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Branching Systems and Spatial Fragmentations

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Alice Callegaro

Declaration of Authorship

I am the author of this thesis, and the work described therein was carried out by myself personally in collaboration with my supervisor Matthew I. Roberts.

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Alice Callegaro

Summary

The largest part of this thesis is concerned with the study of a fragmentation process in which rectangles break up into progressively smaller pieces at rates that depend on their shape. Long, thin rectangles are more likely to break quickly, and are also more likely to split along their longest side.

We are interested in the evolution of the system at large times: how many fragments are there of different shapes and sizes, and how did they reach that state? We give an almost sure growth rate along paths by studying an equivalent branching random walk. Our analysis is highly technical due to the spatial dependence of the rates and the fact that we work under weaker assumptions than the usual large deviations regime for random walks.

In the second part of the thesis we focus on a different, but related problem: estimating the probability that the paths of a random walk stay close to a given function. We prove a small deviation result about the unscaled paths of either a compound Poisson process, or a random walk in discrete time.

Our proof strategy involves a Brownian motion approximation on smaller time intervals, which allows us to take advantage of the sharpest estimates currently available on the probability that a Brownian motion lies in a tube about a given function.

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Chapter 1

Introduction

The main object of study of this thesis is a spatially-dependent fragmentation process, which involves rectangles breaking up into progressively smaller pieces at rates that depend on their shape.

A fragmentation process describes the breaking up of a structure into pieces, and occurs naturally in many situations. Mathematically, fragmentation processes have been a subject of active research in probability for at least 20 years, incorporating several varieties, including homogeneous fragmentations [8], self-similar fragmentations [9], and growth fragmentations [12]. The textbook of Bertoin [10] gives an excellent introduction to this rich mathematical theory. It begins by listing some real-world examples of phenomena that might be considered fragmentation processes, including "stellar fragments in astrophysics, fractures and earthquakes in geophysics, breaking of crystals in crystallography, degradation of large polymer chains in chemistry, DNA fragmentation in biology, fission of atoms in nuclear physics, fragmentation of a hard drive in computer science," and particularly valid from a mathematical point of view, "evolution of blocks of mineral in a crusher."

However, the traditional mathematical definition of a fragmentation process insists that each fragment can be characterized by a real number that should be thought of as its size. This stops us from considering the spatial position of a fragment or further geometrical properties like its shape. In [11], Bertoin does analyse a multitype model where the rates at which the fragments break can depend on one of finitely many types, but this is somewhat restrictive because in applications there is often a continuum of possible shapes.

We consider a spatially-dependent fragmentation process defined as follows. Begin with a square of side length 1. After a random time, the square breaks into two rectangular pieces, uniformly at random. Each of these pieces then repeats this behaviour independently, except that long, thin rectangles break more quickly, and are more likely to break along their longest side.

One of the reasons for having rates depending on the shapes of the fragments is building a more realistic model for a physical crushing process, for example, where long, thin pieces of rock are likely to break more easily than more evenly-proportioned pieces. See Figure 1-1.

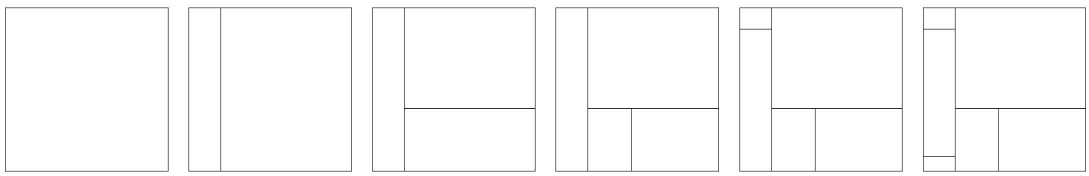

Figure 1-1: *We begin with a square, which splits vertically into two rectangles. One of these then splits horizontally, and the process continues. Thinner rectangles are more likely to split first.*

We work in two dimensions to keep notation manageable, but our proofs should work in three or more dimensions with little additional work. For the same reason, we make a particular choice for the splitting rule—that is, the functions that decide how a fragment's shape affects its branching rate and the direction in which it breaks—but our methods should be adaptable to a variety of spatially-dependent fragmentation models. We propose this model as a proof of concept that spatial fragmentations (with uncountably many types) can be analysed mathematically.

We work in continuous time and begin with a square of side-length 1. At any time, each rectangle of base b and height h independently splits at rate $r(b,h)$ into two smaller rectangles. The probability that it splits vertically is $p(b,h)$, and if so then it splits at a uniform point along its base; otherwise it splits horizontally at a uniform point along its height. The functions r and p are given by

$$r(b,h) = \Big(\frac{1-\log b}{1-\log h}\Big) \vee \Big(\frac{1-\log h}{1-\log b}\Big)$$

and

$$p(b,h) = \frac{1-\log h}{2(1-\log b)}\mathbb{1}_{b\leq h} + \bigg(1 - \frac{1-\log b}{2(1-\log h)}\bigg)\mathbb{1}_{b>h}.$$

It is easy to see that rectangles with either large base relative to their height, or large height relative to their base, split faster, and are more likely to split along their longer side. The appearance of $1-\log b$ and $1-\log h$, rather than b and h, is because splitting events have a multiplicative effect: the distribution of the ratio of each rectangle's measurements to its parent's is invariant. Thus the logarithm of the measurements behaves additively, which ensures that the functions r and p remain non-trivial when we rescale space and time. On the other hand, our choices of r and p are not the only choices with this property, and our methods appear to be fairly robust: it should be possible to adapt them to other sensible splitting rules.

Our main theorem, which we prove in Chapter 3, aims to quantify how many fragments of a given shape there are in a configuration of splitting rectangles at large times and the shape pattern they followed to reach their final base and height. In the next sections we discuss how fragmentation processes can be translated into branching random walks. Finding how many rectangles have a given shape pattern corresponds to determining how many particles in the branching random walk have paths which, when rescaled appropriately, fall within a given set of functions.

1.1. Fragmentations and branching random walks

A key observation in the classical study of mathematical fragmentations is that they satisfy the branching property, in that the future evolution of one fragment, given its current state, does not depend on the other fragments. This enables us to use branching tools in the analysis of fragmentation processes: for example, if we consider the negative logarithm of the sizes of the fragments of a homogeneous fragmentation, then we obtain a continuous-time branching random walk. Bertoin's multitype fragmentation in [11], where the splitting rate can depend on one parameter (varying in a finite set), under the same logarithmic transformation, becomes a multitype branching random walk.

1.1.1 Shape-dependent rates give spatially-dependent walks

Under a logarithmic transformation, the fragmentation process with shape-dependent rates that we defined can also be translated into the language of branching processes. More precisely, for a rectangle v, we denote its base by B_v and its height by H_v. We let $X_v = -\log B_v$ and $Y_v = -\log H_v$. As suggested from the definitions of $r(b,h)$ and $p(b,h)$, X_v and Y_v are more useful parameterisations of size than B_v and H_v from a mathematical point of view, simply because rectangles' sizes will decay exponentially with time. Under this transformation, our fragmentation system has the following alternative description.

Begin with one particle at $(0,0) \in \mathbb{R}^2$. Each particle, when at position (x,y) with $x, y \geq 0$, branches at rate

$$R(x,y) = \frac{x+1}{y+1} \vee \frac{y+1}{x+1}. \tag{1.1}$$

At a branching event, the particle is replaced by two children: letting $\mathcal{U}$ be a uniform random variable on $(0,1)$, independent of everything else, then with probability

$$P(x,y) = \frac{y+1}{2(x+1)} \mathbb{1}_{x \geq y} + \left(1 - \frac{x+1}{2(y+1)}\right) \mathbb{1}_{x<y}$$

the two children have positions $(x - \log \mathcal{U}, y)$ and $(x - \log(1-\mathcal{U}), y)$, and with probability $1 - P(x,y)$ they have positions $(x, y - \log \mathcal{U})$ and $(x, y - \log(1-\mathcal{U}))$.

We let $R_X(x,y) = R(x,y)P(x,y)$ and $R_Y(x,y) = R(x,y)(1 - P(x,y))$. Then R_X and R_Y denote the rates at which a particle at position (x,y) moves in the first spatial dimension, or the second, respectively.

Throughout the thesis, we mostly use the second description, and refer to particles and their positions, rather than rectangles and their sizes. As seen above, the two descriptions are entirely equivalent.

It is clear that the process we defined cannot be analysed with standard tools. Under a negative logarithmic transformation, our system of fragmenting rectangles can also be thought of as a multitype branching random walk, but one with uncountably many types (the type being the ratio x/y). Analysing branching systems with uncountably many types is notoriously difficult. Even multitype Galton-Watson processes with countably many types are beyond the scope of standard tools, hence the restriction to finitely many types in most papers on multitype branching systems, including [11]. Our model includes not just a continuum of types, but a two-dimensional set of possibilities.

This makes our mathematical analysis highly technical. We decided to include Chapter 2 as an intermediate step leading to the long proof that appears in Chapter 3: we discuss the most challenging aspects along the path to determine the growth rates, in the much simpler case in which the branching rate is constant, and so is the probability of a jump in either direction.

1.1.2 Martensitic avalanches

Configurations of splitting rectangles similar to ours have been considered by Cesana and Hambly [23] and Ball, Cesana and Hambly [4]. The authors consider models in which rectangles always split at rates that depend only on their area, with a constant probability p (or $1-p$) of splitting horizontally (or vertically), ensuring that their models, suitably transformed by taking logarithms, fit into the framework of generalised branching random walks. At this point, well-established tools from the broad literature on branching random walks, among which [13, 14, 15, 16], make it possible to try different splitting rule variants and work in both two and three dimensions.

The models introduced in [23], and [4] are motivated by applications to a martensitic phase transition observed in a class of elastic crystals. During a martensitic transformation the system releases energy and the molecules rearrange from the highly regular lattice structure of a crystal to a different configuration with lower symmetry. At this point, an inhomogeneous pattern emerges, resembling the configuration of fragmented rectangles, showing plates separated by sharp interfaces. Motivated by predictions from the physics literature, the authors study the lengths of the horizontal "interfaces" between the fragments, obtaining that in certain cases the total number of interfaces larger than x behaves like a random variable multiplied by an explicit power of x.

Although in principle the same questions could be attempted with our splitting rules, shape-dependent rates lead to a much less flexible model. Chapter 3 shows that even addressing the most natural question on how fast the number of fragments grows requires a significant amount of work. Naturally, a much less tractable mathematical analysis is the price we pay in our model for having shape-dependent rates and splitting probabilities; such dependency in [23] and [4] is lost. See Figure 1-2.

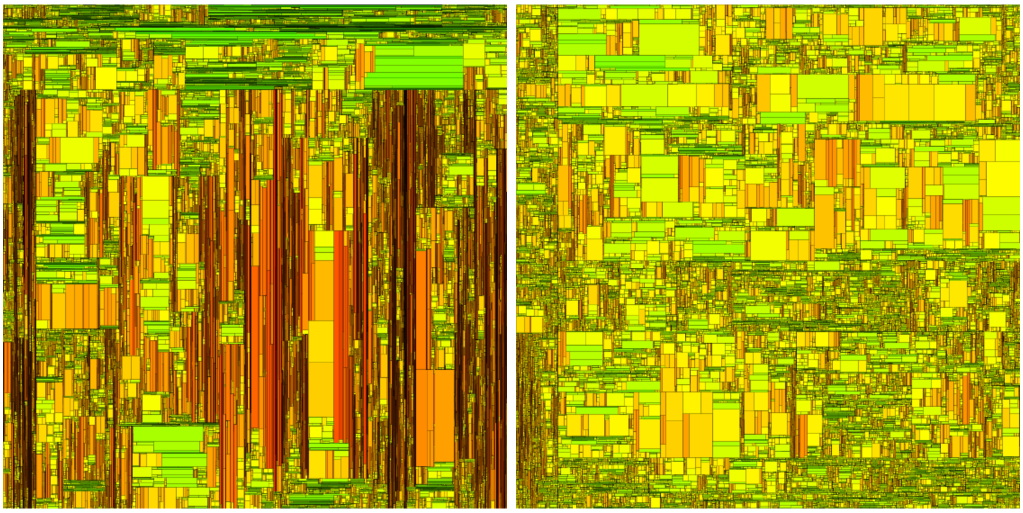

Figure 1-2: *On the left: a homogeneous model, where every rectangle splits at rate* 1 *and splits horizontally or vertically with probability* $1/2$ *each. On the right: our model where long, thin rectangles split faster, and are more likely to split along their longest side. Tall rectangles are coloured red, fat rectangles are coloured green, and squares are coloured yellow.*

1.2. Branching processes with spatial dependencies

By transforming the fragmentation process with shape-dependent rates into an equivalent branching random walk, we obtain a branching system where the rates depend on the particles' positions. Understanding spatially-dependent branching systems is an important problem in its own right, since almost any real-world application of branching tools—from nuclear reactors [31, 33] to the spread of disease [26, 28]—involves spatial inhomogeneity. Another purpose of our work is to contribute new techniques to the rigorous mathematical investigation of spatially-dependent branching structures more generally.

1.2.1 A closely related model with branching Brownian motion

In the recently growing literature on branching processes with spatially dependent rates the most closely related work to our fragmentation model is [7], where the authors determined the growth rates of the number of particles in a branching Brownian motion with inhomogeneous and unbounded branching rate.

The system starts with a single particle at the origin moving as a standard Brownian motion. Each particle at z branches at infinitesimal rate $\beta|z|^p$, where $\beta > 0$ and $p \in (0, 2]$. When it dies, a particle is replaced by a random number of offspring with mean m, which move as Brownian motions starting from the position of their parent.

For a set $F \subset C[0,1]$ let

$$N_T^{BBM}(F) := \#\left\{u \in \mathcal{N}_T : \exists f \in F \text{ width } X_u(sT) = T^{\frac{2}{2-p}} f(s)\ \forall s \in [0,1]\right\}$$

be the number of particles alive at time T, whose rescaled paths belong to F. Denote by H_1 the set of absolutely continuous functions $f : [0,1] \to \mathbb{R}$. Define

$$K_{\mathrm{BBM}}(f,t) = \begin{cases} m\beta \int_0^t |f(s)|^p ds - \frac{1}{2}\int_0^t f'(s)^2 ds & \text{if } f \in H_1 \\ -\infty & \text{otherwise.} \end{cases}$$

Theorem 1.1. *For any closed set $F \subset C[0,1]$,*

$$\limsup_{T\to\infty} \frac{1}{T^{\frac{2+p}{2-p}}} \log N_T^{BBM}(F) \le \sup\{K_{BBM}(f,1) : f \in F,\ K_{BBM}(f,s) > 0\ \forall s \le 1\}$$

almost surely, and for any open set $F \subset C[0,1]$,

$$\liminf_{T\to\infty} \frac{1}{T^{\frac{2+p}{2-p}}} \log N_T^{BBM}(F) \ge \sup\{K_{BBM}(f,1) : f \in F,\ K_{BBM}(f,s) > 0\ \forall s \le 1\}$$

almost surely.

There are three main difficulties in our model relative to that in [7]. Firstly, in the BBM, all particles move as standard Brownian motions, independent of their location and their branching rate, whereas in our model particles jump and branch simultaneously. Indeed, it is worth noting that if the branching Brownian motion in [7] were replaced by an analogous branching random walk, then if we started with one particle at 0, the initial particle would never branch or move; whereas if we started with a particle at any other site, then even with bounded jump sizes, the collection of particles would colonise space dramatically faster than the BBM (subject to the initial population not returning to 0 quickly), since a particle branching at rate $|z|^p$ would also be moving at rate $|z|^p$. This highlights the challenge of controlling the dependencies between particles' positions and the growth of the population.

On top of this initial difference, our branching rate $R(z)$ is much more difficult to control than the smooth, symmetric, monotone (on each half-space) function $|z|^p$. And thirdly, our particles are able to make large jumps, meaning that standard large deviations apparatus is more difficult to apply, and we must use a non-standard topology.

1.2.2 The asymptotic spread of the population

As we already anticipated, in Chapter 3 we find the almost sure growth rates for the branching random walk (with spatially-dependent rates) equivalent to the process of fragmenting rectangles. A closely related question is determining the position of the maximal particle and the typical paths that particles follow to reach a given position.

For homogeneous spatial branching processes, obtaining a full picture of the spread of the population has been a subject of interest for more than 45 years. To give just a few highlights, the position of the extremal particle in BBM was studied by McKean [37] and Bramson [20, 21], with more detailed recent studies on the behaviour near the extremal particle by Aïdékon *et. al.* [2] and Arguin, Bovier and Kistler [3]. For non-lattice branching random walks, Aïdékon [1] proved convergence in law for the recentered position of the extremal particle under fairly weak conditions. Bramson, Ding and Zeitouni [22] gave a shorter proof using a second moment method and indicated that it should be possible to adapt their proof to branching random walks that take values on a lattice.

For the BBM model in [7], after proving Theorem 1.1 the authors analysed their almost-sure growth rate in some detail, giving implicit equations for the optimal paths and the location of the bulk of the population (which became explicit in the cases $p = 0$ and $p = 1$). This was a difficult analytic task even for the relatively simple, monotone growth rate seen in [7]. Our growth rate K is much more complex and it would take a substantial amount of further work to analyse the optimal paths, so we do not attempt this here.

Roberts and Schweinsberg [41] also consider branching Brownian motion in an inhomogeneous potential, this time with a biological application in mind, where the position of a particle represents its fitness and fitter individuals branch more quickly. They use the tools from [7] to give a heuristic explanation of some of their results, but use a more precise truncation argument for their proofs, based on techniques from [5] and [6].

1.3. Small deviation results for random walks

One of the new ingredients that we need to prove in Chapter 3 are the estimates for the probability that the rescaled paths of the branching random walk with spatially dependent rates lie within a function set. In the second part of this thesis we turn away from fragmentation processes to consider a different, but related, question: how accurately can we bound the probability that the path of a stochastic process stays near a given function? The ultimate goal of Chapter 4 is giving finer asymptotics of this probability for a random walk with constant rate. This work is motivated by the fact that, if we consider the random walk from Chapter 3, with the same space-time scaling but narrower tube widths (of order, for example, $T^{1/3}$ instead of T), then the results already available in the literature for the probability that a single path lies in a tube do not apply. Although this is our ultimate goal, the result we prove in Chapter 4 is a preliminary step in this direction and concerns instead the unscaled paths of a random walk with constant rate, as opposed to the rescaled paths of the walk with spatially dependent rates introduced in Chapter 3.

1.3.1 Mogul'skii small deviation theorem

Denote by D the space of càdlàg functions $f : [0, 1] \to \mathbb{R}$. Consider a sequence of independent identically distributed random variables $(\xi_n)_{n\geq 1}$ with $\mathbb{E}[\xi_1] = \mu$ and $\mathbb{E}[\xi_1^2] = \sigma^2$.

For a positive sequence $(x_n)_{n\geq 1}$ such that $x_n \to \infty$, define

$$S_n(t) = \frac{\xi_1 + \cdots + \xi_{\lfloor nt \rfloor} - \mu \lfloor nt \rfloor}{\sigma x_n}, \quad t \in [0, 1].$$

When $x_n = \sqrt{n}$ the Functional Central Limit Theorem ensures that the process $(S_n(t), t \in [0, 1])$ converges in distribution to a standard Brownian motion in the space D equipped with the Skorokhod topology. In [38], Mogul'skii studies the asymptotic behaviour of the sequence $\mathbb{P}(S_n \in G)$ for $G \subseteq D$ in the domain of small deviations, that is when $x_n/\sqrt{n} \to 0$.

Up to a change of measure, we can assume without loss of generality that $\mu = 0$. The following proposition is at the heart of Mogul'skii's Theorem.

Proposition 1.2. *Let $L > 0$, $x \in (-L, L)$ and $\tau \in [0, 1]$. Then*

$$\lim_{n\to\infty} \frac{1}{n^{1/3}} \log \mathbb{P}_x(S_n(t) \in (-L, L)\ \forall t \in [0, \tau]) = -\frac{\pi^2 \tau}{8L^2}.$$

The proof of Proposition 1.2 consists of splitting $[0, \tau]$ into smaller intervals, on which S_n is on the right scale to converge to a Brownian motion; then use the available results for the probability that a Brownian motion stays in a strip and combine together the estimates on these smaller intervals.

In [38] this result is extended to a wider class of subsets of D. For example, for tubes with piecewise constant width, we can apply a version of Proposition 1.2 for non-symmetric tubes on each interval where the boundaries of the tube are constant. On each of these smaller intervals, the process S_n will have different starting and ending positions. Some technical work is required to show that, in the limit as $n \to \infty$, these extra conditions do not affect the probability of S_n staying in a strip. From tubes with piecewise constant boundaries, we can generalise to continuous boundaries by approximating them with piecewise constant functions from above and below. This gives the following proposition.

Proposition 1.3 (Mogul'skii [38])**.** *Take two continuous functions $L^-, L^+ : [0, 1] \to \mathbb{R}$ such that $L^-(t) < L^+(t)$ for every $t \in [0, 1]$. Let $x \in (L^-(0), L^+(0))$. Then*

$$\lim_{n\to\infty} \frac{1}{n^{1/3}} \log \mathbb{P}_x(L^-(t) \le S_n(t) \le L^+(t)\ \forall t \in [0, 1]) = -\frac{\pi^2}{2} \int_0^1 \frac{dt}{(L^+(t) - L^-(t))^2}.$$

Ultimately, the last theorem in [38] extends this even further to more general subsets of D, but the rate of decay is more implicit. Furthermore, we have only considered the random walk S_n as a case study, but a similar statement can be obtained for a wider class of processes with independent increments which converge weakly to a stable distribution with exponent $\alpha \in (0, 2]$.

1.3.2 Tubes about nonlinear functions

With a simple change of measure, it is easy to extend Propositions 1.2 and 1.3 to tubes centred about linear functions $F(s) = zs$. In Chapter 4 we do not rescale the paths of the process and consider functions with a nonlinear component.

Take a compound Poisson process $(X(s),\ s \ge 0)$ with rate r and assume that its jump distribution ξ satisfies $\mathbb{E}[e^{\eta|\xi|}] < \infty$ for some $\eta > 0$.

Let $L > 0$, $p, q \in [-1, 1]$ with $p < q$ be independent of T. We determine the behaviour of

$$\mathbb{P}_{x_T}\big(|X(s) - zs - G(s)| < LT^{1/3}\ \forall s \in [0, T],\ X(T) - zT - G(T) \in (pLT^{1/3}, qLT^{1/3})\big)$$

when T is large where $z \in \mathbb{R}$ is a constant independent of T, $G(s)$ is a twice differentiable function such that $G(0) = 0$ and $x_T \in (-LT^{1/3}, LT^{1/3})$ satisfies $\lim_{T\to\infty} x_T/T^{1/3} = x$, with $x \in (-L, L)$. There are further technical conditions on $G(s)$, which we postpone to Chapter 4, but we essentially require that $|G'|$ decreases fast enough.
Define $\phi(\lambda) = \mathbb{E}[e^{\lambda\xi}]$ and let

$$\Lambda(z) := \sup_{\lambda:\, \phi(\lambda)<\infty} \{\lambda z - \log \mathbb{E}[e^{\lambda X(1)}]\} = \sup_{\lambda:\, \phi(\lambda)<\infty} \{\lambda z - r\phi(\lambda) + r\}$$

be the usual large deviations rate function. Denote by $\lambda(z)$ the value of λ for which the supremum is achieved, so that

$$\Lambda(z) = \lambda(z)z - r\phi(\lambda(z)) + r$$

and $\lambda(z)$ satisfies $\phi'(\lambda(z)) = z/r$.

Let D be the space of càdlàg functions $H : [0,\infty) \to \mathbb{R}$. For $F \in C^2([0,\infty),\mathbb{R})$, let

$$\mathcal{B}(F, L, a, b)|_{[u,t]} = \{H \in D : |H(s) - F(s)| < L \ \forall s \in [u,t],\ H(t) - F(t) \in (aL, bL)\}.$$

The main result we prove in Chapter 4 is the following.

Theorem 1.4. *Let $F_T(s) = zs + G(s) - x_T$, where $G(s)$ and x_T satisfy some extra properties (see* (i)-(v) *from Section 4.1 in Chapter 4). If $z > r\mathbb{E}[\xi]$ then*

$$\lim_{T\to\infty} \frac{1}{T^{1/3}} \Bigg(\log \mathbb{P}_0(X \in \mathcal{B}(F_T, LT^{1/3}, p, q)|_{[0,T]}) + \Lambda(z)T + \lambda(z)G(T) - \frac{1}{2}\int_0^T \frac{G'(s)^2}{r\phi''(\lambda(z))} ds \Bigg) = \lambda(z)(x - pL) - \frac{\pi^2 r\phi''(\lambda(z))}{8L^2}.$$

If $z < r\mathbb{E}[\xi]$ the same result holds but replacing pL with qL on the right-hand side. If $z = r\mathbb{E}[\xi]$ the same holds with $\lambda(z) = 0$, $\Lambda(z) = 0$ and $\phi''(\lambda(z)) = \mathbb{E}[\xi^2]$.

An analogous statement to Theorem 1.4 holds for discrete time random walks.

As we already observed, the key point in the proof of Mogul'skii's Theorem is that, for a process with mean zero, the probability of staying in a strip of constant width is not asymptotically affected by the position at which the process starts and ends at the beginning and at the end of the interval. We can always assume by a simple transformation that our process has mean zero. However, the fact that the values of the process at the endpoints of the smaller intervals give a negligible contribution to the probability that the process stays in a tube is a peculiar feature of the case in which the process has mean zero and the tube is centred around the zero function. As it will appear more explicitly in the proofs in Chapter 4, this is no longer the case when the function is nonlinear.

Further to this difficulty, while Girsanov's Theorem for Brownian motion can change the drift to any continuous curve, for a compound Poisson process, for example, it only allows to consider linear functions on fixed intervals.

Our strategy is similar to the proof of Theorem 1.3, in the sense that we also split $[0,T]$ into subintervals on which we approximate our process with a Brownian motion. To combine these results together, we need to estimate the error in the approximation given by the Functional Central Limit Theorem. An improved version of it, which provides the rate of convergence, is the Komlós-Major-Tusnády Theorem (see [35] for the original paper, and [24] for a more modern formulation). This is one of the steps where the strictest assumptions on G are required. We use sharper estimates of the probability that a Brownian motion stays in a tube with respect to the ones that appear in [38].

We conclude by stressing again that the assumptions we make in Chapter 4 are certainly not optimal and the results we present could be extended, for example, to broader classes of functions and tubes of width varying with time.

Chapter 2

Warming up: splitting at constant rate

In this chapter we work with the same model of fragmenting rectangles described in Chapter 1, except that every rectangle splits at constant rate r and the break occurs horizontally with probability $p \in (0, 1)$, or vertically with probability $1 - p$.

In a similar way to the models in [23] and [4], a logarithmic transformation of this fragmentation process gives a general branching random walk $Z_u(s) = (X_u(s), Y_u(s))$, where every particle branches at rate r; whenever a particle at position (x, y) branches, it is replaced by two children at positions $x - \log \mathcal{U}$ and $x - \log(1 - \mathcal{U})$ with probability p (respectively, at $y - \log \mathcal{U}$ and $y - \log(1 - \mathcal{U})$ with probability $1 - p$), where $\mathcal{U}$ is uniformly distributed on $(0, 1)$. It is clear that $X_u(s)$ and $Y_u(s)$ are independent when the rate and the probability of splitting in either direction are constant. Therefore the mathematical analysis of the fragmentation process reduces to separately considering the two components, that is two branching random walks with rates rp and $r(1 - p)$ respectively.

In this chapter we sketch the proof of the almost sure growth rates for this simplified version of the shape dependent fragmentation process; we prove the same result for the latter in Chapter 3. The tools that appear in the next sections are well-known results for branching random walks, in a technical setting similar to the one that will appear in Chapter 3. This chapter is intended as a warming up, to gradually introduce the notation and to highlight the steps where the spatial dependence of the rates requires a more involved proof. The experienced reader can skip this preliminary example and move on directly to Chapter 3.

2.1. Large deviations and technicalities on function spaces

One of the main ingredients we need in our proof is estimating the probability that a single particle trajectory stays near a given path. Fix T and define the T-rescaled trajectory of a process $(X(t),\, t \geq 0)$ by

$$X^T(t) = \frac{X(tT)}{T}, \quad 0 \leq t \leq 1.$$

For a wide class of processes with constant rate, estimates of the probability that $X^T(\cdot)$ belongs to a given function set are already available in the form of large deviation principles.

Standard large deviation principles assume that the increments of the process satisfy the *strong Cramér condition*, that is $\mathbb{E}[e^{\lambda \xi}] < \infty$ for all $\lambda \in \mathbb{R}$. For random walks in discrete time, for example, the large deviation principle can be established in the space of absolutely continuous functions with the uniform metric, see for example [25]. If the process jumps at random times, the natural choice of the function space is the set of càdlàg functions D, equipped with a topology that makes it a Polish space. Borovkov [17] proved a large deviation principle for compound Poisson processes satisfying the strong Cramér condition, in the space D with the J_1 topology – or Skorokhod topology – that is, the topology induced by the Skorokhod metric.

Our process is more complicated, because the jumps are exponentially distributed and do not satisfy the strong Cramér condition. This means that particles can make macroscopic jumps, in the sense that their rescaled paths will not be continuous. Furthermore, it is possible for particles to make two (or more) macroscopic jumps in quick succession. As a consequence, the J_1 topology is not suitable, because the rescaled set of paths that our particles take will not be compact in this topology.

Instead we use the M_2 topology, induced by the graph distance d_{graph}, which is defined as follows. For $f \in D$ we denote by $\Gamma(f)$ its graph, which is a subset of $\mathbb{R}^2$ containing the following points: if f is continuous at t then $(t, f(t)) \in \Gamma(f)$; if t is a discontinuity point of f then $(t, \alpha) \in \Gamma(f)$ for every α between $f(t-)$ and $f(t+)$. Denote by $\|\cdot\|$ the Euclidean metric in $\mathbb{R}^2$ and define

$$\Gamma(f)_\varepsilon = \{x \in \mathbb{R}^2 : \exists y \in \Gamma(f) : \|x - y\| < \varepsilon\}.$$

Then $d_{graph}(f, g) < \varepsilon$ if and only if $\Gamma(f) \in \Gamma(g)_\varepsilon$ and $\Gamma(g) \in \Gamma(f)_\varepsilon$ *simultaneously.*

The M_2 topology is weaker than the J_1 topology. For example, if we define

$$f_n(x) = \begin{cases} 0 & \text{if } x < 1 - 1/n \\ 1 & \text{if } 1 - 1/n \le x < 1 + 1/n \\ 2 & \text{if } x \ge 1 + 1/n, \end{cases} \qquad f(x) = \begin{cases} 0 & \text{if } x < 1 \\ 2 & \text{if } x \ge 1, \end{cases}$$

then $(f_n)_n$ converges to f in the M_2 topology, but not in the Skorokhod metric, where each f_n has distance 1 from f.

For a detailed analysis of intermediate topologies between J_1 and M_2 we refer to [18] and [42].

2.1.1 Mogul'skii's theorem about large deviations

In [39], Mogul'skii proves a large deviation principle for a compound Poisson process whose increments only satisfy $\mathbb{E}[e^{\lambda|\xi|}] < \infty$ for some $\lambda > 0$ (which is obviously true for the exponential distribution), in the space D with the M_2 topology.

Let $(X(t),\ t \ge 0)$ be a compound Poisson process with rate r and jump distribution ξ such that $\mathbb{E}[e^{\lambda|\xi|}] < \infty$ for some $\lambda > 0$. Let $\lambda^- = \inf\{\lambda : \mathbb{E}[e^{\lambda\xi}] < \infty\} \in [-\infty, 0)$ and $\lambda^+ = \sup\{\lambda : \mathbb{E}[e^{\lambda\xi}] < \infty\} \in (0, +\infty]$.

Define the Legendre transform of $\mathbb{E}[e^{\lambda X(1)}]$ by

$$\Lambda_r(x) = \sup_{\lambda \in (\lambda^-, \lambda^+)} \{\lambda x - \log \mathbb{E}[e^{\lambda X(1)}]\}, \quad x \in \mathbb{R}.$$

When $\lambda \in (\lambda^-, \lambda^+)$ the moment generating function $\phi(\lambda) = \mathbb{E}[e^{\lambda\xi}]$ is well defined and so is $\mathbb{E}[e^{\lambda X(1)}] = e^{r\phi(\lambda) - r}$. Then

$$\Lambda_r(x) = \sup_{\lambda \in (\lambda^-, \lambda^+)} \{\lambda x - r\phi(\lambda) + r\}. \tag{2.1}$$

Denote by D the set of cádlág functions $f : [0, 1] \to \mathbb{R}$. If $f \in D$ is absolutely continuous, for $0 \le a < b \le 1$ let

$$I_r(f, a, b) = \int_a^b \Lambda_r(f'(s))ds$$

and let $I_r(f, a, b) = +\infty$ otherwise. By the Lebesgue decomposition theorem, we can write $f \in D$ as $f = \tilde{f} + \hat{f}$ where $\tilde{f}$ is absolutely continuous and $\hat{f}$ is singular. We can also represent $\hat{f} = \hat{f}^+ - \hat{f}^-$ where $\hat{f}^+, \hat{f}^-$ are non-decreasing functions. Let

$$J_r(f, a, b) = I_r(\tilde{f}, a, b) + \lambda^+(\hat{f}^+(b) - \hat{f}^+(a)) - \lambda^-(\hat{f}^-(b) - \hat{f}^-(a)).$$

Theorem 2.1 (Mogul'skii [39]). *For every $F \subset D$ closed,*

$$\limsup_{T\to\infty} \frac{1}{T} \log \mathbb{P}(X^T \in F) \leq - \inf_{f\in F} J_r(f, 0, 1)$$

and for every $F \subset D$ open

$$\liminf_{T\to\infty} \frac{1}{T} \log \mathbb{P}(X^T \in F) \geq - \inf_{f\in F} J_r(f, 0, 1).$$

Since we are interested in a continuous-time random walk whose jumps are exponentially distributed with mean 1, the trajectories can only have positive jumps, so we can restrict Theorem 2.1 to sets of non-decreasing functions. We denote by E the set of non-decreasing functions in D. The Lévy metric [36] on E is defined by

$$d(f, g) = \inf\{r > 0 : f(x - r) - r < g(x) < f(x + r) + r \ \ \forall x \in [-r, 1 + r]\}, \tag{2.2}$$

where $f(x)$ is interpreted to equal $f(0)$ for $x < 0$ and $f(1)$ for $x > 1$, and similarly for g. The Lévy metric generates Skorokhod's M_2 topology on E, so (E, d) is a subspace of D equipped with the relative topology generated by d_{graph} in D. The metric space (E, d) is complete and separable.

When the branching rate is r and the jump distribution ξ is exponentially distributed with mean 1, standard calculations give that $\mathbb{E}[e^{\lambda\xi}] = 1/(1-\lambda)$ for $\lambda \in (-\infty, 1)$ and that the supremum in (2.1) is achieved when $\lambda = 1 - \sqrt{r/x}$, so

$$\Lambda_r(x) = (\sqrt{r} - \sqrt{x})^2.$$

Using that $\hat{f} = \hat{f}^+$ for every $f \in E$ and that $\lambda^+ = 1$ gives

$$J_r(f, a, b) = \int_a^b \left(\sqrt{r} - \sqrt{f'(s)}\right)^2 ds + \hat{f}(b) - \hat{f}(a). \tag{2.3}$$

2.1.2 Extension to spatially dependent rates: a coupling method

When the rate is dependent on the particles' positions, large deviation results such as Theorem 2.1 are not directly available. In fact, one of the novelties that will appear in Chapter 3 is a method to estimate probabilities such as those in Theorem 2.1 for a process with spatially dependent rates.

Section 3.6 is devoted to rigorously defining a coupling which traps the branching random walk with spatially dependent rates in between two processes with constant rate. We will work on small intervals, on which the rate function can be approximated with its maximum and minimum value. In this way, we translate the probability that a single trajectory of our process stays near a given function into probabilities involving the two bounding processes with constant rates, for which we are able to carry out the calculations.

We then relate these complicated expressions to a more explicit functional, which plays the role of J_r in Theorem 2.1 and to make sure that it satisfies a number of usual properties of large deviation functionals. For example, the lower semicontinuity of $J_r(f)$ with respect to the metric d_{graph} is proved in [19] and we need an analogous result for our rate. Sections 3.A and 3.B of Chapter 3 are devoted to technical checks on semicontinuity.

2.2. Calculating expectations with Many-to-few Lemmas

The Many-to-one Lemma for branching systems is a widely known tool, which reduces an expectation involving all the particles alive at time t to another expectation, under a different probability measure, only involving one process. We state here the result in its simplest version for a branching random walk in which particles die at constant rate r and are replaced by two offspring whose displacements are exponentially distributed with mean 1.

Let $\mathbb{Q}_x$ be a probability measure under which ξ_t is a continuous-time random walk starting from x, with rate $2r$ and exponential jumps with mean 1.

Lemma 2.2 (Many-to-one)**.** *For any measurable function $F : \mathbb{R} \to \mathbb{R}$,*

$$\mathbb{E}_{\mathbb{P}_x}\left[\sum_{v \in \mathcal{N}_t} F(X_v(t))\right] = e^{rt}\mathbb{E}_{\mathbb{Q}_x}[F(\xi_t)].$$

Although it is a relatively simple tool, the Many-to-one Lemma proves to be useful when we deal with space dependent rates. In this case, $e^{rt}\mathbb{E}_{\mathbb{Q}_x}[F(\xi_t)]$ is replaced by

$$\mathbb{E}_{\mathbb{Q}_x}\left[e^{\int_0^t R(\xi_s)ds} F(\xi_t)\right]. \tag{2.4}$$

Take F to be the indicator of events of the form $\{\xi^T(s) \in B(f, \varepsilon)\ \forall s \in [0, t]\}$. On these events, since the trajectory of the process is near f, the branching rate at time sT, $s \leq t$ is approximately $R(Tf(s/T))$. As this quantity is deterministic, the expectation in (2.4) reduces to the product of an exponential, deterministic term and the probability $\mathbb{Q}_x\big(\xi^T(s) \in B(f, \varepsilon)\ \forall s \in [0, t]\big)$.

The Many-to-one Lemma can be proved using spine techniques, see [29], [30] and [32] for a detailed formulation. The same approach of the Many-to-one Lemma can be extended to higher order moments: k-moments can be turned into expectations only involving the correlated paths of k stochastic processes, thus obtaining Many-to-k Lemmas. [32] contains a rigorous construction of the probability measures in terms of Radon-Nykodim derivatives, under more general assumptions on the branching rate, the offspring distribution and the motion of each particle in the system than ours. In particular, in Sections 3.2.1 and 3.3.2 in Chapter 3 we will see in more detail the version of the lemmas in which the branching rate can depend on the position of the particles.

Since our proof relies on the second moment method, we are only interested here in the case $k \in \{1, 2\}$. In the case $k = 2$, we define two continuous-time random walks ξ_t^1, ξ_t^2 and a new probability measure $\mathbb{Q}_x^2$ under which the two processes ξ_t^1, ξ_t^2 behave as follows. Let τ be exponentially distributed with parameter $2r$. Then:

- Conditional on $\tau = u$, we have $\xi_t^1 = \xi_t^2$ for every $t < u$, both processes start from x, jump at rate $2r$ and their jumps are exponentially distributed with mean 1;
- Let $\mathcal{U}$ be uniformly distributed on $(0, 1)$ and independent of everything else. We let $\xi_\tau^1 = \xi_{\tau-}^1 - \log \mathcal{U}$ and $\xi_\tau^2 = \xi_{\tau-}^1 - \log(1 - \mathcal{U})$;

- Conditionally on τ, $(\xi_t^1)_{t\le\tau}$ and $(\xi_t^2)_{t\le\tau}$, the processes $(\xi_t^1)_{t\ge\tau}$ and $(\xi_t^2)_{t\ge\tau}$ behave independently, jumping at rate $2r$ and with jumps exponentially distributed with mean 1.

Lemma 2.3 (Many-to-two)**.** *For any measurable function $F : \mathbb{R}^2 \to \mathbb{R}$,*

$$\mathbb{E}_{\mathbb{P}_x}\left[\sum_{v_1,v_2\in\mathcal{N}_t} F(X_{v_1}(t), X_{v_2}(t))\right] = \mathbb{E}_{\mathbb{Q}_x^2}\left[e^{2rt+r(\tau\wedge t)}F(\xi_t^1,\xi_t^2)\right].$$

2.3. Almost sure growth and growth in expectation

Consider a branching random walk $(X^r(s),\, s\ge 0)$ with rate r and exponentially distributed jumps with mean 1. For $T\ge 0$, we let $\mathcal{N}_T$ be the set of particles that are alive at time T. For $u\in\mathcal{N}_T$ and $t\le T$, let $X_u^r(t)$ be the position of the unique ancestor of u in $\mathcal{N}_t$. For $u\in\mathcal{N}_T$ and $s\in[0,1]$, write

$$X_u^{r,T}(s) = X_u^r(sT)/T.$$

We call $(X_u^{r,T}(s),\, s\in[0,1])$ the T-rescaled path of u. For a given set of functions $F\subseteq E$, define

$$N_T^r(F) = \#\{u\in\mathcal{N}_T : X_u^{r,T}\in F\},$$

the number of particles at time T whose T-rescaled paths have remained within F.

Recall from (2.3) that

$$J_r(f,a,b) = \int_a^b \left(\sqrt{r}-\sqrt{f'(s)}\right)^2 ds + \hat{f}(b) - \hat{f}(a) \tag{2.5}$$

for a random walk with rate r and exponentially distributed jumps with mean 1.

Let

$$\tilde{K}_r(f,a,b) = \begin{cases} r(b-a) - J_{2r}(f,a,b) & \text{if } J_{2r}(f,a,b)<\infty \\ -\infty & \text{otherwise.}\end{cases}$$

The following result, which establishes the expected growth rates for a branching random walk with rate r, is an immediate consequence of the many-to-one Lemma and the large deviations result in Theorem 2.1.

Proposition 2.4. *If $F\subset E$ is closed, then*

$$\limsup_{T\to\infty}\frac{1}{T}\log\mathbb{E}[N_T^r(F)] \le \sup_{f\in F}\tilde{K}_r(f,0,1),$$

and if $F\subset E$ is open, then

$$\liminf_{T\to\infty}\frac{1}{T}\log E[N_T^r(F)] \ge \sup_{f\in F}\tilde{K}_r(f,0,1).$$

We next prove a stronger statement about the growth of the number of particles almost surely. As it often happens, the result that holds in expectation does not reflect the actual behaviour of the system. Define

$$K_r(f) = \begin{cases} \tilde{K}_r(f,0,1) & \text{if } \tilde{K}_r(f,0,s) > 0 \ \forall s \le 1; \\ -\infty & \text{if } \exists s \le 1 \text{ such that } \tilde{K}_r(f,0,s) < 0; \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 2.5. *If $F \subset E$ is closed and $\sup_{f \in F} K_r(f) \neq 0$, then*

$$\limsup_{T\to\infty} \frac{1}{T} \log N_T^r(F) \le \sup_{f\in F} K_r(f) \qquad \textit{almost surely,}$$

and if $F \subset E$ is open and $\sup_{f \in F} K_r(f) \neq 0$, then

$$\liminf_{T\to\infty} \frac{1}{T} \log N_T^r(F) \ge \sup_{f\in F} K_r(f) \qquad \textit{almost surely.}$$

Theorems 2.4 and 2.5 show that if $\tilde{K}_r(f,0,s) > 0$ for all $s \le 1$, then the almost sure number of particles with a rescaled path looking like f up to time 1 matches with the expected number of particles following that path; if there exists a time $s < 1$ at which $\tilde{K}_r(f,0,s) < 0$ then at that time the number of particles along f is zero, that is, the genealogy becomes extinct. Therefore, although we might *expect* to see exponentially many particles at time T near $Tf(1)$, there are almost surely no particles following that path up to time T.

As we already mentioned at the beginning of this chapter, the two-dimensional branching random walk with spatially-dependent rates reduces to two independent components when the branching rate and the probability of splitting in the horizontal (or vertical) direction are constant. From Theorem 2.5, which concerns a one-dimensional branching random walk, we can immediately deduce the growth rates for the branching random walk $Z_u^r(s) = (X_u^{r,p}(s), Y_u^{r,p}(s))$, where $X_u^{r,p}(s)$ and $Y_u^{r,p}(s)$ are independent and branch at rates rp and $r(1-p)$ respectively.

2.4. Sketch proof of Theorem 2.5

In this section we focus on the building blocks necessary to prove Theorem 2.5. Although even shorter arguments than the ones presented here might work, for discussion purposes we decided to follow the same proof structure as in Chapter 3. Some technical details, which we prove in the more complicated case of spatially dependent, are omitted here.

To simplify our notation, until the end of this chapter we will refer to $X_u^r(s)$ and $N_T^r(F)$ as $X_u(s)$ and $N_T(F)$.

2.4.1 Compactness and ruling out difficult paths

As we already noticed in Section 2.2, one of the key ideas we exploit to control the rate of the branching random walk when it depends on the particles' position is looking at the number of particles which have rescaled paths in $B(f,\varepsilon)$. Since the Lévy metric is weaker than the sup-norm ρ on E, for any open set $F \subseteq E$ we can bound $N_T(F)$ from below with $N_T(B_\rho(f,\varepsilon))$, $f \in F$. For an upper bound on $N_T(F)$, we cover all the possible particle trajectories with sets of the form $B(f,\varepsilon)$. We start by ruling out some unlikely paths along which the probability of finding any particle is exponentially small. At the same time, we make sure to reduce the paths of interest to a compact set, for which it is possible to find a finite covering.

Define, for $M > 1$,

$$G_M = \{f \in E : f(s) \le Ms \ \forall s \in [0,1]\}.$$

We want to show that the rescaled path of all particles lie with high probability in G_M, if M is sufficiently large. However, since the particles in the branching random walk have exponentially distributed jumps, it helps to have a larger set of paths that allows for large jumps near the origin: we define, for $M > 1$ and $T > 1$,

$$G_{M,T} := \{f \in E : f(s) \le M(s + 2T^{-2/3}) \ \forall s \in [0,1]\}.$$

The choice of $T^{-2/3}$ is somewhat arbitrary. Any exponent in $(-1, 0)$ would work, but $-2/3$ turns out to be a convenient choice for the calculations in Chapter 3 and so we keep the same definition here.

The next lemma we introduce states that $F \cap G_{M,T}$ is totally bounded. We first need some new definitions.

For $F \subset E$ and $\delta > 0$, let $B_d(F, \delta) = \cup_{f \in F} B_d(f, \delta)$.

Denote by PL_n the set of functions $g : [0,1] \to \mathbb{R}$ which are continuous and piecewise linear on $[i/n, i/n+1/n]$ for $i \in \{0, \dots, n-1\}$. Due to technical reasons, we need to have a better control at how close two functions can be at the endpoints of these intervals, thus we introduce the following distance. For $f, g \in E$, let

$$\Delta_n(f, g) = \max\{|f(i/n) - g(i/n)| : i = 0, \dots, n\}.$$

For $T, M > 1$, $n \in \mathbb{N}$ and $f \in E$, define

$$\Gamma_{M,T}(f, n) = B_{\Delta_n}(f, 1/n^2) \cap B_d(f, 1/n) \cap G_{M,T}.$$

Lemma 2.6. *Let $F \subset E$ and $M > 1$. For any $n \ge 4M$, there exist $N \in \mathbb{N}$ and $g_1, \dots, g_N \in G_{4M} \cap PL_n$ such that*

$$F \cap G_{M,T} \subset \bigcup_{i=1}^{N} \Gamma_{M,T}(g_i, n) \subset B_d(F, 2/n)$$

for all $T \ge (4Mn)^{3/2}$.

We give a proof of Lemma 2.6 in Section 3.B.1 of Chapter 3, so we omit it here.

To complete the previous result, we have to show that with high probability, as T tends to infinity, all the particles have rescaled paths in $G_{M,T}$.

Lemma 2.7. *There exist $M_0, \delta_0, C > 0$ such that for any $T \ge 0$*

$$\mathbb{P}(\exists u \in \mathcal{N}_T : X_u^T \notin G_{M_0,T}) \le Ce^{-\delta_0 T^{1/3}}.$$

We give here a proof of Lemma 2.7 based on a formal construction of the branching random walk introduced in Section 2.3. We define a discrete tree with labels to represent the positions and split times of particles.

Take an infinite binary tree $\mathbb{T}$ and let $\mathbb{T}_n$ be the vertices in the nth generation of $\mathbb{T}$, so that $|\mathbb{T}_n| = 2^n$. Attach to each vertex $v \in \mathbb{T}$ two independent random variables: $\mathcal{U}_v$, which is uniformly distributed on $(0,1)$ and $\mathbb{e}_v$, which is exponentially distributed with parameter r. We define recursively the random variables X_v and T_v for each vertex $v \in \mathbb{T}$, which represent the position of the particle in the branching random walk and its birth time.

If $v \in \mathbb{T}_n$ and $v1$ and $v2$ are the two children of v in generation $n+1$, set

$$X_{v1} = X_v - \log \mathcal{U}_v, \quad X_{v2} = X_v - \log(1-\mathcal{U}_v), \quad T_{v1} = T_{v2} = T_v + \mathbb{e}_v.$$

For each $t \geq 0$, define

$$\mathcal{N}_t = \{v \in \mathbb{T} : T_v \leq t < T_v + \mathbb{e}_v\},$$

the set of particles alive at time t. For $v \in \mathcal{N}_t$, and $s \leq t$, if u is the unique ancestor of v in $\mathbb{T}$ that satisfies $T_u \leq s < T_u + \mathbb{e}_u$, then set $X_v(s) = X_u$. We call $X_v(s)$ the *position* of particle v at time s.

We show the following intermediate result before proving Lemma 2.7.

Lemma 2.8. *There exist $C > 0$ and $M > 1$ such that for any $T \geq 0$*

$$\mathbb{P}(\exists n \geq 0,\ \exists v \in \mathbb{T}_n : X_v > Mn + T^{1/3} \text{ or } T_v < n/M - T^{1/3}) \leq Ce^{-T^{1/3}/2}.$$

Proof. Fix $n \in \mathbb{N}$ and $u \in \mathbb{T}_n$. Then X_u is the sum of n independent exponential random variables with mean 1. Letting $\mathbb{e} \sim \mathrm{Exp}(1)$,

$$\begin{aligned}\mathbb{P}(X_u > Mn + T^{1/3}) \leq \mathbb{E}[e^{X_u/2}]e^{-Mn/2 - T^{1/3}/2} &\leq \mathbb{E}[e^{\mathbb{e}/2}]^n e^{-Mn/2-T^{1/3}/2} \\ &= 2^n e^{-Mn/2-T^{1/3}/2}.\end{aligned}$$

Since there are 2^n particles in $\mathbb{T}_n$, a union bound gives

$$\mathbb{P}(\exists v \in \mathbb{T}_n : X_v > Mn + T^{1/3}) = 2^n \mathbb{P}(X_u > Mn + T^{1/3}) \leq 4^n e^{-Mn/2-T^{1/3}/2},$$

which can be made smaller than $e^{-n-T^{1/3}/2}$ by choosing M large.

Analogous calculations give that $\mathbb{P}(\exists v \in \mathbb{T}_n : T_v < n/M - T^{1/3}) \leq e^{-n-T^{1/3}/2}$ when M is large, and the statement of the lemma follows from a union bound on n. $\square$

Proof of Lemma 2.7. We show that for any $\alpha \in (0,1)$, on the event

$$\{X_v \leq Mn + T^\alpha,\ T_v \geq n/M - T^\alpha\ \forall v \in \mathbb{T}_n,\ \forall n \geq 0\}$$

every particle is in $G_{M^2,T}$, that is for any $u \in \mathcal{N}_T$, $X_u^T(s) \leq M^2(s + 2T^{\alpha-1})\ \forall s \leq 1$.

Let v be the unique ancestor of u in $\mathcal{N}_s$ and $n(v)$ the unique natural number such that $v \in \mathbb{T}_{n(v)}$. Then $X_u(s) = X_v$ and $T_v \leq s < T_v + \mathbb{e}_v$. If $X_v \leq Mn(v) + T^\alpha$ and $T_v \geq n(v)/M - T^\alpha$, which implies $n(v) \leq M(T_v + T^\alpha)$, then

$$X_u(s) = X_v \leq Mn(v) + T^\alpha \leq M^2(T_v + T^\alpha) + T^\alpha \leq M^2(s + T^\alpha) + T^\alpha.$$

Rescaling by T this gives that

$$X_u^T(s) \leq M^2(s + T^{\alpha-1}) + T^{\alpha-1} \leq M^2(s + 2T^{\alpha-1}),$$

as required. Choosing $\alpha = 1/3$, the proof of Lemma 2.7 follows from Lemma 2.8. $\square$

We will use a similar strategy in Section 3.4 of Chapter 3, where the discrete construction of the process will be useful to decouple the dependencies between the jump times and the positions of the particles: by looking at events on which the positions of the particles stay in a cone, we can control the branching rate and therefore the birth times. In Chapter 3 we will have a different definition of G_M, which rules out paths that are too steep but also paths that stay too flat, so that the paths in G_M are those that lie in a cone between two straight lines. Since the rate function is the ratio of the two spatial components, this will give that the rate is bounded along the paths in G_M.

2.4.2 Proof of the upper bound in Theorem 2.5

Recall the definition

$$\Gamma_{M,T}(f,n) = B_{\Delta_n}(f,1/n^2) \cap B_d(f,1/n) \cap G_{M,T}$$

for $T, M > 1$, $n \in \mathbb{N}$ and $f \in E$. Lemma 2.6 shows that we can cover $G_{M,T}$ with the finite union of the sets $\Gamma_{M,T}(g_i, n)$, $i = 1, \dots, N$.

The first proposition we prove concerns the number of particles whose rescaled paths lie in $\Gamma_{M,T}(g_i, n)$ for a fixed g_i. Combining these results together for all the functions g_i, $i \in \{1, \dots, N\}$ we will be able to cover $G_{M,T}$ while, at the same time, we can rule out paths outside $G_{M,T}$ thanks to Lemma 2.7. An upper bound for the number of particles whose paths lie in sets like $\Gamma_{M,T}(g_i, n)$, is easily given by the combination of the Many-to-one Lemma and Theorem 2.1.

If $F \subset E$ and $g : [0,\theta] \to \mathbb{R}$, we say that $g \in F|_{[0,\theta]}$ if there exists a function $h \in F$ such that $h(u) = g(u)$ for all $u \in [0, \theta]$. Define

$$N_T(F,\theta) = \#\{v \in \mathcal{N}_{\theta T} : Z_v^T \in F|_{[0,\theta]}\},$$

the number of particles at time θT whose T-rescaled paths have remained within F up to time θ.

Proposition 2.9. *Suppose that $\theta \in (0, 1]$ and $g \in PL_n$. For any $\kappa > 0$,*

$$\mathbb{P}(N_T(\Gamma_{M,T}(g_i,n),\theta) \geq \kappa) \leq \frac{1}{\kappa} \exp\left(\tilde{K}_r(g,0,\theta)T + \tfrac{8T(\sqrt{r}+1)}{\sqrt{n}}\right)$$

if T is large enough.

Proof. By Markov's inequality, for any $\kappa > 0$

$$\mathbb{P}(N_T(\Gamma_{M,T}(g_i,n),\theta) \geq \kappa) \leq \frac{1}{\kappa}\, \mathbb{E}\left[\sum_{v \in \mathcal{N}_T} \mathbb{1}_{\{X_v^T|_{[0,\theta]} \in \Gamma_{M,T}(g_i,n)|_{[0,\theta]}\}}\right].$$

By the Many-to-one Lemma,

$$\mathbb{E}\left[\sum_{v \in \mathcal{N}_T} \mathbb{1}_{\{X_v^T|_{[0,\theta]} \in \Gamma_{M,T}(g_i,n)|_{[0,\theta]}\}}\right] = e^{rT}\mathbb{Q}\Big(\xi^T|_{[0,\theta]} \in \Gamma_{M,T}(g_i,n)|_{[0,\theta]}\Big), \tag{2.6}$$

where ξ_t branches at rate $2r$ under $\mathbb{Q}$.

By a slight modification of Theorem 2.1,

$$\mathbb{Q}\Big(\xi^T|_{[0,\theta]}\in \Gamma_{M,T}(g_i,n)|_{[0,\theta]}\Big) \le \exp\Big(- \inf_{h\in\overline{\Gamma_{M,T}(g_i,n)|_{[0,\theta]}}} J_{2r}(h,0,\theta)T + \tfrac{T}{\sqrt{n}}\Big)$$

if T is large enough. Since $g_i \in PL_n$, it is easy to see that

$$\inf_{h\in\overline{\Gamma_{M,T}(g_i,n)|_{[0,\theta]}}} J_{2r}(h,0,\theta) \ge J_{2r}(g,0,\theta) - \tfrac{8\sqrt{r}}{\sqrt{n}},$$

and so

$$\mathbb{Q}\Big(\xi^T|_{[0,\theta]}\in \Gamma_{M,T}(g_i,n)|_{[0,\theta]}\Big) \le \exp\Big(- J_{2r}(g,0,\theta)T + \tfrac{8T(\sqrt{r}+1)}{\sqrt{n}}\Big).$$

From this and (2.6) we deduce that

$$\begin{aligned}\mathbb{P}(N_T(\Gamma_{M,T}(g_i,n),\theta) \ge \kappa) &\le \frac{1}{\kappa}\exp\Big(rT - J_{2r}(g,0,\theta)T + \tfrac{8T(\sqrt{r}+1)}{\sqrt{n}}\Big)\\ &= \frac{1}{\kappa}\exp\Big(\tilde{K}_r(g,0,\theta)T + \tfrac{8T(\sqrt{r}+1)}{\sqrt{n}}\Big). \qquad \square\end{aligned}$$

The core of the proof of the upper bound of Theorem 2.5 consists of the following two propositions.

In the first one we show that for any fixed, large time T, the number of particles staying in $F \subset E$ is larger than $\exp\big(\sup_{f\in F\cap G_M} \tilde{K}_r(f,0,1)\big)$ with small probability. The proof uses Lemma 2.6 to cover F with finitely many sets $\Gamma_{M,T}(g_i,n)$, $i \in \{1,\dots,N\}$ and then applies Proposition 2.9. This proposition corresponds to Proposition 3.8 in Chapter 3.

Proposition 2.10. *Recall the definition of M_0 and δ_0 from Lemma 2.7. If $F \subset E$ is closed, then for $M \ge 4M_0$ and $\varepsilon > 0$*

$$\lim_{T\to\infty} \frac{1}{T^{1/3}} \log \mathbb{P}\left(N_T(F) \ge \exp\Big(\sup_{f\in F\cap G_M} \tilde{K}_r(f,0,1)T + \varepsilon T\Big)\right) \le -\delta_0.$$

Proof. By Lemma 2.6, we can choose $N \in \mathbb{N}$ and $g_1,\dots,g_N \in G_{4M} \cap PL_n$ such that

$$F \cap G_{M,T} \subset \bigcup_{i=1}^{N} \Gamma_{M,T}(g_i,n) \subset B_d(F,2/n).$$

Let $A = \sup_{f\in F\cap G_{4M}} \tilde{K}_r(f,0,1) + \varepsilon$. Then

$$\mathbb{P}(N_T(F) \ge e^{AT}) \le \mathbb{P}(\exists v \in \mathcal{N}_T : Z_v^T \notin G_{M,T}) + \sum_{i=1}^{N} \mathbb{P}\Big(N_T(\Gamma_{M,T}(g_i,n)) \ge \tfrac{e^{AT}}{N}\Big).$$

By Lemma 2.7, the first term on the right-hand side is smaller than $Ce^{-\delta_0 T^{1/3}}$, and since $g_i \in PL_n$, we can estimate each term in the sum with Proposition 2.9. If $M \ge M_0$

and n and T are large enough, then

$$\begin{aligned}\mathbb{P}(N_T(F) \geq e^{AT}) &\leq Ce^{-\delta_0 T^{1/3}} + \frac{N}{e^{AT}} \sum_{i=1}^{N} \exp\Big(\tilde{K}_r(g_i, 0, 1)T + \frac{\varepsilon T}{3}\Big) \\ &\leq Ce^{-\delta_0 T^{1/3}} + \frac{N^2}{e^{AT}} \max_{i\in\{1,\dots,N\}} \exp\Big(\tilde{K}_r(g_i, 0, 1)T + \frac{\varepsilon T}{3}\Big). \qquad (2.7)\end{aligned}$$

Since $B(g_i, 1/n) \subset B(F, 2/n)$ for every $i \in \{1, \dots, N\}$,

$$\max_{i\in\{1,\dots,N\}} \exp\Big(\tilde{K}_r(g_i, 0, 1)T + \frac{\varepsilon T}{3}\Big) \leq \sup_{f\in B(F,2/n)\cap G_{4M}} \exp\Big(\tilde{K}_r(f, 0, 1)T + \frac{\varepsilon T}{3}\Big).$$

From the lower semicontinuity of J_r, if $F \subset E$ is closed then

$$\lim_{n\to\infty} \sup_{f\in B(F,2/n)\cap G_{4M}} \tilde{K}_r(f, 0, 1) \leq \sup_{f\in F\cap G_{4M}} \tilde{K}_r(f, 0, 1).$$

Going back to (2.7) and substituting $A = \sup_{f\in F\cap G_{4M}} \tilde{K}_r(f, 0, 1) + \varepsilon$ we obtain

$$\begin{aligned}\mathbb{P}(N_T(F) \geq e^{AT}) &\leq Ce^{-\delta_0 T^{1/3}} + N^2 e^{-AT} \exp\Big(\sup_{f\in F\cap G_{4M}} \tilde{K}_r(f, 0, 1)T + \frac{2\varepsilon T}{3}\Big) \\ &\leq Ce^{-\delta_0 T^{1/3}} + N^2 e^{-\varepsilon T/3},\end{aligned}$$

and so

$$\lim_{T\to\infty} \frac{1}{T^{1/3}} \log \mathbb{P}(N_T(F) \geq e^{AT}) \leq -\delta_0.$$

This is the statement of the proposition with $4M$ instead of M. Since we only assumed in the proof that $M \geq M_0$, the proposition holds also when $M \geq 4M_0$. □

We now prove the second proposition required for the upper bound. We already observed that the paths such that $K_r(f) = -\infty$ but $\tilde{K}_r(f, 0, 1) > 0$ are those for which there exists a time $\theta \in (0, 1)$ with $\tilde{K}_r(f, 0, \theta) < 0$, and therefore the population goes extinct. As a consequence, if F only contains such paths, there will be no particles whose rescaled paths lie in F. For this class of sets, Proposition 2.10 does not provide a useful bound, therefore we need the following result.

Proposition 2.11. *Recall the definition of δ_0 from Lemma 2.7. If $F \subset E$ is closed and $\sup_{f\in F} K_r(f) = -\infty$, then*

$$\lim_{T\to\infty} \frac{1}{T^{1/3}} \log \mathbb{P}(N_T(F) \geq 1) \leq -\delta_0.$$

Proof. We can show that if $F \subset E$ is closed and $\sup_{f\in F} K_r(f) = -\infty$, if $M \geq 4M_0$ there exists n_0 such that

$$\sup_{f\in B_d(F,2/n_0)\cap G_{4M}} \inf_{\theta\in[0,1]} \tilde{K}_r(f, 0, \theta) < 0.$$

For a proof, we refer to the proof to Lemma 3.39 in Chapter 3. Let

$$\eta = -\sup_{f\in B_d(F,2/n_0)\cap G_{4M}} \inf_{\theta\in[0,1]} \tilde{K}_r(f, 0, \theta) > 0.$$

Take $n \geq n_0$ and such that $r/n \leq \eta/2$. By Lemma 2.6, we can choose $N \in \mathbb{N}$ and $g_1, \ldots, g_N \in G_{4M} \cap PL_n$ such that for every $T \geq (4Mn)^{3/2}$

$$F \cap G_{M,T} \subset \bigcup_{i=1}^{N} \Gamma_{M,T}(g_i, n) \subset B_d(F, 2/n).$$

For each $i \in \{1, \ldots, N\}$, since $g_i \in PL_n$, for every $0 \leq s \leq t \leq 1$,

$$\tilde{K}_r(g_i, 0, t) = rt - I_{2r}(g_i, 0, t) \leq rt - I_{2r}(g_i, 0, s) = r(t-s) + \tilde{K}_r(g_i, 0, s). \tag{2.8}$$

In particular, this implies that the function $t \to \tilde{K}_r(g_i, 0, t)$ has only downward jumps, and therefore its infimum is achieved. Let θ_i be such that

$$\tilde{K}_r(g_i, 0, \theta_i) = \inf_{\theta \in [0,1]} \tilde{K}_r(g_i, 0, \theta) \leq -\eta.$$

By Proposition 2.9,

$$\mathbb{P}(N_T(\Gamma_{M,T}(g_i, n), \theta_i) \geq 1) \leq e^{\tilde{K}_r(g_i, 0, \theta_i)T + \eta T/2} \leq e^{-\eta T/2}$$

if n and T are large enough. Since a population that is extinct at time $\theta < 1$ must be extinct also at time 1, then for every $i \in \{1, \ldots, N\}$

$$\mathbb{P}(N_T(\Gamma_{M,T}(g_i, n) \geq 1) \leq \mathbb{P}(N_T(\Gamma_{M,T}(g_i, n), \theta_i) \geq 1) \leq e^{-\eta T/2}. \tag{2.9}$$

From our choice of $g_1, \ldots, g_N$ we have

$$\mathbb{P}(N_T(F) \geq 1) \leq \mathbb{P}(N_T(G^c_{M,T}) \geq 1) + \sum_{i=1}^{N} \mathbb{P}(N_T(\Gamma_{M,T}(g_i, n)) \geq 1).$$

By Lemma 2.7, the first term on the right-hand side is smaller than $Ce^{-\delta_0 T^{1/3}}$ and each one of the terms in the sum is bounded uniformly in i from (2.9). This gives that

$$\mathbb{P}(N_T(F) \geq 1) \leq Ce^{-\delta_0 T^{1/3}} + Ne^{-\eta T/2},$$

from which the statement follows. □

We have now proved Propositions 2.10 and 2.11, which concern the number of particles in the branching system at fixed, large times. We can upgrade these results to *all* large times simultaneously and show that

$$\limsup_{T \to \infty} \frac{1}{T} \log N_T(F) \leq \sup_{f \in F \cap G_M} \tilde{K}_r(f, 0, 1) \quad \text{almost surely}$$

and that, if $\sup_{f \in F} K_r(f) = -\infty$, then $\limsup_{T \to \infty} N_T(F) = 0$ almost surely. The details are rather technical, so we refer to Section 3.7.4 of Chapter 3 for a full proof (but with spatially dependent rates): first we show that rescaling the paths by slightly different values of T does not affect the probability that they lie in a given set of functions, and then we apply the Borel-Cantelli Lemma.

2.4.3 Proof of the lower bound in Theorem 2.5

The standard approach to prove the lower bound in Theorem 2.5 is looking at the descendants of particles alive in an early generation and show that they have paths near f with positive probability. If we can show that there is a large enough number of particles near f in that early generation, then we can improve on this bound and obtain that overall there is a large number of particles whose rescaled paths stay near f.

For $\nu \in [0,1]$ and $u \in \mathcal{N}_{\nu T}$ let

$$N^u_{\nu,T}(F) = \#\{v \in \mathcal{N}_T : u \le v,\ Z^T_v|_{[\nu,1]} \in F|_{[\nu,1]}\}.$$

Fix $u \in \mathcal{N}_{\nu T}$. In the next proposition we show that there is a positive probability that the number of descendants of u whose rescaled paths lie in a given set is large enough. We use a second moment method by calculating the expectations with the Many-to-one and Many-to-two Lemmas.

Proposition 2.12. *Let $n \in \mathbb{N}$, $\nu \in (0,1)$ and $f \in PL_n$ such that $\tilde{K}_r(f,\nu,t) \ge 0$ for all $t \ge \nu$. If $u \in \mathcal{N}_{\nu T}$ and $X^T_u(\nu) = x$, define $f^u(s) = f(s) + x - f(\nu)$ for $s \in [0,1]$. Then, for any $u \in \mathcal{N}_{\nu T}$,*

$$\mathbb{P}\big(N^u_{\nu,T}(B_\rho(f^u,1/n)) \ge e^{\tilde{K}_r(f,\nu,1)T - T/n} | \mathcal{F}_{\nu T}\big) \ge \frac{1}{4r}\exp\Big(-\tfrac{24T(\sqrt{r}+1)}{\sqrt{n}}\Big)$$

if T is large enough.

Proof. The Paley-Zygmund inequality states that for any non-negative random variable X and $\theta \in [0,1]$

$$P(X \ge \theta E[X]) \ge (1-\theta)^2 \frac{E[X]^2}{E[X^2]}.$$

Taking P to be the conditional probability $\mathbb{P}(\,\cdot\,|\mathcal{F}_{\nu T})$ with $X = N^u_{\nu,T}(B_\rho(f^u,1/n))$ and $\theta = 1/2$, we have

$$\begin{aligned}\mathbb{P}\Big(N^u_{\nu,T}(B_\rho(f^u,1/n)) \ge \tfrac{1}{2}\mathbb{E}\big[N^u_{\nu,T}(B_\rho(f^u,1/n))|\mathcal{F}_{\nu T}\big]\ \Big|\mathcal{F}_{\nu T}\Big)& \\ \ge \frac{\mathbb{E}\big[N^u_{\nu,T}(B_\rho(f^u,1/n))|\mathcal{F}_{\nu T}\big]^2}{4\mathbb{E}\big[N^u_{\nu,T}(B_\rho(f^u,1/n))^2|\mathcal{F}_{\nu T}\big]}.& \qquad (2.10)\end{aligned}$$

We find a lower bound for the first moment and an upper bound for the second moment. By Lemma 2.2 and stationarity,

$$\begin{aligned}\mathbb{E}\big[N^u_{\nu,T}(B_\rho(f^u,1/n))|\mathcal{F}_{\nu T}\big] &= e^{rT(1-\nu)}\mathbb{Q}\big(\xi^T|_{[\nu,1]} \in B_\rho(f^u,1/n)|_{[\nu,1]}|\xi^T(\nu) = x\big)\big|_{x=X^T_u(\nu)} \\ &= e^{rT(1-\nu)}\mathbb{Q}_0\big(\xi^T|_{[0,1-\nu]} \in B_\rho(\hat{f},1/n)|_{[0,1-\nu]}\big)\end{aligned}$$

where ξ^T branches at rate $2r$ under $\mathbb{Q}$ and $\hat{f}(s) = f(s+\nu) - f(\nu)$ for $s \in [0,1-\nu]$. A slight modification of Theorem 2.1 gives that

$$\begin{aligned}&\mathbb{Q}_0\big(\xi^T|_{[0,1-\nu]} \in B_\rho(\hat{f},1/n)|_{[0,1-\nu]}\big) \\ &\quad \ge \exp\Big(-\inf_{h \in B_d(f,1/n)} J_{2r}(h,\nu,1)T - T/n\Big) \ge \exp\big(-J_{2r}(f,\nu,1)T - T/n\big)\end{aligned}$$

if T is large, from which it follows that

$$\mathbb{E}[N^u_{\nu,T}(B_\rho(f^u,1/n))|\mathcal{F}_{\nu T}] \\ \geq \exp\big(rT(1-\nu) - J_{2r}(f,\nu,1)T - T/n\big) = \exp\big(\tilde{K}_r(f,\nu,1)T - T/n\big) \quad (2.11)$$

when T is large. On the other hand, by Lemma 2.3

$$\mathbb{E}[N^u_{\nu,T}(B_\rho(f^u,1/n))^2|\mathcal{F}_{\nu T}] \\ = \mathbb{E}_{\mathbb{Q}^2_x}\Big[e^{2rT-3r\nu T+r(\tau\wedge T)}\mathbb{1}_{\{\xi^T_1|_{[\nu,1]},\ \xi^T_2|_{[\nu,1]}\in B_\rho(f^u,1/n)|_{[\nu,1]}\}}\Big|\tau>\nu T,\ \xi^T_1(\nu)=x\Big]\Big|_{x=X^T_u(\nu)}.$$

Using the construction of $\mathbb{Q}^2$ from Section 2.2, and in particular that τ is exponentially distributed with rate $2r$, we get that

$$\mathbb{E}_{\mathbb{Q}^2_x}\Big[e^{2rT-3r\nu T+r(\tau\wedge T)}\mathbb{1}_{\{\xi^T_1|_{[\nu,1]},\ \xi^T_2|_{[\nu,1]}\in B_\rho(f^u,1/n)|_{[\nu,1]}\}}\Big|\tau>\nu T,\ \xi^T_1(\nu)=x\Big]\Big|_{x=X^T_u(\nu)} \\ = e^{3rT(1-\nu)}\mathbb{Q}^2\Big(\xi^T_1|_{[\nu,1]}\in B_\rho(f^u,1/n)|_{[\nu,1]}\Big|\tau>T,\ \xi^T_1(\nu)=x\Big)e^{-2rT(1-\nu)} \\ + \int_\nu^1 \exp\big(2rT-3r\nu T+rsT\big) \\ \cdot\mathbb{Q}^2\Big(\xi^T_1|_{[\nu,1]},\xi^T_2|_{[\nu,1]}\in B_\rho(f^u,1/n)|_{[\nu,1]}\Big|\tau=sT,\ \xi^T_1(\nu)=x\Big)\big(2re^{-2rT(s-\nu)}\big)ds.$$

Since $f\in PL_n$, it is not difficult to prove that

$$\inf_{h\in B_\rho(f,1/n)} J_{2r}(h,\nu,1) \geq J_{2r}(f,\nu,1) - \frac{8\sqrt{r}}{\sqrt{n}},$$

and so by a slight modification of Theorem 2.1

$$\mathbb{Q}^2\Big(\xi^T_1|_{[\nu,1]}\in B_\rho(f^u,1/n)|_{[\nu,1]}\Big|\tau>T,\ \xi^T_1(\nu)=x\Big) \\ \leq \exp\Big(-\inf_{h\in B_\rho(f,1/n)} J_{2r}(h,\nu,1)+\tfrac{T}{n}\Big) \leq \exp\Big(-J_{2r}(f,\nu,1)T+\tfrac{8T\sqrt{r}}{\sqrt{n}}+\tfrac{T}{n}\Big)$$

if T is large. Similarly,

$$\mathbb{Q}^2\Big(\xi^T_1|_{[\nu,1]},\xi^T_2|_{[\nu,1]}\in B_\rho(f^u,1/n)|_{[\nu,1]}\Big|\tau=sT,\ \xi^T_1(\nu)=x\Big) \\ \leq \mathbb{Q}^2\Big(\xi^T_1|_{[\nu,s]}\in B_\rho(f^u,1/n)|_{[\nu,s]}\Big|\tau=sT,\ \xi^T_1(\nu)=x\Big) \\ \cdot \sup_{\|w-f^u(s)\|<1/n}\mathbb{Q}^2\Big(\xi^T_1|_{[s,1]}\in B_\rho(f^u,1/n)|_{[s,1]}\Big|\tau=sT,\ \xi^T_1(u)=w\Big)^2 \\ \leq \exp\Big(-J_{2r}(f,\nu,s)T+\tfrac{8T\sqrt{r}}{\sqrt{n}}+\tfrac{T}{n}\Big)\cdot\exp\Big(-2J_{2r}(f,s,1)T+\tfrac{16T\sqrt{r}}{\sqrt{n}}+\tfrac{2T}{n}\Big).$$

Putting these estimates together gives that

$$\begin{aligned}&\mathbb{E}[N^u_{\nu,T}(B_\rho(f^u,1/n))^2|\mathcal{F}_{\nu T}]\\&\le \exp\Big(K_r(f,\nu,1)T+\tfrac{8T\sqrt{r}}{\sqrt{n}}+\tfrac{T}{n}\Big)\\&\quad+\int_\nu^1 2re^{2rT-\nu T-rsT}\cdot\exp\Big(-J_{2r}(f,\nu,s)T-2J_{2r}(f,s,1)T+\tfrac{24T\sqrt{r}}{\sqrt{n}}+\tfrac{3T}{n}\Big)ds.\end{aligned}$$

Writing

$$2rT-\nu T-rsT-J_{2r}(f,\nu,s)T-2J_{2r}(f,s,1)T=2\tilde{K}_r(f,\nu,1)T-\tilde{K}_r(f,\nu,s)T,$$

we have

$$\begin{aligned}&\mathbb{E}[N^u_{\nu,T}(B_\rho(f^u,1/n))^2|\mathcal{F}_{\nu T}]\\&\quad\le \exp\Big(K_r(f,\nu,1)T+\tfrac{8T\sqrt{r}}{\sqrt{n}}+\tfrac{T}{n}\Big)\\&\qquad+\int_\nu^1 2r\exp\Big(2\tilde{K}_r(f,\nu,1)T-\tilde{K}_r(f,\nu,u)T+\tfrac{24T\sqrt{r}}{\sqrt{n}}+\tfrac{3T}{n}\Big)ds.\end{aligned}$$

Since we assumed that $\tilde{K}_r(f,\nu,t)\ge 0$ for all $t\ge\nu$, this reduces to

$$\mathbb{E}\big[N^u_{\nu,T}(B_\rho(f^u,1/n))^2|\mathcal{F}_{\nu T}\big]\le 4r(1-\nu)\exp\Big(2\tilde{K}_r(f,\nu,1)T+\tfrac{24T\sqrt{r}}{\sqrt{n}}+\tfrac{3T}{n}\Big).\qquad(2.12)$$

Going back to (2.10) and substituting (2.11) and (2.12) gives that

$$\begin{aligned}&\mathbb{P}\Big(N^u_{\nu,T}(B_\rho(f^u,1/n))\ge\tfrac12\mathbb{E}\big[N^u_{\nu,T}(B_\rho(f^u,1/n))|\mathcal{F}_{\nu T}\big]\ \Big|\mathcal{F}_{\nu T}\Big)\\&\qquad\qquad\ge\frac{1}{4r(1-\nu)}\exp\Big(-\tfrac{2T}{n}-\tfrac{24T\sqrt{r}}{\sqrt{n}}-\tfrac{3T}{n}\Big),\end{aligned}$$

and this concludes the proof. □

Proposition 2.12 ensures that for any $u\in\mathcal{N}_{\nu T}$, with positive probability u has a large number of descendants in $\mathcal{N}_T$ whose rescaled path is close to f^u. We now focus on small times, and show that the number of particles near $Tf(\nu)$ at time νT is large with high probability.

In a branching random walk with constant rate the total population grows exponentially fast, but in the spatially-dependent model of Chapter 3 the number of particles in a fixed generation is not easy to calculate, because the rate depends on the path that the particles follow up to that time. To add to this difficulty, the system could have a wild behaviour at small times since, after rescaling, the rate function is discontinuous at 0. This will lead to several new propositions in Section 3.5 of Chapter 3. Our strategy consists of building a piecewise linear function which starts equal to $(s/2,s/2)$ and then gradually changes its drift to $f(s')$ at time $s'>s$. We show that there are exponentially many particles near $(s/2,s/2)$ for small s, and these produce many offspring that end near $f(s')$. By considering paths that stay near this function up to time $s'T$ we can control the rate function along them and therefore the growth of newborn particles.

When the rate is constant, the following cruder estimates are enough.

Proposition 2.13. *Let $f \in PL_n$ and $\nu \in (0,1)$. Define*

$$V_{\nu,T}(f) = \{u \in \mathcal{N}_{\nu T} : |X_u^T(\nu) - f(\nu)| < M\nu\}.$$

If $\eta < \nu$ and M is large enough, there exist $C, k > 0$ such that

$$\mathbb{P}(|V_{\nu,T}(f)| < e^{\eta r T}) \le C e^{-kT}.$$

Proof. Recall that $N(tT)$ denotes the number of particles alive at time tT. Then

$$\mathbb{P}(|V_{\nu,T}(f)| < e^{\eta r T}) \le \mathbb{P}\big(N(\nu T) < e^{\eta r T}\big) + \mathbb{P}\big(\exists u \in \mathcal{N}_{\nu T} : |X_u^T(\nu) - f(\nu)| > M\nu\big). \quad (2.13)$$

We show that both the terms on the right-hand side are exponentially small.

Let $\tau_0 = 0$ and τ_n be the birth time of the nth particle in the system and let $\sigma_n = \tau_n - \tau_{n-1}$. Then $(\sigma_n,\, n \ge 1)$ are independent random variables exponentially distributed with parameter rn. By Markov's inequality,

$$\mathbb{P}(\tau_n > \nu T) = \mathbb{P}\left(\sum_{k=1}^n \sigma_k > \nu T\right) \le \mathbb{E}\Big[e^{(r/2)\sum_{k=1}^n \sigma_k}\Big] e^{-\nu r T/2} = \prod_{k=1}^n \Big(1 - \frac{1}{2k}\Big)^{-1} e^{-\nu r T/2}.$$

Using that $\log(1-x) + x + x^2 \ge 0$ if $x \le 1/2$,

$$\prod_{k=1}^n \Big(1 - \frac{1}{2k}\Big)^{-1} = \exp\left(-\sum_{k=1}^n \log\Big(1 - \frac{1}{2k}\Big)\right) \le \exp\left(\sum_{k=1}^n \Big(\frac{1}{2k} + \frac{1}{4k^2}\Big)\right)$$

and since $\sum_{k=1}^n (1/k) \le \log(n) + 1$, this is smaller than $Cn^{1/2}$ for some constant $C > 0$. Letting $n = \lfloor e^{\eta r T} \rfloor$, we can conclude that

$$\mathbb{P}\big(N(\nu T) < e^{\eta r T}\big) \le \mathbb{P}(\tau_n > \nu T) \le C e^{\eta r T/2 - \nu r T/2},$$

which is exponentially small when $\eta < \nu$.

We now consider the second term in (2.13). Since $f \in PL_n$, we have $f \in G_{M/2}$ for some $M > 0$. By Lemma 2.2,

$$\begin{aligned}\mathbb{P}\big(\exists u \in \mathcal{N}_{\nu T} : |X_u^T(\nu) - f(\nu)| > M\nu\big) &= e^{r\nu T}\mathbb{Q}\big(|\xi^T(\nu) - f(\nu)| > M\nu\big)\\ &\le e^{r\nu T}\mathbb{Q}\big(\xi^T(\nu) > M\nu\big),\end{aligned}$$

where ξ jumps at rate $2r$ under $\mathbb{Q}$. This is smaller than

$$\begin{aligned}e^{r\nu T}\mathbb{E}_{\mathbb{Q}}[e^{\xi(\nu T)/2}]e^{-MT\nu/2} &\le \exp\Big(r\nu T + r\nu T(1 - 1/2)^{-1} - MT\nu/2\Big)\\ &= \exp\Big(3r\nu T - MT\nu/2\Big),\end{aligned}$$

which is exponentially small when $M > 6r$. Since choosing a larger M does not affect the argument, this completes the proof. □

As we anticipated, we can now combine Propositions 2.12 and 2.13 to show that the number of particles whose rescaled paths are in F is smaller than $\exp\big(\sup_{f \in F \cap G_M} \tilde{K}_r(f, 0, 1)\big)$

with exponentially small probability.

Proposition 2.14. *If $F \subset E$ is open and $\sup_{f\in F} K_r(f) \neq 0$, then for every $\varepsilon > 0$ there exists a constant $\kappa > 0$ such that*

$$\lim_{T\to\infty} \frac{1}{T} \log \mathbb{P}\left(N_T(F) < \exp\Big(\sup_{g\in F\cap G_M} \tilde{K}_r(g,0,1)T - \varepsilon T\Big)\right) \leq -\kappa.$$

Proof. Take $\varepsilon > 0$. Since $K_r(f) \in \{-\infty\} \cup [0,\infty)$, if $\sup_{f\in F} K_r(f) \leq 0$ then there is nothing to prove. We may therefore assume that there exists $f \in F$ such that $K_r(f) > 0$. In this case, since F is open we can choose $\varepsilon' \in (0,\varepsilon)$ such that $B_d(f,\varepsilon') \subset F$ and $\tilde{K}_r(f,0,1) \geq \sup_{g\in F\cap G_M} \tilde{K}_r(g,0,1) - \varepsilon'$, so

$$\mathbb{P}\left(N_T(F) < \exp\Big(\sup_{g\in F\cap G_M} \tilde{K}_r(g,0,1)T - 3\varepsilon T\Big)\right) \leq \mathbb{P}\Big(N_T(B_d(f,\varepsilon')) < e^{\tilde{K}_r(f,0,1)T-2\varepsilon T}\Big). \tag{2.14}$$

Denote by $f_n \in PL_n$ the linear interpolation of f. Then $f_n \in B_d(f,\varepsilon'/2)$ if n is large. Furthermore, since f is right-continuous, if ν is small enough then f is continuous on $[0,\nu)$ and $\tilde{K}_r(f,0,\nu) \leq r\nu$, so letting $\nu = 1/n^{1/8}$ we have that $\tilde{K}_r(f,0,\nu) < \varepsilon$ if n is large. We also have that $\tilde{K}_r(f_n,a,b) \geq \tilde{K}_r(f,a,b)$ for any $a,b \in [0,1]$, which implies that $\tilde{K}_r(f_n,\nu,1) \geq \tilde{K}_r(f,\nu,1) \geq \tilde{K}_r(f,0,1) - \varepsilon$ and $K_r(f_n) > 0$.

Since $B_\rho(f_n,\varepsilon'/2) \subseteq B_d(f_n,\varepsilon'/2)$, it follows that (2.14) is smaller than

$$\mathbb{P}\Big(N_T(B_\rho(f_n,\varepsilon'/2)) < e^{\tilde{K}_r(f_n,\nu,1)T-\varepsilon T}\Big).$$

Since $\tilde{K}_r(f_n) > 0$, we have that $f_n \in G_M$ for some $M > 0$.
Recall that $V_{\nu,T}(f_n) = \{u \in \mathcal{N}_{\nu T} : |X_u^T(\nu) - f_n(\nu)|< M\nu\}$. If $u \in V_{\nu,T}(f_n)$ and $X_u^T(\nu) = x$, define $f_n^u(s) = f_n(s) + x - f_n(\nu)$ for $s \in [0,1]$ and note that $\tilde{K}_r(f_n^u,\nu,1) = \tilde{K}_r(f_n,\nu,1)$. Since $|x - f_n(\nu)|< 2M\nu$, conditioning on the system at time νT, when $2M\nu < \varepsilon'/4$

$$\begin{aligned}
&\mathbb{P}\Big(N_T(B_\rho(f_n,\varepsilon'/2)) < e^{\tilde{K}_r(f_n,\nu,1)T-\varepsilon T}\Big) \\
&\qquad = \mathbb{E}\left[\mathbb{P}\left(N_T(B_\rho(f_n,\varepsilon'/2)) < e^{\tilde{K}_r(f_n,\nu,1)T-\varepsilon T} \middle| \mathcal{F}_{\nu T}\right)\right] \\
&\qquad \leq \mathbb{E}\left[\prod_{u\in V_{\nu,T}(f_n)} \mathbb{P}\left(N^u_{\nu,T}(B_\rho(f^u_n,\varepsilon'/4)) < e^{\tilde{K}_r(f_n,\nu,1)T-\varepsilon T} \middle| \mathcal{F}_{\nu T}\right)\right].
\end{aligned}$$

When n is large, by Proposition 2.12, for each $u \in V_{\nu,T}(f_n)$

$$\mathbb{P}\left(N^u_{\nu,T}(B_\rho(f^u_n,\varepsilon'/4)) < e^{\tilde{K}_r(f_n,\nu,1)T-\varepsilon T} \middle| \mathcal{F}_{\nu T}\right) \leq 1 - \frac{1}{4r}\exp\Big(-\tfrac{24T(\sqrt{r}+1)}{\sqrt{n}}\Big)$$

if T is large enough.

With $\nu = 1/n^{1/8}$ and $\eta = 1/n^{1/4}$, Lemma 2.13 ensures that if M is large enough, there exist $C, k > 0$ such that $\mathbb{P}(|V_{\nu,T}(f_n)| \leq e^{rT/n^{1/4}}) \leq Ce^{-kT}$, so

$$\mathbb{P}\Big(N_T(B_\rho(f_n, \varepsilon'/2)) < e^{\tilde{K}_r(f_n,\nu,1)T-\varepsilon T}\Big) \leq \mathbb{E}\left[\prod_{u\in V_{\nu,T}(f)}\left(1-\frac{1}{4r}\exp\Big(-\tfrac{24T(\sqrt{r}+1)}{\sqrt{n}}\Big)\right)\right]$$
$$\leq \left(1-\frac{1}{4r}\exp\Big(-\tfrac{24T(\sqrt{r}+1)}{\sqrt{n}}\Big)\right)^{e^{rT/n^{1/4}}} + Ce^{-kT}.$$

Using that $1-x \leq e^{-x}$ for all x,

$$\mathbb{P}\Big(N_T(B_\rho(f_n, \varepsilon'/2)) < e^{\tilde{K}_r(f_n,\nu,1)T-\varepsilon T}\Big) \leq \exp\Big(-\tfrac{1}{4r}\exp\big(\tfrac{rT}{n^{1/4}} - \tfrac{24T(\sqrt{r}+1)}{\sqrt{n}}\big)\Big) + Ce^{-kT},$$

from which the statement follows. □

To conclude the proof of the lower bound of Theorem 2.5 we can use similar arguments to those used for the upper bound: we can extend Proposition 2.14 to all large times simultaneously by dealing with technicalities about the rescaling and then applying the Borel-Cantelli Lemma. These details are similar to those in Section 3.3 of Chapter 3, so we omit them here.

Chapter 3

A shape-dependent fragmentation process

3.1. The main theorem

In this chapter we consider the spatial fragmentation model defined in Chapter 1 and its equivalent branching random walk description. We recall that this involves a branching random walk in $\mathbb{R}^2$ starting from $(0,0)$ with exponentially distributed jumps, where each particle, when at position (x,y) with $x,y \geq 0$, branches at rate

$$R(x,y) = \frac{x+1}{y+1} \vee \frac{y+1}{x+1}.$$

Let $\mathcal{U}$ be a uniform random variable on $(0,1)$. When a particle branches, it has two children at positions $(x - \log\mathcal{U}, y)$ and $(x - \log(1-\mathcal{U}), y)$ with probability

$$P(x,y) = \frac{y+1}{2(x+1)}\mathbb{1}_{x\geq y} + \left(1 - \frac{x+1}{2(y+1)}\right)\mathbb{1}_{x<y},$$

or at positions $(x, y - \log\mathcal{U})$ and $(x, y - \log(1-\mathcal{U}))$ with probability $1 - P(x,y)$.

We let $R_X(x,y) = R(x,y)P(x,y)$ and $R_Y(x,y) = R(x,y)(1-P(x,y))$. Then R_X and R_Y denote the rates at which a particle at position (x,y) moves in the first spatial dimension, or the second, respectively.

Our main theorem aims to quantify how many particles have paths which, when rescaled appropriately, fall within a given subset of E^2, where E is the set of non-decreasing càdlàg functions $f : [0,1] \to \mathbb{R}$ with $f(0) = 0$.

Set $f'(s) = \infty$ if $f \in E$ is not differentiable at the point $s \in [0,1]$. Recall that with the notation introduced in Chapter 2, we can write any function $f \in E$ as $f = \tilde{f} + \hat{f}$ where $\tilde{f}$ is absolutely continuous and $\hat{f}$ is singular.

Since we are interested in rescaled paths for large times, R and P are essentially governed by the ratios x/y and y/x. We define the functions $R^* : [0,\infty)^2 \to [0,\infty]$ and $P^* : [0,\infty)^2 \to [0,1]$ by

$$R^*(x,y) := \begin{cases} \frac{x}{y} \vee \frac{y}{x} & \text{if } x > 0 \text{ or } y > 0 \\ 1 & \text{if } x = y = 0 \end{cases}$$

and

$$P^*(x,y) := \begin{cases} \frac{y}{2x}\mathbb{1}_{x\geq y} + \left(1 - \frac{x}{2y}\right)\mathbb{1}_{x<y} & \text{if } x > 0 \text{ or } y > 0 \\ 1/2 & \text{if } x = y = 0. \end{cases}$$

Although our splitting rule is described by the functions R and P, which are continuous at 0, at large times the constant terms in those functions become insignificant and the behaviour when the system is rescaled appropriately is captured instead by R^* and P^*. We let

$$R_X^*(x,y) := \begin{cases} R^*(x,y)P^*(x,y) & \text{if } y > 0 \\ 1/2 & \text{if } y = 0 \end{cases}$$

and

$$R_Y^*(x,y) := \begin{cases} R^*(x,y)(1-P^*(x,y)) & \text{if } x>0 \\ 1/2 & \text{if } x=0. \end{cases}$$

Suppose that $f=(f_X,f_Y)\in E^2$ and $0\le a\le b\le 1$. Define the functionals

$$I(f,a,b)=\int_a^b \left(2^{1/2}R_X^*(f(s))^{1/2}-f_X'(s)^{1/2}\right)^2 ds+\int_a^b \left(2^{1/2}R_Y^*(f(s))^{1/2}-f_Y'(s)^{1/2}\right)^2 ds,$$

$$J(f,a,b)=I(f,a,b)+\hat{f}_X(b)-\hat{f}_X(a)+\hat{f}_Y(b)-\hat{f}_Y(a)$$

and

$$\tilde{K}(f,a,b)=\begin{cases} \int_a^b R^*(f(s))ds-J(f,a,b) & \text{if } J(f,a,b)<\infty; \\ -\infty & \text{otherwise.} \end{cases}$$

Note that any $f\in E^2$ is necessarily continuous at 0, and therefore if $\lim_{t\to 0}\tilde{K}(f,0,t)\neq -\infty$ then $\tilde{K}(f,0,t)$ is differentiable (in t) at 0. If $\lim_{t\to 0}\tilde{K}(f,0,t)=-\infty$ then write $\frac{d}{dt}\tilde{K}(f,0,t)|_{t=0}=-\infty$. Define

$$K(f)=\begin{cases} \tilde{K}(f,0,1) & \text{if } \frac{d}{dt}\tilde{K}(f,0,t)|_{t=0}>0 \text{ and } \tilde{K}(f,0,s)>0 \ \ \forall s\le 1; \\ -\infty & \text{if } \exists s\le 1 \text{ such that } \tilde{K}(f,0,s)<0; \\ 0 & \text{otherwise.} \end{cases}$$

These represent the behaviour of $R_X(x,y)$ and $R_Y(x,y)$ at large times. The functional $\tilde{K}(f,0,1)$ will be our expected growth rate, in that the expected number of particles at time T whose paths, when rescaled by a factor T, are "near" f should look something like $e^{\tilde{K}(f,0,1)T}$. However, the actual number of particles behaving in this way will only look like $e^{\tilde{K}(f,0,1)T}$ if $\tilde{K}(f,0,\theta)>0$ for all $\theta\in[0,1]$. If there exists $\theta\in[0,1]$ such that $\tilde{K}(f,0,\theta)<0$ then (with high probability) there will be no particles whose T-rescaled paths look like f, essentially because this point on f acts as a bottleneck; at this point, it is too difficult for particles to follow f, and the population near f dies out.

In order to make this discussion precise we need to specify a topology on our space of functions E^2. As we discussed in Chapter 2, since the jumps in our process are exponentially distributed, the Skorokhod metric is not suitable, and we therefore introduce the Lévy metric on E as defined in (2.2). In an abuse of notation, we will also write d to mean the product metric on E^2 defined by $d((f_X,f_Y),(g_X,g_Y)) = \max\{d(f_X,g_X),d(f_Y,g_Y)\}$.

We recall some notation. For $T\ge 0$, let $\mathcal{N}_T$ be the set of particles that are alive at time T. For $u\in\mathcal{N}_T$ and $t\le T$, let $Z_u(t)=(X_u(t),Y_u(t))$ be the position of the unique ancestor of u in $\mathcal{N}_t$. For $u\in\mathcal{N}_T$ and $s\in[0,1]$, write

$$Z_u^T(s)=Z_u(sT)/T;$$

we call $(Z_u^T(s),\, s\in[0,1])$ the T-rescaled path of u. For a given set of functions F (we will discuss the technicalities on the function spaces later), define

$$N_T(F)=\#\{u\in\mathcal{N}_T : Z_u^T\in F\},$$

the number of particles at time T whose T-rescaled paths have remained within F.

Throughout the chapter, we use the convention that $\inf\emptyset=+\infty$ and $\sup\emptyset=-\infty$.

Theorem 3.1. *If $F \subset E^2$ is closed and $\sup_{f\in F} K(f) \neq 0$, then*

$$\limsup_{T\to\infty} \frac{1}{T} \log N_T(F) \leq \sup_{f\in F} K(f) \quad \textit{almost surely,}$$

and if $F \subset E^2$ is open and $\sup_{f\in F} K(f) \neq 0$, then

$$\liminf_{T\to\infty} \frac{1}{T} \log N_T(F) \geq \sup_{f\in F} K(f) \quad \textit{almost surely.}$$

The case when $\sup_{f\in F} K(f) = 0$ is extremely delicate. If, for some function f, we have $K(f) = 0$ then at some point along the path f, the population remaining near f is "critical", in the sense of a critical branching process. When $\sup_{f\in F} K(f) = 0$, this means that amongst all paths in F, the easiest path for particles to follow is a critical one. In general, critical branching processes are significantly more challenging to analyse than non-critical processes, and our situation is made more complex by the inhomogeneity of our branching system. Indeed, we do not even know if there are open sets $F \subset E^2$ that satisfy $\sup_{f\in F} K(f) = 0$. If not, then the condition that $\sup_{f\in F} K(f) \neq 0$ could essentially be removed, subject to a slight alteration to the definition of $K(f)$.

3.1.1 Heuristics

At a basic level, our theorem says that the number of particles whose T-rescaled paths remain close to a function f is roughly $\exp(K(f)T)$. The growth rate $K(f)$ consists of two parts: the growth of the population along the path, which is simply $\int_0^1 R^*(f(s))ds$, and the cost of a typical particle following the path, which is $J(f, 0, 1)$. However, if the cumulative cost is ever larger than the cumulative growth at any point along the path—that is, if $\tilde{K}(f, 0, s)$ is ever negative—then particles are unable to follow f and therefore $K(f) = -\infty$.

The main strategy for the proof is to break time up into small intervals. On each small interval, we know roughly the location and gradient of f and the rate $R(f(s))$, so we can control both the growth and the cost of following f. We bound the largest and smallest values that $R(z)$ can take when z is within a small ball around $f(s)$, and use a coupling to trap a typical particle in our process between two compound Poisson processes that have jump rates corresponding to these maximum and minimum values of $R(z)$. Fairly standard first and second moment bounds then allow us to translate the behaviour of this typical particle into estimates for the whole branching system.

As mentioned in the introduction, this simple explanation disguises a highly technically demanding proof. One of the difficulties that does not usually appear in work on branching structures is the behaviour at early times. We cannot do a standard approach as the one we illustrated in Chapter 2, instead we are forced to use a discrete-time moment bound to show that there are many particles near one particular path—a straight line corresponding to rectangles that are roughly square—at small times, and then show that this collection of particles can "feed" a population at future times that is easier to control.

Another non-standard element in our proof is the appearance of the Lévy metric. Since our particles take jumps whose sizes are exponentially distributed, there are (many) particles whose T-rescaled paths are not continuous. Indeed, every particle branches at rate at least 1, so at time tT there are at least of order e^{tT} particles, and the probability that one particle performs of a jump larger than aT—which corresponds to size a in the rescaled picture—is e^{-aT}. Thus we expect to see many such jumps when $t > a$. (And since particles can branch faster than rate 1, we will in fact see such jumps significantly earlier.) In order to bound the total number of particles from above, we therefore need to control particles whose paths are discontinuous; hence the appearance of the Lévy metric.

3.1.2 Growth rate in expectation

A relatively minor modification of our proof of Theorem 3.1 would yield the growth rate in expectation mentioned after the definition of $K(f)$, namely that if $F \subset E^2$ is closed then

$$\limsup_{T\to\infty} \frac{1}{T} \log \mathbb{E}[N_T(F)] \leq \sup_{f\in F} \tilde{K}(f,0,1) \tag{3.1}$$

and if $F \subset E^2$ is open then

$$\liminf_{T\to\infty} \frac{1}{T} \log \mathbb{E}[N_T(F)] \geq \sup_{f\in F} \tilde{K}(f,0,1). \tag{3.2}$$

In particular one may note that there are many sets F such that $\sup_{f\in F^\circ} \tilde{K}(f,0,1) > 0$, so the expected number of particles whose rescaled paths fall within F is exponentially large, and yet $\sup_{f\in \bar{F}} K(f) = -\infty$, so almost surely no particles have rescaled paths that fall within F.

We do not include full proofs of (3.1) and (3.2) here, although they are significantly simpler than the proofs of the upper and lower bounds in Theorem 3.1. We will sketch the main points of the arguments in Sections 3.2.3 and 3.3, shortly after the respective proofs of the upper and lower bounds in Theorem 3.1.

3.1.3 Layout of the chapter

We begin, in Sections 3.2 and 3.3, with outlines of the proofs of the upper and lower bounds in Theorem 3.1 respectively. In these sections we state several results that are needed for the proof of the main theorem without proving them. The proofs of these intermediate results are then given in later sections.

In Section 3.4, we give a full construction of our system in terms of a marked binary tree. This discrete setting is useful for decoupling some of the dependency structure between the jump times and jump sizes, and allows us to show that particles remain within some compact set with high probability, which will be an important ingredient, especially for the upper bound in Theorem 3.1.

In Section 3.5 we aim to control the system at small times, which is a difficult task partly due to the discontinuity of R^* at 0. We again use the discrete setup described in Section 3.4, and use moment estimates that take advantage of the fact that our particles prefer to split along their longest edge. This work is used for the proof of the lower bound in Theorem 3.1.

One of the main tools in our proof is a coupling between compound Poisson processes, which we describe in Section 3.6 and then apply to give upper and lower bounds on the probability that a typical particle remains near a given function.

In Section 3.7 we put many of the previous results together, move from lattice times to continuous time, and complete the final details of the proof of the upper bound in Theorem 3.1.

In Appendix 3.A we give deterministic bounds that relate the maximum and minimum of R on small balls to the value of R^* at the centre of the ball, and therefore allow us to link the probabilistic estimates obtained with the coupling in Section 3.6 to our growth rate $\tilde{K}$.

Finally, in Appendix 3.B we carry out some technical work, ensuring that our state space and our growth rate behave sensibly.

3.2. Proof outline for the upper bound in Theorem 3.1

Since the proof of Theorem 3.1 is rather long, we break it into upper and lower bounds. In this section we state a series of results that together enable us to complete the upper bound. We will then prove those results in later sections.

3.2.1 Three probabilistic ingredients

The first step in our proof of the upper bound in Theorem 3.1 is to rule out certain paths that it is difficult for particles to follow, thereby reducing the paths of interest to a compact set. We define, for $M > 1$,

$$G_M = \{f \in E : s/M \leq f(s) \leq Ms \ \forall s \in [0,1]\} \subset E.$$

If $f \in G_M^2$ then we say that f is "M-good". We note that if f is M-good then $R_X^*(f(s)) \leq M^2$ for all $s \in [0,1]$ and similarly for R_Y^*.

We would like to say that the rescaled paths of all particles fall within G_M^2 for sufficiently large M, but there is a complication near $s = 0$ in that particles will not jump immediately and therefore their paths will fall, however briefly, outside G_M^2. Expanding G_M^2 by any fixed distance $\varepsilon > 0$ would not allow us to control the jump rate sufficiently well, and we instead define, for $M > 0$ and $T > 1$,

$$G_{M,T} := \left\{ f \in E : s/M - 2T^{-2/3} \leq f(s) \leq M(s + 2T^{-2/3}) \ \forall s \in [0,1] \right\}.$$

If $f \in G_{M,T}^2$ then we say that f is "(M,T)-good". We can then show that for large M all particles are (M,T)-good with high probability as $T \to \infty$. We note here that the choice of $-2/3$ is not essential; we could choose any power of T in $(-1, -1/2)$.

Lemma 3.2. *There exist $M_0 > 1$ and $\delta_0 > 0$ such that for any sufficiently large T,*

$$\mathbb{P}\big(\exists v \in \mathcal{N}_T : Z_v^T \notin G_{M_0,T}^2\big) \leq e^{-\delta_0 T^{1/3}}.$$

We will prove this lemma in Section 3.4.

Next we give a version of the many-to-one formula, which translates expectations over all particles in our system into calculations involving just one particle. For $z_0 \in [0,\infty)^2$, write $\mathbb{Q}_{z_0}$ for a probability measure under which ξ_t is a Markov process living in $\mathbb{R}^2$, such that

- $\xi_0 = z_0$;
- when the process is in state z, jumps occur at rate $2R(z)$;
- when a jump occurs from state z, it is of the form $(\mathbb{e}, 0)$ with probability $P(z)$ and $(0, \mathbb{e})$ with probability $1 - P(z)$, where $\mathbb{e}$ is an independent exponentially-distributed random variable with parameter 1.

In other words, the process under $\mathbb{Q}_{z_0}$ behaves like a single particle under $\mathbb{P}_{z_0}$ except that it jumps at twice the rate. We write $\mathbb{Q}_{z_0}$ both for the measure and for its corresponding expectation operator. We will often take $z_0 = 0$, and in this case we sometimes write $\mathbb{Q}$ rather than $\mathbb{Q}_0$.

The measure $\mathbb{Q}_{z_0}$ described above is precisely the measure $\mathbb{Q}^1_{z_0}$ that appears in [32]. The following result is [32, Lemma 1] in the case of our model when $k = 1$.

Lemma 3.3 (Many-to-one, Lemma 1 of [32] with $k = 1$)**.** *Suppose that $z \in \mathbb{R}^2$ and $t \geq 0$. For any measurable function $f : \mathbb{R}^2 \to \mathbb{R}$,*

$$\mathbb{E}_z\left[\sum_{u\in\mathcal{N}_t} f(Z_u(t))\right] = \mathbb{Q}_z\left[f(\xi_t)e^{\int_0^t R(\xi_s)ds}\right].$$

This, combined with Markov's inequality, allows us to give upper bounds on the number of particles whose paths fall within a particular set F simply by bounding $R(f)$ over all $f \in F$ and then estimating the probability that ξ falls within F. Estimating this probability will be our next task, but our estimates will not be exactly in terms of the quantities R^*_X and R^*_Y seen in Theorem 3.1. Instead they will involve taking the worst and best possible values of R_X and R_Y over small balls about appropriately chosen functions, during a small time interval. We will need several definitions. The reader may like to think of $F = B(f, \varepsilon)$ for some suitably nice function f and small $\varepsilon > 0$.

For a non-empty interval $I \subset [0, 1]$, $F \subset E^2$ and $T \geq 1$, define

$$R^-_X(I, F, T) = \inf\left\{R_X(Tg(s)) : s \in I,\ g \in F\right\}$$

and

$$R^+_X(I, F, T) = \sup\left\{R_X(Tg(s)) : s \in I,\ g \in F\right\},$$

and similarly for $R^-_Y(I, F, T)$ and $R^+_Y(I, F, T)$. These correspond to the maximal and minimal possible jump rates over the interval I for particles whose T-rescaled paths fall within F. For $s \in [0, 1]$, we also let

$$x^-(s, F) = \inf\{g_X(s) : g \in F\}, \quad x^+(s, F) = \sup\{g_X(s) : g \in F\},$$

and similarly for $y^-(s, F)$ and $y^+(s, F)$.

Writing $|I|$ for the length of I and I^- and I^+ for the infimum and supremum of I respectively, say that we are in the "$X-$ case" if $2R^-_X(I, F, T)|I| > x^+(I^+, F) - x^-(I^-, F)$; and in the "$X+$ case" if $x^-(I^+, F) - x^+(I^-, F) > 2R^+_X(I, F, T)|I|$. Note that these two cases are mutually exclusive, and roughly correspond to whether the drift of the process on the interval I multiplied by the length of the interval is larger or smaller than the distance we would like it to travel. Note also that it is possible to be in neither case. Define

$$\mathcal{E}^+_X(I, F, T) = \begin{cases} \left(\sqrt{2R^-_X(I, F, T)|I|} - \sqrt{x^+(I^+, F) - x^-(I^-, F)}\right)^2 & \text{in the } X- \text{ case;} \\ \left(\sqrt{2R^+_X(I, F, T)|I|} - \sqrt{x^-(I^+, F) - x^+(I^-, F)}\right)^2 & \text{in the } X+ \text{ case;} \\ 0 & \text{otherwise.} \end{cases}$$

Similarly define $\mathcal{E}^+_Y(I, F, T)$. We note that for a single function $f \in E^2$, the quantity $\mathcal{E}^+_X([a, b], \{f\}, T) + \mathcal{E}^+_Y([a, b], \{f\}, T)$ should be an approximation to—but a little bit bigger than—the functional $I(f, a, b)$ seen in Section 3.1. We similarly define a quantity that should be an approximation to $I(f, a, b)$ from below, namely

$$\mathcal{E}_X^-(I, F, T) = \begin{cases} \left(\sqrt{2R_X^+(I, F, T)|I|} - \sqrt{x^-(I^+, F) - x^+(I^-, F)}\right)^2 & \text{in the } X- \text{ case;} \\ \left(\sqrt{2R_X^-(I, F, T)|I|} - \sqrt{x^+(I^+, F) - x^-(I^-, F)}\right)^2 & \text{in the } X+ \text{ case;} \\ 0 & \text{otherwise.} \end{cases}$$

Write $\|z_1 - z_2\| = \max\{|x_1 - x_2|, |y_1 - y_2|\}$ when $z_i = (x_i, y_i) \in \mathbb{R}^2$ for $i = 1, 2$. To help us to break the time interval $[0, 1]$ into smaller chunks, for $n \in \mathbb{N}$ we define a new metric Δ_n on E^2 by

$$\Delta_n(f, g) := \max\left\{\|f(i/n) - g(i/n)\| : i = 0, \ldots, n\right\}.$$

For $T > 1$, $n \in \mathbb{N}$, $M > 1$ and $f \in E^2$, we let

$$\Gamma_{M,T}(f, n) = B_{\Delta_n}(f, 1/n^2) \cap B_d(f, 1/n) \cap G^2_{M,T}$$

and for $j \in \{0, 1, \ldots, n-1\}$, let $I_j = [j/n, (j+1)/n]$.

We also need to extend our rescaling notation to ξ in the natural way. Write ξ^T for the rescaled process $(\xi(sT)/T,\ s \in [0, 1])$, and for $I \subset [0, 1]$ write $\xi^T|_I$ for the restriction to I, $(\xi(sT)/T,\ s \in I)$. If $F \subset E^2$ and a function f is defined on a subinterval I of $[0, 1]$—for example $\xi^T|_{[0,\theta]}$ with $I = [0, \theta]$—then say that $f \in F|_I$ if there exists $g \in F$ such that $f(s) = g(s)$ for all $s \in I$.

Proposition 3.4. *Suppose that $f \in E^2$, $n \in \mathbb{N}$, $T > 1$ and $M > 1$. Then for any $\theta \in (0, 1]$, $i \in \{0, 1, \ldots, \lfloor \theta n \rfloor - 1\}$, and z such that $\|z - f(i/n)\| < 1/n^2$,*

$$\begin{aligned} &\mathbb{Q}\big(\xi^T|_{[i/n,\theta]} \in \Gamma_{M,T}(f, n)\big|_{[i/n,\theta]} \,\big|\, \xi^T_{i/n} = z\big) \\ &\qquad\qquad \le \exp\Bigg(-T \sum_{j=i}^{\lfloor \theta n \rfloor - 1} \big(\mathcal{E}_X^+(I_j, \Gamma_{M,T}(f, n), T) + \mathcal{E}_Y^+(I_j, \Gamma_{M,T}(f, n), T)\big)\Bigg). \end{aligned}$$

The proof of Proposition 3.4 will be the most interesting part of this chapter, and involves coupling the process ξ with two other processes, which—as long as ξ remains within $\Gamma_{M,T}(f, n)$—will stay above and below ξ respectively. We carry out this part of the argument in Section 3.6.

3.2.2 Deterministic bounds

The three results Lemma 3.2, Lemma 3.3 and Proposition 3.4 form the main part of our argument, and contain all of the probability required for the upper bound in Theorem 3.1.

Our next task is to translate the quantities $\mathcal{E}_X^+$ and $\mathcal{E}_Y^+$ into the more palatable rate functions seen in our main theorem. The deterministic arguments required are not particularly interesting. It will sometimes be useful to note that if $\int_a^b R^*(f(s))ds < \infty$, then $\tilde{K}(f,a,b)$ has the following alternative representation:

$$\tilde{K}(f,a,b) = -\int_a^b R^*(f(s))ds + 2\sqrt{2}\int_a^b \sqrt{R_X^*(f(s))f_X'(s)}ds \\ + 2\sqrt{2}\int_a^b \sqrt{R_Y^*(f(s))f_Y'(s)}ds - f_X(b) + f_X(a) - f_Y(b) + f_Y(a). \quad (3.3)$$

This can be seen by expanding out the quadratic terms in the definition of $I(f,a,b)$ and simplifying.

Let PL_n be the subset of functions in E that are linear on each interval $[i/n,(i+1)/n]$ for all $i = 0,\ldots,n-1$ and continuous on $[0,1]$.

Proposition 3.5. *Suppose that $\theta \in (0,1]$, $M > 1$, $n \geq 2M$ and $f \in \mathrm{PL}_n^2 \cap G_M^2$. Then for any $k \in \{\lceil\sqrt{n}\rceil,\ldots,\lfloor\theta n\rfloor - 1\}$,*

$$\sum_{j=k}^{\lfloor\theta n\rfloor-1} \mathcal{E}_X^+(I_j,\Gamma_{M,T}(f,n),T) \geq \int_{k/n}^{\lfloor\theta n\rfloor/n} \Big(\sqrt{2R_X^*(f(s))} - \sqrt{f_X'(s)}\Big)^2 ds - O\Big(\frac{M^4}{n^{1/4}} + \frac{M^3 n}{T^{1/2}}\Big).$$

We do not aim to give best possible bounds on the error term. Similarly for the sum on the left-hand side, small values of j give rise to larger errors, so there should be some cut-off, but the choice of $\lceil\sqrt{n}\rceil$ is convenient rather than optimal. We will prove Proposition 3.5 in Appendix 3.A.1.

We will also need the following bound to control the $\exp(\int_0^t R(\xi_s)ds)$ term seen in Lemma 3.3.

Lemma 3.6. *Suppose that $\theta \in (0,1]$, $M > 1$, $n \geq 2M$, $T^{2/3} \geq 3Mn^{1/2}$, $f \in G_M^2$ and $g \in \Gamma_{M,T}(f,n)$. Then*

$$\int_0^\theta R(Tg(s))ds \leq \int_0^{\lfloor\theta n\rfloor/n} R^*(f(s))ds + \eta(M,n,T)$$

and for any $k \in \{\lceil\sqrt{n}\rceil, \lceil\sqrt{n}\rceil + 1,\ldots,\lfloor\theta n\rfloor\}$,

$$\int_{k/n}^{\lfloor\theta n\rfloor/n} R^*(f(s))ds - \eta(M,n,T) \leq \int_{k/n}^{\theta} R(Tg(s))ds \leq \int_{k/n}^{\lfloor\theta n\rfloor/n} R^*(f(s))ds + \eta(M,n,T)$$

where

$$\eta(M,n,T) = O\Big(\frac{M^4}{n^{1/2}} + \frac{M^3 n}{T^{1/3}}\Big).$$

This result will be proved in Appendix 3.A.2. Again we make little effort to make $\eta(M,n,T)$ the best possible bound.

3.2.3 Completing the proof of the upper bound in Theorem 3.1

Recall that if $F \subset E^2$, and $g : [0,\theta] \to \mathbb{R}^2$, we say that $g \in F|_{[0,\theta]}$ if there exists a function $h \in F$ such that $h(u) = g(u)$ for all $u \in [0,\theta]$. We also generalise our rescaling notation slightly: for $t \in [0,T]$, $v \in \mathcal{N}_t$ and $s \in [0,t/T]$, write

$$Z_v^T(s) = Z_v(sT)/T;$$

again we call $(Z_v^T(s), s \in [0, t/T])$ the T-rescaled path of v (previously this was only defined when $t = T$). We can then define

$$N_T(F, \theta) = \#\{v \in \mathcal{N}_{\theta T} : Z_v^T \in F|_{[0,\theta]}\},$$

the number of particles at time θT whose T-rescaled paths have remained within F up to time θ.

Proposition 3.7. *Suppose that $M > 1$, $\theta \in (0, 1]$, $n \geq 2M$ and $T \geq 6M^{3/2}n^{3/4}$. Then for any $g \in G_M^2 \cap \mathrm{PL}_n^2$ and $\kappa > 0$,*

$$\mathbb{P}\big(N_T(\Gamma_{M,T}(g, n), \theta) \geq \kappa\big) \leq \frac{1}{\kappa} \exp\Big(T\tilde{K}\Big(g, 0, \frac{\lfloor \theta n \rfloor}{n}\Big) + O\Big(\frac{M^4 T}{n^{1/4}} + M^3 n T^{2/3}\Big)\Big).$$

We will prove Proposition 3.7, which forms the heart of the argument to prove the upper bound in Theorem 3.1, in Section 3.7.2.

Our next result applies Proposition 3.7 to show that for $F \subset E^2$, at any large time T, the number of particles whose T-rescaled paths fall within F is unlikely to be much larger than $\exp\big(\sup_{f \in F} \tilde{K}(f, 0, 1)\big)$. Recall the definition of M_0 and δ_0 from Lemma 3.2.

Proposition 3.8. *Suppose that $F \subset E^2$ is closed and $M \geq 4M_0$. Then for any $\varepsilon > 0$,*

$$\lim_{T\to\infty} \frac{1}{T^{1/3}} \log \mathbb{P}\Big(N_T(F) \geq \exp\Big(T \sup_{f \in F \cap G_M^2} \tilde{K}(f, 0, 1) + T\varepsilon\Big)\Big) \leq -\delta_0.$$

The proof of this result will use Lemma 3.2 together with some technical lemmas to ensure that we can cover F with finitely many small balls around piecewise linear functions, and then apply Proposition 3.7. The proof is also in Section 3.7.2.

There are many paths f that satisfy $K(f) = -\infty$ but $\tilde{K}(f, 0, 1) > 0$. These are paths where there exists $\theta \in (0, 1)$ such that $\tilde{K}(f, 0, \theta) < 0$, and therefore the population of particles whose rescaled paths are near f becomes extinct around time θT. Since a population cannot recover once it becomes extinct, no particles follow such paths up to time T even though the expected growth by the end of the path, $\tilde{K}(f, 0, 1)$, can be positive. For sets F that only contain such paths, Proposition 3.8 does not provide a useful bound, and we therefore need a slightly different approach.

Lemma 3.9. *If $F \subset E^2$ is closed and $\sup_{f \in F} K(f) = -\infty$, then*

$$\lim_{T\to\infty} \frac{1}{T^{1/3}} \log \mathbb{P}\big(N_T(F) \geq 1\big) \leq -\delta_0.$$

The proof of Lemma 3.9 is in Section 3.7.3. We can then upgrade Proposition 3.8 and Lemma 3.9, which are both statements about a particular large time T, to get the same result at *all* large times simultaneously.

Proposition 3.10. *Suppose that $F \subset E^2$ is closed and $M \geq 4M_0$. Then*

$$\limsup_{T\to\infty} \frac{1}{T} \log N_T(F) \leq \sup_{f \in F \cap G_M^2} \tilde{K}(f, 0, 1)$$

almost surely. If moreover $\sup_{f \in F} K(f) = -\infty$, then $\limsup_{T\to\infty} N_T(F) = 0$ almost surely.

The proof of this result will appear in Section 3.7.4. We can now complete the proof of the upper bound in our main theorem.

Proof of Theorem 3.1: upper bound. Since $K(f) \in \{-\infty\} \cup [0,\infty)$, if $\sup_{f\in F} K(f) < 0$ then we must have $\sup_{f\in F} K(f) = -\infty$. In this case the second part of Proposition 3.10 tells us that almost surely, $N_T(F) = 0$ for all large T, and therefore $\lim_{T\to\infty} \log N_T(F) = -\infty$. On the other hand if $\sup_{f\in F} K(f) > 0$ then we have

$$\sup_{f\in F} K(f) = \sup_{f\in F} \tilde{K}(f,0,1),$$

and then applying the first part of Proposition 3.10 tells us that

$$\limsup_{T\to\infty} \frac{1}{T} \log N_T(F) \le \sup_{f\in F\cap G^2_M} \tilde{K}(f,0,1) \le \sup_{f\in F} \tilde{K}(f,0,1) = \sup_{f\in F} K(f)$$

almost surely, and the proof is complete. □

Sketch proof of (3.1). The upper bound in expectation (3.1) follows more or less directly from estimates derived above. In particular, much of the proof of Proposition 3.7 involves bounding $\mathbb{E}[N_T(F)]$ from above when F is a small ball around a suitably nice function, and then applying Markov's inequality. From there it is a relatively simple task, similarly to the proof of Proposition 3.8, to apply Lemma 3.2 to reduce F to a compact set, Lemma 3.36 to cover this set with finitely many balls around suitably nice functions, and Corollary 3.38 to check that the resulting bound does not significantly overshoot (3.1). □

3.3. Proof outline for the lower bound in Theorem 3.1

Let ρ be the metric defined by

$$\rho(f,g) = \sup_{s\in[0,1]} \|f(s) - g(s)\| = \sup_{s\in[0,1]} \big\{|f_X(s) - g_X(s)| \vee |f_Y(s) - g_Y(s)|\big\}.$$

Rather than the set $\Gamma_{M,T}(f,n)$ seen in the proof of the upper bound, for the lower bound we will instead often use the set

$$\Lambda_{M,T}(f,n) = B_\rho(f, 1/n^2) \cap G^2_{M,T}.$$

For $F \subset E^2$ and $T > 0$, recall that

$$N_T(F) = \#\{u \in \mathcal{N}_T : Z^T_u \in F\},$$

and for $t \in [0,1]$ and $u \in \mathcal{N}_{tT}$, define

$$N^u_{t,T}(F) = \#\{v \in \mathcal{N}_T : u \le v,\ Z^T_v|_{[t,1]} \in F|_{[t,1]}\}. \tag{3.4}$$

Also let $(\mathcal{F}_t, t \ge 0)$ be the natural filtration for the process.

The main part of our proof relies on a standard second moment argument, and Propositions 3.11 and 3.12 give the first and second moment bounds necessary to carry out that argument. However, this strategy on its own cannot give strong enough estimates to be able to prove an almost sure statement, as required for Theorem 3.1. We therefore give bounds conditionally given $\mathcal{F}_{kT/n}$ for $\sqrt{n} < k \ll n$, with the aim of using the branching structure at time kT/n to increase the accuracy of our estimates.

Proposition 3.11. *Suppose that $M > 1$, $n \geq 6M$, $\sqrt{n} \leq k \leq n$, $T \geq 27M^{3/2}n^{9/2}$ and $f \in \mathrm{PL}_n^2 \cap G_M^2$. Suppose also that $u \in \mathcal{N}_{kT/n}$ satisfies $\|Z_u^T(k/n) - f(k/n)\| \leq \frac{1}{2n^2}$. Then*

$$\mathbb{E}\big[N^u_{k/n,T}(\Lambda_{3M,T}(f,n)) \,\big|\, \mathcal{F}_{kT/n}\big] \geq \exp\Big(T\tilde{K}(f,k/n,1) - O\Big(\frac{M^4T}{n^{1/4}} + M^3nT^{2/3}\Big)\Big).$$

We will prove Proposition 3.11 in Section 3.3.1.

Proposition 3.12. *Suppose that $M > 1$, $n \geq 6M$, $\sqrt{n} \leq k \leq n$, $T \geq 27M^{3/2}n^{9/2}$ and $f \in PL_n^2 \cap G_M^2$. Suppose also that $u \in \mathcal{N}_{kT/n}$ satisfies $\|Z_u^T(k/n) - f(k/n)\| < \frac{1}{n^2}$. Then*

$$\begin{aligned}
&\mathbb{E}\big[N^u_{k/n,T}(\Lambda_{3M,T}(f,n))^2 \,\big|\, \mathcal{F}_{kT/n}\big] \\
&\leq \int_{kT/n}^{T} e^{-T\tilde{K}(f,k/n,t/T)}dt \cdot 12M^2 n \exp\Big(2T\tilde{K}(f,k/n,1) + O\Big(\frac{M^4T}{n^{1/4}} + M^3nT^{2/3}\Big)\Big) \\
&\qquad + \exp\Big(T\tilde{K}(f,k/n,1) + O\Big(\frac{M^4T}{n^{1/4}} + M^3nT^{2/3}\Big)\Big).
\end{aligned}$$

We will prove Proposition 3.12 in Section 3.3.2. We now use a standard second moment method to turn Propositions 3.11 and 3.12 into a lower bound on the probability that the number of particles whose rescaled paths remain near f is roughly $\tilde{K}(f,k/n,1)T$, again conditionally on $\mathcal{F}_{kT/n}$.

Corollary 3.13. *Suppose that $M > 1$, $n \geq 6M$, $k \geq \sqrt{n}$, $T \geq 27M^{3/2}n^{9/2}$ and $f \in PL_n^2 \cap G_M^2$. Suppose also that $u \in \mathcal{N}_{kT/n}$ satisfies $\|Z_u^T(k/n) - f(k/n)\| \leq \frac{1}{2n^2}$, and that $\tilde{K}(f,k/n,t) \geq 0$ for all $t \geq k/n$. Then*

$$\begin{aligned}
\mathbb{P}\big(N^u_{k/n,T}(\Lambda_{3M,T}(f,n)) \geq e^{T\tilde{K}(f,k/n,1) - O(M^4T/n^{1/4} + M^3nT^{2/3})} \,\big|\, \mathcal{F}_{kT/n}\big)& \\
\geq e^{-O(M^4T/n^{1/4} + M^3nT^{2/3})}&.
\end{aligned}$$

Proof. The Paley-Zygmund inequality says that, for any non-negative random variable X and $\theta \in [0,1]$,

$$P\big(X \geq \theta E[X]\big) \geq (1-\theta)^2 \frac{E[X]^2}{E[X^2]}.$$

Taking P to be the conditional probability $\mathbb{P}(\,\cdot \mid \mathcal{F}_{kT/n})$ with $X = N^u_{k/n,T}(\Lambda_{3M,T}(f,n))$ and $\theta = 1/2$, we have

$$\begin{aligned}
\mathbb{P}\big(N^u_{k/n,T}(\Lambda_{3M,T}(f,n)) &\geq (1/2)\mathbb{E}\big[N^u_{k/n,T}(\Lambda_{3M,T}(f,n)) \,\big|\, \mathcal{F}_{kT/n}\big] \,\big|\, \mathcal{F}_{kT/n}\big) \\
&\geq \frac{\mathbb{E}\big[N^u_{k/n,T}(\Lambda_{3M,T}(f,n)) \,\big|\, \mathcal{F}_{kT/n}\big]^2}{4\mathbb{E}\big[N^u_{k/n,T}(\Lambda_{3M,T}(f,n))^2 \,\big|\, \mathcal{F}_{kT/n}\big]}. \qquad (3.5)
\end{aligned}$$

Proposition 3.11 tells us that

$$\mathbb{E}\big[N^u_{k/n,T}(\Lambda_{3M,T}(f,n)) \,\big|\, \mathcal{F}_{kT/n}\big] \geq \exp\Big(T\tilde{K}(f,k/n,1) - O\Big(\frac{M^4T}{n^{1/4}} + M^3nT^{2/3}\Big)\Big)$$

and Proposition 3.12 gives

$$\begin{aligned}
&\mathbb{E}\big[N^u_{k/n,T}(\Lambda_{3M,T}(f,n))^2 \,\big|\, \mathcal{F}_{kT/n}\big] \\
&\le \int_{kT/n}^{T} e^{-T\tilde{K}(f,k/n,t/T)}dt \cdot 12M^2 n \exp\left(2T\tilde{K}(f,k/n,1) + O\Big(\frac{M^4T}{n^{1/4}} + M^3nT^{2/3}\Big)\right) \\
&\qquad\qquad\qquad\qquad + \exp\left(T\tilde{K}(f,k/n,1) + O\Big(\frac{M^4T}{n^{1/4}} + M^3nT^{2/3}\Big)\right).
\end{aligned}$$

and since $\tilde{K}(f,k/n,t) \ge 0$ for all $t \ge k/n$, this reduces to

$$\begin{aligned}
&\mathbb{E}\big[N^u_{k/n,T}(\Lambda_{3M,T}(f,n))^2 \,\big|\, \mathcal{F}_{kT/n}\big] \\
&\qquad\qquad \le 12M^2nT \exp\left(2T\tilde{K}(f,k/n,1) + O\Big(\frac{M^4T}{n^{1/4}} + M^3nT^{2/3}\Big)\right) \\
&\qquad\qquad = \exp\left(2T\tilde{K}(f,k/n,1) + O\Big(\frac{M^4T}{n^{1/4}} + M^3nT^{2/3}\Big)\right).
\end{aligned}$$

Substituting these estimates into (3.5) gives the result. □

By Corollary 3.13, each particle near $Tf(k/n)$ at time kT/n has a not-too-small probability of having roughly $\exp(\tilde{K}(f,k/n,1)T)$ descendants whose rescaled paths remain near f up to time 1. If we can ensure that there is a reasonably large number of particles near $Tf(k/n)$ at time kT/n, then subject to some technicalities (for example Corollary 3.13 assumes that f is piecewise linear, whereas there is no such condition in Theorem 3.1) we will be able to prove the lower bound in Theorem 3.1.

The discontinuity of R^* at 0 makes controlling the growth of the system at small times difficult. The first few particles in the system can have wildly different values of R in different realisations of the process, and it is not *a priori* clear that this cannot have a large effect on the long-term evolution of the system. Our method for showing that particles do in fact spread out in a predictable way is the following. First we show that there are many particles near the line $(s/2, s/2)$ at time s, for suitable values of s. The idea is that our jump distribution prefers to create "almost square" rectangles (since rectangles are more likely to break along their longest side) and therefore we should see many particles near $(s/2, s/2)$. However, since particles away from this line branch and jump more quickly, we use a discrete-time argument to keep control of the dependence between the jump locations and the jump times. A rough estimate using moments in discrete time can then be translated back into continuous time, giving the following result.

Proposition 3.14. *Define*

$$V'_{n,T} = \{u \in \mathcal{N}_{\lceil n^{7/8}\rceil T/n} : \|Z_u(s) - (s/2,s/2)\| \le \tfrac{T}{2n^2} \ \forall s \le \lceil n^{7/8}\rceil T/n\}.$$

There exists a finite constant C such that for any $T \ge Cn^{48}$,

$$\mathbb{P}\big(|V'_{n,T}| < 2^{T/n^{1/8} - 2T/n^2}\big) \le 1/T^{3/2}.$$

We will prove this result in Section 3.5.1. The choice of $\lceil n^{7/8}\rceil$ is somewhat arbitrary, but ensures that there are enough particles at time $\lceil n^{7/8}\rceil T/n$ to outweigh the error arising from Corollary 3.13. The bound of $1/T^{3/2}$ is not the best possible, but is enough to use a Borel-Cantelli argument at the end of the proof of Theorem 3.1. The requirement that $T \geq Cn^{48}$ is also certainly not optimal, but since we will take $T \to \infty$, it is sufficient for our needs.

Once we have shown that there are particles near $(s/2, s/2)$ at small times s, then we need to show that these particles "feed" other directions $(\lambda s', \mu s')$ for suitable λ and μ and $s' > s$. Given $f \in G_M^2$, we will construct a function h that begins by moving along the line $(s/2, s/2)$, so that we can guarantee large numbers of particles near h at small times using Proposition 3.14, but which then gradually changes its gradient to be closer and closer to our given function f. At the same time we will ensure that h is piecewise linear, so that we can then use Corollary 3.13 to ensure appropriate growth of particles along the whole path h. We then show that for $k = \lceil n^{7/8} \rceil < nt$ we have $\tilde{K}(h, k/n, t) \approx \tilde{K}(f, k/n, t)$. This is part of Proposition 3.15 below, which will be proved in Section 3.5.2.

Proposition 3.15. *Suppose that $f \in G_M^2$ satisfies $\frac{d}{dt}\tilde{K}(f, 0, t)|_{t=0} > 0$ and $\tilde{K}(f, 0, t) > 0$ for all $t \in (0, 1]$. Then for any $\varepsilon > 0$ and $n \in \mathbb{N}$, there exists $h_{f,n} \in E^2$ such that*

$$h_{f,n}(s) = (s/2, s/2) \quad \textit{for all } s \leq \lceil n^{7/8} \rceil / n \tag{3.6}$$

and if n is sufficiently large,

$$h_{f,n} \in \mathrm{PL}_n^2 \cap G_M^2 \cap B(f, \varepsilon), \tag{3.7}$$

$$\tilde{K}(h_{f,n}, \lceil n^{7/8} \rceil / n, s) > 0 \quad \textit{for all } s \in (\lceil n^{7/8} \rceil / n, 1] \tag{3.8}$$

and

$$\tilde{K}(h_{f,n}, \lceil n^{7/8} \rceil / n, 1) \geq \tilde{K}(f, 0, 1) - \varepsilon. \tag{3.9}$$

We will prove this in Section 3.5.2. We are now able to finish the proof of our main result.

Proof of Theorem 3.1: lower bound. Fix $\varepsilon > 0$. Recall M_0 from Lemma 3.2. Since $K(f) \in \{-\infty\} \cup [0, \infty)$, if $\sup_{f \in F} K(f) \leq 0$ then there is nothing to prove. We therefore assume that there exists $f \in F$ with $K(f) > 0$. In this case, since F is open and all functions f with $K(f) > 0$ are in G_M^2 for some M, we can choose $M \geq M_0$, $\varepsilon' > 0$ and $f \in G_M^2$ such that $B(f, 2\varepsilon') \subset F$ and

$$K(f) \geq \max\Big\{ \sup_{g \in F} K(g) - \varepsilon,\ \frac{1}{2} \sup_{g \in F} K(g) \Big\} > 0.$$

Since $K(f) > 0$, we have $\frac{d}{dt}\tilde{K}(f, 0, t)|_{t=0} > 0$ and $\tilde{K}(f, 0, t) > 0$ for all $t \in (0, 1]$. Therefore by Proposition 3.15, for all sufficiently large $n \in \mathbb{N}$ the function $h_{f,n}$ satisfies (3.7), (3.8) and (3.9) with $\min\{\varepsilon/2, \varepsilon'/2\}$ in place of ε.

Take $n \in \mathbb{N}$ and write $k = \lceil n^{7/8} \rceil$. From Proposition 3.14, if we define

$$V'_{n,T} = \big\{ u \in \mathcal{N}_{kT/n} : \|Z_u(s) - (s/2, s/2)\| \leq \tfrac{T}{2n^2} \ \ \forall s \leq kT/n \big\},$$

then for $T \geq Cn^{48}$ and C large, we have $\mathbb{P}(|V'_{n,T}| \geq 2^{T/n^{1/8} - 2T/n^2}) \geq 1 - 1/T^{3/2}$.

Since $h_{f,n}$ satisfies (3.7), (3.8) and (3.9) with $\min\{\varepsilon/2, \varepsilon'/2\}$ in place of ε,

$$\begin{aligned}
\mathbb{P}\big(N_T(B(f,\varepsilon')) < e^{(\tilde{K}(f,0,1)-\varepsilon)T}\big)
&\le \mathbb{P}\big(N_T(B(h_{f,n},\varepsilon'/2)) < e^{(\tilde{K}(h_{f,n},k/n,1)-\varepsilon/2)T}\big)\\
&\le \mathbb{E}\Big[\mathbb{P}\Big(N_T(B(h_{f,n},\varepsilon'/2)) < e^{(\tilde{K}(h_{f,n},k/n,1)-\varepsilon/2)T}\,\Big|\,\mathcal{F}_{kT/n}\Big)\Big].
\end{aligned}$$

Recalling the notation (3.4), note that if $u \in V'_{n,T}$ and $N^u_{k/n,T}(\Lambda_{3M,T}(h_{f,n},n)) \ge r$, and n is sufficiently large, then $N_T(B(h_{f,n},\varepsilon'/2)) \ge r$, for any $r \ge 0$. Indeed if $u \in V'_{n,T}$ and $u \le v$ is such that $Z^T_v|_{[k/n,1]} \in \Lambda_{3M,T}(h_{f,n},n)|_{[k/n,1]}$ then using (3.6)

$$\begin{aligned}
\sup_{s\in[0,1]} \big\|Z^T_v(s) - h_{f,n}(s)\big\| &\le \sup_{s\in[0,k/n]} \big\|Z^T_u(s) - (s/2,s/2)\big\| + \sup_{s\in[k/n,1]} \big\|Z^T_v(s) - h_{f,n}(s)\big\|\\
&\le \frac{1}{2n^2} + \frac{1}{n^2},
\end{aligned}$$

so $Z^T_v \in B(h_{f,n},\varepsilon'/2)$ when n is large. Thus

$$\begin{aligned}
&\mathbb{P}\big(N_T(B(f,\varepsilon')) < e^{(\tilde{K}(f,0,1)-\varepsilon)T}\big)\\
&\quad\le \mathbb{E}\Bigg[\prod_{u\in V'_{n,T}} \mathbb{P}\Big(N^u_{k/n,T}(\Lambda_{3M,T}(h_{f,n},n)) < e^{(\tilde{K}(h_{f,n},k/n,1)-\varepsilon/2)T}\,\Big|\,\mathcal{F}_{kT/n}\Big)\Bigg]. \qquad (3.10)
\end{aligned}$$

For n and T sufficiently large, we check that we may apply Corollary 3.13: indeed, by (3.8), we have $\tilde{K}(h_{f,n},k/n,t) \ge 0$ for all $t \ge k/n$, and for $u \in V'_{n,T}$ we have

$$\|Z^T_u(k/n) - h_{f,n}(k/n)\| = \frac{1}{T}\big\|Z_u(kT/n) - (\tfrac{kT}{2n},\tfrac{kT}{2n})\big\| \le \frac{1}{2n^2}.$$

Thus, applying Corollary 3.13 to bound the conditional probability in (3.10) from above, we obtain that

$$\mathbb{P}\big(N_T(B(f,\varepsilon')) < e^{(\tilde{K}(f,0,1)-\varepsilon)T}\big) \le \mathbb{E}\Bigg[\prod_{u\in V'_{n,T}} \Big(1 - e^{-O(M^4T/n^{1/4}+M^3nT^{2/3})}\Big)\Bigg].$$

Recalling that $|V'_{n,T}| \ge 2^{T/n^{1/8}-2T/n^2}$ with probability at least $1 - 1/T^{3/2}$, we get

$$\mathbb{P}\big(N_T(B(f,\varepsilon')) < e^{(\tilde{K}(f,0,1)-\varepsilon)T}\big) \le \big(1 - e^{-O(M^4T/n^{1/4}+M^3nT^{2/3})}\big)^{2^{T/n^{1/8}-2T/n^2}} + 1/T^{3/2}$$

and using that $1 - x \le e^{-x}$ for all x,

$$\begin{aligned}
&\mathbb{P}\big(N_T(B(f,\varepsilon')) < e^{(\tilde{K}(f,0,1)-\varepsilon)T}\big)\\
&\qquad\le \exp\big(-2^{T/n^{1/8}-2T/n^2} e^{-O(M^4T/n^{1/4}+M^3nT^{2/3})}\big) + 1/T^{3/2}. \qquad (3.11)
\end{aligned}$$

By Lemma 3.40 with $s = T$, for $T \ge 3M$, whenever $t - 1 \le T \le t$ we have

$$N_T(B(f,\varepsilon') \cap G^2_{M,T}) \le N_t(B(f,\varepsilon' + 6M/t))$$

and therefore if $T \geq 6M/\varepsilon'$, then we have

$$N_T(B(f,\varepsilon') \cap G^2_{M,T}) \leq \inf_{t\in[T,T+1]} N_t(B(f,2\varepsilon')).$$

Thus

$$\begin{aligned}\mathbb{P}\Big(&\inf_{t\in[T,T+1]} N_t(B(f,2\varepsilon'))e^{-(\tilde{K}(f,0,1)-\varepsilon)t} < 1\Big)\\ &\leq \mathbb{P}\big(N_T(B(f,\varepsilon') \cap G^2_{M,T}) < e^{(\tilde{K}(f,0,1)-\varepsilon)T}\big)\\ &\leq \mathbb{P}\big(N_T(B(f,\varepsilon')) < e^{(\tilde{K}(f,0,1)-\varepsilon)T}\big) + \mathbb{P}\big(N_T((G^2_{M,T})^c) \geq 1\big).\end{aligned}$$

By Lemma 3.2, since $M \geq M_0$, the last term is at most $e^{-\delta_0 T^{1/3}}$, and then applying (3.11), we obtain that

$$\begin{aligned}\mathbb{P}\Big(&\inf_{t\in[T,T+1]} N_t(B(f,2\varepsilon'))e^{-(\tilde{K}(f,0,1)-\varepsilon)t} < 1\Big)\\ &\leq \exp(-2^{T/n^{1/8}-2T/n^2}e^{-O(M^4T/n^{1/4}+M^3nT^{2/3})}) + \frac{1}{T^{3/2}} + e^{-\delta_0 T^{1/3}}.\end{aligned}$$

Taking n large enough that the $2^{T/n^{1/8}}$ term dominates the exponent when T is large, we see that this is summable in T, and therefore by the Borel-Cantelli lemma,

$$\mathbb{P}\Big(\liminf_{t\to\infty} N_t(B(f,2\varepsilon'))e^{-(\tilde{K}(f,0,1)-\varepsilon)t} < 1\Big) = 0.$$

Since $B(f,2\varepsilon') \subset F$ and $\tilde{K}(f,0,1) = K(f) \geq \sup_{g\in F} K(g) - \varepsilon$, the statement of the theorem follows. □

Sketch proof of (3.2). Proving the lower bound in expectation (3.2) involves slightly more work than the upper bound (3.1). Proposition 3.15 creates a function that approximates a given f for much of its path, but begins by following the lead diagonal $(s/2, s/2)$ for a short period. Unfortunately it is designed to work for functions f that satisfy $\frac{d}{dt}\tilde{K}(f,0,t)|_{t=0} > 0$ and $\tilde{K}(f,0,s) > 0$ for all $s \in (0,1]$. To prove (3.2) we cannot make these assumptions on f, but can take a simpler approach than Proposition 3.15. We define a function $\hat{h}_{f,n}$ that follows the lead diagonal $(s/2, s/2)$ until time $\lceil\sqrt{n}\rceil/n$, then satisfies

$$\hat{h}_{f,n}(j/n) = \Big(\frac{\lceil\sqrt{n}\rceil}{2n}, \frac{\lceil\sqrt{n}\rceil}{2n}\Big) + f(j/n) - f(\lceil\sqrt{n}\rceil/n)$$

for every $j \in \{\lceil\sqrt{n}\rceil, \ldots, n\}$, and interpolates linearly between these values. Following a similar proof to that of Proposition 3.33, we can show that

$$\liminf_{n\to\infty} \tilde{K}(\hat{h}_{f,n},0,1) \geq \tilde{K}(f,0,1),$$

and then combining Propositions 3.11 and 3.14 yields (3.2). □

In the proofs of the results above, it will be useful several times to note that since, for any f, n, M and T,

$$\Lambda_{M,T}(f,n) \subset \Gamma_{M,T}(f,n), \tag{3.12}$$

we have

$$R_X^-(I_j,\Gamma_{M,T}(f,n),T) \le R_X^-(I_j,\Lambda_{M,T}(f,n),T) \\ \le R_X^+(I_j,\Lambda_{M,T}(f,n),T) \le R_X^+(I_j,\Gamma_{M,T}(f,n),T) \quad (3.13)$$

and therefore by the deterministic bounds (3.60) in Appendix 3.A, if $M,T>1$, $n\ge 2M$, $f\in G_M^2$, $j\ge n^{1/2}$ and $s\in I_j$,

$$R_X^+(I_j,\Lambda_{M,T}(f,n),T) - \delta_{M,T}(j,n) \\ \le R_X^*(f(s)) \le R_X^-(I_j,\Lambda_{M,T}(f,n),T) + \delta_{M,T}(j,n). \quad (3.14)$$

3.3.1 Lower bound on the first moment: proof of Proposition 3.11

Our aim in this section is to outline a proof of Proposition 3.11. Fix f as in the statement of the proposition. Let $\mathcal{Z}_0 = \{(0,0)\}$ and, for $j\in\{1,\dots,n-1\}$, define

$$\mathcal{Z}_j = \{z\in[0,\infty)^2 : \|z-f(j/n)\| \le \tfrac{1}{2n^2}\}.$$

Lemma 3.3 (Many-to-one), combined with the deterministic bounds on the integral of the rate function from Lemma 3.6, will reduce the problem to bounding

$$\mathbb{Q}\Big(\xi^T|_{[k/n,1]} \in \Lambda_{M,T}(f,n)|_{[k/n,1]} \,\Big|\, \xi^T(k/n)=w\Big)$$

for $w\in\mathcal{Z}_k$, so we concentrate on estimating this quantity.

Fix $n\in\mathbb{N}$ and $M,T>1$, and consider $f\in \mathrm{PL}_n^2\cap G_M^2$ and $j\in\{0,\dots,n-1\}$. We will apply the coupling defined in Section 3.6 with $I=I_j$ and $F=\Lambda_{M,T}(f,n)$. Define

$$q_{n,M,T}^X(z,j,f) = Q_z^{I_j,\Lambda_{M,T}(f,n),T}\Big(\big|X_-(s)-f_X(s)\big| \le \tfrac{1}{n^2}\ \ \forall s\in I_j, \\ \big|X_-(\tfrac{j+1}{n}) - f_X(\tfrac{j+1}{n})\big| \le \tfrac{1}{2n^2},\ X_-|_{I_j}\in G_{M,T}|_{I_j}\Big)$$

and

$$\hat q_{n,M,T}^X(z,j,f) = Q_z^{I_j,\Lambda_{M,T}(f,n),T}\Big(X_+(\tfrac{j+1}{n}) - X_-(\tfrac{j+1}{n}) = 0\Big)$$

and similarly for $q_{n,M,T}^Y(z,j,f)$ and $\hat q_{n,M,T}^Y(z,j,f)$.

Lemma 3.16. *Suppose that $n\ge 3$, $f\in \mathrm{PL}_n^2$ and $T>1$. Then for any $k\in\{0,\dots,n-1\}$ and $w\in\mathcal{Z}_k$,*

$$\mathbb{Q}\Big(\xi^T|_{[k/n,1]} \in \Lambda_{M,T}(f,n)|_{[k/n,1]} \,\Big|\, \xi^T(k/n)=w\Big) \\ \ge \prod_{j=k}^{n-1} \inf_{z\in\mathcal{Z}_j} q_{n,M,T}^X(z,j,f)\,\hat q_{n,M,T}^X(z,j,f)\,q_{n,M,T}^Y(z,j,f)\,\hat q_{n,M,T}^Y(z,j,f).$$

We carry out the proof of Lemma 3.16, which consists of applying the properties of the coupling, in Section 3.6.2. We then need to bound the terms on the right-hand side. Bounding the $\hat q$ terms is fairly straightforward.

Lemma 3.17. *Suppose that $M>1$, $n\ge 2M$, $T>1$ and $f\in \mathrm{PL}_n^2\cap G_M^2$. Then for* *any* $k \geq \lceil n^{1/2} \rceil$,

$$\prod_{j=k}^{n-1} \inf_{z \in \mathcal{Z}_j} \hat{q}^X_{n,M,T}(z,j,f)\hat{q}^Y_{n,M,T}(z,j,f) \geq \exp\Big(- O\Big(\frac{M^4 T}{n^{1/2}} + M^3 n\Big)\Big).$$

Again we will prove Lemma 3.17 in Section 3.6.2. Bounding the q terms is much more delicate. In the following lemma, the precise form of $\Delta(j)$ is not important; we consider it a small term.

Lemma 3.18. *Suppose that* $M > 1$, $n \geq 2M$, $T > 8n^{9/2}M^{3/2}$ *and* $f \in \mathrm{PL}^2_n \cap G^2_M$. *Then for any* $j \in \{\lceil \sqrt{n} \rceil, \ldots, n-1\}$ *and* $z = (x,y) \in \mathcal{Z}_j$,

$$q^X_{n,3M,T}(z,j,f) \geq \exp\Big(- T\int_{j/n}^{(j+1)/n} \Big(\sqrt{2R^*_X(f(s))} - \sqrt{f'_X(s)}\Big)^2 ds - T\Delta(j)\Big)$$

where

$$\Delta(j) = \frac{2(M+1)}{n^{3/2}} + \frac{2\delta_{M,T}(j,n)}{n} + \frac{1}{\sqrt{n}}\sqrt{2\delta_{M,T}(j,n)\big(f_X(\tfrac{j+1}{n}) - f_X(\tfrac{j}{n})\big)}$$

and $\delta_{M,T}(j,n)$ *is defined in Lemma 3.42.*

We again delay the proof of Lemma 3.18 to Section 3.6.2. Putting the above ingredients together and bounding $\sum_{j=\lceil\sqrt{n}\rceil/n}^{n-1} \Delta(j)$ gives us our main bound, which we now state.

Proposition 3.19. *Suppose that* $M > 1$, $n \geq 2M$, $T > 8n^{9/2}M^{3/2}$ *and* $f \in \mathrm{PL}^2_n \cap G^2_M$. *Then for any* $k \geq \lceil \sqrt{n} \rceil$ *and* $w \in \mathcal{Z}_k$,

$$\begin{aligned}\mathbb{Q}\Big(\xi^T|_{[k/n,1]} \in \Lambda_{3M,T}(f,n)|_{[k/n,1]} \,\Big|\, \xi^T(k/n) = w\Big)& \\ \geq \exp\bigg(- TI(f,k/n,1) - O\Big(\frac{M^4 T}{n^{1/4}} + M^3 n T^{1/2}\Big)\bigg)&.\end{aligned}$$

Proof. Combining Lemmas 3.16, 3.17 and 3.18, we have

$$\begin{aligned}\mathbb{Q}\Big(\xi^T|_{[k/n,1]} \in \Lambda_{3M,T}(f,n)|_{[k/n,1]} \,\Big|\, \xi^T(k/n) = w\Big)& \\ \geq \exp\bigg(- TI(f,k/n,1) - 2T\sum_{j=k}^{n-1}\Delta(j) - O\Big(\frac{M^4 T}{n^{1/2}} + M^3 n\Big)\bigg)&.\end{aligned}$$

Recall that

$$\Delta(j) = \frac{2(M+1)}{n^{3/2}} + \frac{2\delta_{M,T}(j,n)}{n} + \frac{1}{\sqrt{n}}\sqrt{2\delta_{M,T}(j,n)\big(f_X(\tfrac{j+1}{n}) - f_X(\tfrac{j}{n})\big)}.$$

By (3.61),

$$\sum_{j=\lceil\sqrt{n}\rceil}^{n-1} \frac{\delta_{M,T}(j,n)}{n} = O\Big(\frac{M^4}{n^{1/2}} + \frac{M^3 n}{T}\Big).$$

By Cauchy-Schwarz,

$$\sum_{j=\lceil\sqrt{n}\rceil}^{n-1} \frac{1}{\sqrt{n}}\sqrt{2\delta_{M,T}(j,n)\big(f_X(\tfrac{j+1}{n})-f_X(\tfrac{j}{n})\big)}$$
$$\leq \Big(\sum_{j=\lceil\sqrt{n}\rceil}^{n-1} \frac{2\delta_{M,T}(j,n)}{n}\Big)^{1/2}\Big(\sum_{j=\lceil\sqrt{n}\rceil}^{n-1} \big(f_X(\tfrac{j+1}{n})-f_X(\tfrac{j}{n})\big)\Big)^{1/2}.$$

Using (3.61) again, together with the fact that $f \in G^2_M$, and that $\sqrt{a+b} \leq \sqrt{a}+\sqrt{b}$ for $a, b \geq 0$, we have

$$\sum_{j=\lceil\sqrt{n}\rceil}^{n-1} \frac{1}{\sqrt{n}}\sqrt{2\delta_{M,T}(j,n)\big(f_X(\tfrac{j+1}{n})-f_X(\tfrac{j}{n})\big)} = O\Big(\frac{M^2}{n^{1/4}}+\frac{M^{3/2}n^{1/2}}{T^{1/2}}\Big)M^{1/2}$$
$$= O\Big(\frac{M^{5/2}}{n^{1/4}}+\frac{M^2 n^{1/2}}{T^{1/2}}\Big).$$

Therefore

$$\sum_{j=k}^{n-1}\Delta(j) \leq \sum_{j=\lceil\sqrt{n}\rceil}^{n-1}\Delta(j) = O\Big(\frac{M}{n^{1/2}}+\frac{M^4}{n^{1/2}}+\frac{M^3 n}{T}+\frac{M^{5/2}}{n^{1/4}}+\frac{M^2 n^{1/2}}{T^{1/2}}\Big).$$

Combining error terms gives the result. □

As promised, we can now easily prove Proposition 3.11, that is the lower bound on the first moment.

Proof of Proposition 3.11. For $u \in \mathcal{N}_{kT/n}$, let $\mathcal{N}_T^{(u)}$ be the set of descendants of u in $\mathcal{N}_T$. Since $u \in \mathcal{N}_{kT/n}$, by the Markov property and Lemma 3.3 (Many-to-one), for any $k \in \{0, \dots, n-1\}$,

$$\mathbb{E}\Big[\sum_{v\in\mathcal{N}_T^{(u)}} \mathbb{1}_{\{Z_v^T|_{[k/n,1]}\in\Lambda_{3M,T}(f,n)|_{[k/n,1]}\}}\Big|\mathcal{F}_{kT/n}\Big]$$
$$= \mathbb{Q}\Big[\mathbb{1}_{\{\xi^T|_{[k/n,1]}\in\Lambda_{3M,T}(f,n)|_{[k/n,1]}\}}e^{\int_{kT/n}^T R(\xi_s)ds}\Big|\xi^T(k/n)=w\Big]\Big|_{w=Z_u^T(k/n)}.$$

Now, since $k \geq \lceil\sqrt{n}\rceil$ and $f \in G^2_M \subset G^2_{3M}$, by (3.12) and the deterministic bounds on the integral of the rate function from Lemma 3.6, if $\xi^T|_{[k/n,1]} \in \Lambda_{3M,T}(f,n)|_{[k/n,1]}$, then

$$\int_{kT/n}^T R(\xi_s)ds = T\int_{k/n}^1 R(T\xi^T(s))ds \geq T\int_{k/n}^1 R^*(f(s))ds - T\eta(3M,n,T),$$

and therefore

$$\mathbb{E}\Bigg[\sum_{v\in\mathcal{N}_T^{(u)}} \mathbb{1}_{\{Z_v^T|_{[k/n,1]}\in\Lambda_{3M,T}(f,n)|_{[k/n,1]}\}}\Bigg|\mathcal{F}_{kT/n}\Bigg]$$
$$\geq \exp\Big(T\int_{k/n}^1 R^*(f(s))ds - T\eta(3M,n,T)\Big)$$
$$\cdot\,\mathbb{Q}\Big(\xi^T|_{[k/n,1]}\in\Lambda_{3M,T}(f,n)|_{[k/n,1]}\,\Big|\,\xi^T(k/n)=w\Big)\Big|_{w=Z_u^T(k/n)}.$$

We also know from Proposition 3.19 that if $w\in\mathcal{Z}_k$, then

$$\mathbb{Q}\Big(\xi^T|_{[k/n,1]}\in\Lambda_{3M,T}(f,n)|_{[k/n,1]}\,\Big|\,\xi^T(k/n)=w\Big)$$
$$\geq \exp\Big(-TI(f,k/n,1)-O\Big(\frac{M^4T}{n^{1/4}}+M^3nT^{1/2}\Big)\Big).$$

Combining these estimates and recalling that $\eta(3M,n,T)=O(M^4n^{-1/2}+M^3nT^{-1/3})$ gives the result. □

3.3.2 Upper bound on the second moment: proof of Proposition 3.12

For our first moment bounds we used the many-to-one lemma, Lemma 3.3, which gives a method for calculating expectations of sums over all the particles in our population at a fixed time. For our second moment bound, we will need an analogue for calculating expectations of squares of sums over particles. This will involve another measure $\mathbb{Q}^2$, whose description is again adapted from [32], this time in the case $k=2$.

Let $\mathbb{Q}^2$ be a probability measure under which ξ_t^1 and ξ_t^2 are Markov processes each living in $\mathbb{R}^2$ constructed in the following way:

- Take an exponential random variable $\mathbb{e}$ of parameter 1.
- Let $(\chi_t, t\geq 0)$ be a pure jump Markov process in $\mathbb{R}^2$ independent of $\mathbb{e}$ such that $\chi_0=0$ and when χ_t is in state z, jumps occur at rate $2R(z)$. When there is a jump from state z, it is of the form $(\mathcal{E},0)$ with probability $P(z)$ and $(0,\mathcal{E})$ with probability $1-P(z)$, where $\mathcal{E}$ is an independent exponentially-distributed random variable with parameter 1.
- Let $\tau=\inf\{t>0:\int_0^t 2R(\chi_s)ds>\mathbb{e}\}$.
- Let $\xi_t^1=\xi_t^2=\chi_t$ for $t<\tau$.
- Let ξ_τ^1 equal χ_τ plus a jump of the form $(-\log\mathcal{U},0)$ with probability $P(\chi_\tau)$ and $(0,-\log\mathcal{U})$ with probability $1-P(\chi_\tau)$, where $\mathcal{U}$ is an independent uniformly distributed random variable on $(0,1)$; let ξ_τ^2 equal χ_τ plus either $(-\log(1-\mathcal{U}),0)$ or $(0,-\log(1-\mathcal{U}))$ respectively.
- Conditionally on τ, $(\xi_t^1)_{t\leq\tau}$ and $(\xi_t^2)_{t\leq\tau}$, the processes $(\xi_{\tau+t}^1, t\geq 0)$ and $(\xi_{\tau+t}^2, t\geq 0)$ behave independently as if under $\mathbb{Q}_{\xi_\tau^1}$ and $\mathbb{Q}_{\xi_\tau^2}$ respectively.

We write $\mathbb{Q}^2$ both for the measure and for its corresponding expectation operator.

Lemma 3.20 (Many-to-two, Lemma 1 of [32] with $k = 2$)**.** *Suppose that $t \geq 0$. For any measurable function $f : (\mathbb{R}^2)^2 \to \mathbb{R}$,*

$$\mathbb{E}\Bigg[\sum_{u_1,u_2\in\mathcal{N}_t} f(Z_{u_1}(t), Z_{u_2}(t))\Bigg] = \mathbb{Q}^2\left[f(\xi_t^1,\xi_t^2)e^{3\int_0^{\tau\wedge t} R(\xi_s^1)ds+\int_{\tau\wedge t}^t R(\xi_s^1)ds+\int_{\tau\wedge t}^t R(\xi_s^2)ds}\right].$$

In fact, by using the description of $\mathbb{Q}^2$ above, the key to the second moment bound will be to estimate terms of the form

$$\mathbb{Q}\big(\xi^T|_{[a,b]} \in \Lambda_{M,T}(f,n)\big|_{[a,b]} \,\big|\, \xi_a^T = z\big)$$

where $\mathbb{Q} = \mathbb{Q}_0$ is the measure seen in Section 3.2.1. The same coupling used for Proposition 3.4 will yield the following result.

Proposition 3.21. *Suppose that $f \in E^2$, $n \in \mathbb{N}$, $T > 1$ and $M > 1$. Then for any $0 \leq a < b \leq 1$ and z such that $\|z - f(a)\| < 1/n^2$,*

$$\begin{aligned}&\mathbb{Q}\big(\xi^T|_{[a,b]} \in \Lambda_{M,T}(f,n)\big|_{[a,b]} \,\big|\, \xi_a^T = z\big)\\ &\leq \exp\Bigg(-T\sum_{j=\lfloor an\rfloor}^{\lceil bn\rceil-1}\big(\mathcal{E}_X^+(I_j\cap[a,b],\Lambda_{M,T}(f,n),T) + \mathcal{E}_Y^+(I_j\cap[a,b],\Lambda_{M,T}(f,n),T)\big)\Bigg).\end{aligned}$$

We postpone the details of the proof to Section 3.6. We then need to relate the right-hand side in Proposition 3.21 to our rate function, in the form of the following lemma.

Lemma 3.22. *Suppose that $M, T > 1$, $n \geq 2M$ and $f \in PL_n^2 \cap G_M^2$. Then for any a, b such that $\lceil\sqrt{n}\rceil/n \leq a < b \leq 1$,*

$$\begin{aligned}&\sum_{j=\lfloor an\rfloor}^{\lceil bn\rceil-1}\Big(\mathcal{E}_X^+(I_j\cap[a,b],\Lambda_{M,T}(f,n),T) + \mathcal{E}_Y^+(I_j\cap[a,b],\Lambda_{M,T}(f,n),T)\Big)\\ &\qquad\qquad \geq I(f,a,b) - O\Big(\frac{M^4}{n^{1/4}} + \frac{M^3 n}{T^{1/2}}\Big).\end{aligned}$$

The proof of Lemma 3.22 is similar to the deterministic bounds required for the upper bound in Section 3.2.2, but also uses the uniform structure of $\Lambda_{M,T}(f,n)$ and therefore requires slightly different estimates. We carry this out in Appendix 3.A.3, and for now continue to the proof of Proposition 3.12, that is the upper bound on the second moment. The proof is fairly long, but uses only the ingredients above together with bounds already developed for the upper bound on the first moment.

Proof of Proposition 3.12. Recall the construction of $\mathbb{Q}^2$ together with the Markov processes ξ^1 and ξ^2 above. For $s \geq 0$ and $T > 0$, write $\xi_s^{1,T} = \xi_{sT}^1/T$ and $\xi_s^{2,T} = \xi_{sT}^2/T$. For $i = 1, 2$, define the event

$$\mathcal{B}_i = \{\xi^{i,T}|_{[k/n,1]} \in \Lambda_{3M,T}(f,n)|_{[k/n,1]}\}$$

and for the single spine ξ defined under $\mathbb{Q}$, define

$$\mathcal{B}(a,b) = \{\xi^T|_{[a,b]} \in \Lambda_{3M,T}(f,n)|_{[a,b]}\}$$

By Lemma 3.20,

$$\begin{aligned}
&\mathbb{E}\left[\left(\sum_{v\in\mathcal{N}_T^{(u)}} \mathbb{1}_{\{Z_v^T|_{[k/n,1]}\in\Lambda_{3M,T}(f,n)|_{[k/n,1]}\}}\right)^2 \,\middle|\, \mathcal{F}_{kT/n}\right] \\
&= \mathbb{Q}^2\left[\mathbb{1}_{\mathcal{B}_1\cap\mathcal{B}_2} e^{3\int_{kT/n}^{T\wedge\tau} R(\xi_s^1)ds + \int_{T\wedge\tau}^T R(\xi_s^1)ds + \int_{T\wedge\tau}^T R(\xi_s^2)ds} \,\middle|\, \tau > \tfrac{kT}{n},\, \xi_{k/n}^{1,T} = z\right]\Big|_{z=Z_u(kT/n)/T}.
\end{aligned}$$

From the construction of $\mathbb{Q}^2$ before Lemma 3.20, it is clear that τ has a density, and that

$$\begin{aligned}
&\mathbb{Q}^2\left[\mathbb{1}_{\mathcal{B}_1\cap\mathcal{B}_2} e^{3\int_{kT/n}^{T\wedge\tau} R(\xi_s^1)ds + \int_{T\wedge\tau}^T R(\xi_s^1)ds + \int_{T\wedge\tau}^T R(\xi_s^2)ds} \,\middle|\, \tau > \tfrac{kT}{n},\, \xi_{k/n}^{1,T} = z\right] \\
&\quad \le \int_{kT/n}^T \mathbb{Q}^2\left[\mathbb{1}_{\mathcal{B}_1\cap\mathcal{B}_2\cap\{\tau\in dt\}} \,\middle|\, \tau > \tfrac{kT}{n},\, \xi_{k/n}^{1,T} = z\right] \\
&\qquad\qquad \cdot \sup_{g\in\Lambda_{3M,T}(f,n)} e^{3\int_{kT/n}^t R(Tg(s/T))ds + 2\int_t^T R(Tg(s/T))ds} \\
&\qquad\qquad + \mathbb{Q}^2\left[\mathbb{1}_{\mathcal{B}_1\cap\mathcal{B}_2\cap\{\tau>T\}} \,\middle|\, \tau > \tfrac{kT}{n},\, \xi_{k/n}^{1,T} = z\right] \sup_{g\in\Lambda_{3M,T}(f,n)} e^{3\int_{kT/n}^T R(Tg(s/T))ds}.
\end{aligned}$$

It also follows from the construction of $\mathbb{Q}^2$ before Lemma 3.20 that

$$\begin{aligned}
&\mathbb{Q}^2\left[\mathbb{1}_{\mathcal{B}_1\cap\mathcal{B}_2\cap\{\tau\in dt\}} \,\middle|\, \tau > \tfrac{kT}{n},\, \xi_{k/n}^{1,T} = z\right] \\
&\le \mathbb{Q}\left[\mathbb{1}_{\mathcal{B}(k/n,t/T)} 2R(\xi_t) e^{-2\int_{kT/n}^t R(\xi_s)ds} dt \,\middle|\, \xi_{k/n}^T = z\right] \\
&\qquad\qquad\qquad\qquad \cdot \sup_{\|w-f(t/T)\|<1/n^2} \mathbb{Q}\left(\mathcal{B}(t/T,1) \,\middle|\, \xi_{t/T}^T = w\right)^2 \\
&\le \mathbb{Q}\left(\mathcal{B}(k/n,t/T) \,\middle|\, \xi_{k/n}^T = z\right) \sup_{\|w-f(t/T)\|<1/n^2} \mathbb{Q}\left(\mathcal{B}(t/T,1) \,\middle|\, \xi_{t/T}^T = w\right)^2 \\
&\qquad\qquad\qquad\qquad \cdot \sup_{h\in\Lambda_{3M,T}(f,n)} 2R(Th(t/T)) e^{-2T\int_{k/n}^{t/T} R(Th(s))ds}
\end{aligned}$$

and that

$$\begin{aligned}
&\mathbb{Q}^2\left[\mathbb{1}_{\mathcal{B}_1\cap\mathcal{B}_2\cap\{\tau>T\}} \,\middle|\, \tau > \tfrac{kT}{n},\, \xi_{k/n}^{1,T} = z\right] \\
&\qquad = \mathbb{Q}\left[\mathbb{1}_{\mathcal{B}(k/n,1)} e^{-2\int_{kT/n}^T R(\xi_s)ds} \,\middle|\, \xi_{k/n}^T = z\right] \\
&\qquad \le \mathbb{Q}\left(\mathcal{B}(k/n,1) \,\middle|\, \xi_{k/n}^T = z\right) \sup_{h\in\Lambda_{3M,T}(f,n)} e^{-2T\int_{k/n}^1 R(Th(s))ds}.
\end{aligned}$$

Combining these bounds, we have shown that

$$
\begin{aligned}
&\mathbb{E}\Bigg[\Bigg(\sum_{v\in\mathcal{N}_T^{(u)}} \mathbb{1}_{\{Z_v^T|_{[k/n,1]}\in\Lambda_{3M,T}(f,n)|_{[k/n,1]}\}}\Bigg)^2\,\Bigg|\,\mathcal{F}_{kT/n}\Bigg]\\
&\quad\le \int_{kT/n}^{T}\Big\{\mathbb{Q}\Big(\mathcal{B}(k/n,t/T)\,\Big|\,\xi_{k/n}^T=z\Big)\Big|_{z=Z_u(kT/n)/T}\\
&\qquad\qquad\cdot\sup_{\|w-f(t/T)\|<1/n^2}\mathbb{Q}\Big(\mathcal{B}(t/T,1)\,\Big|\,\xi_{t/T}^T=w\Big)^2\\
&\qquad\qquad\qquad\cdot\sup_{h\in\Lambda_{3M,T}(f,n)} 2R(Th(t/T))e^{-2T\int_{k/n}^{t/T}R(Th(s))ds}\\
&\qquad\qquad\qquad\qquad\cdot\sup_{g\in\Lambda_{3M,T}(f,n)} e^{3T\int_{k/n}^{t/T}R(Tg(s))ds+2T\int_{t/T}^{1}R(Tg(s))ds}\Big\}dt\\
&\quad+\mathbb{Q}\Big(\mathcal{B}(k/n,1)\,\Big|\,\xi_{k/n}^T=z\Big)\Big|_{z=Z_u(kT/n)/T}\cdot\sup_{h\in\Lambda_{3M,T}(f,n)} e^{-2T\int_{k/n}^{1}R(Th(s))ds}\\
&\qquad\qquad\qquad\qquad\cdot\sup_{g\in\Lambda_{3M,T}(f,n)} e^{3T\int_{k/n}^{1}R(Tg(s))ds}.
\end{aligned}\tag{3.15}
$$

Recall that $k\ge\lceil\sqrt{n}\rceil$. By (3.12) and the deterministic bounds on the integral of the rate function from Lemma 3.6, for any $t\in[kT/n,T]$,

$$\sup_{g\in\Lambda_{3M,T}(f,n)}\int_{k/n}^{t/T}R(Tg(s))ds\le\int_{k/n}^{\lfloor nt/T\rfloor/n}R^*(f(s))ds+\eta(3M,n,T)$$

and

$$\inf_{h\in\Lambda_{3M,T}(f,n)}\int_{k/n}^{t/T}R(Th(s))ds\ge\int_{k/n}^{\lfloor nt/T\rfloor/n}R^*(f(s))ds-\eta(3M,n,T).$$

Thus

$$
\begin{aligned}
&\sup_{h\in\Lambda_{3M,T}(f,n)} e^{-2T\int_{k/n}^{t/T}R(Th(s))ds}\cdot\sup_{g\in\Lambda_{3M,T}(f,n)} e^{3T\int_{k/n}^{t/T}R(Tg(s))ds+2T\int_{t/T}^{1}R(Tg(s))ds}\\
&\qquad\le\exp\Big(-T\int_{k/n}^{\lfloor nt/T\rfloor/n}R^*(f(s))ds+2T\int_{k/n}^{1}R^*(f(s))ds+7T\eta(3M,n,T)\Big).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\sup_{h\in\Lambda_{3M,T}(f,n)} e^{-2T\int_{k/n}^{1}R(Th(s))ds}\cdot\sup_{g\in\Lambda_{3M,T}(f,n)} e^{3T\int_{k/n}^{1}R(Tg(s))ds}\\
&\qquad\qquad\le\exp\Big(T\int_{k/n}^{1}R^*(f(s))ds+5T\eta(3M,n,T)\Big).
\end{aligned}\tag{3.16}
$$

By the definition of $G_{M,T}$, plus the assumption that $T^{2/3}\ge 9Mn^{1/2}$, for any $t\in[kT/n,T]$ we also have

$$\sup_{h\in\Lambda_{3M,T}(f,n)}2R(Th(t/T))\le 2\frac{TM(t/T+2T^{-2/3})+1}{T(t/(MT)-2T^{-2/3})}\le\frac{6MT}{Tk/(2Mn)}\le 12M^2n.$$

The above estimates bound the non-probabilistic terms in (3.15). For the other terms we apply Proposition 3.21 and Lemma 3.22 to obtain the bound

$$\begin{aligned}
&\mathbb{Q}\Big(\mathcal{B}(a,b)\,\Big|\,\xi_a^T = z\Big) \\
&\le \exp\Big(-T\sum_{j=\lfloor an\rfloor}^{\lceil bn\rceil-1}\big(\mathcal{E}_X^+(I_j\cap[a,b],\Lambda_{3M,T}(f,n),T)+\mathcal{E}_Y^+(I_j\cap[a,b],\Lambda_{3M,T}(f,n),T)\big)\Big) \\
&\le \exp\Big(-TI(f,a,b)+O\Big(\frac{M^4T}{n^{1/4}}+M^3nT^{1/2}\Big)\Big).
\end{aligned}$$

Putting all these ingredients together, we obtain that

$$\begin{aligned}
&\mathbb{E}\Bigg[\Bigg(\sum_{v\in\mathcal{N}_T^{(u)}}\mathbb{1}_{\{Z_v^T|_{[k/n,1]}\in\Lambda_{3M,T}(f,n)|_{[k/n,1]}\}}\Bigg)^2\,\Bigg|\,\mathcal{F}_{kT/n}\Bigg] \\
&\le \int_{kT/n}^{T}\exp\Big(-TI(f,k/n,t/T)-2TI(f,t/T,1)+O\Big(\frac{M^4T}{n^{1/4}}+M^3nT^{1/2}\Big)\Big) \\
&\quad\cdot 12M^2n\exp\Big(-T\int_{k/n}^{\lfloor nt/T\rfloor/n}R^*(f(s))ds+2T\int_{k/n}^{1}R^*(f(s))ds+7T\eta(3M,n,T)\Big)dt \\
&+\exp\Big(-TI(f,k/n,1)+O\Big(\frac{M^4T}{n^{1/4}}+M^3nT^{1/2}\Big)\Big) \\
&\qquad\qquad\qquad\cdot\exp\Big(T\int_{k/n}^{1}R^*(f(s))ds+5T\eta(3M,n,T)\Big).
\end{aligned}\tag{3.17}$$

Using that $f\in\mathrm{PL}_n^2$ and therefore is absolutely continuous, we see that

$$\begin{aligned}
&-I(f,k/n,t/T)-2I(f,t/T,1)-\int_{k/n}^{\lfloor nt/T\rfloor/n}R^*(f(s))ds+2\int_{k/n}^{1}R^*(f(s))ds \\
&\qquad\qquad\le 2\tilde{K}(f,k/n,1)-\tilde{K}(f,k/n,t/T)+O(M^2/n).
\end{aligned}$$

The result follows from substituting this into (3.17) and recalling from Lemma 3.6 that

$$\eta(3M,n,T)=O\Big(\frac{M^4}{n^{1/2}}+\frac{M^3n}{T^{1/3}}\Big).\qquad\square$$

3.4. Detailed construction and ruling out difficult paths: proof of Lemma 3.2

In this section, we prove Lemma 3.2, which said that for large M all particles are (M,T)-good with high probability as $T\to\infty$. We will begin by defining a discrete tree with labels to represent the positions and split times of particles, which besides being a necessary step in our proof, also provides a formal construction of the process introduced in Section 3.1.

Take an infinite binary tree $\mathbb{T}$ and let $\mathbb{T}_n$ be the vertices in the nth generation of $\mathbb{T}$, so that $|\mathbb{T}_n|=2^n$. Attach to each vertex $v\in\mathbb{T}$ two independent random variables $\mathcal{U}_v^{\mathrm{split}}$ and $\mathcal{U}_v^{\mathrm{dir}}$, both uniformly distributed on $(0,1)$. Also attach another independent random variable $\mathbb{e}_v$ which is exponentially distributed with parameter 1.

We recursively define random variables B_v, H_v and T_v for each vertex $v\in\mathbb{T}$, which represent the base, height and birth time of the rectangle corresponding to v. Write ρ for the unique vertex in $\mathbb{T}_0$, which we call the root. Under the probability measure

$\mathbb{P}_{a,b}$, set $B_\rho = a$, $H_\rho = b$ and $T_\rho = 0$. We write $\mathbb{P}$ as shorthand for $\mathbb{P}_{1,1}$.

Now take an integer $n \geq 0$ and suppose that we have defined B_u, H_u and T_u for all vertices u in generations $0, \ldots, n$. For a vertex $v \in \mathbb{T}_n$, define

$$D_v = \begin{cases} 1 & \text{if } \mathcal{U}_v^{\text{dir}} \leq P(-\log B_v, -\log H_v) \\ 0 & \text{if } \mathcal{U}_v^{\text{dir}} > P(-\log B_v, -\log H_v). \end{cases}$$

Write $v1$ and $v2$ for the two children of v in generation $n+1$. If $D_v = 1$, then set

$$B_{v1} = \mathcal{U}_v^{\text{split}} B_v, \quad B_{v2} = (1 - \mathcal{U}_v^{\text{split}}) B_v, \quad \text{and} \quad H_{v1} = H_{v2} = H_v;$$

if on the other hand $D_v = 0$, then set

$$H_{v1} = \mathcal{U}_v^{\text{split}} H_v, \quad H_{v2} = (1 - \mathcal{U}_v^{\text{split}}) H_v, \quad \text{and} \quad B_{v1} = B_{v2} = B_v.$$

Then, for each $v \in \mathbb{T}$, define

$$X_v = -\log B_v \quad \text{and} \quad Y_v = -\log H_v.$$

Finally, set

$$T_{v1} = T_{v2} = T_v + \frac{\mathbb{e}_v}{R(X_v, Y_v)}.$$

We now translate this discrete-time process (with continuous labels) into the model in continuous-time described in the introduction. For each $t \geq 0$, define

$$\mathcal{N}_t = \Big\{ v \in \mathbb{T} : T_v \leq t < T_v + \frac{\mathbb{e}_v}{R(X_v, Y_v)} \Big\},$$

the set of particles that are alive at time t. Then for $v \in \mathcal{N}_t$ and $s \leq t$, if u is the unique ancestor of v in $\mathbb{T}$ that satsfies $T_u \leq s < T_u + \mathbb{e}_u / R(X_u, Y_u)$, then set $B_v(s) = B_u$, $H_v(s) = H_u$, $X_v(s) = X_u$ and $Y_v(s) = Y_u$. We call $Z_v(s) = (X_v(s), Y_v(s))$ the *position* of particle v at time s. For $T > 0$, we can also consider particles' paths rescaled by T, by which we mean, for $s \leq t$ and $v \in \mathcal{N}_{tT}$,

$$X_v^T(s) = \frac{X_v(sT)}{T}, \quad Y_v^T(s) = \frac{Y_v(sT)}{T}, \quad Z_v^T(s) = (X_v^T(s), Y_v^T(s)).$$

If we have $v \in \mathcal{N}_T$ then we may refer to X_v^T to mean the function $X_v^T : [0,1] \to \mathbb{R}$, and similarly for Y_v^T and Z_v^T.

Lemma 3.23. *For any $\kappa > 0$, there exists $M > 1$ and $N \in \mathbb{N}$ such that*

$$\mathbb{P}\Big(\exists v \in \mathbb{T}_n : X_v \notin [n/M, Mn] \text{ or } Y_v \notin [n/M, Mn] \\ \text{or } T_v < n/M \text{ or } T_v + \frac{\mathbb{e}_v}{R(X_v, Y_v)} > Mn \Big) \leq e^{-\kappa n}$$

for all $n \geq N$.

Proof. Note that for any $u \in \mathbb{T}_n$, X_u is the sum of n random variables, each of which is (stochastically) bounded above by an independent exponential random variable with parameter 1 (this is the distribution of $-\log U$ when U is $U(0,1)$). Thus, if $E \sim \text{Exp}(1)$,

$$\mathbb{P}(X_u > Mn) \le \mathbb{E}[e^{X_u/2}]e^{-Mn/2} \le \mathbb{E}[e^{E/2}]^n e^{-Mn/2} = 2^n e^{-Mn/2}$$

and, since there are 2^n vertices in $\mathbb{T}_n$, a union bound gives

$$\mathbb{P}(\exists v \in \mathbb{T}_n : X_v > Mn) \le 4^n e^{-Mn/2}.$$

By choosing M large enough, we can make this smaller than $e^{-\kappa n}$. By symmetry we also have

$$\mathbb{P}(\exists v \in \mathbb{T}_n : Y_v > Mn) \le e^{-\kappa n}.$$

For a lower bound on X_v and Y_v, we first give a lower bound on $X_v + Y_v$. Indeed, note that for $u \in \mathbb{T}_n$, $X_u + Y_u$ is a sum of n independent random variables, each of which is exponentially distributed with parameter 1. Thus, for any $\lambda > 0$ and any $u \in \mathbb{T}_n$,

$$\mathbb{P}(X_u + Y_u < n/M) \le \mathbb{E}[e^{-\lambda(X_u+Y_u)}]e^{\lambda n/M} = \mathbb{E}[e^{-\lambda E}]^n e^{\lambda n/M} = \frac{1}{(1+\lambda)^n} e^{\lambda n/M},$$

so that we can choose M_0 large enough that

$$\mathbb{P}(X_u + Y_u < n/M_0) \le 2^{-2n-2}e^{-2\kappa(n+1)}. \tag{3.18}$$

Take $u \in \mathbb{T}_n$, let u' be the unique ancestor of u in $\mathbb{T}_{\lfloor n/2 \rfloor}$ and take $M > M_0$. Note that, applying (3.18), if $n \ge 6$

$$\begin{aligned}
&\mathbb{P}(\exists v \in \mathbb{T}_n : X_v \wedge Y_v < n/M - 1) \\
&\le \mathbb{E}[\#\{v \in \mathbb{T}_n : X_v \wedge Y_v < n/M - 1\}] \\
&= 2^n \mathbb{P}(X_u \wedge Y_u < n/M - 1) \\
&\le 2^n \mathbb{P}(X_u \wedge Y_u < n/M - 1 \text{ and } X_{u'} + Y_{u'} \ge \lfloor n/2 \rfloor / M_0) + 2^n \mathbb{P}(X_{u'} + Y_{u'} < \lfloor n/2 \rfloor / M_0) \\
&\le 2^n \mathbb{P}(X_u \wedge Y_u < n/M - 1 \text{ and } X_{u'} + Y_{u'} \ge n/(3M_0)) + 2^n \cdot 2^{-2\lfloor n/2 \rfloor - 2} e^{-2\kappa(\lfloor n/2 \rfloor + 1)} \\
&\le 2^n \mathbb{P}(X_u \wedge Y_u < n/M - 1 \text{ and } X_{u'} + Y_{u'} \ge n/(3M_0)) + e^{-\kappa n}/2.
\end{aligned} \tag{3.19}$$

Now, if $X_u \wedge Y_u < n/M - 1$ and $X_{u'} + Y_{u'} \ge n/(3M_0)$, then for all vertices v on the path from u' to u, we have

$$\frac{X_v \vee Y_v + 1}{X_v \wedge Y_v + 1} \ge \frac{X_v + Y_v - X_v \wedge Y_v + 1}{X_v \wedge Y_v + 1} \ge \frac{X_v + Y_v}{X_v \wedge Y_v + 1} - 1 \ge \frac{M}{3M_0} - 1.$$

Recalling the definition of P, this means that

$$P(X_v, Y_v) \ge 1 - \frac{X_v \wedge Y_v + 1}{2(X_v \vee Y_v + 1)} \ge 1 - \frac{1}{2M/(3M_0) - 2}$$

and the same holds for $1 - P(X_v, Y_v)$. This means that $X_u \wedge Y_u - X_{u'} \wedge Y_{u'}$ consists of $\lceil n/2 \rceil$ random variables, each of which is (stochastically) bounded below by an independent random variable E' which is zero with probability $1/(2M/(3M_0) - 2)$ and equals an independent copy of E with probability $1 - 1/(2M/(3M_0) - 2)$. Thus, for any $\lambda > 0$,

$$\begin{aligned}
\mathbb{P}(X_u \wedge Y_u < n/M - 1 \text{ and } X_{u'} + Y_{u'} \geq n/(3M_0))& \\
&\leq \mathbb{E}\Big[e^{-\lambda X_u \wedge Y_u} \mathbb{1}_{\left\{\frac{X_v \vee Y_v + 1}{X_v \wedge Y_v + 1} \geq \frac{M}{3M_0} - 1\right\}}\Big] e^{\lambda n/M} \\
&\leq \mathbb{E}\Big[e^{-\lambda (X_u \wedge Y_u - X_{u'} \wedge Y_{u'})} \mathbb{1}_{\left\{\frac{X_v \vee Y_v + 1}{X_v \wedge Y_v + 1} \geq \frac{M}{3M_0} - 1\right\}}\Big] e^{\lambda n/M} \\
&\leq \mathbb{E}[e^{-\lambda E'}]^{\lceil n/2 \rceil} e^{\lambda n/M} \\
&\leq \Big(\frac{1}{2M/(3M_0) - 2} + \mathbb{E}[e^{-\lambda E}]\Big(1 - \frac{1}{2M/(3M_0) - 2}\Big)\Big)^{\lceil n/2 \rceil} e^{\lambda n/M} \\
&\leq \Big(\frac{1}{2M/(3M_0) - 2} + \frac{1}{\lambda + 1}\Big)^{\lceil n/2 \rceil} e^{\lambda n/M}.
\end{aligned}$$

By choosing λ large and then M large, we can ensure that this is smaller than $2^{-n} e^{-\kappa n}/2$, which when combined with (3.19), shows that for n sufficiently large,

$$\mathbb{P}\big(\exists v \in \mathbb{T}_n : X_v \notin [n/M, Mn] \text{ or } Y_v \notin [n/M, Mn]\big) \leq e^{-\kappa n}. \tag{3.20}$$

We now turn to T_v. As for X_v and Y_v, the upper bound is easy: since $R(x, y) \geq 1$ for all x and y, for any fixed $u \in \mathbb{T}_n$ we have

$$\begin{aligned}
\mathbb{P}\Big(T_u + \frac{\mathbb{e}_u}{R(X_u, Y_u)} > Mn\Big) &= \mathbb{P}\Big(\sum_{w \leq u} \frac{\mathbb{e}_w}{R(X_w, Y_w)} > Mn\Big) \\
&\leq \mathbb{P}\Big(\sum_{w \leq u} \mathbb{e}_w > Mn\Big) \leq \mathbb{E}[e^{E/2}]^{n+1} e^{-Mn/2} = 2^{n+1} e^{-Mn/2},
\end{aligned}$$

so a union bound gives

$$\mathbb{P}\Big(\exists v \in \mathbb{T}_n : T_v + \frac{\mathbb{e}_v}{R(X_v, Y_v)} > Mn\Big) \leq 2 \cdot 4^n e^{-Mn/2}$$

which can be made smaller than $e^{-\kappa n}$ by choosing M large.

For a lower bound on T_v, define the event

$$\Upsilon_{n,M} = \{X_v \in [k/M, Mk] \text{ and } Y_v \in [k/M, Mk] \;\; \forall v \in \mathbb{T}_k,\ \forall k \geq n\}.$$

By (3.20), for any $\kappa > 0$, we may choose N and M_0 sufficiently large that

$$\begin{aligned}
\mathbb{P}(\Upsilon^c_{n,M_0}) &\leq \sum_{j=n}^{\infty} \mathbb{P}\big(\exists v \in \mathbb{T}_n : X_v \notin [n/M_0, M_0 n] \text{ or } Y_v \notin [n/M_0, M_0 n]\big) \\
&\leq 2^{-2n-3} e^{-2\kappa(n+1)} \qquad (3.21)
\end{aligned}$$

for all $n \geq N$. Fix $u \in \mathbb{T}_n$ and let $\rho = u_0, u_1, u_2, \ldots, u_n = u$ be the unique path from the root ρ to u in the tree. Then for $n \geq 2N$,

$$\begin{aligned}\mathbb{P}(T_u < n/M) &= \mathbb{P}\Big(\sum_{j=0}^{n-1} \frac{\mathbb{e}_{u_j}}{R(X_{u_j}, Y_{u_j})} < \frac{n}{M}\Big) \\ &\leq \mathbb{P}(\Upsilon^c_{\lfloor n/2\rfloor, M_0}) + \mathbb{P}\Big(\Upsilon_{\lfloor n/2\rfloor, M_0} \cap \Big\{\sum_{j=\lfloor n/2\rfloor}^{n} \frac{\mathbb{e}_{u_j}}{R(X_{u_j}, Y_{u_j})} < \frac{n}{M}\Big\}\Big).\end{aligned} \tag{3.22}$$

Since $n \geq 2N$, we have

$$\mathbb{P}(\Upsilon^c_{\lfloor n/2\rfloor, M_0}) \leq 2^{-2\lfloor n/2\rfloor - 3} e^{-2\kappa(\lfloor n/2\rfloor + 1)}/2 \leq 2^{-n-1} e^{-\kappa n}. \tag{3.23}$$

On the event $\Upsilon_{\lfloor n/2\rfloor, M_0}$, we have

$$R(X_{u_j}, Y_{u_j}) \leq \frac{M_0 j + 1}{j/M_0 + 1} \leq M_0^2$$

for all $j \geq \lfloor n/2\rfloor$; therefore

$$\mathbb{P}\Big(\Upsilon_{\lfloor n/2\rfloor, M_0} \cap \Big\{\sum_{j=\lfloor n/2\rfloor}^{n} \frac{\mathbb{e}_{u_j}}{R(X_{u_j}, Y_{u_j})} < \frac{n}{M}\Big\}\Big) \leq \mathbb{P}\Big(\sum_{j=\lfloor n/2\rfloor}^{n} \frac{\mathbb{e}_{u_j}}{M_0^2} < \frac{n}{M}\Big).$$

But for any $\lambda > 0$,

$$\begin{aligned}\mathbb{P}\Big(\sum_{j=\lfloor n/2\rfloor}^{n} \frac{\mathbb{e}_{u_j}}{M_0^2} < \frac{n}{M}\Big) &= \mathbb{P}\big(e^{-\lambda \sum_{j=\lfloor n/2\rfloor}^{n} \mathbb{e}_{u_j}} > e^{-\lambda M_0^2 n/M}\big) \\ &\leq \mathbb{E}[e^{-\lambda \sum_{j=\lfloor n/2\rfloor}^{n} \mathbb{e}_{u_j}}] e^{\lambda M_0^2 n/M} \\ &\leq \mathbb{E}[e^{-\lambda E}]^{n/2} e^{\lambda M_0^2 n/M} = \frac{1}{(1+\lambda)^{n/2}} e^{\lambda M_0^2 n/M}.\end{aligned}$$

Substituting this and (3.23) into (3.22), we have

$$\mathbb{P}(T_u < n/M) \leq 2^{-n-1} e^{-\kappa n} + \frac{1}{(1+\lambda)^{n/2}} e^{\lambda M_0^2 n/M}.$$

Finally, taking a union bound over all 2^n vertices in $\mathbb{T}_n$, we obtain

$$\mathbb{P}(\exists v \in \mathbb{T}_n : T_v < n/M) \leq e^{-\kappa n}/2 + \Big(\frac{2e^{\lambda M_0^2/M}}{\sqrt{1+\lambda}}\Big)^n$$

which can be made smaller than $e^{-\kappa n}$ by choosing λ large and then M large. □

Fix $\alpha \in (0,1)$ and define the event

$$\begin{aligned}\mathcal{G}_M(T) = \Big\{ & X_v \in \Big[\frac{n}{M} - T^\alpha, Mn + T^\alpha\Big], Y_v \in \Big[\frac{n}{M} - T^\alpha, Mn + T^\alpha\Big], \\ & T_v \geq \frac{n}{M} - T^\alpha \text{ and } T_v + \mathbb{e}_v \leq Mn + T^\alpha \quad \forall v \in \mathbb{T}_n \quad \forall n \geq 0\Big\}.\end{aligned}$$

Corollary 3.24. *There exist $M > 1$ and $\delta > 0$ such that for any $T \geq 0$,*

$$\mathbb{P}(\mathcal{G}_M(T)^c) \leq \exp(-\delta T^\alpha).$$

Proof. By Lemma 3.23 we may choose $M \in (1, \infty)$ such that for all n large enough,

$$\mathbb{P}(\exists v \in \mathbb{T}_n : X_v \notin [n/M, Mn] \text{ or } Y_v \notin [n/M, Mn] \\ \text{ or } T_v < n/M \text{ or } T_v + \mathbb{e}_v > Mn) \leq e^{-n}.$$

Let

$$\mathcal{G}_{M,n}(T) = \Big\{ X_v \in \Big[\frac{n}{M} - T^\alpha, Mn + T^\alpha\Big], Y_v \in \Big[\frac{n}{M} - T^\alpha, Mn + T^\alpha\Big], \\ T_v \geq \frac{n}{M} - T^\alpha \text{ and } T_v + \mathbb{e}_v \leq Mn + T^\alpha \quad \forall v \in \mathbb{T}_n \Big\},$$

so that

$$\mathcal{G}_M(T) = \bigcap_{n=0}^{\infty} \mathcal{G}_{M,n}(T).$$

For $n \leq T^\alpha/M$, since $n/M - T^\alpha \leq 0$, we have

$$\mathcal{G}_{M,n}(T) = \big\{ X_v \leq Mn + T^\alpha, Y_v \leq Mn + T^\alpha \text{ and } T_v + \mathbb{e}_v \leq Mn + T^\alpha \quad \forall v \in \mathbb{T}_n \big\}$$

and therefore

$$\mathcal{G}_{M,n}(T) \supset \big\{ X_v \leq T^\alpha, Y_v \leq T^\alpha \text{ and } T_v + \mathbb{e}_v \leq T^\alpha \quad \forall v \in \mathbb{T}_n \big\}.$$

By monotonicity

$$\bigcap_{n=0}^{\lfloor T^\alpha/M \rfloor} \mathcal{G}_{M,n}(T) \supset \big\{ X_v \leq T^\alpha, Y_v \leq T^\alpha \text{ and } T_v + \mathbb{e}_v \leq T^\alpha \quad \forall v \in \mathbb{T}_{\lfloor T^\alpha/M \rfloor} \big\}.$$

and thus, by our choice of M,

$$\mathbb{P}\Big(\bigcup_{n=0}^{\lfloor T^\alpha/M \rfloor} \mathcal{G}_{M,n}(T)^c \Big) \leq \mathbb{P}\big(\exists v \in \mathbb{T}_{\lfloor T^\alpha/M \rfloor} : X_v > T^\alpha, \text{ or } Y_v > T^\alpha \text{ or } T_v + \mathbb{e}_v > T^\alpha\big) \\ \leq e^{-\lfloor T^\alpha/M \rfloor}.$$

On the other hand, for $n > T^\alpha/M$,

$$\mathcal{G}_{M,n}(T) \supset \Big\{ X_v \in \Big[\frac{n}{M}, Mn\Big], Y_v \in \Big[\frac{n}{M}, Mn\Big], T_v \geq \frac{n}{M} \text{ and } T_v + \mathbb{e}_v \leq Mn \quad \forall v \in \mathbb{T}_n \Big\}$$

so by our choice of M,

$$\mathbb{P}(\mathcal{G}_{M,n}(T)^c) \leq e^{-n}.$$

Combining the bounds for $n \le T^\alpha/M$ and $n > T^\alpha/M$, we have

$$\mathbb{P}\Big(\bigcup_{n=0}^{\infty} \mathcal{G}_{M,n}(T)^c\Big) \le e^{-\lfloor T^\alpha/M \rfloor} + \sum_{n>T^\alpha/M} e^{-n}$$

and choosing $\delta < 1/M$ completes the proof. □

We can now prove our main result for this section, Lemma 3.2.

Proof of Lemma 3.2. For $t \ge 0$, suppose that $u \in \mathcal{N}_t$ and let $n(u)$ be the unique n such that $u \in \mathbb{T}_n$. By the definition of $\mathcal{N}_t$, we have $T_u \le t < T_u + \mathbb{e}_u$. On $\mathcal{G}_M(T)$, we have $T_v + \mathbb{e}_v \le t$ for all $v \in \mathbb{T}_n$ with $Mn + T^\alpha \le t$; so we must have $n(u) > (t - T^\alpha)/M$. Similarly, on $\mathcal{G}_M(T)$, we have $T_v > t$ for all $v \in \mathbb{T}_n$ with $n/M - T^\alpha > t$; so we must have $n(u) \le M(t + T^\alpha)$. Thus, on $\mathcal{G}_M(T)$, we have

$$\frac{t - T^\alpha}{M} < n(u) \le M(t + T^\alpha)$$

and therefore also

$$\frac{t - T^\alpha}{M^2} - T^\alpha < X_u \le M^2(t + T^\alpha) + T^\alpha \quad \text{and} \quad \frac{t - T^\alpha}{M^2} - T^\alpha < Y_u \le M^2(t + T^\alpha) + T^\alpha.$$

Since this holds for any particle $u \in \mathcal{N}_t$ for any $t \ge 0$, taking $\alpha = 1/3$ and rescaling by T we deduce that on $\mathcal{G}_M(T)$, the paths of all particles fall within $G^2_{M^2,T}$, and the result follows from Corollary 3.24. □

3.5. Growth of the population at small times

In this section we prove two results that are essentially concerned with showing that the number of particles near any reasonable straight line $(\lambda s, \mu s)$, $s \ge 0$, grows exponentially fast. The first of these results is Proposition 3.14, which considers the case $\lambda = \mu = 1/2$; the idea in this case is that our rectangles prefer to be "roughly square", and relatively simple moment bounds will show that there are indeed many particles near this line. This will be the content of Section 3.5.1. We then move on to proving Proposition 3.15, which concerns a function that begins by moving along the line $(s/2, s/2)$ but then gradually shifts its gradient towards a general slope $(\lambda s, \mu s)$. This is done in Section 3.5.2.

3.5.1 The lead diagonal: proof of Proposition 3.14

Recall the discrete-time setup from Section 3.4. In order to initially remove the dependence between time and space, let $\tilde{T}_\rho = 0$, and recursively for each $v \in \mathbb{T}$ let $\tilde{T}_{v1} = \tilde{T}_{v2} = \tilde{T}_v + \mathbb{e}_v$.

For $v \in \mathbb{T}_k$ and $j \le k$, write $X_v(j)$ to mean X_u where u is the unique ancestor of v in $\mathbb{T}_j$. Similarly write $Y_v(j)$, $T_v(j)$, $\tilde{T}_v(j)$ and $Z_v(j)$. Also define

$$\Delta_v(j) = X_v(j) - Y_v(j) \quad \text{and} \quad S_v(j) = X_v(j) + Y_v(j) - j,$$

and let $(\mathcal{G}_j, j \ge 0)$ be the natural filtration of the discrete-time process. We begin with sixth moment estimates on $\Delta_v(j)$ and $S_v(j)$. The reason for using the sixth moment is that this eventually gives us a decay of order $1/T$, which will be strong enough for our purposes. We could use higher moments if we wanted to get a better rate of decay.

Lemma 3.25. *There exists a finite constant C such that for any vertex $v \in \mathbb{T}_k$ and $0 \le j \le k$, we have*

$$\mathbb{E}\big[(X_v(j) - Y_v(j))^6\big] \le Cj^3$$

and

$$\mathbb{E}[(X_v(j) + Y_v(j) - j)^6] \le Cj^3.$$

Proof. Let v_j be the vertex in $\mathbb{T}_j$ consisting of all 1s, i.e. $v_j = v_{j-1}1$ for all j. By symmetry it suffices to consider $v = v_k$. Letting $E_j = -\log \mathcal{U}_{v_j}^{\text{split}}$, we see by construction that

$$X_{v_j} - X_{v_{j-1}} = D_{v_{j-1}} E_{j-1} \quad \text{and} \quad Y_{v_j} - Y_{v_{j-1}} = (1 - D_{v_{j-1}}) E_{j-1}.$$

We also note that $\{E_j : j \ge 0\}$ is a collection of independent exponentially distributed random variables of parameter 1, such that E_j is independent of D_{v_j} for each j.

Let $\Delta_j = X_{v_j} - Y_{v_j}$. We begin by bounding the second moment of Δ_j, then the fourth moment, before we tackle the sixth moment. By the above,

$$\begin{aligned}
\mathbb{E}\big[\Delta_j^2 | \mathcal{G}_{j-1}\big] &= \mathbb{E}\big[\big(\Delta_{j-1} + (2D_{v_{j-1}} - 1)E_{j-1}\big)^2 \big| \mathcal{G}_{j-1}\big] \\
&= \Delta_{j-1}^2 + 2\Delta_{j-1}\mathbb{E}[(2D_{v_{j-1}} - 1)E_{j-1} | \mathcal{G}_{j-1}] + \mathbb{E}[(2D_{v_{j-1}} - 1)^2 E_{j-1}^2 | \mathcal{G}_{j-1}] \\
&= \Delta_{j-1}^2 + 2\Delta_{j-1}\mathbb{E}[2D_{v_{j-1}} - 1 | \mathcal{G}_{j-1}] + 2, \qquad (3.24)
\end{aligned}$$

where the last line follows from the independence of E_{j-1} from $D_{v_{j-1}}$ and $\mathcal{G}_{j-1}$ and the fact that $(2D_{v_{j-1}} - 1)^2 = 1$. Now we note that, from the definition of D_{v_j}, if $\Delta_j \ge 0$ then D_{v_j} equals 1 with probability at most $1/2$, whereas if $\Delta_j \le 0$ then D_{v_j} equals 1 with probability at least $1/2$. Thus

$$\Delta_{j-1}\mathbb{E}[2D_{v_{j-1}} - 1 | \mathcal{G}_{j-1}] \le 0, \qquad (3.25)$$

so that (3.24) becomes

$$\mathbb{E}\big[\Delta_j^2 | \mathcal{G}_{j-1}\big] \le \Delta_{j-1}^2 + 2.$$

Taking expectations and summing over $i \le j$, we obtain

$$\mathbb{E}\big[\Delta_j^2\big] \le 2j. \qquad (3.26)$$

We now move on to the fourth moment, following a very similar argument:

$$\begin{aligned}
\mathbb{E}\big[\Delta_j^4 | \mathcal{G}_{j-1}\big] &= \mathbb{E}\big[\big(\Delta_{j-1} + (2D_{v_{j-1}} - 1)E_{j-1}\big)^4 \big| \mathcal{G}_{j-1}\big] \\
&= \Delta_{j-1}^4 + 4\Delta_{j-1}^3\mathbb{E}[(2D_{v_{j-1}} - 1)E_{j-1} | \mathcal{G}_{j-1}] \\
&\quad + 6\Delta_{j-1}^2\mathbb{E}[(2D_{v_{j-1}} - 1)^2 E_{j-1}^2 | \mathcal{G}_{j-1}] \\
&\quad\quad + 4\Delta_{j-1}\mathbb{E}[(2D_{v_{j-1}} - 1)^3 E_{j-1}^3 | \mathcal{G}_{j-1}] + \mathbb{E}[(2D_{v_{j-1}} - 1)^4 E_{j-1}^4 | \mathcal{G}_{j-1}] \\
&= \Delta_{j-1}^4 + 4\Delta_{j-1}^3\mathbb{E}[2D_{v_{j-1}} - 1 | \mathcal{G}_{j-1}] + 6\Delta_{j-1}^2\mathbb{E}[E_{j-1}^2] \\
&\quad\quad + 4\Delta_{j-1}\mathbb{E}[2D_{v_{j-1}} - 1 | \mathcal{G}_{j-1}]\mathbb{E}[E_{j-1}^3] + \mathbb{E}[E_{j-1}^4], \qquad (3.27)
\end{aligned}$$

where again for the last line we used the independence of E_{j-1} from $D_{v_{j-1}}$ and $\mathcal{G}_{j-1}$ and the fact that $(2D_{v_{j-1}}-1)^2 = 1$. By (3.25) and the facts that $\mathbb{E}[E_{j-1}^2] = 2$ and $\mathbb{E}[E_{j-1}^4] = 24$, we obtain

$$\mathbb{E}\big[\Delta_j^4 \big| \mathcal{G}_{j-1}\big] \le \Delta_{j-1}^4 + 12\Delta_{j-1}^2 + 24.$$

Taking expectations and using (3.26), we have

$$\mathbb{E}\big[\Delta_j^4\big] \le \mathbb{E}[\Delta_{j-1}^4] + 24(j-1) + 24 = \mathbb{E}[\Delta_{j-1}^4] + 24j.$$

Summing over $i \le j$, this gives

$$\mathbb{E}\big[\Delta_j^4\big] \le \sum_{i=1}^{j} 24j = 12j(j+1). \tag{3.28}$$

The same strategy works again for the sixth moment, expanding out $\Delta_j^6 = (\Delta_{j-1} + (2D_{v_{j-1}}-1)E_{j-1})^6$ and using the independence of E_{j-1} from $D_{v_{j-1}}$ and $\mathcal{G}_{j-1}$, and then applying (3.25). Omitting the calculations, the upshot is that

$$\mathbb{E}\big[\Delta_j^6 \big| \mathcal{G}_{j-1}\big] \le \Delta_{j-1}^6 + 30\Delta_{j-1}^4 + 360\Delta_{j-1}^2 + 720.$$

Taking expectations and using (3.26) and (3.28), we have

$$\mathbb{E}\big[\Delta_j^6\big] \le \mathbb{E}[\Delta_{j-1}^6] + 360j(j+1) + 720j + 720 = \mathbb{E}[\Delta_{j-1}^6] + 360(j+1)(j+2).$$

Summing over $i \le j$, we have

$$\mathbb{E}\big[\Delta_j^6\big] \le 360\sum_{i=1}^{j}(i+1)(i+2) = 120j(j^2+6j+11)$$

which proves the first part of the lemma.

The second statement of the lemma is much simpler to prove, since $X_{v_j} + Y_{v_j} = \sum_{i=0}^{j-1} E_i$. Either by direct calculation or by using the moment generating function, one may write down an expression for every moment of $X_v(j)+Y_v(j)-j$; in particular one may check that

$$\mathbb{E}[(X_v(j)+Y_v(j)-j)^6] \le Cj^3$$

for some constant C, completing the proof. □

Lemma 3.26. *Let $K(n,T) = \lceil n^{7/8}\rceil T/n + \lceil 2T/n^2\rceil$ and let*

$$U_{n,T} = \Big\{v \in \mathbb{T}_{K(n,T)} : \|Z_v(k)-(k/2,k/2)\| \le \tfrac{T}{32n^4} \text{ and } |\tilde{T}_v(k)-k| \le \tfrac{T}{4n^2}\ \forall k \le K(n,T)\Big\}.$$

Then there exists a finite constant C such that for any $T \ge Cn^{48}$,

$$\mathbb{P}\big(|U_{n,T}| \ge \tfrac{1}{2T^2}2^{K(n,T)}\big) \ge 1 - T^{-3/2}.$$

Proof. Note that, for any $k \geq 0$ and $u \in \mathbb{T}_k$, by the triangle inequality we have

$$|X_u - k/2| = \frac{1}{2}|X_u + Y_u - k + X_u - Y_u| \leq \frac{1}{2}|X_u + Y_u - k| + \frac{1}{2}|X_u - Y_u|,$$

and similarly for $|Y_u - k/2|$. Thus

$$\mathbb{P}\big(\|Z_u - (k/2, k/2)\| > \tfrac{T}{32n^4}\big) \leq \mathbb{P}(|X_u + Y_u - k| > \tfrac{T}{32n^4}) + \mathbb{P}(|X_u - Y_u| > \tfrac{T}{32n^4}).$$

Applying Markov's inequality and the sixth moment estimates from Lemma 3.25, we obtain

$$\begin{aligned}\mathbb{P}\big(\|Z_u - (k/2, k/2)\| > \tfrac{T}{32n^4}\big) &\leq \mathbb{E}\big[|X_u + Y_u - k|^6\big]\big(\tfrac{32n^4}{T}\big)^6 + \mathbb{E}\big[|X_u - Y_u|^6\big]\big(\tfrac{32n^4}{T}\big)^6\\ &\leq 2Ck^3\big(\tfrac{32n^4}{T}\big)^6\end{aligned}$$

where C is a finite constant. Thus, for $K \geq 0$, $v \in \mathbb{T}_K$ and $k \leq K$,

$$\mathbb{P}\big(\|Z_v(k) - (k/2, k/2)\| > \tfrac{T}{32n^4}\big) \leq 2Ck^3\big(\tfrac{32n^4}{T}\big)^6.$$

Now note that $\tilde{T}_v(k)$ is a sum of k independent exponential random variables of parameter 1, and therefore has the same distribution as $X_v(k) + Y_v(k)$. Thus, again by Lemma 3.25,

$$\mathbb{E}\big[|\tilde{T}_v(k) - k|^6\big] \leq Ck^3$$

and therefore

$$\mathbb{P}\big(|\tilde{T}_v(k) - k| > \tfrac{T}{4n^2}\big) \leq \mathbb{E}\big[|\tilde{T}_v(k) - k|^6\big]\big(\tfrac{4n^2}{T}\big)^6 \leq Ck^3\big(\tfrac{4n^2}{T}\big)^6.$$

We deduce that, for some finite constant C',

$$\mathbb{P}\big(\exists k \leq K : \|Z_v(k) - (k/2, k/2)\| > \tfrac{T}{32n^4} \text{ or } |\tilde{T}_v(k) - k| > \tfrac{T}{4n^2}\big) \leq \sum_{k=1}^{K} \frac{C'k^3 n^{24}}{T^6}. \quad (3.29)$$

Summing over k, this is at most $C'K^4 n^{24}/T^6$, and since $K(n,T) = O(n^{-1/8}T) \leq O(T)$, we have

$$\mathbb{P}\big(\exists k \leq K(n,T) : \|Z_v(k) - (k/2, k/2)\| > \tfrac{T}{32n^4} \text{ or } |\tilde{T}_v(k) - k| > \tfrac{T}{4n^2}\big) \leq \frac{C''n^{24}}{T^2}$$

for some finite constant C''.

Converting the above to a statement about $U_{n,T}$, we have shown that

$$\mathbb{P}(v \in U_{n,T}) \geq 1 - \frac{C''n^{24}}{T^2},$$

and since there are $2^{K(n,T)}$ vertices in $\mathbb{T}_{K(n,T)}$,

$$\mathbb{E}\big[|U_{n,T}|\big] \geq 2^{K(n,T)}\Big(1 - \frac{C''n^{24}}{T^2}\Big).$$

Obviously we also have

$$\mathbb{E}\big[|U_{n,T}|^2\big] \le 2^{2K(n,T)},$$

and therefore by the Paley-Zygmund inequality,

$$\begin{aligned}\mathbb{P}\Big(|U_{n,T}| \ge \frac{1}{T^2}2^{K(n,T)}\Big(1-\frac{C''n^{24}}{T^2}\Big)\Big) &\ge \Big(1-\frac{1}{T^2}\Big)^2\frac{\mathbb{E}\big[|U_{n,T}|\big]^2}{\mathbb{E}\big[|U_{n,T}|^2\big]}\\ &\ge \Big(1-\frac{2}{T^2}\Big)\Big(1-\frac{C''n^{24}}{T^2}\Big)^2.\end{aligned}$$

The result follows. □

Lemma 3.27. *Define*

$$V_{n,T} = \big\{v \in \mathbb{T}_{K(n,T)} : \|Z_v(k)-(k/2,k/2)\| \le \tfrac{T}{32n^4} \text{ and } |T_v(k)-k| \le \tfrac{7T}{8n^2} \ \forall k \le K(n,T)\big\},$$

where $K(n,T) = \lceil n^{7/8}\rceil T/n + \lceil 2T/n^2\rceil$ as in Lemma 3.26. Then there exists a finite constant C such that for any $T \ge Cn^{48}$,

$$\mathbb{P}\big(|V_{n,T}| \ge \tfrac{1}{2T^2}2^{K(n,T)}\big) \ge 1 - T^{-3/2}.$$

Proof. We claim that every $v \in U_{n,T}$ is also in $V_{n,T}$. By Lemma 3.26 this is sufficient to complete the proof.

Take $v \in U_{n,T}$. In particular, for each $k \le K(n,T)$, we have $\|Z_v(k)-(k/2,k/2)\| \le \frac{T}{32n^4}$. This ensures that v satisfies the first condition required to be in $V_{n,T}$, but it also implies that

$$R(X_v(k),Y_v(k)) \le \frac{\frac{k}{2}+\frac{T}{32n^4}+1}{\frac{k}{2}-\frac{T}{32n^4}+1} = \frac{1+\frac{T}{32n^4(k/2+1)}}{1-\frac{T}{32n^4(k/2+1)}},$$

and so for $k \ge \frac{T}{4n^2}-2$,

$$R(X_v(k),Y_v(k)) \le \frac{1+\frac{1}{4n^2}}{1-\frac{1}{4n^2}} \le \frac{1}{(1-\frac{1}{4n^2})^2} \le \frac{1}{1-\frac{1}{2n^2}}, \tag{3.30}$$

where we used the fact that $1+x \le 1/(1-x)$ for $x \in [0,1)$.

Now, $\tilde{T}_v(k)$ consists of a sum of k independent exponential random variables of parameter 1, which we call $\mathbb{e}_v(0),\ldots,\mathbb{e}_v(k-1)$. For $k \ge \lfloor T/4n^2\rfloor$ we then have, by definition,

$$T_v(k) = \sum_{i=0}^{k-1}\frac{\mathbb{e}_v(i)}{R(X_v(i),Y_v(i))} \ge \sum_{i=\lfloor T/4n^2\rfloor}^{k-1}\frac{\mathbb{e}_v(i)}{R(X_v(i),Y_v(i))}.$$

Applying (3.30), this is at least

$$\Big(1-\frac{1}{2n^2}\Big)\sum_{i=\lfloor T/4n^2\rfloor}^{k-1}\mathbb{e}_v(i) = \Big(1-\frac{1}{2n^2}\Big)\big(\tilde{T}_v(k)-\tilde{T}_v(\lfloor T/4n^2\rfloor)\big).$$

Since $v \in U_{n,T}$, whenever $k \le K(n,T)$ we have $|\tilde{T}_v(k) - k| \le T/4n^2$, and we obtain

$$T_v(k) \ge \Big(1 - \frac{1}{2n^2}\Big)\big(k - T/4n^2 - (\lfloor T/4n^2 \rfloor + T/4n^2)\big) \ge \Big(1 - \frac{1}{2n^2}\Big)\Big(k - \frac{3T}{4n^2}\Big) \ge k - \frac{7T}{8n^2}.$$

We also obviously have $T_v(k) \ge 0 \ge k - \frac{7T}{8n^2}$ when $k < \lfloor T/4n^2 \rfloor$; and since $R(x,y) \ge 1$ for all x and y, we have $T_v(k) \le \tilde{T}_v(k)$ for all k. Thus we have shown that if $v \in U_{n,T}$ then $|T_v(k) - k| \le 7T/8n^2$ for all $k \le K(n,T)$, and we deduce that also $v \in V_{n,T}$, as required. $\square$

We now want to move from discrete to continuous time. We need some more notation. For $v \in \mathcal{N}_t$ and $s \le t$, let $v(s)$ be the unique ancestor of v that is in $\mathcal{N}_s$. Also let $\mathrm{gen}(v)$ be the unique integer such that $v \in \mathbb{T}_{\mathrm{gen}(v)}$.

Lemma 3.28. *Recall the definition of $V_{n,T}$ from Lemma 3.27. If $v \in V_{n,T}$ then $v \in \mathcal{N}_t$ for some $t \ge \lceil n^{7/8} \rceil T/n + \lceil 2T/n^2 \rceil - T/n^2$, and*

$$\big|\,\mathrm{gen}(v(s)) - s\big| \le \frac{7T}{8n^2} + 1$$

for all $s \le \lceil n^{7/8} \rceil T/n + \lceil 2T/n^2 \rceil - T/n^2 - 1$.

Proof. If $v \in V_{n,T}$ then $|T_v(k) - k| \le 7T/8n^2$ for all $k \le K(n,T)$. In particular

$$T_v \ge K(n,T) - T/n^2 = \lceil n^{7/8} \rceil T/n + \lceil 2T/n^2 \rceil - T/n^2,$$

and therefore $v \in \mathcal{N}_t$ for some $t \ge \lceil n^{7/8} \rceil T/n + \lceil 2T/n^2 \rceil - T/n^2$.

Now, for any $s \le \lceil n^{7/8} \rceil T/n + \lceil 2T/n^2 \rceil - T/n^2 - 1$, since $v \in V_{n,T}$,

$$T_{v(s)} = T_v(\mathrm{gen}(v(s))) \ge \mathrm{gen}(v(s)) - 7T/8n^2,$$

so since $T_{v(s)} \le s$ (because $v(s) \in \mathcal{N}_s$) we have

$$\mathrm{gen}(v(s)) \le s + 7T/8n^2. \tag{3.31}$$

Since $s \le \lceil n^{7/8} \rceil T/n + \lceil 2T/n^2 \rceil - T/n^2 - 1$, the above implies in particular that $\mathrm{gen}(v(s)) + 1 \le K(n,T) = \mathrm{gen}(v)$ and therefore we also have (again since $v \in V_{n,T}$)

$$T_{v(s)} + \mathbb{e}_{v(s)} = T_v(\mathrm{gen}(v(s)) + 1) \le \mathrm{gen}(v(s)) + 1 + 7T/8n^2.$$

Combining this with the fact that $T_{v(s)} + \mathbb{e}_{v(s)} > s$ (because $v(s) \in \mathcal{N}_s$), we obtain

$$s < \mathrm{gen}(v(s)) + 1 + 7T/8n^2,$$

and combining this with (3.31) gives the result. $\square$

We can now prove the main result of this section.

Proof of Proposition 3.14. By Lemma 3.27, with probability at least $1 - 1/T^{3/2}$, we have $|V_{n,T}| \ge \frac{1}{2T^2} 2^{K(n,T)}$. Suppose that $v \in V_{n,T}$ and let $t = \lceil n^{7/8} \rceil T/n$. Then by Lemma 3.28, $\mathrm{gen}(v(t)) \geq t - 7T/8n^2 - 1$, and of course $\mathrm{gen}(v) = K(n,T) = \lceil n^{7/8}\rceil T/n + \lceil 2T/n^2\rceil$. Thus the number of descendants that $v(t)$ has in $V_{n,T}$ is at most

$$2^{\lceil n^{7/8}\rceil T/n + \lceil 2T/n^2\rceil - (\lceil n^{7/8}\rceil T/n - 7T/8n^2 - 1)} \leq 2^{3T/n^2}.$$

We deduce that if $|V_{n,T}| \geq \frac{1}{2T^2} 2^{K(n,T)}$ then the number of *distinct* ancestors of particles in $V_{n,T}$ that are in $\mathcal{N}_t$ must be at least

$$\frac{2^{K(n,T)}}{2T^2 \cdot 2^{3T/n^2}} \geq 2^{T/n^{1/8} - T/n^2 - 2\log_2 T - 1}.$$

For $T \geq Cn^{48}$ and C large the right-hand side is certainly larger than $2^{T/n^{1/8} - 2T/n^2}$.

Now, if $u \in \mathcal{N}_t$ is an ancestor of a particle $v \in V_{n,T}$, and $s \leq t$, then

$$\begin{aligned}
\|Z_u(s) - (s/2, s/2)\| &= \|Z_v(s) - (s/2,s/2)\| \\
&\leq \left\|Z_v(s) - \left(\tfrac{\mathrm{gen}(v(s))}{2}, \tfrac{\mathrm{gen}(v(s))}{2}\right)\right\| + \left\|\left(\tfrac{\mathrm{gen}(v(s))}{2}, \tfrac{\mathrm{gen}(v(s))}{2}\right) - \left(\tfrac{s}{2}, \tfrac{s}{2}\right)\right\| \\
&\leq \frac{T}{32n^4} + \frac{1}{2}|\,\mathrm{gen}(v(s)) - s| \\
&\leq \frac{T}{32n^4} + \frac{1}{2}\Big(\frac{7T}{8n^2} + 1\Big)
\end{aligned}$$

where for the first inequality we used the triangle inequality, for the second we used that $v \in V_{n,T}$, and for the third we again used that $v \in V_{n,T}$ together with Lemma 3.28. For $T \geq Cn^{48}$ and C large this is smaller than $T/2n^2$, which completes the proof. $\square$

3.5.2 From the lead diagonal to other gradients: proof of Proposition 3.15

We will build up to the proof of Proposition 3.15 gradually, first constructing a suitable candidate function $h_{f,n}$, and then proving several lemmas that establish the required properties of $h_{f,n}$.

For $\mu \geq \lambda > 0$ let

$$\kappa(\lambda,\mu) = \frac{\mu}{\lambda} - \Big(\sqrt{2}\Big(\frac{\mu}{\lambda} - \frac{1}{2}\Big)^{1/2} - \lambda^{1/2}\Big)^2 - (1-\mu^{1/2})^2.$$

We have defined κ in such a way that, for $\mu \geq \lambda > 0$, if $g(s) = (\lambda s, \mu s)$ for $s \in [0,1]$ then

$$\tilde{K}(g, 0, t) = \kappa(\lambda,\mu)t.$$

We would like our function $h_{f,n}$ to begin with gradient $(1/2, 1/2)$, but then to transition in small steps to having gradient $(f_X'(0), f_Y'(0))$. In order to ensure that $\tilde{K}(h_{f,n}, 0, t)$ remains positive for all small t, we need to check that $\kappa(\lambda,\mu)$ is strictly positive for all the gradients (λ,μ) that $h_{f,n}$ passes through at small times. If κ was concave (or even concave on the region where it is positive) then this would be trivial since we could ask $h_{f,n}$ to transition linearly. Unfortunately there is a small region on which κ is positive and not concave, so we have to use a more complicated argument. This is done in the following lemma.

Lemma 3.29. *For every $0 < \lambda \leq \mu$ such that $\kappa(\lambda,\mu) > 0$, there exists a path $\gamma(t) = (\gamma_X(t), \gamma_Y(t))$, $t \in [0,1]$ and $\kappa_0 > 0$ such that* *(i)* $(\gamma_X(0), \gamma_Y(0)) = (1/2, 1/2)$ *and* $(\gamma_X(1), \gamma_Y(1)) = (\lambda, \mu)$*;*

(ii) $\kappa(\gamma(t)) \geq \kappa_0 > 0$ *for all* $t \in [0, 1]$*;*

(iii) γ *is piecewise linear and* $|\gamma'_X(t)| \leq 20$ *and* $|\gamma'_Y(t)| \leq 20$ *for all* $t \in [0, 1]$ *such that* γ *is differentiable at* t*;*

(iv) $\gamma_X(t) \in [3/2 - \sqrt{2}, 10]$ *and* $\gamma_Y(t) \in [3/2 - \sqrt{2}, 10]$ *for all* $t \in [0, 1]$.

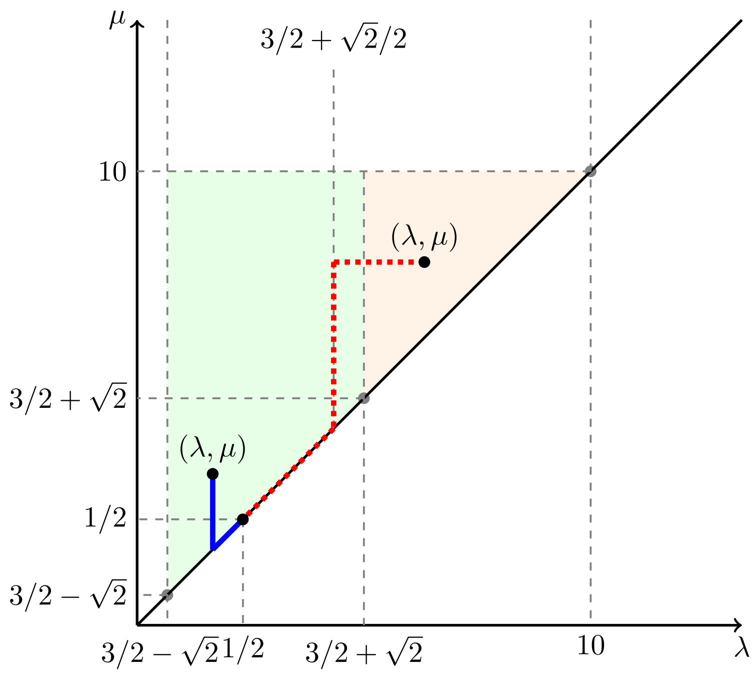

Figure 3-1: *The pale green region is* Υ_1 *and the pale orange region is* Υ_2*. The thick blue (solid) and red (dotted) paths show our definition of* γ *when* (λ, μ) *is in* Υ_1 *and* Υ_2 *respectively.*

Proof. We define $\Upsilon = \Upsilon_1 \cup \Upsilon_2$ where

$$\Upsilon_1 = \{(\lambda, \mu) : \lambda \in (3/2 - \sqrt{2}, 3/2 + \sqrt{2}),\ \mu \in [\lambda, 10)\}$$

and

$$\Upsilon_2 = \{(\lambda, \mu) : \mu \in (3/2 + \sqrt{2}, 10),\ \lambda \in [3/2 + \sqrt{2}, \mu]\}.$$

Figure 3-1 shows Υ_1 and Υ_2 in pale green and pale orange respectively. We show that the statement of the Lemma holds for all the points $(\lambda, \mu) \in \Upsilon$ and that $\kappa(\lambda, \mu) < 0$ if $(\lambda, \mu) \notin \Upsilon$. It is easy to see that

$$\kappa(\lambda, \lambda) = -2\lambda + 4\sqrt{\lambda} - 1 > 0 \quad \text{for} \quad \lambda \in (3/2 - \sqrt{2}, 3/2 + \sqrt{2}). \tag{3.32}$$

and this is concave as a function of λ. Since for $0 < \lambda \leq \mu$ we have

$$\frac{\partial^2 \kappa(\lambda, \mu)}{\partial \lambda^2} = -(1/2)(2\mu - \lambda)^{-3/2} - (2\mu)\lambda^{-3/2} < 0$$

and
$$\frac{\partial^2 \kappa(\lambda,\mu)}{\partial \mu^2} = -2(2\mu-\lambda)^{-3/2} - (1/2)\mu^{-3/2} < 0,$$
the functions $\kappa(\cdot,\mu)$ on $(0,\mu]$ and $\kappa(\lambda,\cdot)$ on $[\lambda,\infty)$ are concave for each fixed λ and μ respectively. This means that if we move parallel to either axis, we have the concave property; and so if for example $\kappa(\lambda_1,\mu) > 0$ and $\kappa(\lambda_2,\mu) > 0$ with $\lambda_1, \lambda_2 \le \mu$, then $\kappa(\lambda,\mu) > 0$ for all $\lambda \in [\lambda_1,\lambda_2]$.

We now take advantage of this concavity parallel to the axes. For every $(\lambda,\mu) \in \Upsilon_1$ such that $\kappa(\lambda,\mu) > 0$ we choose γ to be the union of the linear paths connecting $(1/2,1/2)$ to (λ,λ) and then to (λ,μ). Then clearly γ satisfies (i) and (iv). Since $0 < \lambda \le \mu \le 10$ throughout Υ, the total length of the linear paths described is at most 20, and therefore we may choose a time parameterization of γ such that $|\gamma_X'(t)| \le 20$ and $|\gamma_Y'(t)| \le 20$, so that γ satisfies (iii). We claim that γ also satisfies (ii). Indeed, by (3.32) $\kappa(\gamma(t))$ is positive on the first linear segment; in particular $\kappa(\lambda,\lambda) > 0$, and since by assumption $\kappa(\lambda,\mu) > 0$, by concavity parallel to the axes $\kappa(\gamma(t))$ is positive on the second linear segment too and $\kappa(\lambda,\mu) \ge \kappa_0$ where $\kappa_0 := \min\{\kappa(1/2,1/2), \kappa(\lambda,\lambda), \kappa(\lambda,\mu)\} > 0$.

Now consider $(\lambda,\mu) \in \Upsilon_2$ such that $\kappa(\lambda,\mu) > 0$. Since $\lambda \ge 3/2 + \sqrt{2}/2$ and $\mu < 10$, we have
$$\frac{\partial \kappa(\lambda,\mu)}{\partial \lambda} = \frac{\mu}{\lambda^2} - \frac{1}{\sqrt{2\mu-\lambda}} - 1 \le \frac{4\mu}{(3+\sqrt{2})^2} - \frac{1}{\sqrt{2\mu-\lambda}} - 1 < 0, \tag{3.33}$$
so $\kappa(\lambda',\mu) > \kappa(\lambda,\mu) > 0$ for every $\lambda' \in [3/2+\sqrt{2}/2, \lambda]$. In particular, $\kappa(3/2+\sqrt{2}/2,\mu) > 0$ and $(3/2+\sqrt{2}/2,\mu) \in \Upsilon_1$, so we can define γ as the union of the linear paths connecting $(1/2,1/2)$ to $(3/2+\sqrt{2}/2, 3/2+\sqrt{2}/2)$, then to $(3/2+\sqrt{2}/2,\mu)$, and then to (λ,μ). Then as above, γ clearly satisfies (i) and (iv) and can be parameterized such that it satisfies (iii). Also $\kappa(\gamma(t))$ is positive on the first and second linear segments by the analysis of the $\lambda \in (3/2-\sqrt{2}, 3/2+\sqrt{2})$ case above, it is positive on the third linear segment by (3.33) and $\kappa(\lambda,\mu) \ge \kappa_0$. Thus γ satisfies (ii) too.

To complete our proof, it remains to show that $\kappa(\lambda,\mu) < 0$ for $(\lambda,\mu) \notin \Upsilon$. If $0 < \lambda \le \mu \le 3/2-\sqrt{2}$, this follows from the fact that for every $\lambda \le 3/2-\sqrt{2}$,
$$\kappa(\lambda, 3/2-\sqrt{2}) = -\frac{3/2-\sqrt{2}}{\lambda} - \lambda + 2\sqrt{3-2\sqrt{2}-\lambda} - \frac{3}{2} + \sqrt{2} + 2\sqrt{3/2-\sqrt{2}} < 0, \tag{3.34}$$
and for every $t \in [0, 3/2-\sqrt{2}-\mu]$,
$$\begin{aligned}\frac{d}{dt}\kappa(\lambda+t,\mu+t) &= \left(\frac{\partial \kappa(\lambda,\mu)}{\partial \lambda} + \frac{\partial \kappa(\lambda,\mu)}{\partial \mu}\right)\Bigg|_{(\lambda,\mu)=(\lambda+t,\mu+t)} \\ &= \frac{\mu-\lambda}{(\lambda+t)^2} + \frac{1}{\sqrt{2\mu+t-\lambda}} + \frac{1}{\sqrt{\mu+t}} - 2 \\ &\ge \frac{1}{\sqrt{3/2-\sqrt{2}}} - 2 > 0.\end{aligned}$$

Secondly, if $0 < \lambda \le 3/2 - \sqrt{2} < \mu$, then we use (3.34) plus the fact that

$$\frac{\partial \kappa(\lambda,\mu)}{\partial \mu} = -\frac{1}{\lambda} + \frac{2}{\sqrt{2\mu-\lambda}} + \frac{1}{\sqrt{\mu}} - 1 \le -\frac{1}{3/2-\sqrt{2}} + \frac{3}{\sqrt{3/2-\sqrt{2}}} - 1 < 0.$$

Finally, when $0 < \lambda \le \mu$ and $\mu \ge 10$, the key fact is to observe that for every $\mu \ge 10$

$$\frac{\partial \kappa(\lambda,\mu)}{\partial \mu} = -\frac{1}{\lambda} + \frac{2}{\sqrt{2\mu-\lambda}} + \frac{1}{\sqrt{\mu}} - 1 \le -\frac{1}{\mu} + \frac{3}{\sqrt{\mu}} - 1 < 0. \tag{3.35}$$

Since $\kappa(\lambda, 10) < 0$ for $\lambda \le 10$, (3.35) gives that $\kappa(\lambda,\mu) < 0$ for any $\mu \ge 10$ and $\lambda \le 10$; and since $\kappa(\lambda,\lambda) < 0$ for $\lambda \ge 10$, (3.35) gives that $\kappa(\lambda,\mu) < 0$ whenever $\mu \ge 10$ and $\lambda \ge 10$. This completes the proof. $\square$

Take $f \in G_M^2$ such that $\frac{d}{dt}\tilde{K}(f,0,t)|_{t=0} > 0$ and $\tilde{K}(f,0,t) > 0$ for all $t \in (0,1]$. Also fix $n \in \mathbb{N}$ and $m \in \mathbb{N}$ such that $n \ge m$. We now construct a function $h = h_{f,n,m}$ which depends on n and m; we will later show that for m sufficiently large (and n even larger) the resulting function satisfies the properties of Proposition 3.15.

Let $\tau = m^m \lceil n^{7/8} \rceil / n$. We will eventually choose n much larger than m, so that τ is small. Also let $\lambda = f_X'(0)$ and $\mu = f_Y'(0)$. Take γ as in Lemma 3.29 and for $j \in \{0,1,\dots,m\}$ define

$$\lambda_j = \lambda_j^{(m)} = \gamma_X(j/m), \qquad \mu_j = \mu_j^{(m)} = \gamma_Y(j/m) \quad \text{and} \quad \tau_j = \tau m^{j-m}. \tag{3.36}$$

Begin by defining $h(s) = (s/2, s/2) = (\lambda_0 s, \mu_0 s)$ for $s \le \tau_0$. Then recursively, for each $j = 1,\dots,m$, suppose that $h(s)$ is defined for $s \le \tau_{j-1}$ and set

$$h(s) = h(\tau_{j-1}) + \big(\lambda_j(s-\tau_{j-1}),\, \mu_j(s-\tau_{j-1})\big) \quad \text{for } s \in (\tau_{j-1}, \tau_j].$$

Also define

$$h(s) = h(\tau) + \big(f(2\tau) - h(\tau)\big)\Big(\frac{s-\tau}{\tau}\Big) \quad \text{for } s \in (\tau, 2\tau].$$

Finally, for each $j \in \{2\tau n, 2\tau n + 1, \dots, n\}$ let $h(j/n) = f(j/n)$ and interpolate linearly between these values.

Note that, since $\tilde{K}$ has only downward jumps and $\tilde{K}(f,0,t) > 0$ for all $t \in (0,1]$, we have $\inf_{s\in[\nu,1]} \tilde{K}(f,0,s) > 0$ for every $\nu > 0$. Thus we may choose $\nu = \nu_{f,m} \in (0,1]$ such that

(a) $\|f(s) - (\lambda s, \mu s)\| \le s/m$ for all $s \le \nu$,

(b) $\tilde{K}(f,s,t) > 0$ for all $s \le \nu$ and $t \ge s$, and

(c) $\tilde{K}(f,\nu,1) \ge \tilde{K}(f,0,1) - 1/m$.

Lemma 3.30. *Suppose that $\mu \ge \lambda > 0$ and $\kappa(\lambda,\mu) > 0$. For any $m \ge 2$, $j \in \{1,\dots,m\}$ and any $s \in [\tau_{j-1}, \tau_j]$,*

$$\|h_{f,n,m}(s) - (\lambda_j s, \mu_j s)\| \le 40\tau_{j-1}/m.$$

Moreover, if $2\tau \le \nu$, then for any $s \in [\tau, \nu]$,

$$\|h_{f,n,m}(s) - (\lambda s, \mu s)\| \le 40 s/m.$$

Proof. We begin by noting that for $s \in [\tau_{j-1}, \tau_j]$,

$$h(s) - (\lambda_j s, \mu_j s) = h(\tau_{j-1}) - (\lambda_j \tau_{j-1}, \mu_j \tau_{j-1}), \tag{3.37}$$

so for the first part of the lemma it suffices to show that for any $j \in \{1, \ldots, m\}$,

$$\|h(\tau_{j-1}) - (\lambda_j \tau_{j-1}, \mu_j \tau_{j-1})\| \leq 40\tau_{j-1}/m. \tag{3.38}$$

We prove (3.38) by induction. Recall that for each j, $\lambda_j = \gamma_X(j/m)$, and by Lemma 3.29 (iii), $|\lambda_{j-1} - \lambda_j| \leq 20/m$ and $|\mu_j - \mu_{j-1}| \leq 20/m$. Thus we first have

$$\|h(\tau_0) - (\lambda_1 \tau_0, \mu_1 \tau_0)\| = \max\{|\lambda_0 - \lambda_1|\tau_0, |\mu_0 - \mu_1|\tau_0\} \leq 20\tau_0/m.$$

Suppose that $j \in \{1, \ldots, m-1\}$ and (3.38) holds for j. By the triangle inequality,

$$|h_X(\tau_j) - \lambda_{j+1}\tau_j| \leq |h_X(\tau_j) - \lambda_j \tau_j| + |\lambda_j \tau_j - \lambda_{j+1}\tau_j|$$

and then by (3.37), this equals

$$|h_X(\tau_{j-1}) - \lambda_j \tau_{j-1}| + |\lambda_j - \lambda_{j+1}|\tau_j.$$

Applying (3.38) and using the fact that $|\lambda_{j-1} - \lambda_j| \leq 20/m$, we obtain that

$$|h_X(\tau_j) - \lambda_{j+1}\tau_j| \leq \frac{40\tau_{j-1}}{m} + \frac{20\tau_j}{m} \leq \frac{40\tau_j}{m}.$$

By symmetry we also have $|h_Y(\tau_j) - \mu_{j+1}\tau_j| \leq 40\tau_j/m$. Hence, by induction, (3.38) holds for all $j \in \{1, \ldots, m\}$, proving the first part of the lemma.

For the second part, suppose that $2\tau \leq \nu$. Note first that for $s \in [\tau, 2\tau]$, h is linear and therefore

$$\|h(s) - (\lambda s, \mu s)\| \leq \max\left\{\|h(\tau) - (\lambda\tau, \mu\tau)\|, \|h(2\tau) - (2\lambda\tau, 2\mu\tau)\|\right\}.$$

By the first part of the lemma,

$$\|h(\tau) - (\lambda\tau, \mu\tau)\| \leq 40\tau_{m-1}/m \leq 40\tau/m$$

and since $h(2\tau) = f(2\tau)$, by property (a) of f,

$$\|h(2\tau) - (2\lambda\tau, 2\mu\tau)\| \leq 2\tau/m.$$

This proves the second part of the lemma for $s \in [\tau, 2\tau]$; for $s \in [2\tau, \nu]$, we note that h linearly interpolates between values of f, and therefore for j such that $s \in [j/n, (j+1)/n]$,

$$\begin{aligned} \|h(s) - (\lambda s, \mu s)\| &\leq \max\left\{\|f(\tfrac{j}{n}) - (\lambda\tfrac{j}{n}, \mu\tfrac{j}{n})\|, \|f(\tfrac{j+1}{n}) - (\lambda\tfrac{j+1}{n}, \mu\tfrac{j+1}{n})\|\right\} \\ &\leq \frac{j+1}{nm} \leq \frac{s + 1/n}{m} \leq \frac{2s}{m} \end{aligned}$$

where we used (a) and the fact that $2\tau \geq 1/n$. □

Corollary 3.31. *Suppose that $\mu \geq \lambda > 0$, $\kappa(\lambda, \mu) > 0$ and $m \geq 1600$, $m \in \mathbb{N}$. Then for any $j \in \{1, \dots, m\}$ and any $s \in [\tau_{j-1}, \tau_j]$,*

$$\frac{\mu_j}{\lambda_j}\Big(1 - \frac{1600}{m}\Big) \leq R^*(h_{f,n,m}(s)) \leq \frac{\mu_j}{\lambda_j}\Big(1 + \frac{3200}{m}\Big),$$

and if $2\tau \leq \nu$ then for any $s \in [\tau, \nu]$

$$\frac{\mu}{\lambda}\Big(1 - \frac{1600}{m}\Big) \leq R^*_X(h_{f,n,m}(s)) + 1/2 \leq R^*(h_{f,n,m}(s)) \leq \frac{\mu}{\lambda}\Big(1 + \frac{3200}{m}\Big).$$

Proof. We begin with the lower bound on $R^*(h(s))$ for $s \in [\tau_{j-1}, \tau_j]$. Since $\mu \geq \lambda$, we have $h_Y(s) \geq h_X(s)$, and therefore by the first part of Lemma 3.30,

$$R^*(h(s)) = \frac{h_Y(s)}{h_X(s)} \geq \frac{\mu_j s - 40\tau_{j-1}/m}{\lambda_j s + 40\tau_{j-1}/m} = \frac{\mu_j}{\lambda_j}\Big(\frac{1 - 40\tau_{j-1}/(m\mu_j s)}{1 + 40\tau_{j-1}/(m\lambda_j s)}\Big).$$

Using the fact that $s \geq \tau_{j-1}$, and then that $1/(1+x) \geq 1 - x$ for $x \geq 0$, this is at least

$$\frac{\mu_j}{\lambda_j}\Big(1 - \frac{40}{m\mu_j}\Big)\Big(1 - \frac{40}{m\lambda_j}\Big).$$

By Lemma 3.29 (iv), we have $\lambda_j \geq 3/2 - \sqrt{2} \geq 1/20$ and similarly for μ_j, so the above is at least $\frac{\mu_j}{\lambda_j}(1 - 1600/m)$, and the first lower bound on $R^*(h(s))$ follows. The first upper bound is similar, using that $1/(1-x) \leq 1 + 2x$ for $x \in [0, 1/2]$; since $m \geq 1600$ and $\lambda_j \geq 1/20$ we have $40/(m\lambda_j) \leq 1/2$, and we obtain

$$R^*(h(s)) \leq \frac{\mu_j}{\lambda_j}\Big(1 + \frac{40}{m\mu_j}\Big)\Big(1 + \frac{80}{m\lambda_j}\Big);$$

then since $\lambda_j \geq 1/20$, $\mu_j \geq 1/20$ and $m \geq 1600$, the product of the last two terms reduces to the desired form.

The proof of the second part of the corollary, when $s \in [\tau, \nu]$, is almost identical. Indeed, if $h_Y(s) \geq h_X(s)$ then $R^*_X(h(s)) + 1/2 = R^*(h(s))$ and we use the same argument but apply the second part of Lemma 3.30 rather than the first part. The same applies to the lower bound even when $h_Y(s) < h_X(s)$, since in any case $R^*_X(h(s)) + 1/2 \geq h_Y(s)/h_X(s)$. However, we have to make a slight modification to the upper bound when $h_Y(s) < h_X(s)$; in this case, we instead have $R^*_X(h(s)) + 1/2 \leq R^*(h(s))$ where

$$R^*(h(s)) = \frac{h_X(s)}{h_Y(s)},$$

and then the argument above gives

$$R^*(h(s)) = \frac{h_X(s)}{h_Y(s)} \leq \frac{\lambda s + 40s/m}{\mu s - 40s/m} \leq \frac{\lambda}{\mu}\Big(1 + \frac{40}{\lambda m}\Big)\Big(1 + \frac{80}{\mu m}\Big) \leq \frac{\lambda}{\mu}\Big(1 + \frac{3200}{m}\Big).$$

However, since $\lambda \leq \mu$, we have $\lambda/\mu \leq 1 \leq \mu/\lambda$ and so the same conclusion holds. □

Corollary 3.32. *Suppose that $\mu \geq \lambda > 0$, $\kappa(\lambda,\mu) > 0$ and $m \geq 1600$, $m \in \mathbb{N}$. There exists a finite constant C such that for any $j \in \{1,\dots,m\}$ and any $s,t \in [\tau_{j-1},\tau_j]$ with $s \leq t$, we have*

$$\tilde{K}(h_{f,n,m},s,t) \geq \kappa(\lambda_j,\mu_j)(t-s) - \frac{C}{m}(t-s),$$

and if $2\tau \leq \nu$ then for any $s,t \in [\tau,\nu]$ with $s \leq t$, we have

$$\tilde{K}(h_{f,n,m},s,t) \geq \kappa(\lambda,\mu)(t-s) - \frac{C}{m}(t-s).$$

Proof. We begin with the first statement. Using (3.3), since $h(t)-h(s) = (\lambda_j(t-s),\mu_j(t-s))$, we have

$$\tilde{K}(h,s,t) = -\int_s^t R^*(h(u))du + 2\sqrt{2}\int_s^t \sqrt{R^*_X(h(u))\lambda_j}\,du \\ + 2\sqrt{2}\int_s^t \sqrt{R^*_Y(h(u))\mu_j}\,du - \lambda_j(t-s) - \mu_j(t-s).$$

Since $\mu \geq \lambda$, we have $h_Y(u) \geq h_X(u)$ for all $u \leq \tau$ and therefore $R^*_X(h(u)) = R^*(h(u)) - 1/2$ and $R^*_Y(h(u)) = 1/2$ for all $u \leq \tau$. Thus

$$\tilde{K}(h,s,t) = -\int_s^t R^*(h(u))du + 2\sqrt{2}\int_s^t \sqrt{(R^*(h(u))-1/2)\lambda_j}\,du \\ + 2\sqrt{\mu_j}(t-s) - \lambda_j(t-s) - \mu_j(t-s). \quad (3.39)$$

By Corollary 3.31, for any $u \in [s,t]$ we have

$$\frac{\mu_j}{\lambda_j}\Big(1-\frac{1600}{m}\Big) \leq R^*(h(u)) \leq \frac{\mu_j}{\lambda_j}\Big(1+\frac{3200}{m}\Big)$$

and, using also that $(1-x)^{1/2} \geq 1-x$ for $x \in [0,1)$,

$$\begin{aligned}\sqrt{(R^*(h(u))-1/2)} &\geq \bigg(\frac{\mu_j}{\lambda_j}\Big(1-\frac{1600}{m}\Big)-1/2\bigg)^{1/2} \\ &= \Big(\frac{\mu_j}{\lambda_j}-\frac{1}{2}\Big)^{1/2}\Big(1-\frac{1600\mu_j}{m\lambda_j(\mu_j/\lambda_j-1/2)}\Big)^{1/2} \\ &\geq \Big(\frac{\mu_j}{\lambda_j}-\frac{1}{2}\Big)^{1/2}\Big(1-\frac{3200}{m}\Big).\end{aligned}$$

Substituting these estimates into (3.39), we have

$$\tilde{K}(h,s,t) \geq -\frac{\mu_j}{\lambda_j}\Big(1+\frac{3200}{m}\Big)(t-s) + 2\sqrt{2}\Big(\frac{\mu_j}{\lambda_j}-\frac{1}{2}\Big)^{1/2}\lambda_j^{1/2}\Big(1-\frac{3200}{m}\Big)(t-s) \\ + 2\sqrt{\mu_j}(t-s) - \lambda_j(t-s) - \mu_j(t-s).$$

Recognising that

$$\kappa(\lambda_j,\mu_j) = -\frac{\mu_j}{\lambda_j} + 2\sqrt{2}\Big(\frac{\mu_j}{\lambda_j}-\frac{1}{2}\Big)^{1/2}\lambda_j^{1/2} + 2\mu_j^{1/2} - \lambda_j - \mu_j,$$

we see that

$$\tilde{K}(h,s,t) \geq \kappa(\lambda_j,\mu_j)(t-s) - \frac{3200\mu_j}{\lambda_j m}(t-s) - 2\sqrt{2}\Big(\frac{\mu_j}{\lambda_j} - \frac{1}{2}\Big)^{1/2}\lambda_j^{1/2}\frac{3200}{m}(t-s)$$

and the first part of the result follows using Lemma 3.29 (iv).

The proof of the second part is almost identical, though since for $u \in [\tau,\nu]$ we do not have exactly $h_X'(u) = \lambda$ and $h_Y'(u) = \mu$, we must additionally use the bounds

$$h_X'(u) = \frac{f(2\tau) - h(\tau)}{\tau} \leq \frac{2\lambda\tau + 2\tau/m^2 - \lambda\tau + 40\tau/m}{\tau} \leq \lambda + \frac{42}{m}$$

and similarly

$$h_X'(u) \geq \lambda - \frac{42}{m}$$

and

$$\mu - \frac{42}{m} \leq h_Y'(u) \leq \mu + \frac{42}{m}.$$

With the addition of these estimates, the proof proceeds as before. □

Corollary 3.32 essentially guarantees that $\tilde{K}(h_{f,n,m},s,t)$ is positive for $0 \leq s < t \leq \nu$, provided that $\kappa(\lambda,\mu) > 0$. We now need to show that $\tilde{K}(h_{f,n,m},\nu,t)$ is not too negative for $t \geq \nu$. The following result will be used to check that $\tilde{K}(f,\nu,t)$ is closely approximated by $\tilde{K}(h_{f,n,m},\nu,t)$.

Proposition 3.33. *Suppose that $0 \leq s \leq t \leq 1$ and that $f \in G_M^2$. Let f_n be the function in PL_n constructed by setting $f_n(j/n) = f(j/n)$ for each $j = 0,\dots,n$ and interpolating linearly. Then*

$$\liminf_{n\to\infty} \tilde{K}(f_n,s,t) \geq \tilde{K}(f,s,t).$$

We prove this in Appendix 3.B.2. Later, in Proposition 3.37, we will also show that the opposite inequality holds in certain circumstances. We now have the pieces in place to prove Proposition 3.15.

Proof of Proposition 3.15. As usual let $\lambda = f_X'(0)$, $\mu = f_Y'(0)$ and $\tau = m^m\lceil n^{7/8}\rceil/n$, with λ_j and μ_j as in (3.36) and $\tau_j = \tau m^{j-m}$, for $j \in \{0,1,\dots,m\}$. We will check that $h_{f,n,m}$ satisfies the desired properties when m and n are sufficiently large. Without loss of generality we assume that $\mu \geq \lambda$.

Since $\tau_0 n$ is an integer we have $h_{f,n,m} \in \mathrm{PL}_n^2$, and since $f \in G_M^2$ it is easy to see that $h_{f,n,m} \in G_M^2$ too. Since $\|h_{f,n,m}(s)\| \leq Ms$ and $\|f(s)\| \leq Ms$ for $s \leq 2\tau = 2m^m\lceil n^{7/8}\rceil/n$, and $h_{f,n,m}(j/n) = f(j/n)$ for $j \geq 2\tau n$, by choosing n large enough that $2\tau M \leq \varepsilon$ we have $h_{f,n,m} \in B(f,\varepsilon)$. This proves that $h_{f,n,m}$ satisfies (3.7) when n is large.

For (3.8), note first that $\tau_0 = \lceil n^{7/8}\rceil/n$. Take n large enough that $2\tau < \nu$. Then we claim that since $\lim_{t\to 0}\tilde{K}(f,0,t)/t > 0$, we have $\kappa(\lambda,\mu) > 0$. To see why the claim holds, for small s we have $\|f'(s) - (\lambda,\mu)\| \leq 1/m$ and $\|f(s) - (\lambda s,\mu s)\| \leq s/m$. The same argument as in Corollary 3.31 then shows that for some finite constant C,

$$\frac{\mu}{\lambda}\Big(1-\frac{C}{m}\Big) \le R^*_X(f(s)) + \frac{1}{2} \le R^*(f(s)) \le \frac{\mu}{\lambda}\Big(1+\frac{C}{m}\Big)$$

and plugging these estimates into (3.3) with $a = 0$ and $b = t$ and using standard approximations shows that $\tilde{K}(f,0,t) \le \kappa(\lambda,\mu)t + C't/m$ for some finite constant C'. This implies the claim.

Since $\kappa(\lambda,\mu) > 0$, by Lemma 3.29 we may choose $\kappa_0 > 0$ such that $\kappa(\gamma(t)) \ge \kappa_0$ for all $t \in [0,1]$, and then $\kappa(\lambda_j,\mu_j) \ge \kappa_0$ for all $j \in \{0,\dots,m\}$. Corollary 3.32 then tells us that for $s \in [\tau_{j-1},\tau_j]$ we have

$$\begin{aligned}\tilde{K}(h_{f,n,m},\tau_0,s) &\ge \sum_{i=1}^{j-1}(\kappa(\lambda_i,\mu_i) - C/m)(\tau_i - \tau_{i-1}) + (\kappa(\lambda_j,\mu_j) - C/m)(s - \tau_{j-1})\\ &\ge (\kappa_0 - C/m)(s-\tau_0),\end{aligned}$$

and for $s \in [\tau,\nu]$ we have

$$\begin{aligned}\tilde{K}(h_{f,n,m},\tau_0,s) &\ge \sum_{i=1}^{m}(\kappa(\lambda_i,\mu_i) - C/m)(\tau_i - \tau_{i-1}) + (\kappa(\lambda,\mu) - C/m)(s - \tau)\\ &\ge (\kappa_0 - C/m)(s-\tau_0).\end{aligned}$$

Thus, by choosing m large enough, we may ensure that $\tilde{K}(h_{f,n,m},\tau_0,s) \ge \kappa_0(s-\tau_0)/2$ for all $s \in [\tau_0,\nu]$.

For $s > \nu$, by the above argument we have

$$\begin{aligned}\tilde{K}(h_{f,n,m},\tau_0,s) &\ge \tilde{K}(h_{f,n,m},\tau_0,\nu) + \tilde{K}(h_{f,n,m},\nu,s)\\ &\ge \kappa_0(\nu-\tau_0)/2 + \tilde{K}(h_{f,n,m},\nu,s),\end{aligned} \tag{3.40}$$

and since $\kappa_0(\nu-\tau_0)/2$ increases to $\kappa_0\nu/2$ as $n\to\infty$ and $\tilde{K}(f,\nu,s) > 0$ by (b), to show (3.8) it suffices to show that for large n,

$$\tilde{K}(h_{f,n,m},\nu,s) \ge \tilde{K}(f,\nu,s) - \kappa_0\nu/4.$$

But since h is the piecewise linear interpolation of f on the interval $[\nu,1]$, this follows from Proposition 3.33.

Finally, for (3.9), applying (3.40) with $s = 1$, we certainly have

$$\tilde{K}(h_{f,n,m},\tau_0,1) \ge \tilde{K}(h_{f,n,m},\nu,1);$$

by Proposition 3.33 the right-hand side converges to $\tilde{K}(f,\nu,1)$ as $n\to\infty$; and by (c) we know that $\tilde{K}(f,\nu,1) \ge \tilde{K}(f,0,1) - 1/m$. This completes the proof. □

3.6. Coupling ξ^T with simpler processes

One problem we face is that ξ_X and ξ_Y are not independent, because their jump rates at time t are functions of the pair $(\xi_X(t), \xi_Y(t))$. However, if we already know that ξ^T has remained near a fixed function f, then the jump rates are "almost deterministic" and therefore ξ_X and ξ_Y are "almost independent". In order to take advantage of this idea, we will construct new processes Z_+ and Z_- which have the maximal and minimal jump rates (respectively) that ξ^T may have if it remains near f. We will couple these processes with another process, Z, which will have the same distribution as ξ^T but will be trapped between Z_+ and Z_-, as long as Z remains near f.
Recall the definitions of $R_X^-(I,F,T)$, $R_X^+(I,F,T)$, $R_Y^-(I,F,T)$, $R_Y^+(I,F,T)$, $|I|$, I^+, I^-, $x^-(s,F)$, $x^+(s,F)$, $y^-(s,F)$, $y^+(s,F)$, $\Gamma_{M,T}(f,n)$ and I_j from Section 3.2.1. In what follows, the reader can think of the case $I = I_j$ and $F = \Gamma_{M,T}(f,n)$ for some function f.

Let

$$V(I,F) = [x^-(I^-,F), x^+(I^-,F)] \times [y^-(I^-,F), y^+(I^-,F)].$$

Take $z = (x,y) \in V(I,F)$. Under a probability measure $Q_z = Q_z^{I,F,T}$, let $(X_+(I^- + s), s \in |I|)$ be a compound Poisson process started from x with rate $2R_X^+(I,F,T)T$ and jumps that are exponentially distributed with parameter T, and let $(Y_+(I^- + s), s \in |I|)$ be an independent compound Poisson process started from y with rate $2R_Y^+(I,F,T)T$ and jumps that are exponentially distributed with parameter T. Let $Z_+ = (X_+, Y_+)$.

We now construct—again under $Q_z^{I,F,T}$—two more (pure jump) processes $Z(I^- + s)$ and $Z_-(I^- + s)$ for $s \in |I|$ recursively as follows. Start by setting $Z(I^-) = z$ and $Z_-(I^-) = z$. The jumps of both Z and Z_- are subsets of the jumps of Z_+. Suppose that Z_+ has a jump at time s, and that $Z(s-) = z'$. Let U be an independent Uniform$[0,1]$ random variable. Since X_+ and Y_+ are independent, exactly one of X_+ or Y_+ jumps at time s. Suppose for a moment that X_+ has a jump of size $x' > 0$. Then accept the jump for Z if $U \le R_X(z')/R_X^+(I,F,T)$ and reject it otherwise; in other words, set $Z(s) = z' + (x', 0)$ with probability $R_X(z')/R_X^+(I,F,T)$ and $Z(s) = z'$ otherwise. Accept the jump for Z_- if $U \le R_X^-(I,F,T)/R_X^+(I,F,T)$ and reject it otherwise. Similarly, if Y_+ has a jump of size $y > 0$, then accept the jump for Z if $U \le R_Y(z')/R_Y^+(I,F,T)$, and accept it for Z_- if $U \le R_Y^-(I,F,T)/R_Y^+(I,F,T)$.

Recall that for $F \subset E^2$, $g \in E$ and an interval $I \subset [0,1]$, we say that $g|_I \in F|_I$ if there exists a function $h \in F$ such that $h(u) = g(u)$ for all $u \in I$. Let

$$\mathcal{A}_\xi(I,F,T) = \{\xi^T|_I \in F|_I\}$$

and

$$\mathcal{A}(I,F,T) = \{Z|_I \in F|_I\}.$$

Note that for any $z \in V(I,F)$, on the event $\mathcal{A}(I,F,T)$, under $Q_z^{I,F,T}$ we always have

$$R_X(Z(s)) \in [R_X^-(I,F,T), R_X^+(I,F,T)] \quad \text{and} \quad R_Y(Z(s)) \in [R_Y^-(I,F,T), R_Y^+(I,F,T)],$$

for all $s \in I$. Thus, by our construction:

(i) under $Q_z^{I,F,T}$, on the event $\mathcal{A}(I,F,T)$, we have $X_-(s) \le X(s) \le X_+(s)$ and $Y_-(s) \le Y(s) \le Y_+(s)$ for all $s \in I$;

(ii) the process $(Z(s)\mathbb{1}_{\mathcal{A}(I\cap[0,s],F,T)})_{s\in I}$ under $Q_z^{I,F,T}$ is equal in distribution to the process $(\xi^T(s)\mathbb{1}_{\mathcal{A}_\xi(I\cap[0,s],F,T)})_{s\in I}$ conditionally on $\xi^T(I^-)=z$ under $\mathbb{Q}$;

(iii) under $Q_z^{I,F,T}$, the processes (X_-,X_+) and (Y_-,Y_+) are independent.

Furthermore, by the thinning property of Poisson processes,

(iv) under $Q_z^{I,F,T}$, the processes X_- and X_+-X_- are independent, as are Y_- and Y_+-Y_-.

3.6.1 Applying the coupling to the upper bound: proof of Propositions 3.4 and 3.21

Recall the terminology "$X+$ case" and "$X-$ case" from Section 3.2.1, and the definitions of $\mathcal{E}_X^+(I,F,T)$ and $\mathcal{E}_Y^+(I,F,T)$. The main part of the proof of Proposition 3.4 is the following lemma.

Lemma 3.34. *Suppose that $F\subset E^2$ and $T>1$. Then for any $I\subset[0,1]$ and $z\in V(I,F)$,*

$$\mathbb{Q}\big(\mathcal{A}_\xi(I,F,T)\,\big|\,\xi^T(I^-)=z\big)\le\exp\big(-T\mathcal{E}_X^+(I,F,T)-T\mathcal{E}_Y^+(I,F,T)\big).$$

Proof. For $z\in V(I,F)$, using (ii), (i) and (iii) in that order,

$$\begin{aligned}\mathbb{Q}\big(&\mathcal{A}_\xi(I,F,T)\,\big|\,\xi^T(I^-)=z\big)\\&=Q_z^{I,F,T}\big(\mathcal{A}(I,F,T)\big)\\&\le Q_z^{I,F,T}\big(X_-(I^+)\le x^+(I^+,F),\ Y_-(I^+)\le y^+(I^+,F)\big)\\&=Q_z^{I,F,T}\big(X_-(I^+)\le x^+(I^+,F)\big)Q_z^{I,F,T}\big(Y_-(I^+)\le y^+(I^+,F)\big).\end{aligned}$$

We will apply this bound when we are in the $X-$ and $Y-$ cases. Of course, we were not forced to concentrate on the two upper boundaries $x^+(I^+,F)$ and $y^+(I^+,F)$, and by considering the other permutations of boundaries we obtain upper bounds on the same quantity of the form

$$Q_z^{I,F,T}\big(X_+(I^+)\ge x^-(I^+,F)\big)Q_z^{I,F,T}\big(Y_+(I^+)\ge y^-(I^+,F)\big),$$

$$Q_z^{I,F,T}\big(X_+(I^+)\ge x^-(I^+,F)\big)Q_z^{I,F,T}\big(Y_-(I^+)\le y^+(I^+,F)\big)$$

and

$$Q_z^{I,F,T}\big(X_-(I^+)\le x^+(I^+,F)\big)Q_z^{I,F,T}\big(Y_+(I^+)\ge y^-(I^+,F)\big)$$

which we can apply in other cases as appropriate. Now, for any $\lambda>0$, by Markov's inequality,

$$\begin{aligned}Q_z^{I,F,T}&\big(X_-(I^+)\le x^+(I^+,F)\big)\\&=Q_z^{I,F,T}\big(e^{-\lambda X_-(I^+)}\ge e^{-\lambda x^+(I^+,F)}\big)\\&\le Q_z^{I,F,T}\big[e^{-\lambda(X_-(I^+)-X_-(I^-))}\big]e^{\lambda(x^+(I^+,F)-x^-(I^-,F))}\\&=\exp\Big(-2R_X^-(I,F,T)T|I|\frac{\lambda}{T+\lambda}+\lambda(x^+(I^+,F)-x^-(I^-,F))\Big).\end{aligned}$$

In the $X-$ case we have $2R_X^-(I,F,T)|I| > x^+(I^+,F) - x^-(I^-,F)$, so we can choose the optimal value

$$\lambda = T\sqrt{\frac{2R_X^-(I,F,T)|I|}{x^+(I^+,F) - x^-(I^-,F)}} - T > 0.$$

Simplifying gives

$$\begin{aligned}
&Q_z^{I,F,T}\big(X_-(I^+) \le x^+(I^+,F)\big) \\
&\qquad\qquad \le \exp\Big(- T\Big(\sqrt{2R_X^-(I,F,T)|I|} - \sqrt{x^+(I^+,F) - x^-(I^-,F)}\Big)^2\Big)
\end{aligned}$$

which equals $\exp\big(-T\mathcal{E}_X^+(I,F,T)\big)$ in the $X-$ case. Similarly, in the $X+$ case, by using

$$\begin{aligned}
&Q_z^{I,F,T}\big(X_+(I^+) \ge x^-(I^+,F)\big) \\
&\qquad \le Q_z^{I,F,T}[\exp\big(\mu(X_+(I^+) - X_+(I^-))\big) - \mu(x^-(I^+,F) - x^+(I^-,F))]
\end{aligned}$$

for $\mu > 0$, we obtain

$$\begin{aligned}
&Q_z^{I,F,T}\big(X_+(I^+) \ge x^-(I^+,F)\big) \\
&\qquad\qquad \le \exp\Big(- T\Big(\sqrt{2R_X^+(I,F,T)|I|} - \sqrt{x^-(I^+,F) - x^+(I^-,F)}\Big)^2\Big) \\
&\qquad\qquad = \exp\big(- T\mathcal{E}_X^+(I,F,T)\big)
\end{aligned}$$

and when we are in neither the $X-$ nor $X+$ case we can use a trivial upper bound of 1. By symmetry we obtain the same bounds in terms of Y. Applying these bounds in the appropriate cases completes the proof. $\square$

Our main results in this section are now easy corollaries of Lemma 3.34.

Proof of Proposition 3.4. Recall that $I_j = [j/n, (j+1)/n]$ and let $V(j) = V(I_j, \Gamma_{M,T}(f,n))$. Note that the restrictions on z ensure that $z \in V(i)$, and therefore by the Markov property,

$$\begin{aligned}
&\mathbb{Q}\big(\xi^T|_{[i/n,\theta]} \in \Gamma_{M,T}(f,n)\big|_{[i/n,\theta]} \,\big|\, \xi^T_{i/n} = z\big) \\
&\qquad\qquad\qquad\qquad \le \prod_{j=i}^{\lfloor \theta n\rfloor - 1} \sup_{z'\in V(j)} \mathbb{Q}\big(\xi^T|_{I_j} \in \Gamma_{M,T}(f,n)|_{I_j} \,\big|\, \xi^T_{j/n} = z'\big) \\
&\qquad\qquad\qquad\qquad = \prod_{j=i}^{\lfloor \theta n\rfloor - 1} \sup_{z'\in V(j)} \mathbb{Q}\big(\mathcal{A}_\xi(I_j, \Gamma_{M,T}(f,n), T) \,\big|\, \xi^T_{j/n} = z'\big).
\end{aligned}$$

The result now follows from Lemma 3.34. $\square$

Proof of Proposition 3.21. Let $i = \lfloor an\rfloor$ and $\ell = \lceil bn\rceil$. Let $V_i = \{w : \|w - f(a)\| < 1/n^2\}$, and for $j \in \{i+1, \dots, \ell\}$ let $V_j = \{w : \|w - f(j/n)\| < 1/n^2\}$. Note that, by the Markov property,

$$
\begin{aligned}
\mathbb{Q}\big(\xi^T|_{[a,b]} \in \Lambda_{M,T}(f,n)\big|_{[a,b]} \,\big|\, \xi^T_a = z\big) \\
&\le \prod_{j=i}^{\ell-1} \sup_{w\in V_j} \mathbb{Q}\big(\xi^T|_{I_j\cap[a,b]} \in \Lambda_{M,T}(f,n)\big|_{I_j\cap[a,b]} \,\big|\, \xi^T_{j/n} = w\big) \\
&= \prod_{j=i}^{\ell-1} \sup_{w\in V_j} \mathbb{Q}\big(\mathcal{A}_\xi(I_j\cap[a,b], \Lambda_{M,T}(f,n), T) \,\big|\, \xi^T_{j/n} = w\big).
\end{aligned}
$$

The result now follows from Lemma 3.34, together with (3.12) and (3.13). □

3.6.2 Applying the coupling to the lower bound: proofs of Lemmas 3.16, 3.17 and 3.18

We begin this section with the proof of Lemma 3.16, which links the probability that we want to bound with our coupled compound Poisson processes.

Proof of Lemma 3.16. We begin by splitting $[0,1]$ into its subintervals I_j, $j = 0,\ldots,n-1$. Recall a previous definition: let $\mathcal{Z}_0 = \{(0,0)\}$ and, for $j \in \{1,\ldots,n-1\}$, define

$$\mathcal{Z}_j = \{z \in [0,\infty)^2 : \|z - f(j/n)\| \le \tfrac{1}{2n^2}\}.$$

By applying the Markov property at each time j/n,

$$
\begin{aligned}
&\mathbb{Q}\Big(\xi^T|_{[k/n,1]} \in \Lambda_{M,T}(f,n)|_{[k/n,1]} \,\Big|\, \xi^T(k/n) = w\Big) \\
&\ge \mathbb{Q}\Big(\|\xi^T(s) - f(s)\| < 1/n^2 \;\; \forall s \in I_j,\; \xi^T(\tfrac{j+1}{n}) \in \mathcal{Z}_{j+1}, \\
&\qquad\qquad \xi^T|_{I_j} \in G^2_{M,T}|_{I_j} \;\; \forall j \in \{k,\ldots,n-1\} \,\Big|\, \xi^T(\tfrac{k}{n}) = w\Big) \\
&\ge \prod_{j=k}^{n-1} \inf_{z\in\mathcal{Z}_j} \mathbb{Q}\Big(\big\|\xi^T(s) - f(s)\big\| < 1/n^2 \;\; \forall s \in I_j,\; \xi^T(\tfrac{j+1}{n}) \in \mathcal{Z}_{j+1}, \\
&\qquad\qquad \xi^T|_{I_j} \in G^2_{M,T}|_{I_j} \,\Big|\, \xi^T(\tfrac{j}{n}) = z\Big).
\end{aligned}
$$

It therefore remains to show that for each j and any $z \in \mathcal{Z}_j$,

$$
\begin{aligned}
&\mathbb{Q}\Big(\big\|\xi^T(s) - f(s)\big\| < 1/n^2 \;\; \forall s \in I_j,\; \xi^T(\tfrac{j+1}{n}) \in \mathcal{Z}_{j+1},\; \xi^T|_{I_j} \in G^2_{M,T}|_{I_j} \,\Big|\, \xi^T(\tfrac{j}{n}) = z\Big) \\
&\qquad \ge q^X_{n,M,T}(z,j,f)\, \hat{q}^X_{n,M,T}(z,j,f)\, q^Y_{n,M,T}(z,j,f)\, \hat{q}^Y_{n,M,T}(z,j,f). \qquad (3.41)
\end{aligned}
$$

We now use the coupling from Section 3.6, with $I = I_j$ and $F = \Lambda_{M,T}(f,n)$. We simply write Q_z as shorthand for $Q_z^{I_j,\Lambda_{M,T}(f,n),T}$. By property (ii) of the coupling, we have

$$
\begin{aligned}
&\mathbb{Q}\Big(\big\|\xi^T(s) - f(s)\big\| < 1/n^2 \;\; \forall s \in I_j,\; \xi^T(\tfrac{j+1}{n}) \in \mathcal{Z}_{j+1},\; \xi^T|_{I_j} \in G^2_{M,T}|_{I_j} \,\Big|\, \xi^T(\tfrac{j}{n}) = z\Big) \\
&\qquad = Q_z\Big(\big\|Z(s) - f(s)\big\| \le 1/n^2 \;\; \forall s \in I_j,\; Z(\tfrac{j+1}{n}) \in \mathcal{Z}_{j+1},\; Z|_{I_j} \in G^2_{M,T}|_{I_j}\Big)
\end{aligned}
$$

which, by property (i), is at least

$$Q_z\Big(\big\|Z_-(s)-f(s)\big\|\le \tfrac{1}{n^2}\ \ \forall s\in I_j,\ Z_-(\tfrac{j+1}{n})\in\mathcal{Z}_{j+1},\ Z_-|_{I_j}\in G^2_{M,T}|_{I_j},\\ Z_+(s)-Z_-(s)=0\ \ \forall s\in I_j\Big).$$

By property (iv), this equals

$$Q_z\Big(\big\|Z_-(s)-f(s)\big\|\le \tfrac{1}{n^2}\ \ \forall s\in I_j,\ Z_-(\tfrac{j+1}{n})\in\mathcal{Z}_{j+1},\ Z_-|_{I_j}\in G^2_{M,T}|_{I_j}\Big)\\ \cdot Q_z\Big(Z_+(s)-Z_-(s)=0\ \ \forall s\in I_j\Big)$$

and finally, by property (iii), the above equals

$$Q_z\Big(\big|X_-(s)-f_X(s)\big|\le \tfrac{1}{n^2}\ \ \forall s\in I_j,\ \big|X_-(\tfrac{j+1}{n})-f_X(\tfrac{j+1}{n})\big|\le\tfrac{1}{2n^2},\ X_-|_{I_j}\in G_{M,T}|_{I_j}\Big)\\ \cdot Q_z\Big(\big|Y_-(s)-f_Y(s)\big|\le \tfrac{1}{n^2}\ \ \forall s\in I_j,\ \big|Y_-(\tfrac{j+1}{n})-f_Y(\tfrac{j+1}{n})\big|\le\tfrac{1}{2n^2},\ Y_-|_{I_j}\in G_{M,T}|_{I_j}\Big)\\ \cdot Q_z\Big(X_+(s)-X_-(s)=0\ \ \forall s\in I_j\Big)\cdot Q_z\Big(Y_+(s)-Y_-(s)=0\ \ \forall s\in I_j\Big).$$

Noting that X_+-X_- and Y_+-Y_- are increasing, this is exactly

$$q^X_{n,M,T}(z,j,f)\,\hat{q}^X_{n,M,T}(z,j,f)\,q^Y_{n,M,T}(z,j,f)\,\hat{q}^Y_{n,M,T}(z,j,f).$$

Thus we have shown (3.41) and the proof is complete. □

The proof of Lemma 3.17, which bounds the $\hat{q}$ terms, is elementary.

Proof of Lemma 3.17. Recall that

$$\hat{q}^X_{n,M,T}(z,j,f)=Q_z^{I_j,\Lambda_{M,T}(f,n),T}\Big(X_+(\tfrac{j+1}{n})-X_-(\tfrac{j+1}{n})=0\Big).$$

Also recall that under $Q_z^{I_j,\Lambda_{M,T}(f,n),T}$, the process X_+-X_- jumps at rate

$$2\big(R^+_X(I_j,\Lambda_{M,T}(f,n),T)-R^-_X(I_j,\Lambda_{M,T}(f,n),T)\big)T.$$

Therefore, for each $j\in\{0,\dots,n-1\}$ and $z\in\mathcal{Z}_j$, using (3.13),

$$\hat{q}^X_{n,M,T}(z,j,f)\ge\exp\Big(-2\big(R^+_X(I_j,\Gamma_{M,T}(f,n),T)-R^-_X(I_j,\Gamma_{M,T}(f,n),T)\big)T/n\Big).$$

Since $f\in G^2_M$, for $j\ge\sqrt{n}$, by (3.14) we have

$$\hat{q}^X_{n,M,T}(z,j,f)\ge\exp\big(-4\delta_{M,T}(j,n)T/n\big),$$

and by symmetry

$$\hat{q}^Y_{n,M,T}(z,j,f)\ge\exp\big(-4\delta_{M,T}(j,n)T/n\big).$$

The result then follows from the deterministic bounds from (3.61), in Appendix 3.A. □

The proof of Lemma 3.18 is much more delicate. Our next result provides a bound on compound Poisson processes. This will then be applied to prove Lemma 3.18.

Lemma 3.35. *Suppose that $\delta, t, A > 0$ and $a \in \mathbb{R}$ satisfy $a < tA/2$ and $|a| \leq \delta/2$. Suppose also that $R \geq 1/2$. Let $(X(s), s \geq 0)$ be a compound Poisson process of rate RT whose jumps are exponentially distributed with parameter T.*
Then for $T > \frac{2(A-a/t)^{3/2}(4t+\delta)}{R^{1/2}\delta^2((A-a/t)\wedge 1)^2}$,

$$\begin{aligned}\mathbb{P}(|a + X(s) - As| &< \delta\ \forall s \leq t,\ |a + X(t) - At| < \delta/2) \\ &\geq \frac{1}{2}\exp\Big(-tT(\sqrt{R}-\sqrt{A})^2 - \delta\Big(1+\sqrt{R}\big(\sqrt{2t/\delta}+1/2\big)\Big)T\Big).\end{aligned}$$

Proof. The bulk of the work to prove Lemma 3.35 is done by showing the following intermediate result. For $T > \frac{2A^{3/2}(4t+\delta)}{R^{1/2}\delta^2(A\wedge 1)^2}$,

$$\mathbb{P}(|X(s)-As| < \delta\ \forall s \leq t) \geq \frac{1}{2}\exp\Big(-tT(\sqrt{R}-\sqrt{A})^2 - \delta(A\wedge 1)\big|1-\sqrt{R/A}\big|T\Big). \quad (3.42)$$

For any $q < T$ and $s \geq 0$, we have

$$\mathbb{E}[e^{qX(s)}] = \exp\Big(\frac{Rqs}{1-q/T}\Big).$$

Fix $a = T(1-\sqrt{R/A})$; then elementary calculations show that

$$\frac{\mathbb{E}[X(s)e^{aX(s)}]}{\mathbb{E}[e^{aX(s)}]} = As.$$

Let $(\sigma_s)_{s\geq 0}$ be the natural filtration of X, and define a new probability measure μ by setting

$$\frac{d\mu}{d\mathbb{P}}\Big|_{\sigma_s} = \frac{e^{aX(s)}}{\mathbb{E}[e^{aX(s)}]} = \exp\Big(aX(s) - \frac{Ras}{1-a/T}\Big).$$

Then, by the definition of μ, for any $\delta' > 0$ we have

$$\mathbb{P}(|X(s)-As| < \delta'\ \ \forall s \leq t) = \mu\Big[\exp\Big(-aX(t) + \frac{Rat}{1-a/T}\Big)\mathbb{1}_{\{|X(s)-As|<\delta'\ \ \forall s\leq t\}}\Big]$$

and using the bound $|X(t) - At| < \delta'$ and simplifying we obtain

$$\begin{aligned}\mathbb{P}(|X(s) - As| &< \delta'\ \ \forall s \leq t) \\ &\geq \exp\Big(-tT(\sqrt{R}-\sqrt{A})^2 - |a|\delta'\Big)\mu(|X(s)-As| < \delta'\ \ \forall s \leq t). \quad (3.43)\end{aligned}$$

Taking $\delta' = \delta(A\wedge 1)$, it remains to bound $\mu(|X(s) - As| < \delta(A\wedge 1)\ \ \forall s \leq t)$ from below.

One may easily check that $(X(s) - As, s \geq 0)$ is a martingale under μ, and therefore by Jensen's inequality, $(e^{\nu(X(s)-As)}, s \geq 0)$ is a submartingale under μ for any $\nu < T - a$ (the upper bound on ν is required to ensure that the expectation is finite). By Doob's submartingale inequality, for any $\nu \in (0, T-a)$, we have

$$\begin{aligned}\mu(\exists s \le t : X(s) - As \ge \delta(A\wedge 1)) &= \mu\Big(\sup_{s\le t} e^{\nu(X(s)-As)} \ge e^{\nu\delta(A\wedge 1)}\Big)\\ &\le \mu[e^{\nu(X(t)-At)}]e^{-\nu\delta(A\wedge 1)}\end{aligned} \tag{3.44}$$

and for any $\nu < 0$ we have

$$\begin{aligned}\mu(\exists s \le t : X(s) - As \le -\delta(A\wedge 1)) &= \mu\Big(\sup_{s\le t} e^{\nu(X(s)-As)} \ge e^{-\nu\delta(A\wedge 1)}\Big)\\ &\le \mu[e^{\nu(X(t)-At)}]e^{\nu\delta(A\wedge 1)}.\end{aligned} \tag{3.45}$$

Now, for any $\nu < T-a$,

$$\mu[e^{\nu(X(t)-At)}] = \mathbb{E}[e^{(a+\nu)X(t)}]e^{-Rat/(1-a/T)-A\nu t} = \exp\Big(\frac{R(a+\nu)t}{1-(a+\nu)/T} - \frac{Rat}{1-a/T} - A\nu t\Big),$$

and simplifying we obtain

$$\mu[e^{\nu(X(t)-At)}] = \exp\Big(\frac{A\nu t}{1-\frac{\nu}{T}(A/R)^{1/2}} - A\nu t\Big) = \exp\Big(\frac{A\nu^2 t}{(R/A)^{1/2}T - \nu}\Big).$$

It is then easy to check that for $T > \frac{2A^{3/2}(4t+\delta)}{R^{1/2}\delta^2(A\wedge 1)^2}$, each of the probabilities in (3.44) and (3.45) can be made smaller than $e^{-3/2} < 1/4$ by choosing $\nu = \pm\frac{2}{\delta(A\wedge 1)}$. Thus we have

$$\mu(|X(s) - As| < \delta(A\wedge 1)\ \ \forall s \le t) \ge 1/2$$

for such T. Substituting this into the lower bound in (3.43), using $\delta' = \delta(A\wedge 1)$, gives (3.42).

Lemma 3.35 now requires us to deal with the position of our process at the endpoints of the intervals I_j.

Let $A_t(a) = A - a/t$. Note that if $|X(s) - A_t(a)s| < \delta/2$ for all $s \le t$, then $|a + X(s) - As| < \delta$ for all $s \le t$ and $|a + X(t) - At| < \delta/2$. Thus

$$\mathbb{P}(|a+X(s)-As| < \delta\ \forall s \le t,\ |a+X(t)-At| < \delta/2) \ge \mathbb{P}(|X(s)-A_t(a)s| < \delta/2\ \forall s \le t).$$

Now (3.42) tells us that the latter probability is at least

$$\frac{1}{2}\exp\Bigg(-tT(\sqrt{R}-\sqrt{A_t(a)})^2 - \frac{\delta(A_t(a)\wedge 1)}{2}\Big|1-\Big(\frac{R}{A_t(a)}\Big)^{1/2}\Big|T\Bigg). \tag{3.46}$$

Using the fact that $(1-x)^{1/2} \ge 1 - x^{1/2}$ for $x \in [0,1]$, we have

$$\begin{aligned}t(\sqrt{R}-\sqrt{A_t(a)})^2 &\le \Big(R + A + \frac{\delta}{2t} - 2\sqrt{AR}\Big(1-\frac{\delta}{2tA}\Big)^{1/2}\Big)t\\ &\le (\sqrt{R}-\sqrt{A})^2 t + \frac{\delta}{2} + \sqrt{2\delta R t}.\end{aligned}$$

and
$$\frac{\delta(A_t(a)\wedge 1)}{2}\Big|1-\Big(\frac{R}{A_t(a)}\Big)^{1/2}\Big| \le \frac{\delta}{2}\Big(1+\Big(\frac{R}{A_t(a)}\Big)^{1/2}\Big) \le \frac{\delta}{2}(1+\sqrt{R}).$$

Substituting these estimates into (3.46) gives

$$\frac{1}{2}\exp\bigg(-tT(\sqrt{R}-\sqrt{A})^2-\delta\Big(1+\sqrt{R}\big(\sqrt{2t/\delta}+1/2\big)\Big)T\bigg),$$

as required. □

We now apply Lemma 3.35 to prove Lemma 3.18. However we still need to consider two cases: if f_X does not change much over the interval I_j then we may simply ask our process not to jump over that interval, and a bound similar to that in the proof of Lemma 3.17 is better than the estimate provided by Lemma 3.35.

Proof of Lemma 3.18. Recall that

$$\begin{aligned} q^X_{n,3M,T}(z,j,f) = Q_z^{I_j,\Lambda_{3M,T}(f,n),T}\Big(&\big|X_-(s)-f_X(s)\big| \le \tfrac{1}{n^2}\ \ \forall s\in I_j,\\ &\big|X_-(\tfrac{j+1}{n})-f_X(\tfrac{j+1}{n})\big| \le \tfrac{1}{2n^2},\ X_-|_{I_j}\in G_{3M,T}|_{I_j}\Big) \end{aligned}$$

Write Q_z as shorthand for $Q_z^{I_j,\Lambda_{3M,T}(f,n),T}$.

Since $f\in G^2_M$, $j\ge 1$ and $n\ge 2M$, under Q_z we also have, for any $s\in I_j$,

$$\begin{aligned} X_-(s)\ge x\ge f_X\Big(\frac{j}{n}\Big)-\frac{1}{2n^2}\ge\frac{j}{Mn}-\frac{1}{2n^2}\ge\frac{3j}{3Mn}-\frac{1}{3Mn}\ge\frac{j+2}{3Mn}-\frac{1}{3Mn}&=\frac{j+1}{3Mn}\\ &\ge\frac{s}{3M}, \end{aligned}$$

and if $\big|X_-(\frac{j+1}{n})-f_X(\frac{j+1}{n})\big|\le\frac{1}{2n^2}$ then also

$$X_-(s)\le X_-(\tfrac{j+1}{n})\le f_X(\tfrac{j+1}{n})+\tfrac{1}{2n^2}\le M\tfrac{j+1}{n}+\tfrac{1}{2n^2}\le 3Ms.$$

Thus in fact, under the conditions of the lemma, $X_-|_{I_j}$ is always in $G_{3M,T}|_{I_j}$, so

$$\begin{aligned} &q^X_{n,3M,T}(z,j,f)\\ &\quad= Q_z^{I_j,\Lambda_{3M,T}(f,n),T}\Big(\big|X_-(s)-f_X(s)\big|\le\tfrac{1}{n^2}\ \ \forall s\in I_j,\ \big|X_-(\tfrac{j+1}{n})-f_X(\tfrac{j+1}{n})\big|\le\tfrac{1}{2n^2}\Big). \end{aligned} \tag{3.47}$$

For the remainder of this proof, for $I\subset[0,1]$, we write $\hat{R}^-_X(I)$ as shorthand for $R^-_X(I,\Lambda_{M,T}(f,n),T)$, and similarly for $\hat{R}^+_X(I)$, $\hat{R}^-_Y(I)$ and $\hat{R}^-_Y(I)$. (Recall that we wrote $R^-_X(I)$ in Section 3.6 and Appendix 3.A to mean $R^-_X(I,\Gamma_{M,T}(f,n),T)$.)

Case 1: $f_X(\frac{j+1}{n})\le x+\frac{1}{2n^2}$.
Note that since $z\in\mathcal{Z}_j$, we have $x\le f_X(\frac{j}{n})+\frac{1}{2n^2}\le f_X(\frac{j+1}{n})+\frac{1}{2n^2}$, and therefore $|x-f_X(s)|\le\frac{1}{2n^2}$ for all $s\in I_j$. Thus (3.47) can be bounded in the following trivial way:

$$q^X_{n,3M,T}(z,j,f)\ge Q_z\big(X_-(s)=x\ \ \forall s\in I_j\big)=Q_z\big(X_-(\tfrac{j+1}{n})=x\big).$$

Under Q_z, X_- jumps at rate $2\hat{R}_X^-(j)T$, so we deduce that

$$q^X_{n,3M,T}(z,j,f) \geq \exp\Big(-\frac{2\hat{R}_X^-(j)T}{n}\Big). \tag{3.48}$$

On the other hand we have

$$\begin{aligned}\int_{j/n}^{(j+1)/n} &\Big(\sqrt{2R_X^*(f(s))} - \sqrt{f_X'(s)}\Big)^2 ds \\ &\geq 2\int_{j/n}^{(j+1)/n} R_X^*(f(s))ds - 2\int_{j/n}^{(j+1)/n} \sqrt{2R_X^*(f(s))n\big(f_X(\tfrac{j+1}{n}) - f_X(\tfrac{j}{n})\big)}ds \\ &\qquad + f_X(\tfrac{j+1}{n}) - f_X(\tfrac{j}{n}).\end{aligned}$$

By (3.14),

$$R_X^*(f(s)) \geq \hat{R}_X^+(j) - \delta_{M,T}(j,n) \geq \hat{R}_X^-(j) - \delta_{M,T}(j,n),$$

so since $f_X(\frac{j+1}{n}) - f_X(\frac{j}{n}) \leq \frac{1}{n^2}$ and $R_X^*(f(s)) \leq M$ for all s,

$$\int_{j/n}^{(j+1)/n} \Big(\sqrt{2R_X^*(f(s))} - \sqrt{f_X'(s)}\Big)^2 ds \geq \frac{2\hat{R}_X^-(j) - 2\delta_{M,T}(j,n)}{n} - \frac{2\sqrt{2}M^{1/2}}{n^{3/2}}.$$

The result now follows from this and (3.48).

Case 2: $f_X(\frac{j+1}{n}) > x + \frac{1}{2n^2}$.
Note that X_- jumps at rate $2\hat{R}_X^-(j)T$ and has exponential jumps of parameter T under Q_z. We therefore aim to apply Lemma 3.35, with $A = n(f_X(\frac{j+1}{n}) - f_X(\frac{j}{n}))$, $\delta = 1/n^2$, $t = 1/n$ and $a = x - f_X(j/n)$. We need to check that $a < tA/2$; to see this, note that since $z \in \mathcal{Z}(j)$ and we are in Case 2,

$$2a = 2\big(x - f_X(\tfrac{j}{n})\big) \leq \tfrac{1}{2n^2} + x - f_X(\tfrac{j}{n}) < f_X(\tfrac{j+1}{n}) - f_X(\tfrac{j}{n}) = tA.$$

It is also easy to check that for $T > 8n^{9/2}M^{3/2}$, T is large enough that the conclusion of Lemma 3.35 holds. Thus applying Lemma 3.35 to (3.47) gives

$$\begin{aligned}q^X_{n,3M,T}(z,j,f) \geq \frac{1}{2}\exp\Big(&-\frac{T}{n}\Big(\sqrt{2\hat{R}_X^-(j)} - \sqrt{n\big(f_X(\tfrac{j+1}{n}) - f_X(\tfrac{j}{n})\big)}\Big)^2 \\ &- \frac{1}{n^2}\Big(1 + \sqrt{2\hat{R}_X^-(j)}\Big(\sqrt{2n} + \tfrac{1}{2}\Big)\Big)T\Big).\end{aligned}$$

Since $f \in G_M^2$, we have $\hat{R}_X^-(j) \leq M$ and therefore

$$1 + \sqrt{2\hat{R}_X^-(j)}\big(\sqrt{2n} + 1/2\big) \leq 1 + \sqrt{2M}\big(\sqrt{2n} + 1/2\big) \leq 2(M+1)n^{1/2}.$$

Thus

$$\begin{aligned}&q^X_{n,3M,T}(z,j,f) \\ &\quad\geq \frac{1}{2}\exp\Big(-\frac{T}{n}\Big(\sqrt{2\hat{R}_X^-(j)} - \sqrt{n\big(f_X(\tfrac{j+1}{n}) - f_X(\tfrac{j}{n})\big)}\Big)^2 - \frac{2(M+1)T}{n^{3/2}}\Big).\end{aligned} \tag{3.49}$$

Noting that since $f \in \mathrm{PL}_n^2$ we have $n\big(f_X(\frac{j+1}{n}) - f_X(\frac{j}{n})\big) = f'(s)$ for all $s \in I_j$, and by (3.14)

$$\big(R_X^*(f(s)) - \delta_{M,T}(j,n)\big) \vee 0 \le \hat{R}_X^-(j) \le \hat{R}_X^+(j) \le R_X^*(f(s)) + \delta_{M,T}(j,n),$$

we deduce that

$$\frac{2}{n}\hat{R}_X^-(j) \le \int_{j/n}^{(j+1)/n} 2R_X^*(f(s))ds + \frac{2\delta_{M,T}(j,n)}{n}$$

and using also that $\sqrt{(a-b)\wedge 0} \ge \sqrt{a} - \sqrt{b}$ for $a, b \ge 0$,

$$\begin{aligned}&\frac{1}{n}\sqrt{2\hat{R}_X^-(j)n\big(f_X(\tfrac{j+1}{n}) - f_X(\tfrac{j}{n})\big)} \\ &\qquad \ge \int_{j/n}^{(j+1)/n} \sqrt{2R_X^*(f(s))f_X'(s)}ds - \frac{1}{\sqrt{n}}\sqrt{2\delta_{M,T}(j,n)\big(f_X(\tfrac{j+1}{n}) - f_X(\tfrac{j}{n})\big)}.\end{aligned}$$

Thus

$$\begin{aligned}&\frac{1}{n}\Big(\sqrt{2\hat{R}_X^-(j)} - \sqrt{n\big(f_X(\tfrac{j+1}{n}) - f_X(\tfrac{j}{n})\big)}\Big)^2 \\ &\qquad \le \int_{j/n}^{(j+1)/n} \Big(\sqrt{2R_X^*(f(s))} - \sqrt{f'(s)}\Big)^2 ds + \frac{2\delta_{M,T}(j,n)}{n} \\ &\qquad\qquad + \frac{1}{\sqrt{n}}\sqrt{2\delta_{M,T}(j,n)\big(f_X(\tfrac{j+1}{n}) - f_X(\tfrac{j}{n})\big)}.\end{aligned}$$

This combines with (3.49) to give the result. □

3.7. The final details for the upper bound

3.7.1 Compactness and semicontinuity

There are a few more technical issues that must be resolved in order to complete the proof of the upper bound in Theorem 3.1. One of the remaining ingredients is to prove that the set of functions that we are interested in can be covered by a finite collection of small balls around suitably chosen functions. Recall that PL_n is the subset of functions in E that are linear on each interval $[i/n, (i+1)/n]$ for all $i = 0, \dots, n-1$ and continuous on $[0,1]$. For $F \subset E$ and $r > 0$, write $B_d(F, r) = \bigcup_{f \in F} B_d(f, r)$, where $B_d(f, r)$ is the ball of radius r about f in the metric d.

Lemma 3.36. *Suppose that $F \subset E^2$ and $M > 1$. For any $n \ge 4M$, there exist $N \in \mathbb{N} \cup \{0\}$ and $g_1, \dots, g_N \in G_{4M}^2 \cap \mathrm{PL}_n^2$ such that*

$$F \cap G_{M,T}^2 \subset \bigcup_{i=1}^N \big(B_{\Delta_n}(g_i, 1/n^2) \cap B_d(g_i, 1/n)\big) \subset B_d(F, 2/n)$$

for all $T \ge (4Mn)^{3/2}$.

We will prove this in Appendix 3.B.1.

In order to check that the supremum of our rate function $\tilde{K}$ over $f \in B_d(F, \varepsilon)$ is close to the supremum over $f \in F$ when ε is small, we will need to show that $\tilde{K}$ has some form of upper semi-continuity.

Proposition 3.37. *Suppose that $0 < \theta \le 1$ and there exists $M \in (1, \infty)$ such that $f, f_n \in G^2_M$ for all n. Suppose also that either f is continuous at θ, or $\theta = 1$. If $d(f_n, f) \to 0$ then*

$$\limsup_{n\to\infty} \tilde{K}(f_n, 0, \theta) \le \tilde{K}(f, 0, \theta).$$

The following simple corollary of Proposition 3.37 is written in a more convenient form.

Corollary 3.38. *Suppose that $M \in (1, \infty)$ and $F \subset E^2$ is closed. Then*

$$\lim_{\varepsilon\to 0} \sup_{f\in B_d(F,\varepsilon)\cap G^2_M} \tilde{K}(f, 0, 1) \le \sup_{f\in F\cap G^2_M} \tilde{K}(f, 0, 1).$$

We will prove Proposition 3.37 and Corollary 3.38 in Appendix 3.B.3.

3.7.2 The result for fixed T: proof of Propositions 3.7 and 3.8

Proof of Proposition 3.7. Recall that $g \in G^2_M \cap PL^2_n$. By Markov's inequality, for any $\kappa > 0$,

$$\mathbb{P}\Big(N_T\big(\Gamma_{M,T}(g, n), \theta\big) \ge \kappa\Big) \le \mathbb{E}\Big[\sum_{v\in\mathcal{N}_T} \mathbb{1}_{\{Z^T_v|_{[0,\theta]}\in\Gamma_{M,T}(g,n)|_{[0,\theta]}\}}\Big]\frac{1}{\kappa},$$

and by Lemma 3.3 (Many-to-one),

$$\mathbb{E}\Big[\sum_{v\in\mathcal{N}_T} \mathbb{1}_{\{Z^T_v|_{[0,\theta]}\in\Gamma_{M,T}(g,n)|_{[0,\theta]}\}}\Big] = \mathbb{Q}\big[\mathbb{1}_{\{\xi^T|_{[0,\theta]}\in\Gamma_{M,T}(g,n)|_{[0,\theta]}\}} e^{\int_0^{\theta T} R(\xi_s)ds}\big].$$

Now, if $\xi^T|_{[0,\theta]} \in \Gamma_{M,T}(g, n)|_{[0,\theta]}$, then by Lemma 3.6,

$$\int_0^{\theta T} R(\xi_s)ds = T\int_0^{\theta} R(T\xi^T(s))ds \le T\int_0^{\lfloor\theta n\rfloor/n} R^*(g(s))ds + T\eta(M, n, T),$$

and therefore

$$\mathbb{Q}\big[\mathbb{1}_{\{\xi^T|_{[0,\theta]}\in\Gamma_{M,T}(g,n)|_{[0,\theta]}\}} e^{\int_0^{\theta T} R(\xi_s)ds}\big] \le \mathbb{Q}\big(\xi^T|_{[0,\theta]} \in \Gamma_{M,T}(g, n)|_{[0,\theta]}\big)e^{T\int_0^{\lfloor\theta n\rfloor/n} R^*(g(s))ds+T\eta(M,n,T)}.$$

We also know from Proposition 3.4 that

$$\mathbb{Q}(\xi^T|_{[0,\theta]} \in \Gamma_{M,T}(g, n)|_{[0,\theta]}) \le \exp\Bigg(-T\sum_{j=0}^{\lfloor\theta n\rfloor-1}\big(\mathcal{E}^+_X(I_j, \Gamma_{M,T}(g, n), T) + \mathcal{E}^+_Y(I_j, \Gamma_{M,T}(g, n), T)\big)\Bigg),$$

and by Proposition 3.5 that, if $g = (g_X, g_Y)$,

$$\sum_{j=\lceil\sqrt{n}\rceil}^{\lfloor\theta n\rfloor-1} \mathcal{E}^+_X(I_j, \Gamma_{M,T}(g, n), T) \ge \int_{\lceil\sqrt{n}\rceil/n}^{\lfloor\theta n\rfloor/n}\Big(\sqrt{2R^*_X(g(s))}-\sqrt{g'_X(s)}\Big)^2 ds - O\Big(\frac{M^4}{n^{1/4}}+\frac{M^3 n}{T^{1/2}}\Big).$$

Since $g \in G^2_M$, we also have

$$\int_0^{\lceil\sqrt{n}\rceil/n} \Big(\sqrt{2R^*_X(g(s))} - \sqrt{g'_X(s)}\Big)^2 ds \leq \int_0^{\lceil\sqrt{n}\rceil/n} 2R^*_X(g(s))ds + \int_0^{\lceil\sqrt{n}\rceil/n} g'_X(s)ds$$
$$\leq \frac{2M^2\lceil\sqrt{n}\rceil}{n} + \frac{M\lceil\sqrt{n}\rceil}{n} \leq \frac{4M^2}{\sqrt{n}} \tag{3.50}$$

so

$$\sum_{j=\lceil\sqrt{n}\rceil}^{\lfloor\theta n\rfloor-1} \mathcal{E}^+_X(I_j, \Gamma_{M,T}(g,n), T) \geq \int_0^{\lfloor\theta n\rfloor/n} \Big(\sqrt{2R^*_X(g(s))} - \sqrt{g'_X(s)}\Big)^2 ds - O\Big(\frac{M^4}{n^{1/4}} + \frac{M^3 n}{T^{1/2}}\Big)$$

and by symmetry the same bound holds for Y. Recalling from Lemma 3.6 that $\eta(M,n,T) = O\Big(\frac{M^4}{n^{1/2}} + \frac{M^3 n}{T^{1/3}}\Big)$, we deduce that

$$\begin{aligned}
&\mathbb{P}\Big(N_T\big(\Gamma_{M,T}(g,n),\theta\big) \geq \kappa\Big) \\
&\quad\leq e^{-T\int_0^{\lfloor\theta n\rfloor/n} \big(\sqrt{2R^*_X(g(s))}-\sqrt{g'_X(s)}\big)^2 ds - T\int_0^{\lfloor\theta n\rfloor/n} \big(\sqrt{2R^*_Y(g(s))}-\sqrt{g'_Y(s)}\big)^2 ds} \\
&\qquad\qquad \cdot e^{O\big(\frac{M^4 T}{n^{1/4}} + M^3 n T^{2/3}\big) + T\int_0^{\lfloor\theta n\rfloor/n} R^*(g(s))ds} \cdot \frac{1}{\kappa} \\
&\quad= \frac{1}{\kappa} e^{T\tilde{K}(g,0,\lfloor\theta n\rfloor/n) + O\big(\frac{M^4 T}{n^{1/4}} + M^3 n T^{2/3}\big)}
\end{aligned}$$

as required, where for the last equality we used the fact that $g \in G^2_M \cap \mathrm{PL}^2_n$, and therefore $\tilde{K}(g,0,s) = \int_0^s R^*(g(u))du - I(g,0,s)$ for all s. $\square$

Proposition 3.8 essentially establishes the upper bound in Theorem 3.1 with high probability for a fixed (large) T. The proof mostly involves using Lemma 3.2 and the technical results stated in Section 3.7.1 to ensure that we can cover our set in a suitable way with finitely many balls around piecewise linear functions, and then applying Proposition 3.7.

Proof of Proposition 3.8. Take $M \geq M_0$ and the other parameters as in the statement of the Proposition. By Lemma 3.2,

$$\mathbb{P}(\exists v \in \mathcal{N}_T : Z^T_v \notin G^2_{M,T}) \leq e^{-\delta_0 T^{1/3}}.$$

By Corollary 3.38, since F is closed we may choose n large enough such that $n \geq 4M$ and

$$\sup_{f \in B_d(F,2/n)\cap G^2_{4M}} \tilde{K}(f,0,1) \leq \sup_{f \in F \cap G^2_{4M}} \tilde{K}(f,0,1) + \varepsilon/3.$$

By Lemma 3.36 we may choose $N \in \mathbb{N}$ and $g_1,\ldots,g_N \in G^2_{4M} \cap \mathrm{PL}^2_n$ such that

$$F \cap G^2_{M,T} \subset \bigcup_{i=1}^N \big(B_{\Delta_n}(g_i, 1/n^2) \cap B_d(g_i, 1/n)\big) \subset B_d(F, 2/n)$$

for all $T \geq (4Mn)^{3/2}$. Recall that $\Gamma_{M,T}(g_i,n) = B_{\Delta_n}(g_i,1/n^2) \cap B_d(g_i,1/n) \cap G^2_{M,T}$.

Then for any $A \geq 0$,

$$\begin{aligned}\mathbb{P}\big(N_T(F) \geq e^{AT}\big) &\leq \mathbb{P}\big(\exists v \in \mathcal{N}_T : Z_v^T \notin G^2_{M,T}\big) + \sum_{i=1}^{N} \mathbb{P}\Big(N_T\big(\Gamma_{M,T}(g_i, n)\big) \geq \frac{e^{AT}}{N}\Big) \\ &\leq e^{-\delta_0 T^{1/3}} + \sum_{i=1}^{N} \mathbb{P}\Big(N_T\big(\Gamma_{M,T}(g_i, n)\big) \geq \frac{e^{AT}}{N}\Big). \end{aligned} \tag{3.51}$$

By Proposition 3.7, for each i we have

$$\mathbb{P}\Big(N_T\big(\Gamma_{M,T}(g_i, n)\big) \geq \frac{e^{AT}}{N}\Big) \leq \frac{N}{e^{AT}} \exp\Big(T\tilde{K}(g_i, 0, 1) + O\Big(\frac{M^4 T}{n^{1/4}} + M^3 n T^{2/3}\Big)\Big),$$

and combining this with (3.51) we see that

$$\begin{aligned}&\mathbb{P}\big(N_T(F) \geq e^{AT}\big) \\ &\qquad \leq e^{-\delta_0 T^{1/3}} + \frac{N^2}{e^{AT}} \max_{i\in\{1,\ldots,N\}} \exp\Big(T\tilde{K}(g_i, 0, 1) + O\Big(\frac{M^4 T}{n^{1/4}} + M^3 n T^{2/3}\Big)\Big).\end{aligned}$$

By our choice of $g_1, \ldots, g_N$ and n, we have

$$\max_{i\in\{1,\ldots,N\}} \tilde{K}(g_i, 0, 1) \leq \sup_{f\in B_d(F,2/n)\cap G^2_{4M}} \tilde{K}(f, 0, 1) \leq \sup_{f\in F\cap G^2_{4M}} \tilde{K}(f, 0, 1) + \varepsilon/3$$

and therefore

$$\begin{aligned}&\frac{1}{T^{1/3}} \log \mathbb{P}\big(N_T(F) \geq e^{AT}\big) \\ &\qquad \leq (-\delta_0) \vee \Big(\sup_{f\in F\cap G^2_{4M}} \tilde{K}(f, 0, 1)T^{2/3} - AT^{2/3} + \frac{\varepsilon T^{2/3}}{3} + O\Big(\frac{M^4 T^{2/3}}{n^{1/4}}\Big)\Big).\end{aligned}$$

Increasing n if necessary so that the $O(\frac{M^4 T^{2/3}}{n^{1/4}})$ term is smaller than $\frac{\varepsilon T^{2/3}}{3}$, and choosing

$$A = \sup_{f\in F\cap G^2_{4M}} \tilde{K}(f, 0, 1) + \varepsilon,$$

we have

$$\lim_{T\to\infty} \frac{1}{T^{1/3}} \log \mathbb{P}\big(N_T(F, \theta) \geq e^{AT}\big) \leq -\delta_0.$$

This is precisely the statement of the proposition, but with $4M$ in place of M. Since we only assumed that $M \geq M_0$ in the proof, the proposition holds when $M \geq 4M_0$. □

3.7.3 Paths with $K(f) = -\infty$ are unlikely: proof of Lemma 3.9

Before proving Lemma 3.9, we need to relate K to $\tilde{K}$.

Lemma 3.39. *Suppose that $M > 1$. If $F \subset E^2$ is closed and $\sup_{f\in F} K(f) = -\infty$, then there exists $\varepsilon > 0$ such that*

$$\sup_{f\in B(F,\varepsilon)\cap G^2_{M,1}} \inf_{\theta\in[0,1]} \tilde{K}(f, 0, \theta) < 0.$$

Proof. If the result is not true, then for each $n \in \mathbb{N}$ we may choose $f_n \in B(F, 1/n) \cap G^2_{M,1}$ such that

$$\inf_{\theta\in[0,1]} \tilde{K}(f_n, 0, \theta) \geq -1/n.$$

It is easy to check that $G^2_{M,1}$ is closed and totally bounded. Since (E^2, d) is complete, $G^2_{M,1}$ is compact. We may therefore find a subsequence $(f_{n_j})_{j\geq 1}$ such that $d(f_{n_j}, f_\infty) \to 0$ as $j \to \infty$ for some $f_\infty \in G^2_{M,1}$. Since $F \cap G^2_{M,1}$ is closed, and $d(f_\infty, F \cap G^2_{M,1}) = 0$, we must in fact have $f_\infty \in F \cap G^2_{M,1}$. On the other hand, by Proposition 3.37, for any $\theta \in [0, 1]$ such that f_∞ is continuous at θ,

$$\tilde{K}(f_\infty, 0, \theta) \geq \limsup_{j\to\infty} \tilde{K}(f_{n_j}, 0, \theta) \geq 0.$$

But f_∞ is non-decreasing and therefore continuous almost everywhere, and $t \mapsto \tilde{K}(f, 0, t)$ has only downward jumps, so we must have $\tilde{K}(f_\infty, 0, \theta) \geq 0$ for all $\theta \in [0, 1]$. Thus $K(f_\infty) \geq 0$, which contradicts the hypothesis of the lemma. $\square$

We can now prove Lemma 3.9, which says that if F is closed and $\sup_{f\in F} K(f) = -\infty$, then with high probability $N_T(F)$ is zero.

Proof of Lemma 3.9. Choose $M \geq M_0$. Since $G_{4M} \subset G_{4M,1}$, by Lemma 3.39 we may choose $n_0 \geq 4M$ such that

$$\sup_{f\in B(F,2/n_0)\cap G^2_{4M}} \inf_{\theta\in[0,1]} \tilde{K}(f, 0, \theta) < 0.$$

Let

$$\eta = -\sup_{f\in B(F,2/n_0)\cap G^2_{4M}} \inf_{\theta\in[0,1]} \tilde{K}(f, 0, \theta) > 0. \tag{3.52}$$

Then take $n \geq n_0$ such that the error term in Proposition 3.7 is smaller than $\eta T/3$ for T sufficiently large, and such that $(4M)^2/n \leq \eta/3$.

By Lemma 3.36 we may choose $N \in \mathbb{N} \cup \{0\}$ and $g_1, \ldots, g_N \in G^2_{4M} \cap \mathrm{PL}^2_n$ such that

$$F \cap G^2_{M,T} \subset \bigcup_{i=1}^{N} \big(B_{\Delta_n}(g_i, 1/n^2) \cap B_d(g_i, 1/n)\big) \subset B_d(F, 2/n)$$

for all $T \geq (4Mn)^{3/2}$.

For each $i = 1, \ldots, N$, note that since $g_i \in G^2_{4M}$, by the definition of $\tilde{K}$, for any $0 \leq s \leq t \leq 1$ we have

$$\tilde{K}(g_i, 0, t) \leq \tilde{K}(g_i, 0, s) + (4M)^2(t - s). \tag{3.53}$$

In particular, the function $t \mapsto \tilde{K}(g_i, 0, t)$ has only downward jumps, and therefore its infimum is achieved. Thus, by (3.52), we may choose θ_i such that

$$\tilde{K}(g_i, 0, \theta_i) = \inf_{\theta\in[0,1]} \tilde{K}(g_i, 0, \theta) \leq -\eta.$$

Let $\hat{\theta}_i = \lceil \theta_i n \rceil / n$. Using (3.53) again, we then have

$$\tilde{K}(g_i, 0, \hat{\theta}_i) \leq -\eta + (4M)^2/n \leq -2\eta/3 \tag{3.54}$$

where the last inequality holds because we chose n such that $(4M)^2/n \leq \eta/3$.

Now, by our choice of $g_1, \ldots, g_N$, we have

$$N_T(F) \leq N_T((G^2_{M,T})^c) + \sum_{i=1}^{N} N_T(\Gamma_{M,T}(g_i, n))$$

and therefore

$$\mathbb{P}(N_T(F) \geq 1) \leq \mathbb{P}\big(N_T((G^2_{M,T})^c) \geq 1\big) + \sum_{i=1}^{N} \mathbb{P}\big(N_T(\Gamma_{M,T}(g_i, n)) \geq 1\big). \tag{3.55}$$

By Lemma 3.2, the first term on the right-hand side above is at most $e^{-\delta_0 T^{1/3}}$. Also, since a population that is extinct at time θ must also be extinct at time 1, for each i we have

$$\mathbb{P}\big(N_T(\Gamma_{M,T}(g_i, n)) \geq 1\big) \leq \mathbb{P}\big(N_T(\Gamma_{M,T}(g_i, n), \hat{\theta}_i) \geq 1\big).$$

Since $\hat{\theta}_i$ is an integer multiple of $1/n$, by Proposition 3.7 we have

$$\begin{aligned}\mathbb{P}\big(N_T(\Gamma_{M,T}(g_i, n), \hat{\theta}_i) \geq 1\big) &\leq \exp\left(T\tilde{K}(g_i, 0, \hat{\theta}_i) + O\Big(\frac{M^4 T}{n^{1/4}} + M^3 n T^{2/3}\Big)\right) \\ &\leq \exp(-\eta T/3),\end{aligned}$$

where the last inequality follows from (3.54) and our choice of n. Returning to (3.55), we have shown that

$$\mathbb{P}(N_T(F) \geq 1) \leq e^{-\delta_0 T^{1/3}} + N e^{-\eta T/3},$$

which completes the proof. $\square$

3.7.4 Lattice times to continuous time: proof of Proposition 3.10

Before moving on to the proof of Proposition 3.10, we state and prove two lemmas that will check that paths of particles are not drastically changed by rescaling by a slightly different value of T.

Lemma 3.40. *Suppose that $M > 1$, $t \geq 3M$ and $t - 1 \leq s \leq t$. For any $F \subset E^2$, we have*

$$N_s(F \cap G^2_{M,s}) \leq N_t\big(B(F, 3M/t)\big).$$

Proof. Suppose that $u \in \mathcal{N}_s$ satisfies $Z^s_u \in F \cap G^2_{M,s}$. Take any $v \in \mathcal{N}_t$ such that v is a descendant of u. We claim that $d(X^s_u, X^t_v) \leq 3M/t$, which means that for all $\tau \in [-3M/t, 1 + 3M/t]$,

$$X^t_v(\tau - 3M/t) - 3M/t \leq X^s_u(\tau) \leq X^t_v(\tau + 3M/t) + 3M/t$$

where $f(\tau)$ is interpreted to equal $f(0)$ for $\tau < 0$ and $f(1)$ for $\tau > 1$. Since $Z^s_u \in F$, the claim plus its equivalent Y statement ensure that $Z_v^t \in B(F, 3M/t)$, which is enough to complete the proof.

To prove the claim, first note that it holds when $\tau \le 0$, since in this case $X_u^s(\tau) = X_u^s(0) = X_v^t(0) = X_v^t(\tau)$. If $\tau > 0$, since $s \le t$ and

$$\tau s \ge \tau(t-1) = t(\tau - \tau/t) \ge t\Big(\tau - \frac{1+3M/t}{t}\Big) \ge t\Big(\tau - \frac{3M}{t}\Big),$$

we have

$$X_u^s(\tau) = X_v^s(\tau) \ge X_v^t(\tau - \tfrac{3M}{t}).$$

Also, since $X_u^s \in G^2_{M,s}$, for any $\tau \in [0,1]$ we have

$$\begin{aligned} X_u^s(\tau) = \frac{1}{s}X_u(\tau s) &= \frac{1}{t}X_u(\tau s) + \Big(1 - \frac{s}{t}\Big)X_u^s(\tau) \\ &\le \frac{1}{t}X_v(\tau t) + \Big(\frac{t-s}{t}\Big)M(1+2s^{-2/3}) \\ &\le X_v^t(\tau) + \frac{M}{t}(1+2s^{-2/3}) \le X_v^t(\tau) + \frac{3M}{t} \end{aligned}$$

as required. If $\tau > 1$ then $X_u^s(\tau) = X_u^s(1)$ and then the argument above gives that that $X_u^s(1) \le X_v^t(1) + 3M/t = X_v^t(\tau) + 3M/t$. □

Lemma 3.41. *Suppose that $M > 2$, $T \ge 2$ and $t \in [T-1, T]$. If $N_t((G^2_{M,t})^c) \ge 1$ then either $N_T((G^2_{M/2,T})^c) \ge 1$ or $N_{T-1}((G^2_{M/2,T-1})^c) \ge 1$.*

Proof. Suppose there exists $v \in \mathcal{N}_t$ such that $Z_v^t \in (G^2_{M,t})^c$. It is possible that either X_v^t or Y_v^t (or both) is the reason for Z_v^t falling outside $G^2_{M,t}$; without loss of generality assume that it is X_v^t. Then there exists $s \in [0,1]$ such that either $X_v^t(s) > M(s+2t^{-2/3})$, or $X_v^t(s) < s/M - 2t^{-2/3}$. In the first case, take $w \in \mathcal{N}_T$ such that w is a descendant of v. Then

$$X_w^T(s) = \frac{1}{T}X_w(sT) \ge \frac{t}{T}\frac{1}{t}X_v(st) > \frac{1}{2}M(s+2t^{-2/3}) \ge \frac{M}{2}(s+2T^{-2/3})$$

so $Z_w^T \in (G^2_{M/2,T})^c$. In the second case, let u be the ancestor of v in $\mathcal{N}_{T-1}$. Then

$$\begin{aligned} X_u^{T-1}(s) = \frac{1}{T-1}X_u(s(T-1)) \le \frac{t}{T-1}\frac{1}{t}X_v(st) &< \frac{t}{T-1}\Big(\frac{s}{M} - 2t^{-2/3}\Big) \\ &\le \frac{2s}{M} - 2(T-1)^{-2/3} \end{aligned}$$

so $Z_u^{T-1} \in (G^2_{M/2,T-1})^c$. This completes the proof. □

Proof of Proposition 3.10. We begin with the first part of the result. Take $\varepsilon > 0$. We start by noting that

$$\begin{aligned}
&\mathbb{P}\Big(\exists t \in [T-1,T] : \frac{1}{t}\log N_t(F) \geq \sup_{f\in F\cap G_M^2} \tilde{K}(f,0,1)+\varepsilon\Big)\\
&\qquad\leq \mathbb{P}\Big(\exists t \in [T-1,T] : \frac{1}{t}\log N_t(F\cap G_{M,t}^2) \geq \sup_{f\in F\cap G_M^2} \tilde{K}(f,0,1)+\varepsilon\Big)\\
&\qquad\qquad + \mathbb{P}\big(\exists t\in[T-1,T] : N_t((G_{M,t}^2)^c)\geq 1\big). \quad (3.56)
\end{aligned}$$

We show that the right-hand side is exponentially small in T. By Corollary 3.38, we can choose $\varepsilon' \in (0,1)$ such that

$$\sup_{f\in \overline{B(F,\varepsilon')}\cap G_M^2} \tilde{K}(f,0,1) \leq \sup_{f\in F\cap G_M^2} \tilde{K}(f,0,1)+\varepsilon/3.$$

By Lemma 3.40, provided that $3M/T \leq \varepsilon'$, we have

$$N_t(F\cap G_{M,t}^2) \leq N_T(B(F,\varepsilon'))$$

for all $t\in[T-1,T]$. Therefore for large T

$$\begin{aligned}
&\mathbb{P}\Big(\exists t \in [T-1,T] : \frac{1}{t}\log N_t(F\cap G_{M,t}^2) \geq \sup_{f\in F\cap G_M^2} \tilde{K}(f,0,1)+\varepsilon\Big)\\
&\qquad\leq \mathbb{P}\Big(\frac{1}{T-1}\log N_T(B(F,\varepsilon')) \geq \sup_{f\in F\cap G_M^2} \tilde{K}(f,0,1)+\varepsilon\Big)\\
&\qquad\leq \mathbb{P}\Big(\frac{1}{T}\log N_T(B(F,\varepsilon')) \geq \sup_{f\in \overline{B(F,\varepsilon')}\cap G_M^2} \tilde{K}(f,0,1)+\varepsilon/3\Big).
\end{aligned}$$

Then Proposition 3.8 tells us that this is at most $\exp(-\delta_0 T^{1/3}/2)$ for large T. Substituting this into (3.56), we have

$$\begin{aligned}
&\mathbb{P}\Big(\exists t \in [T-1,T] : \frac{1}{t}\log N_t(F) \geq \sup_{f\in F\cap G_M^2} \tilde{K}(f,0,1)+\varepsilon\Big)\\
&\qquad\leq \exp(-\delta_0 T^{1/3}/2) + \mathbb{P}\big(\exists t\in[T-1,T] : N_t((G_{M,t}^2)^c)\geq 1\big). \quad (3.57)
\end{aligned}$$

For the remaining term, Lemma 3.41 tells us that for $T\geq 2$,

$$\begin{aligned}
&\mathbb{P}(\exists t\in[T-1,T] : N_t((G_{M,t}^2)^c)\geq 1)\\
&\qquad\leq \mathbb{P}(N_T((G_{M/2,T}^2)^c)\geq 1) + \mathbb{P}(N_{T-1}((G_{M/2,T-1}^2)^c)\geq 1).
\end{aligned}$$

By Lemma 3.2, this is at most $2\exp\big(-\delta_0(T-1)^{1/3}\big)$. Returning to (3.57), we have

$$\begin{aligned}
&\mathbb{P}\Big(\exists t \in [T-1,T] : \frac{1}{t}\log N_t(F) \geq \sup_{f\in F\cap G_M^2} \tilde{K}(f,0,1)+\varepsilon\Big)\\
&\qquad\leq \exp(-\delta_0 T^{1/3}/2) + 2\exp\big(-\delta_0(T-1)^{1/3}\big).
\end{aligned}$$

By the Borel-Cantelli lemma,

$$\mathbb{P}\Big(\limsup_{t\to\infty}\frac{1}{t}\log N_t(F) \geq \sup_{f\in F\cap G_M^2}\tilde{K}(f,0,1)+\varepsilon\Big) = 0,$$

and since $\varepsilon > 0$ was arbitrary, we deduce the first part of the result.

The proof when $\sup_{f\in F} K(f) = -\infty$ is very similar. By Lemma 3.39, we may choose $\varepsilon'' > 0$ such that

$$\sup_{f\in \overline{B(F,\varepsilon'')}\cap G_{M,1}^2} K(f) = -\infty. \tag{3.58}$$

Then

$$\begin{aligned}\mathbb{P}(\exists t\in[T-1,T] : N_t(F)\geq 1) \leq \mathbb{P}(\exists t\in[T-1,T] : N_t(F\cap G_{M,t}^2)\geq 1)&\\ + \mathbb{P}(\exists t\in[T-1,T] : N_t((G_{M,t}^2)^c)\geq 1).&\end{aligned} \tag{3.59}$$

As argued above, by Lemmas 3.41 and 3.2 the last term on the right-hand side is at most $2\exp\big(-\delta_0(T-1)^{1/3}\big)$ provided that $T\geq 2$. For the first term on the right-hand side, by Lemma 3.40, provided that $3M/T\leq\varepsilon''$ we have

$$\begin{aligned}&\mathbb{P}(\exists t\in[T-1,T] : N_t(F\cap G_{M,t}^2)\geq 1)\\ &\qquad\leq \mathbb{P}(N_T(\overline{B(F,\varepsilon'')})\geq 1)\\ &\qquad\leq \mathbb{P}(N_T(\overline{B(F,\varepsilon'')}\cap G_{M,1}^2)\geq 1) + \mathbb{P}(N_T((G_{M,1}^2)^c)\geq 1).\end{aligned}$$

Due to (3.58), we can apply Lemma 3.9 to tell us that the first term on the right-hand side above is at most $e^{-\delta_0 T^{1/3}/2}$, and Lemma 3.2 to tell us that the second term on the right-hand side is at most $e^{-\delta_0 T^{1/3}}$. Returning to (3.59), and applying the Borel-Cantelli lemma, we have

$$\mathbb{P}(\limsup_{t\to\infty} N_t(F)\geq 1) = 0.$$

This completes the proof. $\square$

Appendix

3.A. Deterministic bounds on the rate function

We use the same notation as in Section 3.6. Our main aim in this section is to prove Proposition 3.5 and Lemma 3.6, showing that the bounds obtained in Section 3.6, in terms of $\mathcal{E}_X^+(I_j, \Gamma_{M,T}(f,n), T)$, look something like the growth rate seen in our main theorem. This work involves tedious approximations of sums and integrals. Most of the work is in bounding R^* in terms of R^+ and R^-, which is done using the following lemma. Throughout this section we write $R_X^-(j) = R_X^-(I_j, \Gamma_{M,T}(f,n), T)$ and similarly for R_X^+, R_Y^- and R_Y^+.

Lemma 3.42. *Suppose that $M > 1$, $n \geq 2M$, $f|_{I_j} \in G_M^2|_{I_j}$, $j \geq n^{1/2}$ and $s \in I_j$. Then*

$$R_X^+(j) - \delta_{M,T}(j,n) \leq R_X^*(f(s)) \leq R_X^-(j) + \delta_{M,T}(j,n) \tag{3.60}$$

where

$$\delta_{M,T}(j,n) = \big(6M^3 n^{1/2} + \tfrac{2M^2 n}{T}\big)\big(f_X(\tfrac{j+1}{n}) - f_X(\tfrac{j}{n})\big) + Mn^{1/2}\big(f_Y(\tfrac{j+1}{n}) - f_Y(\tfrac{j}{n})\big) + \tfrac{7M^3}{n^{3/2}} + \tfrac{3M^3 n}{T}.$$

Moreover,

$$\sum_{j=\lceil\sqrt{n}\rceil}^{n-1} \frac{\delta_{M,T}(j,n)}{n} \leq \frac{14M^4}{n^{1/2}} + \frac{5M^3 n}{T}. \tag{3.61}$$

Proof. We begin with the upper bound in (3.60), and claim first that for any $j \in \{0,1,\dots,n-1\}$ we have

$$f_Y\Big(\frac{j}{n}\Big) \leq \Big(R_X^-(j) + \frac{1}{2}\Big)\Big(f_X\Big(\frac{j+1}{n}\Big) + \frac{1}{n^2} + \frac{1}{T}\Big) + \frac{1}{n^2}. \tag{3.62}$$

To see why this is true, by the definition of $R_X^-(j)$, for any $\varepsilon > 0$ we may take $g \in \Gamma_{M,T}(f,n)$ and $s \in I_j$ such that

$$R_X(Tg(s)) \leq R_X^-(j) + \varepsilon,$$

and then

$$\frac{g_Y(s) + 1/T}{g_X(s) + 1/T} - \frac{1}{2} \leq R_X(Tg(s)) \leq R_X^-(j) + \varepsilon.$$

Noting that $g_Y(s) \geq g_Y(\frac{j}{n}) \geq f_Y(\frac{j}{n}) - 1/n^2$ and $g_X(s) \leq g_X(\frac{j+1}{n}) \leq f_X(\frac{j+1}{n}) + 1/n^2$, we see that

$$\frac{f_Y(\frac{j}{n}) - 1/n^2 + 1/T}{f_X(\frac{j+1}{n}) + 1/n^2 + 1/T} - \frac{1}{2} \leq R_X^-(j) + \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, the left-hand side must in fact be at most $R_X^-(j)$, and then rearranging gives (3.62).

We now aim to bound $R_X^*(f(s))$ above for $s \in I_j$. We concentrate first on the case that $f_Y(s) > f_X(s)$. Whenever this holds, using (3.62),

$$\begin{aligned}R_X^*(f(s)) &= \frac{f_Y(s)}{f_X(s)} - \frac{1}{2} = \frac{f_Y(\frac{j}{n})}{f_X(s)} - \frac{1}{2} + \frac{f_Y(s) - f_Y(\frac{j}{n})}{f_X(s)} \\ &\leq \frac{\big(R_X^-(j) + 1/2\big)\big(f_X(\frac{j+1}{n}) + 1/n^2 + 1/T\big) + \frac{1}{n^2}}{f_X(s)} - \frac{1}{2} + \frac{f_Y(s) - f_Y(\frac{j}{n})}{f_X(s)}.\end{aligned}$$

Writing

$$\begin{aligned}&\Big(R_X^-(j) + \frac{1}{2}\Big)\Big(f_X(\tfrac{j+1}{n}) + \frac{1}{n^2} + \frac{1}{T}\Big) \\ &\qquad = \Big(R_X^-(j) + \frac{1}{2}\Big)f_X(s) + \Big(R_X^-(j) + \frac{1}{2}\Big)\Big(f_X(\tfrac{j+1}{n}) - f_X(s) + \frac{1}{n^2} + \frac{1}{T}\Big)\end{aligned}$$

and substituting this into the bound above, we have (for $f_Y(s) > f_X(s)$)

$$R_X^*(f(s)) \leq R_X^-(j) + \frac{\big(R_X^-(j) + \frac{1}{2}\big)\big(f_X(\frac{j+1}{n}) - f_X(s) + \frac{1}{n^2} + \frac{1}{T}\big) + \frac{1}{n^2} + f_Y(s) - f_Y(\frac{j}{n})}{f_X(s)}.$$

The first term on the right-hand side is the important one, and we now aim to bound the other terms. Since $f|_{I_j} \in G_M^2|_{I_j}$, we have $f_X(s) \geq s/M$, and since also $f \in \Gamma_{M,T}(f,n)$,

$$R_X^-(j) \leq R_X(Tf(s)) \leq \frac{Ms + 1/T}{s/M} - \frac{1}{2} = M^2 + \frac{M}{sT} - \frac{1}{2}, \tag{3.63}$$

so

$$R_X^*(f(s)) \leq R_X^-(j) + \frac{\big(M^2 + \frac{M}{sT}\big)\big(f_X(\frac{j+1}{n}) - f_X(\frac{j}{n}) + \frac{1}{n^2} + \frac{1}{T}\big) + \frac{1}{n^2} + f_Y(\frac{j+1}{n}) - f_Y(\frac{j}{n})}{s/M}.$$

This is true in the case $f_Y(s) > f_X(s)$, but when $f_Y(s) \leq f_X(s)$ we have $R_X^*(f(s)) = 1/2 \leq R_X^-(j)$, so the inequality above trivially holds in that case too. Taking $s \geq j/n \geq n^{-1/2}$, and combining some of the terms, we obtain the upper bound in (3.60).

The lower bound in (3.60) is similar. We choose $g \in \Gamma_{M,T}(f,n)$ and $s \in [\frac{j}{n}, \frac{j+1}{n}]$ such that $R_X(Tg(s)) \geq R_X^+(j) - 1/n^2$. If $R_X(Tg(s)) = 1/2$ then $R_X^+(j) \leq 1/2 + 1/n^2$, so the lower bound on that interval is trivial; we may therefore assume that $R_X(Tg(s)) > 1/2$ and then we have a similar bound to (3.62):

$$f_Y\Big(\frac{j+1}{n}\Big) \geq \Big(R_X^+(j) + \frac{1}{2}\Big)\Big(f_X\Big(\frac{j}{n}\Big) - \frac{1}{n^2}\Big) - \frac{1}{n^2} - \frac{1}{T}. \tag{3.64}$$

We then apply this essentially as in the proof of the upper bound to obtain

$$R_X^*(f(s)) \geq R_X^+(j) - \frac{\big(R_X^+(j) + \frac{1}{2}\big)\big(f_X(s) - f_X(\frac{j}{n}) + \frac{1}{n^2}\big) + \frac{1}{n^2} + \frac{1}{T} + f_Y(\frac{j+1}{n}) - f_Y(s)}{f_X(s)}.$$

In place of (3.63) we must use the slightly more involved bound, for $j \geq \sqrt{n}$ and $n \geq 2M$,

$$R_X^+(j) \leq \frac{M\frac{j+1}{n} + \frac{1}{n^2} + \frac{1}{T}}{\frac{1}{M}\frac{j}{n} - \frac{1}{n^2} + \frac{1}{T}} - \frac{1}{2} \leq \frac{3M\frac{j}{n} + \frac{1}{T}}{\frac{j}{2Mn}} - \frac{1}{2} = 6M^2 + \frac{2Mn}{jT} - \frac{1}{2} \leq 6M^2 + \frac{2M\sqrt{n}}{T} - \frac{1}{2}.$$

Applying this and taking $s \geq j/n \geq n^{-1/2}$, and combining terms, gives the lower bound in (3.60).

To prove (3.61), summing over $j \geq n^{1/2}$ and telescoping gives

$$\sum_{j=\lceil\sqrt{n}\rceil}^{n-1} \frac{\delta_{M,T}(j,n)}{n} \leq \Big(\frac{6M^3}{n^{1/2}} + \frac{2M^2}{T}\Big) f_X(1) + \frac{M}{n^{1/2}} f_Y(1) + \frac{7M^3}{n^{3/2}} + \frac{3M^3 n}{T}$$

Using that $f_X(1) \leq M$ and $f_Y(1) \leq M$, and combining terms, gives the result. $\square$

3.A.1 Proof of Proposition 3.5

We first give a lemma which handles the cross-term that appears when multiplying out the quadratics involved in Proposition 3.5.

Lemma 3.43. *Suppose that $f \in \mathrm{PL}_n^2 \cap G_M^2$, $\theta \in (0,1]$, $M > 1$ and $n \geq 2M$. Then for any $k \in \{\lceil\sqrt{n}\rceil, \ldots, \lfloor\theta n\rfloor - 1\}$,*

$$\int_{k/n}^{\lfloor\theta n\rfloor/n} \sqrt{R_X^*(f(s)) f_X'(s)} ds \geq \sum_{j=k}^{\lfloor\theta n\rfloor-1} \sqrt{\frac{R_X^+(j)}{n}(x_{j+1}^+ - x_j^-)} - \frac{8M^{5/2}}{n^{1/4}} - \frac{4M^2 n^{1/2}}{T^{1/2}}.$$

Proof. Take $s \in [\frac{j}{n}, \frac{j+1}{n}]$ for some $j \in \{0, 1, \ldots, n-1\}$. Note that since $f \in \mathrm{PL}_n$, we have

$$f_X'(s) = n(f(\tfrac{j+1}{n}) - f(\tfrac{j}{n})) \geq n\big(x_{j+1}^+ - 1/n^2 - x_j^- - 1/n^2\big) \vee 0.$$

Using the elementary inequality $\sqrt{(a-b) \vee 0} \geq a^{1/2} - b^{1/2}$ valid for all $a, b \geq 0$, we obtain

$$\sqrt{f_X'(s)} \geq n^{1/2}(x_{j+1}^+ - x_j^-)^{1/2} - \sqrt{2}n^{-1/2}.$$

Thus

$$\int_{j/n}^{(j+1)/n} \sqrt{R_X^*(f(s)) f_X'(s)} ds \geq \big((x_{j+1}^+ - x_j^-)^{1/2} - \sqrt{2}/n\big) \int_{j/n}^{(j+1)/n} \sqrt{n R_X^*(f(s))} ds$$

and since $f \in G_M^2$, $R_X^*(f(s)) \leq M^2$ for all $s > 0$, so

$$\int_{j/n}^{(j+1)/n} \sqrt{R_X^*(f(s)) f_X'(s)} ds \geq (x_{j+1}^+ - x_j^-)^{1/2} \int_{j/n}^{(j+1)/n} \sqrt{n R_X^*(f(s))} ds - \sqrt{2} M n^{-3/2}.$$

We now use the lower bound in (3.60) to see that for $j \geq \sqrt{n}$,

$$\begin{aligned}\int_{j/n}^{(j+1)/n} \sqrt{n R_X^*(f(s))} ds &\geq \int_{j/n}^{(j+1)/n} \sqrt{(n R_X^+(j) - n\delta_{M,T}(j,n)) \vee 0}\, ds \\ &= \sqrt{\Big(\frac{R_X^+(j)}{n} - \frac{\delta_{M,T}(j,n)}{n}\Big) \vee 0}\end{aligned}$$

Again using $\sqrt{(a-b)\vee 0} \geq a^{1/2} - b^{1/2}$, we therefore have, for $j \geq \sqrt{n}$,

$$\int_{j/n}^{(j+1)/n} \sqrt{R_X^*(f(s))f_X'(s)}ds$$
$$\geq \sqrt{\frac{R_X^+(j)}{n}(x_{j+1}^+ - x_j^-)} - \frac{\sqrt{2}M}{n^{3/2}} - \sqrt{\frac{\delta_{M,T}(j,n)}{n}}(x_{j+1}^+ - x_j^-)^{1/2}. \quad (3.65)$$

By the Cauchy-Schwartz inequality,

$$\sum_{j=k}^{n-1} \sqrt{\frac{\delta_{M,T}(j,n)}{n}}(x_{j+1}^+ - x_j^-)^{1/2} \leq \Big(\sum_{j=k}^{n-1} \frac{\delta_{M,T}(j,n)}{n} \sum_{i=k}^{n-1}(x_{i+1}^+ - x_i^-)\Big)^{1/2}$$
$$\leq \Big(\sum_{j=k}^{n-1} \frac{\delta_{M,T}(j,n)}{n}(f_X(1) + 1/n)\Big)^{1/2}$$

and applying (3.61), together with the fact that $f_X(1) \leq M$, gives

$$\Big(\sum_{j=k}^{n-1} \frac{\delta_{M,T}(j,n)}{n}(f(1)+1/n)\Big)^{1/2} \leq \Big(\frac{28M^5}{n^{1/2}} + \frac{10M^4 n}{T}\Big)^{1/2} \leq \frac{6M^{5/2}}{n^{1/4}} + \frac{4M^2 n^{1/2}}{T^{1/2}}.$$

Summing (3.65) over $k \leq j \leq \lfloor \theta n \rfloor - 1$ and substituting the above bound gives the result. $\square$

We can now prove our main proposition for this section.

Proof of Proposition 3.5. We first claim that for each $j = 0, \ldots, n-1$ we have

$$\mathcal{E}_X^+(I_j, \Gamma_{M,T}(f,n), T) \geq \frac{2R_X^-(j)}{n} - 2\sqrt{\frac{2R_X^+(j)}{n}(x_{j+1}^+ - x_j^-)} + x_{j+1}^- - x_j^+ - \frac{4}{n^2}. \quad (3.66)$$

Indeed, in either the $X+$ case or the $X-$ case, this follows directly from the definition of $\mathcal{E}_X^+$, even without the $4/n^2$ error term on the right-hand side. If we are in neither the $X+$ nor the $X-$ case, then $2R_X^-(j)/n \leq x_{j+1}^+ - x_j^-$ and $2R_X^+(j)/n \geq x_{j+1}^- - x_j^+$, so

$$\frac{2R_X^-(j)}{n} - 2\sqrt{\frac{2R_X^+(j)}{n}(x_{j+1}^+ - x_j^-)} + x_{j+1}^- - x_j^+$$
$$\leq x_{j+1}^+ - x_j^- - 2\sqrt{\big((x_{j+1}^- - x_j^+)\vee 0\big)(x_{j+1}^+ - x_j^-)} + \big(x_{j+1}^- - x_j^+\big)\vee 0$$
$$= \Big(\sqrt{x_{j+1}^+ - x_j^-} - \sqrt{\big(x_{j+1}^- - x_j^+\big)\vee 0}\Big)^2$$

and using $\sqrt{(a-b)\vee 0} \geq a^{1/2} - b^{1/2}$ we have

$$\sqrt{\big(x_{j+1}^- - x_j^+\big)\vee 0} \geq \sqrt{\big(x_{j+1}^+ - x_j^- - 4/n^2\big)\vee 0} \geq \sqrt{x_{j+1}^+ - x_j^-} - 2/n,$$

so in this case

$$\frac{2R_X^-(j)}{n} - 2\sqrt{\frac{2R_X^+(j)}{n}(x_{j+1}^+ - x_j^-)} + x_{j+1}^- - x_j^+$$
$$\leq 4/n^2 \leq \mathcal{E}_X^+(I_j, \Gamma_{M,T}(f,n), T) + 4/n^2$$

and the claim is proved.

Now write $K = \lfloor \theta n \rfloor$. By the upper bound in (3.60), for any $j \geq n^{1/2}$,

$$\int_{\frac{j}{n}}^{\frac{j+1}{n}} R_X^*(f(s))ds \leq \frac{R_X^-(j)}{n} + \frac{\delta_{M,T}(j,n)}{n}.$$

Summing over $k \leq j \leq K-1$ and applying (3.61) gives

$$\int_{k/n}^{K/n} R_X^*(f(s))ds \leq \sum_{j=k}^{K-1} \frac{R_X^-(j)}{n} + \frac{14M^4}{n^{1/2}} + \frac{5M^3 n}{T}.$$

Lemma 3.43 gives that

$$\int_{k/n}^{K/n} \sqrt{R_X^*(f(s))f_X'(s)}ds \geq \sum_{j=k}^{K-1} \sqrt{\frac{R_X^+(j)}{n}(x_{j+1}^+ - x_j^-)} - \frac{8M^{5/2}}{n^{1/4}} - \frac{4M^2 n^{1/2}}{T^{1/2}}.$$

Also

$$\int_{k/n}^{K/n} f_X'(s)ds \leq \sum_{j=k}^{K-1} \left(f(\tfrac{j+1}{n}) - f(\tfrac{j}{n})\right) \leq \sum_{j=k}^{K-1} \left(x_{j+1}^- + 1/n^2 - x_j^+ + 1/n^2\right)$$
$$\leq \sum_{j=k}^{K-1} \left(x_{j+1}^- - x_j^+\right) + 2/n.$$

Putting these bounds together with (3.66), and combining error terms, we obtain

$$\int_{k/n}^{K/n} \left(\sqrt{2R_X^*(f(s))} - \sqrt{f_X'(s)}\right)^2 ds \leq \sum_{j=k}^{K-1} \mathcal{E}_X^+(I_j, \Gamma_{M,T}(f,n), T) + O\Big(\frac{M^4}{n^{1/4}} + \frac{M^3 n}{T^{1/2}}\Big),$$

completing the proof. □

3.A.2 Proof of Lemma 3.6

The proof of Lemma 3.6 is relatively straightforward. The upper and lower bounds are very similar, but quite lengthy, so we separate them out into two proofs.

Proof of Lemma 3.6: upper bound. Write $K = \lfloor \theta n \rfloor$. We split the integral from 0 to θ into three parts:

$$\int_0^\theta R(Tg(s))ds \leq \int_0^{3MT^{-2/3}} R(Tg(s))ds + \int_{3MT^{-2/3}}^{\lceil\sqrt{n}\rceil/n} R(Tg(s))ds + \int_{\lceil\sqrt{n}\rceil/n}^{\theta} R(Tg(s))ds. \tag{3.67}$$

For the first term on the right-hand side, note that for any $g \in G^2_{M,T}$ and $s \le 3MT^{-2/3}$,

$$R(Tg(s)) \le \frac{MT(s + 2T^{-2/3}) + 1}{1} \le MT(3M + 2)T^{-2/3} + 1 \le 6M^2T^{1/3}.$$

For the second term on the right-hand side of (3.67), we note that for $s > 3MT^{-2/3}$ we have $2T^{-2/3} \le \frac{2}{3}s/M$ and therefore, since $g \in G^2_{M,T}$,

$$\begin{aligned} R(Tg(s)) \le \frac{MT(s + 2T^{-2/3}) + 1}{T(s/M - 2T^{-2/3})} \le \frac{MT(s + \frac{2s}{3M}) + 1}{T\frac{s}{3M}} &\le 3M^2\Big(1 + \frac{2}{3M}\Big) + \frac{3M}{Ts} \\ &\le 6M^2 + T^{-1/3}. \end{aligned}$$

We now consider the last term in (3.67), but work with any $k \ge \lceil \sqrt{n} \rceil$; since $g \in \Gamma_{M,T}(f, n)$, by definition of R^+_X and R^+_Y we have

$$\begin{aligned} \int_{k/n}^{\theta} R(Tg(s))ds &= \int_{k/n}^{\theta} R_X(Tg(s))ds + \int_{k/n}^{\theta} R_Y(Tg(s))ds \\ &\le \sum_{j=k}^{K} \int_{\frac{j}{n}}^{\frac{j+1}{n}} R^+_X(I_j, \Gamma_{M,T}(f, n), T)ds + \sum_{j=k}^{K} \int_{\frac{j}{n}}^{\frac{j+1}{n}} R^+_Y(I_j, \Gamma_{M,T}(f, n), T)ds \\ &= \sum_{j=k}^{K} \frac{R^+_X(j)}{n} + \sum_{j=k}^{K} \frac{R^+_Y(j)}{n}. \end{aligned} \tag{3.68}$$

By the lower bound in (3.60), for any $s \in [\frac{j}{n}, \frac{j+1}{n}]$,

$$R^+_X(j) \le R^*_X(f(s)) + \delta_{M,T}(j, n),$$

so using (3.61) and the fact that f is M-good,

$$\begin{aligned} \sum_{j=k}^{K} \frac{R^+_X(j)}{n} &\le \int_{k/n}^{(K+1)/n} R^*_X(f(s))ds + \sum_{j=k}^{K} \frac{\delta_{M,T}(j, n)}{n} \\ &\le \int_{k/n}^{(K+1)/n} R^*_X(f(s))ds + O\Big(\frac{M^4}{n^{1/2}} + \frac{M^3n}{T}\Big) \\ &\le \int_{k/n}^{K/n} R^*_X(f(s))ds + O\Big(\frac{M^4}{n^{1/2}} + \frac{M^3n}{T}\Big). \end{aligned}$$

By symmetry we also have

$$\sum_{j=k}^{K} \frac{R^+_Y(j)}{n} \le \int_{k/n}^{K/n} R^*_Y(f(s))ds + O\Big(\frac{M^4}{n^{1/2}} + \frac{M^3n}{T}\Big).$$

Substituting these bounds into (3.68) gives the upper bound in the second part of the lemma. For the first part of the lemma, returning to (3.67) and substituting in our estimates above for the three terms on the right-hand side, we have

$$\int_0^{\theta} R(Tg(s))ds \le \int_{k/n}^{K/n} R^*_X(f(s))ds + O\Big(\frac{M^3}{T^{1/3}} + \frac{M^4}{n^{1/2}} + \frac{1}{T^{1/3}n^{1/2}} + \frac{M^3n}{T}\Big).$$

Since $R_X^*(f(s)) \geq 0$ for all s, the result follows. □

Proof of Lemma 3.6: lower bound. Note that, again writing $K = \lfloor \theta n \rfloor$ but now with any k satisfying $\lceil \sqrt{n} \rceil \leq k \leq K$,

$$\int_{k/n}^{\theta} R(Tg(s))ds \geq \int_{k/n}^{K/n} R(Tg(s))ds.$$

Since $g \in \Gamma_{M,T}(f,n)$, by definition of R_X^- and R_Y^- we have

$$\begin{aligned}
\int_{k/n}^{\theta} R(Tg(s))ds &= \int_{k/n}^{\theta} R_X(Tg(s))ds + \int_{k/n}^{\theta} R_Y(Tg(s))ds \\
&\geq \sum_{j=k}^{K-1} \int_{\frac{j}{n}}^{\frac{j+1}{n}} R_X^-(I_j, \Gamma_{M,T}(f,n), T)ds + \sum_{j=k}^{K-1} \int_{\frac{j}{n}}^{\frac{j+1}{n}} R_Y^-(I_j, \Gamma_{M,T}(f,n), T)ds \\
&= \sum_{j=k}^{K-1} \frac{R_X^-(j)}{n} + \sum_{j=k}^{K-1} \frac{R_Y^-(j)}{n}.
\end{aligned}$$

By the upper bound in (3.60), for any $s \in [\frac{j}{n}, \frac{j+1}{n}]$, we have $R_X^-(j) \geq R_X^*(f(s)) - \delta_{M,T}(j,n)$, so using (3.61) and the fact that f is M-good,

$$\begin{aligned}
\sum_{j=k}^{K-1} \frac{R_X^-(j)}{n} &\geq \int_{k/n}^{K/n} R_X^*(f(s))ds - \sum_{j=k}^{K-1} \frac{\delta_{M,T}(j,n)}{n} \\
&\geq \int_{k/n}^{K/n} R_X^*(f(s))ds - O\Big(\frac{M^4}{n^{1/2}} + \frac{M^3 n}{T}\Big).
\end{aligned}$$

By symmetry we also have

$$\sum_{j=k}^{K-1} \frac{R_Y^-(j)}{n} \geq \int_{k/n}^{K/n} R_Y^*(f(s))ds - O\Big(\frac{M^4}{n^{1/2}} + \frac{M^3 n}{T}\Big).$$

Combining these bounds gives the result. □

3.A.3 Proof of Lemma 3.22

The main difference between Lemma 3.22 and our previous deterministic bounds on the rate function is that it requires us to consider more general time intervals than those of the form $[j/n, (j+1)/n]$. Lemma 3.44 will do most of the work required, and uses the uniform structure of $\Lambda_{M,T}(f,n)$ to get better bounds than are possible for $\Gamma_{M,T}(f,n)$.

Lemma 3.44. *Suppose that $M, T > 1$, $n \geq 2M$ and $f \in PL_n^2 \cap G_M^2$. Then for any $j \in \{\lceil \sqrt{n} \rceil, \dots, n-1\}$ and u, v such that $\frac{j}{n} \leq u < v \leq \frac{j+1}{n}$,*

$$\begin{aligned}
&\int_u^v \Big(\sqrt{2R_X^*(f(s))} - \sqrt{f_X'(s)}\Big)^2 ds \\
&\leq \mathcal{E}_X^+\big([u,v], \Lambda_{M,T}(f,n), T\big) + \frac{6\delta_{M,T}(j,n)}{n} + 2\sqrt{\frac{2\delta_{M,T}(j,n)}{n}\big(f_X(\tfrac{j+1}{n}) - f_X(\tfrac{j}{n})\big)} + \frac{14M}{n^{3/2}}.
\end{aligned}$$

Proof. As in the proof of Lemma 3.18, for $I \subset [0,1]$ we write $\hat{R}_X^-(I)$ as shorthand for

$R_X^-(I, \Lambda_{M,T}(f,n), T)$, and similarly for $\hat{R}_X^+(I)$, $\hat{R}_Y^-(I)$ and $\hat{R}_Y^+(I)$. We also write, for $s \in [0,1]$,

$$x^-(s) = x^-(s, \Lambda_{M,T}(f,n)) = \inf\{g_X(s) : g \in \Lambda_{M,T}(f,n)\}$$

and similarly for $x^+(s)$, $y^-(s)$ and $y^+(s)$.

By (3.14) and the fact that f is linear on I_j (and therefore on $[u,v]$), we have

$$\begin{aligned}
&\int_u^v \left(\sqrt{2R_X^*(f(s))} - \sqrt{f_X'(s)}\right)^2 ds \\
&= 2\int_u^v R_X^*(f(s))ds + \int_u^v f_X'(s)ds - 2\int_u^v \sqrt{2R_X^*(f(s))f_X'(s)}ds \\
&\leq 2\hat{R}_X^-(I_j)(v-u) + 2\delta_{M,T}(j,n)(v-u) + f_X(v) - f_X(u) \\
&\qquad\qquad - 2\int_u^v \sqrt{2R_X^*(f(s))\frac{f_X(v)-f_X(u)}{v-u}}ds. \qquad (3.69)
\end{aligned}$$

Applying (3.14) and using the elementary inequality $\sqrt{(a-b)\vee 0} \geq \sqrt{a} - \sqrt{b}$, valid for all $a, b \geq 0$, for any $s \in I_j$ we have

$$\begin{aligned}
\sqrt{R_X^*(f(s))} &\geq \sqrt{\left(\hat{R}_X^+(I_j) - \delta_{M,T}(j,n)\right) \vee 0} \\
&\geq \sqrt{\hat{R}_X^+(I_j)} - \sqrt{\delta_{M,T}(j,n)} \geq \sqrt{\hat{R}_X^+([u,v])} - \sqrt{\delta_{M,T}(j,n)}
\end{aligned}$$

so we have

$$\begin{aligned}
&\int_u^v \sqrt{2R_X^*(f(s))\frac{f_X(v)-f_X(u)}{v-u}}ds \\
&\geq \int_u^v \left(\sqrt{2\hat{R}_X^+([u,v])} - \sqrt{2\delta_{M,T}(j,n)}\right)\sqrt{\frac{f_X(v)-f_X(u)}{v-u}}ds \\
&\geq \sqrt{2\hat{R}_X^+([u,v])(f_X(v)-f_X(u))(v-u)} - \sqrt{\frac{2\delta_{M,T}(j,n)}{n}\left(f_X(\tfrac{j+1}{n}) - f_X(\tfrac{j}{n})\right)}. \qquad (3.70)
\end{aligned}$$

Using again that $\sqrt{(a-b)\vee 0} \geq \sqrt{a} - \sqrt{b}$ we have

$$\begin{aligned}
\sqrt{f_X(v) - f_X(u)} &\geq \sqrt{(x^+(v) - 1/n^2 - (x^-(u) + 1/n^2)) \vee 0} \\
&\geq \sqrt{x^+(v) - x^-(u)} - \sqrt{2/n^2},
\end{aligned}$$

and since f is M-good, and therefore by (3.14) $R_X^+([u,v]) \leq M^2 + \delta_{M,T}(j,n)$, we deduce that

$$\begin{aligned}
&\sqrt{2\hat{R}_X^+([u,v])(f_X(v)-f_X(u))(v-u)} \\
&\qquad\qquad \geq \sqrt{2\hat{R}_X^+([u,v])(x^+(v)-x^-(u))(v-u)} - \frac{2\sqrt{M^2+\delta_{M,T}(j,n)}}{n^{3/2}}.
\end{aligned}$$

Substituting this into (3.70), and using that

$$\sqrt{M^2 + \delta_{M,T}(j,n)} \leq \sqrt{M^2} + \sqrt{\delta_{M,T}(j,n)} \leq M + \delta_{M,T}(j,n) + 1$$

gives that

$$\int_u^v \sqrt{2R_X^*(f(s))\frac{f_X(v)-f_X(u)}{v-u}}ds$$
$$\geq \sqrt{2\hat{R}_X^+([u,v])(v-u)(x^+(v)-x^-(u))} - \sqrt{\frac{2\delta_{M,T}(j,n)}{n}\big(f_X(\tfrac{j+1}{n})-f_X(\tfrac{j}{n})\big)}$$
$$- \frac{2(M+\delta_{M,T}(j,n)+1)}{n^{3/2}}.$$

Substituting this bound into (3.69) and using that $\hat{R}_X^-(I_j) \leq \hat{R}_X^-([u,v])$ and $v-u \leq 1/n$, we obtain

$$\int_u^v \left(\sqrt{2R_X^*(f(s))} - \sqrt{f_X'(s)}\right)^2 ds$$
$$\leq 2\hat{R}_X^-([u,v])(v-u) + x^-(v) - x^+(u) - 2\sqrt{2\hat{R}_X^+([u,v])(v-u)(x^+(v)-x^-(u))}$$
$$+ \frac{6\delta_{M,T}(j,n)}{n} + \frac{2}{n^2} + 2\sqrt{\frac{2\delta_{M,T}(j,n)}{n}\big(f_X(\tfrac{j+1}{n})-f_X(\tfrac{j}{n})\big)} + \frac{8M}{n^{3/2}}.$$

It then remains to note that, following exactly the same argument as (3.66),

$$\mathcal{E}_X^+\big([u,v],\Lambda_{M,T}(f,n),T\big)$$
$$\geq 2\hat{R}_X^-([u,v])(v-u)+x^-(v)-x^+(u)-2\sqrt{2\hat{R}_X^+([u,v])(v-u)(x^+(v)-x^-(u))}-4/n^2.$$

Combining error terms gives the result. □

It is now a relatively simple task to apply Lemma 3.44 to complete the proof of Lemma 3.22.

Proof of Lemma 3.22. By symmetry it suffices to show that

$$\sum_{j=\lfloor an\rfloor}^{\lceil bn\rceil-1} \mathcal{E}_X^+(I_j\cap[a,b],\Lambda_{M,T}(f,n),T) \geq \int_a^b \left(\sqrt{2R_X^*(f(s))}-\sqrt{f_X'(s)}\right)^2 ds - O\Big(\frac{M^4}{n^{1/4}}+\frac{M^3 n}{T^{1/2}}\Big).$$

By Lemma 3.44,

$$\sum_{j=\lfloor an\rfloor}^{\lceil bn\rceil-1} \mathcal{E}_X^+(I_j\cap[a,b],\Lambda_{M,T}(f,n),T)$$
$$\geq \sum_{j=\lfloor an\rfloor}^{\lceil bn\rceil-1} \int_{I_j\cap[a,b]} \left(\sqrt{2R_X^*(f(s))} - \sqrt{f_X'(s)}\right)^2 ds$$
$$- \sum_{j=\lfloor an\rfloor}^{\lceil bn\rceil-1} \left(\frac{6\delta_{M,T}(j,n)}{n} + 2\sqrt{\frac{2\delta_{M,T}(j,n)}{n}\big(f_X(\tfrac{j+1}{n})-f_X(\tfrac{j}{n})\big)} + \frac{14M}{n^{3/2}}\right).$$

Note that

$$\sum_{j=\lfloor an\rfloor}^{\lceil bn\rceil-1} \int_{I_j\cap[a,b]} \left(\sqrt{2R_X^*(f(s))} - \sqrt{f_X'(s)}\right)^2 ds = \int_a^b \left(\sqrt{2R_X^*(f(s))} - \sqrt{f_X'(s)}\right)^2 ds,$$

and by (3.61)

$$\sum_{j=\lfloor an\rfloor}^{\lceil bn\rceil-1}\Big(\frac{6\delta_{M,T}(j,n)}{n}+\frac{14M}{n^{3/2}}\Big)=O\Big(\frac{M^4}{n^{1/2}}+\frac{M^3 n}{T}\Big).$$

Finally, by Cauchy-Schwarz,

$$\sum_{j=\lfloor an\rfloor}^{\lceil bn\rceil-1}\sqrt{\frac{2\delta_{M,T}(j,n)}{n}\big(f_X(\tfrac{j+1}{n})-f_X(\tfrac{j}{n})\big)}$$
$$\le\Big(\sum_{j=\lfloor an\rfloor}^{\lceil bn\rceil-1}\frac{2\delta_{M,T}(j,n)}{n}\Big)^{1/2}\Big(\sum_{j=\lfloor an\rfloor}^{\lceil bn\rceil-1}\big(f_X(\tfrac{j+1}{n})-f_X(\tfrac{j}{n})\big)\Big)^{1/2},$$

and using (3.61) and the fact that $f\in G^2_M$, we see that

$$\sum_{j=\lfloor an\rfloor}^{\lceil bn\rceil-1}\sqrt{\frac{2\delta_{M,T}(j,n)}{n}\big(f_X(\tfrac{j+1}{n})-f_X(\tfrac{j}{n})\big)}=O\Big(\frac{M^{5/2}}{n^{1/4}}+\frac{M^2 n^{1/2}}{T^{1/2}}\Big).$$

Combining these estimates completes the proof. □

3.B. Proofs of compactness and semicontinuity

3.B.1 Compactness of $G^2_{M,T}$: proof of Lemma 3.36

The proof of Lemma 3.36, which says that for any $F\subset E^2$ we can cover $F\cap G^2_{M,T}$ in a nice way with small balls around piecewise linear functions, is straightforward. We directly construct piecewise linear approximations to an arbitrary function within $F\cap G^2_{M,T}$.

Proof of Lemma 3.36. Suppose that $T\ge(4Mn)^{3/2}$ and take $h\in F\cap G^2_{M,T}$. Then define a function $g\in \mathrm{PL}^2_n$ by interpolating linearly between the values

$$g(j/n)=\lfloor n^2h(j/n)\rfloor/n^2,\qquad j=0,1,\dots,n.$$

Then clearly

$$\Delta_n(g,h)<1/n^2.$$

We claim that $d(g,h)\le 1/n$. To see this, take $s\in[0,1]$, and then fix $j\in\{0,1,\dots,n-1\}$ such that $s\in[j/n,(j+1)/n]$. Then

$$g(s)\le g(\tfrac{j+1}{n})\le h(\tfrac{j+1}{n})$$

and

$$g(s)\ge g(\tfrac{j}{n})\ge h(\tfrac{j}{n})-1/n^2$$

which, by the definition of d, establishes the claim.

Next we claim that $g\in G^2_{4M}$. Since $h\in G^2_{M,T}$ we know that for any $j=1,2,\dots,n$,

$$\frac{j}{Mn}-2T^{-2/3}\le h(j/n)\le M\Big(\frac{j}{n}+2T^{-2/3}\Big)$$

and since $T \geq (4Mn)^{3/2}$ we obtain

$$\frac{j}{2Mn} \leq \frac{j - 1/2}{Mn} \leq h(j/n) \leq \frac{Mj + 1/2}{n} \leq \frac{2Mj}{n}.$$

But $h(j/n) - 1/n^2 \leq g(j/n) \leq h(j/n)$ so

$$\frac{j}{2Mn} - \frac{1}{n^2} \leq g(j/n) \leq \frac{2Mj}{n}$$

and since $n \geq 4M$, $j/(2Mn) - 1/n^2 \geq j/(4Mn)$, which, given that g interpolates linearly between these values, proves the claim.

Since the functions g created in this way can take only finitely many values (namely integer multiples of $1/n^2$ with a maximum of at most $2M$) at the times $0, 1/n, 2/n, \ldots, 1$, and interpolate linearly between these values, there are only finitely many possible such functions, and therefore the proof is complete. $\square$

3.B.2 Partial lower semi-continuity of $\tilde{K}$: proof of Proposition 3.33

To complete the proof of the lower bound in Section 3.3, we need to prove a partial semi-continuity result about $\tilde{K}$, which was stated in Proposition 3.33. We begin with a useful lemma which states that given continuity, convergence under d implies convergence pointwise.

Lemma 3.45. *If $f \in E$ is continuous at s and $d(f_n, f) \to 0$, then $f_n(s) \to f(s)$. Moreover, if $d(f_n, f) \to 0$, then $f_n(1) \to f(1)$ (regardless of whether f is continuous at 1).*

Proof. Fix $\varepsilon > 0$ and $s \in [0, 1]$ such that f is continuous at s. Then we can find $\delta > 0$ such that $|f(u) - f(s)| < \varepsilon/2$ for any $u \in [s - \delta, s + \delta] \cap [0, 1]$. Choose N such that $d(f_n, f) < (\varepsilon/2) \wedge \delta$ for all $n \geq N$. By the definition of d, this means that

$$f((s - \delta) \vee 0) - \varepsilon/2 \leq f_n(s) \leq f((s + \delta) \wedge 1) + \varepsilon/2.$$

Then we have

$$f(s) - \varepsilon \leq f((s - \delta) \vee 0) - \varepsilon/2 \leq f_n(s) \leq f((s + \delta) \wedge 1) + \varepsilon/2 \leq f(s) + \varepsilon$$

and since $\varepsilon > 0$ was arbitrary, we have shown that $f_n(s) \to f(s)$.

For the second part of the lemma, simply note that by the definition of d, if $d(f_n, f) < \varepsilon$ then $|f_n(1) - f(1)| < \varepsilon$. $\square$

We now show that when f_n is the piecewise linear interpolation to f, the cross-terms that appear when multiplying out the quadratic terms in $\tilde{K}$ satisfy a semicontinuity property.

Lemma 3.46. *Suppose that $0 \leq a < b \leq 1$ and that $f \in G_M^2$ for some M. Let f_n be the function in PL_n constructed by setting $f_n(j/n) = f(j/n)$ for each $j = 0, \ldots, n$ and* *interpolating linearly. Then*

$$\liminf_{n\to\infty} \int_a^b \sqrt{R_X^*(f_n(s))f'_{n,X}(s)}ds \geq \int_a^b \sqrt{R_X^*(f(s))f'_X(s)}ds$$

where we write $f_n(s) = (f_{n,X}(s), f_{n,Y}(s))$.

Proof. We carry out the proof when $a = 0$ and $b = 1$; the general case follows by including $\mathbb{1}_{\{s\in[a,b]\}}$ throughout.

Note that

$$\begin{aligned}
&\int_0^1 \sqrt{R_X^*(f_n(s))f'_{n,X}(s)}ds \\
&\qquad = \int_0^1 \sum_{i=1}^n \mathbb{1}_{\{s\in[\frac{i-1}{n},\frac{i}{n})\}} \sqrt{R_X^*(f_n(s))}\sqrt{n(f_{n,X}(\tfrac{i}{n}) - f_{n,X}(\tfrac{i-1}{n}))}ds \\
&\qquad = \int_0^1 \sum_{i=1}^n \mathbb{1}_{\{s\in[\frac{i-1}{n},\frac{i}{n})\}} \sqrt{R_X^*(f_n(s))}\sqrt{n(f_X(\tfrac{i}{n}) - f_X(\tfrac{i-1}{n}))}ds \\
&\qquad \geq \int_0^1 \sum_{i=1}^n \mathbb{1}_{\{s\in[\frac{i-1}{n},\frac{i}{n})\}} \inf_{u\in[\frac{i-1}{n},\frac{i}{n}]} \sqrt{R_X^*(f_n(u))}\sqrt{n(f_X(\tfrac{i}{n}) - f_X(\tfrac{i-1}{n}))}\, ds.
\end{aligned}$$

Since f is continuous almost everywhere, by Lemma 3.45, $f_n(u) \to f(u)$ almost everywhere. Since f is M-good, and R_X^* is continuous away from 0, $R_X^*(f_n(u)) \to R_X^*(f(u))$ for almost every $u \in [0,1]$. Since f is differentiable almost everywhere, we deduce that the integrand above converges to $\sqrt{R_X^*(f(s))f'_X(s)}$ for almost every $s \in [0,1]$. It is also bounded above by

$$F_n(s) = \sum_{i=1}^n \mathbb{1}_{\{s\in[\frac{i-1}{n},\frac{i}{n})\}} M\Big(n\Big(f_X\Big(\frac{i}{n}\Big) - f_X\Big(\frac{i-1}{n}\Big)\Big) + 1\Big)$$

which is integrable and whose integral equals $M(f_X(1) + 1)$ for each n, which is also the integral of $\lim_{n\to\infty} F_n(s)$. Therefore, by the generalised dominated convergence theorem, the integral converges to

$$\int_0^1 \sqrt{R_X^*(f(s))f'_X(s)}\, ds$$

and the proof is complete. □

It is then a simple task to prove Proposition 3.33, which shows that $\tilde{K}(f,0,t)$ can be bounded above by taking piecewise linear approximations to f.

Proof of Proposition 3.33. By (3.3), for any $f \in E^2$,

$$\begin{aligned}
\tilde{K}(f,0,t) = -\int_0^t R^*(f(s))ds + 2\sqrt{2}\int_0^t \sqrt{R_X^*(f(s))f'_X(s)}ds - f_X(t) \\
+ 2\sqrt{2}\int_0^t \sqrt{R_Y^*(f(s))f'_Y(s)}ds - f_Y(t).
\end{aligned}$$

It therefore suffices, by symmetry, to show that

$$\limsup_{n\to\infty}\int_0^t R^*(f_n(s))ds \le \int_0^t R^*(f(s))ds,$$

$$\liminf_{n\to\infty}\int_0^t \sqrt{R_X^*(f_n(s))f'_{n,X}(s)}ds \ge \int_0^t \sqrt{R_X^*(f(s))f'_X(s)}ds$$

and

$$\limsup_{n\to\infty} f_{n,X}(t) \le f_X(t).$$

The first of these statements follows from Lemma 3.45 and the continuity and boundedness of R^* away from 0, using the fact that f, and therefore f_n, is good. The second follows from Lemma 3.46. For the third, we observe that since f is increasing and right-continuous,

$$f_{n,X}(t) \le f_X\big(\tfrac{\lceil nt\rceil}{n}\big) \to f_X(t),$$

which completes the proof. □

3.B.3 Upper semi-continuity of $\tilde{K}$: proofs of Proposition 3.37 and Corollary 3.38

The following consequence of the Cauchy-Schwarz inequality is the key to proving Proposition 3.37.

Lemma 3.47. *Suppose that $0 \le a < b \le 1$ and $f, f_n \in G_M^2$ for all n. If f is differentiable on $[a,b]$, and $d(f_n, f) \to 0$, then*

$$\limsup_{n\to\infty}\int_a^b \sqrt{R_X^*(f_n(s))f'_{n,X}(s)}ds \le \int_a^b \sqrt{R_X^*(f(s))f'_X(s)}ds$$

where we write $f_{n,X}$ for the x-component of f_n.

Proof. We carry out the proof when $a = 0$ and $b = 1$; the general case follows by including $\mathbb{1}_{\{s\in[a,b]\}}$ throughout. By the Cauchy-Schwarz inequality, for any $m \in \mathbb{N}$,

$$\begin{aligned}\int_0^1 \sqrt{R_X^*(f_n(s))f'_{n,X}(s)}ds &= \sum_{i=1}^m \int_{(i-1)/m}^{i/m} \sqrt{R_X^*(f_n(s))f'_{n,X}(s)}ds\\ &\le \sum_{i=1}^m \Big(\int_{(i-1)/m}^{i/m} R_X^*(f_n(s))ds\Big)^{1/2}\Big(\int_{(i-1)/m}^{i/m} f'_{n,X}(s)ds\Big)^{1/2}\\ &\le \sum_{i=1}^m \Big(\int_{(i-1)/m}^{i/m} R_X^*(f_n(s))ds\Big)^{1/2}\big(f_{n,X}\big(\tfrac{i}{m}\big) - f_{n,X}\big(\tfrac{i-1}{m}\big)\big)^{1/2}\end{aligned}$$

where the last inequality is not an equality since we do not know whether $f_{n,X}$ is absolutely continuous. Since f is continuous, by Lemma 3.45 we know that $f_n(s) \to f(s)$ for every s. Thus, using that $f, f_n \in G_M^2$ and R_X^* is continuous away from 0, by bounded convergence the right-hand side above converges to

$$\sum_{i=1}^m \Big(\int_{(i-1)/m}^{i/m} R_X^*(f(s))ds\Big)^{1/2}\Big(f_X\Big(\frac{i}{m}\Big) - f_X\Big(\frac{i-1}{m}\Big)\Big)^{1/2}$$

which is at most

$$\sum_{i=1}^{m} \frac{1}{m} \Big(\sup_{u \in [\frac{i-1}{m}, \frac{i}{m}]} R_X^*(f(u)) \Big)^{1/2} \Big(m \Big(f_X\Big(\frac{i}{m}\Big) - f_X\Big(\frac{i-1}{m}\Big) \Big) \Big)^{1/2}. \tag{3.71}$$

We claim that (3.71) converges, as $m \to \infty$, to $\int_0^1 \sqrt{R_X^*(f(s)) f_X'(s)} ds$. To prove this we can follow almost exactly the same argument as in the proof of Lemma 3.46, writing (3.71) in the form

$$\int_0^1 \sum_{i=1}^{m} \mathbb{1}_{\{s \in [\frac{i-1}{m}, \frac{i}{m})\}} \Big(\sup_{u \in [\frac{i-1}{m}, \frac{i}{m}]} R_X^*(f(u)) \Big)^{1/2} \Big(m \Big(f_X\Big(\frac{i}{m}\Big) - f_X\Big(\frac{i-1}{m}\Big) \Big) \Big)^{1/2} ds$$

and applying the generalised dominated convergence theorem since the integrand evaluated at s converges as $m \to \infty$ to $\sqrt{R_X^*(f(s)) f_X'(s)}$ for almost every $s \in [0, 1]$, and can be bounded above by

$$F_m(s) = \sum_{i=1}^{m} \mathbb{1}_{\{s \in [\frac{i-1}{m}, \frac{i}{m})\}} M \Big(m \Big(f_X\Big(\frac{i}{m}\Big) - f_X\Big(\frac{i-1}{m}\Big) \Big) + 1 \Big).$$

This completes the proof. □

The next step is to extend the previous lemma to functions that are not necessarily continuous.

Lemma 3.48. *Suppose that $0 \leq a < b \leq 1$ and $f, f_n \in G_M^2$ for all n. If $d(f_n, f) \to 0$, then*

$$\limsup_{n \to \infty} \int_a^b \sqrt{R_X^*(f_n(s)) f_{n,X}'(s)} ds \leq \int_a^b \sqrt{R_X^*(f(s)) f_X'(s)} ds$$

where we write $f_{n,X}$ for the x-component of f_n.

Proof. Fix $\varepsilon \in (0, 6M)$. Let $S \subset (0, 1)$ be the set of points (in $(0, 1)$) at which f is not differentiable. Since f is increasing, S has zero Lebesgue measure, and can therefore be covered by a finite collection $(s_1^-, s_1^+), \ldots, (s_N^-, s_N^+)$ of open intervals whose total length is at most ε^2/M^3. Let $S' = \bigcup_{i=1}^N (s_i^-, s_i^+)$. Then by Lemma 3.47, since $[a, b] \setminus S'$ is a finite union of closed intervals on which f is absolutely continuous, we have

$$\begin{aligned} \limsup_{n \to \infty} \int_{[a,b] \setminus S'} \sqrt{R_X^*(f_n(s)) f_{n,X}'(s)} ds &\leq \int_{[a,b] \setminus S'} \sqrt{R_X^*(f(s)) f_X'(s)} ds \\ &\leq \int_a^b \sqrt{R_X^*(f(s)) f_X'(s)} ds. \end{aligned}$$

It therefore suffices to show that

$$\limsup_{n \to \infty} \int_{[a,b] \cap S'} \sqrt{R_X^*(f_n(s)) f_{n,X}'(s)} ds \leq \varepsilon. \tag{3.72}$$

However, since $f_n \in G_M^2$, we have

$$\int_{[a,b] \cap S'} \sqrt{R_X^*(f_n(s)) f_{n,X}'(s)} ds \leq \int_{[a,b] \cap S'} M \sqrt{f_{n,X}'(s)} ds,$$

and by Jensen's inequality, this is at most

$$M\sqrt{|[a,b]\cap S'|}\left(\int_{[a,b]\cap S'} f'_{n,X}(s)ds\right)^{1/2} \leq M\sqrt{|S'|}(f_{n,X}(b) - f_{n,X}(a))^{1/2}$$

where $|S'|$ denotes the Lebesgue measure of S'. Since $f_n \in G^2_M$, this is at most $M^{3/2}\sqrt{|S'|}$, which is smaller than ε by construction. Thus (3.72) holds and the proof is complete. $\square$

The proof of Proposition 3.37 is now a simple consequence of the results above.

Proof of Proposition 3.37. We use the alternative form of $\tilde{K}$ mentioned in (3.3), i.e.

$$\begin{aligned}\tilde{K}(f,0,\theta) = -\int_0^\theta R^*(f(s))ds + 2\sqrt{2}\int_0^\theta \sqrt{R^*_X(f(s))f'_X(s)}ds \\ + 2\sqrt{2}\int_0^\theta \sqrt{R^*_Y(f(s))f'_Y(s)}ds - f_X(\theta) - f_Y(\theta).\end{aligned} \tag{3.73}$$

Since either f is continuous at θ, or $\theta = 1$, by Lemma 3.45 we have

$$f_{n,X}(\theta) + f_{n,Y}(\theta) \to f_X(\theta) + f_Y(\theta).$$

Since f is continuous almost everywhere, by Lemma 3.45 and the continuity of R^* away from 0 (using the fact that $f_n, f \in G^2_M$), we have

$$\int_0^\theta R^*(f_n(s))ds \to \int_0^\theta R^*(f(s))ds.$$

The result then follows from Lemma 3.48 and the symmetry between the X and Y components. $\square$

Corollary 3.38 follows easily from Proposition 3.37.

Proof of Corollary 3.38. For each $n \in \mathbb{N}$, take $f_n \in B_d(F, 1/n) \cap G^2_M$ such that

$$\tilde{K}(f_n, 0, 1) \geq \sup_{f\in B_d(F,1/n)\cap G^2_M} \tilde{K}(f, 0, 1) - 1/n.$$

By Lemma 3.36 we know that $G^2_{M,T}$ is totally bounded, and since $G^2_M \subset G^2_{M,T}$ and is closed, and (E^2, d) is complete, we deduce that G^2_M is compact under d. Therefore there exists a subsequence $(f_{n_j})_{j\geq 1}$ such that $d(f_{n_j}, f_\infty) \to 0$ as $j \to \infty$ for some $f_\infty \in G^2_M$. Since $d(f_{n_j}, f_\infty) \to 0$, and F is closed, we also have $f_\infty \in F$. By Proposition 3.37

$$\limsup_{j\to\infty} \tilde{K}(f_{n_j}, 0, 1) \leq \tilde{K}(f_\infty, 0, 1).$$

Then by our choice of f_n,

$$\begin{aligned}\limsup_{j\to\infty} \sup_{f\in B_d(F,1/n_j)\cap G^2_M} &\tilde{K}(f,0,1)\\ &\le \limsup_{j\to\infty}(\tilde{K}(f_{n_j},0,1)+1/n_j) \le \tilde{K}(f_\infty,0,1) \le \sup_{f\in F\cap G^2_M} \tilde{K}(f,0,1)\end{aligned}$$

which completes the proof. □

Chapter 4

A small deviations result for the unscaled paths of random walks

4.1. Introduction

In many applications it is of interest to estimate the probability that the path of a process is close to a given function. In this chapter, we consider this problem for a compound Poisson process $(X(s),\, s \geq 0)$ starting from the origin, which jumps at rate r and has jump distribution ξ satisfying $\mathbb{E}[e^{\eta|\xi|}] < \infty$ for some $\eta > 0$.

Let $L > 0$, $p, q \in [-1, 1]$ with $p < q$ be constants. We define $L_T := LT^{1/3}$, but our proof in principle could be extended to $L_T = LT^{\gamma}$ with $\gamma < 1/2$. We determine the behaviour of

$$\mathbb{P}_0\big(|X(s) - F_T(s)| < L_T \;\forall s \in [0, T],\; X(T) - F_T(T) \in (pL_T, qL_T)\big)$$

when T is large, for a specific class of functions. We assume that $F_T : [0, \infty) \to \mathbb{R}$ is of the form $F_T(s) = zs + G_T(s)$, where $z \in \mathbb{R}$ is a constant independent of T, $G_T(s) = G(s) - x_T$ is a twice differentiable function such that $G(0) = 0$ and $x_T \in (-L_T, L_T)$ satisfies $x_T/T^{1/3} \to x$, with $x \in (-L, L)$.

We stress that G is a function independent of T and satisfies some extra properties, which we postpone for now, but essentially require that $|G'|$ decreases fast enough at large times. Our result can also be extended to the case in which L, p and q depend on T, but converge to constants as T tends to infinity.

The greatest challenge in this work is that we consider tubes centred about functions with a nonlinear component. Using a standard change of measure, we can deduce the result from Mogul'skiii [38] in the case $F(s) = zs$. As we have mentioned in Chapter 1, in the proof of Theorem 1.3 the problem is reduced to estimating the probability that a process with mean zero stays in a strip of constant width around the zero function, and this is not strongly affected by the position at which the process starts and ends at the beginning and at the end of the interval. However, this is no longer the case when $F(s)$ is nonlinear.

Our proof is based on a Brownian motion approximation on smaller intervals covering $[0, T]$. This way, we translate our problem into the estimate of probabilities that a Brownian motion lies in a tube about a given function, which is a much simpler task. By putting together the approximations on the smaller intervals we find our result on the probability that $X(s)$ stays near $F_T(s)$ on $[0, T]$. We now introduce our definitions more formally.

Let $T_0 = T^{1/3-\varepsilon}$, with $\varepsilon \in (0, 1/3)$.

We split $[T_0, T]$ into $N_T := \lfloor T^{1/3-\nu} \rfloor$ intervals for $\nu \in (0, 1/3)$, so that each interval has length $\Delta_T = (T - T_0)/N_T$ satisfying $\lim_{T\to\infty} \Delta_T / T^{2/3+\nu} = 1$.

These choices for T_0 and N_T are motivated by the fact that we are considering the probability that a process stays in a tube of width L_T, and the Brownian motion approximation of a compound Poisson process is accurate when the length of the time interval, divided by the square of the tube width, tends to infinity. In our case, this means that we need $\Delta_T / L_T^2 \to \infty$. This condition is satisfied for any N_T smaller than $\lfloor T^{1/3-\nu} \rfloor$, even $N_T = 1$, and in fact for some choices of $G(s)$ our theorem could be proved without splitting $[T_0, T]$ into smaller intervals. However, some technical conditions on $G(s)$ are needed to make the Brownian approximation precise, which are only required locally on each interval, so that having a large number of them ultimately allows to consider a wider class of functions. More details about this will appear later, in Section 4.7.

On a different note, we will need a different argument to estimate the probability that $X(s) - zs$ stays near $G_T(s)$ at small times, up to time T_0. Using the fact that $G(s)$ does not grow wildly, we will bound from below this probability using the fact that T_0 is much smaller compared to the tube width, which is guaranteed by our choice of $T_0 = T^{1/3-\varepsilon}$.

We can now state the properties required on $G(s)$, which we need in different parts of our proof. These assumptions ensure small errors in the approximation of the probability that $X(s)-zs$ stays near $G_T(s)$ with the probability that a Brownian motion stays near $G_T(s)$ on each interval of length Δ_T, and also guarantee that combining these errors on the smaller intervals gives an error which is small enough.

We assume that it is possible to find a sequence $\delta_T \in (0,1)$ with $\lim_{T\to\infty} \delta_T = 0$, such that

(i) $\lim\limits_{T\to\infty} \delta_T^2 N_T = \infty$, $\lim\limits_{T\to\infty} \dfrac{\log(T)}{L_T \delta_T} = 0$ and $\left| \dfrac{x_T}{L_T} - \dfrac{x}{L} \right| < \dfrac{\sqrt{\delta_T}}{8}$;

(ii) $\lim\limits_{T\to\infty} \dfrac{1}{\delta_T} \sup\limits_{s\in[T_0,T]} |G'(s)| = 0$;

(iii) $\lim\limits_{T\to\infty} \sqrt{\delta_T} \sum\limits_{i=1}^{N_T} |G'(T_i)| = 0$;

(iv) $\lim\limits_{T\to\infty} \sup\limits_{1\le i\le N_T} \dfrac{1}{L_T \delta_T} \displaystyle\int_{T_{i-1}}^{T_i} G'(s)^2 ds = 0$;

(v) $\lim\limits_{T\to\infty} \displaystyle\int_{T_0}^{T} |G''(s)| ds = 0$.

We state the following property separately for future reference, since some of our results, in particular Proposition 4.12, will require this assumption:

(vi) there exists a constant $M > 0$, independent of T, such that $|G'(s)| \le M \ \forall s \ge 0$.

Note that this is a consequence of (ii). Indeed, (ii) implies that there exists a constant $M' > 0$ independent of T such that

$$\sup_{s\in[T_0,T]} |G'(s)| \le M' \delta_T.$$

This in particular gives that $|G'(T)| \le M' \delta_T$ and so $\lim_{T\to\infty} |G'(T)| = 0$, from which (vi) follows using that G is continuously differentiable and independent of T.

The requirements on $G(s)$ implied by the assumptions (i)-(v) seem to be implicit. To clarify this, we mention that one of the possible applications of our result is related to the problem of the consistent maximal displacement, that is, to determine how closely the particles in a branching system can travel to the path of the rightmost particle. This problem has been considered for example in [27] for branching random walks and in [40] for Branching Brownian motion. Existing results can be refined by showing that there are particles above curves that are very close below the rightmost particle path,

for example above curves of the form $zs - bs^\alpha$.
With this in mind, we show that when $G(s) = (s+1)^\alpha$ with $\alpha \in (0, 7/10]$, it is possible to choose δ_T in such a way that the assumptions (i)-(v) are satisfied. We postpone this discussion to Section 4.7 at the end of the chapter and we now proceed towards the statement of our main theorem.
Define $\phi(\lambda) = \mathbb{E}[e^{\lambda\xi}]$ and let

$$\Lambda(z) := \sup_{\lambda:\, \phi(\lambda)<\infty} \{\lambda z - \log \mathbb{E}[e^{\lambda X(1)}]\} = \sup_{\lambda:\, \phi(\lambda)<\infty} \{\lambda z - r\phi(\lambda) + r\}$$

be the usual large deviations rate function. Denote by $\lambda(z)$ the value of λ for which the supremum is achieved, so that

$$\Lambda(z) = \lambda(z)z - r\phi(\lambda(z)) + r$$

and $\lambda(z)$ satisfies $\phi'(\lambda(z)) = z/r$.

Let D be the space of càdlàg functions $H : [0,\infty) \to \mathbb{R}$. For $F \in C^2([0,\infty), \mathbb{R})$, let

$$\mathcal{B}(F, L, a, b)|_{[u,t]} = \{H \in D : |H(s) - F(s)| < L \ \forall s \in [u,t],\ H(t) - F(t) \in (aL, bL)\}.$$

Theorem 4.1. *Let $F_T(s) = zs + G(s) - x_T$ where $G(s)$ and x_T satisfy the properties* (i)-(v)*. If $z > r\mathbb{E}[\xi]$ then*

$$\lim_{T\to\infty} \frac{1}{T^{1/3}} \Bigg(\log \mathbb{P}_0(X \in \mathcal{B}(F_T, L_T, p, q)|_{[0,T]}) + \Lambda(z)T + \lambda(z)G(T) - \frac{1}{2}\int_0^T \frac{G'(s)^2}{r\phi''(\lambda(z))} ds \Bigg) = \lambda(z)(x - pL) - \frac{\pi^2 r\phi''(\lambda(z))}{8L^2}.$$

If $z < r\mathbb{E}[\xi]$ the same result holds but replacing pL with qL on the right-hand side. If $z = r\mathbb{E}[\xi]$ the same holds with $\lambda(z) = 0$, $\Lambda(z) = 0$ and $\phi''(\lambda(z)) = \mathbb{E}[\xi^2]$.

An identical statement to Theorem 4.1 holds for a discrete time random walk.
Let $\Lambda_d(z) = \lambda_d(z)z - \log\phi(\lambda_d(z))$ where λ_d satisfies $\phi'(\lambda_d(z)) = z$. For $u, t \in \mathbb{N}$ let

$$\mathcal{B}_d(F, L, a, b)|_{[u,t]} = \{H \in D : |H(k) - F(k)| < L \ \forall k \in [u,t] \cap \mathbb{N},\ H(t) - F(t) \in (aL, bL)\}.$$

Theorem 4.2. *Let $S_n = \xi_1 + \ldots, \xi_n$, $n \in \mathbb{N}$ with $\xi_k \sim \xi$. Let $F_n(s) = zs + G(s) - x_n$, where $G(s)$ and x_n satisfy the properties* (i)-(v)*. If $z > \mathbb{E}[\xi]$ then*

$$\lim_{n\to\infty} \frac{1}{n^{1/3}} \Bigg(\log \mathbb{P}_0(S \in \mathcal{B}_d(F_n, L_n, p, q)|_{[0,n]}) + \Lambda_d(z)n + \lambda_d(z)G(n) - \frac{1}{2}\int_0^n \frac{G'(s)^2}{r\phi''(\lambda_d(z))} ds \Bigg) = \lambda_d(z)(x - pL) - \frac{\pi^2 r\phi''(\lambda_d(z))}{8L^2}.$$

If $z < \mathbb{E}[\xi]$ the same result holds but replacing pL with qL on the right-hand side. If $z = \mathbb{E}[\xi]$ the same holds with $\lambda_d(z) = 0$, $\Lambda_d(z) = 0$ and $\phi''(\lambda_d(z)) = \mathbb{E}[\xi^2]$.

We only prove Theorem 4.1, but the proof of Theorem 4.2 is identical and in fact simpler because the discretisation argument to transform $Y(s)$ into a process in discrete time from Lemma 4.12 is not necessary.

The remaining of the chapter is structured as follows. In Section 4.2.1 we deal with the linear term zs with a standard change of measure, thereby reducing our problem to estimating the probability that a process $Y(s)$ with mean 0 stays near $G(s) - x_T$.

In Section 4.2.2 we prove some elementary estimates about the probability that a Brownian motion stays in a tube about a given function.

In Section 4.3.1 we show how to split the probability that $Y(s)$ stays near $G(s)-x_T$ in $[0,T]$ into probabilities on smaller time intervals.

In Section 4.3.2 we approximate each one of these with the probability that a Brownian motion stays near $G(s) - x_T$, using the Komlós-Major-Tusnády Theorem. Proposition 4.13 in Section 4.3.3, in which we split the probability that $Y(s)$ stays near $G(s) - x_T$ on $[0,T]$ into probabilities that a Brownian motion stays near $G(s) - x_T$ on smaller intervals, will be at the core of the proof of Theorem 4.1.

In Section 4.4 we give some technical results on how to combine together terms coming from different intervals. Finally, in Sections 4.5 and 4.6 we prove the upper and lower bound of Theorem 4.1, respectively.

4.2. Elementary bounds on compound Poisson processes and Brownian motion

4.2.1 Changing the measure to deal with the linear component of F

The probability that a compound Poisson process stays near a linear function zs can be easily addressed with a Girsanov-type change of measure, so that under the new probability measure the process has drift zs.

For $s \leq T$, let $\mathcal{F}_s$ be the filtration generated by the process $(X(u) : u \leq s)$. Define a new measure by

$$\left.\frac{\mathrm{d}\mathbb{Q}^{\lambda(z)}}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_s} := e^{\lambda(z)X(s)-rs(\phi(\lambda(z))-1)}, \quad s \in [0,T]. \tag{4.1}$$

Lemma 4.3. *Under $\mathbb{Q}^{\lambda(z)}$, $(X(s), s \geq 0)$ has mean zs and variance $\phi''(\lambda(z))rs$. In particular, if we let*

$$Y(s) = \frac{X(s) - zs}{\sqrt{r\phi''(\lambda(z))}}, \quad s \geq 0, \tag{4.2}$$

then $\mathbb{E}_{\mathbb{Q}^{\lambda(z)}}\big[Y(j+1) - Y(j)\big] = 0$ and $\mathbb{E}_{\mathbb{Q}^{\lambda(z)}}\big[(Y(j+1) - Y(j))^2\big] = 1$ for every $j \in \mathbb{N}$.

Proof. Let $S_n = \xi_1 + \cdots + \xi_n$, $n \in \mathbb{N}$ be the sum of n independent copies of ξ. Conditioning to the number of jumps of $X(s)$ up to time s and using the independence of the increments, we get

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}^{\lambda(z)}}\big[X(s)\big] &= \mathbb{E}_{\mathbb{P}}\big[e^{\lambda(z)X(s)-rs\phi(\lambda(z))+rs}X(s)\big] \\
&= \sum_{n=0}^{\infty} \mathbb{E}_{\mathbb{P}}\big[e^{\lambda(z)S_n}S_n\big]e^{-rs\phi(\lambda(z))+rs}e^{-rs}\frac{(rs)^n}{n!} \\
&= \sum_{n=0}^{\infty} n\mathbb{E}_{\mathbb{P}}\big[e^{\lambda(z)\xi}\xi\big]\mathbb{E}_{\mathbb{P}}\big[e^{\lambda(z)\xi}\big]^{n-1}e^{-rs\phi(\lambda(z))}\frac{(rs)^n}{n!}.
\end{aligned}$$

Noting that $\mathbb{E}_{\mathbb{P}}\big[e^{\lambda(z)\xi}\xi\big] = \phi'(\lambda(z))$, the above gives that

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}^{\lambda(z)}}\big[X(s)\big] &= \sum_{n=0}^{\infty} n\phi'(\lambda(z))\phi(\lambda(z))^{n-1}e^{-rs\phi(\lambda(z))}\frac{(rs)^n}{n!} \\
&= \phi'(\lambda(z))e^{-rs\phi(\lambda(z))}rs\sum_{n=1}^{\infty}\phi(\lambda(z))^{n-1}\frac{(rs)^{n-1}}{(n-1)!} \\
&= rs\phi'(\lambda(z)),
\end{aligned}$$

and since $\lambda(z)$ satisfies $\phi'(\lambda(z)) = z/r$, we have that $\mathbb{E}_{\mathbb{Q}^{\lambda(z)}}[X(s)] = zs$. Similarly,

$$\mathbb{E}_{\mathbb{Q}^{\lambda(z)}}\big[X(s)^2\big] = \mathbb{E}_{\mathbb{P}}\big[e^{\lambda(z)X(s)-rs\phi(\lambda(z))+rs}X(s)^2\big] = \sum_{n=0}^{\infty}\mathbb{E}_{\mathbb{P}}\big[e^{\lambda(z)S_n}S_n^2\big]e^{-rs\phi(\lambda(z))}\frac{(rs)^n}{n!}.$$

Using that

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}}\big[e^{\lambda(z)S_n}S_n^2\big] &= \sum_{i=1}^{n}\mathbb{E}_{\mathbb{P}}\big[e^{\lambda(z)S_n}\xi_i^2\big] + 2\sum_{i<j}\mathbb{E}_{\mathbb{P}}\big[e^{\lambda(z)S_n}\xi_i\xi_j\big] \\
&= n\mathbb{E}_{\mathbb{P}}\big[e^{\lambda(z)\xi}\xi^2\big]\mathbb{E}_{\mathbb{P}}\big[e^{\lambda(z)\xi}\big]^{n-1} + n(n-1)\mathbb{E}_{\mathbb{P}}\big[e^{\lambda(z)\xi}\xi\big]^2\mathbb{E}_{\mathbb{P}}\big[e^{\lambda(z)\xi}\big]^{n-2} \\
&= n\phi''(\lambda(z))\phi(\lambda(z))^{n-1} + n(n-1)\phi'(\lambda(z))^2\phi(\lambda(z))^{n-2}.
\end{aligned}$$

Substituting this, we obtain

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}^{\lambda(z)}}\big[X(s)^2\big] &= \sum_{n=1}^{\infty}\phi''(\lambda(z))\phi(\lambda(z))^{n-1}e^{-rs\phi(\lambda(z))}\frac{(rs)^n}{(n-1)!} \\
&\quad + \sum_{n=2}^{\infty}\phi'(\lambda(z))^2\phi(\lambda(z))^{n-2}e^{-rs\phi(\lambda(z))}\frac{(rs)^n}{(n-2)!} \\
&= \phi''(\lambda(z))rs + \phi'(\lambda(z))^2(rs)^2,
\end{aligned}$$

so $\mathbb{E}_{\mathbb{Q}^{\lambda(z)}}\big[X(s)^2\big] - \mathbb{E}_{\mathbb{Q}^{\lambda(z)}}[X(s)]^2 = \phi''(\lambda(z))rs$. From the definition of $Y(s)$ in (4.2), it is clear that $\mathbb{E}_{\mathbb{Q}^{\lambda(z)}}[Y(j)] = 0$ for every $j \in \mathbb{N}$. Since

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}^{\lambda(z)}}\big[Y(j+1)Y(j)\big] &= \mathbb{E}_{\mathbb{Q}^{\lambda(z)}}\big[(Y(j+1)-Y(j)+Y(j))Y(j)\big] \\
&= \mathbb{E}_{\mathbb{Q}^{\lambda(z)}}\big[Y(j+1)-Y(j)\big]\mathbb{E}_{\mathbb{Q}^{\lambda(z)}}\big[Y(j)\big] + \mathbb{E}_{\mathbb{Q}^{\lambda(z)}}\big[Y(j)^2\big] \\
&= \mathbb{E}_{\mathbb{Q}^{\lambda(z)}}\big[Y(j)^2\big]
\end{aligned}$$

and

$$\mathbb{E}_{\mathbb{Q}^{\lambda(z)}}\big[Y(j)^2\big] = \frac{\mathbb{E}_{\mathbb{Q}^{\lambda(z)}}\big[(X(j)-zj)^2\big]}{r\phi''(\lambda(z))} = \frac{\mathbb{E}_{\mathbb{Q}^{\lambda(z)}}\big[(X(j)-\mathbb{E}_{\mathbb{Q}^{\lambda(z)}}[X(j)])^2\big]}{r\phi''(\lambda(z))} = j,$$

we can conclude that

$$\begin{aligned}
&\mathbb{E}_{\mathbb{Q}^{\lambda(z)}}\big[(Y(j+1)-Y(j))^2\big] \\
&= \mathbb{E}_{\mathbb{Q}^{\lambda(z)}}\big[Y(j+1)^2\big] + \mathbb{E}_{\mathbb{Q}^{\lambda(z)}}\big[Y(j)^2\big] - 2\mathbb{E}_{\mathbb{Q}^{\lambda(z)}}\big[Y(j+1)Y(j)\big] = (j+1)+j-2j = 1.
\end{aligned}$$

This completes the proof. $\square$

Since $\mathbb{E}_{\mathbb{P}}[X(s)] = rs\mathbb{E}_{\mathbb{P}}[\xi]$, when $z = r\mathbb{E}_{\mathbb{P}}[\xi]$ the process $(X(s) - zs,\ s \geq 0)$ already has mean 0 under $\mathbb{P}$. In this case, it is easy to see that $\lambda(z) = 0$, $\phi(\lambda(z)) = 1$, $\phi'(\lambda(z)) = \mathbb{E}_{\mathbb{P}}[\xi]$ and $\phi''(\lambda(z)) = \mathbb{E}_{\mathbb{P}}[\xi^2]$. Therefore, from (4.1), we deduce that $\mathbb{Q}^{\lambda(z)} = \mathbb{P}$. In this case, Lemma 4.3 still holds, although there is actually no change of measure involved.

The next lemma uses the change of measure introduced in Lemma 4.3 to transform the probability that $X(s)$ stays in a tube about $F_T(s) = zs + G_T(s)$ into the probability that $Y(s)$ stays in a tube about a rescaled version of $G_T(s)$, under the new probability measure $\mathbb{Q}^{\lambda(z)}$.

Lemma 4.4. *Recall that* $F_T(s) = zs + G_T(s)$, *where* $G_T(s) = G(s) - x_T$. *Define* $\tilde{G}_T(s) = G_T(s)\big(r\phi''(\lambda(z))\big)^{-1/2}$ *and* $\tilde{L}_T = L_T\big(r\phi''(\lambda(z))\big)^{-1/2}$.
If $z > r\mathbb{E}[\xi]$, *for any* $\varepsilon > 0$ *small enough*

$$\begin{aligned}&\exp\big(-\Lambda(z)T - \lambda(z)(G(T) - x_T + (p+\varepsilon)L_T)\big)\mathbb{Q}_0^{\lambda(z)}\big(Y \in \mathcal{B}(\tilde{G}_T, \tilde{L}_T, p, p+\varepsilon)|_{[0,T]}\big)\\ &\quad \leq \mathbb{P}_0\big(X \in \mathcal{B}(F_T, L_T, p, q)|_{[0,T]}\big)\\ &\qquad \leq \exp\big(-\Lambda(z)T - \lambda(z)(G(T) - x_T + pL_T)\big)\mathbb{Q}_0^{\lambda(z)}\big(Y \in \mathcal{B}(\tilde{G}_T, \tilde{L}_T, p, q)|_{[0,T]}\big)\end{aligned}$$

and if $z < r\mathbb{E}[\xi]$, *for any* $\varepsilon > 0$ *small enough*

$$\begin{aligned}&\exp\big(-\Lambda(z)T - \lambda(z)(G(T) - x_T + (q+\varepsilon)L_T)\big)\mathbb{Q}_0^{\lambda(z)}\big(Y \in \mathcal{B}(\tilde{G}_T, \tilde{L}_T, q-\varepsilon, q)|_{[0,T]}\big)\\ &\quad \leq \mathbb{P}_0\big(X \in \mathcal{B}(F_T, L_T, p, q)|_{[0,T]}\big)\\ &\qquad \leq \exp\big(-\Lambda(z)T - \lambda(z)(G(T) - x_T + qL_T)\big)\mathbb{Q}_0^{\lambda(z)}\big(Y \in \mathcal{B}(\tilde{G}_T, \tilde{L}_T, p, q)|_{[0,T]}\big).\end{aligned}$$

When $z = r\mathbb{E}[\xi]$,

$$\mathbb{P}_0\big(X \in \mathcal{B}(F_T, L_T, p, q)|_{[0,T]}\big) = \mathbb{P}_0\big(Y \in \mathcal{B}(\tilde{G}_T, \tilde{L}_T, p, q)|_{[0,T]}\big).$$

Proof. Let $A_T = \big\{X \in \mathcal{B}(F_T, L_T, p, q)|_{[0,T]}\big\}$. We note that $\phi'(0) = \mathbb{E}_{\mathbb{P}}[\xi]$ and $\phi''(\lambda) = \mathbb{E}\big[e^{\lambda\xi}\xi^2\big] > 0\ \forall\lambda \in \mathbb{R}$. Since $\lambda(z)$ satisfies $\phi'(\lambda(z)) = z/r$, then $\lambda(z) > 0$ if and only if $z > r\mathbb{E}[\xi]$. It follows that when $z > r\mathbb{E}[\xi]$, since $X(T) \geq zT + G(T) - x_T + pL_T$ on A_T, then

$$\begin{aligned}\mathbb{P}_0\big(A_T\big) &= \mathbb{Q}_0^{\lambda(z)}\big[\exp\big(-\lambda(z)X(T) + rT(\phi(\lambda(z) - 1)\big)\mathbb{1}_{A_T}\big]\\ &\leq \exp\big(-\lambda(z)(zT + G(T) - x_T + pL_T) + rT\phi(\lambda(z)) - rT\big)\mathbb{Q}_0^{\lambda(z)}(A_T)\\ &= \exp\big(-\Lambda(z)T - \lambda(z)(G(T) - x_T + pL_T)\big)\mathbb{Q}_0^{\lambda(z)}(A_T).\end{aligned}$$

Recalling the definition of $Y(s)$ from Lemma 4.3, we can write

$$\mathbb{Q}_0^{\lambda(z)}(A_T) = \mathbb{Q}_0^{\lambda(z)}(Y \in \mathcal{B}(\tilde{G}_T, \tilde{L}_T, p, q)|_{[0,T]},$$

so the upper bound is proved. For any $\varepsilon > 0$ let $A_T^{\varepsilon} = \big\{X \in \mathcal{B}(F_T, L_T, p, p+\varepsilon)|_{[0,T]}\big\}$. Then the lower bound can be proved in the same way using that

$$\mathbb{P}_0\big(X \in \mathcal{B}(F_T, L_T, p, q)|_{[0,T]}\big) \geq \mathbb{P}_0(A_T^{\varepsilon}).$$

The case $z < r\mathbb{E}[\xi]$ is analogous, and when $z = r\mathbb{E}[\xi]$ we already noticed that there is no change of measure, the drift is already 0 so the only effect is dividing all quantities by $\sqrt{r\phi''(\lambda(z))} = \sqrt{r\mathbb{E}[\xi^2]}$. $\square$

From now on, we denote by $\mathbb{Q}_0$ the probability measure $\mathbb{Q}_0^{\lambda(z)}$ and all the results we present involve the process $Y(s)$ under $\mathbb{Q}_0$. We will switch back to $X(s)$ when we prove Theorem 4.1 in Sections 4.5 and 4.6.

4.2.2 Detailed estimates for the probability that a Brownian motion stays in a tube about a function f

In this section we prove some sharp estimates for Brownian motion, most of which can be almost directly deduced from [40], although we give here more explicit error bounds.

Our first lemma concerns the probability that a Brownian motion starting from x stays in a symmetric strip of constant width L up to time t.

Proposition 4.5. *Fix $x \in (-L, L)$ and $-1 \le p < q \le 1$. There exists a constant $C > 0$ such that if $t \ge 8L^2$*

$$\left(e^{-\frac{\pi^2 t}{8L^2}} \cos\left(\frac{\pi x}{2L}\right) \int_p^q \cos\left(\frac{\pi \nu}{2}\right) d\nu\right) (1 - Ce^{-\frac{\pi^2 t}{L^2}})$$
$$\le \mathbb{P}_x(B(s) \in (-L, L)\ \forall s \in [0, t],\ B(t) \in (pL, qL))$$
$$\le \left(e^{-\frac{\pi^2 t}{8L^2}} \cos\left(\frac{\pi x}{2L}\right) \int_p^q \cos\left(\frac{\pi \nu}{2}\right) d\nu\right) (1 + Ce^{-\frac{\pi^2 t}{L^2}}).$$

Proof. From standard results (see for example Problem 1.7.8 in [34]) we have that

$$\mathbb{P}_x(B(s) \in (-L, L)\ \forall s \in [0, t],\ B(t) \in (pL, qL))$$
$$= \sum_{n=1}^{\infty} e^{-n^2 \frac{\pi^2 t}{8L^2}} \sin\left(\frac{\pi n}{2}\right)^2 \cos\left(\frac{\pi n x}{2L}\right) \int_p^q \cos\left(\frac{\pi n \nu}{2}\right) d\nu.$$

Since all the even terms are zero, this can be rewritten as

$$\begin{aligned}
&\mathbb{P}_x(B(s) \in (-L, L)\ \forall s \in [0, t],\ B(t) \in (pL, qL)) \\
&= \sum_{k=0}^{\infty} e^{-(2k+1)^2 \frac{\pi^2 t}{8L^2}} \cos\left(\frac{\pi(2k+1)x}{2L}\right) \int_p^q \cos\left(\frac{\pi(2k+1)\nu}{2}\right) d\nu \\
&= e^{-\frac{\pi^2 t}{8L^2}} \cos\left(\frac{\pi x}{2L}\right) \int_p^q \cos\left(\frac{\pi \nu}{2}\right) d\nu \\
&\qquad + \sum_{k=1}^{\infty} e^{-(2k+1)^2 \frac{\pi^2 t}{8L^2}} \cos\left(\frac{\pi(2k+1)x}{2L}\right) \int_p^q \cos\left(\frac{\pi(2k+1)\nu}{2}\right) d\nu.
\end{aligned}$$

The lemma is proved if we show that there exists a constant $C > 0$ such that

$$\left|\sum_{k=1}^{\infty} e^{-(2k+1)^2 \frac{\pi^2 t}{8L^2}} \cos\left(\frac{\pi(2k+1)x}{2L}\right) \int_p^q \cos\left(\frac{\pi(2k+1)\nu}{2}\right) d\nu\right|$$
$$\le Ce^{-\frac{9\pi^2 t}{8L^2}} \cos\left(\frac{\pi x}{2L}\right) \int_p^q \cos\left(\frac{\pi \nu}{2}\right) d\nu. \quad (4.3)$$

Using that for every $k \geq 1$ and $\alpha \in (-1,1)$

$$\left|\cos\left(\frac{\pi(2k+1)\alpha}{2}\right)\right| \leq (2k+1)\cos\left(\frac{\pi\alpha}{2}\right)$$

we obtain

$$\begin{aligned}
&\left|\sum_{k=1}^{\infty} e^{-(2k+1)^2\frac{\pi^2 t}{8L^2}} \cos\left(\frac{\pi(2k+1)x}{2L}\right)\int_p^q \cos\left(\frac{\pi(2k+1)\nu}{2}\right)d\nu\right| \\
&\qquad \leq \sum_{k=1}^{\infty} e^{-(2k+1)^2\frac{\pi^2 t}{8L^2}} \left|\cos\left(\frac{\pi(2k+1)x}{2L}\right)\right| \int_p^q \left|\cos\left(\frac{\pi(2k+1)\nu}{2}\right)\right| d\nu \\
&\qquad \leq \left(\sum_{k=1}^{\infty}(2k+1)^2 e^{-(2k+1)^2\frac{\pi^2 t}{8L^2}}\right)\cos\left(\frac{\pi x}{2L}\right)\int_p^q \cos\left(\frac{\pi\nu}{2}\right)d\nu.
\end{aligned}$$

When $t \geq 8L^2$,

$$\begin{aligned}
\sum_{k=1}^{\infty}(2k+1)^2 e^{-(2k+1)^2\frac{\pi^2 t}{8L^2}} &= 9e^{-\frac{9\pi^2 t}{8L^2}} + \sum_{k=2}^{\infty}(2k+1)^2 e^{-(2k+1)^2\frac{\pi^2 t}{8L^2}} \\
&= e^{-\frac{9\pi^2 t}{8L^2}}\left(9 + \sum_{k=2}^{\infty}(2k+1)^2 e^{-((2k+1)^2-9)\frac{\pi^2 t}{8L^2}}\right) \\
&\leq e^{-\frac{9\pi^2 t}{8L^2}}\left(9 + \sum_{k=2}^{\infty}(2k+1)^2 e^{-4k^2\pi^2}\right) \leq Ce^{-\frac{9\pi^2 t}{8L^2}}.
\end{aligned}$$

This proves (4.3) and completes the proof. □

We now extend Proposition 4.5 and consider the probability that a Brownian motion lies in a tube of constant width centred about a given function f. Again, the proof is adapted from [40], but we give explicit error bounds.

Proposition 4.6. *Let $(B(s),\ s \geq 0)$ be a standard Brownian motion. Let $x \in (-L,L)$ and $f : [0,\infty) \to \mathbb{R}$ be a twice differentiable function such that $f(0) = -x$. If $f'(t) > 0$, then*

$$\begin{aligned}
&e^{-f'(t)qL - \frac{1}{2}\int_0^t f'(s)^2 ds + f'(0)x - \frac{\pi^2 t}{8L^2} - L\int_0^t |f''(s)|ds}\left(1 - Ce^{-\frac{\pi^2 t}{L^2}}\right)\cos\left(\frac{\pi x}{2L}\right)\int_p^q \cos\left(\frac{\pi\nu}{2}\right)d\nu \\
&\leq \mathbb{P}(B(s) - f(s) \in (-L,L)\ \forall s \in [0,t],\ B(t) - f(t) \in (pL, qL)) \\
&\ \leq e^{-f'(t)pL - \frac{1}{2}\int_0^t f'(s)^2 ds + f'(0)x - \frac{\pi^2 t}{8L^2} + L\int_0^t |f''(s)|ds}\left(1 + Ce^{-\frac{\pi^2 t}{L^2}}\right)\cos\left(\frac{\pi x}{2L}\right)\int_p^q \cos\left(\frac{\pi\nu}{2}\right)d\nu.
\end{aligned}$$

If $f'(t) < 0$, the same inequalities hold but swapping $f'(t)qL$ with $f'(t)pL$.

Proof. The proof is a standard application of Girsanov's theorem, combined with Proposition 4.5. Define a new measure $\hat{\mathbb{P}}_x$ by

$$\left.\frac{\mathrm{d}\hat{\mathbb{P}}_x}{\mathrm{d}\mathbb{P}}\right|_t = e^{\int_0^t f'(s)dB(s) - \frac{1}{2}\int_0^t f'(s)^2 ds}, \quad s \in [0,t].$$

Under $\hat{\mathbb{P}}_x$, the process $\tilde{B}(s) = B(s) - f(s)$, $s \in [0,t]$ is a Brownian motion starting from x. Let

$$\mathcal{A}_t := \{\tilde{B}(s) \in (-L, L)\ \forall s \in [0,t],\ \tilde{B}(t) \in (pL, qL)\}.$$

Then

$$\hat{\mathbb{P}}_x(\mathcal{A}_t) = \mathbb{E}_{\mathbb{P}}\left[e^{\int_0^t f'(s)dB(s) - \frac{1}{2}\int_0^t f'(s)^2 ds}\mathbb{1}_{\mathcal{A}_t}\right].$$

We write the exponential martingale in the change of measure in terms of $\tilde{B}$. The stochastic integration by parts formula gives that

$$\int_0^t f'(s)dB(s) = f'(t)B(t) - \int_0^t f''(s)B(s)ds.$$

Substituting $B(s) = \tilde{B}(s) + f(s)$ for all $s \in [0,t]$ and using that (from the deterministic integration by parts formula)

$$\int_0^t f''(s)f(s)ds = f'(t)f(t) + f'(0)x - \int_0^t f'(s)^2 ds,$$

we get

$$\begin{aligned}
\int_0^t f'(s)dB(s) &= f'(t)\tilde{B}(t) + f'(t)f(t) - \int_0^t f''(s)\tilde{B}(s)ds - \int_0^t f''(s)f(s)ds \\
&= f'(t)\tilde{B}(t) - \int_0^t f''(s)\tilde{B}(s)ds - f'(0)x + \int_0^t f'(s)^2 ds.
\end{aligned}$$

This, together with the bounds on $\tilde{B}$ on the event $\mathcal{A}_t$, gives that

$$\begin{aligned}
\hat{\mathbb{P}}_x(\mathcal{A}_t) &= \mathbb{E}_{\mathbb{P}}\left[e^{\int_0^t f'(s)dB(s) - \frac{1}{2}\int_0^t f'(s)^2 ds}\mathbb{1}_{\mathcal{A}_t}\right] \\
&= \mathbb{E}_{\mathbb{P}}\left[e^{f'(t)\tilde{B}(t) - f'(0)x - \int_0^t f''(s)\tilde{B}(s)ds + \frac{1}{2}\int_0^t f'(s)^2 ds}\mathbb{1}_{\mathcal{A}_t}\right] \\
&\geq e^{f'(t)pL - f'(0)x - L\int_0^t |f''(s)|ds + \frac{1}{2}\int_0^t f'(s)^2 ds}\mathbb{P}(\mathcal{A}_t).
\end{aligned}$$

The upper bound on $\mathbb{P}(\mathcal{A}_t)$ follows by estimating $\hat{\mathbb{P}}(\mathcal{A}_t)$ with Proposition 4.5. The proof of the lower bound is identical. $\square$

In [40], Proposition 4.6 is extended even further, to the case in which the tube has width $L(s)$, which varies with time. The result with tubes of constant width is enough for our purposes, so we refer to [40] for further details.

We introduce more notation, which will be more convenient later on in our proofs. Until the remaining of this section, $(B(s),\ s \geq 0)$ is a standard Brownian motion and $H : [0, \infty) \to \mathbb{R}$ is a twice differentiable function such that $H(0) = -w$, where $w \in (-L, L)$.

For $a, b \in [-1, 1]$ with $a < b$ define

$$\theta_t^+(H, L, a, b) = \begin{cases} aL & \text{if } H'(t) > 0 \\ bL & \text{if } H'(t) < 0 \end{cases} \qquad \theta_t^-(H, L, a, b) = \begin{cases} bL & \text{if } H'(t) > 0 \\ aL & \text{if } H'(t) < 0. \end{cases}$$

Define also

$$\kappa_{u,t}^{+}(H,w,L,a,b)$$
$$= -\frac{1}{2}\int_u^t H'(s)^2 ds + H'(u)w - \frac{\pi^2(t-u)}{8L^2} - \theta_t^{+}(H,L,a,b)H'(t) + L\int_u^t |H''(s)|ds,$$

$$\kappa_{u,t}^{-}(H,w,L,a,b)$$
$$= -\frac{1}{2}\int_u^t H'(s)^2 ds + H'(u)w - \frac{\pi^2(t-u)}{8L^2} - \theta_t^{-}(H,L,a,b)H'(t) - L\int_u^t |H''(s)|ds$$

and

$$\Gamma(y,a,b) = \cos\Big(\frac{\pi y}{2}\Big)\int_a^b \cos\Big(\frac{\pi\nu}{2}\Big)d\nu.$$

We can rewrite Proposition 4.6 as follows.

Proposition 4.7. *Let $t \geq 0$ and $a, b \in [-1,1]$ with $a < b$. There exists a constant $C > 0$ such that*

$$\begin{aligned}\exp\big(\kappa_{0,t}^{-}(H,w,L,a,b)\big)\Gamma(w/L,a,b)\big(1-Ce^{-\pi^2 t/L^2}\big)\\ \leq \mathbb{Q}_0(|B(s)-H(s)|< L\ \forall s\in[0,t],\ B(t)-H(t)\in(aL,bL))\\ \leq \exp\big(\kappa_{0,t}^{+}(H,w,L,a,b)\big)\Gamma(w/L,a,b)\big(1+Ce^{-\pi^2 t/L^2}\big).\end{aligned}$$

We notice that the previous proposition gives a sharp estimate for the probability that $B(s)$ stays near a flat function $H(s) = -w$ only when $t/L^2 \to \infty$. When $t/L^2 \to 0$ instead, $B(s)$ stays near its starting position with probability close to 1. We make this assertion precise in the next lemmas, which we will need later, in Section 4.6, when we prove the lower bound of Theorem 4.1.

Lemma 4.8. *Let $x \in (-L,L)$ and $\varepsilon > 0$ such that $-L+\varepsilon L < x < L-\varepsilon L$. Then*

$$\mathbb{Q}_x(|B(s)|< L\ \forall s\in[0,t],\ |B(t)-x|<\varepsilon L) \geq 1 - \frac{2}{\sqrt{\pi}}\Big(\frac{\sqrt{2t}}{\varepsilon L}\Big)\exp\Big(-\frac{\varepsilon^2 L^2}{2t}\Big).$$

Proof. Using that $-L+\varepsilon L < x < L-\varepsilon L$, by standard properties of the Brownian motion we have

$$\begin{aligned}&\mathbb{Q}_x(|B(s)|< L\ \forall s\in[0,t],\ |B(t)-x|<\varepsilon L)\\ &\quad\geq \mathbb{Q}_x(|B(s)-x|<\varepsilon L\ \forall s\in[0,t],\ |B(t)-x|<\varepsilon L)\\ &\quad= \mathbb{Q}_0(|B(s)|<\varepsilon L\ \forall s\in[0,t]) = 1-\mathbb{Q}_0(\exists s\in[0,t] : |B(s)|\geq\varepsilon L).\end{aligned}$$

From the reflection principle,

$$\mathbb{Q}_0(\exists s\in[0,t] : |B(s)|\geq\varepsilon L) \leq 2\mathbb{Q}_0(\exists s\in[0,t] : B(s)\geq\varepsilon L) = 4\mathbb{Q}_0(B(t)\geq\varepsilon L).$$

Using the tail estimates

$$\frac{1}{w\sqrt{2\pi}}\Big(1-\frac{1}{w^2}\Big)\exp\Big(-\frac{w^2}{2}\Big) \leq \mathbb{Q}(W>w) \leq \frac{1}{w\sqrt{2\pi}}\exp\Big(-\frac{w^2}{2}\Big) \tag{4.4}$$

where $W \sim \mathcal{N}(0,1)$ and $w > 0$, we obtain

$$4\mathbb{Q}_0(B(t) \geq \varepsilon L) \leq \frac{2}{\sqrt{\pi}}\Big(\frac{\sqrt{2t}}{\varepsilon L}\Big) \exp\Big(-\frac{\varepsilon^2 L^2}{2t}\Big). \qquad \square$$

It is easy to extend the previous lemma to the probability that $B(s)$ is near a function $H(s)$ with a Girsanov change of measure:

Lemma 4.9. *Let $x \in (-L, L)$ and $\varepsilon > 0$ such that $-L + \varepsilon L < x < L - \varepsilon L$. For any $H : [0,\infty) \to \mathbb{R}$ such that $H'(t) > 0$,*

$$\mathbb{Q}_x\Big(B \in \mathcal{B}\big(H, L, \tfrac{x}{L} - \varepsilon, \tfrac{x}{L} + \varepsilon\big)\big|_{[0,t]}\Big)$$
$$\geq \exp\Big(-H'(t)(x+\varepsilon L)+H'(0)x-L\int_0^t |H''(s)|ds-\frac{1}{2}\int_0^t H'(s)^2 ds\Big)\Big(1-\frac{2\sqrt{2t}}{\sqrt{\pi}\varepsilon L}e^{-\frac{\varepsilon^2 L^2}{2t}}\Big).$$

Proof. Define a new measure $\hat{\mathbb{Q}}_x$ by

$$\frac{\mathrm{d}\hat{\mathbb{Q}}_x}{\mathrm{d}\mathbb{Q}_x}\bigg|_t = e^{\int_0^t H'(s)dB(s)-\frac{1}{2}\int_0^t H'(s)^2 ds}, \quad s \in [0,t].$$

Under $\hat{\mathbb{Q}}_x$, the process $\tilde{B}(s) = B(s) - H(s)$, $s \in [0,t]$ is a Brownian motion starting from x. Let

$$\mathcal{A}_t := \{\tilde{B}(s) \in (-L, L)\ \forall s \in [0,t],\ \tilde{B}(t) \in (x - \varepsilon L, x + \varepsilon L)\}.$$

Then, with the same steps as in the proof of Proposition 4.7 we can show that

$$\hat{\mathbb{Q}}_x(\mathcal{A}_t) \leq \exp\Big(H'(t)(x+\varepsilon L) - H'(0)x + L\int_0^t |H''(s)|ds + \frac{1}{2}\int_0^t H'(s)^2 ds\Big)\mathbb{Q}_x(\mathcal{A}_t).$$

Estimating $\hat{\mathbb{Q}}_x(\mathcal{A}_t)$ with Lemma 4.8 gives the result. $\square$

4.3. Proof outline for Theorem 4.1

4.3.1 Splitting $[0,T]$ into smaller intervals

As we anticipated in Section 4.1, our proof consists of splitting $[0,T]$ into smaller intervals, on which we approximate the compound Poisson process with a Brownian motion. The next lemma shows how we split the time intervals, going backwards and separating the final part.

For $a, b \in [-1,1]$ and $\varepsilon < (b-a)/4$ define

$$\mathcal{B}^+_\varepsilon(H, L, a, b)|_{[0,t]} = \mathcal{B}\Big(H, (1+\varepsilon)L, \frac{a-\varepsilon}{1+\varepsilon}, \frac{b+\varepsilon}{1+\varepsilon}\Big)\Big|_{[0,t]} \tag{4.5}$$

and

$$\mathcal{B}^-_\varepsilon(H, L, a, b)|_{[0,t]} = \mathcal{B}\Big(H, (1-\varepsilon)L, \frac{a+\varepsilon}{1-\varepsilon}, \frac{b-\varepsilon}{1-\varepsilon}\Big)\Big|_{[0,t]}. \tag{4.6}$$

Lemma 4.10. *Denote by $\mathbb{Q}_0$ the probability measure $\mathbb{Q}_0^{\lambda(z)}$ defined in Lemma 4.3. Let $a, b \in [-1,1]$ and $n \in \mathbb{N}$ such that $1/n^2 \leq (b-a)/4$. Define $k_n^+ = (k+1/2)/n^2$ for*

$k \in \{-n^2, \dots, n^2-1\}$. Let

$$G_{i,T}^{k,n}(t) = G(t+T_{i-1}) - G(T_{i-1}) - k_n^+ L_T$$

for $t \in [0, T_i - T_{i-1}]$ and $i \in \{1, \dots, N_T\}$. Then

$$\begin{aligned}
&\mathbb{Q}_0\big(Y \in \mathcal{B}(G_T, L_T, a, b)|_{[0,T_i]}\big) \\
&\quad \leq \sum_{k=-n^2}^{n^2-1} \mathbb{Q}_0\Big(Y \in \mathcal{B}\big(G_T, L_T, \tfrac{k}{n^2}, \tfrac{k+1}{n^2}\big)\big|_{[0,T_{i-1}]}\Big)\mathbb{Q}_0\big(Y \in \mathcal{B}^+_{1/(2n^2)}(G_{i,T}^{k,n}, L_T, a, b)|_{[0,\Delta_T]}\big)
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{Q}_0\big(Y \in \mathcal{B}(G_T, L_T, a, b)|_{[0,T_i]}\big) \\
&\quad \geq \sum_{k=-n^2+4}^{n^2-5} \mathbb{Q}_0\Big(Y \in \mathcal{B}\big(G_T, L_T, \tfrac{k}{n^2}, \tfrac{k+1}{n^2}\big)\big|_{[0,T_{i-1}]}\Big)\mathbb{Q}_0\big(Y \in \mathcal{B}^-_{1/(2n^2)}(G_{i,T}^{k,n}, L_T, a, b)|_{[0,\Delta_T]}\big).
\end{aligned}$$

Proof. We let $I_k := (kL_T/n^2, (k+1)L_T/n^2)$ for $k \in \{-n^2, \dots, n^2-1\}$ and write $(-L_T, L_T)$ as the disjoint union $\bigcup_{k=-n^2}^{n^2-1} I_k$. Splitting the last time interval we obtain

$$\begin{aligned}
&\mathbb{Q}_0(Y \in \mathcal{B}(G_T, L_T, a, b)|_{[0,T_i]}) \\
&\quad \leq \sum_{k=-n^2}^{n^2-1} \mathbb{Q}_0\Big(Y \in \mathcal{B}\big(G_T, L_T, \tfrac{k}{n^2}, \tfrac{k+1}{n^2}\big)\big|_{[0,T_{i-1}]}\Big) \\
&\qquad\qquad \cdot \sup_{x \in I_k} \mathbb{Q}_{G(T_{i-1})+x}\big(Y \in \mathcal{B}(G_T, L_T, a, b)|_{[T_{i-1},T_i]}\big).
\end{aligned}$$

Using that $|x - k_n^+ L_T| \leq L_T/(2n^2)$ for every $x \in I_k$ and stationarity,

$$\begin{aligned}
&\sup_{x \in I_k} \mathbb{Q}_{G(T_{i-1})+x}(Y \in \mathcal{B}(G_T, L_T, a, b)|_{[T_{i-1},T_i]}) \\
&\quad \leq \mathbb{Q}_{G(T_{i-1})+k_n^+ L_T}(Y \in \mathcal{B}^+_{1/(2n^2)}(G_T, L_T, a, b)|_{[T_{i-1},T_i]}) \\
&\quad = \mathbb{Q}_0(Y \in \mathcal{B}^+_{1/(2n^2)}(G_{i,T}^{k,n}, L_T, a, b)|_{[0,T_i-T_{i-1}]}).
\end{aligned}$$

This concludes the proof of the upper bound. On the other hand, using that

$$\begin{aligned}
&\inf_{x \in I_k} \mathbb{Q}_{G(T_{i-1})+x}\Big(Y \in \mathcal{B}(G_T, L_T, a, b)|_{[T_{i-1},T_i]}\Big) \\
&\quad \geq \mathbb{Q}_0\Big(Y \in \mathcal{B}^-_{1/(2n^2)}(G_{i,T}^{k,n}, L_T, a, b)|_{[0,T_i-T_{i-1}]}\Big)
\end{aligned}$$

we can prove the lower bound in the same way. □

4.3.2 Brownian motion approximation: the KMT Theorem

We now proceed with the approximation of $Y(s)$ with a standard Brownian motion on each interval of length Δ_T. Under suitable rescaling, the Functional Central Limit Theorem ensures that the processes converge in distribution. However, since the number of intervals depends on T, we need to quantify the error in this approximation.

To achieve this, we use the Komlós-Major-Tusnády theorem (KMT), which was first introduced in [35], but we state here an equivalent version taken from [24]:

Theorem 4.11 (KMT). *Let $Z_1, Z_2, \dots$ be i.i.d. random variables with $\mathbb{E}(Z_1) = 0$, $\mathbb{E}(Z_1)^2 = 1$ and $\mathbb{E}(e^{\theta|Z_1|}) < \infty$ for some $\theta > 0$. Let $S(j) := Z_1 + \cdots + Z_j$.*
Then for any $\tau \in \mathbb{N}$, it is possible to construct a version of $S(j)$, $j \in \{0, \dots, \tau\}$ and a standard Brownian motion $(B(t), t \le \tau)$ on the same probability space such that for every $z \ge 0$,

$$\mathbb{Q}_0\left(\max_{j\le\tau}|S(j) - B(j)|\ge C\log\tau + z\right) \le Ke^{-\mu z},$$

where C, K, μ are positive constants which do not depend on τ.

We want to apply Theorem 4.11 with $S(j) = Y(j)$, where $Z_k = Y(k) - Y(k-1)$ satisfies $\mathbb{E}_{\mathbb{Q}^{\lambda(z)}}\big[Y(k) - Y(k-1)\big] = 0$ and $\mathbb{E}_{\mathbb{Q}^{\lambda(z)}}\big[(Y(k) - Y(k-1))^2\big] = 1$ for $k \in \mathbb{N}$, from Lemma 4.3. The following lemma adapts Theorem 4.11 to the approximation of $Y(s)$ with a Brownian motion in continuous time.

Lemma 4.12. *Assume that $H : [0, \infty) \to \mathbb{R}$ satisfies $|H'(s)|\le M$ for every $s \ge 0$, where $M > 0$ is a constant independent of T. Then there exist two constants $\mu', K' > 0$, also independent of T, such that*

$$\begin{aligned}
&\mathbb{Q}_0\big(B \in \mathcal{B}^-_{3\delta_T}(H, L_T, a_T, b_T)|_{[0,\Delta_T]}\big) - K'e^{-\mu' L_T\delta_T} \\
&\quad\le \mathbb{Q}_0\big(Y \in \mathcal{B}(H, L_T, a_T, b_T)|_{[0,\Delta_T]}\big) \\
&\qquad\le \mathbb{Q}_0\big(B \in \mathcal{B}^+_{3\delta_T}(H, L_T, a_T, b_T)|_{[0,\Delta_T]}\big) + K'e^{-\mu' L_T\delta_T},
\end{aligned}$$

for every T large enough, where $a_T, b_T \in [-1, 1]$ and δ_T is a sequence satisfying conditions (i)-(iv) *and such that $8\delta_T \le b_T - a_T$.*

Proof. In the first part of the proof we show that we can change the probability that a Brownian motion stays near a function $H(s)$ on $[0, t]$ into the probability that the process is close to $H(s)$ at discrete times up to t, with an exponentially small additive error.

Let $\tau = \lfloor \Delta_T \rfloor$.

For $F, H \in D$, define $d(F, H)|_{[0,\Delta_T]}= \max\{|F(j) - H(j)|: \ j = 0, \dots, \tau\}$ and let

$$\mathcal{B}_d(H, L, a, b)|_{[0,\Delta_T]}= \{F \in D : d(F, H)|_{[0,\Delta_T]}< L, \ aL < F(\tau) - H(\tau) < bL\}$$

and for ε such that $8\varepsilon \le b - a$ define $\mathcal{B}^+_{d,\varepsilon}(H, L, a, b)|_{[0,\Delta_T]}$ and $\mathcal{B}^-_{d,\varepsilon}(H, L, a, b)|_{[0,\Delta_T]}$ analogously to (4.5) and (4.6). We start by showing that if $B(s)$ is a standard Brownian motion, then there exist $C > 0$ and $\nu > 0$ independent of T such that

$$\begin{aligned}
&\mathbb{Q}_0\big(B \in \mathcal{B}^-_{d,\delta_T}(H, L_T, a_T, b_T)|_{[0,\Delta_T]}\big) - Ce^{-\nu L_T\delta_T} \\
&\quad\le \mathbb{Q}_0\big(B \in \mathcal{B}(H, L_T, a_T, b_T)|_{[0,\Delta_T]}\big) \\
&\qquad\le \mathbb{Q}_0\big(B \in \mathcal{B}^+_{d,\delta_T}(H, L_T, a_T, b_T)|_{[0,\Delta_T]}\big) + Ce^{-\nu L_T\delta_T}. \qquad (4.7)
\end{aligned}$$

Writing

$$
\begin{aligned}
&\mathbb{Q}_0\big(B \in \mathcal{B}^-_{d,\delta_T}(H, L_T, a_T, b_T)|_{[0,\Delta_T]}\big) \\
&\qquad \le \mathbb{Q}_0\big(B \in \mathcal{B}(H, L_T, a_T, b_T)|_{[0,\Delta_T]}\big) \\
&\qquad\qquad + \mathbb{Q}_0\big(\exists j \le \tau,\ \exists s \in [j, j+1] : |B(s) - H(s) - (B(j) - H(j))| > L_T\delta_T\big),
\end{aligned}
$$

we get the lower bound in (4.7) if we show that

$$
\mathbb{Q}_0\Big(\exists j \le \tau,\ \exists s \in [j, j+1] : |B(s) - H(s) - (B(j) - H(j))| > L_T\delta_T\Big) \le Ce^{-\nu L_T\delta_T}. \quad (4.8)
$$

From a union bound and using stationarity, we get

$$
\begin{aligned}
&\mathbb{Q}_0\big(\exists j \le \tau,\ \exists s \in [j, j+1] : |B(s) - H(s) - (B(j) - H(j))| > L_T\delta_T\big) \\
&\quad \le \sum_{j=0}^{\tau} \mathbb{Q}_0\big(\exists s \in [0, 1] : |B(s) - (H(s+j) - H(j))| > L_T\delta_T\big) \\
&\quad \le \sum_{j=0}^{\tau} \mathbb{Q}_0\Big(\sup_{s\in[0,1]} |B(s)| > L_T\delta_T - \sup_{s\in[0,1]} |H(s+j) - H(j)|\Big).
\end{aligned}
$$

We now have

$$
\sup_{j\le\tau}\sup_{s\in[0,1]} |H(s+j) - H(j)| \le \sup_{j\le\tau}\sup_{s\in[j,j+1]} |H'(s)| \le \sup_{s\in[0,\Delta_T]} |H'(s)| \le M,
$$

which is smaller than $L_T\delta_T/2$ for every T large enough. Using this, together with the reflection principle and (4.4)

$$
\begin{aligned}
&\mathbb{Q}_0\big(\exists j \le \tau,\ \exists s \in [j, j+1] : |B(s) - H(s) - (B(j) - H(j))| > L_T\delta_T\big) \\
&\quad \le 2\tau\mathbb{Q}_0\Big(\sup_{s\in[0,1]} |B(s)| > \frac{L_T\delta_T}{2}\Big) \le 8\tau\mathbb{Q}_0\Big(B(1) > \frac{L_T\delta_T}{2}\Big) \\
&\quad \le \frac{8\tau}{\sqrt{2\pi}}\Big(\frac{2}{L_T\delta_T}\Big)\exp\Big(-\frac{L_T^2\delta_T^2}{8}\Big)
\end{aligned}
$$

so since from (i) we have that $\lim_{T\to\infty} \frac{\log(T)}{L_T\delta_T} = 0$, (4.8) is proved.

For the upper bound in (4.7) we only have to check that the condition at the end of the interval can be moved from t to τ. From

$$
\begin{aligned}
&\mathbb{Q}_0\big(B \in \mathcal{B}(H, L_T, a_T, b_T)|_{[0,\Delta_T]}\big) \\
&\qquad \le \mathbb{Q}_0\big(B \in \mathcal{B}^+_{d,\delta_T}(H, L_T, a_T, b_T)|_{[0,\Delta_T]}\big) \\
&\qquad\qquad + \mathbb{Q}_0\big(\exists s \in [\tau, t] : |B(s) - H(s) - (B(\tau) - H(\tau))| > L_T\delta_T\big),
\end{aligned}
$$

the result is proved by bounding the second term with the probability in (4.8).

We also note that (4.7) holds if we replace $B(s)$ with $Y(s)$. The only change in the previous calculations for a Brownian motion is when we estimate

$$
2\tau\mathbb{Q}_0\Big(\sup_{s\in[0,1]} |Y(s)| > \frac{L_T\delta_T}{2}\Big).
$$

Recall that the increment distribution ξ of $X(s)$ satisfies $\mathbb{E}_{\mathbb{P}}[e^{\eta^*|\xi|}] < \infty$ for some $\eta^* > 0$. Note that $Y(s)$, $s \geq 0$ is a martingale under $\mathbb{Q}_0^{\lambda(z)}$ and so the exponential of $|Y(s)|$ is a positive submartingale. By Doob's maximal inequality, for any $\eta < \eta^*$, we have

$$\begin{aligned}\mathbb{Q}_0\Big(\sup_{s\in[0,1]}|Y(s)|> \tfrac{L_T\delta_T}{2}\Big) &\leq \mathbb{E}_{\mathbb{Q}}\Big[\exp\Big(\eta\sqrt{r\phi''(\lambda(z))}|Y(1)|\Big)\Big]\exp\Big(-\frac{\eta}{2}\sqrt{r\phi''(\lambda(z))}L_T\delta_T\Big)\\ &= \mathbb{E}_{\mathbb{Q}}\big[\exp\big(\eta|X(1)-z|\big)\big]\exp\Big(-\frac{L_T\delta_T}{2}\cdot\eta\sqrt{r\phi''(\lambda(z))}\Big).\end{aligned}$$

Since $|X(1)|$ is smaller than the sum of $N \sim Pois(r)$ independent copies of $|\xi|$, we have

$$\mathbb{E}_{\mathbb{P}}\big[e^{\kappa|X(1)|}\big] \leq \mathbb{E}_{\mathbb{P}}\big[e^{\kappa\sum_{k=1}^N|\xi_k|}\big] = \exp\big(r\mathbb{E}_{\mathbb{P}}\big[e^{\kappa|\xi|}\big]-r\big),$$

and so $\mathbb{E}_{\mathbb{P}}\big[e^{\kappa|X(1)|}\big] < \infty$ for every κ such that $\mathbb{E}_{\mathbb{P}}\big[e^{\kappa|\xi|}\big] < \infty$. We then have that

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}\big[\exp\big(\eta|X(1)-z|\big)\big] &= \mathbb{E}_{\mathbb{P}}\big[\exp\Big(\lambda(z)X(1)-r\phi(\lambda(z))+r\big)\exp\big(\eta|X(1)-z|\big)\big]\\ &\leq \mathbb{E}_{\mathbb{P}}\big[\exp\big(\lambda(z)X(1)+\eta|X(1)|\big)\big]\exp\big(-r\phi(\lambda(z))+r+\eta|z|\big),\end{aligned}$$

where the expectation is finite if η is small enough, since by definition of $\lambda(z)$ we have that $\mathbb{E}[e^{\lambda(z)X(1)}] < \infty$, and $\{\lambda : \phi(\lambda) < \infty\}$ is an open set.

Now that we can change the probabilities on the continuous-time interval $[0,t]$, for both a Brownian motion and a compound Poisson process, into probabilities involving the discrete times on $[0,\Delta_T]$, we can bound the error in the approximation of $Y(j)$ with $B(j)$ for $j \leq \tau$ with Theorem 4.11.

Theorem 4.11 says that we can build a coupling of $Y(j)$ for $1 \leq j \leq \tau$ with a standard Brownian motion $B(s)$, defined under the same probability measure $\hat{\mathbb{Q}}$, such that

$$\hat{\mathbb{Q}}_0\Big(\exists j \leq \tau : |Y(j)-B(j)|\geq L_T\delta_T\Big) \leq K\exp\Big(-\frac{\mu L_T\delta_T}{2}\Big),$$

where μ and K are constants that do not depend on T. Therefore

$$\begin{aligned}&\mathbb{Q}_0\big(Y \in \mathcal{B}^+_{d,\delta_T}(H,L_T,a_T,b_T)|_{[0,\Delta_T]}\big)\\ &\quad\leq \mathbb{Q}_0\big(B \in \mathcal{B}^+_{d,2\delta_T}(H,L_T,a_T,b_T)|_{[0,\Delta_T]}\big) + \hat{\mathbb{Q}}_0\big(\exists j \leq \tau : |Y(j)-B(j)|\geq L_T\delta_T\big)\\ &\quad\leq \mathbb{Q}_0\big(B \in \mathcal{B}^+_{d,2\delta_T}(H,L_T,a_T,b_T)|_{[0,\Delta_T]}\big) + K\exp\Big(-\frac{\mu L_T\delta_T}{2}\Big).\end{aligned}$$

Combining this with the discrete approximations of $B(s)$ and $Y(s)$ from (4.7) gives

$$\begin{aligned}&\mathbb{Q}_0\big(Y \in \mathcal{B}(H,L_T,a_T,b_T)|_{[0,\Delta_T]}\big)\\ &\quad\leq \mathbb{Q}_0\big(Y \in \mathcal{B}^+_{d,\delta_T}(H,L_T,a_T,b_T)|_{[0,\Delta_T]}\big) + Ce^{-\nu L_T\delta_T}\\ &\quad\leq \mathbb{Q}_0\big(B \in \mathcal{B}^+_{d,2\delta_T}(H,L_T,a_T,b_T)|_{[0,\Delta_T]}\big) + Ke^{-\mu L_T\delta_T/2} + Ce^{-\nu L_T\delta_T}\\ &\quad\leq \mathbb{Q}_0\big(B \in \mathcal{B}^+_{3\delta_T}(H,L_T,a_T,b_T)|_{[0,\Delta_T]}\big) + Ke^{-\mu L_T\delta_T/2} + 2Ce^{-\nu L_T\delta_T},\end{aligned}$$

which proves the upper bound in the statement of the lemma. The lower bound can be obtained in the same way. $\square$

4.3.3 Switching back to $Y(s)$

Lemma 4.12 bounds the probability that $Y(s)$ stays in a tube with the sum of two terms, one involving the probability that a Brownian motion stays in a slightly modified tube and an error term. We can apply Proposition 4.7 to estimate the Brownian motion probability. Our next step is changing the additive error term in Lemma 4.12 into a multiplicative error.

In order to use the results from Lemma 4.10 and Lemma 4.12 in an optimal way, we need to choose n, a and b dependent of T.

Recall the definition of δ_T and the conditions (i)-(v) from Section 4.1. Let

(a) $n_T = \lfloor \delta_T^{-1/2} \rfloor$

(b) $r_T = 4\delta_T$.

If $b_T - a_T = 1/n_T$, then clearly $4/n_T^2 \leq b_T - a_T$ and

$$b_T - a_T = \lfloor \delta_T^{-1/2} \rfloor^{-1} \geq \delta_T^{1/2} \geq 8\delta_T$$

if T is large, so the assumptions in Lemma 4.10 and Lemma 4.12 are fulfilled. Furthermore, $r_T = 4\delta_T \geq 1/(2n_T^2) + 3\delta_T$ and it is easy to check that

$$\frac{\sqrt{r_T}}{2} \leq \frac{1}{n_T} \leq \frac{\sqrt{r_T}}{2} \cdot \frac{1}{1 - \sqrt{r_T}/2} \leq \sqrt{r_T}.$$

From now on, whenever we use n and r we always consider them to be chosen as in (a) and (b), although we sometimes omit the dependence on T as a shorthand.

Proposition 4.13. *Take $a_T, b_T \in [-1,1]$ and $n_T = \lfloor \delta_T^{-1/2} \rfloor$ such that $b_T - a_T = 1/n_T$. Let $r_T = 4\delta_T$. Recall the definitions of k_n^+ and $G_{i,T}^{k,n}$ from Lemma 4.10 and $\kappa^+_{T_{i-1},T_i}$, $\kappa^-_{T_{i-1},T_i}$ and Γ before Lemma 4.7.*

There exists $T' > 0$ such that when $T \geq T'$, for every $i \in \{1, \ldots, N_T\}$, for every $k \in \{-n^2, \ldots, n^2 - 1\}$

$$\begin{aligned}
&\mathbb{Q}_0\Big(Y \in \mathcal{B}^+_{1/(2n^2)}(G_{i,T}^{k,n}, L_T, a_T, b_T)\big|_{[0,\Delta_T]}\Big) \\
&\quad \leq \exp\left(\kappa^+_{T_{i-1},T_i}\Big(G, k_n^+ L_T, (1+r_T)L_T, \tfrac{a_T - r_T}{1+r_T}, \tfrac{b_T + r_T}{1+r_T}\Big)\right)\Gamma\Big(\tfrac{k_n^+}{1+r_T}, \tfrac{a_T - r_T}{1+r_T}, \tfrac{b_T + r_T}{1+r_T}\Big)\Psi_T^+,
\end{aligned}$$

and for every $k \in \{-n^2 + 4, \ldots, n^2 - 5\}$

$$\begin{aligned}
&\mathbb{Q}_0\Big(Y \in \mathcal{B}^-_{1/(2n^2)}(G_{i,T}^{k,n}, L_T, a_T, b_T)\big|_{[0,\Delta_T]}\Big) \\
&\quad \geq \exp\left(\kappa^-_{T_{i-1},T_i}\Big(G, k_n^+ L_T, (1-r_T)L_T, \tfrac{a_T + r_T}{1-r_T}, \tfrac{b_T - r_T}{1-r_T}\Big)\right)\Gamma\Big(\tfrac{k_n^+}{1-r_T}, \tfrac{a_T + r_T}{1-r_T}, \tfrac{b_T - r_T}{1-r_T}\Big)\Psi_T^-,
\end{aligned}$$

where

$$\psi_T^+ = \Big(1 + C\exp\Big(-\tfrac{\pi^2 \Delta_T}{L_T^2(1+r_T)^2}\Big)\Big)\Big(1 + K'\exp\big(-\tfrac{\mu'}{2}L_T\delta_T\big)\Big)$$

and

$$\psi_T^- = \Big(1 - C\exp\Big(-\tfrac{\pi^2 \Delta_T}{L_T^2(1-r_T)^2}\Big)\Big)\Big(1 - K'\exp\big(-\tfrac{\mu'}{2}L_T\delta_T\big)\Big).$$

Before proving Proposition 4.13 we state an easy result which we will need to estimate from below the cosine terms in the proof.

Lemma 4.14. *Recall from Lemma 4.7 the definition*

$$\Gamma(y,a,b)=\cos\left(\frac{\pi y}{2}\right)\int_a^b\cos\left(\frac{\pi\nu}{2}\right)d\nu.$$

Let $a,b\in[-1,1]$ and $n\in\mathbb{N}$ such that $1/n=b-a$. Recall from Lemma 4.10 that $k_n^+=(k+1/2)/n^2$ for $k\in\{-n^2,\dots,n^2-1\}$.
If r is small enough and $\frac{\sqrt{r}}{2}\le\frac{1}{n}\le\frac{\sqrt{r}}{2(1-\sqrt{r}/2)}$, for every $k\in\{-n^2+4,\dots,n^2-5\}$

$$0<c_1(r)\le\Gamma\Big(\frac{k_n^+}{1-r},\frac{a+r}{1-r},\frac{b-r}{1-r}\Big)\tag{4.9}$$

and for every $k\in\{-n^2,\dots,n^2-1\}$

$$0<c_1(r)\le\Gamma\Big(\frac{k_n^+}{1+r},\frac{a-r}{1+r},\frac{b+r}{1+r}\Big),\tag{4.10}$$

where $c_1(r)$ satisfies

$$\lim_{T\to\infty}\frac{-\log(c_1(\delta_T))}{L_T\delta_T}=0.\tag{4.11}$$

Proof. When $k\in\{-n^2+4,\dots,n^2-5\}$ we have $|k_n^+|\le 1-\frac{9}{2n^2}\le 1-\frac{9r}{8}$. This gives, writing $\frac{1}{1-r}=1+\frac{r}{1-r}$, that

$$\left|\frac{\pi k_n^+}{2(1-r)}\right|\le\frac{\pi}{2}\Big(1+\frac{r}{1-r}\Big)\Big(1-\frac{9r}{8}\Big)=\frac{\pi}{2}\Big(1-\frac{\pi r}{8(1-r)},\Big)$$

and so

$$\cos\left(\frac{\pi k_n^+}{2(1-r)}\right)\ge\cos\left(\frac{\pi}{2}-\frac{\pi r}{16(1-r)}\right)=\sin\left(\frac{\pi r}{16(1-r)}\right).$$

Using the parity of the cosine,

$$\begin{aligned}\int_{\frac{a+r}{1-r}}^{\frac{b-r}{1-r}}\cos\left(\frac{\pi\nu}{2}\right)d\nu\ge\int_{1-\frac{(b-a)}{1-r}+\frac{2r}{1-r}}^{1}\cos\left(\frac{\pi\nu}{2}\right)d\nu&=\frac{2}{\pi}-\frac{2}{\pi}\sin\left(\frac{\pi}{2}-\frac{\pi}{2}\cdot\frac{b-a-2r}{1-r}\right)\\&=\frac{2}{\pi}-\frac{2}{\pi}\cos\left(\frac{\pi}{2}\cdot\frac{b-a-2r}{1-r}\right).\end{aligned}$$

Using that $b-a=1/n\ge\sqrt{r}/2$,

$$\frac{b-a-2r}{1-r}\ge\frac{\sqrt{r}/2-2r}{1-r}\ge\sqrt{r}/4$$

if r is small enough, so since $\cos\left(\frac{\pi x}{2}\right)$ decreasing when $x\in(0,1)$,

$$\cos\left(\frac{\pi}{2}\cdot\frac{b-a+2r}{1-r}\right)\le\cos\left(\pi\sqrt{r}/2\right).$$

It follows that

$$\int_{\frac{a+r}{1-r}}^{\frac{b-r}{1-r}}\cos\left(\frac{\pi\nu}{2}\right)d\nu\ge\frac{2}{\pi}-\frac{2}{\pi}\cos\left(\frac{\pi}{2}\cdot\frac{b-a+2r}{1-r}\right)\ge\frac{2}{\pi}\big(1-\cos(\pi\sqrt{r}/2)\big).$$

This shows that we can choose

$$c_1(r) = \frac{2}{\pi}\sin\Big(\frac{\pi r}{16(1-r)}\Big)\big(1-\cos(\pi\sqrt{r}/2)\big).$$

To show (4.10), using that $|k_n^+|\le 1$ we have $\frac{|k_n^+|}{1+r} \le \frac{1}{1+r} = 1 - \frac{r}{1+r}$, so

$$\cos\Big(\frac{\pi k_n^+}{2(1+r)}\Big) \ge \cos\Big(\frac{\pi}{2} - \frac{\pi r}{2(1+r)}\Big) = \sin\Big(\frac{\pi r}{2(1+r)}\Big).$$

It is easy to check that $\frac{\pi r}{2(1+r)} \ge \frac{\pi r}{16(1-r)}$ when $r \le 7/9$, so

$$\sin\Big(\frac{\pi r}{2(1+r)}\Big) \ge \sin\Big(\frac{\pi r}{16(1-r)}\Big).$$

We also have that

$$\int_{\frac{a-r}{1+r}}^{\frac{b+r}{1+r}} \cos\Big(\frac{\pi\nu}{2}\Big)d\nu \ge \int_{\frac{a+r}{1-r}}^{\frac{b-r}{1-r}} \cos\Big(\frac{\pi\nu}{2}\Big)d\nu$$

because the interval on the left-hand side is bigger and the integrating function is always positive on $[-1,1]$. It follows that for every $k\in\{-n^2,\dots,n^2-1\}$

$$\cos\left(\frac{\pi k_n^+}{2(1+r)}\right)\int_{\frac{a-r}{1+r}}^{\frac{b+r}{1+r}} \cos\Big(\frac{\pi\nu}{2}\Big)\,d\nu \ge \sin\left(\frac{\pi r}{16(1-r)}\right)\int_{\frac{a+r}{1-r}}^{\frac{b-r}{1-r}} \cos\Big(\frac{\pi\nu}{2}\Big)d\nu$$

and so (4.10) is a consequence of the bound in (4.9).

We are left to prove (4.11). It suffices to show that

$$\lim_{T\to\infty} \log\big(\sin(\delta_T)\big)\big(L_T\delta_T\big)^{-1} + \log\big(1-\cos(\sqrt{\delta_T})\big)\big(L_T\delta_T\big)^{-1} = 0.$$

Recall that $\lim_{T\to\infty} N_T\delta_T^2 = \infty$ by (i), $N_T \le T^{1/3}$ and $\delta_T < 1$, so we also have $\lim_{T\to\infty} L_T\delta_T = \infty$. Using that $\lim_{x\to 0}\frac{\sin(x)}{x} = 1$ and $\lim_{x\to0}\frac{1-\cos(x)}{x^2} = 1/2$, our limit is equivalent to

$$\lim_{T\to\infty}\log(\delta_T)\big(L_T\delta_T\big)^{-1} = 0.$$

This can be rewritten as

$$\lim_{T\to\infty}\delta_T\log\big(\delta_T\big)\big(L_T\delta_T^2\big)^{-1},$$

which is 0 since $\lim_{x\to 0^+} x\log x = 0$ and $\lim_{T\to\infty} L_T\delta_T^2 = \infty$ from (i). $\square$

Proof of Proposition 4.13. In order to estimate the probability that $Y(s)$ is in a tube about $G_{i,T}^{k,n}$ we apply Lemma 4.12 starting from $\mathcal{B}^+_{1/(2n_T)}(G^{k,n}_{i,T}, L_T, a_T, b_T)|_{[0,\Delta_T]}$.

Recall that $r_T = 4\delta_T \ge 1/(2n_T^2) + 3\delta_T$.

Then for every T large enough

$$\begin{aligned}\mathbb{Q}_0\Big(Y\in\mathcal{B}^+_{1/(2n_T^2)}\big(G^{k,n}_{i,T}, L_T, a_T, b_T\big)\big|_{[0,\Delta_T]}\Big)\\ \le \mathbb{Q}_0\Big(B\in\mathcal{B}^+_{r_T}\big(G^{k,n}_{i,T}, L_T, a_T, b_T\big)\big|_{[0,\Delta_T]}\Big) + K'\exp\big(-\mu' L_T\delta_T\big).\end{aligned}\tag{4.12}$$

Noting that $(G^{k,n}_{i,T})'(s) = G'(s+T_{i-1})\ \forall s\in[0,\Delta_T]$ and $G^{k,n}_{i,T}(0) = -k_n^+L_T$, Proposition

4.7 gives that

$$\mathbb{Q}_0\Big(B \in \mathcal{B}^+_{r_T}\big(G^{k,n}_{i,T}, L_T, a_T, b_T\big)\big|_{[0,\Delta_T]}\Big)$$
$$\le \exp\left(\kappa^+_{0,\Delta_T}\Big(G^{k,n}_{i,T}, k^+_n L_T, (1+r_T)L_T, \frac{a_T - r_T}{1+r_T}, \frac{b_T + r_T}{1+r_T}\Big)\right)$$
$$\cdot\,\Gamma\Big(\frac{k^+_n}{1+r_T}, \frac{a_T - r_T}{1+r_T}, \frac{b_T + r_T}{1+r_T}\Big)\Big(1 + Ce^{-\pi^2\Delta_T/L_T^2}\Big), \quad (4.13)$$

where

$$\kappa^+_{u,t}(H, w, L, a, b)$$
$$= -\frac{1}{2}\int_u^t H'(s)^2 ds + H'(u)w - \frac{\pi^2(t-u)}{8L^2} - \theta^+_t(H, L, a, b)H'(t) + L\int_u^t |H''(s)| ds,$$

and

$$\Gamma(y, a, b) = \cos\Big(\frac{\pi y}{2}\Big)\int_a^b \cos\Big(\frac{\pi \nu}{2}\Big) d\nu.$$

Denote by A the right-hand side in (4.13). Then, by shifting time to $[T_{i-1}, T_i]$ we have

$$A = \exp\left(\kappa^+_{T_{i-1},T_i}\Big(G, k^+_n L_T, (1+r_T)L_T, \frac{a_T - r_T}{1+r_T}, \frac{b_T + r_T}{1+r_T}\Big)\right)$$
$$\cdot\,\Gamma\Big(\frac{k^+_n}{1+r_T}, \frac{a_T - r_T}{1+r_T}, \frac{b_T + r_T}{1+r_T}\Big)\Big(1 + Ce^{-\pi^2\Delta_T/L_T^2}\Big).$$

The proposition is proved if we show that the second term in (4.12) satisfies

$$K' \exp\big(-\mu' L_T \delta_T\big) \le A \cdot K' \exp\Big(-\frac{\mu'}{2} L_T \delta_T\Big) \quad (4.14)$$

for $T \ge T'$. By (4.10) in Lemma 4.14

$$0 < c_1(r_T) \le \Gamma\Big(\frac{k^+_n}{1+r_T}, \frac{a_T - r_T}{1+r_T}, \frac{b_T + r_T}{1+r_T}\Big).$$

Using this, together with the fact that $r_T \le 4$, $|\theta^+_{T_i} G'(T_i)| \le 2|G'(T_i)|L_T$ and $|k^+_n| \le 1$, we have

$$A \ge c_1(r_T)\exp\Big(-2|G'(T_i)|L_T - \frac{1}{2}\int_{T_{i-1}}^{T_i} G'(s)^2 ds - |G'(T_{i-1})|L_T - \frac{\pi^2\Delta_T}{8L_T^2}\Big),$$

so (4.14) is proved if we show that for T large

$$c_1(r_T)\exp\Big(-2|G'(T_i)|L_T - \frac{1}{2}\int_{T_{i-1}}^{T_i} G'(s)^2 ds - |G'(T_{i-1})|L_T - \frac{\pi^2\Delta_T}{8L_T^2}\Big)$$
$$\ge \exp\Big(-\frac{\mu'}{2} L_T \delta_T\Big).$$

Taking the logarithm of both terms and changing sign gives

$$-\log(c_1(r_T)) + 2|G'(T_i)|L_T + \frac{1}{2}\int_{T_{i-1}}^{T_i} G'(s)^2 ds + |G'(T_{i-1})|L_T + \frac{\pi^2\Delta_T}{8L_T^2} \le \frac{\mu'}{2} L_T \delta_T,$$

and dividing by $L_T\delta_T$ we obtain

$$\frac{-\log(c_1(r_T))}{L_T\delta_T}+\frac{2}{\delta_T}|G'(T_i)|+\frac{1}{2L_T\delta_T}\int_{T_{i-1}}^{T_i}G'(s)^2ds \\ +\frac{1}{\delta_T}|G'(T_{i-1})|+\frac{\pi^2\Delta_T}{8L_T^3\delta_T}\leq\frac{\mu'}{2}. \quad (4.15)$$

Recalling that $\Delta_T=(T-T_0)/N_T$ and $L_T=LT^{1/3}$, we have

$$\frac{\Delta_T}{L_T^3\delta_T}=\frac{T-T_0}{L^3T}\cdot\frac{1}{N_T\delta_T},$$

which tends to 0 as $T\to\infty$ since $\lim_{T\to\infty}\delta_T N_T=\infty$ from (i). By Lemma 4.14,

$$\lim_{T\to\infty}-\log(c_1(r_T))\big(L_T\delta_T\big)^{-1}=0.$$

We also have that

$$\frac{2}{\delta_T}|G'(T_i)|+\frac{1}{2L_T\delta_T}\int_{T_{i-1}}^{T_i}G'(s)^2ds+\frac{1}{\delta_T}|G'(T_{i-1})| \\ \leq\frac{3}{\delta_T}\sup_{s\in[T_0,T]}|G'(s)|+\frac{1}{2L_T\delta_T}\int_{T_{i-1}}^{T_i}G'(s)^2ds,$$

which tends to 0 for every $i\in\{1,\dots,N_T\}$ by (ii) and (iv). We thus get that all the terms on the left-hand side of (4.15) tend to 0 when T is large.

This concludes the proof of (4.14) and therefore the proof of the upper bound in the statement of the proposition. The lower bound can be obtained with identical calculations, using (4.9) in Lemma 4.14. □

4.4. Proof of Theorem 4.1

The strategy of the proof of Theorem 4.1 is combining Lemma 4.10 with Proposition 4.13: we split a time interval of length Δ_T from the end, then we estimate the probability that $Y(s)$ stays near $G(s)$ on this interval, and then we keep going backwards.

In order to iterate this procedure, we need an intermediate step. Indeed, Lemma 4.10 separates two time intervals in such a way that the endpoints of the first interval are L_T/n^2 apart and the tube on the next interval is widened or narrowed by L_T/n^2. However, the lemma requires that $L_T/n^2\leq(bL_T-aL_T)/4$, and this assumption is not satisfied at the endpoints of the first interval, since $b-a=1/n^2$. Therefore, before applying the lemma again, we use a union bound to get the endpoints at the end of the first interval L_T/n apart. This is summarised in the following proposition.

Proposition 4.15. *Let $n\geq 4$. We use N as a shorthand for the number of intervals N_T. Recall that $r_T=4\delta_T$. Let $c_2(n)$ be defined as in Lemma 4.16,*

$$\psi_T'^{+}=\Big(1+C\exp\Big(-\tfrac{\pi^2\Delta_T}{L_T^2(1+r_T)^2}\Big)\Big)\Big(1+K'\exp\big(-\tfrac{\mu'}{2}L_T\delta_T\big)\Big)\big(1+c_2(n)\big)$$

and

$$\psi_T'^{-} = \Big(1 - C\exp\Big(-\tfrac{\pi^2\Delta_T}{L_T^2(1-r_T)^2}\Big)\Big)\Big(1 - K'\exp\big(-\tfrac{\mu'}{2}L_T\delta_T\big)\Big)\big(1-c_2(n)\big).$$

Then

$$\begin{aligned}
&\mathbb{Q}_0\big(Y \in \mathcal{B}(G_T, L_T, p, q)|_{[0,T]}\big)\\
&\le \sum_{j_N=\lfloor pn\rfloor}^{\lceil qn-1\rceil}\sum_{j_{N-1}=-n}^{n-1}\cdots\sum_{j_0=-n}^{n-1}\mathbb{Q}_0\Big(Y\in\mathcal{B}\big(G_T, L_T, \tfrac{j_0}{n}, \tfrac{j_0+1}{n}\big)|_{[0,T_0]}\Big)\\
&\quad\cdot\prod_{i=1}^{N}\Big\{\exp\Big(\kappa^+_{T_{i-1},T_i}\Big(G, j^+_{i-1,n}L_T, (1+r_T)L_T, \tfrac{j_i/n-r_T}{1+r_T}, \tfrac{j_i/n+1/n+r_T}{1+r_T}\Big)\Big)\Big\}\\
&\qquad\cdot\prod_{i=1}^{N}\Big\{\Gamma\left(\frac{j_{i-1}+1/2}{n(1+r_T)}, \frac{j_i/n-r_T}{1+r_T}, \frac{j_i/n+1/n+r_T}{1+r_T}\right)\Psi_T'^{+}\Big\}. \qquad (4.16)
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{Q}_0\big(Y \in \mathcal{B}(G_T, L_T, p, q)|_{[0,T]}\big)\\
&\ge \sum_{j_{N-1}=-n+1}^{n-2}\cdots\sum_{j_1=-n+1}^{n-2}\mathbb{Q}_0\Big(Y\in\mathcal{B}\big(G_T, L_T, \tfrac{x}{L}-\tfrac{1}{2n_T}, \tfrac{x}{L}+\tfrac{1}{2n_T}\big)|_{[0,T_0]}\Big)\\
&\quad\cdot\prod_{i=1}^{N}\Big\{\exp\Big(\kappa^-_{T_{i-1},T_i}\Big(G, j^-_{i-1,n}L_T, (1-r_T)L_T, \tfrac{j_i/n+r_T}{1-r_T}, \tfrac{j_i/n+1/n-r_T}{1-r_T}\Big)\Big)\Big\}\\
&\qquad\cdot\prod_{i=1}^{N}\Big\{\Gamma\left(\frac{j_{i-1}+1/2}{n(1-r_T)}, \frac{j_i/n+r_T}{1-r_T}, \frac{j_i/n+1/n-r_T}{1-r_T}\right)\Psi_T'^{-}\Big\}. \qquad (4.17)
\end{aligned}$$

where $j^+_{i,n}$ and $j^-_{i,n}$ for $i \in \{0,\dots,N-1\}$ are defined by

$$j^+_{i,n} = \begin{cases} j_i/n + 1/n & \text{if } G'(T_i) > 0\\ j_i/n - 1/n & \text{if } G'(T_i) < 0\end{cases} \qquad j^-_{i,n} = \begin{cases} j_i/n - 1/n & \text{if } G'(T_i) > 0\\ j_i/n + 1/n & \text{if } G'(T_i) < 0,\end{cases}$$

and in (4.17) *we define, for convenience of notation, $j_0 := x/L$ and $j_N := pn$.*

Before proving Proposition 4.15, we state separately some easy approximations of the terms involving the cosine.

Lemma 4.16. *Let $n \ge 4$ and $\frac{\sqrt{r}}{2} \le \frac{1}{n} \le \frac{\sqrt{r}}{2(1-\sqrt{r}/2)}$. Recall that for $k \in \{-n^2,\dots,n^2-1\}$ we defined $k^+_n = (k+1/2)/n^2$.*
If $j \in \{-n,\dots,n-1\}$ and $k \in \{jn,\dots,jn+n-1\}$,

$$\cos\left(\frac{\pi k^+_n}{2(1+r)}\right) \le \cos\left(\frac{\pi(j+1/2)}{2n(1+r)}\right)(1+c_2(n)) \qquad (4.18)$$

and if $j \in \{-n+1,\dots,n-2\}$ and $k \in \{jn,\dots,jn+n-1\}$,

$$\cos\left(\frac{\pi k^+_n}{2(1-r)}\right) \ge \cos\left(\frac{\pi(j+1/2)}{2n(1-r)}\right)(1-c_2(n)), \qquad (4.19)$$

where $c_2(n) = \frac{\pi}{4n}\sin\left(\frac{\pi}{4n}\right)^{-1}$.

Proof. Using that $|k-jn-n/2+1/2|\leq n/2-1/2 \leq n/2$ for every $k \in \{jn, \dots, jn+n-1\}$, Taylor's formula gives

$$\left|\cos\left(\frac{\pi k_n^+}{2(1+r)}\right) - \cos\left(\frac{\pi(j+1/2)}{2n(1+r)}\right)\right| \leq \left|\frac{\pi(k+1/2)}{2n^2(1+r)} - \frac{\pi(jn+n/2)}{2n^2(1+r)}\right| \leq \frac{\pi}{4n(1+r)}$$

and similarly

$$\left|\cos\left(\frac{\pi k_n^+}{2(1-r)}\right) - \cos\left(\frac{\pi(j+1/2)}{2n(1-r)}\right)\right| \leq \frac{\pi}{4n(1-r)}.$$

Since for every $-n \leq j \leq n-1$

$$\left|\frac{\pi(j+1/2)}{2n(1+r)}\right| \leq \frac{\pi(n-1/2)}{2n(1+r)} \leq \frac{\pi(n-1/2)}{2n} = \frac{\pi}{2} - \frac{\pi}{4n}$$

and the cosine is even, we have

$$\cos\left(\frac{\pi(j+1/2)}{2n(1+r)}\right) \geq \cos\left(\frac{\pi}{2} - \frac{\pi}{4n}\right), \tag{4.20}$$

which gives (4.18). Similarly, for every $-n+1 \leq j \leq n-2$, using that $|j+1/2|\leq n-3/2$ and $\sqrt{r} \leq 2/n$

$$\begin{aligned}\left|\frac{\pi(j+1/2)}{2n(1-r)}\right| \leq \frac{\pi(n-3/2)}{2n(1-r)} = \frac{\pi}{2(1-r)} - \frac{3\pi}{4n(1-r)} &= \frac{\pi}{2} + \frac{\pi r}{2(1-r)} - \frac{3\pi}{4n(1-r)} \\ &\leq \frac{2\pi}{n^2(1-r)} - \frac{3\pi}{4n(1-r)}.\end{aligned}$$

Since for $n \geq 4$

$$\frac{2\pi}{n^2(1-r)} - \frac{3\pi}{4n(1-r)} = \frac{\pi}{n(1-r)}\left(\frac{2}{n} - \frac{3}{4}\right) \leq \frac{\pi}{n(1-r)}\left(\frac{1}{2} - \frac{3}{4}\right) = -\frac{\pi}{4n(1-r)} \leq -\frac{\pi}{4n},$$

it follows that

$$\cos\left(\frac{\pi(j+1/2)}{2n(1-r)}\right) \geq \cos\left(\frac{\pi}{2} - \frac{\pi}{4n}\right),$$

and so (4.19) is proved. □

Proof of Proposition 4.15. We start by applying Lemma 4.10 once, to separate the last time interval. This gives

$$\begin{aligned}&\mathbb{Q}_0\big(Y \in \mathcal{B}(G_T, L_T, p, q)|_{[0,T_N]}\big) \\ &\leq \sum_{k=-n^2}^{n^2-1} \mathbb{Q}_0\Big(Y \in \mathcal{B}\big(G_T, L_T, \tfrac{k}{n^2}, \tfrac{k+1}{n^2}\big)\big|_{[0,T_{N-1}]}\Big)\mathbb{Q}_0\big(Y \in \mathcal{B}^+_{1/(2n^2)}(G^{k,n}_{N,T}, L_T, p, q)|_{[0,\Delta_T]}\big).\end{aligned}$$

By Proposition 4.13,

$$\begin{aligned}&\mathbb{Q}_0\Big(Y \in \mathcal{B}^+_{1/(2n^2)}\big(G^{k,n}_{N,T}, L_T, p, q\big)\big|_{[0,\Delta_T]}\Big)\\&\quad\le \exp\Big(\kappa^+_{T_{N-1},T_N}\Big(G, k^+_n L_T, (1+r_T)L_T, \tfrac{p-r_T}{1+r_T}, \tfrac{q+r_T}{1+r_T}\Big)\Big)\,\Gamma\Big(\tfrac{k^+_n}{1+r_T}, \tfrac{p-r_T}{1+r_T}, \tfrac{q+r_T}{1+r_T}\Big)\Psi^+_T,\end{aligned}$$

where

$$\psi^+_T = \Big(1 + C\exp\Big(-\tfrac{\pi^2\Delta_T}{L_T^2(1+r_T)^2}\Big)\Big)\Big(1 + K'\exp\big(-\tfrac{\mu'}{2}L_T\delta_T\big)\Big).$$

Then

$$\begin{aligned}&\mathbb{Q}_0\big(Y \in \mathcal{B}(G_T, L_T, p, q)|_{[0,T_N]}\big)\\&\le \sum_{k=-n^2}^{n^2-1}\mathbb{Q}_0\Big(Y \in \mathcal{B}\big(G_T, L_T, \tfrac{k}{n^2}, \tfrac{k+1}{n^2}\big)|_{[0,T_{N-1}]}\Big)\\&\quad\cdot\exp\Big(\kappa^+_{T_{N-1},T_N}\Big(G, k^+_n L_T, (1+r_T)L_T, \tfrac{p-r_T}{1+r_T}, \tfrac{q+r_T}{1+r_T}\Big)\Big)\,\Gamma\Big(\tfrac{k^+_n}{1+r_T}, \tfrac{p-r_T}{1+r_T}, \tfrac{q+r_T}{1+r_T}\Big)\Psi^+_T.\end{aligned}$$

We split the sum over k into groups of n indices, thus obtaining

$$\begin{aligned}&\mathbb{Q}_0\big(Y \in \mathcal{B}(G_T, L_T, p, q)|_{[0,T_N]}\big)\\&\le \sum_{j=-n}^{n-1}\sum_{k=jn}^{jn+n-1}\mathbb{Q}_0\Big(Y \in \mathcal{B}\big(G_T, L_T, \tfrac{k}{n^2}, \tfrac{k+1}{n^2}\big)|_{[0,T_{N-1}]}\Big)\\&\quad\cdot\exp\Big(\kappa^+_{T_{N-1},T_N}\Big(G, k^+_n L_T, (1+r_T)L_T, \tfrac{p-r_T}{1+r_T}, \tfrac{q+r_T}{1+r_T}\Big)\Big)\,\Gamma\Big(\tfrac{k^+_n}{1+r_T}, \tfrac{p-r_T}{1+r_T}, \tfrac{q+r_T}{1+r_T}\Big)\Psi^+_T.\end{aligned}$$

We now change the terms that depend on k into quantities involving j.

Let $c_2(n) = \frac{\pi}{4n}\sin\big(\frac{\pi}{4n}\big)^{-1}$. By (4.18) in Lemma 4.16, If $j \in \{-n, \dots, n-2\}$ and $k \in \{jn, \dots, jn+n-1\}$,

$$\cos\left(\frac{\pi k^+_n}{2(1+r)}\right) \le \cos\left(\frac{\pi(j+1/2)}{2n(1+r)}\right)(1 + c_2(n)).$$

Furthermore, $k^+_n - j/n = (k + 1/2 - jn)/n^2 \le 1/n$ when $k \in \{jn, \dots, jn+n-1\}$. Using this and the definition of $j^+_{N-1,n}$, we have

$$\begin{aligned}&\mathbb{Q}_0\big(Y \in \mathcal{B}(G_T, L_T, p, q)|_{[0,T]}\big)\\&\le \sum_{j=-n}^{n-1}\mathbb{Q}_0\Big(Y \in \mathcal{B}\big(G, L_T, \tfrac{j}{n}, \tfrac{j+1}{n}\big)|_{[0,T_{N-1}]}\Big)\\&\cdot\exp\Big(\kappa^+_{T_{N-1},T_N}\Big(G, j^+_{N-1,n}L_T, (1+r_T)L_T, \tfrac{p-r_T}{1+r_T}, \tfrac{q+r_T}{1+r_T}\Big)\Big)\,\Gamma\Big(\tfrac{j+1/2}{n(1+r_T)}, \tfrac{p-r_T}{1+r_T}, \tfrac{q+r_T}{1+r_T}\Big)\Psi'^+_T.\end{aligned}$$

where $\Psi'^+_T = \big(1 + c_2(n)\big)\Psi^+_T$.

According to the notation introduced in the statement of the proposition, we denote the index j in the sum above by $j_{N-1,n}$, since it is the index determining the endpoint of the process at time T_{N-1}. We repeat this N times where, for $0 \le i \le N-1$, we substitute k with k_i to denote the index for splitting the tube width at time T_i. This proves the upper bound in the statement of the proposition.

The lower bound can be proved in the same way, using the lower bounds in Lemma 4.10, Proposition 4.13, Lemma 4.16 and $j^-_{N-1,n}$ instead of $j^+_{N-1,n}$. $\square$

Before we move to the next section and prove Theorem 4.1, we prove another elementary result which will be useful in the proof to deal with the cosine terms that appear combining all the Brownian motion probabilities.

Lemma 4.17. *Recall that for $y \in (-1,1)$ and $a, b \in (-1,1)$ we defined*

$$\Gamma(y,a,b) = \cos\left(\frac{\pi y}{2}\right)\int_a^b \cos\left(\frac{\pi \nu}{2}\right) d\nu.$$

For every $n \geq 4$ and $\frac{\sqrt{r}}{2} \leq \frac{1}{n} \leq \frac{\sqrt{r}}{2(1-\sqrt{r}/2)}$,

$$\sum_{j=-n}^{n-1} \Gamma\left(\frac{j+1/2}{n(1+r)}, \frac{j/n - r}{1+r}, \frac{j/n + 1/n + r}{1+r}\right) \leq 1 + c_3(n) \tag{4.21}$$

and

$$\sum_{j=-n+1}^{n-2} \Gamma\left(\frac{j+1/2}{n(1-r)}, \frac{j/n + r}{1-r}, \frac{j/n + 1/n - r}{1-r}\right) \geq 1 - c_3(n), \tag{4.22}$$

where $c_3(n) > 0$ and $\lim_{n\to\infty} c_3(n) = 0$.

Proof. We only prove the lower bound. Letting $\beta = \nu(1-r) - j/n$, we have

$$\int_{\frac{j/n+r}{1-r}}^{\frac{j/n+1/n-r}{1-r}} \cos\left(\frac{\pi\nu}{2}\right) d\nu = \frac{1}{1-r}\int_r^{1/n-r} \cos\left(\frac{\pi(\beta + j/n)}{2(1-r)}\right) d\beta.$$

For every $\beta \in (r, 1/n - r)$, a Taylor expansion and then using that $|\beta| \leq 1/n + r \leq 3/n$ gives

$$\left|\cos\left(\frac{\pi(j/n + 1/(2n))}{2(1-r)}\right) - \cos\left(\frac{\pi(\beta + j/n)}{2(1-r)}\right)\right| \leq \frac{\pi|1/(2n) - \beta|}{2(1-r)} \leq \frac{2\pi}{n(1-r)},$$

so

$$\cos\left(\frac{\pi(j/n + 1/(2n))}{2(1-r)}\right) \geq \sup_{\beta\in(r,1/n-r)} \cos\left(\frac{\pi(\beta + j/n)}{2(1-r)}\right) - \frac{2\pi}{n(1-r)}.$$

Substituting this and changing back to the original variable in the integral, we obtain

$$\begin{aligned}
&\cos\left(\frac{\pi(j+1/2)}{2n(1-r)}\right)\int_{\frac{j/n+r}{1-r}}^{\frac{j/n+1/n-r}{1-r}} \cos\left(\frac{\pi\nu}{2}\right) d\nu \\
&\geq \left(\sup_{\beta\in(r,1/n-r)} \cos\left(\frac{\pi(\beta + j/n)}{2(1-r)}\right) - \frac{2\pi}{n(1-r)}\right)\int_r^{1/n-r} \cos\left(\frac{\pi(\beta + j/n)}{2(1-r)}\right)\frac{d\beta}{1-r} \\
&\geq \int_r^{1/n-r} \cos^2\left(\frac{\pi(\beta + j/n)}{2(1-r)}\right)\frac{d\beta}{1-r} - \frac{2\pi}{n(1-r)}\left(\frac{1/n}{1-r}\right) \\
&= \int_{\frac{j/n+r}{1-r}}^{\frac{j/n+1/n-r}{1-r}} \cos^2\left(\frac{\pi\nu}{2}\right) d\nu - \frac{2\pi}{n^2(1-r)^2}.
\end{aligned}$$

Taking the sum of the integrals over j, we obtain

$$\begin{aligned}\sum_{j=-n+1}^{n-2} \int_{\frac{j/n+r}{1-r}}^{\frac{j/n+1/n-r}{1-r}} \cos^2\left(\frac{\pi\nu}{2}\right) d\nu &= \int_{-1+\frac{1}{n(1-r)}}^{1-\frac{1}{n(1-r)}} \cos^2\left(\frac{\pi\nu}{2}\right) d\nu \\ &= \int_{-1}^{1} \cos^2\left(\frac{\pi\nu}{2}\right) d\nu - 2\int_{1-\frac{1}{n(1-r)}}^{1} \cos^2\left(\frac{\pi\nu}{2}\right) d\nu \\ &= 1 - 2\int_{1-\frac{1}{n(1-r)}}^{1} \cos^2\left(\frac{\pi\nu}{2}\right) d\nu \geq 1 - \frac{2}{n(1-r)},\end{aligned}$$

and so using that $\frac{1}{1-r} \leq \frac{1}{1-4/n^2} = \frac{n^2}{n^2-4}$ we ultimately get

$$\begin{aligned}\sum_{j=-n+1}^{n-2} \cos\left(\frac{\pi(j+1/2)}{2n(1-r)}\right) \int_{\frac{j/n+r}{1-r}}^{\frac{j/n+1/n-r}{1-r}} \cos\left(\frac{\pi\nu}{2}\right) &\geq 1 - \frac{2}{n(1-r)} - 2n \cdot \frac{2\pi}{n^2(1-r)^2} \\ &\geq 1 - \frac{2n}{n^2-4} - \frac{4\pi n^3}{(n^2-4)^2}. \qquad \square\end{aligned}$$

4.5. Proof of the upper bound in Theorem 4.1

The proof of the upper bound in Theorem 4.1 is a consequence of Proposition 4.15 and the assumptions we made on $G(s)$.

Proof of the upper bound in Theorem 4.1. In this proof, we use N as a shorthand for the number of intervals N_T. From (4.16) in Proposition 4.15,

$$\begin{aligned}&\mathbb{Q}_0\big(Y \in \mathcal{B}(G_T, L_T, p, q)|_{[0,T]}\big) \\ &\leq \sum_{j_N=\lfloor pn \rfloor}^{\lceil qn-1 \rceil} \sum_{j_{N-1}=-n}^{n-1} \cdots \sum_{j_0=-n}^{n-1} \mathbb{Q}_0\Big(Y \in \mathcal{B}\big(G_T, L_T, \tfrac{j_0}{n}, \tfrac{j_0+1}{n}\big)|_{[0,T_0]}\Big) \\ &\quad \cdot \prod_{i=1}^{N} \left\{ \exp\left(\kappa^+_{T_{i-1},T_i}\Big(G, j^+_{i-1,n} L_T, (1+r_T)L_T, \tfrac{j_i/n - r_T}{1+r_T}, \tfrac{j_i/n+1/n+r_T}{1+r_T}\Big)\right) \right\} \\ &\qquad \cdot \prod_{i=1}^{N} \left\{ \Gamma\left(\frac{j_{i-1}+1/2}{n(1+r_T)}, \frac{j_i/n - r_T}{1+r_T}, \frac{j_i/n+1/n+r_T}{1+r_T}\right) \Psi'^{+}_{T} \right\}, \qquad (4.23)\end{aligned}$$

where

$$\begin{aligned}&\kappa^+_{u,t}(H, w, L, a, b) \\ &\quad = -\frac{1}{2}\int_u^t H'(s)^2 ds + H'(u)w - \frac{\pi^2(t-u)}{8L^2} - \theta^+_t(H, L, a, b)H'(t) + L\int_u^t |H''(s)| ds\end{aligned}$$

and

$$\Gamma(y, a, b) = \cos\left(\frac{\pi y}{2}\right) \int_a^b \cos\left(\frac{\pi\nu}{2}\right) d\nu.$$

The rest of the proof essentially consists of combining together the terms in the products in (4.23). We first deal with the exponential terms involving $\kappa^+_{T_{i-1},T_i}$. We have

$$\prod_{i=1}^{N}\exp\left(-\frac{1}{2}\int_{T_{i-1}}^{T_i}G'(s)^2ds-\frac{\pi^2(T_i-T_{i-1})}{8L_T^2(1+r_T)^2}+L_T(1+r_T)\int_{T_{i-1}}^{T_i}|G''(s)|ds\right)$$
$$=\exp\left(-\frac{1}{2}\int_{T_0}^{T}G'(s)^2ds+L_T(1+r_T)\int_{T_0}^{T}|G''(s)|ds-\frac{\pi^2(T-T_0)}{8L_T^2(1+r_T)^2}\right). \quad (4.24)$$

We show that the terms involving $\theta_{T_i}^+$ and $j_{i-1,n}^+$ can be rewritten as a telescopic sum where all the terms are small, except from the first and the last one. Recall the definitions

$$\theta_{T_i}^+(G,L,a,b)=\begin{cases}aL & \text{if } G'(T_i)>0\\ bL & \text{if } G'(T_i)<0,\end{cases}\qquad j_{i,n}^+=\begin{cases}j_i/n+1/n & \text{if } G'(T_i)>0\\ j_i/n-1/n & \text{if } G'(T_i)<0.\end{cases}$$

Shifting the index i in the $j_{i-1,n}^+$ gives that

$$\prod_{i=1}^{N}\exp\left(-\theta_{T_i}^+\left(G,(1+r_T)L_T,\tfrac{j_i/n-r_T}{1+r_T},\tfrac{j_i/n+1/n+r_T}{1+r_T}\right)G'(T_i)+j_{i-1,n}^+L_TG'(T_{i-1})\right)$$
$$=\left\{\prod_{i=1}^{N-1}\exp\left(-\theta_{T_i}^+\left(G,(1+r_T)L_T,\tfrac{j_i/n-r_T}{1+r_T},\tfrac{j_i/n+1/n+r_T}{1+r_T}\right)G'(T_i)+j_{i,n}^+L_TG'(T_i)\right\}$$
$$\cdot\exp\left(-\theta_{T_N}^+\left(G,(1+r_T)L_T,\tfrac{j_N/n-r_T}{1+r_T},\tfrac{j_N/n+1/n+r_T}{1+r_T}\right)G'(T_N)+j_{0,n}^+L_TG'(T_0)\right).$$

If $G'(T_i)>0$, using that $\frac{1}{1-\sqrt{r}/2}\leq 2$ for every $r<1$, and so $\frac{1}{n}\leq\frac{\sqrt{r}}{2}\cdot\frac{1}{1-\sqrt{r}/2}\leq\sqrt{r}$, we have

$$\exp\left(-\theta_{T_i}^+\left(G,(1+r_T)L_T,\tfrac{j_i/n-r_T}{1+r_T},\tfrac{j_i/n+1/n+r_T}{1+r_T}\right)G'(T_i)+j_{i,n}^+L_TG'(T_i)\right)$$
$$=\exp\left(-G'(T_i)\Big(\frac{j_i}{n}-r_T\Big)L_T+G'(T_i)\Big(\frac{j_i}{n}+\frac{1}{n}\Big)L_T\right)$$
$$=\exp\left(G'(T_i)\Big(r_T+\frac{1}{n}\Big)L_T\right)\leq\exp\left(3\sqrt{r_T}|G'(T_i)|L_T\right),$$

and it is easy to check that the same holds when $G'(T_i)<0$. Substituting this in the above and using that

$$\exp\left(-\theta_{T_N}^+\left(G,(1+r_T)L_T,\tfrac{j_N/n-r_T}{1+r_T},\tfrac{j_N/n+1/n+r_T}{1+r_T}\right)G'(T_N)+j_{0,n}^+L_TG'(T_0)\right)$$
$$\leq\exp\left(2|G'(T_N)|L_T+|G'(T_0)|L_T\right),$$

we obtain that

$$\prod_{i=1}^{N}\exp\left(-\theta_{T_i}^+\left(G,(1+r_T)L_T,\tfrac{j_i/n-r_T}{1+r_T},\tfrac{j_i/n+1/n+r_T}{1+r_T}\right)G'(T_i)+j_{i-1,n}^+L_TG'(T_{i-1})\right)$$
$$\leq\exp\left(3\sqrt{r_T}L_T\sum_{i=1}^{N-1}|G'(T_i)|+2|G'(T_N)|L_T+|G'(T_0)|L_T\right). \quad (4.25)$$

Going back to (4.23) and substituting (4.24) and (4.25), we have

$$\begin{aligned}
&\mathbb{Q}_0(Y \in \mathcal{B}(G_T, L_T, p, q)|_{[0,T]}) \\
&\qquad \le \exp\Big(-\frac{1}{2}\int_{T_0}^{T} G'(s)^2 ds + L_T(1+r_T)\int_{T_0}^{T}|G''(s)|ds - \frac{\pi^2(T-T_0)}{8L_T^2(1+r_T)^2}\Big) \\
&\qquad\qquad \cdot \exp\Big(3\sqrt{r_T}L_T\sum_{i=1}^{N-1}|G'(T_i)|+2|G'(T_N)|L_T + |G'(T_0)|L_T\Big) \\
&\qquad\qquad \cdot \sum_{j_N=\lfloor pn\rfloor}^{\lceil qn-1\rceil}\sum_{j_{N-1}=-n}^{n-1}\cdots\sum_{j_0=-n}^{n-1}\mathbb{Q}_0\Big(Y\in\mathcal{B}\big(G_T, L_T, \tfrac{j_0}{n}, \tfrac{j_0+1}{n}\big)|_{[0,T_0]}\Big) \\
&\qquad\qquad \cdot \prod_{i=1}^{N}\Big\{\Gamma\Big(\frac{j_{i-1}+1/2}{n(1+r_T)}, \frac{j_i/n - r_T}{1+r_T}, \frac{j_i/n+1/n+r_T}{1+r_T}\Big)\Psi_T'^{+}\Big\}. \qquad (4.26)
\end{aligned}$$

We now deal with the sums in the last two lines of (4.26). The error terms

$$\Psi_T'^{+} = \big(1+c_2(n)\big)\Big(1+C\exp\Big(-\tfrac{\pi^2\Delta_T}{L_T^2(1+r_T)^2}\Big)\Big)\Big(1+K'e^{-\frac{\mu'}{2}L_T\delta_T}\Big)$$

are independent of i and $j_0,\ldots,j_{N-1}$. Recall that we defined

$$\Gamma(y,a,b) = \cos\Big(\frac{\pi y}{2}\Big)\int_a^b \cos\Big(\frac{\pi\nu}{2}\Big)d\nu.$$

With a shift and then using that $\Gamma\Big(\frac{j_0+1/2}{n(1+r_T)}, \frac{j_N/n-r_T}{1+r_T}, \frac{j_N/n+1/n+r_T}{1+r_T}\Big) \le \frac{3}{n_T}$, we can write

$$\begin{aligned}
&\prod_{i=1}^{N}\Gamma\Big(\frac{j_{i-1}+1/2}{n(1+r_T)}, \frac{j_i/n-r_T}{1+r_T}, \frac{j_i/n+1/n+r_T}{1+r_T}\Big) \\
&= \Gamma\Big(\tfrac{j_0+1/2}{n(1+r_T)}, \tfrac{j_N/n-r_T}{1+r_T}, \tfrac{j_N/n+1/n+r_T}{1+r_T}\Big)\prod_{i=1}^{N-1}\Gamma\Big(\frac{j_i+1/2}{n(1+r_T)}, \frac{j_i/n-r_T}{1+r_T}, \frac{j_i/n+1/n+r_T}{1+r_T}\Big) \\
&\le \frac{3}{n_T}\prod_{i=1}^{N-1}\Gamma\Big(\frac{j_i+1/2}{n(1+r_T)}, \frac{j_i/n-r_T}{1+r_T}, \frac{j_i/n+1/n+r_T}{1+r_T}\Big).
\end{aligned}$$

Using that $\lceil qn-1\rceil - \lfloor pn\rfloor \le qn - pn$, this gives that

$$\begin{aligned}
&\sum_{j_N=\lfloor pn\rfloor}^{\lceil qn-1\rceil}\sum_{j_{N-1}=-n}^{n-1}\cdots\sum_{j_0=-n}^{n-1}\mathbb{Q}_0\Big(Y\in\mathcal{B}\big(G_T, L_T, \tfrac{j_0}{n}, \tfrac{j_0+1}{n}\big)|_{[0,T_0]}\Big) \\
&\qquad \cdot\prod_{i=1}^{N}\Big\{\Gamma\Big(\frac{j_{i-1}+1/2}{n(1+r_T)}, \frac{j_i/n-r_T}{1+r_T}, \frac{j_i/n+1/n+r_T}{1+r_T}\Big)\Psi_T'^{+}\Big\} \\
&\qquad\qquad \le 3(q-p)\big(\Psi_T'^{+}\big)^N\sum_{j_0=-n}^{n-1}\mathbb{Q}_0\Big(Y\in\mathcal{B}\big(G_T, L_T, \tfrac{j_0}{n}, \tfrac{j_0+1}{n}\big)|_{[0,T_0]}\Big) \\
&\qquad\qquad\qquad \cdot\Big\{\prod_{i=1}^{N-1}\sum_{j_i=-n}^{n-1}\Gamma\Big(\frac{j_i+1/2}{n(1+r_T)}, \frac{j_i/n-r_T}{1+r_T}, \frac{j_i/n+1/n+r_T}{1+r_T}\Big)\Big\}.
\end{aligned}$$

By Lemma 4.17, for every $n \geq 4$ and $\frac{\sqrt{r}}{2} \leq \frac{1}{n} \leq \frac{\sqrt{r}}{2(1-\sqrt{r}/2)}$,

$$\sum_{j=-n}^{n-1} \Gamma\left(\frac{j+1/2}{n(1+r)}, \frac{j/n - r}{1+r}, \frac{j/n+1/n+r}{1+r}\right) \leq 1 + c_3(n),$$

so the above is smaller than

$$3(q-p)\big(\Psi_T'^{+}\big)^N \sum_{j_0=-n}^{n-1} \mathbb{Q}_0\Big(Y \in \mathcal{B}\big(G_T, L_T, \tfrac{j_0}{n}, \tfrac{j_0+1}{n}\big)|_{[0,T_0]}\Big) \cdot (1+c_3(n))^{N-1}.$$

After taking the sum over j_0 and bounding the probability with 1, since $q - p \leq 2$ this is again smaller than

$$6\big(\Psi_T'^{+}\big)^N \big(1 + c_3(n)\big)^{N-1}.$$

Returning to (4.26), we deduce that

$$\begin{aligned}
&\mathbb{Q}_0(Y \in \mathcal{B}(G_T, L_T, p, q)|_{[0,T]}) \\
&\leq \exp\left(-\frac{1}{2}\int_{T_0}^T G'(s)^2 ds + L_T(1+r_T)\int_{T_0}^T |G''(s)| ds - \frac{\pi^2(T-T_0)}{8L_T^2(1+r_T)^2}\right) \\
&\cdot \exp\left(3\sqrt{r_T}L_T\sum_{i=1}^{N-1}|G'(T_i)| + 2|G'(T_N)|L_T + |G'(T_0)|L_T\right) \cdot 6\big(\Psi_T'^{+}\big)^N\big(1+c_3(n)\big)^{N-1}.
\end{aligned} \tag{4.27}$$

Recall that we are interested in $\mathbb{P}_0\big(X \in \mathcal{B}(F_T, L_T, p, q)|_{[0,T]}\big)$, and in Lemma 4.4 we showed that when $z > r\mathbb{E}[\xi]$

$$\begin{aligned}
&\mathbb{P}_0\big(X \in \mathcal{B}(F_T, L_T, p, q)|_{[0,T]}\big) \\
&\qquad \leq \exp\big(-\Lambda(z)T - \lambda(z)(G(T) - x_T + pL_T)\big)\mathbb{Q}_0^{\lambda(z)}\big(Y \in \mathcal{B}(\tilde{G}_T, \tilde{L}_T, p, q)|_{[0,T]}\big),
\end{aligned}$$

where $\tilde{G}_T(s) = G_T(s)\big(r\phi''(\lambda(z))\big)^{-1/2}$ and $\tilde{L}_T = L_T\big(r\phi''(\lambda(z))\big)^{-1/2}$.

An upper bound for $\mathbb{Q}_0^{\lambda(z)}\big(Y \in \mathcal{B}(\tilde{G}_T, \tilde{L}_T, p, q)|_{[0,T]}\big)$ is given by (4.27) with $\tilde{G}_T$ and $\tilde{L}_T$ instead of G_T and L_T, so combining this with Lemma 4.4 gives that

$$\begin{aligned}
&\mathbb{P}_0\big(X \in \mathcal{B}(F_T, L_T, p, q)|_{[0,T]}\big) \\
&\leq \exp\Big(-\Lambda(z)T - \lambda(z)(G(T) - x_T + pL_T)\Big) \cdot 6\big(\Psi_T'^{+}\big)^N\big(1+c_3(n)\big)^{N-1} \\
&\cdot \exp\left(-\frac{1}{2}\int_{T_0}^T \frac{G'(s)^2}{r\phi''(\lambda(z))} ds + L_T(1+r_T)\int_{T_0}^T \frac{|G''(s)|}{r\phi''(\lambda(z))} ds - \frac{\pi^2(T-T_0)r\phi''(\lambda(z))}{8L_T^2(1+r_T)^2}\right) \\
&\cdot \exp\left(3\sqrt{r_T}L_T\sum_{i=1}^{N-1}\frac{|G'(T_i)|}{r\phi''(\lambda(z))} + \frac{2|G'(T_N)|L_T}{r\phi''(\lambda(z))} + \frac{|G'(T_0)|L_T}{r\phi''(\lambda(z))}\right).
\end{aligned}$$

We take logarithms on both sides and divide by $T^{1/3}$, thus getting

$$\begin{aligned}
&\frac{1}{T^{1/3}}\log\mathbb{P}_0\big(X\in\mathcal{B}(F_T,L_T,p,q)|_{[0,T]}\big)\\
&\le\frac{1}{T^{1/3}}\big(-\Lambda(z)T-\lambda(z)G(T)\big)+\frac{\lambda(z)x_T}{T^{1/3}}-\lambda(z)pL+\frac{\log(6)}{T^{1/3}}+\frac{N_T}{T^{1/3}}\log\left(\Psi_T'^{+}\right)\\
&\quad+\frac{N_T-1}{T^{1/3}}\log\big(1+c_3(n_T)\big)-\frac{1}{2T^{1/3}}\int_{T_0}^{T}\frac{G'(s)^2}{r\phi''(\lambda(z))}ds+L(1+r_T)\int_{T_0}^{T}\frac{|G''(s)|}{r\phi''(\lambda(z))}ds\\
&\qquad-\frac{\pi^2(T-T_0)r\phi''(\lambda(z))}{8L^2T(1+r_T)^2}+3\sqrt{r_T}L\sum_{i=1}^{N-1}\frac{|G'(T_i)|}{r\phi''(\lambda(z))}+\frac{2|G'(T_N)|L}{r\phi''(\lambda(z))}+\frac{|G'(T_0)|L}{r\phi''(\lambda(z))}.
\end{aligned}$$

Writing

$$-\frac{1}{2T^{1/3}}\int_{T_0}^{T}\frac{G'(s)^2}{r\phi''(\lambda(z))}ds=-\frac{1}{2T^{1/3}}\int_{0}^{T}\frac{G'(s)^2}{r\phi''(\lambda(z))}ds+\frac{1}{2T^{1/3}}\int_{0}^{T_0}\frac{G'(s)^2}{r\phi''(\lambda(z))}ds$$

and rearranging terms, this is equivalent to

$$\begin{aligned}
&\frac{1}{T^{1/3}}\left(\log\mathbb{P}_0(X\in\mathcal{B}(F_T,L_T,p,q)|_{[0,T]})+\Lambda(z)T+\lambda(z)G(T)+\frac{1}{2}\int_0^T\frac{G'(s)^2}{r\phi''(\lambda(z))}ds\right)\\
&\le\frac{\lambda(z)x_T}{T^{1/3}}-\lambda(z)pL+\frac{\log(6)}{T^{1/3}}+\frac{N_T}{T^{1/3}}\log\left(\Psi_T'^{+}\right)+\frac{N_T-1}{T^{1/3}}\log\big(1+c_3(n_T)\big)\\
&\quad+\frac{1}{2T^{1/3}}\int_0^{T_0}\frac{G'(s)^2}{r\phi''(\lambda(z))}ds+L(1+r_T)\int_{T_0}^{T}\frac{|G''(s)|}{r\phi''(\lambda(z))}ds-\frac{\pi^2(T-T_0)r\phi''(\lambda(z))}{8L^2T(1+r_T)^2}\\
&\qquad+3\sqrt{r_T}L\sum_{i=1}^{N-1}\frac{|G'(T_i)|}{r\phi''(\lambda(z))}+\frac{2|G'(T_N)|L}{r\phi''(\lambda(z))}+\frac{|G'(T_0)|L}{r\phi''(\lambda(z))}.
\end{aligned}$$

We now take the limit as T tends to infinity. For the first term, recall that $x_T/T^{1/3}\to x$ with $x\in(-L,L)$. We defined

$$\Psi_T'^{+}=\big(1+c_2(n_T)\big)\Big(1+C\exp\Big(-\tfrac{\pi^2\Delta_T}{L_T^2(1+r_T)^2}\Big)\Big)\Big(1+K'e^{-\frac{\mu'}{2}L_T\delta_T}\Big).$$

Since $\lim_{T\to\infty}\Delta_T/L_T^2=\infty$, $\lim_{T\to\infty}L_T\delta_T=\infty$ and $\lim_{T\to\infty}N_T/T^{1/3}=0$, we have

$$\lim_{T\to\infty}\frac{N_T}{T^{1/3}}\log\Big(1+C\exp\Big(-\tfrac{\pi^2\Delta_T}{L_T^2(1+r_T)^2}\Big)+\lim_{T\to\infty}\frac{N_T}{T^{1/3}}\log\Big(1+K'e^{-\frac{\mu'}{2}L_T\delta_T}\Big)=0.$$

Recalling that $c_2(n)=\frac{\pi}{4n}\sin\left(\frac{\pi}{4n}\right)^{-1}$, and so $\lim_{T\to\infty}c_2(n_T)=1$, we also have

$$\lim_{T\to\infty}\frac{N_T}{T^{1/3}}\log\big(1+c_2(n_T)\big)=0,$$

which ultimately shows that

$$\lim_{T\to\infty}\frac{N_T}{T^{1/3}}\log\left(\Psi_T'^{+}\right)=0.$$

Lemma 4.17 ensures that $\lim_{T\to\infty}c_3(n_T)=0$, so

$$\lim_{T\to\infty}\frac{N_T-1}{T^{1/3}}\log\big(1+c_3(n_T)\big)=0.$$

Recall that $T_0 = T^{1/3-\varepsilon}$ and that, as we showed in (vi), there exists $M > 0$ such that $|G'(s)| \leq M$ for every $s \geq 0$. Then

$$\lim_{T\to\infty} \frac{1}{T^{1/3}} \int_0^{T_0} \frac{G'(s)^2}{r\phi''(\lambda(z))} ds \leq \lim_{T\to\infty} \frac{M^2 T_0}{r\phi''(\lambda(z)) T^{1/3}} = 0.$$

By assumption (v),

$$\lim_{T\to\infty} \frac{L(1+r_T)}{r\phi''(\lambda(z))} \int_{T_0}^{T} |G''(s)| ds = 0.$$

We also have that

$$3\sqrt{r_T} L \sum_{i=1}^{N_T-1} \frac{|G'(T_i)|}{r\phi''(\lambda(z))} + \frac{2|G'(T_N)|L}{r\phi''(\lambda(z))} + \frac{|G'(T_0)|L}{r\phi''(\lambda(z))}$$

tends to 0 by (iii) and the fact that $\lim_{T\to\infty} |G'(T)| = 0$. Putting all these results together, we conclude that

$$\lim_{T\to\infty} \frac{1}{T^{1/3}} \Bigg(\log \mathbb{P}_0(X \in \mathcal{B}(F_T, L_T, p, q)|_{[0,T]}) + \Lambda(z)T$$
$$+ \lambda(z)G(T) + \frac{1}{2} \int_0^T \frac{G'(s)^2}{r\phi''(\lambda(z))} ds \Bigg) \leq \lambda(z)x - \lambda(z)pL - \frac{\pi^2 r\phi''(\lambda(z))}{8L^2}. \quad \square$$

4.6. Proof of the lower bound in Theorem 4.1

In all the lemmas that led to the proof of Theorem 4.1 we proved upper bounds and (most of the times, symmetric) lower bounds, which ensure that we can replicate the same proof structure we used for the upper bound to prove the lower bound as well. The only substantial technicality arises in the lower bound of the probability that $Y(s)$ stays in the tube about $G_T(s)$ at small times.

From the calculations in Section 4.7, we see that the less strict condition on $G(s)$ is achieved by choosing T_0 as large as possible, provided that $\lim_{T\to\infty} \delta_T L_T^2 / T_0 = \infty$, which gives $T_0 = T^{1/3-\varepsilon}$. We use a two stage argument to bound from below the probability that $Y(s)$ stays near $G_T(s)$ on $[0, T_0]$: up to small times $T_0' = T^\nu$, we use the fact that the contribution of $G(s)$ is negligible compared to the size of the tube for such a small time. After T_0', which however small, tends to infinity as T tends to infinity, we can use that the derivatives of G tend to zero to show that the probability that $Y(s)$ stays near $G_T(s)$ tends to one exponentially fast, and fast enough to overrule the error terms.

Lemma 4.18. *Let $T_0' = T^\nu$ and $T_0 = T^{1/3-\varepsilon}$, with $\varepsilon \in (0, 1/3)$. Recall that we defined $n_T = \lfloor \delta_T^{-1/2} \rfloor$ and we assumed that $|x_T/L_T - x/L| < \sqrt{\delta_T}/8$. Then*

$$\lim_{T\to\infty} \frac{1}{L_T} \log \mathbb{Q}_0 \Big(Y \in \mathcal{B}\big(G_T, L_T, \tfrac{x}{L} - \tfrac{1}{2n_T}, \tfrac{x}{L} + \tfrac{1}{2n_T}\big)\big|_{[0,T_0]} \Big) = 0.$$

Proof. Since $\delta_T < 8/n_T$, by Lemma 4.12 we have

$$\begin{aligned}
&\mathbb{Q}_0\Big(Y \in \mathcal{B}\big(G_T, L_T, \tfrac{x}{L} - \tfrac{1}{2n_T}, \tfrac{x}{L} + \tfrac{1}{2n_T}\big)\big|_{[0,T_0]}\Big) \\
&\qquad \geq \mathbb{Q}_0\Big(B \in \mathcal{B}^-_{3\delta_T}\big(G_T, L_T, \tfrac{x}{L} - \tfrac{1}{2n_T}, \tfrac{x}{L} + \tfrac{1}{2n_T}\big)\big|_{[0,T_0]}\Big) - K' \exp\big(-\mu' L_T \delta_T\big) \qquad (4.28)
\end{aligned}$$

for T large enough. Using standard properties of Brownian motion, for large T we can write

$$\begin{aligned}
&\mathbb{Q}_0\Big(B \in \mathcal{B}^-_{3\delta_T}\big(G_T, L_T, \tfrac{x}{L} - \tfrac{1}{2n_T}, \tfrac{x}{L} + \tfrac{1}{2n_T}\big)\big|_{[0,T_0]}\Big) \\
&\quad \geq \mathbb{Q}_0\left(B \in \mathcal{B}\Big(G_T, (1-3\delta_T)L_T, \tfrac{x_T/L_T - 1/(2n_T^2)}{1-3\delta_T}, \tfrac{x_T/L_T + 1/(2n_T^2)}{1-3\delta_T}\Big)\Big|_{[0,T_0']}\right) \\
&\qquad \cdot \mathbb{Q}_{x_T}\Big(B \in \mathcal{B}^-_{3\delta_T + 1/(2n_T^2)}\big(\tilde{G}, L_T, \tfrac{x}{L} - \tfrac{1}{2n_T}, \tfrac{x}{L} + \tfrac{1}{2n_T}\big)\big|_{[0,T_0-T_0']}\Big), \qquad (4.29)
\end{aligned}$$

where $\tilde{G}(s) = G(s + T_0') - G(T_0')$, $s \in [0, T_0 - T_0']$. We start by finding a lower bound on the first term. Recall that $G_T(s) = G(s) - x_T$. As we deduced in (vi), we have $|G'(s)| \leq M$ $\forall s \geq 0$, where M is a constant independent of T. The mean value Theorem gives that

$$\sup_{s \in [0,T_0']} |G(s)| \leq \sup_{s \in [0,T_0']} |G'(s)| T_0' \leq M T_0'.$$

Since $\frac{T_0'}{L_T \delta_T^2} \asymp \frac{1}{T^{1/3-\nu}\delta_T^2}$ and this tends to zero by assumption (i), then $MT_0'/L_T \leq \delta_T^2$ when T is large enough. Using this, and that $\frac{1}{2n_T^2} - \delta_T^2 \geq \frac{\delta_T}{2} - \delta_T^2 \geq \frac{\delta_T}{4}$ for large T, we have

$$\begin{aligned}
&\mathbb{Q}_0\left(B \in \mathcal{B}\Big(G_T, (1-3\delta_T)L_T, \tfrac{x_T/L_T - 1/(2n_T^2)}{1-3\delta_T}, \tfrac{x_T/L_T + 1/(2n_T^2)}{1-3\delta_T}\Big)\Big|_{[0,T_0']}\right) \\
&\qquad \geq \mathbb{Q}_{x_T}\left(B \in \mathcal{B}\Big(H, (1-4\delta_T)L_T, \tfrac{x_T/L_T - \delta_T/4}{1-4\delta_T}, \tfrac{x_T/L_T + \delta_T/4}{1-4\delta_T}\Big)\Big|_{[0,T_0']}\right)
\end{aligned}$$

where $H(s) \equiv 0$. Lemma 4.9 states that if $x \in (-L, L)$ and $-L + \varepsilon L < x < L - \varepsilon L$, for any $H : [0, \infty) \to \mathbb{R}$

$$\begin{aligned}
&\mathbb{Q}_x\Big(B \in \mathcal{B}\big(H, L, \tfrac{x}{L} - \varepsilon, \tfrac{x}{L} + \varepsilon\big)\big|_{[0,t]}\Big) \\
&\geq \exp\Big(-H'(t)(x+\varepsilon L) + H'(0)x - L\int_0^t |H''(s)|ds - \frac{1}{2}\int_0^t H'(s)^2 ds\Big)\Big(1 - \frac{2\sqrt{2t}}{\sqrt{\pi}\varepsilon L}e^{-\frac{\varepsilon^2 L^2}{2t}}\Big).
\end{aligned}$$

This, with $H(s) \equiv 0$ and $\varepsilon L_T = \delta_T L_T/4$, gives

$$\begin{aligned}
&\mathbb{Q}_{x_T}\left(B \in \mathcal{B}\Big(H, (1-4\delta_T)L_T, \tfrac{x_T/L_T - \delta_T/4}{1-4\delta_T}, \tfrac{x_T/L_T + \delta_T/4}{1-4\delta_T}\Big)\Big|_{[0,T_0']}\right) \\
&\qquad \geq 1 - \frac{2}{\sqrt{\pi}} \cdot \frac{4\sqrt{2T_0'}}{\delta_T L_T} \exp\left(-\frac{\delta_T^2 L_T^2}{4^2 \cdot 2T_0'}\right).
\end{aligned}$$

Since $\delta_T^2 L_T^2 (T_0')^{-1} \asymp \delta_T^2 T^{2/3-\varepsilon}$, and this tends to infinity by (i), the right-hand side can be made bigger than $1/2$ by choosing T large enough.

We now consider the second term in (4.29). since $1/n_T \leq 2\delta_T$, we have $3\delta_T + \frac{1}{2n_T^2} \leq$

$8\delta_T$ for large T. Furthermore, using that $\frac{1}{2n_T} - 8\delta_T \geq \frac{\sqrt{\delta_T}}{4}$ for T large enough and $|\frac{x_T}{L_T} - \frac{x}{L}| < \frac{\sqrt{\delta_T}}{8}$, we have

$$\big(\tfrac{x_T}{L_T} - \tfrac{\sqrt{\delta_T}}{8}, \tfrac{x_T}{L_T} + \tfrac{\sqrt{\delta_T}}{8}\big) \subseteq \big(\tfrac{x}{L} - \tfrac{1}{2n_T} + 8\delta_T, \tfrac{x}{L} + \tfrac{1}{2n_T} - 8\delta_T\big).$$

Then

$$\begin{aligned}&\mathbb{Q}_{x_T}\Big(B \in \mathcal{B}^-_{3\delta_T + 1/(2n_T^2)}\big(\tilde{G}_T, L_T, \tfrac{x}{L} - \tfrac{1}{2n_T}, \tfrac{x}{L} + \tfrac{1}{2n_T}\big)\big|_{[0,T_0-T_0']}\Big)\\ &\qquad \geq \mathbb{Q}_{x_T}\left(B \in \mathcal{B}\Big(\tilde{G}_T, (1-8\delta_T)L_T, \tfrac{x_T/L_T - \sqrt{\delta_T}/8}{1-8\delta_T}, \tfrac{x_T/L_T + \sqrt{\delta_T}/8}{1-8\delta_T}\Big)\Big|_{[0,T_0-T_0']}\right).\end{aligned}$$

Using Lemma 4.9 with $H(s) = \tilde{G}(s)$ and $\varepsilon L_T = \sqrt{\delta_T} L_T/8$, this is bigger than

$$\begin{aligned}A := &\exp\Big(- G'(T_0)\Big(x_T + \tfrac{\sqrt{\delta_T}L_T}{8}\Big) + G'(T_0')x_T - L_T(1-8\delta_T)\int_{T_0'}^{T_0} |G''(s)|ds\Big)\\ &\cdot \exp\Big(-\frac{1}{2}\int_{T_0'}^{T_0} G'(s)^2 ds\Big)\left(1 - \frac{2}{\sqrt{\pi}}\cdot\frac{8\sqrt{2(T_0-T_0')}}{\sqrt{\delta_T}L_T}\exp\Big(-\tfrac{\delta_T L_T^2}{8^2\cdot 2(T_0-T_0')}\Big)\right).\end{aligned}$$

Then

$$\begin{aligned}\frac{\log A}{L_T} &= -G'(T_0)\Big(\tfrac{x_T}{L_T} + \tfrac{\sqrt{\delta_T}}{8}\Big) + G'(T_0')\tfrac{x_T}{L_T} - (1-8\delta_T)\int_{T_0'}^{T_0}|G''(s)|ds\\ &- \frac{1}{2L_T}\int_{T_0'}^{T_0} G'(s)^2 ds + \frac{1}{L_T}\log\left(1 - \frac{2}{\sqrt{\pi}}\cdot\frac{8\sqrt{2(T_0-T_0')}}{\sqrt{\delta_T}L_T}\exp\Big(-\tfrac{\delta_T L_T^2}{8^2\cdot 2(T_0-T_0')}\Big)\right).\end{aligned}$$

Note that $\delta_T L_T^2 T_0^{-1} \asymp \delta_T T^{1/3}$, which tends to infinity by (i). We also have that $\lim_{T\to\infty} x_T/L_T = x/L \in (-1,1)$ and $\lim_{T\to\infty} G'(T_0') = 0$. Furthermore,

$$\frac{1}{L_T}\int_{T_0'}^{T_0} G'(s)^2 ds \leq \frac{M^2 T_0}{L_T},$$

which tends to 0. Using all these facts, and assumption (v) for the integral of the second derivative, we obtain

$$\lim_{T\to\infty}\frac{\log A}{L_T} = 0. \tag{4.30}$$

Going back to (4.29), we have thus shown that

$$\mathbb{Q}_0\Big(B \in \mathcal{B}^-_{3\delta_T}\big(G_T, L_T, \tfrac{x}{L} - \tfrac{1}{2n_T}, \tfrac{x}{L} + \tfrac{1}{2n_T}\big)\big|_{[0,T_0]}\Big) \geq A/2$$

for T large. Therefore, substituting this in (4.28), we obtain

$$\mathbb{Q}_0\Big(Y \in \mathcal{B}\big(G_T, L_T, \tfrac{x}{L} - \tfrac{1}{2n_T}, \tfrac{x}{L} + \tfrac{1}{2n_T}\big)\big|_{[0,T_0]}\Big) \geq A/2 - K'\exp\big(-\mu' L_T\delta_T\big).$$

Then

$$\frac{1}{L_T}\log \mathbb{Q}_0\Big(Y \in \mathcal{B}\big(G_T, L_T, \tfrac{x}{L} - \tfrac{1}{2n_T}, \tfrac{x}{L} + \tfrac{1}{2n_T}\big)\big|_{[0,T_0]}\Big)$$
$$\geq \frac{1}{L_T}\log\Big(\frac{A}{2}\Big) + \frac{1}{L_T}\log\Big(1 - \frac{2K'}{A}\exp\big(-\mu' L_T\delta_T\big)\Big),$$

which tends to 0 by (4.30) and the fact that

$$\frac{1}{L_T}\log\Big(\frac{1}{A}\cdot\exp\big(-\mu' L_T\delta_T\big)\Big) = -\frac{\log(A)}{L_T} - \mu'\delta_T$$

is negative and tends to 0. □

Lemma 4.18 was the only extra ingredient required for the lower bound, which we are now ready to prove.

Proof of the lower bound in Theorem 4.1. From (4.17) in Proposition 4.15 and analogous considerations on the exponential terms as those we have used for the upper bound, we have

$$\begin{aligned}
&\mathbb{Q}_0(Y \in \mathcal{B}(G_T, L_T, p, q)|_{[0,T]})\\
&\quad\geq \exp\Big(-\frac{1}{2}\int_{T_0}^{T} G'(s)^2 ds - L_T(1-r_T)\int_{T_0}^{T}|G''(s)|ds - \frac{\pi^2(T-T_0)}{8L_T^2(1-r_T)^2}\Big)\\
&\qquad\cdot\exp\Big(-3\sqrt{r_T}L_T\sum_{i=1}^{N-1}|G'(T_i)| - 2|G'(T_N)|L_T - |G'(T_0)|L_T\Big)\\
&\qquad\cdot\sum_{j_{N-1}=-n+1}^{n-2}\cdots\sum_{j_1=-n+1}^{n-2}\mathbb{Q}_0\Big(Y \in \mathcal{B}\big(G_T, L_T, \tfrac{x}{L} - \tfrac{1}{2n_T}, \tfrac{x}{L} + \tfrac{1}{2n_T}\big)\big|_{[0,T_0]}\Big)\\
&\qquad\cdot\prod_{i=1}^{N}\Big\{\Gamma\Big(\frac{j_{i-1}+1/2}{n(1-r_T)}, \frac{j_i/n + r_T}{1-r_T}, \frac{j_i/n + 1/n - r_T}{1-r_T}\Big)\Psi_T'^{-}\Big\}.
\end{aligned}\tag{4.31}$$

where $j_0 := x$ and $j_N := pn$. Writing

$$\begin{aligned}
&\prod_{i=1}^{N}\Gamma\Big(\frac{j_{i-1}+1/2}{n(1-r)}, \frac{j_i/n + r}{1-r}, \frac{j_i/n + 1/n - r}{1-r}\Big)\\
&\quad= \Gamma\Big(\frac{x+1/2}{n(1-r)}, \frac{p+r}{1-r}, \frac{p+1/n-r}{1-r}\Big)\prod_{i=1}^{N-1}\Gamma\Big(\frac{j_i+1/2}{n(1-r)}, \frac{j_i/n+r}{1-r}, \frac{j_i/n+1/n-r}{1-r}\Big),
\end{aligned}$$

we have that

$$\begin{aligned}
&\sum_{j_{N-1}=-n+1}^{n-2} \cdots \sum_{j_1=-n+1}^{n-2} \mathbb{Q}_0\Big(Y \in \mathcal{B}\big(G_T, L_T, \tfrac{x}{L} - \tfrac{1}{2n}, \tfrac{x}{L} + \tfrac{1}{2n}\big)\big|_{[0,T_0]}\Big) \\
&\qquad\qquad \cdot \prod_{i=1}^{N} \Big\{ \Gamma\Big(\frac{j_{i-1}+1/2}{n(1-r)}, \frac{j_i/n + r}{1-r}, \frac{j_i/n + 1/n - r}{1-r}\Big) \Psi_T'^{-} \Big\} \\
&\geq \mathbb{Q}_0\Big(Y \in \mathcal{B}\big(G_T, L_T, \tfrac{x}{L} - \tfrac{1}{2n}, \tfrac{x}{L} + \tfrac{1}{2n}\big)\big|_{[0,T_0]}\Big) \Gamma\Big(\frac{x+1/2}{n(1-r)}, \frac{p+r}{1-r}, \frac{p+1/n-r}{1-r}\Big) \\
&\qquad\qquad \cdot \Big(\Psi_T'^{-}\Big)^N \cdot \prod_{i=1}^{N-1} \sum_{j_i=-n+1}^{n-2} \Gamma\Big(\frac{j_i+1/2}{n(1-r)}, \frac{j_i/n+r}{1-r}, \frac{j_i/n+1/n-r}{1-r}\Big).
\end{aligned}$$

From (4.22) in Lemma 4.17, for every $n \geq 4$ and $\frac{\sqrt{r}}{2} \leq \frac{1}{n} \leq \frac{\sqrt{r}}{2(1-\sqrt{r}/2)}$,

$$\sum_{j=-n+1}^{n-2} \Gamma\left(\frac{j+1/2}{n(1-r)}, \frac{j/n+r}{1-r}, \frac{j/n+1/n-r}{1-r}\right) \geq 1 - c_3(n),$$

so going back to (4.31), this gives that

$$\begin{aligned}
&\mathbb{Q}_0(Y \in \mathcal{B}(G_T, L_T, p, q)|_{[0,T]}) \\
&\quad \geq \exp\left(-\frac{1}{2}\int_{T_0}^{T} G'(s)^2 ds - L_T(1-r_T)\int_{T_0}^{T} |G''(s)| ds - \frac{\pi^2(T-T_0)}{8L_T^2(1-r_T)^2}\right) \\
&\quad\ \cdot \exp\Big(-3\sqrt{r_T} L_T \sum_{i=1}^{N-1} |G'(T_i)| - 2|G'(T_N)| L_T - |G'(T_0)| L_T\Big) \\
&\quad\ \cdot \mathbb{Q}_0\Big(Y \in \mathcal{B}\big(G_T, L_T, \tfrac{x}{L} - \tfrac{1}{2n_T}, \tfrac{x}{L} + \tfrac{1}{2n_T}\big)\big|_{[0,T_0]}\Big) \\
&\quad\ \cdot \Gamma\Big(\frac{x+1/2}{n_T(1-r_T)}, \frac{p+r_T}{1-r_T}, \frac{p+1/n_T-r_T}{1-r_T}\Big) \left(\Psi_T'^{-}\right)^N \big(1 - c_3(n_T)\big)^{N-1}. \qquad (4.32)
\end{aligned}$$

Lemma 4.4 ensures that, if $z > r\mathbb{E}[\xi]$, for any $\varepsilon > 0$ small enough

$$\begin{aligned}
&\mathbb{P}_0(X \in \mathcal{B}(F_T, L_T, p, q)|_{[0,T]}) \\
&\qquad \geq e^{-\Lambda(z)T - \lambda(z)G(T) + \lambda(z)x_T - \lambda(z)(p+\varepsilon)L_T} \mathbb{Q}^{\lambda(z)}(Y \in \mathcal{B}(\tilde{G}_T, \tilde{L}_T, p, p+\varepsilon)|_{[0,T]}),
\end{aligned}$$

where $\tilde{G}_T(s) = G_T(s)\big(r\phi''(\lambda(z))\big)^{-1}$ and $\tilde{L}_T = L_T\big(r\phi''(\lambda(z))\big)^{-1}$. A lower bound for $\mathbb{Q}^{\lambda(z)}\big(Y \in \mathcal{B}(\tilde{G}_T, \tilde{L}_T, p, p+\varepsilon)|_{[0,T]}\big)$ is given by (4.32) with G_T, L_T and q replaced by

$\tilde{G}_T$, $\tilde{L}_T$ and $p+\varepsilon$. This gives

$$\begin{aligned}
&\mathbb{P}_0\big(X\in\mathcal{B}(F_T,L_T,p,q)|_{[0,T]}\big)\\
&\geq \exp\Big(-\Lambda(z)T-\lambda(z)G(T)+\lambda(z)x_T-\lambda(z)(p+\varepsilon)L_T\Big)\\
&\quad\cdot\exp\left(-\frac{1}{2}\int_{T_0}^{T}\frac{G'(s)^2}{r\phi''(\lambda(z))}ds-L_T(1-r_T)\int_{T_0}^{T}\frac{|G''(s)|}{r\phi''(\lambda(z))}ds-\frac{\pi^2(T-T_0)r\phi''(\lambda(z))}{8L_T^2(1-r_T)^2}\right)\\
&\quad\cdot\exp\left(-3\sqrt{r_T}L_T\sum_{i=1}^{N-1}\frac{|G'(T_i)|}{r\phi''(\lambda(z))}-\frac{2|G'(T_N)|L_T}{r\phi''(\lambda(z))}-\frac{|G'(T_0)|L_T}{r\phi''(\lambda(z))}\right)\\
&\quad\cdot\mathbb{Q}_0\Big(Y\in\mathcal{B}\big(G,L_T,\tfrac{x}{L}-\tfrac{1}{2n_T},\tfrac{x}{L}+\tfrac{1}{2n_T}\big)\big|_{[0,T_0]}\Big)\\
&\quad\cdot\Gamma\Big(\frac{x+1/2}{n_T(1-r_T)},\frac{p+r_T}{1-r_T},\frac{p+1/n_T-r_T}{1-r_T}\Big)\left(\Psi_T'^{-}\right)^N\big(1-c_3(n_T)\big)^{N-1}.
\end{aligned}\tag{4.33}$$

Write

$$-\frac{1}{2T^{1/3}}\int_{T_0}^{T}\frac{G'(s)^2}{r\phi''(\lambda(z))}ds=-\frac{1}{2T^{1/3}}\int_{0}^{T}\frac{G'(s)^2}{r\phi''(\lambda(z))}ds+\frac{1}{2T^{1/3}}\int_{0}^{T_0}\frac{G'(s)^2}{r\phi''(\lambda(z))}ds.$$

Taking logarithms on both sides, dividing by $T^{1/3}$ and rearranging (in the same way we have done for the upper bound) gives that

$$\begin{aligned}
&\frac{1}{T^{1/3}}\left(\log\mathbb{P}_0(X\in\mathcal{B}(F_T,L_T,p,q)|_{[0,T]})+\Lambda(z)T+\lambda(z)G(T)+\frac{1}{2}\int_0^T\frac{G'(s)^2}{r\phi''(\lambda(z))}ds\right)\\
&\geq\frac{\lambda(z)x_T}{T^{1/3}}-\lambda(z)(p+\varepsilon)L-\frac{1}{2T^{1/3}}\int_0^{T_0}\frac{G'(s)^2}{r\phi''(\lambda(z))}ds-L(1-r_T)\int_{T_0}^{T}\frac{|G''(s)|}{r\phi''(\lambda(z))}ds\\
&\quad-\frac{\pi^2(T-T_0)r\phi''(\lambda(z))}{8L^2T(1-r_T)^2}-3\sqrt{r_T}L\sum_{i=1}^{N-1}\frac{|G'(T_i)|}{r\phi''(\lambda(z))}-\frac{2|G'(T_N)|L}{r\phi''(\lambda(z))}-\frac{|G'(T_0)|L}{r\phi''(\lambda(z))}\\
&\quad+\frac{1}{T^{1/3}}\log\mathbb{Q}_0\Big(Y\in\mathcal{B}\big(G_T,L_T,\tfrac{x}{L}-\tfrac{1}{2n_T},\tfrac{x}{L}+\tfrac{1}{2n_T}\big)\big|_{[0,T_0]}\Big)+\frac{N_T-1}{T^{1/3}}\log\big(1-c_3(n_T)\big)\\
&\quad+\frac{1}{T^{1/3}}\log\Gamma\Big(\frac{x+1/2}{n_T(1-r_T)},\frac{p+r_T}{1-r_T},\frac{p+1/n_T-r_T}{1-r_T}\Big)+\frac{N_T}{T^{1/3}}\log\big(\Psi_T'^{-}\big).
\end{aligned}$$

Lemma 4.18 gives that

$$\lim_{T\to\infty}\frac{1}{T^{1/3}}\log\mathbb{Q}_0\Big(Y\in\mathcal{B}\big(G_T,L_T,\tfrac{x}{L}-\tfrac{1}{2n_T},\tfrac{x}{L}+\tfrac{1}{2n_T}\big)\big|_{[0,T_0]}\Big)=0.$$

Recall that we defined

$$\psi_T'^{-}=\Big(1-C\exp\big(-\tfrac{\pi^2\Delta_T}{L_T^2(1-r_T)^2}\big)\Big)\Big(1-K'\exp\big(-\tfrac{\mu'}{2}L_T\delta_T\big)\Big)\big(1-c_2(n)\big),$$

where $c_2(n)=\frac{\pi}{4n}\sin\big(\frac{\pi}{4n}\big)^{-1}$. We have

$$\lim_{T\to\infty}\frac{N_T}{T^{1/3}}\log\big(1-c_2(n_T)\big)=0.$$

With analogous considerations for the other terms as the ones we made at the end of the proof of the upper bound when we take the limit as T tends to infinity we get

$$\lim_{T\to\infty} \frac{1}{T^{1/3}}\Bigg(\log \mathbb{P}_0(X \in \mathcal{B}(F_T, L_T, p, q)|_{[0,T]} + \Lambda(z)T \\ + \lambda(z)G(T) + \frac{1}{2}\int_0^T \frac{G'(s)^2}{r\phi''(\lambda(z))}ds\Bigg) \geq \lambda(z)x - \lambda(z)(p+\varepsilon)L - \frac{\pi^2 r\phi''(\lambda(z))}{8L^2}.$$

Since ε was arbitrary, this completes the proof. $\square$

4.7. Example of a relevant choice of $G(s)$

In this section we show that functions of the form $G(s) = A(s+1)^\alpha$ satisfy the conditions (i)-(v) for every $A \in \mathbb{R}$ and $\alpha \in (0, 7/10]$. Note that $G'(s) = \alpha A(s+1)^{\alpha-1}$ is a positive, decreasing function and $\lim_{T\to\infty} G'(T) \to 0$. Since $G''(s) = \alpha(\alpha-1)A(s+1)^{\alpha-2} < 0$,

$$\int_u^t |G''(s)|ds = G'(u) - G'(t)$$

so clearly (v) is satisfied as the endpoints of the interval tend to infinity.
We take $\delta_T = T^{-\eta}$, where η will be chosen later. We write $h(t) \asymp g(t)$ if there exist two constants $c, C > 0$ such that $ch(t) \leq g(t) \leq Ch(t)$ for every t.

Then (i) requires $1/3 - \nu - 2\eta > 0$, that is

$$\eta < 1/6 - \nu/2. \tag{4.34}$$

For (ii), since

$$\frac{1}{\delta_T} \sup_{s\in[T_0,T]} |G'(s)| = T^\eta G'(T_0) \asymp T^{\eta+(\alpha-1)(1/3-\varepsilon)},$$

we need

$$\eta < 1/3 - \alpha/3 - \varepsilon(1-\alpha). \tag{4.35}$$

We now move to (iii). Recall that $\Delta_T \asymp T^{2/3+\nu}$, so since $T_i = T_0 + i\Delta_T$ for $i \in \{1, \dots, N_T\}$ we have $T_i \asymp iT^{2/3+\nu}$. Then, using that for $\alpha \in (0,1)$

$$\sum_{i=1}^{N_T} i^{\alpha-1} \leq \sum_{i=1}^{N_T}\int_{i-1}^{i} x^{\alpha-1}dx = \int_0^{N_T} x^{\alpha-1}dx = \frac{N_T^\alpha}{\alpha} \leq \frac{T^{\alpha/3-\alpha\nu}}{\alpha},$$

we have

$$\begin{aligned}\sum_{i=1}^{N_T} |G'(T_i)| \asymp \sum_{i=1}^{N_T} T_i^{\alpha-1} \asymp T^{(2/3+\nu)(\alpha-1)} \sum_{i=1}^{N_T} i^{\alpha-1} &\leq T^{(2/3+\nu)(\alpha-1)} \frac{T^{\alpha/3-\alpha\nu}}{\alpha} \\ &\asymp T^{2\alpha/3-2/3+\alpha\nu-\nu+\alpha/3-\alpha\nu} \\ &= T^{\alpha-2/3-\nu}\end{aligned}$$

and so to get (iii) we need $\alpha - 2/3 - \nu - \eta/2 < 0$, that is

$$\eta > 2\alpha - 4/3 - 2\nu. \tag{4.36}$$

Finally, we need to compute the integral of the derivative squared in (iv). Note that for any $\alpha \neq 1/2$

$$\int_u^t G'(s)^2 ds = \alpha^2 A^2 \int_u^t (s+1)^{2\alpha-2} = \frac{\alpha^2 A^2}{|2\alpha-1|}|(t+1)^{2\alpha-1} - (u+1)^{2\alpha-1}|.$$

When $\alpha < 1/2$ we have

$$\begin{aligned}\lim_{T\to\infty} \sup_{1\le i\le N_T} \int_{T_{i-1}}^{T_i} G'(s)^2 ds &\le \lim_{T\to\infty} \sup_{1\le i\le N_T} \frac{\alpha^2 A^2}{|2\alpha-1|} \cdot 2(T_{i-1}+1)^{2\alpha-1} \\ &\le \lim_{T\to\infty} \frac{\alpha^2 A^2}{|2\alpha-1|} \cdot 2(T_0+1)^{2\alpha-1} = 0,\end{aligned}$$

and since $\delta_T T^{1/3} \ge \delta_T^2 N_T$, where the latter tends to infinity by (i), then (iv) is proved. The same argument works when $\alpha = 1/2$, since for every $i \in \{1\dots, N_T\}$

$$\int_{T_{i-1}}^{T_i} G'(s)^2 ds = \alpha^2 A^2 (\log(T_i+1) - \log(T_{i-1}+1)) \le \alpha^2 A^2 \log(T+1)$$

and $\delta_T T^{1/3} \ge \delta_T^2 N_T = T^{1/3-\nu+2\eta}$, which tends to infinity as a power of T.
When $\alpha > 1/2$ we have

$$\begin{aligned}\big(1+T_i\big)^{2\alpha-1} - \big(1+T_{i-1}\big)^{2\alpha-1} &= \big(1+T_0+i\Delta_T\big)^{2\alpha-1} - \big(1+T_0+(i-1)\Delta_T\big)^{2\alpha-1} \\ &\asymp \big(i\Delta_T\big)^{2\alpha-1} - \big((i-1)\Delta_T\big)^{2\alpha-1} \\ &\asymp \big(i\Delta_T\big)^{2\alpha-1}\left(1 - \left(1-\frac{1}{i}\right)^{2\alpha-1}\right) \asymp i^{2\alpha-2}\Delta_T^{2\alpha-1},\end{aligned}$$

so

$$\int_{T_{i-1}}^{T_i} G'(s)^2 ds = \frac{\alpha^2 A^2}{2\alpha-1}\Big((1+T_i)^{2\alpha-1} - (1+T_{i-1})^{2\alpha-1}\Big) \asymp i^{2\alpha-2}\Delta_T^{2\alpha-1}.$$

It follows that

$$\frac{1}{\delta_T T^{1/3}} \int_{T_{i-1}}^{T_i} G'(s)^2 ds \asymp T^{\eta-1/3}\, i^{2\alpha-2} T^{4\alpha/3-2/3+\nu(2\alpha-1)} \le T^{4\alpha/3-1+\nu(2\alpha-1)+\eta},$$

and so (iv) requires that $4\alpha/3 - 1 + \nu(2\alpha-1) + \eta < 0$, that is

$$\eta < 1 - 4\alpha/3 - \nu(2\alpha-1). \tag{4.37}$$

In order to show that the assumptions (i)-(iv) are fulfilled, we need to put together the conditions (4.34)-(4.37), which give the system

$$\begin{cases} \eta < 1/6 - \nu/2 \\ \eta < 1/3 - \alpha/3 - \varepsilon(1-\alpha) \\ \eta > 2\alpha - 4/3 - 2\nu \\ \eta < 1 - 4\alpha/3 - \nu(2\alpha-1). \end{cases}$$

It is easy to check that when $\alpha \in (0, 7/10]$, for every ν small enough and $\varepsilon =$

$\nu/(1-\alpha)$ we have

$$2\alpha - 4/3 - 2\nu < \min\{1/6 - \nu/2, 1/3 - \alpha/3, 1 - 4\alpha/3 - \nu(2\alpha - 1)\},$$

which means that it is possible to choose η such that the system has a solution, and therefore the conditions (4.34)-(4.37) are satisfied. Indeed

- $2\alpha - 4/3 - 2\nu < 1/6 - \nu/2$ if and only if $\alpha < 3/4 + 3\nu/2$;
- $2\alpha - 4/3 - 2\nu < 1/3 - \alpha/3 - \varepsilon(1-\alpha)$ if and only if $\alpha < 5/7 + 6\nu/7 - (3/7)\varepsilon(1-\alpha)$. If $\varepsilon = \nu/(1-\alpha)$, this gives $\alpha < 5/7 + 3\nu/7$;
- $2\alpha - 4/3 - 2\nu < 1 - 4\alpha/3 - \nu(2\alpha - 1)$ if and only if $\alpha < 7/10 + (3/10)(3 - 2\alpha)\nu$.

Since $7/10 < 5/7 < 3/4$, if ν is small enough then the most restrictive condition is $\alpha < 7/10 + (3/10)(3 - 2\alpha)\nu$, which ultimately gives the upper bound $\alpha \le 7/10$.

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