes W_n^*.$$

This is a (weak) factorization structure. Indeed, it is an injection of $(m+n+1)$-dimensional vector space into a tensor product of $m+n$ 2-dimensional vector spaces, and hence dimensions follow Definition 3.1.1. Furthermore, as each intersection of the $(\langle S \rangle, \langle T \rangle)$-product with $\Sigma_{j,\ell_j}^0 \leq V_1^* \otimes \cdots \otimes V_m^* \otimes W_1^* \otimes \cdots \otimes W_n^*$ for $j \in \{1, \ldots, m+n\}$ reduces to an intersection with either $\varphi(\mathfrak{h})$ or $\chi(\mathfrak{g})$, depending on the index j, it is clear that the $(\langle S \rangle, \langle T \rangle)$-product satisfies (3.1.3) of Definition 3.1.1 or its stronger form in the case of factorization structures.

For example, the product Segre-Veronese factorization structure can be written as a product of two factorization structures in multiple ways. Let $I \subset \{1, \ldots, k\}$ be a non-trivial subset. Then we have decomposition

$$\left(\sum_{j\in I} ins_j\left(S^{d_j}W_j^*\otimes\bigotimes_{\substack{b\in I\\ b\neq j}}\langle v_b\rangle^{\otimes d_b}\right)\right)\otimes\bigotimes_{b\in I^c}\langle v_b\rangle^{\otimes d_b}+ \tag{3.1.8}$$

$$+\bigotimes_{b\in I}\langle v_b\rangle^{\otimes d_b}\otimes\left(ins_j\left(\sum_{j\in I^c}S^{d_j}W_j^*\otimes\bigotimes_{\substack{b\in I^c\\ b\neq j}}\langle v_b\rangle^{\otimes d_b}\right)\right) \tag{3.1.9}$$

where I^c stands for the complement of I.

Note that all these examples of weak factorization structures are in fact factorization structures. As we shall see in the of the following section this is not a coincidence. It turns out that any weak factorization structures is a factorization structure.

Using topological arguments we deduce that the generic condition (3.1.3) holds globally. More precisely we have

Lemma 3.1.2. *If $m\geq 2$, then for every $k\in\{1,\cdots,m\}$ the condition* (3.1.3) *from Definition 3.1.1 holds on the whole $\mathbb{P}(V_k)$ and d_k is constant on an open nonempty subset of $\mathbb{P}(V_k)$.*

Proof. For $m=2$ this was solved in Example 3.1.1.
Suppose $m=3$. At first, we shall show that the nested sets $U_1\supset U_2\supset\cdots$ defined by

$$U_d := \{[\ell_k]\in\mathbb{P}(V_k) : |\varphi(\mathfrak{h})\cap\Sigma^0_{k,\ell_k}|\geq d\}$$

are closed. Note if $d>m+1$, then $U_d=\emptyset$ and hence closed. Suppose now $1\leq d\leq m+1$. The assumptions of Lemma 2.2.1 are satisfied and hence

$$\mathcal{U}_d=\{\Lambda\in Gr(2^{m-1},V^*)\mid|\varphi(\mathfrak{h})\cap\Lambda|\geq d\} \tag{3.1.10}$$

is closed. Note that U_d is the preimage of $\mathcal{U}_d$ via regular map (polynomial mapping) $\mathbb{P}(V_k)\to Gr(2^{m-1},V^*)$ defined by $\ell_k\mapsto\Sigma^0_{k,\ell_k}$, and hence closed.

The set U_1 is open and non-empty by the definition of weak factorization structure, so $U_1=\mathbb{P}(V_k)$, and hence the condition (3.1.3) from Definition 3.1.1 holds on the whole $\mathbb{P}(V_k)$.
Define

$$U^d := U_d\backslash U_{d+1}=\{[\ell_k]\in\mathbb{P}(V_k) : |\varphi(\mathfrak{h})\cap\Sigma^0_{k,\ell_k}|=d\} \tag{3.1.11}$$

The set $U^1=U_1\backslash U_2=\mathbb{P}(V_k)\backslash U_2$ is open as U_2 is closed. Thus, if there exists $\ell_k\in\mathbb{P}(V_k)$ such that $|\varphi(\mathfrak{h})\cap\Sigma^0_{k,\ell_k}|=1$, i.e. $U^1\neq\emptyset$, then $|\varphi(\mathfrak{h})\cap\Sigma^0_{k,\ell_k}|=1$ holds for every $\ell_k\in U^1$, i.e. for a generic ℓ_k. This means that d_k is constant on an open nonempty set and has value 1.
However, if the set U^1 is empty, then $U_1\subset U_2$ which amounts to $\mathbb{P}(V_k)=U_1=U_2$. Now, the set $U^2=U_2\backslash U_3=\mathbb{P}(V_k)\backslash U_3$ is open as U_3 is closed. Again, if there exists

$\ell_k \in \mathbb{P}(V_k)$ such that $|\varphi(\mathfrak{h}) \cap \Sigma^0_{k,\ell_k}| = 2$, then $|\varphi(\mathfrak{h}) \cap \Sigma^0_{k,\ell_k}| = 2$ holds for a generic ℓ_k which means d_k is constant on an open nonempty set and has value 2.
Since $\mathfrak{h}$ is a weak factorization structure this process gives the claim before d exceeds $\dim(\mathfrak{h}) = m+1$. □

3.2 Weak factorization structures are factorization structures

The main motivation for quotient weak factorization structures, apart from the general concept of a quotient structure, is that inductive arguments with respect to the dimension of a weak factorization structure show that weak factorization structures are factorization structures, and, in particular, establish the notion of quotient factorization structure.

Examples from the previous section show that for a fixed j the contraction of a 1-dimensional subspace $\ell \subset V_j$ with $\varphi(\mathfrak{h}) \leq V_1^* \otimes \cdots \otimes V_m^*$ at the jth slot results in a factorization structure. For example, the contraction of $\ell \subset W$ with any slot of $S^m W^* \leq \otimes^m W^*$ is isomorphic with $S^{m-1}W^*$. There are multiple ways how to formalise this idea in the setting of weak factorization structures and we adopt the one which regards the aforementioned contractions as quotients. Based on these observations we shall define an object $\varphi_j(\ell_j) : \mathfrak{h}_j(\ell_j) \to \hat{V}_j^* \otimes \ell_j^*$ which represents a quotient weak factorization structure. However, it takes some time to show that this object is an actual weak factorization structure.

Let $\varphi : \mathfrak{h} \to V^*$ be a weak factorization structure. The following inclusion of short exact sequences defines $\varphi_j(\ell_j) : \mathfrak{h}_j(\ell_j) \to \hat{V}_j^* \otimes \ell_j^*$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \varphi^{-1}(\varphi(\mathfrak{h}) \cap \Sigma^0_{j,\ell_j}) & \longrightarrow & \mathfrak{h} & \xrightarrow{\pi_j^{\ell_j}} & \mathfrak{h}_j(\ell_j) & \longrightarrow & 0 \\ & & \big\downarrow \varphi & & \big\downarrow \varphi & & \big\downarrow \varphi_j(\ell_j) & & \\ 0 & \longrightarrow & \Sigma^0_{j,\ell_j} & \longrightarrow & V^* & \xrightarrow{\rho_j^{\ell_j}} & ins_j(\ell_j^* \otimes \hat{V}_j^*) & \longrightarrow & 0 \end{array} \tag{3.2.1}$$

where $V^*/\Sigma^0_{j,\ell_j} \cong \hat{V}_j^* \otimes (V_j^*/\ell_j^0) \cong \hat{V}_j^* \otimes \ell_j^*$ and $|\varphi(\mathfrak{h}) \cap \Sigma^0_{j,\ell_j}| = 1$. In the next exposition φ_j is used instead of $\varphi_j(\ell_j)$. Note that taking quotient by Σ^0_{j,ℓ_j} as well as making the contraction with $\ell_j \subset V_j$ at the jth slot gives the same result up to remembering ℓ_j^*.

Remark 3.2.1. *We intend $\varphi_j(\ell_j) : \mathfrak{h}_j(\ell_j) \to ins_j(\hat{V}_j^* \otimes \ell_j^*)$ to be a weak factorization structure. This means that it is an inclusion of m-dimensional vector space into a tensor product of $m-1$ 2-dimensional vector spaces with desired property on intersections (see Definition 3.1.1). At this point it is not clear what 2-dimensional spaces we work with since an extra ℓ_j^* is attached to $\hat{V}_j^*$. We fix this by defining a new 2-dimensional vector space $V_{j-1}^* \otimes \ell_j^*$ if the index $j-1$ is valid, otherwise the new vector space is defined to* *be* $V_{j+1}^* \otimes \ell_j^*$. *In what follows we keep the notation* $\hat{V}_j^* \otimes \ell_j^*$ *with the aforementioned meaning and drop the operator* ins_j.

An equivalent approach for fixing this issue would be to tensor (3.2.1) *with* ℓ_j *and consider the inclusion* $\mathfrak{h}_j(\ell_j) \otimes \ell_j \to \hat{V}_j^*$.

In order to understand if the quotient $\varphi_j(\ell_j) : \mathfrak{h}_j(\ell_j) \to \hat{V}_j^* \otimes \ell_j^*$ is a weak factorization structure the dimension of $\varphi_j(\mathfrak{h}_j) \cap \Sigma_{k,\ell_k}^{\wedge_j 0} \otimes \ell_j^*$ needs to be understood, where

$$\Sigma_{k,\ell_k}^{\wedge_j 0} \otimes \ell_j^* = V_1^* \otimes \cdots \otimes V_{k-1}^* \otimes \ell_k^0 \otimes V_{k+1}^* \otimes \cdots \otimes V_{j-1}^* \otimes \ell_j^* \otimes V_{j+1}^* \otimes \cdots \otimes V_m^*, \tag{3.2.2}$$

i.e. $\Sigma_{k,\ell_k}^{\wedge_j 0} \otimes \ell_j^*$ is Σ_{k,ℓ_k}^0 with ℓ_j^* in the j-th slot. These intersections are analogous to intersections $\varphi(\mathfrak{h}) \cap \Sigma_{k,\ell_k}^0$. Notice that for $k \neq j$, and any $\ell_k \in \mathbb{P}(V_k)$

$$\rho_j^{\ell_j}(\varphi(\mathfrak{h}) \cap \Sigma_{k,\ell_k}^0) \subset \rho_j^{\ell_j}(\varphi(\mathfrak{h})) \cap \rho_j^{\ell_j}(\Sigma_{k,\ell_k}^0) = \varphi_j(\mathfrak{h}_j) \cap \Sigma_{k,\ell_k}^{\wedge_j 0} \otimes \ell_j^* \tag{3.2.3}$$

If moreover $p \neq j$ and $\ell_p \in \mathbb{P}(V_p)$, then

$$\rho_j^{\ell_j}(\varphi(\mathfrak{h}) \cap \Sigma_{k,\ell_k}^0 \cap \Sigma_{p,\ell_p}^0) \subset \varphi_j(\mathfrak{h}_j) \cap \Sigma_{k,\ell_k}^{\wedge_j 0} \otimes \ell_j^* \cap \Sigma_{p,\ell_p}^{\wedge_j 0} \otimes \ell_j^* \tag{3.2.4}$$

Using rank-nullity theorem and the fact that $Ker\ \rho_j^{\ell_j} = \Sigma_{j,\ell_j}^0$, the equation (3.2.3) implies

$$|\varphi_j(\mathfrak{h}_j) \cap \Sigma_{k,\ell_k}^{\wedge_j 0} \otimes \ell_j^*| \geq |\rho_j^{\ell_j}(\varphi(\mathfrak{h}) \cap \Sigma_{k,\ell_k}^0)| = |\varphi(\mathfrak{h}) \cap \Sigma_{k,\ell_k}^0| - |\varphi(\mathfrak{h}) \cap \Sigma_{k,\ell_k}^0 \cap \Sigma_{j,\ell_j}^0| \tag{3.2.5}$$

Hence, $\varphi_j(\ell_j) : \mathfrak{h}_j(\ell_j) \to \hat{V}_j^* \otimes \ell_j^*$ is a weak factorization structure if the RHS is at least one. This amounts to knowing that any of $\varphi(\mathfrak{h}) \cap \Sigma_{k,\ell_k}^0, \varphi(\mathfrak{h}) \cap \Sigma_{j,\ell_j}^0$ is not a subspace of the other.

Lemma 3.2.1. *Let* $\varphi : \mathfrak{h} \to (V_1 \otimes \cdots \otimes V_m)^*$ *be a weak factorization structure and fix* $j \in \{1, \ldots, m\}$. *Then there exist an open nonempty set* $A_j \subset \mathbb{P}(V_j)$ *such that for each* $\ell_j \in A_j$ *and each* $k \in \{1, \ldots, m\} \backslash \{j\}$ *there exists an open nonempty set* $A_k \subset \mathbb{P}(V_k)$ *such that for each* $\ell_k \in A_k$

$$\varphi(\mathfrak{h}) \cap \Sigma_{j,\ell_j}^0 \not\subset \varphi(\mathfrak{h}) \cap \Sigma_{k,\ell_k}^0 \tag{3.2.6}$$

and

$$\varphi(\mathfrak{h}) \cap \Sigma_{k,\ell_k}^0 \not\subset \varphi(\mathfrak{h}) \cap \Sigma_{j,\ell_j}^0 \tag{3.2.7}$$

Proof. Fix $\ell_j \in \mathbb{P}(V_j)$ and $k \neq j$. Let $U_k \subset \mathbb{P}(V_k)$ be the open nonempty set where d_k attains its minimal value, say d. Such U_k exists by Lemma 3.1.2. By Lemma 2.2.1 the set

$$\mathcal{U}_d = \{\Lambda \in Gr(2^{m-1}, V^*) \mid |(\varphi(\mathfrak{h}) \cap \Sigma^0_{j,\ell_j}) \cap \Lambda| \geq d\} \tag{3.2.8}$$

is closed. We define

$$c_k(\ell_j) = \{\ell_k \in U_k \;\mid\; |\varphi(\mathfrak{h}) \cap \Sigma^0_{k,\ell_k} \cap \Sigma^0_{j,\ell_j}| \geq d\} \tag{3.2.9}$$

$$= \{\ell_k \in U_k \;\mid\; \varphi(\mathfrak{h}) \cap \Sigma^0_{k,\ell_k} \subset \varphi(\mathfrak{h}) \cap \Sigma^0_{j,\ell_j}\} \tag{3.2.10}$$

Note $c_k(\ell_j)$ is the preimage of $\mathcal{U}_d$ via regular map $\mathbb{P}(V_k) \to Gr(2^{m-1}, V^*)$ defined by $\ell_k \mapsto \Sigma^0_{k,\ell_k}$ and hence closed. Thus $c_k(\ell_j)$ is either equal to U_k, or it is closed and proper in U_k.

For each $k \neq j$ there is at most one $\ell_j \in \mathbb{P}(V_j)$ such that $c_k(\ell_j) = U_k$. Indeed, if there would exist two distinct $\ell_j, \hat{\ell}_j \in \mathbb{P}(V_j)$ such that $c_k(\ell_j) = U_k = c_k(\hat{\ell}_j)$, then for $\ell_k \in U_k$ we have $0 \neq \varphi(\mathfrak{h}) \cap \Sigma^0_{k,\ell_k} \subset \varphi(\mathfrak{h}) \cap \Sigma^0_{j,\ell_j} \cap \Sigma^0_{j,\hat{\ell}_j} = 0$ which is a contradiction.

Therefore there is an open nonempty set $A_j \subset \mathbb{P}(V_j)$ such that for each $\ell_j \in A_j$ and each $k \neq j$ the set

$$C_k(\ell_j) = \{\ell_k \in \mathbb{P}(V_k) \;\mid\; \varphi(\mathfrak{h}) \cap \Sigma^0_{k,\ell_k} \subset \varphi(\mathfrak{h}) \cap \Sigma^0_{j,\ell_j}\} \tag{3.2.11}$$

is closed and proper in $\mathbb{P}(V_k)$.

Finally, for a fixed $\ell_j \in A_j$ the set

$$B_k(\ell_j) = \{\ell_k \in \mathbb{P}(V_k) \;\mid\; \varphi(\mathfrak{h}) \cap \Sigma^0_{k,\ell_k} \supset \varphi(\mathfrak{h}) \cap \Sigma^0_{j,\ell_j}\} \tag{3.2.12}$$

is at most singleton and thus $A_k := \mathbb{P}(V_k) \setminus (C_k(\ell_j) \cup B_k(\ell_j))$, $k \neq j$ are the desired open and nonempty sets from the statement of this lemma. □

Corollary 3.2.1.1. *Let $\mathfrak{h}$ be a weak factorization structure. Then, for a fixed $j \in \{1, \ldots, m\}$ there exist an open nonempty set $A_j \subset \mathbb{P}(V_j)$ such that for any $\ell_j \in A_j$ and any $k \neq j$*

$$|\varphi(\mathfrak{h}) \cap \Sigma^0_{k,\ell_k}| - 1 \geq |\varphi(\mathfrak{h}) \cap \Sigma^0_{k,\ell_k} \cap \Sigma^0_{j,\ell_j}| \tag{3.2.13}$$

holds for generic $\ell_k \in \mathbb{P}(V_k)$.

Proof. As $\varphi(\mathfrak{h}) \cap \Sigma^0_{k,\ell_k}$ and $\varphi(\mathfrak{h}) \cap \Sigma^0_{j,\ell_j}$ are not in an inclusion relation, their intersection $\varphi(\mathfrak{h}) \cap \Sigma^0_{k,\ell_k} \cap \Sigma^0_{j,\ell_j}$ has dimension at least one less than dimension of $\varphi(\mathfrak{h}) \cap \Sigma^0_{k,\ell_k}$. □

Corollary 3.2.1.2. *Let $\mathfrak{h}$ be a factorization structure. Then, for a fixed $j \in \{1, \ldots, m\}$ there exist an open nonempty set $A_j \subset \mathbb{P}(V_j)$ such that for any $\ell_j \in A_j$ and any $k \neq j$*

$$|\varphi(\mathfrak{h}) \cap \Sigma^0_{k,\ell_k} \cap \Sigma^0_{j,\ell_j}| = 0 \tag{3.2.14}$$

holds for generic $\ell_k \in \mathbb{P}(V_k)$.

Proof. As $\mathfrak{h}$ is a factorization structure, $|\varphi(\mathfrak{h}) \cap \Sigma^0_{k,\ell_k}| = 1$ holds for generic values of ℓ_k. Substituting this into (3.2.13) gives the claim. □

Finally, we are able to show that our definition of the quotient object $\mathfrak{h}_j(\ell_j)$ makes sense in the framework of weak factorization structures.

Proposition 3.2.1. *For a generic $\ell_j \in \mathbb{P}(V_j)$ the quotient $\mathfrak{h}_j(\ell_j)$ of a weak factorization structure $\mathfrak{h}$ is a weak factorization structure.*

Proof. Let A_j be as in the Lemma 3.2.1 and let $\ell_j \in A_j$. By Corollary 3.2.1.1

$$|\varphi(\mathfrak{h}) \cap \Sigma^0_{k,\ell_k}| - |\varphi(\mathfrak{h}) \cap \Sigma^0_{k,\ell_k} \cap \Sigma^0_{j,\ell_j}| \geq 1$$

holds for any $k \neq j$ and generic ℓ_k. No that

$$|\rho_j^{\ell_j}(\varphi(\mathfrak{h}) \cap \Sigma^0_{k,\ell_k})| = |\varphi(\mathfrak{h}) \cap \Sigma^0_{k,\ell_k}| - |\varphi(\mathfrak{h}) \cap \Sigma^0_{k,\ell_k} \cap \Sigma^0_{j,\ell_j}| \tag{3.2.15}$$

is the dimension of image $\rho_j^{\ell_j}(\varphi(\mathfrak{h}) \cap \Sigma^0_{k,\ell_k})$ of $\varphi(\mathfrak{h}) \cap \Sigma^0_{k,\ell_k}$ under the linear map $\rho_j^{\ell_j}$ since $\ker \rho_j^{\ell_j} = \Sigma^0_{j,\ell_j}$.

Using (3.2.3) this results in

$$|\varphi_j(\mathfrak{h}_j) \cap \Sigma^{\wedge_j 0}_{k,\ell_k} \otimes \ell_j^*| \geq |\rho_j^{\ell_j}(\varphi(\mathfrak{h}) \cap \Sigma^0_{k,\ell_k})| = |\varphi(\mathfrak{h}) \cap \Sigma^0_{k,\ell_k}| - |\varphi(\mathfrak{h}) \cap \Sigma^0_{k,\ell_k} \cap \Sigma^0_{j,\ell_j}| \geq 1 \tag{3.2.16}$$

for generic ℓ_k and $k \neq j$, which implies that the quotient $\mathfrak{h}_j(\ell_j)$ is a wfs. □

Observe that if a quotient $\varphi_j(\ell_j) : \mathfrak{h}_j(\ell_j) \to \hat{V}_j^* \otimes \ell_j^*$ of a weak factorization structure $\varphi : \mathfrak{h} \to V^*$ is a factorization structure and if $\varphi(\mathfrak{h}) \cap \Sigma^0_{k,\ell_k}$ meets $Ker\ \rho_j^{\ell_j} = \Sigma^0_{j,\ell_j}$ trivially, then

$$1 = |\varphi_j(\mathfrak{h}_j) \cap \Sigma^{\wedge_j 0}_{k,\ell_k} \otimes \ell_j^*| \geq |\rho_j^{\ell_j}(\varphi(\mathfrak{h}) \cap \Sigma^0_{k,\ell_k})| = \underbrace{|\varphi(\mathfrak{h}) \cap \Sigma^0_{k,\ell_k}|}_{\geq 1} - \underbrace{|\varphi(\mathfrak{h}) \cap \Sigma^0_{k,\ell_k} \cap \Sigma^0_{j,\ell_j}|}_{0} \tag{3.2.17}$$

by rank-nullity theorem and (3.2.3). In other words, this forces $\varphi(\mathfrak{h}) \cap \Sigma^0_{k,\ell_k}$ to be one-dimensional. Formalising this idea gives a sufficient condition when a weak factorization structure is a factorization structure in terms of its quotients.

Lemma 3.2.2. *Let $\mathfrak{h}$ be a weak factorization structure. Let $\mathfrak{h}_{j_1}(\ell_{j_1})$ and $\mathfrak{h}_{j_2}(\ell_{j_2})$ be two quotients of $\mathfrak{h}$ which are factorization structures, $j_1 \neq j_2$. If for all $k \in \{1, \ldots, m\} \backslash \{j_1\}$ the dimension of $\varphi(\mathfrak{h}) \cap \Sigma^0_{k,\ell_k} \cap \Sigma^0_{j_1,\ell_{j_1}}$ is zero for generic values of ℓ_k, and if the dimension of $\varphi(\mathfrak{h}) \cap \Sigma^0_{j_1,\ell} \cap \Sigma^0_{j_2,\ell_{j_2}}$ is zero for generic values of ℓ then $\mathfrak{h}$ is a factorization structure.*

Proof. Rename j_1 to be j. By assumptions for any $k \in \{1, \cdots, m\} \backslash \{j\}$ there exists an open nonempty set

- $A_k \subset \mathbb{P}(V_k)$, where $|\varphi(\mathfrak{h}) \cap \Sigma^0_{k,\ell_k}| \geq 1$ for $\ell_k \in A_k$; ($\mathfrak{h}$ is wfs)
- $B_k \subset \mathbb{P}(V_k)$, where $|\varphi(\mathfrak{h}) \cap \Sigma^0_{k,\ell_k} \cap \Sigma^0_{j,\ell_j}| = 0$ for $\ell_k \in B_k$;

- $C_k \subset \mathbb{P}(V_k)$, where $|\varphi_j(\mathfrak{h}_j) \cap \Sigma^{\wedge_j 0}_{k,\ell_k} \otimes \ell_j^*| = 1$ for $\ell_k \in C_k$. $\qquad (\mathfrak{h}_{j_1}(\ell_{j_1})$ is fs)

For $\ell_k \in A_k \cap B_k \cap C_k$ one has

$$1 = |\varphi_j(\mathfrak{h}_j) \cap \Sigma^{\wedge_j 0}_{k,\ell_k} \otimes \ell_j^*| \geq |\rho_j^{\ell_j}(\varphi(\mathfrak{h}) \cap \Sigma^0_{k,\ell_k})| = \underbrace{|\varphi(\mathfrak{h}) \cap \Sigma^0_{k,\ell_k}|}_{\geq 1} - \underbrace{|\varphi(\mathfrak{h}) \cap \Sigma^0_{k,\ell_k} \cap \Sigma^0_{j,\ell_j}|}_{0} \tag{3.2.18}$$

which amounts to

$$|\varphi(\mathfrak{h}) \cap \Sigma^0_{k,\ell_k}| = 1$$

for a generic ℓ_k and for $k \neq j = j_1$, where a reasoning behind (3.2.18) is as in the proof of Proposition 3.2.1.

To understand the dimension $|\varphi(\mathfrak{h}) \cap \Sigma^0_{j_1,\ell}|$ one uses the same approach as above with j_2 in place of j and j_1 in place of k. This results to $|\varphi(\mathfrak{h}) \cap \Sigma^0_{j_1,\ell_{j_1}}| = 1$ for a generic ℓ_{j_1}.

As $|\varphi(\mathfrak{h}) \cap \Sigma^0_{k,\ell_k}| = 1$ for all k and for generic values of ℓ_k, $\mathfrak{h}$ is a factorization structure.

□

As a consequence we get

Proposition 3.2.2. *Let $\mathfrak{h}$ be a weak factorization structure ($m \geq 3$) and let for all $j = 1, \dots, m$ and generic ℓ_j the quotient $\mathfrak{h}_j(\ell_j)$ be factorization structure. Then $\mathfrak{h}$ is a factorization structure.*

Proof. In this proof we shall check that assumptions of Lemma 3.2.2 are satisfied which proves the claim.

For fixed $a \in \{1, \cdots, m\}$ we shall prove the existence of $L_a \in \mathbb{P}(V_a)$ such that the quotient factorization structure $\mathfrak{h}_a(L_a)$ exist and $\varphi(\mathfrak{h}) \cap \Sigma^0_{a,L_a} \cap \Sigma^0_{j,\ell_j} = 0$ for every $j \neq a$ and generic $\ell_j \in \mathbb{P}(V_j)$.
Let $r \in \{1, \dots, m\} \backslash \{a\}$ and choose $\ell_r, \bar{\ell}_r \in \mathbb{P}(V_r)$ such that $\ell_r \neq \bar{\ell}_r$ and quotients $\rho_r^{\ell_r}(\varphi(\mathfrak{h}))$ and $\rho_r^{\bar{\ell}_r}(\varphi(\mathfrak{h}))$ exist. These are factorization structures by assumptions, and hence for $\ell \in \{\ell_r, \bar{\ell}_r\}$ we have by Corollary 3.2.1.2 that there exist an open nonempty set $A_\ell \in \mathbb{P}(V_a)$ such that for any $\ell_a(\ell) \in A_\ell$ and for any $b \in \{1, \dots, m\} \backslash \{r, a\}$ there exist an open nonempty set $B_\ell \subset \mathbb{P}(V_b)$ such that

$$\varphi_r(\mathfrak{h}_r(\ell)) \cap \Sigma^{\wedge_r 0}_{a,\ell_a} \otimes \ell^* \cap \Sigma^{\wedge_r 0}_{b,\ell_b} \otimes \ell^* = 0 \tag{3.2.19}$$

holds for all $\ell_b \in B_\ell$. Note that here we used $m \geq 3$. Therefore, eq. (3.2.4) implies

$$\varphi(\mathfrak{h}) \cap \Sigma^0_{a,\ell_a} \cap \Sigma^0_{b,\ell_b} \subset \text{ Ker } (\rho_r^\ell) = \Sigma^0_{r,\ell} \tag{3.2.20}$$

Now, for $\ell_a \in A_{\ell_r} \cap A_{\bar{\ell}_r} \subset \mathbb{P}(V_a)$ we have

$$\varphi(\mathfrak{h}) \cap \Sigma^0_{a,\ell_a} \cap \Sigma^0_{b,\ell_b} \subset \Sigma^0_{r,\ell_r} \cap \Sigma^0_{r,\bar{\ell}_r} = 0 \tag{3.2.21}$$

for any $\ell_b \in B_{\ell_r} \cap B_{\bar{\ell}_r}$, where $b \in \{1, \dots, m\} \backslash \{r, a\}$.

Now, let $q \in \{1, \dots, m\} \backslash \{r, a\}$ and again consider two quotients at distinct points $\ell_q, \bar{\ell}_q \in \mathbb{P}(V_q)$. Similarly, we get the existence of an open nonempty set $A_{\ell_q} \cap A_{\bar{\ell}_q} \subset \mathbb{P}(V_a)$ such that for any $\bar{\ell}_a \in A_{\ell_q} \cap A_{\bar{\ell}_q}$ we have

$$\varphi(\mathfrak{h}) \cap \Sigma^0_{a,\bar{\ell}_a} \cap \Sigma^0_{r,\ell_r} = 0 \tag{3.2.22}$$

for generic $\ell_r \in \mathbb{P}(V_r)$ as before. Finally, choosing $L_a \in A_{\ell_r} \cap A_{\bar{\ell}_r} \cap A_{\ell_q} \cap A_{\bar{\ell}_q} \subset \mathbb{P}(V_a)$ such that quotient $\mathfrak{h}_a(L_a)$ exist we get get the claim.

Since $a \in \{1, \dots, m\}$ was arbitrary the claim must hold in every slot and hence the assumptions of Lemma 3.2.2 are satisfied. □

At last we obtain

Theorem 3.2.3. *A weak factorization structure is a factorization structure.*

Proof. Use Proposition 3.2.2 inductively on m where the base case is $m = 2$ (see Lemma 3.1.1). □

Corollary 3.2.3.1. *For each $j \in \{1, \dots, m\}$ a generic quotient $\mathfrak{h}_j(\ell_j)$ of a factorization structure $\mathfrak{h}$ is a factorization structure.*

Proof. As $\mathfrak{h}$ is a factorization structure it is also a weak factorization structure. By Proposition 3.2.1, $\mathfrak{h}_j(\ell_j)$ is a weak factorization structure for a generic ℓ_j. Finally, by Theorem 3.2.3, $\mathfrak{h}_j(\ell_j)$ is a factorization structure. □

Equipped with the notion of quotient factorization structure we can say more about tensors Γ_i occurring in the example of the Segre-Veronese factorization structure. Recall that $\langle\Gamma_i\rangle \subset \bigotimes_{\substack{j=1 \\ j\neq i}}^{k} (W_j^*)^{\otimes d_j}$, $i \in \{1, \dots, k\}$, define the Segre-Veronese factorization structure if

$$\varphi(\mathfrak{h}) = \sum_{i=1}^{k} ins_i \left(S^{d_i} W_i^* \otimes \langle\Gamma_i\rangle \right) \tag{3.2.23}$$

has dimension $m + 1$, where $\sum_{i=1}^{k} d_i = m$, $d_i \geq 1$.

Lemma 3.2.4. *In the notation established above, if $\{\langle\Gamma_j\rangle\}_{j=1}^{k}$ define the Segre-Veronese factorization structure, then $\langle\Gamma_j\rangle \subset \bigotimes_{\substack{i=1 \\ i\neq j}}^{k} S^{d_i} W_i^*$.*

Proof. By induction on k.

When $k = 1$ Segre-Veronese factorization structure reduces to Veronese factorization structure which does not contain any Γs and there is nothing to prove.

Let $k = 2$. Then

$$S^{d_1} W_1^* \otimes \langle\Gamma_1^2\rangle + \langle\Gamma_2^2\rangle \otimes S^{d_2} W_2^* \tag{3.2.24}$$

has dimension $d_1 + d_2 + 1$ if and only if $\langle\Gamma_1^2\rangle \subset S^{d_2} W_2^*$ and $\langle\Gamma_2^2\rangle \subset S^{d_1} W_1^*$.

Suppose the claim holds for $k \geq 2$ and fix $j \in \{1, \ldots, k+1\}$. For $r_1 \in \{1, \ldots, d_j\}$ choose $\ell_{r_1} \in \mathbb{P}(W_j)$ such that the quotient of

$$\varphi(\mathfrak{h}) = \sum_{i=1}^{k+1} ins_i \left(S^{d_i} W_i^* \otimes \langle \Gamma_i \rangle \right) \tag{3.2.25}$$

by ℓ_{r_1} exists. Now, choose $r_2 \in \{1, \ldots, d_j\} \backslash \{r_1\}$ such that quotient of (3.2.25) by ℓ_{r_2} exists. Repeat this process for each $r \in \{1, \ldots, d_j\}$. This way we eliminated all grouped j-slots and we end with a factorization structure similar to (3.2.25) but without grouped j-slots:

$$\sum_{\substack{i=1 \\ i \neq j}}^{k+1} ins_i \left(S^{d_i} W_i^* \otimes \langle L \lrcorner \Gamma_i \rangle \right) \tag{3.2.26}$$

where $L = \ell_1 \otimes \cdots \otimes \ell_{d_j}$ and $\lrcorner$ denotes contraction in the jth slots. By construction, $\langle \Gamma_j \rangle$ belongs to (3.2.26). By the induction assumption we have

$$\langle L \lrcorner \Gamma_i \rangle \subset \bigotimes_{\substack{b=1 \\ b \neq i,j}}^{k+1} S^{d_b} W_b^* \tag{3.2.27}$$

for $i \in \{1, \ldots, k+1\} \backslash \{j\}$ and hence for any $j \in \{1, \ldots, k+1\}$ we have $\langle \Gamma_j \rangle \subset (3.2.26) \subset \bigotimes_{\substack{b=1 \\ b \neq j}}^{k+1} S^{d_b} W_b^*$. □

3.3 Curves

In this section we shall associate m curves, called factorization curves, to each m-dimensional factorization structure. We show that these are given by homogeneous polynomials of the same degree in homogeneous coordinates on $\mathbb{P}^1$, and establish the notion of degree. We prove that these curves are injective and we analyse behaviour of their degrees in relation to quotient factorization structures.

3.3.1 Definition of curves. Lemma 3.1.2 allows us to define $\mathbb{P}(\mathfrak{h})$-valued curves for any factorization structure $\varphi : \mathfrak{h} \to V^*$ on an open nonempty subset U_k of $\mathbb{P}(V_k)$, where d_k is constant and equals one, by

$$\begin{aligned} &U_k \to \mathbb{P}(\mathfrak{h}) \\ &\ell_k \mapsto \psi_k(\ell_k) := \varphi^{-1} \left(\varphi(\mathfrak{h}) \cap \Sigma^0_{k,\ell_k} \right) \end{aligned} \tag{3.3.1}$$

We have $\varphi \circ \psi_k(\ell_k) = \varphi(\mathfrak{h}) \cap \Sigma^0_{k,\ell_k} \in \mathbb{P}(\varphi(\mathfrak{h}))$.

Example 3.3.1. In the case of Segre-Veronese we generically have

$$\varphi \circ \psi_j(\ell_j) = ins_j\left((\ell_j^0)^{\otimes d_j} \otimes \langle \Gamma_j \rangle\right) \in \mathbb{P}\left(\sum_{j=1}^{k} ins_j\left(S^{d_j} W_j^* \otimes \langle \Gamma_j \rangle\right)\right) \tag{3.3.2}$$

Note that some curves coincide. To clarify this and why they are only defined generically consider the following example.

Example 3.3.2. The inclusion

$$S^2 W_1^* \otimes \langle \Gamma_1 \rangle + \langle \Gamma_2 \rangle \otimes W_2^* \hookrightarrow W_1^* \otimes W_1^* \otimes W_2^* \tag{3.3.3}$$

defines a factorization structure if $\langle \Gamma_1 \rangle$ and $\langle \Gamma_2 \rangle$ are one dimensional subspaces in W_2^* and $S^2 W_1^*$, respectively. If there exist $\langle v \rangle \subset W_1^*$ so that $\langle \Gamma_2 \rangle = \langle v \otimes v \rangle$ then

$$\begin{aligned} \varphi \circ \psi_1(\ell) = \varphi \circ \psi_2(\ell) = \ell^0 \otimes \ell^0 \otimes \langle \Gamma_1 \rangle \qquad & \forall \ell \in \mathbb{P}(W) \backslash \{\langle v \rangle\} \\ \varphi \circ \psi_3(\ell_3) = \langle v \otimes v \rangle \otimes \ell_3^0 \qquad & \forall \ell_3 \in \mathbb{P}(W_2^*) \end{aligned} \tag{3.3.4}$$

Thus, the curves ψ_1 and ψ_2 coincide and are defined generically: away from $\langle v \rangle$. Clearly, the intersections

$$\varphi(\mathfrak{h}) \cap \langle v \rangle \otimes W_1^* \otimes W_2^* = \varphi(\mathfrak{h}) \cap W_1^* \otimes \langle v \rangle \otimes W_2^* = \langle v \otimes v \rangle \otimes W_2^* \tag{3.3.5}$$

are 2-dimensional and do not define a point in $\mathbb{P}(\varphi(\mathfrak{h}))$. This demonstrates the genericness.

On the other hand, if $\langle \Gamma_2 \rangle$ is indecomposable, then both curves are defined on whole projective lines.

Suppose again $\langle \Gamma_2 \rangle = \langle v \otimes v \rangle$ and choose bases for W_1 and W_2 so that $v = (v_1, v_2)$ and $\Gamma_1 = (u_1, u_2)$. This allows us to represent curves (3.3.4) in homogeneous coordinates

$\forall [x : y] \in \mathbb{P}(W) \backslash \{\langle (v_1, v_2) \rangle\}$:

$$\varphi \circ \psi_1([x : y]) = \varphi \circ \psi_2([x : y]) = \tag{3.3.6}$$

$$[u_1 y^2 : u_2 y^2 : -u_1 xy : -u_2 xy : -u_1 xy : -u_2 xy : u_1 x^2 : u_2 x^2] \tag{3.3.7}$$

$\forall [x : y] \in \mathbb{P}(W_2^*)$:

$$\varphi \circ \psi_3([x : y]) = [-v_1^2 y : v_1^2 x : -v_1 v_2 y : v_1 v_2 x : -v_2 v_1 y : v_2 v_1 x : -v_2^2 y : v_2^2 x] \tag{3.3.8}$$

Note that on the domain of definition of these curves they are defined by homogeneous polynomials of the same degree in homogeneous coordinates. Furthermore, they clearly extend to whole projective line by the same formulae. Although we have not defined degrees of curves yet, we can intuitively conclude $deg(\varphi \circ \psi_1) = deg(\varphi \circ \psi_2) = 2$ and $deg(\varphi \circ \psi_3) = 1$ by regarding degrees of curves as the degrees of the polynomials which define them. In addition, note that curves (3.3.4) are rational normal curves in subspaces $\mathbb{P}(S^2 W_1^* \otimes \langle \Gamma_1 \rangle)$ and $\mathbb{P}(\langle v \otimes v \rangle \otimes W_2^*)$ of $\mathbb{P}(\varphi(\mathfrak{h}))$, respectively.

Observe that for more complicated examples of Segre-Veronese factorization structure curves may not be defined in finitely many points which agrees with the fact that they are only defined generically.

Motivated by the analysis above we shall show that a curve defined by (3.3.1) is given by homogeneous polynomials of the same degree in homogeneous coordinates on U_k, i.e. it is a regular map on U_k, thus, by definition, it is a rational map $\mathbb{P}(V_k) \dashrightarrow \mathbb{P}(\varphi(\mathfrak{h}))$. In addition we shall recall from algebraic geometry that such a rational map is uniquely extendable to a regular map on the whole of $\mathbb{P}(V_k)$. Then, after establishing the notion of degree of these curves we define complexified factorization structures which help us to understand the behaviour of degrees in quotient factorization structures.

We start with showing that $\varphi \circ \psi_k$ is a rational map.

Lemma 3.3.1. *Let $\mathfrak{h}$ be a factorization structure of dimension m. Then, the factorization curve $\varphi \circ \psi_k$ is a regular map on an open and non-empty subset of $U_k = \{\ell_k \in \mathbb{P}(V_k) \mid |\varphi(\mathfrak{h}) \cap \Sigma^0_{k,\ell_k}| = 1\}$ of degree at most m.*

Proof. In order to show that $\varphi \circ \psi_k$ is given by homogeneous polynomials of the same degree in homogeneous coordinates on U_k let $\ell \in U_k$ and let $c_{a_1 \cdots a_m}$ be standard coordinates on $V^* = V_1^* \otimes \cdots \otimes V_m^*$ with respect to bases of V_j^*'s, where $a_j \in \{1, 2\}$. The subspace $\Sigma^0_{k,\ell}$ in V^* can be described by 2^{m-1} independent linear equations

$$x c_{a_1 \cdots a_{k-1} 1 a_{k+1} \cdots a_m} = -y c_{a_1 \cdots a_{k-1} 2 a_{k+1} \cdots a_m} \tag{3.3.9}$$

where $\ell^0 = [-y : x]$ and for $j \neq k$ we have $a_j \in \{1, 2\}$. These are also homogeneous polynomials of degree one in $\ell = [x : y]$ with coefficients $c_{\dots}$'s. The subspace $\varphi(\mathfrak{h})$ in V^* can be described via $2^m - (m+1)$ independent linear equations which do not depend on ℓ, call that system (E). Finally, the subspace $\varphi(\mathfrak{h}) \cap \Sigma^0_{k,\ell}$, which is one-dimensional for a fixed ℓ, is the solution to the system of $2^{m-1} + 2^m - (m+1)$ linear equations (3.3.9) and (E). Clearly, there are only $2^m - 1$ independent equations, and hence the system (3.3.9) and (E) is equivalent to (E) and another m independent linear equations from (3.3.9). The latter stay independent on an open subset $U(\ell) \subset \mathbb{P}(V_k)$ containing ℓ. Thus, knowing $\varphi(\mathfrak{h}) \cap \Sigma^0_{k,\ell}$ is equivalent to a system of $2^m - 1$ independent linear equations of out which m are homogeneous of degree one in ℓ and the others do not depend on ℓ. Using Cramer's rule to solve the system of $2^m - 1$ equations in 2^m variables and the definition of determinant shows that $\varphi(\mathfrak{h}) \cap \Sigma^0_{k,\ell}$ depends on ℓ in a homogeneous way and the degree of homogeneity is at most m which, for example, is attained in the case when $\varphi(\mathfrak{h}) = S^m W^*$. This shows that $\varphi \circ \psi_k$ is defined by homogeneous polynomials of the same degree in homogenous variables on $U(\ell) \cap U_k$, i.e. it is a regular map on $U(\ell) \cap U_k$. Now, observe that by varying ℓ the open sets $U(\ell)$ cover U_k, and since $\varphi \circ \psi_k$ is regular on $U(\ell) \cap U_k$ for a generic ℓ, it is regular on U_k. □

In fact, rational maps from $\mathbb{P}^1$ to $\mathbb{P}^m$ uniquely extend as explained in

Lemma 3.3.2. *Let U be an open non-empty subset of $\mathbb{P}^1$. A regular map $f : U \to \mathbb{P}^n$ extends uniquely to $\mathbb{P}^1$.*

Proof. The map f is given by $f([x:y]) = [f_0(x,y) : \cdots : f_n(x,y)]$ where f_j are homogeneous polynomials of the same degree. The expression $[f_0(x,y) : \cdots : f_n(x,y)]$ fails to define a point in $\mathbb{P}^n$ if and only if all f_j vanish at $[x:y]$. However, this means that all f_j have a factor in common which can be removed. This way it extends to whole $\mathbb{P}^1$. $\square$

This allows us to define factorization curves, the extension of (3.3.1).

Definition 3.3.1. Let $\varphi : \mathfrak{h} \to V^*$ be a real/complex factorization structure. For each $j \in \{1, \ldots, m\}$ we define *factorization curve* $\psi_j : \mathbb{P}(V_j) \to \mathbb{P}(\mathfrak{h})$ as follows. On an open and non-empty subset of $\{\ell_j \in \mathbb{P}(V_j) \mid |\varphi(\mathfrak{h}) \cap \Sigma^0_{j,\ell_j}| = 1\}$ where the curve is regular by Lemma 3.3.1 we set $\psi_j(\ell_j) = \varphi^{-1}\left(\varphi(\mathfrak{h}) \cap \Sigma^0_{j,\ell_j}\right) \in \mathbb{P}(\mathfrak{h})$ and extend it to the whole $\mathbb{P}(V_j)$ by Lemma 3.3.2.

3.3.2 Degree. In what follows we shall define a notion of degree for factorization curves and investigate its behaviour with respect to quotient factorization structures. To do so we shall need complexified factorization structures as the proof of Lemma 3.3.7 requires to work over an algebraically closed field.

Consider the complexification $\varphi^{\mathbb{C}} : \mathfrak{h} \otimes \mathbb{C} \to V^* \otimes \mathbb{C}$ of a real factorization structure $\varphi : \mathfrak{h} \to V^*$, and denote

$$(V_1^* \otimes \mathbb{C}) \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} (V_{k-1}^* \otimes \mathbb{C}) \otimes_{\mathbb{C}} L_k \otimes_{\mathbb{C}} (V_{k+1}^* \otimes \mathbb{C}) \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} (V_m^* \otimes \mathbb{C})$$

by $\mathbb{C}\Sigma^0_{k,L_k}$ for any complex 1-dimensional subspace $L_k \subset V_k^* \otimes \mathbb{C}$. Such a complexification is called a complexified factorization structure which is justified by

Lemma 3.3.3. *A map $\varphi : \mathfrak{h} \to V^*$ is a real factorization structure iff its complexification $\varphi^{\mathbb{C}} : \mathfrak{h} \otimes \mathbb{C} \to V^* \otimes \mathbb{C}$ is a complex factorization structure.*

Proof. For $L_k \in \mathbb{P}(V_k \otimes \mathbb{C})$ such that

$$|\varphi^{\mathbb{C}}(\mathfrak{h} \otimes \mathbb{C}) \cap \mathbb{C}\Sigma^0_{k,L_k}| = 1 \tag{3.3.10}$$

define $\varphi^{\mathbb{C}} \circ \psi_k^{\mathbb{C}}(L_k)$ to be the one-dimensional intersection. Note that on an open nonempty subset of $\mathbb{P}(V_k)$

$$\begin{aligned} \varphi \circ \psi_k(\ell_k) \otimes \mathbb{C} &= (\varphi(\mathfrak{h}) \cap \Sigma^0_{k,\ell_k}) \otimes \mathbb{C} = \varphi(\mathfrak{h}) \otimes \mathbb{C} \cap \mathbb{C}\Sigma^0_{k,\ell_k \otimes \mathbb{C}} \\ &= \varphi^{\mathbb{C}}(\mathfrak{h} \otimes \mathbb{C}) \cap \mathbb{C}\Sigma^0_{k,\ell_k \otimes \mathbb{C}} = \varphi^{\mathbb{C}} \circ \psi_k^{\mathbb{C}}(\ell_k \otimes \mathbb{C}) \end{aligned} \tag{3.3.11}$$

holds.

Now, if φ is a real factorization structure, then there exist $\ell_k \in \mathbb{P}(V_k)$ such that (3.3.2) holds, i.e. the open set $O := \{L_k \in \mathbb{P}(V_k \otimes \mathbb{C}) \mid |\varphi^{\mathbb{C}}(\mathfrak{h} \otimes \mathbb{C}) \cap \mathbb{C}\Sigma^0_{k,L_k}| = 1\}$ is nonempty as $\ell_k \otimes \mathbb{C}$ belongs there, and hence $\varphi^{\mathbb{C}} : \mathfrak{h} \otimes \mathbb{C} \to V^* \otimes \mathbb{C}^*$ is complex factorization structure.

On the other hand, if $\varphi^{\mathbb{C}} : \mathfrak{h} \otimes \mathbb{C} \to V^* \otimes \mathbb{C}$ is a complex factorization structure, then O is nonempty and has nontrivial intersection with $P := \{\ell_k \otimes \mathbb{C} \in \mathbb{P}(V_k \otimes \mathbb{C}) \mid \ell_k \in$

$\mathbb{P}(V_k)\}$. Indeed, O is $\mathbb{P}(V_k \otimes \mathbb{C})$ minus a finite set of points, however P is infinite and hence $P \cap Q \neq \emptyset$. This means that the open set $\{\ell_k \in \mathbb{P}(V_k) \mid |\varphi(\mathfrak{h}) \cap \Sigma^0_{k,\ell_k}| = 1\}$ is nonempty by (3.3.2), and therefore $\varphi : \mathfrak{h} \to V^*$ is a real factorization structure. $\square$

To motivate the definition of degree we reinterpret ψ_k from Definition 3.3.1. By pulling the tautological section $\tau : \mathbb{P}(\mathfrak{h}) \to \mathcal{O}_{\mathfrak{h}}(1) \otimes \mathfrak{h}$ back via a factorization curve ψ_k we interpret the curve as a section of $\mathcal{O}_{V_k}(e_k) \otimes \mathfrak{h}$,

$$\begin{array}{ccc} \mathcal{O}_{V_k}(e_k) \otimes \mathfrak{h} & \longrightarrow & \mathcal{O}_{\mathfrak{h}}(1) \otimes \mathfrak{h} \\ \uparrow & & \uparrow \tau \\ \mathbb{P}(V_k) & \xrightarrow{\psi_k} & \mathbb{P}(\mathfrak{h}) \end{array},$$

where e_k is determined as follows. The pullback $(\psi_k)^*\mathcal{O}_{\mathfrak{h}}(1)$ is a line bundle over $\mathbb{P}(V_k)$, and thus the classification of line bundles over projective spaces implies $(\psi_k)^*\mathcal{O}_{\mathfrak{h}}(1) \cong \mathcal{O}_{V_k}(e_k)$ for some $e_k \in \mathbb{Z}$.

Now we define degree for factorization curves.

Definition 3.3.2. Let $\mathfrak{h}$ be a real/complex factorization structure. The degree $\deg \psi_k$ of ψ_k is defined to be $\deg \psi_k = e_k$, where e_k is such that $(\psi_k)^*\mathcal{O}_{\mathfrak{h}}(1) \cong \mathcal{O}_{V_k}(e_k)$.

We recall a lemma from algebraic algebraic geometry which discusses uniqueness of overlapping curves.

Lemma 3.3.4. *Suppose that two curves $\varphi \circ \psi_i : \mathbb{P}^1 \to \mathbb{P}(\varphi(\mathfrak{h}))$ and $\varphi \circ \psi_j : \mathbb{P}^1 \to \mathbb{P}(\varphi(\mathfrak{h}))$ are defined on the same projective line and that they agree on an open nonempty subset U. Then they agree everywhere.*

Proof. Since projective spaces are separated the set $\{\ell \in \mathbb{P}^1 \mid \varphi \circ \psi_i(\ell) = \varphi \circ \psi_j(\ell)\}$ is closed. Furthermore, it contains U, and therefore, since the only closed set which contain an open nonempty set is the whole $\mathbb{P}^1$, it must be $\mathbb{P}^1$. $\square$

The following lemma shows that complexification does not change degrees of curves.

Lemma 3.3.5. *Let $\varphi \circ \psi_k$ be a curve in a real factorization structure $\varphi : \mathfrak{h} \to V^*$. Then $deg(\varphi \circ \psi_k) = deg(\varphi^{\mathbb{C}} \circ \psi_k^{\mathbb{C}})$, where $\varphi^{\mathbb{C}} \circ \psi_k^{\mathbb{C}}$ is a k-th curve in the complexified factorization structure.*

Proof. Observe that equalities (3.3.2) can be extended to the whole $\mathbb{P}(V_k)$ by Lemma 3.3.2 and Lemma 3.3.4, and that they show $\big((\varphi^{\mathbb{C}} \circ \psi_k^{\mathbb{C}})^*\mathcal{O}_{\varphi^{\mathbb{C}}(\mathfrak{h}\otimes\mathbb{C})}(-1)\big)_{\ell_k \otimes \mathbb{C}} = \{\ell_k \otimes \mathbb{C}, \varphi \circ \psi_k(\ell_k) \otimes \mathbb{C}\}$. By definition of degree, the fibre $\big((\varphi \circ \psi_k)^*\mathcal{O}_{\varphi(\mathfrak{h})}(-1)\big)_{\ell_k} = \{\ell_k, \varphi \circ \psi_k(\ell_k)\}$ is isomorphic to a $deg(\varphi \circ \psi_k)$-fold product of ℓ_k^0's for any $\ell_k \in \mathbb{P}(V_k)$. Thus $(\varphi^{\mathbb{C}} \circ \psi_k^{\mathbb{C}})^*\mathcal{O}_{\varphi^{\mathbb{C}}(\mathfrak{h}\otimes\mathbb{C})}(-1)$ is a line bundle over a projective line with the fibre over $\ell_k \otimes \mathbb{C}$ isomorphic with $deg(\varphi \circ \psi_k)$-fold product of $\ell_k^0 \otimes \mathbb{C}$. We have $(\varphi^{\mathbb{C}} \circ \psi_k^{\mathbb{C}})^*\mathcal{O}_{\varphi^{\mathbb{C}}(\mathfrak{h}\otimes\mathbb{C})}(-1) \cong \mathcal{O}_{V_k \otimes \mathbb{C}}(deg(\varphi \circ \psi_k))$ and therefore $deg(\varphi \circ \psi_k) = deg(\varphi^{\mathbb{C}} \circ \psi_k^{\mathbb{C}})$. $\square$

The following lemma helps us prove that factorization curves are injective.

Lemma 3.3.6. *Let* $\mathfrak{h}$ *be a real/complex factorization structure. Then* $\forall \ell_k \in \mathbb{P}(V_k)$ $\exists T \in \hat{V}_k^*$ *such that* $\varphi \circ \psi_k(\ell_k) = \ell_k^0 \otimes T$.

Proof. On an open nonempty subset $U \subset \mathbb{P}(V_k)$ one has $\varphi \circ \psi_k(\ell_k) = \varphi(\mathfrak{h}) \cap \Sigma^0_{k,\ell_k}$, $\ell_k \in U$, and hence for such ℓ_k there exist $T \in \hat{V}_k^*$ as required. Suppose $deg(\varphi \circ \psi_k) = d$ so the curve is given by homogeneous polynomials of degree d in variables x, y, where $\ell_k^0 = [x : y]$, i.e. for $\ell_k \in U$, $\varphi \circ \psi_k(\ell_k) = \ell_k^0 \otimes T(\ell_k)$ and T is given by homogeneous polynomials of degree $d-1$. Therefore the map $T : U \subset \mathbb{P}(V_k) \to \mathbb{P}(\hat{V}_k^*)$ is regular, so by Lemma 3.3.2 it extends to a map $T : \mathbb{P}(V_k) \to \mathbb{P}(\hat{V}_k^*)$ which defines a curve $\mathcal{C} : \mathbb{P}(V_k) \to \mathbb{P}(V^*)$ by $\mathcal{C}(\ell_k) = \ell_k^0 \otimes T(\ell_k)$. Since $\mathbb{P}(\varphi(\mathfrak{h})) \hookrightarrow \mathbb{P}(V^*)$, $\varphi \circ \psi_k : \mathbb{P}(V_k) \to \mathbb{P}(\varphi(\mathfrak{h}))$ can be thought as a map with values in $\mathbb{P}(V^*)$, and then $\mathcal{C}(\ell_k) = \varphi \circ \psi_k(\ell_k)$ on U. By Lemma 3.3.4 curves $\mathcal{C}$ and $\varphi \circ \psi_k$ coincide, so for $\ell_k \in \mathbb{P}(V_k)$ one has $\varphi \circ \psi_k(\ell_k) = \ell_k^0 \otimes T(\ell_k) \in \mathbb{P}(\varphi(\mathfrak{h}))$. □

Corollary 3.3.6.1. *Let* $\mathfrak{h}$ *be a real/complex factorization structure. Curves are injective.*

Proof. Supposing $\varphi \circ \psi_k(\ell_k) = \varphi \circ \psi_k(\tilde{\ell}_k)$ we get $\ell_k^0 \otimes T(\ell_k) = \tilde{\ell}_k^0 \otimes T(\tilde{\ell}_k)$ which implies $\ell_k^0 = \tilde{\ell}_k^0$, and thus $\ell_k = \tilde{\ell}_k$. □

We establish behaviour of degree of a factorization curve in quotients by

Lemma 3.3.7. *Let* $\mathfrak{h}$ *be a complex factorization structure and let* $\mathfrak{h}_j(\ell_j)$ *be its quotient. Let* $k \neq j$ *and let* ψ_k *and* ζ_k *be* k*-th curves in* $\mathfrak{h}$ *and* $\mathfrak{h}_j(\ell_j)$*, respectively. Then*

$$deg(\varphi_j^{\ell_j} \circ \zeta_k^{\ell_j}) = \begin{cases} deg(\varphi \circ \psi_k) - 1 & \textit{if } \exists\, \ell_k \in \mathbb{P}(V_k) : \varphi \circ \psi_k(\ell_k) = \varphi \circ \psi_j(\ell_j) \\ deg(\varphi \circ \psi_k) & \textit{otherwise} \end{cases}$$

Proof. Since $(\varphi \circ \psi_k)^* \mathcal{O}_{\varphi(\mathfrak{h})}(-1)$ is uniquely determined by $\varphi \circ \psi_k$ (and its image), as it can be seen from

$$(\varphi \circ \psi_k)^* \mathcal{O}_{\varphi(\mathfrak{h})}(-1) = \{(\ell_k, (\ell, v)) \in \mathbb{P}(V_k) \times \mathcal{O}_{\varphi(\mathfrak{h})}(-1) \mid \varphi \circ \psi_k(\ell_k) = \ell \in \mathbb{P}(\varphi(\mathfrak{h}))\}, \tag{3.3.12}$$

we consider the map $\varphi \circ \psi_k : \mathbb{P}(V_k) \to \mathbb{P}(Im(\varphi \circ \psi_k))$ without losing track of degree. And similarly for the curve $\varphi_j^{\ell_j} \circ \zeta_k^{\ell_j}$.

We define a bundle map

$$\begin{array}{ccc} \mathcal{O}_{Im(\varphi\circ\psi_k)}(-1) & \xrightarrow{\gamma_j(\ell_j)} & \mathcal{O}_{Im(\varphi_j^{\ell_j}\circ\zeta_k^{\ell_j})}(-1) \\ \downarrow & & \downarrow \\ \mathbb{P}(Im(\varphi \circ \psi_k)) & \xrightarrow{\Gamma_j(\ell_j)} & \mathbb{P}(Im(\varphi_j^{\ell_j} \circ \zeta_k^{\ell_j})) \end{array}.$$

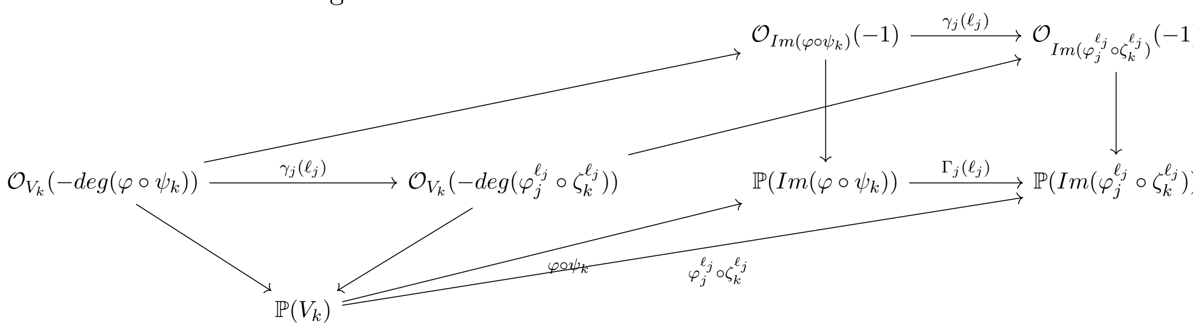
Since curves are injective (Corollary 3.3.6.1) we can define $\Gamma_j(\ell_j)$ as the composition $(\varphi_j^{\ell_j} \circ \zeta_k^{\ell_j}) \circ (\varphi \circ \psi_k)^{-1}$ which means $\Gamma_j(\ell_j)((\varphi \circ \psi_k)(\ell_k)) = (\varphi_j^{\ell_j} \circ \zeta_k^{\ell_j})(\ell_k)$. Note that on an open nonempty subset of $\mathbb{P}(V_k)$ the map $\Gamma_j(\ell_j)$ agrees with the restriction of $\rho_j^{\ell_j}$ to $Im(\varphi \circ \psi_k)$ (see (3.2.1)), where we have $\rho_j^{\ell_j}(\varphi(\mathfrak{h}) \cap \Sigma^0_{k,\ell_k}) = \varphi_j(\mathfrak{h}_j) \cap \Sigma^{\wedge_j 0}_{k,\ell_k} \otimes \ell_j^*$.

The map $\gamma_j(\ell_j)$ defined by $(\varphi \circ \psi_k(\ell_k), v) \mapsto (\Gamma(\ell_j)(\varphi \circ \psi_k(\ell_k)), v \bmod \Sigma^0_{j,\ell_j})$ covers the map $\Gamma(\ell_j)$. Since fibres are one-dimensional $\gamma_j(\ell_j)$ is an isomorphism on fibres unless there exist $\ell_k \in \mathbb{P}(V_k)$ such that $\varphi \circ \psi_k(\ell_k) \subset \Sigma^0_{j,\ell_j}$. This condition is equivalent to existence of $\ell_k \in \mathbb{P}(V_k)$ such that $0 \neq \varphi \circ \psi_k(\ell_k) \cap \Sigma^0_{j,\ell_j}$. Note

$$\varphi \circ \psi_k(\ell_k) \cap \Sigma^0_{j,\ell_j} = \varphi \circ \psi_k(\ell_k) \cap \varphi(\mathfrak{h}) \cap \Sigma^0_{j,\ell_j} = \varphi \circ \psi_k(\ell_k) \cap \varphi \circ \psi_j(\ell_j),$$

where the last equality comes from the fact that $\varphi(\mathfrak{h}) \cap \Sigma^0_{j,\ell_j}$ is one-dimensional since ℓ_j is such that quotient exist (see Definition 3.3.1 and (3.2.1)). To summarise, $\gamma_j(\ell_j)$ is a linear isomorphism on fibres unless there exist $\ell_k \in \mathbb{P}(v_k)$ such that $\varphi \circ \psi_k(\ell_k) = \varphi \circ \psi_j(\ell_j)$. Now, as (j, ℓ_j) is fixed there exist at most one such ℓ_k. Indeed, if ℓ_k, ℓ_k' are such that $\varphi \circ \psi_k(\ell_k) = \varphi \circ \psi_j(\ell_j) = \varphi \circ \psi_k(\ell_k')$, then since curves are injective (Corollary 3.3.6.1) we have $\ell_k = \ell_k'$.

Pulling $\gamma_j(\ell_j)$ back to $\mathbb{P}(V_k)$ along maps $\varphi \circ \psi_k$ and $\varphi_j^{\ell_j} \circ \zeta_k^{\ell_j}$ as shown in the following three-dimensional diagram



allows us to interpret $\gamma_j(\ell_j)$ as a global section of

$$(\mathcal{O}_{V_k}(-deg(\varphi \circ \psi_k)))^* \otimes \mathcal{O}_{V_k}(-deg(\varphi_j^{\ell_j} \circ \zeta_k^{\ell_j})) \cong \mathcal{O}_{V_k}\left(deg(\varphi \circ \psi_k) - deg(\varphi_j^{\ell_j} \circ \zeta_k^{\ell_j})\right) \tag{3.3.13}$$

thanks to the isomorphism $\mathcal{O}_{V_k}(d)^* \cong \mathcal{O}_{V_k}(-d)$, where $\mathcal{O}_{V_k}(d)^*$ is the dual line bundle to $\mathcal{O}_{V_k}(d)$.

Finally, the global section $\gamma_j(\ell_j)$ has either none or one single zero as established above. Since we work over an algebraically closed field, this means that it must be a section of $\mathcal{O}_{V_k}(0)$ or $\mathcal{O}_{V_k}(1)$ which shows $deg(\varphi \circ \psi_k) - deg(\varphi_j^{\ell_j} \circ \zeta_k^{\ell_j}) \in \{0, 1\}$ depending on the kernel. □

Corollary 3.3.7.1. *Lemma 3.3.7 holds for a real factorization structure $\mathfrak{h}$ as well.*

Proof. Observe that complexification commutes with taking quotients,

$$\left(\rho_j^{\ell_j}\right)^{\mathbb{C}} \circ \varphi^{\mathbb{C}}(\mathfrak{h} \otimes \mathbb{C}) = \left(\rho_j^{\ell_j}\right)^{\mathbb{C}}(\varphi(\mathfrak{h}) \otimes \mathbb{C}), \tag{3.3.14}$$

which, by definition of complexification of a linear map, equals to $(\rho_j^{\ell_j} \circ \varphi(\mathfrak{h})) \otimes \mathbb{C}$. Since complexification does not change the degree of curves (Lemma 3.3.5) one may consider the complexification of $\mathfrak{h}$ and apply Lemma 3.3.7 to get the claim. $\square$

3.4 Decomposability of curves

We define decomposability of curves, show its relation to the complexification, and prove that top degree curves are decomposable, essentially unique, and determine the whole factorization structure to be of Veronese type. In the case when all curves in a given factorization structure $\mathfrak{h}$ of dimension m are decomposable we show that $\mathfrak{h}$ must be of Segre-Veronese type. We conjecture that factorization curves are decomposable. To establish these ideas we use an equivalence relation on factorization curves (equivalently on partitions of $\{1, \dots, m\}$):

Definition 3.4.1. Two factorization curves are equivalent if they have the same image.

Recall that the Segre-Veronese factorization structure has form

$$\varphi(\mathfrak{h}) = \sum_{i=1}^{k} ins_i \left(S^{d_i} W_i^* \otimes \langle \Gamma_i \rangle \right), \tag{3.4.1}$$

and the first factorization curve is

$$\varphi \circ \psi_1(\ell) = \underbrace{\ell^0 \otimes \cdots \otimes \ell^0}_{d_1 \text{ times}} \otimes \langle \Gamma_1 \rangle \tag{3.4.2}$$

and has degree d_1. We note that $\varphi \circ \psi_1 = \cdots = \varphi \circ \psi_{d_1}$ and thus each of them has degree d_1. However, we wish to interpret this as one curve $\ell \mapsto \ell^0 \otimes \cdots \otimes \ell^0 \otimes \langle \Gamma_1 \rangle$ of degree d_1 with the property that in the each of the first d_1 slots it is a factorization curve. A way how to do this is to define an equivalence relation on factorization curves and work with equivalence classes instead of curves: two curves are equivalent if they have the same image. In addition, we can define a degree for any class of factorization curves as two curves with the same image must have the same degree. For example, (3.4.1) has k factorization curves classes with degrees $d_1, \dots, d_k$ which add up to m.

We also observe from (3.4.2) that the tensor depending on ℓ in $\varphi \circ \psi_1(\ell)$, namely $\ell^0 \otimes \cdots \otimes \ell^0 \leq (W_1^*)^{\otimes d_1}$, is decomposable in every slot. For example, for a fixed 2-dimensional vector space W consider the curve $\mathcal{C}_1 : \mathbb{P}^1 \to \mathbb{P}(W \otimes W)$ given by

$$\mathcal{C}_1([x : y]) = [y^2 : -xy : -xy : x^2]. \tag{3.4.3}$$

This is a coordinate expression for the curve $\ell \mapsto \ell^0 \otimes \ell^0$. Clearly the tensor $\ell^0 \otimes \ell^0 = [y^2 : -xy : -xy : x^2]$ is decomposable in $W \otimes W$. On the other hand, the curve

$$\mathcal{C}_2([x : y]) = [y^2 : -xy : -xy : -x^2] \tag{3.4.4}$$

is given by homogeneous polynomials of the same degree 2, but it is not decomposable as can be checked by the determinant. In the case of tensor product of more than two vector spaces the situation may get more complicated. Some slots could be decomposable and some not. For a general factorization curve $\varphi \circ \psi_j$ it is not plain to see if the tensor in $\varphi \circ \psi_j(\ell)$ depending on ℓ is decomposable as in (3.4.2).

Note that decomposability is tied with the aforementioned equivalence relation. Suppose for $k \neq j$ we have $\mathrm{image}(\varphi \circ \psi_j) = \mathrm{image}(\varphi \circ \psi_k)$. This implies that every curve in the class where they belong is decomposable in jth and kth slot because $\varphi \circ \psi_j(\ell)$ must have ℓ^0 in the jth slot and similarly for $\varphi \circ \psi_k$. We formalise this in

Definition 3.4.2. Let $\varphi : \mathfrak{h} \to V^*$ be a factorization structure of dimension m. A factorization curve ψ_j of degree $d \geq 1$ is called decomposable if the number of factorization curves which have the same image as ψ_j is d. A factorization curve of degree m is called a top degree factorization curve.

With respect to the equivalence relation, a curve is decomposable if the size of its class equals to its degree. As an example we can take (3.4.2) which defines a class of size d_1, and thus is decomposable as expected.

Now we prove a lemma which helps us establish decomposability of top degree factorization curves and give an alternative description of decomposability.

Lemma 3.4.1. *Let $\varphi : \mathfrak{h} \to V^*$ be a complex factorization structure. The following are equivalent.*

(i) Factorization curves $\varphi \circ \psi_k$ and $\varphi \circ \psi_j$ are equivalent.

(ii) There exists an open non-empty subset $U \subset \mathbb{P}(V_k)$ such that for every $\ell_k \in U$ there exists $\ell_j \in \mathbb{P}(V_j)$ such that $\varphi \circ \psi_k(\ell_k) = \varphi \circ \psi_j(\ell_j)$.

(iii) There exists an invertible projective transformation $f : \mathbb{P}(V_k) \to \mathbb{P}(V_j)$ such that $\varphi \circ \psi_k(\ell_k) = \varphi \circ \psi_j(f(\ell_k))$.

Furthermore, if there exist $\ell \in \mathbb{P}(V_j)$ such that $\varphi_j^\ell \circ \zeta_k^\ell$ is decomposable, then $\varphi \circ \psi_k$ and $\varphi \circ \psi_j$ are decomposable.

Proof. Clearly, (i) implies (ii), and (iii) implies (i). If we prove that (ii) implies (iii) we show the equivalence.

Suppose (ii) holds. Injectivity of curves allows us to describe the assignment $\ell_k \mapsto \ell_j$ from (ii) by $f(\ell_k) = (\varphi \circ \psi_j)^{-1} \circ (\varphi \circ \psi_k|_U)(\ell_k)$,

$$\begin{array}{ccc} U & & \\ {\scriptstyle f}\downarrow & \searrow^{\varphi\circ\psi_k|_U} & \\ \mathbb{P}(V_j) & \xrightarrow[\varphi\circ\psi_j]{} & \mathbb{P}(\varphi(\mathfrak{h})) \end{array}$$

We show that f is an invertible projective transformation. To this end, we note that since $\varphi \circ \psi_j$ and $\varphi \circ \psi_k$ are injective and regular, they are also bijective and rational as maps onto their images, hence birational (see [39, 50, 32]). Thus, the map f is rational and can be extended (Lemma 3.3.2) to the whole $\mathbb{P}(V_k)$. We call the extension again f and prove that the extended f is an injective function. Let $\ell_0, \ell_1 \in \mathbb{P}(V_k)$ be such that $f(\ell_0) = f(\ell_1)$. Then $\varphi \circ \psi_j(f(\ell_0)) = \varphi \circ \psi_j(f(\ell_1))$ and the commutativity of the diagram yields $\varphi \circ \psi_k(\ell_0) = \varphi \circ \psi_k(\ell_1)$. The injectivity of curves forces $\ell_0 = \ell_1$. Now, since f is an injective birational maps between projective lines it must be an invertible projective linear transformation between $\mathbb{P}(V_k)$ and $\mathbb{P}(V_j)$ which completes the proof of the equivalence.

To address the decomposability note that Lemma 3.3.6 allows us to find for each $\ell_k \in \mathbb{P}(V_k)$ a one-dimensional subspace $\langle S \rangle$ such that $\varphi \circ \psi_k(\ell_k) = \varphi \circ \psi_j(\ell_j) = \ell_k^0 \otimes \ell_j^0 \otimes \langle S \rangle$ up to permutation of slots, where $\ell_j = f(\ell_k)$. By assumptions, the curve $\varphi_j^\ell \circ \zeta_k^\ell(\ell_k) = \ell_k^0 \otimes \ell^* \otimes \langle S \rangle$ is decomposable which forces decomposability of $\langle S \rangle$ and so of $\varphi \circ \psi_k$. □

We establish a characterisation of decomposable factorization curves in

Lemma 3.4.2. *Let $\varphi : \mathfrak{h} \to V^*$ be a factorization structure of dimension m. A factorization curve $\varphi \circ \psi_j$ of degree $d \geq 1$ is decomposable if and only if there exist pair-wise distinct $i_1, \ldots, i_d \in \{1, \ldots, m\}$ such that for each $r \in \{1, \ldots, d\}$ there exists invertible $g_r \in Hom(\mathbb{P}(V_j^*), \mathbb{P}(V_{i_r}^*))$ such that for all $\ell \in \mathbb{P}(V_j)$: $\varphi \circ \psi_j(\ell)$ has $g_r \ell^0$ in the i_r-th slot. In other words, for some Γ_j we have*

$$\varphi \circ \psi_j(\ell) = g_1 \ell^0 \otimes \cdots \otimes g_d \ell^0 \otimes \langle \Gamma_j \rangle \tag{3.4.5}$$

up to permutation of slots. Clearly, one of these indices must be j for which g_j must be the identity.

Proof. The only if part is obvious. For the other implication note that since $\varphi \circ \psi_j$ is decomposable of degree d there must exist d pair-wise distinct indices $i_1, \ldots, i_d \in \{1, \ldots, m\}$ such that for each $\ell \in \mathbb{P}(V_j)$ there exist $\ell_{i_r} \in \mathbb{P}(V_{i_r})$ such that $\varphi \circ \psi_j(\ell) = \varphi \circ \psi_{i_r}(\ell_{i_r})$ for all $r = 1, \ldots, d$. Using Lemma 3.4.1 one can see that the assignments $\ell \mapsto \ell_{i_r}$ are invertible projective transformations g_r. We note that Γ_j cannot depend on ℓ because it would contradict the degree. This proves the claim. □

Now we focus on proving that top degree curves are decomposable.

Lemma 3.4.3. *Let $\mathfrak{h}$ be a complex factorization structure of dimension m. If top degree curves in factorization structures of dimension $m-1$ are decomposable, then top degree curves in $\mathfrak{h}$ are decomposable.*

Proof. Let $\varphi \circ \psi_k$ be a top degree curve in $\mathfrak{h}$, i.e. $deg(\varphi \circ \psi_k) = m$. Let (j, ℓ_j), $\ell_j \in \mathbb{P}(V_j)$, be such that a quotient $\mathfrak{h}_j(\ell_j)$ exist, where $j \neq k$. By Lemma 3.3.7, the projected curve $\varphi_j^{\ell_j} \circ \zeta_k^{\ell_j}$ has necessarily degree $m-1$ as a curve of degree m does not exist in m-factorization structure $\mathfrak{h}_j(\ell_j)$. Again by Lemma 3.3.7, there exist $\ell_k \in \mathbb{P}(V_k)$ such that $\varphi \circ \psi_k(\ell_k) = \varphi \circ \psi_j(\ell_j)$. Therefore, for any ℓ_j from an open nonempty subset U of $\mathbb{P}(V_j)$ where a quotient exist, there exist $\ell_k \in \mathbb{P}(V_k)$ such that $\varphi \circ \psi_k(\ell_k) = \varphi \circ \psi_j(\ell_j)$. Lemma 3.4.1 implies the claim. □

Using this result inductively, starting from $m = 2$, we show that top degree curves are decomposable. Note that for $m = 2$ a top degree curve occurs only in Veronese factorization structure. In a sense it is one decomposable curve of degree 2.

Corollary 3.4.3.1. *Top degree curves are decomposable.*

Proof. We prove this by induction of the dimension of a factorization structure involved starting with the base case when $m = 2$. In such a case, a top degree curve exists in Veronese factorization structure only,

$$S^2W^* \to W^* \otimes W^*, \tag{3.4.6}$$

it is one these

$$\begin{aligned} \varphi \circ \psi_1(\ell) &= \ell^0 \otimes \ell^0 \\ \varphi \circ \psi_2(\ell) &= \ell^0 \otimes \ell^0, \end{aligned} \tag{3.4.7}$$

and has degree 2. Either of them is decomposable as both have the same image (see Definition 3.4.2).

To progress in the induction argument we assume that any top degree curve in a factorization structure of dimension $m-1$ is decomposable. Now applying Lemma 3.4.3 gives the claim. □

In fact, a factorization structure with a decomposable top degree curve is of Veronese type, and thus has only one equivalence class of factorization curves.

Corollary 3.4.3.2. *If a factorization structure contains a decomposable top degree curve $\varphi \circ \psi_j$, then it is isomorphic to the Veronese factorization structure.*

Proof. As the curve is of the top degree and decomposable we can use the characterisation (3.4.5) from Lemma 3.4.2 to conclude

$$\varphi \circ \psi_j(\ell) = g_1\ell^0 \otimes \cdots \otimes g_m\ell^0, \tag{3.4.8}$$

where $g_r \in \mathrm{Hom}(\mathbb{P}(V_j^*), \mathbb{P}(V_r^*))$, $g_r\ell^0 = \ell_r^0$, are invertible projective transformations and g_j is the identity. Under the isomorphism of factorization structures provided by $g_1^{-1} \otimes \cdots \otimes g_m^{-1}$, $\varphi \circ \psi_j$ becomes the rational normal curve $\ell \mapsto (\ell^0)^{\otimes m}$, and hence $\varphi \circ \psi_j$ spans $(m+1)$-dimensional space: the whole $\varphi(\mathfrak{h}) = S^mV_j^*$. □

The next result shows that factorization curves are decomposable if and only if their complexifications are.

Lemma 3.4.4. *A factorization curve in a real factorization structure is decomposable if and only if its complexification in the complexified factorization structure is decomposable.*

Proof. Fix a real factorization structure $\mathfrak{h}$ and a degree $d \geq 1$ of a curve $\varphi \circ \psi_k$, and consider their complexifications $\mathfrak{h} \otimes \mathbb{C}$ and $\varphi^{\mathbb{C}} \circ \psi_k^{\mathbb{C}}$. Suppose $\varphi^{\mathbb{C}} \circ \psi_k^{\mathbb{C}}$ is decomposable. Then, the equation (3.3.2) shows that for any real line $\ell \in \mathbb{P}(V_k)$:

$$\begin{aligned} &\varphi \circ \psi_k(\ell) \otimes \mathbb{C} = \varphi^{\mathbb{C}} \circ \psi_k^{\mathbb{C}}(\ell \otimes \mathbb{C}) = \\ &ins_k\Big((\ell^0 \otimes \mathbb{C}) \otimes_{\mathbb{C}} g\,\underbrace{\big((\ell^0 \otimes \mathbb{C}) \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} (\ell^0 \otimes \mathbb{C})\big)}_{(d-1)-times} \otimes \langle T \rangle\Big) \end{aligned} \tag{3.4.9}$$

up to a permutation of slots, where $g = (g_1, \ldots, g_{d-1})$ are the corresponding transformations of projective lines. The expression (3.4.9) is a complexification of a real 1-dimensional vector space, and hence there exists a real subspace $\langle t \rangle$ such that

$$\varphi \circ \psi_k(\ell) \otimes \mathbb{C} = \ell^0 \otimes g(\ell^0 \otimes \cdots \otimes \ell^0) \otimes \langle t \rangle \otimes \mathbb{C} \tag{3.4.10}$$

which shows that ψ_k is decomposable. On the other hand, if ψ_k is decomposable, then (3.4.10) together with (3.3.2) give the claim. □

Now we show that if a factorization structure of dimension m contains only decomposable curves, then it must of Segre-Veronese shape and the corresponding degrees sum to m.

Lemma 3.4.5.

(i) Suppose that any factorization structure of dimension $m-1$ with decomposable factorization curves is of Segre-Veronese type. Then, any factorization structure of dimension m with decomposable curves is of Segre-Veronese type.

(ii) A factorization structure with decomposable curves is of Segre-Veronese type.

Proof. We start with proving (i). Suppose we are not in the Veronese situation as this was dealt with in Corollary 3.4.3.2. Since every curve is decomposable we have by Lemma 3.4.2

$$\varphi \circ \psi_j(\ell) = ins_j \left(\otimes g_1 \ell^0 \otimes \cdots \otimes g_{d_j} \ell^0 \otimes \langle \Gamma_j \rangle \right) \tag{3.4.11}$$

up to a permutation of slots, for some $g_r \in \mathrm{Hom}(\mathbb{P}(V_j^*), \mathbb{P}(V_{i_r}^*))$, $r = 1, \ldots, d_j$. Clearly, the span of $\{\varphi \circ \psi_j(\ell) \mid \ell \in \mathbb{P}(V_j)\}$ is the jth summands of (3.4.1), and hence the sum over $j = 1, \ldots, m$, which results in (3.4.1), is a subspace of $\varphi(\mathfrak{h})$. We claim that this subspace, whose preimage under φ we call $\mathcal{SV}$, is the whole $\varphi(\mathfrak{h})$. Equivalently, $\mathcal{SV} = \mathfrak{h}$. To show so, we take $v \in \mathfrak{h}$. Observe that factorization curves ψ_r, $r = 1, \ldots, m$, from $\mathfrak{h}$ are mapped onto all factorization curves in a quotient $\pi : \mathfrak{h} \to \mathfrak{h}^{(j,\ell_j)}$, which must be decomposable since ψ_j are. Therefore, by the assumption from the statement of this lemma, $\mathfrak{h}^{(j,\ell_j)}$ is of Segre-Veronese type and hence equals to the span of its factorization curves. Thus, $\pi(v)$ can be expressed as a linear combination of points lying on factorization curves. In addition, $\pi^{-1}(\pi(v))$ is spanned by a lift of this linear combination to a linear combination of points lying on ψ_r, $r \neq j$, thus belonging to $\mathcal{SV}$, and $\ker \pi$. It follows

$$v \in \pi^{-1}(\pi(v)) \subset \mathcal{SV} + \ker \pi = \mathcal{SV} \tag{3.4.12}$$

as $\ker \pi = \psi_j(\ell_j) \subset \mathcal{SV}$.

To show (ii) use induction with respect to the dimension of a factorization structure with the base case $m = 2$, where from their classification we know that both are of Segre-Veronese type. □

Conjecture 3.4.1. *Factorization curves are decomposable.*

Chapter 4

Separable geometries and the extremality equation

At the beginning of this chapter we establish the notion of separable toric Kähler geometries: a factorization structure determines a class of separable CR structures whose Sasaki-Reeb quotients are separable toric Kähler geometries. Then we calculate the Laplace operator on functions and scalar curvature for separable geometries corresponding to a general Segre-Veronese factorization structure. The presence of factorization structures makes these calculations elegant and easy to perform. We formulate the extremality equation as a functional system of ODEs, and derive families of ODEs which must be satisfied by its solutions A_{pq}. The ODEs which a fixed A_{pq} satisfies are parametrised by values of all other variables x_{ir}, $i \neq p$ or $r \neq q$. Most of these ODEs are derived via an extension of a differential identity used to calculate the scalar curvature.

In the case of separable toric Kähler geometries corresponding to the decomposable Segre-Veronese factorization structure,

$$\varphi(\mathfrak{h}) = \sum_{i=1}^{k} ins_i \left(S^{d_i} W_i^* \otimes \langle \Gamma_i \rangle \right), \quad \Gamma_i = \bigotimes_{\substack{b=1 \\ b \neq i}}^{k} (\epsilon_i^b)^{\otimes d_b}, \quad \epsilon_j^b \in W_b^*,$$

and a decomposable Sasaki structure β, i.e. $\varphi(\beta)$ is a decomposable tensor, all the ODEs are easy to solve and show that solutions A_{pq}, $p = 1, \ldots, k$, $q = 1, \ldots, d_p$, of the extremality equation take form

$$A_{pq}(x_{pq}) = \frac{pol_p(x_{pq}) + (a + bx_{pq})^{m+1}(\nu_{pq}^0 + \nu_{pq}^1 x_{pq})}{\prod_{\substack{j=1 \\ j \neq p}}^{k} \langle (1, x_{pq}), \epsilon_j^b \rangle^{d_j}}, \tag{4.0.1}$$

where pol_p is an univariate q-independent polynomial with degree depending on k, $d_1, \ldots, d_k$ and vectors ϵ_j^b, and $\nu_{pq}^1, \nu_{pq}^2 \in \mathbb{R}$. However, if $\varphi(\beta)$ does not decompose and $d_p \geq 3$ then solutions are as in (4.0.1) with $\nu_{pq}^1 = \nu_{pq}^2 = 0$, while for $d_p = 2$ they satisfy ODE (4.4.59) which we do not solve here.

For the product Segre-Veronese factorization structure, i.e. $\epsilon_j^b = (1,0)$ for all j and b, with a decomposable Sasaki structure

$$\varphi(\beta) = ins_p\Big((a,b)^{\otimes d_p} \otimes (1,0) \otimes \cdots \otimes (1,0)\Big), \qquad a,b \in \mathbb{R} : ab \neq 0,$$

we see that solutions (4.0.1) become polynomials. In this case we use previously derived generalised Vandermonde identities to directly verify that these polynomials satisfy the extremality equation, and hence we obtain its complete set of solutions. As a by-product we find the expression for the scalar curvature as a linear combination of momentum coordinates which we make use of in the next chapter.

4.1 Separable Sasaki and Kähler geometries

Let $(N, \mathcal{D}, \mathfrak{h}/2\pi\Lambda)$ be a toric contact $(2m+1)-$manifold with the momentum section $\mu : N \to (TN/\mathcal{D}) \otimes \mathfrak{h}^*$ and angle coordinates $\tau : N^0 \to \mathfrak{h}/2\pi\Lambda$, where $\iota : N^0 \subset N$ is the dense open set where the action is free (see Section 2.1.4). We call N *separable* if there is a factorization structure $\varphi : \mathfrak{h} \to V_1^* \otimes \cdots \otimes V_m^*$ and coordinates $[\mathbf{x}_j] : N^0 \to I_j \subset \mathbb{P}(V_j)$, $j = 1, \ldots, m$, such that

$$\mu = \varphi^T \mathbf{x} : N^0 \to (TN^0/\iota^*\mathcal{D}) \otimes \mathfrak{h}^*, \tag{4.1.1}$$

where

$$\mathbf{x} = \mathbf{x}_1 \otimes \cdots \otimes \mathbf{x}_m, \tag{4.1.2}$$

$$\mathbf{x}_j : N^0 \to [\mathbf{x}_j]^* \mathcal{O}_{V_j}(1) \otimes V_j, \tag{4.1.3}$$

and $\mathbf{x}_j = [\mathbf{x}_j]^* z_j$ is the pullback of the tautological section $z_j : \mathbb{P}(V_j) \to \mathcal{O}_{V_j}(1) \otimes V_j$, which assigns to each class $[v]$ the canonical inclusion $\mathrm{span}\{v\} \hookrightarrow V_j$ viewed as an element of $\mathrm{span}\{v\}^* \otimes V_j$.

In addition to separable toric contact geometries we shall consider separable toric CR and Sasaki geometries. In the spirit of orthotoric geometry ([4, 11]) we wish to define a complex structure J on $\mathcal{D}$ by

$$J d\tau\big|_{\mathcal{D}} = \sum_{j=1}^{m} \zeta_j([\mathbf{x}_j]) d[\mathbf{x}_j]\big|_{\mathcal{D}}, \tag{4.1.4}$$

where $d[\mathbf{x}_j] \in \Omega^1_N([\mathbf{x}_j]^* \mathcal{O}(2) \otimes \Lambda^2 V_j^*)$ and ζ_j are sections of $\mathcal{O}_{V_j}(-2) \otimes \Lambda^2 V_j$. Applying J onto the identity $0 = \eta_{\mathcal{D}}|_{\mathcal{D}} = \langle \mu, d\tau|_{\mathcal{D}} \rangle$ we get

$$0 = \langle \mu, \zeta_j([\mathbf{x}_j]) \rangle = \langle x, \varphi \circ \zeta_j([\mathbf{x}_j]) \rangle \tag{4.1.5}$$

Now for any $j \in \{1, \ldots, m\}$, $\mathbf{x}$ takes values in $V_1 \otimes \cdots \otimes \langle \mathbf{x}_j \rangle \otimes \cdots \otimes V_m$. Fixing a value of $\mathbf{x}_j$ and letting vary $\mathbf{x}_i$, $i \neq j$, (4.1.5) shows $\varphi \circ \zeta_j([\mathbf{x}_j]) \in \varphi(\mathfrak{h}) \cap V_1^* \otimes \cdots \otimes \mathbf{x}_j^0 \otimes \cdots \otimes V_m^*$ which, by the definition of factorization structure, is generically 1-dimensional and spanned by $\varphi \circ \psi_j([\mathbf{x}_j])$ (see Definition 3.1.1 and Definition 3.3.1). Thus $\zeta_j([\mathbf{x}_j])$ must be a point-wise scalar multiple of $\psi_j([\mathbf{x}_j]) \in C^\infty(N^0, [\mathbf{x}_j]^*\mathcal{O}_{V_j}(d_j) \otimes \mathfrak{h})$ and we write

$$\zeta_j([\mathbf{x}_j]) = -\frac{\psi_j([\mathbf{x}_j])}{A_j([\mathbf{x}_j])}, \tag{4.1.6}$$

where A_j is a non-vanishing section of $\mathcal{O}_{V_j}(d_j+2)$ over I_j.

From now on we shall use charts on $\mathbb{P}(V_j)$, $j = 1, \ldots, m$, to trivialise all line bundles over $\mathbb{P}(V_j)$ as well as we shall choose an area form on V_j to trivialise $\bigwedge^2 V_j$ and its dual for all j. More explicitly, we introduce an affine chart on each V_j so that $\mathbf{x}_j = (1, x_j)$ for some real-valued function x_j on N^0.
We notice that by differentiating the identity

$$\langle \mu, \psi_k(x_k) \rangle = \langle x, \varphi \circ \psi_k(x_k) \rangle = 0 \tag{4.1.7}$$

we obtain

$$\langle \partial_{x_j}\mu, \psi_k(x_k) \rangle = -\delta_j^k \langle x, \partial_{x_j}\varphi \circ \psi_k(x_k) \rangle = \delta_j^k \langle \partial_{x_j} x, \varphi \circ \psi_j(x_j) \rangle, \tag{4.1.8}$$

which is an essential identity used in many calculations to follow.

If J is to be a complex structure we must define

$$J dx_j|_{\mathcal{D}} = \frac{A_j(x_j)}{\langle \partial_{x_j}\mu, \psi_j(x_j) \rangle} \langle \partial_{x_j}\mu, d\tau \rangle\big|_{\mathcal{D}} \tag{4.1.9}$$

Indeed, assuming $J^2 = -Id_{\mathcal{D}}$ and applying J on the identity

$$\langle \partial_{x_j}\mu, J d\tau|_{\mathcal{D}} \rangle = -\left\langle \partial_{x_j}\mu, \frac{\psi_j(x_j)}{A_j(x_j)} \right\rangle dx_j|_{\mathcal{D}} \tag{4.1.10}$$

gives the claim, where (4.1.8) was used. To check J is an almost complex structure denote J_x and J_τ the corresponding parts of J which act on the two distributions generated by $dx_j|_{\mathcal{D}}$, $j = 1, \ldots, m$, and by $d\tau|_{\mathcal{D}}$, respectively. We want

$$\begin{bmatrix} 0 & J_x \\ J_\tau & 0 \end{bmatrix}^2 = -Id_{\mathcal{D}} \tag{4.1.11}$$

The equation $J_\tau J_x = -Id$ is easy to see. Indeed,

$$J^2 dx_j|_{\mathcal{D}} = \frac{A_j(x_j)}{\langle \partial_{x_j}\mu, \psi_j(x_j) \rangle} \left\langle \partial_{x_j}\mu, -\sum_{r=1}^m \frac{\psi_r(x_r)}{A_r(x_r)} dx_r|_{\mathcal{D}} \right\rangle = -dx_j|_{\mathcal{D}}. \tag{4.1.12}$$

In the ring of m-by-m matrices, a left inverse is also a right inverse, thus $J_x J_\tau = -Id$. This shows that J is an almost complex structure, and furthermore we get

$$d\tau|_{\mathcal{D}} = \sum_{r=1}^m \frac{\psi_r(x_r)}{\langle \partial_{x_r}\mu, \psi_r(x_r) \rangle} \langle \partial_{x_r}\mu, d\tau|_{\mathcal{D}} \rangle \tag{4.1.13}$$

Now we verify the integrability of J.

Lemma 4.1.1. *J is an integrable almost complex structure.*

Proof. To show that J is integrable we use the following characterisation: J is integrable if and only if for any one-form $\alpha \in T^*N$ such that $\alpha|_{\mathcal{D}}$ is of type $(1,0)$ we have that $d\alpha$ is in the differential ideal generated by such forms. Note that $\alpha|_{\mathcal{D}}$ must be a linear combination of coordinates of $d\tau|_{\mathcal{D}} + iJd\tau|_{\mathcal{D}}$. However,

$$d(d\tau|_{\mathcal{D}} + iJd\tau|_{\mathcal{D}}) = -i\sum_{r=1}^{m} d\left(\frac{\psi_r(x_r)}{A_r(x_r)}dx_r|_{\mathcal{D}}\right) = 0 \tag{4.1.14}$$

which gives the claim. $\square$

Now we explain how to obtain *separable Kähler geometry* from a separable Sasaki geometry with Sasaki structure $\beta \in \mathfrak{h}$. We note

$$Jdx_j|_{\mathcal{D}} = \frac{A_j(x_j)\langle\mu,\beta\rangle}{\langle\partial_{x_j}\mu,\psi_j(x_j)\rangle}\frac{\langle\partial_{x_j}\mu,d\tau\rangle}{\langle\mu,\beta\rangle}\bigg|_{\mathcal{D}} = \frac{A_j(x_j)\langle\mu,\beta\rangle}{\langle\partial_{x_j}\mu,\psi_j(x_j)\rangle}\mathcal{L}_{\partial_{x_j}}\frac{\langle\mu,d\tau\rangle}{\langle\mu,\beta\rangle}\bigg|_{\mathcal{D}} \tag{4.1.15}$$

since $\langle\mu,d\tau\rangle|_{\mathcal{D}} = \eta_{\mathcal{D}}|_{\mathcal{D}} = 0$. Therefore

$$\mathcal{L}_{\partial_{x_j}}\frac{\langle\mu,d\tau\rangle}{\langle\mu,\beta\rangle} = \mathcal{L}_{\partial_{x_j}}\eta_\beta \tag{4.1.16}$$

is a basic 1-form with respect to the Sasaki-Reeb vector field X_β of β because it is invariant and

$$(\mathcal{L}_{\partial_{x_j}}\eta_\beta)(X_\beta) = \underbrace{\mathcal{L}_{\partial_{x_j}}(\eta_\beta(X_\beta))}_{0} - \underbrace{\eta_\beta(\mathcal{L}_{\partial_{x_j}}(X_\beta))}_{0} = 0 \tag{4.1.17}$$

since $\eta_\beta(X_\beta) = 1$, X_β is a contact vector field, and $\partial_{x_j} \in C^\infty(N,\mathcal{D})$. Hence $\mathcal{L}_{\partial_{x_j}}\eta_\beta = \pi_\beta^*\theta_j$ for some 1-form θ_j on the quotient by X_β. Thus $(d\eta_\beta, J)$ is the pullback of the Kähler structure

$$\omega_\beta = \sum_{j=1}^{m} dx_j \wedge \theta_j, \qquad J_\beta dx_j = \frac{A_j(x_j)\langle\mu,\beta\rangle}{\langle\partial_{x_j}\mu,\psi_j(x_j)\rangle}\theta_j, \qquad \pi_\beta^*\theta_j = \mathcal{L}_{\partial_{x_j}}\frac{\langle\mu,d\tau\rangle}{\langle\mu,\beta\rangle} \tag{4.1.18}$$

on the (local) Sasaki-Reeb quotient $\pi_\beta : N^0 \to M_\beta^0$ by the Sasaki-Reeb vector field X_β. The associated Kähler metric is

$$g_\beta = \sum_{j=1}^{m}\left(\frac{\langle\partial_{x_j}\mu,\psi_j(x_j)\rangle}{A_j(x_j)\langle\mu,\beta\rangle}dx_j^2 + \frac{A_j(x_j)\langle\mu,\beta\rangle}{\langle\partial_{x_j}\mu,\psi_j(x_j)\rangle}\theta_j^2\right). \tag{4.1.19}$$

Let $\mathfrak{t} = \mathfrak{h}/\langle\beta\rangle$, and observe that

$$\partial_{x_j}\frac{\mu}{\langle\mu,\beta\rangle} = \frac{\langle\mu,\beta\rangle\partial_{x_j}\mu - \langle\partial_{x_j}\mu,\beta\rangle\mu}{\langle\mu,\beta\rangle^2} \tag{4.1.20}$$

is in $\beta^0 \cong \mathfrak{t}^*$, and that is X_β-invariant. Furthermore, since $d\tau \bmod \beta$ is basic, it descends to $\mathfrak{t}$-valued one-form dt on the quotient by X_β. Thus we may write

$$\theta_j = \langle \partial_{x_j} \mu_\beta, dt \rangle, \qquad \text{where} \qquad \mu_\beta = \frac{\mu}{\langle \mu, \beta \rangle}, \tag{4.1.21}$$

and

$$\begin{aligned} g_\beta &= \sum_{j=1}^m \left(\frac{\langle \partial_{x_j} \mu, \psi_j(x_j) \rangle}{A_j(x_j) \langle \mu, \beta \rangle} dx_j^2 + \frac{A_j(x_j) \langle \mu, \beta \rangle}{\langle \partial_{x_j} \mu, \psi_j(x_j) \rangle} \langle \partial_{x_j} \mu_\beta, dt \rangle^2 \right) \\ \omega_\beta &= \sum_{j=1}^m dx_j \wedge \langle \partial_{x_j} \mu_\beta, dt \rangle \\ J_\beta dx_j &= \frac{A_j(x_j) \langle \mu, \beta \rangle}{\langle \partial_{x_j} \mu, \psi_j(x_j) \rangle} \langle \partial_{x_j} \mu_\beta, dt \rangle \qquad J_\beta dt = -\sum_{j=1}^m \frac{\psi_j(x_j) \bmod \beta}{A_j(x_j)} dx_j \end{aligned} \tag{4.1.22}$$

In the case of the Segre-Veronese factorization structure,

$$\varphi(\mathfrak{h}) = \sum_{j=1}^k ins_j \left(S^{d_j} W_j^* \otimes \langle \Gamma_j \rangle \right), \tag{4.1.23}$$

we adapt labelling of variables to grouped slots; the variables corresponding to jth grouped slot are $x_{j1}, \ldots, x_{jd_j}$. In this notation, the identity (4.1.8) takes form

$$\langle \partial_{x_{ir}} \mu, \psi_{js}(x_{js}) \rangle = \delta_i^j \delta_r^s \langle \hat{\mathbf{x}}_i, \Gamma_i \rangle \Delta_{ir}, \tag{4.1.24}$$

where

$$\hat{\mathbf{x}}_i = \bigotimes_{\substack{j=1 \\ j \neq i}}^k \bigotimes_{s=1}^{d_j} (1, x_{js}), \tag{4.1.25}$$

$$\psi_{ir}(x_{ir}) = \left\langle ins_i \left((x_{ir}, -1)^{\otimes d_i} \otimes \Gamma_i \right) \right\rangle, \tag{4.1.26}$$

$$\Delta_{ir} = \prod_{\substack{s=1 \\ s \neq r}}^{d_i} (x_{is} - x_{ir}), \tag{4.1.27}$$

$$\tag{4.1.28}$$

δ is the Kronecker delta, and if $d_i = 1$ then Δ_{ir} is defined to be 1. In the Veronese case (4.1.24) yields also Vandermonde identities (see Remark 4.1.1). Furthermore,

$$\langle \partial_{x_{ir}} \mu_\beta, \psi_{js}(x_{js}) \bmod \beta \rangle = \delta_i^j \delta_r^s \frac{\langle \hat{\mathbf{x}}_i, \Gamma_i \rangle \Delta_{ir}}{\langle \mu, \beta \rangle} \tag{4.1.29}$$

as the contraction on $\mathfrak{t}^* \otimes \mathfrak{t}$, since $\langle \mu, \psi_{js}(x_{js}) \rangle = 0$ and $\partial_{x_{ir}} \mu_\beta \in \beta^0$.

From now on we shall work with a general Segre-Veronese factorization structure. Thus, an explicit form of separable Kähler geometry is given by

$$
\begin{aligned}
g_\beta &= \sum_{i=1}^{k}\sum_{r=1}^{d_i}\left(\frac{\Delta_{ir}\langle \hat{\mathbf{x}}_i,\Gamma_i\rangle}{A_{ir}(x_{ir})\langle \mu,\beta\rangle}dx_{ir}^2+\frac{A_{ir}(x_{ir})\langle \mu,\beta\rangle}{\Delta_{ir}\langle \hat{\mathbf{x}}_i,\Gamma_i\rangle}\langle \partial_{x_{ir}}\mu_\beta, dt\rangle^2\right)\\
\omega_\beta &= \sum_{i=1}^{k}\sum_{r=1}^{d_i} dx_{ir}\wedge\langle \partial_{x_{ir}}\mu_\beta, dt\rangle\\
J_\beta dx_{ir} &= \frac{A_{ir}(x_{ir})\langle \mu,\beta\rangle}{\Delta_{ir}\langle \hat{\mathbf{x}}_i,\Gamma_i\rangle}\langle \partial_{x_{ir}}\mu_\beta, dt\rangle \qquad J_\beta dt = -\sum_{i=1}^{k}\sum_{r=1}^{d_i}\frac{\psi_{ir}(x_{ir}) \bmod \beta}{A_{ir}(x_{ir})}dx_{ir}
\end{aligned}
\tag{4.1.30}
$$

Let $T\in C^\infty(M^0_\beta, TM\otimes\mathfrak{t}^*)$ be the angular vector fields dual to dt, i.e. $dt(T)=Id_{\mathfrak{t}}$ and $\langle dt,T\rangle = Id_{TM^0_\beta}$. Using (4.1.29) we compute

$$
\begin{aligned}
g_\beta^{-1} &= \sum_{i=1}^{k}\sum_{r=1}^{d_i}\left(\frac{A_{ir}(x_{ir})\langle \mu,\beta\rangle}{\Delta_{ir}\langle \hat{\mathbf{x}}_i,\Gamma_i\rangle}(\partial_{ir})^2+\frac{1}{A_{ir}(x_{ir})}\langle \psi_{ir}(x_{ir}) \bmod \beta, T\rangle^2\right)\\
\omega_\beta^{-1} &= \sum_{i=1}^{k}\sum_{r=1}^{d_i}\frac{\langle \mu,\beta\rangle\partial_{ir}\wedge\langle \psi_{ir}(x_{ir}) \bmod \beta, T\rangle}{\Delta_{ir}\langle \hat{\mathbf{x}}_i,\Gamma_i\rangle}\\
J_\beta T &= -\sum_{i=1}^{k}\sum_{r=1}^{d_i}\frac{A_{ir}(x_{ir})\langle \mu,\beta\rangle}{\Delta_{ir}\langle \hat{\mathbf{x}}_i,\Gamma_i\rangle}\partial_{x_{ir}}\mu_\beta \qquad J_\beta\partial_{ir} = \frac{\langle \psi_{ir}(x_{ir}) \bmod \beta, T\rangle}{A_{ir}(x_{ir})}
\end{aligned}
\tag{4.1.31}
$$

We prepare this lemma for further use in the following sections.

Lemma 4.1.2. *For each $i\in\{1,\ldots,k\}$ and $r\in\{1,\ldots,d_i\}$, we have*

$$
\sum_{j=1}^{k}\sum_{s=1}^{d_j}\left\langle \partial_{x_{ir}}\partial_{x_{js}}\boldsymbol{x}, \frac{\varphi\circ\psi_{js}(x_{js}) \ mod\ \varphi(\beta)}{\Delta_{js}\langle \hat{\boldsymbol{x}}_j,\Gamma_j\rangle}\right\rangle = \sum_{j=1}^{k}\sum_{s=1}^{d_j}\left\langle \partial_{x_{ir}}\partial_{x_{js}}\boldsymbol{x}, \frac{\varphi\circ\psi_{js}(x_{js})}{\Delta_{js}\langle \hat{\boldsymbol{x}}_j,\Gamma_j\rangle}\right\rangle = \sum_{j=1}^{k} d_j\frac{\partial_{x_{ir}}\langle \hat{\boldsymbol{x}}_j,\Gamma_j\rangle}{\langle \hat{\boldsymbol{x}}_j,\Gamma_j\rangle}+\sum_{\substack{s=1\\ s\neq r}}^{d_i}\frac{\partial_{x_{ir}}\Delta_{is}}{\Delta_{is}}
\tag{4.1.32}
$$

In addition,

$$
\sum_{\substack{s=1\\ s\neq r}}^{d_i}\frac{\partial_{x_{ir}}\Delta_{is}}{\Delta_{is}} = \frac{\partial_{x_{ir}}\Delta_{ir}}{\Delta_{ir}}.
\tag{4.1.33}
$$

Proof. Observe

$$
\langle \partial_{x_{ir}}\partial_{x_{js}}\hat{\mathbf{x}}, \varphi\circ\psi_{js}(x_{js})\rangle = \Delta_{js}\langle \partial_{x_{ir}}\hat{\mathbf{x}}_j,\Gamma_j\rangle + \delta_i^j(1-\delta_r^s)\langle \hat{\mathbf{x}}_j,\Gamma_j\rangle\partial_{x_{ir}}\Delta_{js}
\tag{4.1.34}
$$

the first term being zero if $i = j$. Indeed, if $i = j$ and $r = s$, then both sides are zero as $\partial_{x_{ir}}\partial_{x_{js}}\mathbf{x} = 0$. Other cases come from differentiation of (4.1.29) and the fact $\partial_{x_{ir}}\varphi \circ \psi_{js} \mod \varphi(\beta) = 0$. Now divide by $\Delta_{js}\langle \hat{\mathbf{x}}_j, \Gamma_j\rangle$ and sum over j, s.

The other identity follows from the fact that $(\prod_{s\neq r} \Delta_{is})/\Delta_{ir}$ is independent of x_{ir}, and thus by taking the $\partial_{x_{ir}}$-logarithmic derivative we prove the claim. □

The next remark connects (4.1.29) and (4.1.24) for the case of Veronese factorization structure to Vandermonde identities which we use for calculations in the following sections.

Remark 4.1.1. *In the case of the Veronese factorization structure $\varphi(\mathfrak{h}) = S^m W^*$ we write x_j instead of x_{1j} and Δ_j instead of Δ_{1j}. Using (4.1.24) for Veronese factorization structure with $\varphi(\beta) = (1,0)^{\otimes m}$ we get*

$$\sum_{r=1}^{m} W_{ir}V_{rj} = \delta_{ij}, \tag{4.1.35}$$

where

$$W = \begin{bmatrix} \frac{1}{\Delta_1} & \frac{\sigma_1(\hat{x}_1)}{\Delta_1} & \cdots & \frac{\sigma_{m-1}(\hat{x}_1)}{\Delta_1} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{1}{\Delta_m} & \frac{\sigma_1(\hat{x}_m)}{\Delta_m} & \cdots & \frac{\sigma_{m-1}(\hat{x}_m)}{\Delta_m} \end{bmatrix} \qquad V = \begin{bmatrix} x_1^{m-1} & \cdots & x_m^{m-1} \\ -x_1^{m-2} & \cdots & -x_m^{m-2} \\ \vdots & \cdots & \vdots \\ (-1)^{m-1} & \cdots & (-1)^{m-1} \end{bmatrix},$$

σ_r is the rth elementary symmetric polynomial in variables $x_1, \ldots, x_m$, and $\sigma_{r-1}(\hat{x}_j) := \partial_{x_j}\sigma_r$, i.e. $\sigma_{r-1}(\hat{x}_j)$ is the $(r-1)$st elementary symmetric polynomial in variables $x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_m$. Observe, $\sigma_r = \sigma_r(\hat{x}_j) + x_j\sigma_{r-1}(\hat{x}_j)$ with $\sigma_0 = 1$.

Reading (4.1.35) as $VW = id$ provides us with Vandermonde identity

$$\sum_{j=1}^{m} \frac{x_j^{m-s}\sigma_{r-1}(\hat{x}_j)}{\Delta_j} = (-1)^{s-1}\delta_{rs} \qquad \text{for any} \quad r, s = 1, \ldots, m. \tag{4.1.36}$$

This identity extends (see Appendix B in [6])

$$\sum_{j=1}^{m} \frac{x_j^{m+k}\sigma_{r-1}(\hat{x}_j)}{\Delta_j} = \sum_{s=0}^{k}(-1)^s h_{k-s}\sigma_{r+s}, \tag{4.1.37}$$

where k is a non-negative integer, $r = 1, \ldots, m$ and h_k is the kth complete homogeneous symmetric polynomial ($h_0 = 1$). In particular, for $r = 1$ we have

$$\sum_{j=1}^{m} \frac{x_j^{m-1+p}}{\Delta_j} = h_p, \tag{4.1.38}$$

where p is a nonnegative integer. In addition, the transformation $x_j \mapsto 1/x_j$ for $r = 1$

in (4.1.36) *and for* $p=1$ *in* (4.1.38), $j=1,\dots,m$, *gives*

$$\begin{aligned} \sum_{j=1}^{m}\frac{x_j^{s-2}}{\Delta_j} &= (-1)^{m-1}\frac{\delta_{s1}}{\sigma_m}, \qquad s=1,\dots,m \\ \sum_{j=1}^{m}\frac{x_j^{-2}}{\Delta_j} &= (-1)^{m-1}\frac{\sigma_{m-1}}{\sigma_m^2}. \end{aligned} \tag{4.1.39}$$

Finally, the transformation $x_j \mapsto x_j + t$ *for* $s=1$ *in* (4.1.39) *gives*

$$\begin{aligned} \sum_{j=1}^{m}\frac{(x_j+t)^{-1}}{\Delta_j} &= \frac{(-1)^{m-1}}{\prod_{j=1}^{m}(x_j+t)} \\ \sum_{j=1}^{m}\frac{(x_j+t)^{-2}}{\Delta_j} &= (-1)^{m-1}\frac{\sum_{i=1}^{m}\prod_{\substack{j=1\\ j\neq i}}^{m}(x_j+t)}{\prod_{j=1}^{m}(x_j+t)^2}, \end{aligned} \tag{4.1.40}$$

for $r=1$ *in* (4.1.36)

$$\sum_{j=1}^{m}\frac{(x_j+t)^{m-s}}{\Delta_j} = \delta_{1s} \qquad \textit{for any} \quad s=1,\dots,m, \tag{4.1.41}$$

and for $p=1$ *in* (4.1.38)

$$\sum_{j=1}^{m}\frac{(x_j+t)^{m}}{\Delta_j} = \sigma_1 + mt. \tag{4.1.42}$$

4.2 The Laplacian

Using (2.1.2), (2.1.3), (4.1.30) and (4.1.31) we calculate

$$df = \sum_{i=1}^{k}\sum_{r=1}^{d_i}(\partial_{x_{ir}} f)dx_{ir} \tag{4.2.1}$$

$$Jdf = \sum_{i=1}^{k}\sum_{r=1}^{d_i}\frac{A_{ir}(x_{ir})\langle\mu,\beta\rangle}{\Delta_{ir}\langle\hat{\mathbf{x}}_i,\Gamma_i\rangle}(\partial_{x_{ir}} f)\langle\partial_{x_{ir}}\mu_\beta, dt\rangle \tag{4.2.2}$$

$$\begin{aligned} dJdf =& \sum_{i=1}^{k}\sum_{r=1}^{d_i}\sum_{j=1}^{k}\sum_{s=1}^{d_i}\partial_{x_{js}}\left(\frac{A_{ir}(x_{ir})}{\langle\hat{\mathbf{x}}_i,\Gamma_i\rangle}(\partial_{x_{ir}} f)\right)dx_{js}\wedge\left\langle\ \frac{\langle\mu,\beta\rangle\partial_{x_{ir}}\mu_\beta}{\Delta_{ir}}, dt\right\rangle + \\ &+\sum_{i=1}^{k}\sum_{r=1}^{d_i}\frac{A_{ir}(x_{ir})}{\langle\hat{\mathbf{x}}_i,\Gamma_i\rangle}(\partial_{x_{ir}} f)\sum_{j=1}^{k}\sum_{s=1}^{d_j}dx_{js}\wedge\left\langle\partial_{x_{js}}\frac{\langle\mu,\beta\rangle\partial_{x_{ir}}\mu_\beta}{\Delta_{ir}}, dt\right\rangle \end{aligned} \tag{4.2.3}$$

$$\begin{aligned} \Delta f =& \omega_\beta^{-1}(dJdf) = \sum_{i=1}^{k}\sum_{r=1}^{d_i}\frac{\langle\mu,\beta\rangle}{\Delta_{ir}\langle\hat{\mathbf{x}}_i,\Gamma_i\rangle}\partial_{x_{ir}}\left(A_{ir}(x_{ir})(\partial_{x_{ir}} f)\right) + \\ &+\sum_{i=1}^{k}\sum_{r=1}^{d_i}\frac{A_{ir}(x_{ir})}{\langle\hat{\mathbf{x}}_i,\Gamma_i\rangle}(\partial_{x_{ir}} f)\sum_{j=1}^{k}\sum_{s=1}^{d_j}\frac{\langle\mu,\beta\rangle}{\Delta_{js}\langle\hat{\mathbf{x}}_j,\Gamma_j\rangle}\left\langle\partial_{x_{js}}\frac{\langle\mu,\beta\rangle\partial_{x_{ir}}\mu_\beta}{\Delta_{ir}}, \psi_{js}(x_{js}) \bmod \beta\right\rangle \end{aligned} \tag{4.2.4}$$

Remark 4.2.1. *Note*

$$\begin{aligned} &\sum_{i=1}^{k}\sum_{r=1}^{d_i}\frac{A_{ir}(x_{ir})}{\langle\hat{\boldsymbol{x}}_i,\Gamma_i\rangle}(\partial_{x_{ir}} f)\sum_{j=1}^{k}\sum_{s=1}^{d_j}\frac{\langle\mu,\beta\rangle}{\Delta_{js}\langle\hat{\boldsymbol{x}}_j,\Gamma_j\rangle}\left\langle\partial_{x_{js}}\frac{\langle\mu,\beta\rangle\partial_{x_{ir}}\mu_\beta}{\Delta_{ir}}, \psi_{js}(x_{js})\ mod\ \beta\right\rangle = \\ &= \sum_{i=1}^{k}\sum_{r=1}^{d_i}A_{ir}(x_{ir})(\partial_{x_{ir}} f)\sum_{j=1}^{k}\sum_{s=1}^{d_j}\frac{\langle\mu,\beta\rangle}{\Delta_{js}\langle\hat{\boldsymbol{x}}_j,\Gamma_j\rangle}\left\langle\partial_{x_{js}}\frac{\langle\mu,\beta\rangle\partial_{x_{ir}}\mu_\beta}{\Delta_{ir}\langle\hat{\boldsymbol{x}}_i,\Gamma_i\rangle}, \psi_{js}(x_{js})\ mod\ \beta\right\rangle \end{aligned} \tag{4.2.5}$$

Indeed, $\langle\hat{\boldsymbol{x}}_i,\Gamma_i\rangle$ is constant with respect to x_{is} for any $s = 1,\ldots,d_i$. On the other hand, if $i \neq j$ then

$$\left\langle\left(\partial_{x_{js}}\frac{1}{\langle\hat{\boldsymbol{x}}_i,\Gamma_i\rangle}\right)\frac{\langle\mu,\beta\rangle\partial_{x_{ir}}\mu_\beta}{\Delta_{ir}}, \psi_{js}(x_{js})\ mod\ \beta\right\rangle = 0 \tag{4.2.6}$$

by (4.1.29).

To simplify the contraction in Δf, using (4.1.7), (4.1.8) and (4.1.29) we calculate

$$\begin{aligned}
&\left\langle \partial_{x_{js}}(\langle \mu, \beta \rangle \partial_{x_{ir}} \mu_\beta), \psi_{js}(x_{js}) \bmod \beta \right\rangle = \\
&= \left\langle \partial_{x_{ir}} \partial_{x_{js}} \mathbf{x} - \frac{\langle \partial_{x_{ir}} \mu, \beta \rangle \partial_{x_{js}} \mathbf{x}}{\langle \mu, \beta \rangle}, \varphi \circ \psi_{js}(x_{js}) \bmod \varphi(\beta) \right\rangle = \\
&= \left\langle \partial_{x_{ir}} \partial_{x_{js}} \mathbf{x}, \varphi \circ \psi_{js}(x_{js}) \bmod \varphi(\beta) \right\rangle - \frac{\langle \partial_{x_{ir}} \mu, \beta \rangle \langle \hat{\mathbf{x}}_j, \Gamma_j \rangle \Delta_{js}}{\langle \mu, \beta \rangle}
\end{aligned} \tag{4.2.7}$$

so Lemma 4.1.2 implies

$$\begin{aligned}
&\sum_{j=1}^{k} \sum_{s=1}^{d_j} \left\langle \partial_{x_{js}}(\langle \mu, \beta \rangle \partial_{x_{ir}} \mu_\beta), \frac{\psi_{js}(x_{js}) \bmod \varphi(\beta)}{\Delta_{js} \langle \hat{\mathbf{x}}_j, \Gamma_j \rangle} \right\rangle = \\
&= \sum_{j=1}^{k} d_j \frac{\langle \mu, \beta \rangle}{\langle \hat{\mathbf{x}}_j, \Gamma_j \rangle} \partial_{x_{ir}} \frac{\langle \hat{\mathbf{x}}_j, \Gamma_j \rangle}{\langle \mu, \beta \rangle} + \sum_{\substack{s=1 \\ s \neq r}}^{d_i} \frac{\partial_{x_{ir}} \Delta_{is}}{\Delta_{is}}
\end{aligned} \tag{4.2.8}$$

Hence

$$\sum_{j=1}^{k} \sum_{s=1}^{d_j} \left\langle \Delta_{ir} \partial_{x_{js}} \frac{\langle \mu, \beta \rangle \partial_{x_{ir}} \mu_\beta}{\Delta_{ir} \langle \hat{\mathbf{x}}_i, \Gamma_i \rangle}, \frac{\psi_{js}(x_{js}) \bmod \varphi(\beta)}{\Delta_{js} \langle \hat{\mathbf{x}}_j, \Gamma_j \rangle} \right\rangle = \frac{1}{\langle \hat{\mathbf{x}}_i, \Gamma_i \rangle} \sum_{j=1}^{k} d_j \frac{\langle \mu, \beta \rangle}{\langle \hat{\mathbf{x}}_j, \Gamma_j \rangle} \partial_{x_{ir}} \frac{\langle \hat{\mathbf{x}}_j, \Gamma_j \rangle}{\langle \mu, \beta \rangle} \tag{4.2.9}$$

which can be seen as follows. When the numerator in the first slot is differentiated, the contraction is given by (4.2.8), while if the denominator is differentiated, the contraction is nonzero iff $j = i$ and $s = r$ due to (4.1.29) which leaves us with $-\frac{1}{\langle \hat{\mathbf{x}}_i, \Gamma_i \rangle} \frac{\partial_{x_{ir}} \Delta_{ir}}{\Delta_{ir}}$ as $\langle \hat{\mathbf{x}}_i, \Gamma_i \rangle$ is constant in this case. By (4.1.33) this gives the claim.

We conclude

$$\sum_{j=1}^{k} \sum_{s=1}^{d_j} \frac{\langle \mu, \beta \rangle}{\Delta_{js} \langle \hat{\mathbf{x}}_j, \Gamma_j \rangle} \left\langle \partial_{x_{js}} \frac{\langle \mu, \beta \rangle \partial_{x_{ir}} \mu_\beta}{\Delta_{ir} \langle \hat{\mathbf{x}}_i, \Gamma_i \rangle}, \psi_{js}(x_{js}) \bmod \beta \right\rangle = \tag{4.2.10}$$

$$= \frac{\langle \mu, \beta \rangle}{\Delta_{ir} \langle \hat{\mathbf{x}}_i, \Gamma_i \rangle} \sum_{j=1}^{k} d_j \frac{\langle \mu, \beta \rangle}{\langle \hat{\mathbf{x}}_j, \Gamma_j \rangle} \partial_{x_{ir}} \frac{\langle \hat{\mathbf{x}}_j, \Gamma_j \rangle}{\langle \mu, \beta \rangle} = \tag{4.2.11}$$

$$= \frac{\langle \mu, \beta \rangle}{\Delta_{ir} \langle \hat{\mathbf{x}}_i, \Gamma_i \rangle} \frac{\partial_{x_{ir}} H}{H} \tag{4.2.12}$$

where

$$H = \frac{\prod_{j=1}^{k} \langle \hat{\mathbf{x}}_j, \Gamma_j \rangle^{d_j}}{\langle \mu, \beta \rangle^m} \tag{4.2.13}$$

Finally

$$\Delta f =$$

$$= \sum_{i=1}^{k}\sum_{r=1}^{d_i} \frac{\langle \mu, \beta\rangle}{\Delta_{ir}\langle \hat{\mathbf{x}}_i, \Gamma_i\rangle} \partial_{x_{ir}} \left(A_{ir}(x_{ir})(\partial_{x_{ir}} f)\right) + \sum_{i=1}^{k}\sum_{r=1}^{d_i} \frac{\langle \mu, \beta\rangle A_{ir}(x_{ir})}{\Delta_{ir}\langle \hat{\mathbf{x}}_i, \Gamma_i\rangle} \frac{\partial_{x_{ir}} H}{H} \partial_{x_{ir}} f =$$

$$\sum_{i=1}^{k}\sum_{r=1}^{d_i} \frac{\langle \mu, \beta\rangle}{\Delta_{ir}\langle \hat{\mathbf{x}}_i, \Gamma_i\rangle H} \partial_{x_{ir}} \left(A_{ir}(x_{ir}) H (\partial_{x_{ir}} f)\right) \tag{4.2.14}$$

4.3 Scalar curvature

We shall use the expression (2.1.10) to calculate scalar curvature of separable Kähler geometry. For the symplectic volume we have

$$\wedge^m \omega_\beta = det(\partial_{x_{ir}} \mu_\beta) \wedge_{i,r} dx_{ir} \wedge \wedge^m dt, \tag{4.3.1}$$

while the holomorphic volume defined by $dt - iJ_\beta dt$ is

$$\frac{det(\psi_{ir}(x_{ir}) \bmod \beta)}{\prod_{i=1}^{k}\prod_{r=1}^{d_i} A_{ir}(x_{ir})} \wedge_{i,r} dx_{ir} \wedge \wedge^m dt \tag{4.3.2}$$

Furthermore, (4.1.29) implies

$$det(\partial_{x_{ir}} \mu_\beta)\, det(\psi_{ir}(x_{ir}) \bmod \beta) = \tag{4.3.3}$$

$$= det\left((\partial_{x_{ir}} \mu_\beta)(\psi_{ir}(x_{ir}) \bmod \beta)\right) = \frac{\prod_{i=1}^{k} \langle \hat{\mathbf{x}}_i, \Gamma_i\rangle^{d_i}}{\langle \mu, \beta\rangle^m} \prod_{i=1}^{k}\prod_{r=1}^{d_i} \Delta_{ir} \tag{4.3.4}$$

Therefore, the Ricci potential is $-1/2$ times logarithm of the ratio of these volumes, the last being

$$\frac{det\left(\langle \mu, \beta\rangle \partial_{x_{ir}} \mu_\beta\right)^2 \left(\prod_{i=1}^{k}\prod_{r=1}^{d_i} A_{ir}(x_{ir})\right)}{\langle \mu, \beta\rangle^m \left(\prod_{j=1}^{k} \langle \hat{\mathbf{x}}_j, \Gamma_j\rangle^{d_j}\right)\left(\prod_{i=1}^{k}\prod_{r=1}^{d_i} \Delta_{ir}\right)} \tag{4.3.5}$$

where we changed the determinant $det(\partial_{x_{ir}} \mu_\beta)$ by $\langle \mu, \beta\rangle$ for convenience in the following calculations. In order to use the formula (4.2.14) for Laplace and calculate the scalar curvature we need to understand $\partial_{x_{ir}}$-log derivatives of (4.3.5). We have

$$\partial_{x_{ir}} ln \prod_{j=1}^{k} \prod_{s=1}^{d_j} A_{js}(x_{js}) = \frac{\partial_{x_{ir}} A_{ir}(x_{ir})}{A_{ir}(x_{ir})} \tag{4.3.6}$$

$$\partial_{x_{ir}} ln \langle \mu, \beta \rangle = \frac{\langle \partial_{x_{ir}} \mu, \beta \rangle}{\langle \mu, \beta \rangle} \tag{4.3.7}$$

$$\partial_{x_{ir}} ln \prod_{j=1}^{k} \langle \hat{\mathbf{x}}_j, \Gamma_j \rangle^{d_j} = \sum_{j=1}^{k} \frac{\langle \partial_{x_{ir}} \hat{\mathbf{x}}_j, \Gamma_j \rangle}{\langle \hat{\mathbf{x}}_j, \Gamma_j \rangle} \tag{4.3.8}$$

$$\partial_{x_{ir}} ln \prod_{j=1}^{k} \prod_{s=1}^{d_j} \Delta_{js} = \sum_{s=1}^{d_j} \frac{\partial_{x_{ir}} \Delta_{is}}{\Delta_{is}} = 2 \sum_{\substack{s=1 \\ s \neq r}} \frac{\partial_{x_{ir}} \Delta_{is}}{\Delta_{is}} \tag{4.3.9}$$

where the last equality follows from (4.1.33). The last factor to $\partial_{x_{ir}}$-log differentiate is $det\left(\langle \mu, \beta \rangle \partial_{x_{ir}} \mu_\beta\right)$. To this end, we use the general formula $d \; log \; det \; A = trA^{-1} dA$ and (4.1.29) to get

$$\partial_{x_{ir}} ln \; det\left(\langle \mu, \beta \rangle \partial_{x_{ir}} \mu_\beta\right) = \sum_{j=1}^{k} \sum_{s=1}^{d_j} \left\langle \partial_{x_{ir}} \partial_{x_{js}} \left(\langle \mu, \beta \rangle \partial_{x_{ir}} \mu_\beta\right), \frac{\psi_{js}(x_{js}) \bmod \beta}{\langle \hat{\mathbf{x}}_j, \Gamma_j \rangle \Delta_{js}} \right\rangle = \tag{4.3.10}$$

$$= \left\langle \partial_{x_{ir}} \partial_{x_{js}} \mathbf{x} - \frac{\langle \partial_{x_{js}} \mu, \beta \rangle \partial_{x_{ir}} \mathbf{x}}{\langle \mu, \beta \rangle}, \frac{\varphi \circ \psi_{js}(x_{js}) \bmod \beta}{\langle \hat{\mathbf{x}}_j, \Gamma_j \rangle \Delta_{js}} \right\rangle = \tag{4.3.11}$$

$$= \left\langle \partial_{x_{ir}} \partial_{x_{js}} \mathbf{x}, \frac{\varphi \circ \psi_{js}(x_{js}) \bmod \beta}{\langle \hat{\mathbf{x}}_j, \Gamma_j \rangle \Delta_{js}} \right\rangle - \frac{\langle \partial_{x_{ir}} \mu, \beta \rangle}{\langle \mu, \beta \rangle} \tag{4.3.12}$$

which by Lemma 4.1.2 is

$$\sum_{j=1}^{k} d_j \frac{\partial_{x_{ir}} \langle \hat{\mathbf{x}}_j, \Gamma_j \rangle}{\langle \hat{\mathbf{x}}_j, \Gamma_j \rangle} + \sum_{\substack{s=1 \\ s \neq r}}^{d_i} \frac{\partial_{x_{ir}} \Delta_{is}}{\Delta_{is}} - \frac{\langle \partial_{x_{ir}} \mu, \beta \rangle}{\langle \mu, \beta \rangle} \tag{4.3.13}$$

Moreover, since

$$\partial_{x_{ir}} ln \; det^2\left(\langle \mu, \beta \rangle \partial_{x_{ir}} \mu_\beta\right) = 2 \sum_{j=1}^{k} d_j \frac{\partial_{x_{ir}} \langle \hat{\mathbf{x}}_j, \Gamma_j \rangle}{\langle \hat{\mathbf{x}}_j, \Gamma_j \rangle} + 2 \sum_{\substack{s=1 \\ s \neq r}}^{d_i} \frac{\partial_{x_{ir}} \Delta_{is}}{\Delta_{is}} - 2 \frac{\langle \partial_{x_{ir}} \mu, \beta \rangle}{\langle \mu, \beta \rangle} = \tag{4.3.14}$$

$$= \partial_{x_{ir}} ln \frac{\left(\prod_{j=1}^{k} \langle \hat{\mathbf{x}}_j, \Gamma_j \rangle^{d_j}\right)^2 \left(\prod_{j=1}^{k} \prod_{s=1}^{d_j} \Delta_{js}\right)}{\langle \hat{\mathbf{x}}, \beta \rangle^2} \tag{4.3.15}$$

for all $i = 1, \ldots, k$ and $r = 1, \ldots, d_i$, where the second sum was split and sum via (4.1.33), we obtain that $det^2\left(\langle \mu, \beta \rangle \partial_{x_{ir}} \mu_\beta\right)$ is a constant multiple of

$$\frac{\left(\prod_{j=1}^{k}\langle\hat{\mathbf{x}}_j,\Gamma_j\rangle^{d_j}\right)^2\left(\prod_{j=1}^{k}\prod_{s=1}^{d_j}\Delta_{js}\right)}{\langle\hat{\mathbf{x}},\beta\rangle^2} \tag{4.3.16}$$

and hence the Ricci potential reads (up to a constant multiple)

$$\frac{\left(\prod_{j=1}^{k}\langle\hat{\mathbf{x}}_j,\Gamma_j\rangle^{d_j}\right)\left(\prod_{j=1}^{k}\prod_{s=1}^{d_j}A_{js}(x_{js})\right)}{\langle\hat{\mathbf{x}},\beta\rangle^{m+2}} \tag{4.3.17}$$

whose $\partial_{x_{ir}}$-log derivative is

$$\begin{gathered}\sum_{j=1}^{k}d_j\frac{\partial_{x_{ir}}\langle\hat{\mathbf{x}}_j,\Gamma_j\rangle}{\langle\hat{\mathbf{x}}_j,\Gamma_j\rangle}+\frac{\partial_{x_{ir}}A_{ir}(x_{ir})}{A_{ir(x_{ir})}}-(m+2)\frac{\langle\partial_{x_{ir}}\mu,\beta\rangle}{\langle\mu,\beta\rangle}= \\ \frac{\langle\mu,\beta\rangle^{m+2}}{A_{ir}(x_{ir})}\frac{1}{\prod_{j=1}^{k}\langle\hat{\mathbf{x}}_j,\Gamma_j\rangle^{d_j}}\partial_{x_{ir}}\left(\frac{A_{ir}(x_{ir})}{\langle\mu,\beta\rangle^{m+2}}\prod_{j=1}^{k}\langle\hat{\mathbf{x}}_j,\Gamma_j\rangle^{d_j}\right)\end{gathered} \tag{4.3.18}$$

Thus the scalar curvature is a constant multiple of

$$\sum_{i=1}^{k}\sum_{r=1}^{d_i}\frac{\langle\mu,\beta\rangle^{m+1}}{\Delta_{ir}\langle\hat{\mathbf{x}}_i,\Gamma_i\rangle\left(\prod_{p=1}^{k}\langle\hat{\mathbf{x}}_p,\Gamma_p\rangle^{d_p}\right)}\partial_{x_{ir}}\left(\langle\mu,\beta\rangle^2\partial_{x_{ir}}\left(\frac{A_{ir}(x_{ir})}{\langle\mu,\beta\rangle^{m+2}}\prod_{j=1}^{k}\langle\hat{\mathbf{x}}_j,\Gamma_j\rangle^{d_j}\right)\right) \tag{4.3.19}$$

Now if $f''(x)=0$ then

$$\frac{d}{dx}\left(f^2(x)\frac{d}{dx}\frac{h(x)}{f^{m+2}(x)}\right)= \tag{4.3.20}$$

$$=\frac{h''(x)}{f^m(x)}-2(m+1)\frac{f'(x)h'(x)}{f^{m+1}(x)}+(m+1)(m+2)\frac{(f'(x))^2h(x)}{f^{m+2}(x)}= \tag{4.3.21}$$

$$=f(x)\frac{d^2}{dx^2}\frac{h(x)}{f^{m+1}(x)} \tag{4.3.22}$$

Thus we have

Theorem 4.3.1. *The scalar curvature of separable Kähler geometry* (4.1.30) *is given by*

$$\sum_{i=1}^{k}\sum_{r=1}^{d_i}\frac{\langle\mu,\beta\rangle^{m+2}}{\Delta_{ir}\langle\hat{\boldsymbol{x}}_i,\Gamma_i\rangle\left(\prod_{p=1}^{k}\langle\hat{\boldsymbol{x}}_p,\Gamma_p\rangle^{d_p}\right)}\partial_{x_{ir}}^2\left(\frac{A_{ir}(x_{ir})}{\langle\mu,\beta\rangle^{m+1}}\prod_{j=1}^{k}\langle\hat{\boldsymbol{x}}_j,\Gamma_j\rangle^{d_j}\right), \tag{4.3.23}$$

and the extremality equation reads

$$\sum_{i=1}^{k}\sum_{r=1}^{d_i}\frac{\langle\mu,\beta\rangle^{m+3}}{\Delta_{ir}\langle\hat{\boldsymbol{x}}_i,\Gamma_i\rangle\left(\prod_{p=1}^{k}\langle\hat{\boldsymbol{x}}_p,\Gamma_p\rangle^{d_p}\right)}\partial^2_{x_{ir}}\left(\frac{A_{ir}(x_{ir})}{\langle\mu,\beta\rangle^{m+1}}\prod_{j=1}^{k}\langle\hat{\boldsymbol{x}}_j,\Gamma_j\rangle^{d_j}\right)=\langle\mu,\alpha\rangle \quad (4.3.24)$$

for some $\alpha\in\mathfrak{h}$.

Remark 4.3.1. *In the case of the product Segre-Veronese factorization structure the extremality equation reads*

$$\sum_{i=1}^{k}\sum_{r=1}^{d_i}\frac{\langle\mu,\beta\rangle^{m+3}}{\Delta_{ir}}\partial^2_{x_{ir}}\left(\frac{A_{ir}(x_{ir})}{\langle\mu,\beta\rangle^{m+1}}\right)=\langle\mu,\alpha\rangle \quad (4.3.25)$$

In particular, the Segre and Veronese factorization structures recover extremality equations formulated in [4] for twisted Kähler products and twisted orthotoric geometries, respectively. In the case when $k=1$ and $m=2$ or when $k=2$ and $d_1=d_2=1$ we recover the corresponding equations for ambitoric geometries (see [8, 7]).

4.4 Shape of solutions of the (separable) extremality equation

In this section we determine the shape of solutions of the separable extremality equation corresponding to the Segre-Veronese factorization structure of dimension m using two methods. In the first method we show that a solution A_{pq} satisfies a family of ODEs parametrised by values of variables x_{ir}, where $(i,r)\neq(p,q)$. In the case of decomposable factorization structure we solve these and as a by-product we find that A_{pq} is a rational function and that A_{pq}, $q=1,\dots,d_p$, have the same polynomial function as the denominator,

$$A_{pq}(x_{pq})=\frac{pol_{pq}(x_{pq})}{\prod_{\substack{j=1\\ j\neq p}}^{k}\langle(1,x_{pq}),\epsilon_j^p\rangle^{d_j}},$$

where pol_{pq} is an univariate polynomial and ϵ_j^p are the vectors determining the decomposable factorization structure. The degree and shape of pol_{pq} depend on a particular choice of the factorization structure and decomposability of the tensor $\varphi(\beta)$ determining the separable Kähler geometry.

In the second method, when $d_p\geq 2$, we show that solutions A_{pq}, $q=1,\dots,d_p$, are even more related. In general, for $q\neq r$ the difference of A_{pq} and A_{pr} restricted to the diagonal, $A_{pq}(x)-A_{pr}(x)$, satisfies a family of ODEs parametrised by values of x_{ir}, where $(i,r)\neq(p,q)$ and $(i,r)\neq(p,r)$. For decomposable factorization structure we find that $\varphi(\beta)$ is either indecomposable and $A_{pq}(x)-A_{pr}(x)=0$ which means that A_{pq}, $q=1,\dots,d_p$, are the same rational function, or decomposable, $\varphi(\beta)=$

$ins_p\left((a,b)^{d_p}\otimes(1,0)^{m-d_p}\right)$, and we get

$$A_{pq}(x_{pq}) = \frac{pol_p(x_{pq}) + (\nu^0_{pq} + \nu^1_{pq}x_{pq})(a+bx_{pq})^{m+1}}{\prod_{\substack{j=1\\ j\neq p}}^k \langle (1,x_{pq}), \epsilon^p_j\rangle^{d_j}}, \qquad \nu^1_{pq}, \nu^2_{pq} \in \mathbb{R},$$

where pol_p is a q-independent univariate polynomial whose degree depends on a particular choice of the factorization structure and b.

4.4.1 Method I. In the previous section the extremality equation (4.3.24) for separable Kähler geometry was derived. Now we obtain necessary conditions for its solutions A_{pq}, $p=1,\ldots,k$, $q=1,\ldots,d_p$. The idea is to multiply the extremality equation with a term such that all summands but one become polynomials in a fixed variable x_{pq}. Thus, when the equation is differentiated enough times with respect to x_{pq}, it yields a family of ODEs in a single unknown A_{pq} parametrised by values of the other variables.

First, we recall conventions that empty product is defined to be $1\in\mathbb{R}$ and empty sum equals to $0\in\mathbb{R}$. We assume $k\geq 1$ and $d_j\geq 1$ for all $j\in\{1,\ldots,k\}$. We fix $p\in\{1,\ldots,k\}$ and multiply the extremality equation (4.3.24) by

$$\Delta^p_\Gamma := \left(\prod_{1\leq a<b\leq d_p}(x_{pa}-x_{pb})\right)\prod_{b=1}^k \langle \hat{\mathbf{x}}_b, \Gamma_b\rangle^{d_b}, \quad \text{with convention } \prod_{b=1}^1 \langle \hat{\mathbf{x}}_b, \Gamma_b\rangle^{d_b} = 1, \tag{4.4.1}$$

to get

$$\sum_{i=1}^k\sum_{r=1}^{d_i} \pm\hat{\Delta}_{ir}\langle\mu,\beta\rangle^{m+3}\langle \hat{\mathbf{x}}_i, \Gamma_i\rangle^{d_i-1}\partial^2_{x_{ir}}\left(\frac{A_{ir}(x_{ir})}{\langle\mu,\beta\rangle^{m+1}}\prod_{\substack{b=1\\ b\neq i}}^k\langle \hat{\mathbf{x}}_b, \Gamma_b\rangle^{d_b}\right) = \langle\mu,\alpha\rangle\Delta^p_\Gamma \tag{4.4.2}$$

where

$$\hat{\Delta}_{ir} = \frac{\Delta^p_\Gamma}{\Delta_{ir}\left(\prod_{p=1}^k\langle \hat{\mathbf{x}}_p, \Gamma_p\rangle^{d_p}\right)} = \begin{cases} \prod\limits_{\substack{1\leq a<b\leq d_p\\ a\neq r\\ b\neq r}} (x_{pa}-x_{pb}), & \text{if } i=p\\ \dfrac{\prod_{1\leq a<b\leq d_p}(x_{pa}-x_{pb})}{\Delta_{ir}}, & \text{if } i\neq p\end{cases} \tag{4.4.3}$$

The term $\langle \hat{\mathbf{x}}_i, \Gamma_i\rangle^{d_i-1}$ comes from the cancellation of $\langle \hat{\mathbf{x}}_i, \Gamma_i\rangle^{d_i}$ from the product behind the differentiation sign in (4.3.24) against $\langle \hat{\mathbf{x}}_i, \Gamma_i\rangle$, since $\langle \hat{\mathbf{x}}_i, \Gamma_i\rangle$ does not depend on x_{ir}.

Remark 4.4.1. *When $k=1$ (4.4.2) reads*

$$\sum_{r=1}^m \pm\hat{\Delta}_r\langle\mu,\beta\rangle^{m+3}\partial^2_{x_r}\left(\frac{A_r(x_r)}{\langle\mu,\beta\rangle^{m+1}}\right) = \langle\mu,\alpha\rangle\prod_{1\leq a<b\leq m}(x_a-x_b), \tag{4.4.4}$$

where we used notation from Remark 4.1.1.

We show that the degree with respect to x_{pq} of the RHS and of the coefficients at $A_{ir}(x_{ir}), \partial_{x_{ir}} A_{ir}(x_{ir})$ and $\partial^2_{x_{ir}} A_{ir}(x_{ir})$, $(i,r) \neq (p,q)$, in (4.4.2) is at most m which further implies

Lemma 4.4.1.

$$\forall p \in \{1,\dots,k\}\ \forall q \in \{1,\dots,d_p\}: \qquad \partial^{m+1}_{x_{pq}} \left(\langle \mu,\beta\rangle^{m+3} \partial^2_{x_{pq}} \left(\frac{A_{pq}(x_{pq})}{\langle \mu,\beta\rangle^{m+1}} \prod_{j=1}^{k} \langle \hat{\boldsymbol{x}}_j, \Gamma_j\rangle^{d_j} \right)\right) = 0. \tag{4.4.5}$$

Proof. To this end we observe

$$\begin{aligned} deg_{x_{pq}}\left(\langle\mu,\beta\rangle^2\right) = deg_{x_{pq}}\left(\langle \mu,\beta\rangle^{m+3}\partial_{x_{ir}}\frac{1}{\langle\mu,\beta\rangle^{m+1}}\right) &= \\ deg_{x_{pq}}\left(\langle \mu,\beta\rangle^{m+3}\partial^2_{x_{ir}}\frac{1}{\langle\mu,\beta\rangle^{m+1}}\right) &\leq 2 \\ deg_{x_{pq}}\left(\prod_{\substack{j=1\\ j\neq i}}^{k} \langle \hat{\mathbf{x}}_j,\Gamma_j\rangle^{d_j}\right) &\leq \begin{cases} m-d_p, & i=p \\ m-d_i-d_p, & i\neq p\end{cases} \\ \text{if } d_p \geq 2, \text{ then } deg_{x_{pq}}\left(\hat{\Delta}_{ir}\right) &\leq \begin{cases} d_p-2, & i=p \\ d_p-1, & i\neq p\end{cases}, \\ \text{if } d_p = 1, \text{ then } deg_{x_{pq}}\left(\hat{\Delta}_{ir}\right) &= 0 \\ deg_{x_{pq}}\left(\langle\hat{\mathbf{x}}_i,\Gamma_i\rangle^{d_i-1}\right) &\leq \begin{cases} 0, & i=p \\ d_i-1, & i\neq p\end{cases} \end{aligned} \tag{4.4.6}$$

and

$$\begin{aligned} deg_{x_{pq}}\left(\langle\mu,\alpha\rangle\right) &\leq 1 \\ deg_{x_{pq}}\left(\Delta^p_\Gamma\right) &\leq m-1. \end{aligned} \tag{4.4.7}$$

Thus, when $i=p$ the degree is $2+m-d_p+d_p-2+0=m$, while if $i\neq p$ we get $2+m-d_i-d_p+d_p-1+d_i-1=m$ as claimed. □

To deal with (4.4.5) note by direct calculation that operators $D_{j,p}$ defined by

$$D_{j,p}g = f^{p+1}\partial_{x_j}\frac{g}{f^p} \qquad p=0,1,\dots \tag{4.4.8}$$

commute $[D_{j,p},D_{j,q}]=0$, where f and g are multivariate functions and f is such that

$\partial^2_{x_j} f = 0$. Applying this with $f = \langle \mu, \beta \rangle$ we have

$$
\begin{aligned}
&\partial^l_{x_{pq}} \left(\langle \mu, \beta \rangle^{m+3} \partial^2_{x_{pq}} \left(\frac{A_{pq}(x_{pq})}{\langle \mu, \beta \rangle^{m+1}} \prod_{j=1}^k \langle \hat{\mathbf{x}}_j, \Gamma_j \rangle^{d_j} \right) \right) = \\
&= \frac{1}{\langle \mu, \beta \rangle^l} D_{pq,l-1} \circ \cdots \circ D_{pq,0} \circ D_{pq,m+2} \circ D_{pq,m+1} \left(A_{pq}(x_{pq}) \prod_{j=1}^k \langle \hat{\mathbf{x}}_j, \Gamma_j \rangle^{d_j} \right) = \\
&= \frac{1}{\langle \mu, \beta \rangle^l} D_{pq,m+2} \circ D_{pq,m+1} \circ D_{pq,l-1} \circ \cdots \circ D_{pq,0} \left(A_{pq}(x_{pq}) \prod_{j=1}^k \langle \hat{\mathbf{x}}_j, \Gamma_j \rangle^{d_j} \right) = \\
&= \langle \mu, \beta \rangle^{m+3-l} \partial^2_{x_{pq}} \left(\frac{1}{\langle \mu, \beta \rangle^{m+1-l}} \partial^l_{x_{pq}} \left(A_{pq}(x_{pq}) \prod_{j=1}^k \langle \hat{\mathbf{x}}_j, \Gamma_j \rangle^{d_j} \right) \right)
\end{aligned}
\tag{4.4.9}
$$

In particular, for $l = m+1$ we reformulated (4.4.5) and see that $A_{pq}(x_{pq}) \prod_{j=1}^k \langle \hat{\mathbf{x}}_j, \Gamma_j \rangle^{d_j}$ is a polynomial in x_{pq} of degree at most $m+2$. As not all factorization structures of Segre-Veronese type attain the bounds in degree estimates (4.4.6) and (4.4.7) (see e.g. Remark 4.4.2), we describe how the exact values of these degrees shape solutions A_{pq}. Since $\Gamma_j \in \bigotimes_{\substack{i=1 \\ i \neq j}}^k S^{d_i} W_i^*$ (see Lemma 3.2.4), we note that $deg_{x_{pa}}(\langle \hat{\mathbf{x}}_j, \Gamma_j \rangle) = deg_{x_{pb}}(\langle \hat{\mathbf{x}}_j, \Gamma_j \rangle)$ for any $a, b = 1, \ldots, d_p$ and $j = 1, \ldots, k$, i.e. the degree is the same with respect to any variable belonging to grouped p-slots. This and the exact values of degrees in (4.4.6) and (4.4.7) allow us to find the smallest $\ell_p \in \{1, \ldots, m+1\}$ such that

$$
\partial^{l_p}_{x_{pq}} \left(\langle \mu, \beta \rangle^{m+3} \partial^2_{x_{pq}} \left(\frac{A_{pq}(x_{pq})}{\langle \mu, \beta \rangle^{m+1}} \prod_{j=1}^k \langle \hat{\mathbf{x}}_j, \Gamma_j \rangle^{d_j} \right) \right) = 0 \qquad \forall q = 1, \ldots, d_p, \tag{4.4.10}
$$

similarly as in (4.4.5) where the upper estimate, $\ell_p = m+1$, was used. Using (4.4.9) we conclude (4.4.10) is equivalent to

$$
\partial^2_{x_{pq}} \left(\frac{1}{\langle \mu, \beta \rangle^{m+1-l_p}} \partial^{l_p}_{x_{pq}} \left(A_{pq}(x_{pq}) \prod_{j=1}^k \langle \hat{\mathbf{x}}_j, \Gamma_j \rangle^{d_j} \right) \right) = 0 \qquad \forall q = 1, \ldots, d_p \tag{4.4.11}
$$

Notation 4.4.1. *From now on for $p \in \{1, \ldots, k\}$ let $l_p \in \{1, \ldots, m+1\}$ be the smallest value such that* (4.4.11) *holds.*

Remark 4.4.2. *In the case of the product Segre-Veronese factorization structure we have $\langle \hat{\boldsymbol{x}}_j, \Gamma_j \rangle = 1$ for all $j = 1, \ldots, k$ which allows us to improve estimates* (4.4.6) *and* (4.4.7)*. For $d_p \geq 2$ the degree of the coefficients at $A_{ir}(x_{ir}), \partial_{x_{ir}} A_{ir}(x_{ir})$ and $\partial^2_{x_{ir}} A_{ir}(x_{ir})$ with respect to x_{pq}, $(i, r) \neq (p, q)$, behaves in the same way and can be* *summarised as follows;*

$$\begin{aligned} &\textit{if } i = p, \textit{ then } deg_{x_{pq}} \leq d_p \\ &\textit{if } i \neq p, \textit{ then } deg_{x_{pq}} \leq d_p + 1, \end{aligned} \tag{4.4.12}$$

while

$$deg_{x_{pq}} \left(\langle \mu, \alpha \rangle \Delta_\Gamma^p \right) \leq d_p. \tag{4.4.13}$$

This case corresponds to $l_p = d_p + 2$ *in (4.4.11). If we restrict ourselves further and consider* $\langle \mu, \beta \rangle$ *to be constant, then* $l_p = d_p + 1$*. In particular, these estimates hold if* $k = 1$*.*

Thus if $\varphi(\beta) = ins_j \left((a,b)^{\otimes d_j} \otimes (1,0) \otimes \cdots \otimes (1,0) \right)$ *for some* $j \in \{1, \ldots, k\}$ *and* $(a,b) \in W_j^*$*, then*

$$deg_{x_{pq}} \left(\langle \mu, \beta \rangle \right) = \begin{cases} 1, & p = j \\ 0, & p \neq j \end{cases} \tag{4.4.14}$$

and we get $l_p = d_p + 1$ *if* $j \neq p$*, and* $l_p = d_p + 2$ *if* $j = p$*.*

Finally, if $d_p = 1$ *we get* $l_p = 3$ *if* $\langle \mu, \beta \rangle$ *depends on* x_{p1} *and* $l_p = 2$ *otherwise.*

Remark 4.4.3. *Since* $\varphi(\beta) \in \otimes_{b=1}^k S^{d_b} W_b^*$ *(see Lemma 3.2.4) we note that if* $\varphi(\beta)$ *decomposes in* (p,q)*th slot, then it decomposes in the grouped p-slots. Furthermore,* $\varphi(\beta)$ *decomposes in* (p,q)*th slot if and only if* $\langle \mu, \beta \rangle = \kappa \prod_{r=1}^{d_p} \langle (1, x_{pr}), (a,b) \rangle$*, where* κ *does not depend on any* $x_{p1}, \ldots, x_{pd_p}$ *and* $(a,b) \in W_p^*$*.*

Recall $\Gamma_j \in \bigotimes_{\substack{b=1 \\ b \neq j}}^k S^{d_b} W_b^*$ (see Lemma 3.2.4). If $\Gamma_j = \bigotimes_{\substack{b=1 \\ b \neq j}}^k (\epsilon_j^b)^{\otimes d_b}$ for some $\epsilon_j^b \in W_b^*$, $j = 1, \ldots, k$, i.e. all Γ_js are decomposable, then we solve (4.4.11) as follows.

When $l_p = m + 1$ we immediately get

$$\forall q \in \{1, \ldots, d_p\}: \quad A_{pq}(x_{pq}) = \frac{pol_{pq}^{m+2}(x_{pq})}{\prod_{\substack{j=1 \\ j \neq p}}^k \langle (1, x_{pq}), \epsilon_j^p \rangle^{d_j}} \tag{4.4.15}$$

for any $\langle \mu, \beta \rangle$, where pol_{pq}^{m+2} is an univariate polynomial of degree at most $m + 2$.

Now, when $l_p < m + 1$ the equation (4.4.11) gives

$$\partial_{x_{pq}}^{l_p} \left(A_{pq}(x_{pq}) \prod_{\substack{j=1 \\ j \neq p}}^k \langle (1, x_{pq}), \epsilon_j^p \rangle^{d_j} \right) = (a_{pq}^0 + a_{pq}^1 x_{pq}) \langle \mu, \beta \rangle^{m+1-l_p}, \quad \text{where } a_{pq}^i \in \mathbb{R}. \tag{4.4.16}$$

As the LHS of (4.4.16) depends on x_{pq} only we infer either $\varphi(\beta)$ decomposes in the (p,q)th slot, or does not decompose and $a_{pq}^0 = a_{pq}^1 = 0$.

If $\varphi(\beta)$ does not decompose, then

$$\forall q \in \{1,\dots,d_p\}: \quad A_{pq}(x_{pq}) = \frac{pol_{pq}^{l_p-1}(x_{pq})}{\prod_{\substack{j=1\\ j\neq p}}^{k} \langle (1,x_{pq}), \epsilon_j^p\rangle^{d_j}}, \tag{4.4.17}$$

where $pol_{pq}^{l_p-1}$ is an univariate polynomials of degree at most $l_p - 1$.

If $\varphi(\beta)$ decomposes in the (p,q)th slot then by Remark 4.4.3 (4.4.11) reduces to

$$\partial_{x_{pq}}^{l_p} \left(A_{pq}(x_{pq}) \prod_{\substack{j=1\\ j\neq p}}^{k} \langle (1,x_{pq}), \epsilon_j^p\rangle^{d_j} \right) = (a_{pq}^0 + a_{pq}^1 x_{pq}) \langle (1,x_{pq}), (a,b)\rangle^{m+1-l_p}, \tag{4.4.18}$$

where $a_{pq}^i \in \mathbb{R}$. Thus

$$\begin{aligned} A_{pq}(x_{pq}) \prod_{\substack{j=1\\ j\neq p}}^{k} \langle (1,x_{pq}), \epsilon_j^p\rangle^{d_j} = \\ = \underbrace{\int \cdots \int}_{l_p} (a_{pq}^0 + a_{pq}^1 x_{pq}) \langle (1,x_{pq}), (a,b)\rangle^{m+1-l_p} \underbrace{dx_{pq} \cdots dx_{pq}}_{l_p} \end{aligned} \tag{4.4.19}$$

Now, if $\langle \mu, \beta\rangle$ is constant with respect to $x_{p1}, \dots, x_{pd_p}$, equivalently wrt some x_{pq}, then $b = 0$ and (4.4.19) gives

$$\forall q \in \{1,\dots,d_p\}: \quad A_{pq}(x_{pq}) = \frac{pol_{pq}^{l_p+1}(x_{pq})}{\prod_{\substack{j=1\\ j\neq p}}^{k} \langle (1,x_{pq}), \epsilon_j^p\rangle^{d_j}}. \tag{4.4.20}$$

On the other hand, if $\langle \mu, \beta\rangle$ is not constant with respect to $x_{p1}, \dots, x_{pd_p}$, equivalently wrt some x_{pq}, then $b \neq 0$ and (4.4.19) gives

$$\forall q \in \{1,\dots,d_p\}: \quad A_{pq}(x_{pq}) = \frac{pol_{pq}^{l_p-1}(x_{pq}) + (a + b x_{pq})^{m+1}(\nu_{pq}^0 + \nu_{pq}^1 x_{pq})}{\prod_{\substack{j=1\\ j\neq p}}^{k} \langle (1,x_{pq}), \epsilon_j^p\rangle^{d_j}}, \tag{4.4.21}$$

where $\nu_{pq}^i \in \mathbb{R}$.

Lemma 4.4.2. *Let A_{ir} be solutions of the extremality equation (4.3.24) with $\Gamma_j = \bigotimes_{\substack{b=1\\ b\neq j}}^{k} (\epsilon_j^b)^{\otimes d_b}$, $j = 1,\dots,k$, i.e. all Γ_js are decomposable. Let $p \in \{1,\dots,k\}$ and let $l_p - 1$ be the highest degree with respect to x_{pq} of coefficients in (4.4.2) for some $q = 1,\dots,d_p$, equivalently for all $q = 1,\dots,d_p$ (see Notation 4.4.1 and the discussion above). Let pol_{pq}^d denote an univariate polynomial of degree at most d. Then if*

(I) $l_p - 1 = m$*, then*

$$\forall q \in \{1, \dots, d_p\}: \;\; A_{pq}(x_{pq}) = \frac{pol_{pq}^{m+2}(x_{pq})}{\prod_{\substack{j=1 \\ j \neq p}}^{k} \langle (1, x_{pq}), \epsilon_j^p \rangle^{d_j}} \tag{4.4.22}$$

(II) $l_p - 1 < m$*, then*

(IIa) either $\varphi(\beta)$ *does not decompose and we have*

$$\forall q \in \{1, \dots, d_p\}: \;\; A_{pq}(x_{pq}) = \frac{pol_{pq}^{l_p-1}(x_{pq})}{\prod_{\substack{j=1 \\ j \neq p}}^{k} \langle (1, x_{pq}), \epsilon_j^p \rangle^{d_j}}, \tag{4.4.23}$$

(IIb) or $\varphi(\beta)$ *decomposes in* (p, q)*-slot for some* $q \in \{1, \dots, d_p\}$ *and we distinguish:*

(IIbi) $\langle \mu, \beta \rangle$ *is constant wrt* $x_{p1}, \dots, x_{pd_p}$ *(equivalently wrt some* x_{pr}*)*

$$\forall q \in \{1, \dots, d_p\}: \;\; A_{pq}(x_{pq}) = \frac{pol_{pq}^{l_p+1}(x_{pq})}{\prod_{\substack{j=1 \\ j \neq p}}^{k} \langle (1, x_{pq}), \epsilon_j^p \rangle^{d_j}} \tag{4.4.24}$$

(IIbii) $\langle \mu, \beta \rangle$ *is not constant wrt* $x_{p1}, \dots, x_{pd_p}$ *(equivalently wrt some* x_{pr}*)*

$$\forall q \in \{1, \dots, d_p\}: \;\; A_{pq}(x_{pq}) = \frac{pol_{pq}^{l_p-1}(x_{pq}) + (a + bx_{pq})^{m+1}(\nu_{pq}^0 + \nu_{pq}^1 x_{pq})}{\prod_{\substack{j=1 \\ j \neq p}}^{k} \langle (1, x_{pq}), \epsilon_j^p \rangle^{d_j}}, \tag{4.4.25}$$

where $\nu_{pq}^i \in \mathbb{R}$*.*

4.4.2 Method II. If $d_p \geq 2$ for some p, then we can derive more information from the extremality equation on the solutions $A_{p1}, \dots, A_{pd_p}$. In Section 4.4.1 we found that all solutions A_{ir} are rational functions (for decomposable case see Lemma 4.4.2). This allows us to interpret the extremality equation as an equation in the field of rational functions where our formal analysis takes place. Thus we can fix $p \in \{1, \dots, k\}$ and multiply (4.3.24) by $x_{pq} - x_{pr}$, where $q, r \in \{1, \dots, d_p\}$ are distinct. The resulting equation does not have poles at $x_{pq} = x_{pr}$ and hence we can evaluate at $x_{pq} = x_{pr} = x$ to get

$$\begin{gathered} A\Big[\langle \mu, \beta \rangle^2 \mathcal{P}'' - 2(m+1)\langle \mu, \beta \rangle \langle \mu, \beta \rangle' \mathcal{P}' + (m+1)(m+2)(\langle \mu, \beta \rangle')^2 \mathcal{P}\Big] + \\ 2A'\Big[\langle \mu, \beta \rangle^2 \mathcal{P}' - (m+1)\mathcal{P}\langle \mu, \beta \rangle \langle \mu, \beta \rangle'\Big] + \\ A''\mathcal{P}\langle \mu, \beta \rangle^2 = 0, \end{gathered} \tag{4.4.26}$$

where

$$A = A_{pq}(x) - A_{pr}(x) \tag{4.4.27}$$

$$\mathcal{P}^{(l)} = \partial^l_{x_{pq}}\Big|_{x_{pq}=x_{pr}=x} \prod_{j=1}^{k} \langle \hat{\mathbf{x}}_j, \Gamma_j\rangle^{d_j} = \partial^l_{x_{pr}}\Big|_{x_{pq}=x_{pr}=x} \prod_{j=1}^{k} \langle \hat{\mathbf{x}}_j, \Gamma_j\rangle^{d_j}, \quad l = 0,1,2 \tag{4.4.28}$$

$$\langle \mu, \beta\rangle' = \partial_{x_{pq}}\Big|_{x_{pq}=x_{pr}=x} \langle \mu, \beta\rangle = \partial_{x_{pr}}\Big|_{x_{pq}=x_{pr}=x} \langle \mu, \beta\rangle. \tag{4.4.29}$$

These equalities are well-defined since $\langle \mu, \beta\rangle$ and $\langle \hat{\mathbf{x}}_j, \Gamma_j\rangle$, $j \in \{1, \dots, k\}$, are symmetric in variables $x_{p1}, \dots x_{pd_p}$ (see Lemma 3.2.4). Note this also implies that (4.4.26) takes the same shape for any distinct $q, r \in \{1, \dots, d_p\}$.

Note that for a general $\langle \mu, \beta\rangle$ and Γ_i, $i = 1, \dots, k$, (4.4.26) depends on variables other than x. A direct observation reveals that if either $\{\langle \mu, \beta\rangle^2, \langle \mu, \beta\rangle\langle \mu, \beta\rangle', (\langle \mu, \beta\rangle')^2\}$ or $\{\mathcal{P}, \mathcal{P}', \mathcal{P}''\}$ is a linearly independent set in the vector space $F^p_{q,r}$ of rational functions depending on all variables x_{js} except x_{pq} and x_{pr} with coefficients in the field of rational functions $\mathbb{R}(x)$ of variable x, then (4.4.26) has trivial solutions only, i.e. $A_{pq} = A_{pr}$, $q, r = 1, \dots, d_p$, subjected to (4.4.11) and the degrees of $\langle \hat{\mathbf{x}}_i, \Gamma_i\rangle$, $i \neq p$.

We shall focus again on the case when $\Gamma_b = \bigotimes_{\substack{i=1\\ i\neq b}}^{k} (\epsilon^i_b)^{\otimes d_i}$, $b = 1, \dots, k$, i.e. all Γ_bs are decomposable. Then dividing (4.4.26) by $\langle \mu, \beta\rangle^2 \mathcal{P}$ yields

$$\begin{gathered} A\left[\mathcal{S}^2 + \mathcal{S}' - 2(m+1)\frac{\langle \mu, \beta\rangle'}{\langle \mu, \beta\rangle}\mathcal{S} + (m+1)(m+2)\left(\frac{\langle \mu, \beta\rangle'}{\langle \mu, \beta\rangle}\right)^2\right] + \\ 2A'\left[\mathcal{S} - (m+1)\frac{\langle \mu, \beta\rangle'}{\langle \mu, \beta\rangle}\right] + \\ A'' = 0, \end{gathered} \tag{4.4.30}$$

where

$$S = \sum_{\substack{j=1\\ j\neq p}}^{k} d_j \frac{\langle (0,1), \epsilon^p_j\rangle}{\langle (1,x), \epsilon^p_j\rangle} \in \mathbb{R}(x) \tag{4.4.31}$$

is the $\partial_{x_{pq}}$-logarithmic derivative of $\prod_{j=1}^{k} \langle \hat{\mathbf{x}}_j, \Gamma_j\rangle^{d_j}$ evaluated at $x_{pq} = x_{pr} = x$, i.e. $\mathcal{P}' = \mathcal{P}\mathcal{S}$. We also note

$$S = \partial_x \ln\left[\prod_{\substack{j=1\\ j\neq p}}^{k} \langle (1,x), \epsilon^b_j\rangle^{d_j}\right] \tag{4.4.32}$$

First we deal with a special case of (4.4.30)

Remark 4.4.4. *If $\langle \mu, \beta \rangle' = 0$, then* (4.4.30) *reduces to*

$$A'' + 2A'\mathcal{S} + A(\mathcal{S}^2 + \mathcal{S}') = 0, \tag{4.4.33}$$

where, this time, the prime differentiation agrees with differentiation with respect to x. Hence (4.4.33) *is equivalent with*

$$\left(e^{\int \mathcal{S}} A\right)'' = 0 \tag{4.4.34}$$

whose solutions are

$$A = \frac{\beta^0 + \beta^1 x}{e^{\int \mathcal{S}}}. \tag{4.4.35}$$

Using (4.4.32) *we see that $e^{\int S}$ is*

$$\prod_{\substack{j=1\\ j\neq p}}^{k} \langle (1, x), \epsilon_j^b \rangle^{d_j}, \tag{4.4.36}$$

up to a constant. Thus, since we already know the shape of A_{pq}s ((4.4.22), (4.4.24)*) and their differences at the diagonal (*(4.4.35)*) we conclude (seeNotation 4.4.1)*

(i) if $l_p - 1 = m$, then

$$\forall q \in \{1, \ldots, d_p\}: \;\; A_{pq}(x_{pq}) = \frac{pol_p^{m+2}(x_{pq}) + \beta^0_{pq} + \beta^1_{pq} x_{pq}}{\prod_{\substack{j=1\\ j\neq p}}^{k} \langle (1, x_{pq}), \epsilon_j^p \rangle^{d_j}}, \tag{4.4.37}$$

where $\beta^i_{pq} \in \mathbb{R}$ and pol_p^{m+2} is q-independent polynomial of degree at most $m+2$,

(ii) if $l_p - 1 < m$, then

$$\forall q \in \{1, \ldots, d_p\}: \;\; A_{pq}(x_{pq}) = \frac{pol_p^{l_p+1}(x_{pq}) + \beta^0_{pq} + \beta^1_{pq} x_{pq}}{\prod_{\substack{j=1\\ j\neq p}}^{k} \langle (1, x_{pq}), \epsilon_j^p \rangle^{d_j}}, \tag{4.4.38}$$

where $\beta^i_{pq} \in \mathbb{R}$ and $pol_p^{l_p+1}$ is q-independent polynomial of degree at most $l_p + 1$.

Observe that linear independence of $\{1, \frac{\langle \mu,\beta \rangle'}{\langle \mu,\beta \rangle}, \left(\frac{\langle \mu,\beta \rangle'}{\langle \mu,\beta \rangle}\right)^2\}$ in $F^p_{q,r}$ is equivalent to linear independence of $\langle \mu, \beta \rangle$ and $\langle \mu, \beta \rangle'$. From this and the fact that (4.4.30) is a linear combination 1, $\frac{\langle \mu,\beta \rangle'}{\langle \mu,\beta \rangle}$ and $\left(\frac{\langle \mu,\beta \rangle'}{\langle \mu,\beta \rangle}\right)^2$ we conclude that (4.4.30) has trivial solutions if and only if $\langle \mu, \beta \rangle$ and $\langle \mu, \beta \rangle'$ are linearly independent in $F^p_{q,r}$ over $\mathbb{R}(x)$. In order to understand this linear independence we derive

Lemma 4.4.3. *Let $d_j \geq 3$ and let $\Gamma \in S^{d_j} W^* \otimes \bigotimes_{\substack{b=1\\ b\neq j}}^{a} \bigotimes_{r=1}^{d_b} V^*_{br}$, $i \in \{1, \ldots, k\}$, where W and V_{br} are two dimensional spaces. If we denote $\boldsymbol{x}_b = (1, x_{b1}) \otimes \cdots \otimes (1, x_{bd_b})$,*

$b = 1, \dots, a$, *and*

$$\gamma := \langle \Gamma, \boldsymbol{x}_j \otimes \bigotimes_{\substack{b=1 \\ b \neq j}}^{a} \boldsymbol{x}_b \rangle, \tag{4.4.39}$$

then

$$\partial_{x_{jp}} \Big|_{x_{jp}=x_{jq}=x} \gamma = \partial_{x_{jq}} \Big|_{x_{jp}=x_{jq}=x} \gamma \tag{4.4.40}$$

Furthermore

$$\gamma \Big|_{x_{jp}=x_{jq}=x} \qquad \text{and} \qquad \partial_{x_{jp}} \Big|_{x_{jp}=x_{jq}=x} \gamma \tag{4.4.41}$$

are linearly dependent over rational functions $\mathbb{R}(x)$ *if and only if*

$$\gamma = \kappa^j \prod_{r=1}^{d_j} (a + b x_{jr}), \tag{4.4.42}$$

where κ^j *does not depend on* $x_{j1}, \dots, x_{jd_j}$ *and* $a, b \in \mathbb{R}$.

Proof. Note that γ is symmetric polynomial in $x_{j1}, \dots, x_{jd_j}$.

For σ^r the rth elementary symmetric polynomial in variables $x_{j1}, \dots, x_{jd_j}$ denote $\partial_{x_{jp}} \sigma^r$ by $\sigma^{r-1}(\hat{x}_{jp})$, i.e. $(r-1)$st elementary symmetric polynomial in variables $x_{j1}, \dots, x_{j(p-1)}, x_{j(p+1)}, \dots, x_{jd_j}$. Note since $\sigma^r = \sigma^r(\hat{x}_{jp}) + x_{jp}\sigma^{r-1}(\hat{x}_{jp})$ one has

$$\begin{aligned} \gamma = \sum_{r=0}^{d_j} \alpha_r \sigma_r = \alpha_0 + \alpha_1 \left[\sigma^1(\hat{x}_{jp}) + x_{jp}\right] + \alpha_2 \left[\sigma^2(\hat{x}_{jp}) + x_{jp}\sigma^1(\hat{x}_{jp})\right] + \cdots \\ + \alpha_{d_j-1} \left[\sigma^{d_j-1}(\hat{x}_{jp}) + x_{jp}\sigma^{d_j-2}(\hat{x}_{jp})\right] + \alpha_{d_j} x_{jp} \sigma^{d_j-1}(\hat{x}_{jp}), \end{aligned} \tag{4.4.43}$$

where α_r does not depend on $x_j^1, \dots, x_j^{d_j}$ for each r. Applying (4.4.43) twice yields

$$\gamma = \sum_{r=0}^{d_j-1} (\alpha_r + x_{jp}\alpha_{r+1}) \sigma^r(\hat{x}_{jp}) = \sum_{r=0}^{d_j-2} \left[\alpha_r + (x_{jp} + x_{jq})\alpha_{r+1} + x_{jp}x_{jq}\alpha_{r+2}\right] \sigma^r(\hat{x}_{jp}, \hat{x}_{jq}) \tag{4.4.44}$$

The statement (4.4.40) follows from

$$\begin{aligned} \partial_{x_{jp}} \gamma &= \sum_{r=0}^{d_j-2} \left[\alpha_{r+1} + x_{jq}\alpha_{r+2}\right] \sigma^r(\hat{x}_{jp}, \hat{x}_{jq}) \\ \partial_{x_{jq}} \gamma &= \sum_{r=0}^{d_j-2} \left[\alpha_{r+1} + x_{jp}\alpha_{r+2}\right] \sigma^r(\hat{x}_{jp}, \hat{x}_{jq}) \end{aligned} \tag{4.4.45}$$

Now we look for coefficients $\{\alpha_r\}_{r=0}^{d_j}$ so that (4.4.44) and (4.4.45) are linearly dependent when evaluated at $x_{jp} = x_{jq} = x$. The dependence relation is given by existence of rational functions $\bar{s}, \bar{t} \in \mathbb{R}(x)$, not both zero, so that

$$\forall r \in \{0, \ldots, d_j - 2\} : \qquad \bar{t}(\alpha_{r+1} + x\alpha_{r+2}) = \bar{s}(\alpha_r + 2x\alpha_{r+1} + x^2\alpha_{r+2}) \tag{4.4.46}$$

which, after clearing the denominators, is equivalent with

$$\forall r \in \{0, \ldots, d_j - 2\} : \qquad t(\alpha_{r+1} + x\alpha_{r+2}) = s(\alpha_r + 2x\alpha_{r+1} + x^2\alpha_{r+2}), \tag{4.4.47}$$

where $s, t \in \mathbb{R}[x]$.

If $t = 0$, then $\alpha_r + 2x\alpha_{r+1} + x^2\alpha_{r+2} = 0$ for all $r = 0, \ldots, d_j - 2$. Taking two derivatives in x shows $\alpha_2 = \cdots = \alpha_{d_j} = 0$. Plugging back to the equation for $r = 0$ gives $\alpha_0 + 2x\alpha_1 = 0$. Taking one derivative in x shows $\alpha_1 = 0$ which further implies $\alpha_0 = 0$. Hence $\gamma = 0$.

If $s = 0$, then $\alpha_{r+1} + x\alpha_{r+2} = 0$ for all $r = 0, \ldots, d_j - 2$. Proceeding analogously to the previous paragraph we get $\alpha_1 = \cdots = \alpha_{d_j} = 0$, and thus $\gamma = \alpha_0$, i.e. γ does not depends on $x_{j1}, \ldots, x_{jd_j}$.

If $t \neq 0$ and $s \neq 0$, then considering (4.4.47) as an equality of two polynomials in the variable x we compare coefficients at the top and the second top degree. This gives $s_{\text{top}} = s_{\text{top}} =: c \neq 0$ and $t_{\text{top}}\alpha_{r+1} + t_{\text{top}-1}\alpha_{r+2} = 2\alpha_{r+1}s_{\text{top}} + s_{\text{top}-1}\alpha_{r+2}$. Thus, if $t_{\text{top}-1} = s_{\text{top}-1}$, then $\alpha_1 = \cdots = \alpha_{d_j-1} = 0$ and plugging back to (4.4.47) for $r = 0$ forces $\alpha_0 = 0$. The solution is $\alpha_r = 0$ for $r = 0, \ldots, d_j - 1$ and α_{d_j} can be arbitrary. This solution corresponds to $\gamma = \kappa^j \prod_{r=1}^{d_j} x_{jr}$, where $\kappa_j = \alpha_{d_j}$.
Finally, if $t_{\text{top}-1} \neq s_{\text{top}-1}$

$$k\alpha_{r+1} = \alpha_{r+2}, \qquad r = 0, \ldots, d_j - 2, \tag{4.4.48}$$

where

$$k = \frac{c}{t_{\text{top}-1} - s_{\text{top}-1}} \neq 0. \tag{4.4.49}$$

So,

$$\alpha_r = k^{r-1}\alpha_1, \qquad r = 1, \ldots, d_j. \tag{4.4.50}$$

In order to determine α_0 we consider (4.4.47) for $r = 0$ and $r = 1$. The latter gives

$$\alpha_1 kt(1 + kx) = \alpha_1 s(1 + kx)^2. \tag{4.4.51}$$

If $\alpha_1 = 0$, then $\alpha_r = 0$ for $r = 1, \ldots, d_j$ and (4.4.47) for $r = 0$ forces $\alpha_0 = 0$. Thus $\gamma = 0$. However, if $\alpha_1 \neq 0$, then

$$t/s = (1 + kx)/k \tag{4.4.52}$$

which shows ((4.4.47) for $r = 0$)

$$\alpha_1 \frac{(1+kx)^2}{k} = \alpha_0 + 2x\alpha_1 + x^2 k\alpha_1 \tag{4.4.53}$$

and $\alpha_r = k^r \alpha_0$ for $r = 0, \ldots, d_j$. This solution corresponds to $\gamma = \kappa^j \prod_{r=1}^{d_j}(\frac{1}{k} + x_{jr})$. $\square$

Remark 4.4.5. *In the case when $d_j = 2$ an analogous but rather lengthy proof shows $\gamma = \kappa^j(a + b(x_{j1} + x_{j2}) + cx_{j1}x_{j2})$, where $a, b, c \in \mathbb{R}$, i.e. γ splits but $(a + b(x_{j1} + x_{j2}) + cx_{j1}x_{j2})$-part does not have to be a product of linear terms.*

Remark 4.4.6. *Thus, if $d_j \geq 3$, then $\langle\mu, \beta\rangle\big|_{x_{pq}=x_{pr}=x}$ and $\partial_{x_{pq}}\big|_{x_{pq}=x_{pr}=x}\langle\mu, \beta\rangle$ are linearly dependent over $\mathbb{R}(x)$ if and only if $\langle\mu, \beta\rangle = \kappa^p \prod_{s=1}^{d_p}(a + bx_{ps})$, where κ^p does not depend on $x_{p1}, \ldots, x_{pd_p}$. This means that $\varphi(\beta)$ is decomposable in grouped p-slots.*

In order to get non-trivial solutions of (4.4.30) we suppose $\langle\mu, \beta\rangle\big|_{x_{pq}=x_{pr}=x}$ and $\partial_{x_{pq}}\big|_{x_{pq}=x_{pr}=x}\langle\mu, \beta\rangle$ are linearly dependent over $\mathbb{R}(x)$, i.e. $\frac{\partial_{x_{pq}}\big|_{x_{pq}=x_{pr}=x}\langle\mu,\beta\rangle}{\langle\mu,\beta\rangle\big|_{x_{pq}=x_{pr}=x}} \in \mathbb{R}(x)$. For $d_j \geq 3$ we have

$$\partial_x ln(a + bx) = \frac{\partial_{x_{pq}}\big|_{x_{pq}=x_{pr}=x}\langle\mu, \beta\rangle}{\langle\mu, \beta\rangle\big|_{x_{pq}=x_{pr}=x}} \tag{4.4.54}$$

while for $d_j = 2$

$$\partial_x ln(a + 2bx + cx^2) = 2\frac{\partial_{x_{pq}}\big|_{x_{pq}=x_{pr}=x}\langle\mu, \beta\rangle}{\langle\mu, \beta\rangle\big|_{x_{pq}=x_{pr}=x}} \tag{4.4.55}$$

Therefore for $d_j \geq 3$ since (4.4.30) is an ODE of type (4.4.33) with $\mathcal{S}$ replaced by $\mathcal{S} - (m+1)\partial_x ln(a + bx)$ the solutions are

$$A = (\beta^0 + \beta^1 x)\frac{(a + bx)^{m+1}}{\prod_{\substack{j=1\\j\neq p}}^{k}\langle(1, x), \epsilon_j^p\rangle^{d_j}}, \tag{4.4.56}$$

where we assume $b \neq 0$ as the other case was solved in Remark 4.4.4, and thus (see Notation 4.4.1)

(i) if $l_p - 1 = m$, then

$$\forall q \in \{1, \ldots, d_p\}: \quad A_{pq}(x_{pq}) = \frac{pol_p^{m+2}(x_{pq}) + (a + bx_{pq})^{m+1}(\beta_{pq}^0 + \beta_{pq}^1 x_{pq})}{\prod_{\substack{j=1\\j\neq p}}^{k}\langle(1, x_{pq}), \epsilon_j^p\rangle^{d_j}}, \tag{4.4.57}$$

where $\beta_{pq}^i \in \mathbb{R}$ and pol_p^{m+2} is q-independent polynomial of degree at most $m + 2$.

(ii) if $l_p - 1 < m$, then

$$\forall q \in \{1,\dots,d_p\}: \quad A_{pq}(x_{pq}) = \frac{pol_p^{l_p-1}(x_{pq}) + (a+bx_{pq})^{m+1}(\beta^0_{pq} + \beta^1_{pq}x_{pq})}{\prod_{\substack{j=1\\ j\neq p}}^k \langle (1,x_{pq}), \epsilon_j^p\rangle^{d_j}}, \tag{4.4.58}$$

where $\beta^i_{pq} \in \mathbb{R}$ and $pol_p^{l_p-1}$ is q-independent polynomial of degree at most $l_p - 1$.

For $d_j = 2$ we rewrite (4.4.30) as

$$A\left[\mathcal{S}^2 + \mathcal{S}' - (m+1)\mathcal{S}\partial_x ln(a+2bx+cx^2) + \frac{(m+1)(m+2)}{4}\left(\partial_x ln(a+2bx+cx^2)\right)^2\right] + 2A'\Big[\mathcal{S} - \frac{m+1}{2}\partial_x ln(a+2bx+cx^2)\Big] + A'' = 0, \tag{4.4.59}$$

This ODE is not of type (4.4.33) and we do not solve it in this text.

Note that the tensor Γ which gives rise to the decomposition (4.4.42) has form $\Gamma = ins_j((a,b)^{\otimes d_j} \otimes K)$. To obtain more information on K we derive

Lemma 4.4.4. *Let T be a non-zero element of $(\langle A\rangle, \langle B\rangle)$-product factorization structure $\varphi(\mathfrak{h}) \otimes \langle B\rangle + \langle A\rangle \otimes \chi(\mathfrak{g})$ (see Example 3.1.5). Then*

$$T = \iota \otimes \kappa \tag{4.4.60}$$

for some $\iota \in \varphi(\mathfrak{h})$ and some $\kappa \in \chi(\mathfrak{g})$ if and only if

$$\langle T\rangle = \langle \iota \otimes B\rangle \quad \text{or} \quad \langle T\rangle = \langle A \otimes \kappa\rangle. \tag{4.4.61}$$

Proof. Note that any element of the product factorization structure can be written as $\tau_1 \otimes B + A \otimes \tau_2$, where $\tau_1 \in \varphi(\mathfrak{h})$ and $\tau_2 \in \chi(\mathfrak{g})$. Thus we need to solve

$$\tau_1 \otimes B + A \otimes \tau_2 = \iota \otimes \kappa \tag{4.4.62}$$

for τ_1 and τ_2. Cases when $\tau_1 \in \text{span}\{A\}$ or $\tau_2 \in \text{span}\{B\}$ readily satisfy (4.4.61). The cases when $\tau_1 \notin \text{span}\{A\}$ and $\tau_2 \notin \text{span}\{B\}$ lead to contradiction as illustrated in the following.

Supposing that ι is in the 2-dimensional space $\text{span}\{\tau_1, A\}$ we can express it as $\iota = a_1\tau_1 + a_2 A$ for some constants $a_1, a_2 \in \mathbb{R}$. This transforms (4.4.62) into

$$\tau_1 \otimes (B - a_1\kappa) + A \otimes (\tau_2 - a_2\kappa) = 0, \tag{4.4.63}$$

which is a contradiction as τ_1 and A are independent.

Suppose τ_1, A and ι span 3-dimensional space and complete them into a basis. Then, the contraction of (4.4.62) with any of the dual vectors to τ_1, A or ι, leads to a contradiction. □

As the product Segre-Veronese factorization structure is a product factorization structure in various ways (see Example 3.1.5) we have

Corollary 4.4.4.1. *A non-zero element $\varphi(\beta)$ of the product Segre-Veronese factorization structure is decomposable in (p,q)th slot if and only if it is decomposable in the grouped p-slots if and only if*

(i) $\varphi(\beta) = ins_p\left((a,b)^{\otimes d_p} \otimes \bigotimes_{\substack{b=1\\ b\neq p}}^{k} (1,0)^{\otimes d_b}\right)$ *for some* $(a,b) \in W_p^*$*, or*

(ii) there exists a non-trivial subset $I \subset \{1,\ldots,k\}$ *such that* $\varphi(\beta) = \bigotimes_{i\in I}(1,0)^{\otimes d_p} \otimes T$*, where T cannot be split as in (4.4.60) into a tensor product of two factors coming from a product factorization structure*

Gathering all results together we get

Lemma 4.4.5. *Let A_{pq} be a solution of the extremality equation (4.3.24) with* $\Gamma_j = \bigotimes_{\substack{b=1\\ b\neq j}}^{k} (\epsilon_j^b)^{\otimes d_b}$*, $j = 1,\ldots,k$, i.e. all Γ_js are decomposable. Let $p \in \{1,\ldots,k\}$ and let $l_p - 1$ be the highest degree with respect to x_{pq} of coefficients in (4.4.2) for some $q = 1,\ldots,d_p$, equivalently for all $q = 1,\ldots,d_p$ (see Notation 4.4.1 and the discussion above). Let pol_p^d denote a q-independent univariate polynomial of degree at most d. If*

(I) $l_p - 1 = m$*, then*

(Ia) either $\varphi(\beta)$ decomposes in (p,q)-slot for some $q \in \{1,\ldots,d_p\}$, $d_p \geq 1$, (this case includes $\langle\mu,\beta\rangle$ being constant) we have

$$\forall q \in \{1,\ldots,d_p\}: \quad A_{pq}(x_{pq}) = \frac{pol_p^{m+2}(x_{pq}) + (a + bx_{pq})^{m+1}(\beta_{pq}^0 + \beta_{pq}^1 x_{pq})}{\prod_{\substack{j=1\\ j\neq p}}^{k}\langle(1,x_{pq}),\epsilon_j^p\rangle^{d_j}}, \tag{4.4.64}$$

where $a, b, \beta_{pq}^i \in \mathbb{R}$*,*

(Ib) or $\varphi(\beta)$ does not decompose and $d_p \geq 3$, then solutions are of the form (4.4.64) with $\beta_{pq}^0 = \beta_{pq}^1 = 0$.

(Ic) or $\varphi(\beta)$ does not decompose and $d_p = 2$, then solutions satisfy (4.4.59) and Lemma 4.4.2

(II) $l_p - 1 < m$*, then*

(IIa) either $\varphi(\beta)$ does not decompose and $d_p \geq 3$, then we have

$$\forall q \in \{1,\ldots,d_p\}: \quad A_{pq}(x_{pq}) = \frac{pol_p^{l_p-1}(x_{pq})}{\prod_{\substack{j=1\\ j\neq p}}^{k}\langle(1,x_{pq}),\epsilon_j^p\rangle^{d_j}}, \tag{4.4.65}$$

(IIb) or $\varphi(\beta)$ does not decompose and $d_p = 2$, then solutions satisfy (4.4.59) and Lemma 4.4.2

(IIc) or $\varphi(\beta)$ decomposes in (p,q)-slot for some $q \in \{1,\dots,d_p\}$, $d_p \geq 1$, and we distinguish:

(IIci) $\langle \mu, \beta \rangle$ is constant wrt $x_{p1},\dots,x_{pd_p}$ (equivalently wrt some x_{pr})

$$\forall q \in \{1,\dots,d_p\}: \quad A_{pq}(x_{pq}) = \frac{pol_p^{l_p+1}(x_{pq}) + \beta^0_{pq} + \beta^1_{pq} x_{pq}}{\prod_{\substack{j=1\\ j\neq p}}^k \langle (1,x_{pq}), \epsilon^p_j \rangle^{d_j}}, \quad (4.4.66)$$

where $\beta^i_{pq} \in \mathbb{R}$,

(IIcii) $\langle \mu, \beta \rangle$ is not constant wrt $x_{p1},\dots,x_{pd_p}$ (equivalently wrt some x_{pr})

$$\forall q \in \{1,\dots,d_p\}: \quad A_{pq}(x_{pq}) = \frac{pol_p^{l_p-1}(x_{pq}) + (a+bx_{pq})^{m+1}(\nu^0_{pq} + \nu^1_{pq} x_{pq})}{\prod_{\substack{j=1\\ j\neq p}}^k \langle (1,x_{pq}), \epsilon^p_j \rangle^{d_j}}, \quad (4.4.67)$$

where $\nu^i_{pq} \in \mathbb{R}$.

4.5 Product Segre-Veronese extremality equation

In this section we shall describe and verify the complete set of solution of the product Segre-Veronese extremality equation with $\varphi(\beta) = ins_p\left((a,b)^{\otimes d_p} \otimes \bigotimes_{\substack{b=1\\ b\neq p}}^k (1,0)^{\otimes d_b}\right)$ for some $(a,b) \in W^*_p$, and characterise the corresponding affine extremal functions. To this end, we recall that the extremality equation in this situation reads

$$\sum_{q=1}^{d_p} \frac{\langle \mu,\beta\rangle^2 (a+bx_{pq})^{m+1}}{\Delta_{pq}} \partial^2_{x_{pq}} \left(\frac{A_{pq}(x_{pq})}{(a+bx_{pq})^{m+1}} \right) + \sum_{\substack{i=1\\ i\neq p}}^k \sum_{r=1}^{d_i} \frac{\langle \mu,\beta\rangle^2}{\Delta_{ir}} \partial^2_{x_{ir}} A_{ir}(x_{ir}) = \langle \mu, \alpha \rangle \quad (4.5.1)$$

since $\langle \mu,\beta\rangle = \prod_{q=1}^{d_p}(a+bx_{pq})$. By counting degrees we see that when $b \neq 0$ and $k \geq 2$ possible solutions are A_{pq} of the form (4.4.67) for $l = d_p + 2$ and $q = 1,\dots,d_p$, and A_{ir} of the form (4.4.66) for $l = d_i + 1$, where $i \neq p$ and $r = 1,\dots,d_i$. When $b = 0$ and $k \geq 2$ possible solutions A_{ir} are of the form (4.4.66) for $l = d_i + 1$ as before. Finally, if $k = 1$, i.e. $l - 1 = m$, then possible solutions are of the form (4.4.64).

Now we test whether functions of this form really are solutions. First, we focus on evaluating the first sum in (4.5.1). We note that $(a+bx_{pq})^{m+1}(\beta^0_{pq} + \beta^1_{pq}x_{pq})$-part of solutions does not contribute. In order to evaluate

$$\sum_{q=1}^{d_p} \frac{\langle \mu,\beta\rangle^2 (a+bx_{pq})^{m+1}}{\Delta_{pq}} \partial^2_{x_{pq}} \left(\frac{pol_p^d(x_{pq})}{(a+bx_{pq})^{m+1}} \right), \quad d \in \{d_p+1, m+2\}, \quad (4.5.2)$$

when $b \neq 0$ we express pol_p^d in powers of $(a+bx_{pq})$. To do so we use Taylor expansion

(binomial formula)

$$x^t = \sum_{l=0}^{t} \binom{t}{l} c^{t-l}(x-c)^l, \quad \text{where } t = 0, 1, \ldots \tag{4.5.3}$$

which gives

$$\begin{gathered} pol_p^d(x_{pq}) = \sum_{t=0}^{d} \alpha_t^p (x_{pq})^t = \\ = \sum_{t=0}^{d} \sum_{l=0}^{t} \binom{t}{l} c^{t-l} \alpha_t^p (x-c)^l = \sum_{l=0}^{d} \sum_{t=l}^{d} \binom{t}{l} c^{t-l} \alpha_t^p (x-c)^l \end{gathered} \tag{4.5.4}$$

For fixed $l \in \{0, \ldots, d_p + 1\}$, $k \geq 2$ and $d_p \geq 2$ we calculate

$$\sum_{q=1}^{d_p} \frac{(\frac{a}{b} + x_{pq})^{m+1}}{\Delta_{pq}} \partial_{x_j}^2 \left(\frac{(\frac{a}{b} + x_{pq})^l}{(\frac{a}{b} + x_{pq})^{m+1}} \right) = \tag{4.5.5}$$

$$= (m+1-l)(m+2-l) \sum_{q=1}^{d_p} \frac{(\frac{a}{b} + x_{pq})^{l-2}}{\Delta_{pq}} = \tag{4.5.6}$$

$$= \begin{cases} (m+1)(m+2)(-1)^{d_p-1} \dfrac{\sum_{\nu=1}^{d_p} \prod_{\substack{q=1 \\ q \neq \nu}}^{d_p} (\frac{a}{b} + x_{pq})}{\prod_{q=1}^{d_p} (\frac{a}{b} + x_{pq})^2}, & \text{if } l = 0 \\ \dfrac{(-1)^{d_p-1} m(m+1)}{\prod_{q=1}^{d_p} (\frac{a}{b} + x_{pq})}, & \text{if } l = 1 \\ 0, & \text{if } 2 \leq l \leq d_p \\ (m-d_p)(m+1-d_p), & \text{if } l = d_p + 1 \end{cases} \tag{4.5.7}$$

using Vandermonde identities from Remark 4.1.1. When $k = 1$ we observe that $l = m+1$ and $l = m+2$ evaluate to zero in (4.5.5), while for $l = 0$, $l = 1$ and $2 \leq l \leq m$

we get results as in (4.5.7) with $d_p = m$. Thus for $c = -a/b$, $k \geq 2$ and $d_p \geq 2$ we have

$$\sum_{q=1}^{d_p} \frac{(\frac{a}{b} + x_{pq})^{m+1}}{\Delta_{pq}} \partial_{x_j}^2 \left(\frac{pol_p^{d_p+1}(x_{pq})}{(\frac{a}{b} + x_{pq})^{m+1}} \right) = \tag{4.5.8}$$

$$= \sum_{l=0}^{d_p+1} \sum_{t=l}^{d_p+1} \binom{t}{l} \left(-\frac{a}{b}\right)^{t-l} \alpha_t^p (m+1-l)(m+2-l) \sum_{q=1}^{d_p} \frac{(\frac{a}{b} + x_{pq})^{l-2}}{\Delta_{pq}} = \tag{4.5.9}$$

$$= (m+2)(m+1)(-1)^{d_p-1} \frac{\sum_{\nu=1}^{d_p} \prod_{\substack{q=1 \\ q \neq \nu}}^{d_p} (\frac{a}{b} + x_{pq})}{\prod_{q=1}^{d_p} (\frac{a}{b} + x_{pq})^2} \sum_{t=0}^{d_p+1} \left(-\frac{a}{b}\right)^t \alpha_t^p +$$

$$+ (m+1)m \frac{(-1)^{d_p-1}}{\prod_{q=1}^{d_p} (\frac{a}{b} + x_{pq})} \sum_{t=1}^{d_p+1} t \left(-\frac{a}{b}\right)^{t-1} \alpha_t^p + (m+1-d_p)(m-d_p)\alpha_{d_p+1}^p, \tag{4.5.10}$$

Although a separate but straightforward computation for $k \geq 2$ and $d_p = 1$ is needed, the result is the same as the expression (4.5.10) for $d_p = 1$. For $k = 1$ a similar calculation yields

$$(m+2)(m+1)(-1)^{m-1} \frac{\sum_{\nu=1}^{m} \prod_{\substack{q=1 \\ q \neq \nu}}^{m} (\frac{a}{b} + x_q)}{\prod_{q=1}^{m} (\frac{a}{b} + x_q)^2} \sum_{t=0}^{m+2} \left(-\frac{a}{b}\right)^t \alpha_t +$$

$$+ (m+1)m \frac{(-1)^{m-1}}{\prod_{q=1}^{m} (\frac{a}{b} + x_q)} \sum_{t=1}^{m+2} t \left(-\frac{a}{b}\right)^{t-1} \alpha_t \tag{4.5.11}$$

Thus when $k \geq 2$ (4.5.2) equals

$$(m+2)(m+1)(-1)^{d_p-1} b^{2d_p} \left(\sum_{\nu=1}^{d_p} \prod_{\substack{q=1 \\ q \neq \nu}}^{d_p} (\frac{a}{b} + x_{pq}) \right) \sum_{t=0}^{d_p+1} \left(-\frac{a}{b}\right)^t \alpha_t^p +$$

$$+ (m+1)m(-1)^{d_p-1} b^{d_p} \left(\prod_{q=1}^{d_p} (a + bx_{pq}) \right) \sum_{t=1}^{d_p+1} t \left(-\frac{a}{b}\right)^{t-1} \alpha_t^p +$$

$$+ (m+1-d_p)(m-d_p)\alpha_{d_p+1}^p \prod_{q=1}^{d_p} (a + bx_{pq})^2, \tag{4.5.12}$$

while for $k = 1$ we have

$$(m+2)(m+1)(-1)^{m-1} b^{2m} \left(\sum_{\nu=1}^{m} \prod_{\substack{q=1 \\ q \neq \nu}}^{m} (\frac{a}{b} + x_q) \right) \sum_{t=0}^{m+2} \left(-\frac{a}{b}\right)^t \alpha_t +$$

$$+(m+1)m(-1)^{m-1}b^m\left(\prod_{q=1}^{m}(a+bx_q)\right)\sum_{t=1}^{m+2}t\left(-\frac{a}{b}\right)^{t-1}\alpha_t \tag{4.5.13}$$

In order to evaluate

$$\sum_{\substack{i=1\\i\neq p}}^{k}\sum_{r=1}^{d_i}\frac{\langle\mu,\beta\rangle^2}{\Delta_{ir}}\partial^2_{x_{ir}}\left(pol_i^{d_i+2}(x_{ir})+\beta^0_{ir}+\beta^1_{ir}x_{ir}\right) \tag{4.5.14}$$

for $k\geq 2$ with $pol_i^{d_i+2}(x_{ir})=\sum_{t=0}^{d_i+2}\alpha^i_t(x_{ir})^t$ we calculate for $d_i\geq 1$

$$\sum_{r=1}^{d_i}\frac{\partial^2_{x_{ir}}pol_i^{d_i+2}(x_{ir})}{\Delta_{ir}}=\alpha^i_{d_i+1}d_i(d_i+1)+\alpha^i_{d_i+2}(d_i+2)(d_i+1)\sigma_1(x_{i1},\ldots,x_{id_i}), \tag{4.5.15}$$

where for $d_i\geq 2$ we used Vandermonde identities Remark 4.1.1. Thus (4.5.14) equals

$$\langle\mu,\beta\rangle^2\sum_{\substack{i=1\\i\neq p}}^{k}\alpha^i_{d_i+1}d_i(d_i+1)+\alpha^i_{d_i+2}(d_i+2)(d_i+1)\sigma_1(x_{i1},\ldots,x_{id_i}) \tag{4.5.16}$$

In order to ensure that the sum of (4.5.2) and (4.5.14) is affine linear in μ as required by the extremality equation (4.5.1) we are forced $\alpha_{d_i+2}=0$ for all $i\neq p$ and

$$(m+1-d_p)(m-d_p)\alpha^p_{d_p+1}+\sum_{\substack{i=1\\i\neq p}}^{k}\alpha^i_{d_i+1}d_i(d_i+1)=0 \tag{4.5.17}$$

When $b=0$ the situation is slightly different and governed by (4.5.16) for any $k\geq 1$. Thus we have

Theorem 4.5.1. *Let pol^d denote an univariate polynomial of degree at most d with coefficients α_r, $r=0,\ldots,d$. The complete set of solutions of the product Segre-Veronese extremality equations with $\varphi(\beta)=ins_p\left((a,b)^{\otimes d_p}\otimes\bigotimes_{\substack{b=1\\b\neq p}}^{k}(1,0)^{\otimes d_b}\right)$ for some $(a,b)\in W_p^*$ are as follows.*

$k=1$ *:*

$b=0$

$$\forall j\in\{1,\ldots,m\}:\ \ A_j(x_j)=pol^{m+2}(x_j)+\beta^0_j+\beta^1_jx_j \tag{4.5.18}$$

with the extremal affine function

$$m(m+1)a^2\alpha_{m+1}+(m+1)(m+2)a^2\alpha_{m+2}\sigma_1(x_1,\ldots,x_m) \tag{4.5.19}$$

$b \neq 0$

$$\forall j \in \{1,\ldots,m\}: \;\; A_j(x_j) = pol^{m+2}(x_j) + (a+bx_j)^{m+1}(\beta_j^0 + \beta_j^1 x_j) \tag{4.5.20}$$

with the extremal affine function

$$(m+2)(m+1)(-1)^{m-1}b^{2m}\left(\sum_{\nu=1}^{m}\prod_{\substack{q=1\\q\neq\nu}}^{m}(\frac{a}{b}+x_q)\right)\sum_{t=0}^{m+2}\left(-\frac{a}{b}\right)^t\alpha_t +$$

$$+(m+1)m(-1)^{m-1}b^m\left(\prod_{q=1}^{m}(a+bx_q)\right)\sum_{t=1}^{m+2}t\left(-\frac{a}{b}\right)^{t-1}\alpha_t \tag{4.5.21}$$

$k \geq 2$ *:*

$b = 0$

$$\forall i \in \{1,\ldots,k\} \;\; \forall r \in \{1,\ldots,d_i\}: \;\; A_{ir}(x_{ir}) = pol_i^{d_i+2} + \beta_{ir}^0 + \beta_{ir}^1 x_{ir} \tag{4.5.22}$$

with the extremal affine function

$$a^2\sum_{i=1}^{k} d_i(d_i+1)\alpha_{d_i+1}^i + (d_i+1)(d_i+2)\alpha_{d_i+2}^i\sigma_1(x_{i1},\ldots,x_{id_i}) \tag{4.5.23}$$

$b \neq 0$

$$\forall q \in \{1,\ldots,d_p\}: \;\; A_{pq}(x_{pq}) = pol_p^{d_p+1} + (a+bx_{pq})^{m+1}(\beta_{pq}^0 + \beta_{pq}^1 x_{pq}) \tag{4.5.24}$$

$$\forall i \in \{1,\ldots,k\}\backslash\{p\} \;\; \forall r \in \{1,\ldots,d_i\}: \;\; A_{ir}(x_{ir}) = pol_i^{d_i+1} + \beta_{ir}^0 + \beta_{ir}^1 x_{ir}, \tag{4.5.25}$$

where

$$(m+1-d_p)(m-d_p)\alpha_{d_p+1}^p + \sum_{\substack{i=1\\i\neq p}}^{k}\alpha_{d_i+1}^i d_i(d_i+1) = 0, \tag{4.5.26}$$

with the extremal affine function

$$(m+2)(m+1)(-1)^{d_p-1}b^{2d_p}\left(\sum_{\nu=1}^{d_p}\prod_{\substack{q=1\\q\neq\nu}}^{d_p}(\frac{a}{b}+x_{pq})\right)\sum_{t=0}^{d_p+1}\left(-\frac{a}{b}\right)^t\alpha_t^p+$$

$$+(m+1)m(-1)^{d_p-1}b^{d_p}\left(\prod_{q=1}^{d_p}(a+bx_{pq})\right)\sum_{t=1}^{d_p+1}t\left(-\frac{a}{b}\right)^{t-1}\alpha_t^p \quad (4.5.27)$$

For latter purposes we expand products from the above theorem in

Lemma 4.5.2. *For $d \geq 1$ we have*

$$\sum_{\nu=1}^{d}\prod_{\substack{q=1\\q\neq\nu}}^{d}(\frac{a}{b}+x_{pq})=\sum_{t=0}^{d-1}(d-t)\left(\frac{a}{b}\right)^{d-t-1}\sigma_t \quad (4.5.28)$$

Proof. Taking $\partial_{x_{p\nu}}$-derivative of the identity

$$\prod_{q=1}^{d}(\frac{a}{b}+x_{pq})=\sum_{t=0}^{d}\left(\frac{a}{b}\right)^{d-t}\sigma_t(x_{p1},\ldots,x_{pd_p}) \quad (4.5.29)$$

we find

$$\prod_{\substack{q=1\\q\neq\nu}}^{d}(\frac{a}{b}+x_{pq})=\sum_{t=0}^{d-1}\left(\frac{a}{b}\right)^{d-t-1}\sigma_t(\hat{x}_{p\nu}) \quad (4.5.30)$$

We note

$$\sum_{\nu=1}^{d}\sigma_t(\hat{x}_{p\nu})=k\sigma_t \quad \text{for some } k\in\mathbb{R} \quad (4.5.31)$$

since the LHS of (4.5.31) is symmetric in $x_{p1},\ldots,x_{pd}$ and has degree t. To specify k we restrict (4.5.31) to the diagonal, i.e. $x_{p1}=\cdots=x_{pd}=x$, and get $k=d-t$. Thus summing (4.5.30) over $\nu=1,\ldots,d$ we get (4.5.28). □

Chapter 5

Compactification

The purpose of this chapter is to outline a geometrical characterisation of compactifications of separable toric Kähler geometries of the product Segre-Veronese type. The main idea is to describe these compactifications via Delzant polytopes with extremal affine functions belonging to a particular family. This approach originates in [42] where equipoised extremal affine functions were considered. Here we work with Delzant polytopes compatible with the factorization structure. Such a polytope is the image of the m-cube in separable coordinates and has the number of facets between $m+1$ and $2m$.

We derive a version of compactification theorem (Theorem 2.1.3) which shows that a separable geometry compactifies only if boundary conditions are satisfied. These form a system (B) of linear equations which involves functions A_{ir} defining the metric and scales of normals of a Delzant polytope where the geometry compactifies. In the case of extremal metric, (B) is over-determined. In order to characterise compactifications of extremal geometries via extremal affine functions we consider a linear system (E) which expresses the fact that the extremal affine function of a Delzant polytope Δ is the L^2-projection of the scalar curvature of the corresponding extremal metric. Furthermore, Proposition 2.1.1 reveals that if an extremal metric g compactifies, i.e. (B) is satisfied, then $\mathrm{Scal}(g)$ is the extremal affine function, and hence (E) is satisfied.

For extremal separable toric Kähler metrics corresponding to the product Segre-Veronese factorization structure with a decomposable Sasaki structure we characterise their extremal affine functions which, with the use of system (E), imposes constraints on scales, like the system (B). We show that if (B) and (E) have both full ranks, then compactifications of such a geometry can be characterised via Delzant polytopes with extremal affine functions belonging to a particular family.

We start with a description of compatible (Delzant) polytopes.

5.1 Delzant polytopes and factorization structure

For a toric symplectic geometry $(M, \mathbb{T}^m)$ the corresponding Delzant polytope $\bar{\Delta} \subset \mathfrak{t}^*$, $\mathfrak{t} = \mathrm{Lie}(\mathbb{T}^m)$, associated via Delzant correspondence ([28, 36]) is, in particular, a set of integral vectors $u_j \in \mathfrak{t}$, $j = 1, \dots, n$, together with constants $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ such that

$$\bar{\Delta} = \{x \in \mathfrak{t}^* \mid L_j(x) \geq 0, \ \ j = 1, \dots, n\} \tag{5.1.1}$$

where the affine functions L_j are given by

$$L_j(x) = \langle u_j, x \rangle + \lambda_j. \tag{5.1.2}$$

Moreover, $\bar{\Delta}$ is the image of the momentum map $\mu_M : M \to \mathfrak{t}^*$ of M and the vectors u_j are understood to be normals of affine hyperplanes bounding $\bar{\Delta}$. On the other hand, according to [44], the image of the momentum map $\mu_N : N \to \mathfrak{h}^*$ of a toric contact geometry $(N, \mathbb{T}^{m+1})$, $\mathrm{Lie}(\mathbb{T}^{m+1}) = \mathfrak{h}$, is a good convex polyhedral cone Δ in $\mathfrak{h}^*$ which is associated to N via the contact version of Delzant correspondence. Such a cone is given by a set of integral vectors $v_j \in \mathfrak{h}$ which are normals to the hyperplane bounding the cone.

Furthermore, if M is the quotient of N by a Reeb vector field X_β, $\beta \in \mathfrak{h}$, then there is a geometric correspondence between $\Delta = \mathrm{im}(\mu_N)$ and $\bar{\Delta} = \mathrm{im}(\mu_M)$. In fact, we have

$$\mu_M = \frac{\mu_N}{\langle \mu_N, \beta \rangle}$$

showing that $\bar{\Delta}$ is the intersection of Δ with the affine chart $\mathcal{A}_\beta$ given by β (see Section 2.1.4). The linear forms on $\mathfrak{h}^*$ corresponding to normals $v_j \in \mathfrak{h}$ restrict to affine functions on $\mathcal{A}_\beta$ which correspond to the affine functions L_j in an identification of $\mathfrak{t}^*$ with $\mathcal{A}_\beta$.

Remark 5.1.1. *The following discussion works for a general factorization structures too, but we restrict ourselves to the product Segre-Veronese factorization structure in order to establish notation for what follows.*

Recall that if N and M are separable then their momentum maps satisfy $\langle \mu_N, \psi_{ir} \rangle = 0$ and $\langle \mu_M, \psi_{ir} \bmod \beta \rangle = 0$ for $i = 1, \dots, k$, $r = 1, \dots, d_i$ (see (4.1.7)), where ψ_{ir} represents a 1-parametric family of hyperplanes in $\mathfrak{h}^*$. We require Δ to be compatible with the underlying factorization φ in the following sense. We say that the product Segre-Veronese factorization structure φ is *compatible* with Δ if the product of intervals $I := \times_{i=1}^{k} \times_{r=1}^{d_i} I_{ir}$ maps bijectively onto Δ via the composition of the Segre embedding

$$\begin{aligned} \mathbb{P}(V_1) \times \cdots \times \mathbb{P}(V_m) &\to \mathbb{P}(V) \\ ([v_1], \dots, [v_m]) &\mapsto [v_1 \otimes \cdots \otimes v_m], \end{aligned} \tag{5.1.3}$$

with the dual projection $\mathbb{P}(V) \dashrightarrow \mathbb{P}(\mathfrak{h}^*)$ induced by the transpose φ^T (see Appendix A in [7]). Thus, when φ and Δ are compatible, the boundary of I is mapped onto the boundary of Δ, so the number of facets for Δ is at most $2m$, and $\{\psi_{ir}(\lambda_{ir}^{\nu})\}$ represent these as can be seen from (4.1.7), where λ_{ir}^1 and λ_{ir}^1 are endpoints of I_{ir}. A Delzant polytope $\bar{\Delta}$ is called compatible with φ if it is an affine slice of a compatible Δ by $\mathcal{A}_\beta$.

As an example of compatible polytopes we describe a class of quadrilaterals corresponding to ambitoric compactifications studied in [8]. These are compatible with the

Veronese factorization structure $\varphi(\mathfrak{h}) = S^2W^*$ and originate from a projective quadrilateral $\Delta \subset \mathbb{P}(\mathfrak{h}^*)$ given by 4 points on a rational normal curve of degree 2, i.e. the factorization curve, corresponding to the projective normals of facets of compatible Δ. A rational normal curve of degree 2 is determined by 5 points, or, equivalently by a single quadric, and hence there is a family of Veronese factorization structures compatible with such a polytope. On the other hand, the local classification of ambitoric geometries reveals that the corresponding extremal affine functions must be of particular shape which imposes a linear relation on quadrics and fixes a compatible Veronese factorization structure uniquely.

Now we describe a class of projective polytopes Δ which are compatible with the product Segre-Veronese factorization structure of dimension m with k factorization curves. We start with cases when this compatibility is determined uniquely. These can be described as a choice of $d_j + 3 + n_j$ points on the factorization curve ψ_j for each $j = 1, \dots, k$, which represent projective normals of hyperplanes bounding Δ, where $d_j - 3 \geq n_j \geq 0$ are integers and ψ_j represents the curve $\psi_{j1} = \cdots = \psi_{jd_j}$. Indeed, since ψ_j, $j = 1, \dots, k$, is a rational normal curve of degree d_j, it is determined by $d_j + 3$ points, which, in the end, fixes the factorization structure. To illustrate how this projective polytope Δ looks we take the affine slice of Δ given by $\varphi(\beta) = (1, 0)^{\otimes m+1} \in \varphi(\mathfrak{h})$ which results in a product of k polytopes with $d_j + 3 + n_j$ facets. This can be seen from the shape of momentum map in this case (see (5.3.2)). The reason for n_j to be bounded from above by $d_j - 3$ is related with domains of definition for ψ_{ir}, or, equivalently, A_{ir}. Recall that normals of facets are of the form $\psi_{ir}(\lambda_{ir}^\nu)$. For a fixed i there are at most $2d_i$ of them and at least $d_i + 1$ which happens when $I_{ir} \cap I_{i,r+1} = \{\lambda_{ir}^2\} = \{\lambda_{i,r+1}^1\}$ for all $r = 1, \dots, d_i$, hence the bound on n_j.

In general, if there is an underdetermined factorization curve, then the corresponding polytope is compatible with families of factorization structures. Sometimes, however, this freedom can be compensated by restricting the class of extremal affine functions as in ambitoric situation.

5.2 Boundary conditions

We derive a version of Theorem 2.1.3 for separable Kähler metrics corresponding to the product Segre-Veronese factorization structure $\mathfrak{h}$. Compactifications of such a metric are defined for Delzant polytopes compatible with the factorization structure; these are images of compatible projective polytopes in the affine chart determined by $\beta \in \mathfrak{h}$. Thus, a compatible Delzant polytope has normals of the form

$$C_{\lambda_{ir}^\nu} \psi_{ir}(\lambda_{ir}^\nu) \bmod \beta$$

for some $C_{\lambda_{ir}^\nu} \in \mathbb{R}^\times$.

Lemma 5.2.1. *Let (M, ω) be a toric compact symplectic geometry and the corresponding Delzant polytope Δ be compatible with the product Segre-Veronese factorization* *structure. Let* $\boldsymbol{H}\colon \Delta^0 \to S^2\mathfrak{t}^*$ *defined by*

$$\boldsymbol{H} = \sum_{i=1}^{k}\sum_{r=1}^{d_i} \frac{A_{ir}(x_{ir})\langle \mu, \beta\rangle}{\langle \partial_{x_{ir}}\mu, \psi_{ir}(x_{ir})\rangle} \langle \partial_{x_{ir}}\mu_\beta, dt\rangle^2, \tag{5.2.1}$$

be positive definite. Then $\boldsymbol{H}$ *is the torus part of a* $\mathbb{T}^m$*-invariant,* ω*-compatible Kähler metric on* M *via* (4.1.22) *if and only if*

- [smoothness] A_{ir} *is a smooth function on* I_{ir} *for all* $i = 1, \dots, k$ *and all* $r = 1, \dots, d_i$*;*
- [boundary values] $\forall\, i = 1, \dots, k\ \forall\, r = 1, \dots, d_i\ \forall \nu = 1, 2:$

$$A_{ir}(\lambda_{ir}^\nu) = 0 \tag{5.2.2}$$

and

$$C_{\lambda_{ir}^\nu} A'_{ir}(\lambda_{ir}^\nu) = 2; \tag{5.2.3}$$

- [positivity] *for any point* y *in interior of a face* $F \subset \Delta$*,* $\boldsymbol{H}_y(-,-)$ *is positive definite when viewed as a smooth function with values in* $S^2(\mathfrak{t}/\mathfrak{t}_F)^*$*, where* $\mathfrak{t}_F \subset \mathfrak{t}$ *is the vector subspace spanned by the inward normals* $u_j \in \mathfrak{t}$ *to all codimension one faces of* Δ *containing* F*.*

Proof. We check that formulation of this lemma fits into the framework of Theorem 2.1.3.
The first thing to check is smoothness. Using $\mu = \varphi^T x$, $\langle \mu, \beta\rangle > 0$ and $\langle \partial_{x_{ir}}\mu, \psi_{ir}(x_{ir})\rangle = \Delta_{ir}$ we see that all expressions in (5.2.1) are explicit. The only part to comment on is the denominator Δ_{ir} as it may get zero on facets. However, the terms which are zero on facets cancel against the numerator since $A_{ir}(\lambda_{ir}^\nu) = 0$ as we shall see now.

At the facet corresponding to $\psi_{js}(\lambda_{js}^\nu)$ (i.e. at $x_{js} = \lambda_{js}^\nu$) we have

$$0 = H(C_{\lambda_{js}^\nu}\psi_{js}(\lambda_{js}^\nu) \bmod \beta, -) = C_{\lambda_{js}^\nu} A_{js}(\lambda_{js}^\nu)\langle \mu, \beta\rangle \langle \partial_{x_{js}}\mu_\beta, dt\rangle\big|_{x_{js}=\lambda_{js}^\nu} \tag{5.2.4}$$

by (4.1.29). We claim $A_{js}(\lambda_{js}^\nu) = 0$. Indeed, since $\langle \partial_{x_{js}}\mu_\beta, dt\rangle\big|_{x_{js}=\lambda_{js}^\nu} = 0$ if and only if

$$\langle \mu, \beta\rangle \partial_{x_{js}}\mu = \langle \partial_{x_{js}}\mu, \beta\rangle \mu \qquad \text{at } x_{js} = \lambda_{js}^\nu \tag{5.2.5}$$

we conclude that the contraction of (5.2.5) with $\psi_{js}(\lambda_{js}^\nu)$ gives

$$\langle \mu, \beta\rangle \Delta_{js} = 0 \qquad \text{at } x_{js} = \lambda_{js}^\nu \tag{5.2.6}$$

which is a contradiction as $\langle \mu, \beta\rangle > 0$ and Δ_{js} at $x_{js} = \lambda_{js}^\nu$ cannot be zero for all values of x_{jr} where $r \neq s$.

By (5.2.1) we have

$$H(C_{\lambda_{js}^\nu}\psi_{js}(\lambda_{js}^\nu) \bmod \beta, C_{\lambda_{js}^\nu}\psi_{js}(\lambda_{js}^\nu) \bmod \beta) = \\ = C^2_{\lambda_{js}^\nu}\sum_{i=1}^{k}\sum_{r=1}^{d_i}\frac{A_{ir}(x_{ir})\langle\mu,\beta\rangle}{\langle\partial_{x_{ir}}\mu,\psi_{ir}(x_{ir})\rangle}\langle\partial_{x_{ir}}\mu_\beta,\psi_{js}(\lambda_{js}^\nu) \bmod \beta\rangle^2 \tag{5.2.7}$$

Taking the exterior derivative at $x_{js}=\lambda_{js}^\nu$ and make use of (4.1.29) we get

$$C^2_{\lambda_{js}^\nu}A'_{js}(\lambda_{js}^\nu)\langle\partial_{x_{js}}\big|_{x_{js}=\lambda_{js}^\nu}\mu_\beta,\psi_{js}(\lambda_{js}^\nu) \bmod \beta\rangle dx_{js} = \\ = C^2_{\lambda_{js}^\nu}A'_{js}(\lambda_{js}^\nu)d\big|_{x_{js}=\lambda_{js}^\nu}\langle\mu_\beta,\psi_{js}(\lambda_{js}^\nu) \bmod \beta\rangle \tag{5.2.8}$$

since $A_{js}(\lambda_{js}^\nu)=0$. Computing the differential of the affine function yields

$$d\big|_{x_{js}=\lambda_{js}^\nu}\langle\mu_\beta, C_{\lambda_{js}^\nu}\psi_{js}(\lambda_{js}^\nu) \bmod \beta\rangle = C_{\lambda_{js}^\nu}\psi_{js}(\lambda_{js}^\nu) \bmod \beta, \tag{5.2.9}$$

which further gives

$$C_{\lambda_{js}^\nu}A'_{js}(\lambda_{js}^\nu)=2. \tag{5.2.10}$$

□

Corollary 5.2.1.1. *If* $\boldsymbol{H}$ *is positive definite, then* A_{ir} *does not change sign on its domain of definition* I_{ir}, $i=1,\dots,k$, $r=1,\dots,d_i$.

Proof. Working on the interior of intervals I_{ir}, i.e. $x_{ir}\in I^0_{ir}$, $i=1,\dots,k$, $r=1,\dots,d_i$, we find

$$0 < H(\psi_{js}(x_{js}) \bmod \beta, \psi_{js}(x_{js}) \bmod \beta) = \\ = A_{js}(x_{js})\langle\partial_{x_{js}}\mu_\beta,\psi_{js}(x_{js}) \bmod \beta\rangle = A_{js}(x_{js})\Delta_{js} \tag{5.2.11}$$

Since A_{js} is continuous we see that A_{js} does not change sign on I_{js}. □

As we see from Lemma 5.2.1 a separable Kähler metric corresponding to the product Segre-Veronese factorization structure compactifies only if the system of $4m$ equations (5.2.3) and (5.2.2) is satisfied. When the system has full rank we get

Lemma 5.2.2. *Let* Δ *be an* n*-facets polytope compatible with the product Segre-Veronese factorization structure with* k *factorization curves. Suppose that the system of all boundary conditions* (5.2.3) *and* (5.2.2) *has the full rank and that the positivity condition from Lemma 5.2.1 holds. Then for the separable Kähler metric corresponding to*

1. *Veronese factorization structure with* $\varphi(\beta)=(a,b)^{\otimes m}$ *such that* $\langle\mu,\beta\rangle>0$ *there exist* $(n-m+1)$*-parametric family of compactifications,*

2. $\varphi(\beta)=ins_p\left((a,0)^{\otimes d_p}\otimes\bigotimes_{\substack{b=1\\b\neq p}}^{k}(1,0)^{\otimes d_b}\right)$, $k\geq 2$, *such that* $\langle\mu,\beta\rangle>0$ *there exist* $(n-m+k)$*-parametric family of compactifications,*

3. $\varphi(\beta) = ins_p\left((a,b)^{\otimes d_p} \otimes \bigotimes_{\substack{b=1\\b\neq p}}^{k}(1,0)^{\otimes d_b}\right)$, *$b \neq 0$ and $k \geq 2$, such that $\langle \mu, \beta\rangle > 0$ there exist $(n-m+1)$-parametric family of compactifications.*

Furthermore, their extremal affine functions belong to one of the four families of extremal affine functions as described in Theorem 4.5.1.

Proof. Since the polytope has n facets we need to determine n scales $C_{\lambda^{\nu}_{js}}$ corresponding to normals of the facets in order to describe a compactification (see Lemma 5.2.1). We start with counting free parameters in solutions to the extremality equation in each case (see (4.5.1))

1. Veronese case has $3m+1$ parameters
2. $k \geq 2$ and $b = 0$ case has $\sum_{i=1}^{k} d_i + 1 + 2d_i = 3m + k$ parameters
3. $k \geq 2$ and $b \neq 0$ case has $-1 + d_p + 2 + 2d_p + \sum_{\substack{i=1\\i\neq p}}^{k} d_i + 2d_i = 3m+1$ parameters, where the -1 comes from the linear constrain (4.5.26).

Thus, supposing the boundary system (5.2.3) and (5.2.2) consisting of $4m$ equations has the full rank we find that the n scales satisfies $m-1$, $m-k$ and $m-1$ equations, respectively, which shows the claim. □

Remark 5.2.1. *Some equations in* (5.2.3) *and* (5.2.2) *can be naturally combined using the nature of solutions of the product Segre-Veronese extremality equation (see Section 4.4.2). If we consider a compatible polytope which has strictly less than $2m$ facets, then necessarily there exist $i \in \{1, \dots, k\}$ and $r \in \{1, \dots, d_i\}$ such that $\lambda^2_{ir} = \lambda^1_{i,r+1} =: \lambda$. Then we have*

$$\begin{gathered}A'_{ir}(\lambda^2_{ir}) - A'_{i,r+1}(\lambda^1_{i,r+1}) = (A_{ir} - A_{i,r+1})'(\lambda) = \\ (m+1)(a+b\lambda)^m b(\beta^0 + \beta^1\lambda) + (a+b\lambda)^{m+1}\beta^1 = \\ (a+b\lambda)^m\left((m+1)b(\beta^0+\beta^1\lambda)+\beta^1\right)\end{gathered} \tag{5.2.12}$$

for some β^0 and β^1 which can be found explicitly, and similarly for the difference $A_{ir}(\lambda^2_{ir}) - A_{i,r+1}(\lambda^1_{i,r+1})$.

In Section 5.4 we explicitly derive the boundary conditions system in the case of the product Segre-Veronese factorization structure with more than one curve.

5.3 The generalised equipoised condition

We know that the compactifications described in Lemma 5.2.2 have extremal affine functions belonging to one of the four types described in Theorem 4.5.1. In compactifications, these become extremal affine functions and we discuss if the class of Delzant polytopes with extremal affine functions from one of these types comes from such compactifications.

First we derive a characterisation of the aforementioned types of extremal affine functions. To this end we fix coordinates on the product Segre-Veronese factorization structure $\mathfrak{h}$ by pulling back the coordinates on its image

$$\sum_{j=1}^{k} ins_j\left(S^{d_j}W_j^* \otimes \langle \bigotimes_{\substack{b=1\\ b\neq j}}^{k}(1,0)^{\otimes d_b}\rangle\right)$$

corresponding to the basis consisting of elements

$$\begin{aligned} &(1,0)^{\otimes\, m+1} \\ &ins_j\left(\epsilon_j^s \otimes \bigotimes_{\substack{b=1\\ b\neq j}}^{k}(1,0)^{\otimes d_b}\right) \qquad \text{for } s=1,\dots,d_j \text{ and } j=1,\dots,k \end{aligned} \tag{5.3.1}$$

via $\varphi:\mathfrak{h}\to V^*$, where ϵ_j^s, $s=0,\dots,d_j$, is the standard basis of $S^{d_j}W_j^*$. Since $\mu=\varphi^T x$ we get

$$\begin{aligned} \mu_0 &= 1 \\ \mu_{js} &= \sigma_s(x_{j1},\dots,x_{jd_j}) \end{aligned} \tag{5.3.2}$$

For i,j such that $d_i,d_j\geq 2$ and for all $r=1,\dots,d_i-1$ and $s=1,\dots,d_j-1$ we define $\varphi^*Q_{ir,js}\in S^2\mathfrak{h}^*$ to be the φ-pullback of the bilinear forms

$$Q_{ir,js}(u,v)=\frac{1}{2}\left(u_{ir}v_{js}+u_{js}v_{ir}-u_{i,r-1}v_{j,s+1}-u_{j,s+1}v_{i,r-1}\right), \tag{5.3.3}$$

where we adopt convention $u_{i0}=u_0$ for all $i=1,\dots,k$. Note that in the case when $k=1$ and $m\geq 2$ the corresponding quadratic forms define the factorization curve $\psi_1=\dots=\psi_m$ which is a rational normal curve of degree m.

Using notation for separable Kähler geometries as quotients of Sasaki geometries, recall that affine functions in the affine chart given by $\beta\in\mathfrak{h}$, and in particular affine functions on Δ_β, are given by elements of $\mathfrak{h}$. This way any bilinear form on $\mathfrak{h}$ provides a notion of orthogonality for affine functions on Δ_β.

Lemma 5.3.1. *For the product Segre-Veronese factorization structure with* $\varphi(\beta) = ins_p\left((a,b)^{\otimes d_p}\otimes\bigotimes_{\substack{b=1\\ b\neq p}}^{k}(1,0)^{\otimes d_b}\right)$ *for some* $(a,b)\in W_p^*$ *we have:*
An affine function on Δ_β *given by* $\zeta\in\mathfrak{h}$ *is an extremal affine function if and only if*

$b=0$*:*

$$\varphi^*Q_{ir,js}(\beta,\zeta)=0$$

for all $i,j=1,\dots,k$*,* $r=1,\dots,d_i-1$*,* $s=1,\dots,d_j-1$

$a=0$*:*

$$\varphi^*Q_{pr,ps}(\beta,\zeta)=0$$

for all $r,s=1,\dots,d_p-1$ *and* $\zeta_{ia}=0$ *for all* $i\neq p$ *and all* $a=1,\dots,d_i$*.*

$a \neq 0$ and $b \neq 0$:

$$\varphi^* Q_{pr,ps}(\beta, \zeta) = 0$$

for all $r, s = 1, \dots, d_p - 1$ and $\zeta_{ia} = 0$ for all $i \neq p$ and all $a = 1, \dots, d_i$.

Proof. With respect to the coordinate system defined above β has coordinates as follows: $\beta_0 = a^{d_p}$, $\beta_{pq} = a^{d_p - q} b^q$ for $q = 1, \dots, d_p$, and $\beta_{ir} = 0$ otherwise.

Note that equations characterising ζ are valid if the corresponding degree is at least 2. Thus, it places no conditions in the degree 1 situation.
We start with examining what ζ satisfy $\varphi^* Q_{pr,ps}(\beta, \zeta) = 0$ when $d_p \geq 2$. There are a few cases to consider.

$b = 0$: The only non-trivial equations are

$$\forall j \in \{1, \dots, k\} : d_j \geq 2 \;\; \forall s \in \{1, \dots, d_j - 1\} : \;\; -\beta_0 \zeta_{j,s+1} = 0 \tag{5.3.4}$$

and thus $\zeta_0, \zeta_{j1} \in \mathbb{R}$ and $\zeta_{ir} = 0$ otherwise. Comparing this with Theorem 4.5.1 shows that it completely characterises extremal affine functions when $b = 0$.

$a = 0$: The only non-trivial equations are

$$\forall r \in \{1, \dots, d_p - 1\} : \;\; -\beta_{pd_p} \zeta_{p,r-1} = 0 \tag{5.3.5}$$

which shows $\zeta_{p,d_p-1}, \zeta_{p,d_p} \in \mathbb{R}$ and $\zeta_{pr} = 0$ for $r = 0, \dots, d_p - 2$. Now Theorem 4.5.1 shows that ζ is an extremal affine function iff $\zeta_{ia} = 0$ for all $i \neq p$ and all $a = 1, \dots, d_i$, which proves the claim.

$a \neq 0$ and $b \neq 0$: If $d_p \geq 2$, then equations with indices $p = i = j$ and $q = r = s$ yield

$$\zeta_{pq} = -(q-1) \left(\frac{b}{a}\right)^q \zeta_0 + q \left(\frac{b}{a}\right)^{q-1} \zeta_{p1}, \quad \text{where } q = 0, 1, \dots, d_p \;\text{ and }\; \zeta_0, \zeta_{p1} \in \mathbb{R}. \tag{5.3.6}$$

A straightforward calculation shows that these values satisfy the equations for indices $p = i = j$ and $r, s = 1, \dots, d_p$. Once again, comparing with Theorem 4.5.1 gives the claim.

□

Now we use this characterisation to describe compactifications of separable toric Kähler geometries of the product Segre-Veronese type with decomposable Sasaki structure. Let Δ_β be a Delzant polytope compatible with the product Segre-Veronese factorization structure. Recall that Δ_β has a unique extremal affine function ζ which is also the $L^2(\Delta_\beta)$-projection of $\mathrm{Scal}(g)$ to the space of affine-linear functions (see [42] and Section 2.1.3), i.e.

$$W\zeta = Z$$

$$\text{with} \quad W_{ij} = \int_{\Delta_\beta} \mu_i \mu_j dv \quad \text{and} \quad Z_i = 2 \int_{\partial \Delta_\beta} \mu_i d\sigma \tag{5.3.7}$$

Let $F_1, \ldots, F_n$ be all facets of Δ_β, and since Δ_β is compatible with the factorization structure, these are given by normals $\psi_{ir}(\lambda_{ir}^\nu) \mod \beta$, where only n of them are unique. Notice that if we rescale the normals by $C_{\lambda_{ir}^\nu}$, then the matrix W does not depend on scales, while the right hand side depends linearly on inverse scales. Indeed, if we call the unique scales $C_1, \ldots, C_n$, then we can write $Z_i = \sum_{j=1}^n a_{ij} r_j$, where $a_{ij} = \int_{F_j} \mu_i d\sigma$ and $r_j = 2/C_j$ (see [42]). Thus the system (5.3.7) is equivalent to

$$\zeta = W^{-1} A r, \tag{5.3.8}$$

where $A_{ij} = a_{ij}$. The characterisation of extremal affine functions from Lemma 5.3.1 (namely (5.3.4),(5.3.5),(5.3.6)) impose linear relations on the inverse scales via (5.3.8). These systems can be summarised as follows: if

- $[k = 1$ and $b = 0]$ then $\zeta_p = 0$, $p = 2, \ldots, m$,
- $[k = 1$ and $b \neq 0]$ then ζ satisfies (5.3.6)
- $[k \geq 2$ and $b = 0]$ then $\zeta_{ir} = 0$ for $i = 1, \ldots, k$ and $r = 2, \ldots, d_i$,
- $[k \geq 2$ and $b \neq 0]$ then ζ_{pq} satisfies (5.3.6) for $q = 1, \ldots, d_p$ and $\zeta_{ir} = 0$ for $i \neq p$ and $r = 1, \ldots, d_i$,

In the case when this system has the full rank we have

Lemma 5.3.2. *Let Δ be an n-facets polytope compatible with the product Segre-Veronese factorization structure with k curves and decomposable Sasaki structure, i.e. k and b are fixed. Suppose that the system (B) and the corresponding system on extremal affine functions described above have both full ranks. Then they have the same solution set. In other words, compactifications of such a separable toric Kähler geometry can be described via Delzant polytopes with underlying polytope Δ and with extremal affine functions belonging to the corresponding family of extremal affine functions described above.*

Proof. We treat all the cases at the same time. We see that system described above the lemma consist of

- $m - 1$ equations
- $m - 1$ equations
- $m - k$ equations
- $m - 1$ equations

respectively. These are equations in the inverse scales and we assume these systems have full ranks. Comparing with Lemma 5.2.2 we see that solution sets for the inverse scales have the same dimension. In addition, we know that any solution of (B) is a solution for (E), thus in these cases, these systems have the same solution sets. □

5.4 Appendix calculations

We describe explicitly the boundary conditions for the extremal separable Kähler metrics corresponding to the product Segre-Veronese factorization structure with k factorization curves of degrees $d_1, \ldots, d_k$, and

$$\varphi(\beta) = ins_p\big((a,b)^{\otimes d_p} \otimes \bigotimes_{\substack{b=1 \\ b\neq p}}^{k} (1,0)^{\otimes d_b}\big), \tag{5.4.1}$$

where $k \geq 2$ and $m \geq 3$. Recall that such an extremal metric compactifies only if there exist polynomials A_{ir}, $i = 1, \ldots, k$, $r = 1, \ldots, d_i$, of degree at most $m+2$ such that for all indices we have

$$A_{ir}(\lambda_{ir}^{\nu}) = 0 \tag{5.4.2}$$

$$C_{\lambda_{ir}^{\nu}} A'_{ir}(\lambda_{ir}^{\nu}) = 2 \tag{5.4.3}$$

$$A_{ir} \text{ satisfies the extremality equation,} \tag{5.4.4}$$

see Theorem 4.5.1 and Lemma 5.2.1. The polynomial

$$A_{ir}(x_{ir}) = (x_{ir} - \lambda_{ir}^1)(x_{ir} - \lambda_{ir}^2) \sum_{q=0}^{m} \alpha_{ir}^q x_{ir}^q, \tag{5.4.5}$$

solves (5.4.2). Comparing coefficients in

$$(x_{ir} - \lambda_{ir}^1)(x_{ir} - \lambda_{ir}^2) \sum_{q=0}^{m} \alpha_{ir}^q x_{ir}^q = pol_i(x_{ir}) + (a + bx_{ir})^{m+1}(\beta_{ir}^0 + \beta_{ir}^1 x) \tag{5.4.6}$$

and solving for $\{\alpha_{ir}^q\}_{q=0}^m$ shows what polynomials (5.4.5) solve the extremality equation, where $d = \deg(pol_i) \in \{d_i + 1, d_i + 2, \}$. The other-degree cases can be done similarly as in what follows. In turns out that the expression for α_{ir}^q depends linearly on the coefficients of pol_i, and λ_{ir}^1 and λ_{ir}^2 occur as rational functions in the complete homogenous polynomials in variables λ_{ir}^1 and λ_{ir}^2. Furthermore, such A_{ir} solve (5.4.3) if and only if the coefficients α_{ir}^q satisfy

$$\frac{2}{C_{\lambda_{ir}^1}} = (\lambda_{ir}^1 - \lambda_{ir}^2) \sum_{q=0}^{m} \alpha_{ir}^q (\lambda_{ir}^1)^q, \tag{5.4.7}$$

$$\frac{2}{C_{\lambda_{ir}^2}} = (\lambda_{ir}^2 - \lambda_{ir}^1) \sum_{q=0}^{m} \alpha_{ir}^q (\lambda_{ir}^2)^q. \tag{5.4.8}$$

Now we compare coefficients in (5.4.6)

5.4.1 Polynomials satisfying the product Segre-Veronese extremality equation explicitly. Since the conditions (5.4.2)-(5.4.4) concern a single polynomial A_{ir}

we simplify notation where possible. First we note

$$(a+bx)^{m+1}(\beta^0+\beta^1 x) = R_0 + \sum_{r=1}^{m+1} x^r R_r + R_{m+2}x^{m+2}, \tag{5.4.9}$$

where

$$R_0 = \beta^0 a^{m+1} \tag{5.4.10}$$
$$R_r = b^{r-1}a^{m+1-r}\left(b\beta^0\binom{m+1}{r} + a\beta^1\binom{m+1}{r-1}\right), \quad r=1,\dots,m+1 \tag{5.4.11}$$
$$R_{m+2} = \beta^1 b^{m+1}. \tag{5.4.12}$$

Furthermore,

$$A_{pq}(x) = (x-\lambda_1)(x-\lambda_2)\sum_{r=0}^{m}\alpha_r x^r =$$
$$\lambda_1\lambda_2\alpha_0 + (-(\lambda_1+\lambda_2)\alpha_0 + \lambda_1\lambda_2\alpha_1)\,x + \sum_{n=2}^{m}(\alpha_{n-2} - (\lambda_1+\lambda_2)\alpha_{n-1} + \lambda_1\lambda_2\alpha_n)\,x^n +$$
$$+ (\alpha_{m-1} - (\lambda_1+\lambda_2)\alpha_m)\,x^{m+1} + \alpha_m x^{m+2} \tag{5.4.13}$$

for some $\alpha_0,\dots,\alpha_m$, where λ_1 and λ_2 are fixed roots. By comparing coefficients in $A_{pq}(x) = pol(x) + (a+bx)^{m+1}(\beta^0+\beta^1 x)$ we get

$$p_0 + R_0 = \lambda_1\lambda_2\alpha_0 \tag{5.4.14}$$
$$p_1 + R_1 = -(\lambda_1+\lambda_2)\alpha_0 + \lambda_1\lambda_2\alpha_1 \tag{5.4.15}$$

$\forall\, r \in \{2,\dots,m\}$:

$$p_r + R_r = \alpha_{r-2} - (\lambda_1+\lambda_2)\alpha_{r-1} + \lambda_1\lambda_2\alpha_r \tag{5.4.16}$$

and

$$b^m\left(\beta^0 b + (m+1)\beta^1 a\right) = \alpha_{m-1} - (\lambda_1+\lambda_2)\alpha_m \tag{5.4.17}$$
$$\beta^1 b^{m+1} = \alpha_m, \tag{5.4.18}$$

where $p_i = 0$ for $d+1 \le i \le m$. Thus we have $m+3$ equations which we solve for $\alpha_0,\dots,\alpha_m,\beta^0,\beta^1$. Using (5.4.14)-(5.4.16) we find the solution for $\alpha_0,\dots,\alpha_m$,
$\forall\, r \in \{0,\dots,m\}$:

$$\alpha_r = \sum_{i=0}^{r}\frac{h_{r-i}(\lambda_1,\lambda_2)}{(\lambda_1\lambda_2)^{r+1-i}}\left(p_i + R_i\right), \tag{5.4.19}$$

where $h_j(\lambda_1, \lambda_2)$ is the jth complete homogenous polynomials in two variables λ_1 and λ_2, and $p_i = 0$ for $d+1 \leq i \leq m$.

In order to solve for β^0 and β^1 we use equivalent formulation of (5.4.17) and (5.4.18),

$$\beta^0 b^{m+1} + \beta^1 \left((m+1)ab^m + (\lambda_1+\lambda_2)b^{m+1}\right) = \alpha_{m-1} \tag{5.4.20}$$

$$\beta^1 b^{m+1} = \alpha_m. \tag{5.4.21}$$

Combining the above with (5.4.19) yields

$$\begin{aligned}\beta^0 \left(-b^{m+1} + \sum_{i=0}^{m-1} \frac{h_{m-1-i}}{(\lambda_1\lambda_2)^{m-i}} b^i a^{m+1-i} \binom{m+1}{i}\right) + \\ +\beta^1 \left(-(\lambda_1+\lambda_2)b^{m+1} - (m+1)ab^m + \sum_{r=1}^{m-1} \frac{h_{m-1-r}}{(\lambda_1\lambda_2)^{m-r}} b^{r-1} a^{m+2-r} \binom{m+1}{r-1}\right) = \\ = -\sum_{i=0}^{d} \frac{h_{m-1-i}}{(\lambda_1\lambda_2)^{m-i}} p_i\end{aligned} \tag{5.4.22}$$

$$\begin{aligned}\beta^0 \sum_{i=0}^{m} \frac{h_{m-i}}{(\lambda_1\lambda_2)^{m+1-i}} b^i a^{m+1-i} \binom{m+1}{i} + \\ +\beta^1 \left(-b^{m+1} + \sum_{r=1}^{m} \frac{h_{m-r}}{(\lambda_1\lambda_2)^{m+1-r}} b^{r-1} a^{m+2-r} \binom{m+1}{r-1}\right) = -\sum_{i=0}^{d} \frac{h_{m-i}}{(\lambda_1\lambda_2)^{m+1-i}} p_i\end{aligned} \tag{5.4.23}$$

We start with summing coefficients for these equations. First we change variables in

$$\sum_{r=1}^{m-1} \frac{h_{m-1-r}}{(\lambda_1\lambda_2)^{m-r}} b^{r-1} a^{m+2-r} \binom{m+1}{r-1} = \sum_{i=0}^{m-2} \frac{h_{m-2-i}}{(\lambda_1\lambda_2)^{m-1-i}} b^i a^{m+1-i} \binom{m+1}{i} \tag{5.4.24}$$

and in

$$\sum_{r=1}^{m} \frac{h_{m-r}}{(\lambda_1\lambda_2)^{m+1-r}} b^{r-1} a^{m+2-r} \binom{m+1}{r-1} = \sum_{i=0}^{m-1} \frac{h_{m-1-i}}{(\lambda_1\lambda_2)^{m-i}} b^i a^{m+1-i} \binom{m+1}{i} \tag{5.4.25}$$

Now for $l = 0, 1, 2$ we evaluate

$$\sum_{i=0}^{m-l} \frac{h_{m-l-i}}{(\lambda_1\lambda_2)^{m-l+1-i}} b^i a^{m+1-i} \binom{m+1}{i} \tag{5.4.26}$$

Since $h_{d-i} = \frac{\lambda_1^{d+1-i} - \lambda_2^{d+1-i}}{\lambda_1 - \lambda_2}$, (5.4.26) equals

$$\sum_{i=0}^{m-l} \frac{\lambda_1^{m-l+1-i} - \lambda_2^{m-l+1-i}}{(\lambda_1\lambda_2)^{m-l+1-i}(\lambda_1 - \lambda_2)} b^i a^{m+1-i} \binom{m+1}{i} = \tag{5.4.27}$$

$$= \frac{1}{\lambda_1 - \lambda_2} \sum_{i=0}^{m-l} \left(\frac{\lambda_2^l}{\lambda_2^{m+1-i}} - \frac{\lambda_1^l}{\lambda_1^{m+1-i}} \right) b^i a^{m+1-i} \binom{m+1}{i} = \tag{5.4.28}$$

$$= \frac{\lambda_2^l}{\lambda_1 - \lambda_2} \left[\left(\frac{a}{\lambda_2} + b \right)^{m+1} - \sum_{r=m+1-l}^{m+1} b^r \left(\frac{a}{\lambda_2} \right)^{m+1-r} \binom{m+1}{r} \right] - \tag{5.4.29}$$

$$- \frac{\lambda_1^l}{\lambda_1 - \lambda_2} \left[\left(\frac{a}{\lambda_1} + b \right)^{m+1} - \sum_{r=m+1-l}^{m+1} b^r \left(\frac{a}{\lambda_1} \right)^{m+1-r} \binom{m+1}{r} \right] \tag{5.4.30}$$

Now, if $l = 0$, then the result is

$$\frac{1}{\lambda_1 - \lambda_2} \left[\left(\frac{a}{\lambda_2} + b \right)^{m+1} - \left(\frac{a}{\lambda_1} + b \right)^{m+1} \right] \tag{5.4.31}$$

If $l = 1$,

$$\frac{1}{\lambda_1 - \lambda_2} \left[\lambda_2 \left(\frac{a}{\lambda_2} + b \right)^{m+1} - \lambda_1 \left(\frac{a}{\lambda_1} + b \right)^{m+1} + b^{m+1}(\lambda_1 - \lambda_2) \right] \tag{5.4.32}$$

If $l = 2$,

$$\frac{1}{\lambda_1 - \lambda_2} \left[\lambda_2^2 \left(\frac{a}{\lambda_2} + b \right)^{m+1} - \lambda_1^2 \left(\frac{a}{\lambda_1} + b \right)^{m+1} + b^{m+1}(\lambda_1^2 - \lambda_2^2) + (m+1)ab^m(\lambda_1 - \lambda_2) \right] \tag{5.4.33}$$

Therefore, the system (5.4.22) and (5.4.23) is equivalent to

$$\left[\begin{array}{cc|c} \lambda_2\ell_2 - \lambda_1\ell_1 & \lambda_2^2\ell_2 - \lambda_1^2\ell_1 & -(\lambda_1 - \lambda_2)\sum_{i=0}^{d} \frac{h_{m-1-i}}{(\lambda_1\lambda_2)^{m-i}} p_i \\ \ell_2 - \ell_1 & \lambda_2\ell_2 - \lambda_1\ell_1 & -(\lambda_1 - \lambda_2)\sum_{i=0}^{d} \frac{h_{m-i}}{(\lambda_1\lambda_2)^{m+1-i}} p_i \end{array} \right], \tag{5.4.34}$$

where $\ell_1 = \left(\frac{a}{\lambda_1} + b\right)^{m+1}$ and $\ell_2 = \left(\frac{a}{\lambda_2} + b\right)^{m+1}$. For the determinant D of (5.4.34) we have

$$D = (\lambda_1\ell_1 + \lambda_2\ell_2)^2 - \left[(\lambda_2\ell_2)^2 + \left(\lambda_1^2 + \lambda_2^2\right)\ell_1\ell_2 + (\lambda_1\ell_1)^2\right] = -(\lambda_1 - \lambda_2)^2\ell_1\ell_2 \tag{5.4.35}$$

Now we use Cramer's rule to solve this system which yields

$$-(\lambda_1-\lambda_2)^2\ell_1\ell_2\beta^0 =$$

$$-(\lambda_1-\lambda_2)\left(\lambda_2\ell_2-\lambda_1\ell_1\right)\sum_{i=0}^{d}\frac{h_{m-1-i}}{(\lambda_1\lambda_2)^{m-i}}p_i+(\lambda_1-\lambda_2)\left(\lambda_2^2\ell_2-\lambda_1^2\ell_2\right)\sum_{i=0}^{d}\frac{h_{m-i}}{(\lambda_1\lambda_2)^{m+1-i}}p_i \tag{5.4.36}$$

and

$$-(\lambda_1-\lambda_2)^2\ell_1\ell_2\beta^1 =$$

$$-(\lambda_1-\lambda_2)\left(\lambda_2\ell_2-\lambda_1\ell_1\right)\sum_{i=0}^{d}\frac{h_{m-i}}{(\lambda_1\lambda_2)^{m+1-i}}p_i+(\lambda_1-\lambda_2)\left(\ell_2-\ell_1\right)\sum_{i=0}^{d}\frac{h_{m-1-i}}{(\lambda_1\lambda_2)^{m-i}}p_i \tag{5.4.37}$$

We observe

$$\frac{h_{m-1-i}}{(\lambda_1\lambda_2)^{m-i}}-\lambda_2\frac{h_{m-i}}{(\lambda_1\lambda_2)^{m+1-i}}=\frac{1}{\lambda_1-\lambda_2}\left(\frac{\lambda_1^{m-i}-\lambda_2^{m-i}}{(\lambda_1\lambda_2)^{m-i}}-\lambda_2\frac{\lambda_1^{m+1-i}-\lambda_2^{m+1-i}}{(\lambda_1\lambda_2)^{m+1-i}}\right)=$$

$$=\frac{\lambda_1^i}{\lambda_1^{m+1}},\qquad \text{for } i=0,\dots,d \tag{5.4.38}$$

Thus (5.4.36) and (5.4.37) become

$$\ell_1\ell_2\beta^0=(\lambda_1-\lambda_2)^{-1}\sum_{i=0}^{d}\left(\lambda_2\frac{\lambda_1^i}{\lambda_1^{m+1}}\ell_2-\lambda_1\frac{\lambda_2^i}{\lambda_2^{m+1}}\ell_1\right)p_i \tag{5.4.39}$$

and

$$\ell_1\ell_2\beta^1=(\lambda_1-\lambda_2)^{-1}\sum_{i=0}^{d}\left(-\frac{\lambda_1^i}{\lambda_1^{m+1}}\ell_2+\frac{\lambda_2^i}{\lambda_2^{m+1}}\ell_1\right)p_i \tag{5.4.40}$$

Remark 5.4.1. *Plugging β^0 and β^1 back into the expression* (5.4.19) *for α_r and rewriting* (5.4.7) *and* (5.4.8) *in these terms yields a linear system of equations in α_{ir}^q whose solutions determine polynomials A_{ir} satisfying* (5.4.2)-(5.4.4)*. The system can be written explicitly which may shed light on its rank.*

5.4.2 Summing some sums. Here we address the condition (5.2.3) in a different way. The main purpose of this part is to show how to sum certain coefficients naturally occurring in the computation of boundary conditions. We use expression

$$A_{pq}(x)=(x-\lambda_1)(x-\lambda_2)\sum_{r=0}^{m}\alpha_r x^r \tag{5.4.41}$$

as before and investigate the equation $A'_{pq}(\lambda_1) = 2\kappa(\lambda_1)$, where $\kappa(\lambda_1)$ is a scale depending on the endpoint λ_1 (as in (5.4.3)). We have

$$\frac{2\kappa(\lambda_1)}{\lambda_1-\lambda_2} =$$

$$= \sum_{r=0}^{m} \alpha_r \lambda_1^r = \sum_{r=0}^{m} \lambda_1^r \sum_{i=0}^{r} \frac{h_{r-i}}{(\lambda_1\lambda_2)^{r+1-i}} (p_i + R_i) = \sum_{i=0}^{m} (p_i + R_i) \sum_{r=i}^{m} \frac{h_{r-i}}{(\lambda_1\lambda_2)^{r+1-i}} \lambda_1^r =$$

$$= \sum_{i=0}^{d} p_i \sum_{r=i}^{m} \frac{h_{r-i}}{(\lambda_1\lambda_2)^{r+1-i}} \lambda_1^r + \beta^0 \left[\sum_{i=0}^{m} b^i a^{m+1-i} \binom{m+1}{i} \sum_{r=i}^{m} \frac{h_{r-i}}{(\lambda_1\lambda_2)^{r+1-i}} \lambda_1^r \right] +$$

$$+\beta^1 \left[\sum_{j=1}^{m} b^{j-1} a^{m+2-j} \binom{m+1}{j-1} \sum_{l=j}^{m} \frac{h_{l-j}}{(\lambda_1\lambda_2)^{l+1-j}} \lambda_1^l \right]. \tag{5.4.42}$$

First we sum coefficients at β^0 and β^1. To this end, we find

$$\sum_{r=i}^{m} \frac{h_{r-i}}{(\lambda_1\lambda_2)^{r+1-i}} \lambda_1^r = \frac{\lambda_1^{i-1}}{\lambda_1-\lambda_2} \sum_{r=i}^{m} \frac{\lambda_1^{r+1-i} - \lambda_2^{r+1-i}}{\lambda_2^{r+1-i}} =$$

$$= \frac{\lambda_1^{i-1}}{\lambda_1-\lambda_2} \sum_{r=i}^{m} \left[\left(\frac{\lambda_1}{\lambda_2}\right)^{r+1-i} - 1 \right] = \frac{\lambda_1^{i-1}}{\lambda_1-\lambda_2} \left[-(m+1-i) + \frac{\lambda_1}{\lambda_2} \sum_{l=0}^{m-i} \left(\frac{\lambda_1}{\lambda_2}\right)^l \right] =$$

$$= \frac{\lambda_1^{i-1}}{\lambda_1-\lambda_2} \left[-(m+1-i) + \frac{\lambda_1}{\lambda_2} \frac{1-\left(\frac{\lambda_1}{\lambda_2}\right)^{m+1-i}}{1-\frac{\lambda_1}{\lambda_2}} \right] =$$

$$\frac{1}{\lambda_1-\lambda_2} \left[-(m+1-i)\lambda_1^{i-1} - \frac{1}{\lambda_1-\lambda_2} \left(\lambda_1^i - \left(\frac{\lambda_1}{\lambda_2}\right)^{m+1} \lambda_2^i \right) \right]. \tag{5.4.43}$$

We recall that by differentiating the binomial identity we obtain

$$(m+1)(p+q)^m = \sum_{i=0}^{m} p^i q^{m-i} (m+1-i) \binom{m+1}{i}. \tag{5.4.44}$$

Thus the coefficient at β^0 is

$$\sum_{i=0}^{m} b^i a^{m+1-i} \binom{m+1}{i} \sum_{r=i}^{m} \frac{h_{r-i}}{(\lambda_1\lambda_2)^{r+1-i}} \lambda_1^r =$$

$$-\frac{m+1}{\lambda_1-\lambda_2} \frac{a}{\lambda_1} (a+\lambda_1 b)^m - \frac{1}{(\lambda_1-\lambda_2)^2} \left((a+\lambda_1 b)^{m+1} - (\lambda_1 b)^{m+1} \right) +$$

$$+\frac{1}{(\lambda_1-\lambda_2)^2} \left(\frac{\lambda_1}{\lambda_2}\right)^{m+1} \left((a+\lambda_2 b)^{m+1} - (\lambda_2 b)^{m+1} \right) =$$

$$-\frac{m+1}{\lambda_1-\lambda_2} \frac{a}{\lambda_1} (a+\lambda_1 b)^m + \frac{\lambda_1^{m+1}}{(\lambda_1-\lambda_2)^2} (\ell_2 - \ell_1). \tag{5.4.45}$$

Furthermore, the coefficient at β^1 can be expressed as

$$\sum_{j=1}^{m} b^{j-1} a^{m+2-j} \binom{m+1}{j-1} \sum_{l=j}^{m} \frac{h_{l-j}}{(\lambda_1\lambda_2)^{l+1-j}} \lambda_1^l =$$

$$= \sum_{i=0}^{m-1} b^i a^{m+1-i} \binom{m+1}{i} \left[\left(\sum_{l=i}^{m} \frac{h_{l-j}}{(\lambda_1\lambda_2)^{l+1-j}} \lambda_1^l \right) - \frac{\lambda_1^i}{\lambda_1\lambda_2} \right] =$$

$$- \frac{m+1}{\lambda_1 - \lambda_2} \frac{a}{\lambda_1} \left((a+\lambda_1 b)^m - (\lambda_1 b)^m \right) -$$

$$- \frac{1}{(\lambda_1-\lambda_2)^2} \left((a+\lambda_1 b)^{m+1} - (\lambda_1 b)^{m+1} - (m+1) a (\lambda_1 b)^m \right) +$$

$$+ \frac{1}{(\lambda_1-\lambda_2)^2} \left(\frac{\lambda_1}{\lambda_2} \right)^{m+1} \left((a+\lambda_2 b)^{m+1} - (\lambda_2 b)^{m+1} - (m+1) a (\lambda_2 b)^m \right) -$$

$$- \frac{1}{\lambda_1\lambda_2} \left((a+\lambda_1 b)^{m+1} - (\lambda_1 b)^{m+1} - (m+1) a (\lambda_1 b)^m \right) =$$

$$- \frac{m+1}{\lambda_1 - \lambda_2} \frac{a}{\lambda_1} (a+\lambda_1 b)^m + \frac{\lambda_1^{m+1}}{(\lambda_1-\lambda_2)^2} (\ell_2 - \ell_1) - \frac{\lambda_1^m}{\lambda_2} \left(\ell_1 - b^{m+1} \right). \tag{5.4.46}$$

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