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## Imagery and the Mental Manipulation of Knots

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## Imagery and the

## Mental Manipulation of Knots

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Thesis submitted for the degree of
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# Imagery and the Mental Manipulation of Knots 


#### Abstract

The objective of this study is to establish the strategies which are used in spatial tasks and to determine whether and how the visualisation skills used in the performance of such tasks can be taught.

The primary focus is on measuring the level of complexity when comparing different knot shapes and on analysing the subjects' verbal and visual strategies. To address these issues, quantitative and qualitative studies are carried out, firstly to identify the factors which affect complexity when comparing knots and, secondly, to identify the strategies that are most effective.

The first phase of research measures two complexity indicators: the time taken to perform the task and the accuracy of response when comparing pairs of knots at varying orientations in order to determine whether the two knots are the same or different. The endogenous factors associated with the level of difficulty are knot shape, crossing number, rotation and pair type, and the exogenous factors considered in the statistical analysis are gender and educational background. Six different knot shapes are used and the results show that some knot shapes are processed more efficiently than others, a finding similar to that in psychology where images which have a characteristic 'foundation part' are more easily stored and accessed in the brain. The results also reveal that there are differences in time and accuracy for topologically different pair types and that knot rotation causes an increase in decision times and error rate.

During the experiment, subjects show a significant improvement in their ability to perform the knot tasks, indicating that imagery and spatial skills are enhanced as a result. In the second phase of research, where subjects explain their changes of strategy whilst viewing a recording of the experiment, evidence is provided concerning the way in which the performance of the knot tasks improves spatial and visualisation skills. It is shown how the nature and deformability of knots allow strategies other than the self-evident one of mental rotation, and the research therefore establishes alternative methods that may be employed to develop imagery in mathematics.


## Introduction

The importance of visualisation and imagery in the teaching and learning of mathematics is now widely recognised, not only with regard to the spatial aspects of mathematics but also with regard to general problem solving. One of the difficulties encountered by teachers in encouraging learners to use imagery is that a knowledge of the complexities involved in processing images is necessary so that activities can be devised which develop visualisation skills sequentially. This research study focuses on this issue and tries to establish some of the strategies used by learners in carrying out spatial tasks and some of the obstacles which they experience with visualising.

The question 'what is research in mathematics education?' opens the discussion in Chapter 1 and some views on what the research domain should include, as well as how it can be used to inform teaching, are given. Other questions which are considered are: 'How do children learn?' and, more specifically, 'How do children learn mathematics?' To begin, a review of the background research in cognitive development focuses on mathematical development starting with the work of Piaget. There is shown to be a wide range of theories of learning, and the differing approaches are described. The seminal contribution of Piaget is outlined first, followed by that of his critics such as Donaldson, Vygotsky, Bruner and others. An outline of the 'constructivist' and 'information processing' schools of thought is included together with various notions of quantifying cognitive development by means of stage descriptors or mental effort statistics. The research on different mathematical abilities identified by Krutetskii and by Sternberg acts as a useful starting point to focus on spatial ability and imagery.

Many research studies on imagery have taken place in the field of psychology, and there has been some published work involving imagery in mathematics (for example Krutetskii 1976, Battista et al. 1982, Battista 1990). Chapter 2 provides a review of the literature concerned specifically
with imagery and mathematical ability. A definition of imagery and spatial ability is given and the problems faced by researchers in carrying out studies in visual imagery are described.

The well documented dichotomy between verbal thinkers and visual thinkers has been given much consideration by many authors (for example Krutetskii 1976, Clements 1981) and an overview is given in Chapter 2. This existence of different types of thinkers helps to explain the difficulty some learners may have when a particular preferred cognitive style is not consistent with the method of instruction.

Evidence is presented from previous research which suggests that imagery is a useful skill in problem-solving in mathematics. Research studies which consider some of the difficulties experienced by the learner in solving mathematical tasks which require imagery are described. The chapter closes with a look at some innovative school materials for improving imagery in mathematics.

Chapter 3 begins with a description of some of the early work by psychologists on imagery, research which was mainly concerned with how imagery can aid short term memory. The focus then turns to research concerning the mental rotation of objects and the quest to discover how the brain deals with such operations. The imagery debate, concerning the nature of a mental image and whether it is essentially pictorial or verbal, is examined and research supporting the various opinions is reported. It is shown how the psychology literature can inform our ideas about the theory of mental imagery, how it is related to visual perception and how it may relate to mathematics learning.

Knots provide a new and potentially interesting stimulus for research on mental manipulation and Chapter 4 deals with the history and mathematical theory of knots. Primitive humans had the ability to tie knots and devise intricate designs for knots. The earliest evidence for knot tying is presented and some typical ancient knots are shown. In spite of this ancient knowledge about knot tying and decoration, the
mathematics of knots is a recent field of mathematical knowledge and research. The sequence of discoveries and theories about knots which bring us to the present state of understanding of their properties is described.

Chapter 5 sets out the motivation for the research reported in this thesis and the main issues to be investigated. Research in mathematics education usually has as one of its aims the improvement of mathematics teaching. One aim of this study is to look for ways of developing or enabling certain mental capacities. The fact that imagery is useful for doing mathematics discussed in Chapter 2 and the need for information on how it may be encouraged in pupils is analysed. The different skills employed by pupils need to be recognised and opportunity given in the classroom for their practice and development. Some questions regarding the ability and inclination of individuals to use imagery and how this may or may not be nurtured is investigated in this research.

Chapter 6 describes the first of three experimental studies using knots as stimuli for mental manipulation. Twenty one subjects were tested and a detailed description of the experiment and an analysis of the results is given. Subjects performed a series of spatial tests involving the comparison of diagrams of interlaced ropes, in different shapes and at varying orientations. Consideration of the results of this initial study led to two further studies, one quantitative and the other qualitative. These are described in Chapters 7 and 8.

The second quantitative study was based upon the pilot study but with adaptations designed to refine the experiment and to introduce additional explanatory variables. In particular, given the results of the first study, it was considered desirable to modify the test instrument and to use a larger sample that would allow for the introduction of two more variables, gender and academic background. The results of this second study receive thorough analysis, both in terms of the individual variables under consideration, and the interactions between these

Introduction
variables. The results give some indication of the strategies which were used to solve certain items and also give an indication of the relative complexity of the tasks.

The qualitative phase of the research is discussed in Chapter 8, where strategies for manipulating knot shapes are investigated. Details of extended interviews with subjects are discussed and the verbal confirmation by subjects of strategies and difficulties is seen to support the deduced results from the quantitative studies described in Chapters 6 and 7. This chapter also analyses the modes of thinking used by the subjects and contains a detailed summary of each of the interviews where it can be seen that many ideas were expressed during performance of the tasks and afterwards during the discussion. These are reviewed and details are given of five principal domains, namely; strategies, use of a foundation part, changes of strategy, learning of skills and difficulties which subjects experienced. The complete transcripts for the interviews are provided as an appendix.

The concluding chapter revisits the previous research and discusses how it relates to the results of the current study. The implications of the research findings for teaching are considered and some recommendations are given.

## Chapter 1

## An overview of Research in Mathematics Education and Cognitive Development

### 1.1 What is Mathematics Education Research?

The teaching of mathematics in schools came under intense scrutiny in the early 1980s when both curriculum content and teaching methods were under review. The result was the report of the Cockcroft Committee of Inquiry entitled 'Mathematics Counts' (CITMS, 1982). One of the main contributions to the writing of the report was the body of evidence gathered from research in the field of mathematics education. Two questions which needed to be asked then, and are still valid questions today, were 'What is research in mathematics education?' and furthermore 'How can findings from such research be used?'.

These questions were considered important enough to justify an entire conference in Washington DC in 1994 dedicated to their resolution. In fact, preparation started in Quebec in 1992 at the seventh International Congress on Mathematical Education (ICME), where a discussion document was distributed with a call for papers for the forthcoming Study Conference which was sponsored by the International Commission on Mathematical Instruction (ICMI). At the Washington study conference, the mathematical philosopher Paul Ernest considered the nature of research in mathematics education, arguing for 'a multiplicity and variety of viewpoints, theories, frameworks, methodologies and interests' (Ernest, 1998, page 71). At the same time, as in other disciplines, research in mathematics education should involve systematic and critical enquiry, and, Ernest (Ernest, 1998, page 76) continues, 'should:

- link with and build on existing knowledge in the relevant educational research literature, thus adding to the body of
knowledge, fitting into the 'system of knowledge';
- use organised processes of enquiry, systematic methods of research, linked to existing methodology, providing a justification for knowledge claims;
- result in a systematically organised text, document or other public communicative form, so that others can assess the results of the educational research;
- possibly engage in theory-building resulting in the construction of some systematically organised system of reflective knowledge.'

At the same conference, Bell (1994) stressed that education has the general aim of developing mental capacities, and that mathematics education has the aim of developing or enabling the practice of mathematics. Bell (1994) believes that research in mathematics education involves the recognition of misconceptions and the design and trialling of specific treatments, the underlying sciences being developmental psychology and learning theory. Making the case for action research, Bell (1994, page 50) states that:
'The core of mathematical education research is the study of the interaction of teachers, learners and learning environments with a view to better understanding of learning processes and to the improvement of learning. Immediate pedagogical questions such as 'What do students find difficult about decimals and how can they be helped?' figure quite strongly in existing mathematical education research.'

In other words, as well as building on existing knowledge of learning processes by investigation using existing methodologies, research in mathematics education should investigate the difficulties experienced by students and generate strategies to address the problems involved.

In this context, another contributor to the Washington conference, Presmeg (1998, page 60), highlighted the important link between these two aims in mathematics education research as follows:
'In addition to the scientific goal of theory-building which is an
important aspect of research in any discipline, mathematics education research has had and will continue to have a pragmatic goal, namely, the improvement of teaching and learning mathematics at all levels. . . Each aim is sterile without the other'

In an earlier attempt to address the question 'What is research in mathematics education?', Bishop (1992, page 711) suggested that three essential components should be present, i.e.:

- Enquiry, which concerns the reason for the research activity. It represents the systematic quest for knowledge, the search for understanding, and gives dynamism to the activity. Research must be intentional enquiry.
- Evidence, which is necessary in order to relate the research to the reality of the mathematical education situation under study, be it classrooms, syllabuses, textbooks, or historical documents. Evidence samples the reality on which the theorising is focused.
- Theory, which recognises the existence of value assumptions and generalised relationships. It is the way in which we represent knowledge and understanding that comes from any particular research study. Theory is the essential product of the research activity, and theorising is, therefore its essential goal.

Relating this specifically to the aims of mathematics education research, Bishop (1992, page 712) continued:
'If the object of research is the improvement of mathematics teaching then it clearly makes sense to examine the normal activities associated with mathematics teaching to see if they are creating the obstacles to improvement, that is to what extent are they part of the solution rather than just part of the problem?'
Bishop's analysis underpinned Presmeg's (1998) argument for the integration of normative research, research which is concerned with theoretical deduction from first principles, and positive research which is research concerned with how teachers and students function in the classroom and which offers potential for insight into successful
teaching. Presmeg (1998) emphasised the complementary aims of constructing theory and of improving practice, action research which studies 'normal' teaching activities alone does not constitute 'research' without the presence of the three components outlined above.

In addition to these philosophical considerations, another aspect of mathematics education research is the scope of the mathematics curriculum itself. Although research in mathematics education usually focuses upon what aspects are difficult to teach and to learn and what methods facilitate either or both, it is also concerned with what mathematics is taught in schools. This implies the continual review of curriculum content and whether that content is appropriate both in terms of contemporary knowledge of cognitive development of the child and technological developments in society.

Research in cognitive development has a direct impact on mathematics education since, in order to teach mathematics, it is essential to know at what age or stage of development pupils are ready to learn a particular mathematical process or understand an element of mathematical content. Some of the most important studies in cognitive development were those carried out by Piaget and the Geneva school, for example, Piaget, Inhelder and Szeminska (1960), Piaget \& Inhelder (1971). Piaget and his colleagues made a major contribution to knowledge of child development. Piaget's theories stimulated other researchers such as Bruner (1963) and Donaldson (1979) who were to challenge some of Piaget's findings and his experiments and procedures. Thus, although some of Piaget's conclusions are no longer accepted, much of the work to extend the body of knowledge on cognitive development stems from these early studies by the Geneva school. One such advance relates to the development of spatial ability, the particular concern of this thesis.

The evolution of Piaget's work and that of his successors is summarised later in this chapter. First, however, we look at the special case of mathematics education, the need for a greater understanding of the matching of teaching style and cognitive development, and the
implications for education policy.

### 1.2 The difficulty of teaching mathematics

It is accepted that teaching mathematics is difficult (Cockcroft CITMS, 1982) and that many pupils under-achieve in mathematics. One of the recommendations made by the Cockcroft Report was that pupils should be taught how to apply mathematics to unfamiliar problems and to investigate mathematical situations. This gives rise to a number of questions:- What are the best ways to achieve this? How do pupils learn to think mathematically? What does research tell us about the skills which enable children to solve non-routine problems?

The ability to solve non-routine problems has been explored by several researchers, for example Wheatley (1991), Reynolds \& Wheatley (1997). This ability seems to depend upon the right hemisphere of the brain and correlates with good spatial ability ${ }^{1}$. In this context, Sharma (1979, page 63) had already identified two types of mathematical learning personalities, one displaying more aptitude for creative problem solving: ' $(A)$. . . one who has left hemispheric orientation . . . is good in language and verbal expressions, is good in solving those problems bit by bit . . . is good in quantifying and in quantitative operations which build up sequentially, such as counting, addition and multiplication. This child when given a word problem looks for a familiar algorithm to solve the problem.
(B) . . . one who has right hemispheric orientation . . . looks at the problem holistically and explores global approaches to solutions .. . . is good in identifying patterns - both spatial and symbolic, is more creative and faster in solving 'real life' problems. This child when given a word problem seems to play with the problem in a non-directed metaphoric way before he (sic) begins to solve it.'

It has also been argued by Dickson et al. (1984), on the basis of increasing physiological evidence, that the two different aspects of

1 For a more detailed discussion of the activity of the two hemispheres of the brain see Chapter 3 section 3.4
learning mathematics, visual and verbal, may be linked to the level of activity in the different hemispheres of the brain. If the ability to solve non-routine tasks depends upon developing the right hemisphere activity, then how might this be achieved, and how can it be encouraged in natural left hemisphere learner types? In particular, what range of teaching styles will be required?

The style and teaching methods employed by teachers have an effect upon the kind of knowledge which a child acquires about mathematics (Ernest, 1989a). Furthermore, different teaching styles seem to suit different preferred cognitive styles, (Presmeg, 1985). If right hemispheric thinkers are better problem solvers, what are the implications for teaching? The components of the curriculum which are described in the current statutory order for mathematics (DFE, 1995) include problem solving and investigating, in other words, solving unfamiliar problems using processes in mathematics as well as mathematical content and skills. According to the document, individuals need to learn how to apply mathematics to everyday situations and to think mathematically as well as to have core skills and content knowledge. With regard to the acquisition of these skills, Bell et al. (1983) stated,
'Specifically, one must attempt to establish a structure of ideas which will facilitate the assimilation of further knowledge, and teach actual skills, strategies and attitudes needed for the acquisition of this knowledge.'

A requirement for teaching non-routine problem solving is to pace the level of difficulty of the material. A child will not make progress in solving a problem if it is not accessible, if she has no idea how to begin. The complexity of tasks must be known and pitched at a level so as to promote learning. Therefore detailed knowledge of what adds to complexity should be available in order to negotiate the successful sequencing of tasks. Much research has been carried out, for example by APU (1980), to discover this information. However, regarding the method of collection and the kind of data collected, Bell et al. stated (1983,
page 10)
'This (research) should develop away from the simple collection of empirical results, as in the first APU surveys, towards the identification of general factors determining levels of difficulty'

A further factor affecting the level of difficulty might be the matching or otherwise of the cognitive style implicit in a problem and the preferred cognitive style of the child. The cognitive style which a child constructs is formed during the cognitive development of the individual, but the factors which affect the preferred style that emerges are unknown. Whichever cognitive style a learner prefers, two types classified by Sharma (1979) as A and B were described earlier, understanding develops slowly and teaching needs to be planned so as to revisit ideas in a variety of contexts so as to promote both of these styles (Bell et al., 1983). It will be shown later, in Chapter 2, that verbal styles have received more emphasis and have been more valued yet visual imagery is a skill which has been found to be present early on in the child's cognitive development. In order to enhance learning of all cognitive types and especially visualisers, a variety of ways of using imagery in mathematics needs to be presented.

In summary, the teaching of mathematics is a complex task and one which needs constant evaluation and reflective thought. Research can give guidance on the nature of mathematical abilities and on how to improve pupils' learning by making teaching methods more effective.

### 1.3 The work of Piaget

Piaget's important contribution to cognitive development research is acknowledged by all in the mathematics education community. Within the teaching community, the Piagetian stages of cognitive development have informed curriculum design for many years and followers of Piaget's theories agree that in the teaching of mathematics it is essential to dovetail the current Programmes of Study (DFE, 1995) for each Key Stage with the Piagetian stage of development of the child. Table 1.1 indicates the Piagetian stages and the age at which it can be expected that
the child reaches each stage. ${ }^{2}$

With regard to identifying the age at which children in different environments and with different experiences reach Piaget's stages, Sutherland (1992) found that the stage of operational thought is achieved at different ages in different cultures. For example, in Iranian villages it is 2 years later, and in Martinique 4 years later, as compared to French speaking Switzerland. In some cultures, where abstract thinking is not required, the last stage of formal operational thought does not exist, that Piagetian level is never achieved. The implication seems to be that the child should be given experiences, should be challenged, but most importantly the child should be provided with suitable learning experiences at the appropriate time in her development.

## TABLE 1.1

Piagetian Stages and M values

| Piagetian Stage | Age in years | M value |
| :--- | :---: | :---: |
| Early pre-operations | $3-4$ | $a+1$ |
| Late pre-operations - <br> early concrete <br> Mid-concrete operations | $7-8$ | $a+2$ |
| Late concrete operations | $9-10$ | $a+3$ |
| Early formal operations | $11-12$ | $a+5$ |
| Mid-formal operations | $13-14$ | $a+6$ |
| Late formal operations | $15-16$ | $a+7$ |

The other data given alongside Piaget's stages in Table 1.1 are examples of another statistic which has been proposed by a number of psychologists, for example Pascual-Leone (1976). These neo-Piagetian

[^0]theories attempt to explain Piagetian stages in terms of the variation with age of the number of items which can be stored in the child's working memory. This M statistic, proposed by Pascual-Leone (1976), relates Piagetian stages to mental processing complexities and may be helpful in determining the appropriate time to provide the child with certain learning experiences. For any task it is possible to calculate a mental effort statistic regarding the processing involved, specifically, this can be regarded as the number of items of information which must be held and coordinated in working memory at the same time. In early concrete operational thought (age 7-8) this statistic is a +33 , that is, the child of 7 or 8 can only deal with 3 different schemes or concepts at once. The notion is also supported by similar research carried out by Case (1975, reported in Bell et al., 1983).


Source: Bell et al. (1983)
Figure 1.1 Maze puzzle

[^1]Case (see Bell et al., 1983) gave an example of a spatial task given to a 5 year old child. The maze shown in Figure 1.1 (Bell et al., 1983, page 31) was given to the child as a puzzle to find the shortest route from the point $X$ at the base of the picture to the house. The child avoided the blind alley but found a longer possible route. However, when asked to choose the shortest routes of three offered by the interviewer, the child was able to choose the correct one. The child understood what it means to pick the shortest path and also how to avoid dead ends but could not hold both ideas in her mind at the same time while searching for the route.

Bell et al. (1983) claimed that these theories imply that in order to teach a complex concept or skill, not only is it necessary to ensure that the child has acquired the prerequisite concepts or skills, but also that the number of these which must be integrated in the task does not exceed this mental effort statistic.

Returning to Piaget's definitions, Dickson et al. (1984) reiterate Piaget's claim that children at the 'concrete operations' stage cannot deal with symbols which are not firmly related to physical actions. They state that since most children have not progressed beyond Piaget's 'concrete operations' stage by the age of 16 then they are likely to be dependent upon spatial concepts for their understanding in all areas of mathematics. These claims obviously have implications for the teaching of the subject and could explain why mathematics is notoriously problematic for some children.

### 1.4 Vygotsky

Bruner and Vygotsky were among the first to challenge Piaget's ideas. Vygotsky was a critic of Piaget's work believing that it is the teacher, not just the child, who has the all important role (Sutherland, 1992). Vygotsky's work, in contrast to Piaget's, was theoretical. Whereas Piaget focused on detailed clinical observations of the child, Vygotsky focused on the teacher and the factors which lead to successful learning, on the
theoretical overall process of education rather than empirical studies on cognitive development.

Vygotsky (1978) also saw activity as central to learning but believed the role of the teacher to be more important. Vygotsky placed great emphasis upon social and linguistic influences on learning. He believed that the teacher could 'scaffold' a pupil to competence in any skill. The central role of the teacher is characterised by Vygotsky's 'zone of proximal development' - the area within which the child can work with support from the teacher (or more capable peers). A child capable of operating alone at one level can be extended to the next level by teacher intervention. Vygotsky defined this zone as being the distance between the actual development level as determined by independent problem solving and the level of potential development through collaboration. He expected that some children would have more restricted zones than others.

Jaworski (1994, page 31) considered the metaphor of scaffolding to be a useful one but warned:
'An important question seems to be what sort of scaffold would be appropriate in general problem solving terms? The act of scaffolding could result in creating dependency if the child became too reliant on the tutor's management. An extreme of the scaffolding principle is that the child never experiences the bewilderment of tackling a problem alone, and so is totally unprepared for any new task for which the tutor is not present.' Jaworski (1994) suggested that a better form of scaffolding might be the offering of strategies for thinking and learning.

### 1.5 Bruner

Bruner $(1963,1986)$ also stressed the need for teachers to intervene actively in the process of cognitive development. Bruner was an interventionist and he desired to optimise the child's full potential. He defined three stages:
enactive - learning by doing,
iconic - learning by means of images and pictures, and
symbolic - learning by means of words or numbers.
Bruner did not believe it necessary to wait for the child to be ready intellectually but that the teacher should take the initiative to stimulate the child to readiness, and should take a forceful interventionist role. Clements and Battista (1992, page 426) support Bruner's view that it is not necessary to wait for the child to be ready intellectually and stated:
'Progress from one . . . (van Hiele). . level to the next is more dependent on instruction than on age or biological maturation.'

### 1.6 Donaldson

Donaldson (1978) also believed that the teacher has a direct role and in the 1970s Donaldson reshaped our understanding of what children can do. Language emerged as the major factor in helping or hindering children to understand. She found that children could do what Piaget said they could not. Donaldson's (1978) experiments highlighted some of the difficulties which arose from language rather than from conceptual difficulties, and showed that it was linguistic complexity and context which was determining whether a child could perform Piaget's tasks. Donaldson (1978) set up tasks equivalent in conceptual difficulty to Piaget's but with contexts which had more meaning for the child and with the result that the child was then able to carry them out.

### 1.7 The Constructivists

Another school of thought which exists is the constructivist school (see for example von Glasersfeld 1987, Jaworski 1994). Jaworski (1994) believes that modern constructivism derives directly from Piaget's work and reminds us that Piaget's stages of development of logical thinking formed the basis of the widely used Nuffield Mathematics Project for primary schools published in the 1960s. Constructivists generally reject Piaget's concept of stage theory but accept his theory of the acquisition of knowledge (Sutherland 1992). They would not agree that Piaget's stages
of thinking are so clear cut and precise, rather that the child's thinking is simply more effective when she is older and at a later stage of development.

Constructivists see the teacher as playing an enabling role whereas Piaget preferred the child to discover everything herself through her own experience. Constructivists in science and mathematics also argue that the child learns primarily from practical experiences but see cognitive development as a gradual process of modifying existing concepts rather than one involving radical breakthroughs.

A primary consideration in planning learning experiences must always be the one originally articulated by the educational psychologist Ausubel (1968), that the most important single factor influencing learning is what the learner already knows. Once this is ascertained, the way forward is clear. However, Ausubel (1968) also argued that we learn largely by means of language, a view not shared by everyone (Sutherland 1992). The constructivist view is that children build up their concepts from experiences in the world, that learning takes place as a response to the environment and has little to do with genetic input. The constructivist believes that the child constructs her own version of reality from her own unique experiences. Therefore the teacher must know each child's initial knowledge and previous learning experiences including preferred learning strategies in order to plan a learning programme. The alternative is to go through the motions of teaching a prescribed syllabus regardless of whether or not pupils are learning anything.

The constructivist message then is that teachers should be aware of the learning strategies which children already have and ways must be found of using all the mathematical abilities and prior knowledge which children bring with them to school.

### 1.8 Information Processing (IP) school

The information processing school of thought differs from the Piagetian school in focussing on a single act of learning taking place at a time. The information processors look for quantitative rather than qualitative change. Whereas Piagetian research does not try to give an account of what happens at one moment of learning, IP's primary research method is one of experiment resulting in a quantitative account of a single learning experience.

Short term memory is of great importance in the IP approach. If a child cannot solve a problem, IP theory argues that the demands of the task are greater than the child's processing capacity, whereas the Piagetian approach explains a child's inability to perform certain tasks in terms of the child not yet being at the appropriate stage.

The child's capacity is determined by proficiency at certain skills. One of the main advocates of IP, Sternberg (1977), described six factors:

1. Spatial ability. The ability to visualise a problem spatially in all its details.
2. Perceptual speed. The ability to grasp a new visual field (or view) quickly.
3. Inductive reasoning
4. Verbal comprehension ability
5. Memory. The ability to store visual material in the brain
6. Number ability

The teacher who favours the IP approach would need to find ways of increasing the child's ability in each of the six areas described above.

### 1.9 Krutetskii

Krutetskii (1976) carried out a vast study of children's mathematical abilities and of how able children solved mathematical problems. One of the abilities which Krutetskii considered important is rapid and stable remembering of mathematical material and he noted that able pupils are able to remember schemas rather than remembering facts or specific lists.
of data. Skemp (1971) too points out that a schema reduces cognitive strain and that a suitable schema increases both retention of material and understanding.

Krutetskii (1976) organised problems into different types and tried to identify specific abilities which came into play in solving the different types of problem. The five mathematical abilities which Krutetskii (1976, page 351) identified are as follows ${ }^{4}$.

1. The swiftness of mental processes and the ability to reflect deliberately and profoundly.
2. Computational ability.
3. Memory for symbols, numbers and formulae
4. Spatial ability
5. Visualisation of abstract mathematical relationships and dependencies

Krutetskii (1976) stated that these abilities can be demonstrated in specific activities and can also be created and developed by an activity. The latter point will be returned to later in this thesis.

Krutetskii described two types ${ }^{5}$ of thinkers which represent two distinct outlooks on mathematics, a geometric type (visual/pictorial) and an analytic type (verbal/logical)6. Krutetskii (1976) found that most gifted pupils could think in either way and could vary their approach according to the problem. Some of the differences between these two modes of thinking were highlighted by Skemp (1971) who wrote about two kinds of symbol which can be used in mathematics, visual and verbal. These are closely related to the characteristics of Krutetskii's two types of thinkers and are summarised in Table 1.2 (Skemp 1971, page 111)

[^2]The ability to visualise is considered by Krutetskii to be a form of support which can be useful in solving problems. However, he found that pupils who were analytic type thinkers did not generally need this visual support. Analytic types tended to analyse concepts rather than situations. The geometric type on the other hand was characterised by a tendency to interpret visually any mathematical relationship or situation. Krutetskii (1976) noted a correlation between the analytic type and success in learning algebra and, correspondingly, between the geometric type and success in learning geometry. Krutetskii (1976) claimed that spatial ability is not a necessary ingredient of mathematical giftedness, but that it influences the mathematical cast of mind and characterises a particular type of mathematical giftedness.

TABLE 1.2
Skemp's summary of properties of two kinds of symbolism

| Visual | Verbal-algebraic |
| :--- | :--- |
| Abstracts spatial properties, <br> such as shape, position. | Abstract properties which are <br> independent of spatial <br> configuration, such as number. |
| Harder to communicate. | Easier to communicate. |
| May represent more individual <br> thinking. | May represent more socialised <br> thinking. |
| Integrative, showing structure. | Analytic, showing detail. |
| Simultaneous. | Sequential. |
| Intuitive. | Logical. |

Teaching methods which emerged following Krutetskii's work included; suggesting strategies to pupils such as the notions of reversing problems, asking pupils to invent 'similar' problems, choosing simple numbers instead of hard ones, moving from materials to symbolisations and drawing diagrams. The expectation was that these strategies would
be learned and eventually the pupils would be able to apply the strategies for themselves.

### 1.10 The van Hiele levels

The van Hieles (1986) put forward a theory of spatial development which claimed that learning is a discontinuous process and they described five stages or levels. A description of their work can be found in Clements and Battista (1992), Hoffer (1983) or Dickson et al. (1984). The levels are defined as follows:
Level 1. Visual: Figures are distinguished in terms of their individual shapes as a whole and relationships are not seen between these shapes or their parts. They often use visual prototypes such as 'door shaped' for a rectangle. They do not attend to the geometric properties
Level 2. Descriptive/Analytic: There begins a development of an awareness of parts of figures and the child can characterise them by their properties. These properties become realised through observations during such practical work as measuring, drawing model making etc. They do not see relationships between classes of figures.
Level 3. Abstract/Relational: Relationships and definitions are beginning to be clarified. The square is now seen as a special case of a rectangle which is a particular instance of a parallelogram. The student understands that one property can follow from another and distinguishes between necessary and sufficient conditions.
Level 4. Formal Deduction: The student can see the role of axioms and how possibilities exist for developing a theory proceeding from various premises. The student can apply deductive reasoning and establish theorems within an axiomatic system.
Level 5. Rigour: The student can reason formally by manipulating axioms or theorems without the presence of reference models or concrete interpretation. Theory construction can be completely abstract.

The higher levels relate to the student's ability to understand the notion of proof, with level 3 seeing the initial stages developing and level 4 indicating mastery of proof.

The implication is that secondary teachers of mathematics need to find ways of helping children over the discontinuity from one van Hiele level into the next level.

### 1.11 Curriculum Development and Educational Research

During the 1960s there was much activity in curriculum development in schools, both here in Britain and in the US. The so-called 'new mathematics' in the US or 'modern mathematics' in Britain was beginning and issues such as readiness for learning and learning by discovery were at the fore. There was a need for a solid research community to be established to assist in this development and to monitor change. The very existence of this ferment in the curriculum fostered an acceleration in research (Kilpatrick, 1992).

In addition, the interest in the connection between psychology and mathematics teaching was growing. A conference organised by the Committee on Intellective Processes Research of the Social Sciences Research Council was held in 1962 which brought together psychologists and mathematics educators but there was some difficulty in the two groups communicating with one another (Kilpatrick, 1992). At the time this problem prevented much dialogue from taking place and many subsequent conferences focussed only upon the views of mathematics educators. It was not until 1977 that the first meeting of PME (Psychology of Mathematics Education) was organised in Utrecht.

Prior to the formation of PME, in 1969, the first International Congress on Mathematical Education (ICME) took place in Lyons, France. In the closing address Begle stated (1969 page 110),
'I see little hope for any further substantial improvement in mathematics education until we turn mathematics education into an experimental science, until we abandon our reliance on philosophical discussion based on dubious assumptions and instead follow a carefully correlated pattern of observation and
speculation, the pattern so successfully employed by the physical and natural scientist.'

Fortunately, many researchers took up the challenge to make mathematics education research a scientific endeavour, incorporating methodologies from such disciplines as psychology.

### 1.12 The psychology input

The interest shown by some mathematics educators in psychology and the psychology literature did have the effect of stimulating more research. This broadening of perspective informed research method and opened up the different methodologies for which Begle (1969) was appealing in the quote above.

Problem solving tasks and most learning tasks involve holding information in short term memory while scanning for helpful relationships. Knowledge about memory inevitably impinges upon mathematics teaching since in order for knowledge of ideas or concepts to pass into long term memory, there must be connectivity to other (mathematical) knowledge. Memory is about attaching meaning to information or adding it to a schema rather than simply about mental lists. Imagery has been shown to aid memory and much research has been carried out in the psychology literature on the nature of imagery. This literature on imagery and visualisation is given further attention in Chapter 3.

### 1.13 Action Research

This term is commonly used to refer to research done by practitioners. It was defined by Romberg (1992, page 57)) to be a 'research strategy used to investigate schooling situations where the researcher assumes 'wise practice' that needs to be documented and understood has evolved in schools or classrooms.' Kent and Hedger (1980) carried out an important research study of this kind describing teaching situations and several case studies of individual pupils. Their stated aims as teachers were 'to
cultivate those mental powers characteristic in mathematicians and . . . to create situations which allow these powers to develop.' (1980, page 139). Kent and Hedger's (1980) work demonstrates Bishop's (1992) view that the power of action research lies in its richness rather than any succinct mathematical construct. Bishop (1992, page 718) commented:
'The pedagogical wisdom of Polya or Freudenthal cannot be understood by terms such as 'discovery learning' or 'learner centred instruction'. The wisdom of the pedagogue is revealed in the well-theorised and articulated innovative practice, in the empathetic elaborations of the educational situation, and in the insightful analysis of experience.'

Polya (1957) considered problem solving to be the focus of mathematical instruction. Polya believed that the student's experience of mathematics must be consistent with the way in which mathematicians work. This means, among other things, making conjectures and then trying to prove them (guessing and testing!).

There is also the view today that mathematics learning is a culture specific experience. The term ethno-mathematics has entered the vocabulary, meaning the mathematics which is embedded in the particular culture and whose purpose is other than for 'doing mathematics' (Nunes, 1992). On the importance of the school situations occurring in a culture, Schoenfeld (1992, page 341) stated:
'The lessons students learn about mathematics in our current classrooms are broadly cultural, extending far beyond the scope of the mathematical facts and procedures that they study.'
Emphasising this importance, Resnick stated (1989, page 58):
'Becoming a good mathematical problem solver - becoming a good thinker in any domain - may be as much a matter of acquiring the habits and dispositions of interpretation and sense making as of acquiring any particular set of skills, strategies or knowledge. If this is so, we may do well to conceive of mathematics education less as an instructional process (in the sense of teaching specific, well defined skills or items of knowledge), than as a socialisation
process.'

### 1.14 Conclusion

There are certain mathematical skills which need to be developed in children and these have been classified by various authors. Krutetskii (1976) defined five specific abilities and Sternberg (1977) defined six.

There exist two extreme modes of mathematical thinking based upon either verbal or visual strategies and these are linked to the left and the right hemispheres of the brain respectively. Pupils may incline towards one or other cognitive style and if the teaching style which they experience is restricted to just one approach, it can have the effect of hindering the pupils' learning.

Piaget described distinct developmental phases which the child must pass through and estimated the ages at which these phases are reached. Other researchers (Case 1974 and Pascual-Leone 1976) describe a mental effort statistic attached to Piaget's stages which could further define the level at which a child is working.

Van Hiele studied spatial ability and listed five discontinuous hierarchical levels of spatial development. The challenge which this presents to teachers is how to advance pupils over the discontinuity from one level to the next.

Many schools of thought favour the alternative view that cognitive development does not occur in discrete stages, but that it is a gradual and continuous process and intervention by the teacher can enhance this progression immensely. Vygotsky's 'zone of proximal development' puts the teacher centre stage in the intellectual progression of the child whereas the constructivists see the teacher with an important but less central, more enabling, role.

The study of imagery and visualisation in the practice of mathematics is the main interest here. Skemp (1971), Krutetskii (1976)

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and others have identified the different ways in which imagery can be used in doing mathematics.

The areas of mathematics education research which are relevant are those concentrating on the ability to visualise, the effects of different kinds of imagery, what tasks enable/effect learning and what abilities may be needed and/or used in doing certain tasks in mathematics, and finally what strategies and/or processes are used.

## Chapter 2

## Spatial ability, visualisation and imagery in mathematical thinking

### 2.0 Introduction

This chapter describes the research on imagery and visualisation carried out by mathematics educators from 1960 to the present. Research by psychologists in the general area of imagery will be discussed in Chapter 3. A characteristic of much of the work described in this chapter is that it has been carried out with the role of the teacher in mind and satisfies some, if not all, of the requirements set out by Ernest (1998), Bell (1994) and Bishop (1992) considered in Chapter 1.

The motivation for research in the mathematics education discipline is necessarily disparate from that in the psychology discipline, since the mathematics teacher's prime concern is with the improvement of mathematical thinking and learning, whereas the psychology researcher is concerned with understanding the processes involved. The different objectives of mathematics educators, as distinct from psychologists, have tended to result in different methodologies (and research paradigms) being employed in the two disciplines. However, certain studies and research findings in the psychology literature are of interest to mathematics educators and, as a result, some mathematics educators have been attracted by the methodologies which are characteristic of research in that discipline. This broadening of perspective has informed recent research in mathematics education and possibly had the effect of encouraging more research. Therefore, in this chapter, although we shall look mainly at research aimed at improving mathematical ability, we shall also consider research on how children think about and learn mathematics.

The extensive work of Piaget (for example 1960, 1967, 1971) and his co-workers in the Geneva school set the standard for developmental research in mathematical thinking and this work has been outlined in Chapter 1. Since Piaget, much mathematical research has reported upon case studies and has given descriptions of episodes in classrooms, some of which has focused on the child's ability to visualise. The review of the literature in this chapter covers research which has sought to extend our knowledge of the visualisation process both by qualitative case studies as well as by the experimental quantitative studies activated by the broader range of interdisciplinary methodologies.

### 2.1 The ability to visualise

The view that individuals have the ability to visualise and to manipulate images in some form follows from the empirical work commencing in the late 1960s on mental imagery. Many mathematics educationists (for example, Gattegno 1965, 1971, Mason 1988) base much of their work upon the assumption that all individuals have the ability to visualise, to work on images, to hold an image, and to rotate or manipulate it in the mind. Gattegno (1965, page 24) wrote:
'Because imagery is one of the attributes of the mind that everyone brings to school with himself (sic) it becomes today possible to see mathematics education as part of a true education of the whole self for a fuller use of oneself in a more abundant life.'

Mason (1991, page 84) described it thus: 'that amazing facility which we all share, the ability to form and manipulate mental images' but he admitted that the skill may need awakening. This might be achieved with the use of film, animations or a series of diagrams. Mason (1988, page 300) stated 'Animations are an excellent way to support dynamic imagery by providing experience to refer back to.'

During public lectures and in his writings, John Mason (see for example Mason, 1988) frequently engages his audience with mental acrobatics, inviting them to visualise and to manipulate images. A
typical request is the following (Mason 1991, page 83):
'Imagine a triangle. Imagine a circle tucked in one corner, touching the two sides. It grows in size always touching the two sides of the triangle, getting bigger and bigger, until it touches the third side as well. Let it grow even bigger, until it is entirely outside the triangle, but touches all three sides (two will need to be extended). Then let it shrink back down until it is tucked back in the same (starting) corner.

Now let it grow again, but this time let its centre leave a trace or track. Let the circle grow until it touches the third side of the triangle as well. Since it touches all three sides, we can allow it to 'let go' of a different side, and then shrink until it is tucked in a different corner. Experiment with that for a while.

Allow it to grow until it is outside the triangle but touching all three sides, then let go of a different side and shrink. Where can its centre go to in all this movement?'

The visualisation exercise described in the quote above provides a vivid mental experience of constraints and illustrates that there can be some personal discovery of a theorem. Mason continued (1991, page 83):
'Any statement of what you are seeing is a geometrical theorem. The significance of your theorem will depend on whether others can recognise what you are seeing, and agree to its generality, to its applying to any triangle.'

In speaking about his experience of working on imagery with large groups of people Mason (1992, page 13) commented,
'For years I have noticed that it is possible to generate a collective state in a room which is quite different to the usual bustle of a classroom . . . . It is as if they have been transported, as if they are not really present in the room but rather in a world entered through a fusion of physical and mental screen.'

Mason's experiments extend the earlier work of Gattegno who

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believed in allowing children to explore their own imagination. Gattegno's research (for example, 1965), reported a similar experience. Gattegno (1965, page 22) wrote,
'In a number of experiments I have asked my classes to consider with their eyes shut some situation in their mind which I generated by instructing them to produce some images and act mentally upon them. . . .a number of distinctive features which when talked about sounded strangely like the geometrical statements we read in books as theorems. . . . . they actually believed they were making a statement which was universally valid, the more so because everyone in the group got the same insight into the situation.'

Gattegno was interested in identifying mental structures and the activities needed for the production of mathematical structures. Gattegno (1965, page 22) believed imagery to be important and stated:
'In geometry it is visual imagery that is used. But the dynamics of the mind when formalised produces all the conceivable algebras. Algebra differs from geometry in that the first describes mental dynamics while the other uses mental content, imagery.'

Many other teachers and mathematics educationists believe that practicing this skill of imagery improves spatial ability and that it is beneficial to mathematical (and algebraic) thinking. However, it is not an easy task to develop the skill fully. This view has been confirmed by Dawe (1993, page 62) who, commenting upon pupils' visualising abilities pointed out:
'Although students may readily summon up an image of a circle for example, it is another matter to control it for some mathematical purpose.'

### 2.2 A definition of imagery

Different kinds of imagery exist and psychologists do not agree on the nature of a mental image, whether it is visual or spatial or some
combination of the two ${ }^{1}$. It is relevant to define what is meant here by imagery and to give some of the definitions used by other researchers working in this area.

Wheatley (1990) summarised the three elements of imagery which Kosslyn (1983) had identified in an earlier work, these are:
(i) the construction of images from directly viewing the objects, the images that are constructed may be concrete and limiting, or dynamic and abstract;
(ii) re-presentation of the image at some time after its original construction; and
(iii) transformation of an image as, for example, in the rotation of one image to facilitate comparison with another.
Wheatley (1991, page 34) further stated,
'constructing an image from pictures, words, or thoughts; representing the image as needed; and transforming that image . . . . The need to make such comparisons (mental rotations) occurs frequently in mathematics and is facilitated by the use of visual imagery.'

Presmeg (1986), in her study of the efficacy of employing visualisation in high school mathematics, focused on visual imagery and described five kinds of visual imagery, including dynamic imagery. Her definition, (1986, page 42) 'a visual image is a mental scheme depicting visual or spatial information' is broad enough to include shape, pattern and form and not simply the 'picture in the mind' interpretation.

The imagery of particular interest here is what Presmeg (1986) described as 'dynamic imagery' which involves moving and transforming images. However, whilst this type is reported to be 'potentially effective' in the solution of the mathematical problems which Presmeg posed, Presmeg (1986) found that it was used very infrequently by subjects in her study, ( $<4 \%$ ). She also mentions (1986, page 42) that 'a pupil may be highly successful in the learning of school

[^3]mathematics without needing to resort to visual thinking'. This finding is discussed further in section 2. 5 where Battista (1994) reflects upon Presmeg's conclusion.

In spite of the accepted importance of imagery in thinking and learning, it is difficult to proceed with research involving the concept of imagery without a clear working definition of what research on imagery in mathematics might be. Bishop (1989) believes that the psychometric approach is not helpful in getting at the processes involved in visualisation. However, whilst mathematics educators must be concerned with the processes involved in imagery (which are discussed in Chapter 3) and with the stages of development of these processes, they are more concerned with the interventions which the teacher might design to provide experiences to enhance learning. Mathematics educators may also be concerned with placing learning in chosen contexts. These concerns might suggest that one direction for imagery research should be to investigate those kinds of mental images which can play a role in the enhancement of mathematics learning and thus could provide information on the kinds of interventions which are needed.

### 2.3 Linking visual images to concrete experiences

Piaget and Inhelder's (1971) ideas on imagery suggest that a child cannot perform manipulations of images in the mind until she has experienced those events with real objects. Piaget and Inhelder (1971) distinguish between 'reproductive' images which correspond to the known objects and 'anticipatory' images in which an object not previously perceived is represented. Implicit in Piaget and Inhelder's work is the notion of an image as a picture in the mind although no formal theory of imagery is presented. Dickson et al. (1984) stated that the child's power to reconstruct spatial images starts at around two years old and is perfected at around seven years old.

Sutherland (1992), in discussing Piaget's work and the 'vertical lag' between Piagetian stages, gave an example which is more specific about
the imagery issue. The example describes the situation of a toddler (preoperational) being able to find her way around the rooms of a house but who will only be able to imagine the experience in her brain when she reaches the operational period (age 5 according to Sutherland and age 7 according to Piaget). The child will then be able to represent those sensorimotor experiences in the brain as an image or diagram. At the age of 5 to 7 years, when the child embarks upon the operational period, she becomes more able to imagine scenes but her ability to manipulate images will be dependent upon when appropriate experiences with real objects commence. According to the theories put forward by Case (1974) of a mental effort statistic and by Pascual-Leone (1976) of an M power, we must conclude that the imagery tasks which a child of 5 to 7 years can perform are limited.

With regard to cognitive development, Battista (1994) stated that the learner acquires facts about a new situation which are concrete, then routes and relationships are established connecting these facts and these are assimilated into a schema. With time, new facts at increasingly higher levels of interrelationships are established and these become integrated into the schema.

Campbell et al. (1995) distinguished between 'rich' images, which are particular concrete images of objects, and image schemata which are more general. An image schema of a triangle, for example, is not a completely abstract propositional structure, but neither is it a single specific triangle. It is not simply a concept of a triangle as a three sided plane closed figure but it can be different triangles. An image schema carries visual information but the imager distinguishes between an image schema of a triangle and the image of a particular triangle. Image schemata operate at a level of generality somewhere between abstract propositional structures and specific mental pictures. They form a bridge between abstract logical structures and particular concrete experiences.

Imagery is often a precursor to drawing a diagram. Wheatley (1991) suggested that teachers should encourage pupils to draw pictures
and stated that imagery is particularly useful in solving non routine problems. However, care should be taken that too much emphasis is not placed upon the static diagram at the expense of a dynamic mental image. Mason (1988, page 297) pointed out:
'The diagram can act as scaffolding or support for the mental screen, stabilising the image, but if the diagram becomes the sole object of attention then it can hinder rather than help thinking.'

The ability to manipulate images and 'play' with different possibilities can be a very productive activity in mathematics, and of course in many other disciplines. For instance, Dawe (1993) cited the use by athletes of visualising the sequence of movements in their routines so as to enhance performance. With regard to mathematics learning, Dawe (1993, page 73) reinforced Piaget and Inhelder's (1971) view which suggested that a child cannot perform mental manipulations of images until they have experienced those events with real objects, and commented:
'In the case of young children constructing mathematical ideas there is good evidence to suggest that teachers should consciously create mathematical learning environments which enable children to link visual images, verbal propositions and memories of activities involving the manipulation of physical objects.'
(Dawe, 1993, page 73)

Clements and Del Campo (1989) also stressed that children should be encouraged to link visual images with verbal knowledge thus linking images with concepts in mathematics learning. They gave fraction tasks both pictorially and symbolically and found that pupils sometimes had difficulty with 'expressive' tasks (pupils had to express the answer themselves perhaps with the aid of simple equipment) rather than 'receptive' (pupils had to identify the correct response) tasks. Subjects could identify pictures of thirds and quarters but had more difficulty cutting the fractions from a circular card. Examples of the tasks are shown in Figure 2.1.


Source: Clements and Del Campo (1989)
Figure 2.1 Fractions tasks comparing pictorial and symbolic representations

A pertinent example of linking visual images to concrete experiences in a method used for calculation can be can be observed in Japanese abacus (soroban) users who, after many years of practice, no longer need the instrument but simply see it in their mind and can perform mentally all kinds of complex calculations in a way which derives from the use of the equipment.

### 2.4 What is spatial ability?

There is evidence to suggest that 'the quintessential attribute of someone with high spatial ability is a good spatial memory' (Smith 1991, page 2). Lean \& Clements' (1981) view is that spatial ability means the ability to formulate mental images and to manipulate these images in the mind. With regard to formulating mental images, according to Descartes (in Muldane and Ross 1968), we can easily imagine the difference between a pentagon and a hexagon, in a way whereby they are introspectively distinguishable. Something rather more difficult to perform would be to imagine a chiliagon (a 1000-sided figure) and to 'see' how it is different
from a 999 sided figure. Although the two are conceivable, we cannot see how one differs from the other. Similarly, imagining some unfamiliar animal such as a zebra feels different from merely thinking of a zebra without forming any image ${ }^{2}$.

Descartes concluded that imagining is not the same as thinking, imagining involves images, and the images are of the 'picture in the mind' type. Thinking, in the context described by Descartes, seems similar to von Glasersfeld's (1987) interpretation of a 'mental representation' which von Glasersfeld called a 'conception'. Von Glasersfeld (1987b, page 220) gave the example:
'One's mental representation of, say, one hundred will be the either the numeral ' 100 ' or ' $C$ ', or a specific lot of unitary items whose count is presumed to yield the number word 'hundred', or an arrangement of specific lots according to a transform derived from the accepted symbol system, such as ' $10 \times 10^{\prime}$.'

Following on from these imaginings we might ask, 'What is the relationship between the ability to imagine a pentagon or a hexagon and spatial ability?'. Is spatial ability a skill which enables the possibility for shapes or objects to be manipulated in the imagination? Or is it the inverse, is imagery ability a skill which enhances spatial ability?

Different forms of spatial ability have been described. Bishop (1983, page 182) summarised McGee's (1979) two types of spatial ability:

1. Spatial visualisation $(\mathrm{Vz})$, which involves the ability to mentally manipulate, rotate, twist or invert a pictorially presented stimulus object, and
2. Spatial orientation (SR-O), which involves the comprehension of the arrangement of elements within a visual stimulus pattern and the aptitude to remain unconfused by the changing orientations in which a spatial configuration may be presented.

Bishop (1983, page 184) also proposed two different types of ability

[^4]constructs of his own:

1. The ability for interpreting information (IFI). This ability involves understanding the visual representations and spatial vocabulary used in geometric work, graphs, charts, and diagrams of all types.
2. The ability for visual processing (VP), This ability involves visualisation and the translation of abstract relationships and nonfigural information into visual terms, It also includes the manipulation and transformation of visual representations and visual imagery.

With regard to spatial ability and its correlation with mathematical problem solving ability, Wheatley and Brown (1989) reported that students who achieved above average scores on standard mathematics tests but who had low spatial ability were poor at problem solving. In a later paper Wheatley (1991, page 35) stated:
'students with high spatial ability whose performance was average or below on standardized mathematics tests and in school mathematics class had an excellent grasp of mathematical ideas and were able to solve non-routine problems, often creatively.' Wheatley (1991) interpreted the findings to indicate that 'spatial ability lies at the heart of meaningfulness' in mathematics.

Brown and Presmeg (1993, page 137), in their study of types of imagery used by secondary school students, made a similar claim that, 'students do use imagery in the construction of mathematical meaning.' It seems likely that if spatial ability and imagery can be nurtured and developed, then pupils' mathematical understanding and ability to solve non-standard mathematical problems may be enhanced.

Different approaches to research into spatial ability
In 1983, Alan Bishop reported that there was not an extensive or comprehensive corpus of research on which to draw ideas for teaching geometry and that very little work on geometry learning was being produced. In a survey of articles in the Journal for Research in

Mathematics Education for the year 1980, he found only 10 out of 161 entries concerned with space and geometry in some way. Most research at the time seemed to focus on number. Bishop (1983, page 200) commented, 'We are still relatively ignorant of spatial and geometric ideas'. Much has been done since this statement was made but still no clear definitions and teaching techniques have emerged.

Numerous studies (Smith 1964, Guay and McDaniel 1977, Fennema and Sherman 1977, Battista et al. 1982, Brown and Wheatley 1997) have tried to show that spatial ability is positively correlated with mathematical ability. For instance, Guay and McDaniel (1977) studied the relationship between mathematical ability and spatial ability and in their study, high mathematics achievers did better on spatial tests. They noted that high spatial ability was characterised as requiring the visualisation of 3-D configurations and the mental manipulation of these visual images.

Fennema and Sherman (1977) studied gender differences in spatial ability and found, firstly, that spatial ability is positively related to mathematical ability and secondly, that there was no difference between males and females when controlled for background.

Battista et al. (1982) were also interested in the relationship between spatial ability and achievement in mathematics. They studied the role that the study of geometry plays in spatial visualisation. As part of their study, they used the Purdue Spatial Visualisation Test on mental rotations which had been shown to be a valid measure of spatial ability (Guay, 1977). A typical task involving the mental rotation of a solid object is shown in Figure 2.2.


Source: Battista et al. (1982)
Figure 2.2 Spatial test item involving mental rotation

Battista (1994) claimed that the relationship between spatial ability and mathematical ability is based upon the fact that operations performed while interacting with mental models in mathematics are often the same as those used to operate in spatial environments. He also declared a verbal function as familiarity with the tasks increased commenting that as learners become proficient at manipulating mental models they begin to use words as 'pointers' to important operations and to think without re-presenting the operations (1994, page 93):
'familiar problems might be solved by referring to verbally encoded propositions or procedures, by-passing the spatial like thinking required to use the underlying mental model.'
but emphasised:
'However, even though such thinking may appear strictly verbal, for it to be conceptually meaningful and powerful enough to encompass novel situations, it must be based - at some level- on operations with mental models.'

There have been different emphases in research in the area of spatial ability. Some developmental research focused upon what kind of tasks can be performed and also upon what is easy and what is difficult. Other research has dealt with individual differences and why there are differences (due to gender, cultural background, prior experience etc.). Yet another approach has been to apply factor analysis in the hope of identifying what constituent abilities make up 'mathematical ability'. Spatial ability has emerged as one such factor ${ }^{3}$. However, Bishop (1980) has commented that the factor analysis approach produces findings which are unclear and inconclusive. Bishop (1983) prefers the methodology of studying clinical cases and learning about strategies in that way, not least because he believes that the only way to test VP is by individual clinical procedures, this being due to the personal and idiosyncratic nature of VP. A further drawback of the factor analyst's approach, according to Bishop (1983), is that it makes no reference to individuals and rarely pursues how an individual approaches a solution to a problem.

Krutetskii (1976, page 14), when discussing some research carried out with 130 mathematically able adolescents is rather dismayed at the lack of collection of qualitative data and argued:
'Kennedy (1963) . . . computed, for 130 mathematically gifted adolescents, their scores on different kinds of tests and studied the correlation between them, . . . . . The process of solution did not interest the investigator. But what rich material could be provided by a study of the processes of mathematical thinking in 130 mathematically able adolescents.'

With regard to quantitative versus qualitative research, Bishop (1983) recommended paying attention to the individual subject's data rather than to overall means of the data in order to ascertain which processes are actually being used in tasks. Lean and Clements (1981)

[^5]agreed and urged researchers to pay less attention to testing and more attention to clinical investigations, which concentrate on the extent to which spatial ability is used by students attempting different kinds of mathematical problems, only thus can relationships between spatial ability and mathematical performance be established.

A complication which arises with experimental testing is that different tests have been assumed to be testing the same cognitive ability. It has been assumed that a single test will measure the same thing for every individual. However, most tests measure acquired spatial skills rather than innate ability. It is also the case that many spatial tests can be solved using different strategies - different subjects often use different strategies for the same tasks, possibly confounding the results in some cases (Clements, 1983) ${ }^{4}$.

A sensible approach would seem to be to use a combination of experimental testing with clinical interviews so that both quantitative an qualitative data can be collected. In this way curricular sequences to improve spatial ability through mental manipulation skills may then be developed.

In summary, whether spatial ability is an innate ability or can be taught is an important question. It is innate ability which frequently interests psychologists, whereas the acquisition of spatial skills is possibly relatively of more interest to mathematics educators as it is more likely to be supported by schooling.

### 2.5 Visualisation and mathematical ability

The importance of visualisation and imagery in the learning of mathematics has been the subject of a number of research studies. However, it has been noted (Clements, 1981) that one of the problems in

[^6]imagery investigations is finding an appropriate research methodology. Different researchers have advocated a range of different methodologies as mentioned above, from developmental psychology, to individual difference clinical case studies, to teaching programmes, testing and factor analysis.

Brown and Wheatley (1997) adopted the case study approach and identified two components of imagery capability which seem particularly important in mathematical understanding, (i) decomposition/ recombination and (ii) transformation, and stated that there are individual differences in performance of the components. Brown and Wheatley (1997) went so far as to propose that the difficulty which one of their subjects had with mathematical reasoning resulted directly from poor imagery ability and, furthermore, stated (1997, page 69):
'We believe that forming images of mathematical relationships is essential for effective problem solving.'

This hypothesis, that imagery aids creative problem solving in unfamiliar problems, is supported in the psychology literature (reviewed in Chapter 3). Kaufmann (1985, page 58) stated:
'It may now be argued that the location of verbal and visual symbolic representation on the two dimensions of 'level of processing' and 'type of processing' may be seen to point in the same direction in relation to the novelty parameter in problem solving. Linguistic representation is the more appropriate and economical the higher the degree of task familiarity. With increasing situational novelty, the functional significance of visual imagery will increase.'
Kaufmann (1985) further stated that imagery has its most important function in the initial phase of the problem solving process.

In their paper, Reynolds and Wheatley (1997) described the ability of a particular child to solve problems using imagery thus:
'Elaine was able to do this problem because of the images that she formed, which were expressed in her drawings, particularly
rectangles; her metaphors of stretching, chopping and covering; and her hand movements. Elaine's imaging activity was the result of her intention to make sense of relationships.'

Reynolds and Wheatley (1997, page 104) went further in describing their subject:
'Her well developed spatial sense has given Elaine great mathematical power enabling her to construct, examine, and reconstruct some complex mathematical relationships.'
Reynolds and Wheatley (1997, page 104) strongly believe that engaging pupils in spatial activities fosters mathematical reasoning and claimed that the case of Elaine 'highlights the need for classrooms to focus on activities that encourage students to develop their spatial sense in a variety of ways.'

In a study of the use of dynamic imagery by high ability children, Thomas and Mulligan (1995) found a higher percentage using dynamic imagery than for average/low ability children they had observed in earlier studies. Furthermore those children with dynamic visualisations were shown to have higher achievement on the numeration tasks than those with simply static visualisations.

Some studies, for example Pesci (1995), have attempted to find out what interventions involving imagery are successful in mathematics. Rather than taking the factor analyst's approach or interviewing individuals as case studies, Pesci (1995) carried out experiments in entire teaching programmes. Pesci's study took place over a period of four years and involved 21 experimental classes and 7 control classes. A teaching programme was delivered, tests administered and solution strategies of pupils were analysed. Pesci (1995) concluded that diagrammatic models helped mental models and that providing students with visual means for synthesising data is helpful, particularly for the weaker pupils.

Although it is difficult to study the way in which individuals
process ideas and whether or not imagery is used, Moses (1977) provided some novel proposals on how to ascertain subjects' modes of thinking in their solution of problems. Moses (1977) collected written responses to problems and noted any prevalence of diagrams. Suwarsono (1982) continued this theme and developed a series of tasks which subjects were first of all required to solve and then were asked to choose from a list of typical solutions the one which corresponded closely to their method of solution. Suwarsono's (1982) method was an improvement on Moses' (1977) since analysis of written solutions, even when these solutions do include diagrams, does not always give a proper indication of the extent to which visual solution processes have been used.


Source: Presmeg (1986)

Figure 2.3 Parallelogram task utilising dynamic imagery

Presmeg (1986) used a modified version of Suwarsono's tasks and also interviewed her subjects thus providing still richer data. An example of one of Presmeg's tasks is shown in Figure 2.3. However, Presmeg (1986) did raise a warning regarding the use of imagery in such a problem. When discussing this example involving the area of a parallelogram, one of her subjects used imagery to slide the
parallelogram AECF into a rectangle. This happened to give the correct solution because the height was kept constant and the areas were therefore equal, but Presmeg (1986) was concerned about vague manipulation of an image which, if not coupled with rigorous analytical thought processes, may be unhelpful.

The advent of dynamic geometry packages to support pupils' personal imagery may be useful in this respect by correcting misconceptions, but Mason (1988) remarks that it is essential that pupils enter into the situation by owning their images and that they do not keep the mathematics at arm's length inside the computer program.

Alternatively, imagery could be developed by working with 'common-sense' objects such as folding paper or untying knots 5 where the personal experience of the learner provides the knowledge of what is, and what is not, possible. Wheatley (1991) commented that our experience of seeing may depend very much on what we know about what we are looking at. Kosslyn (1983, page 189) made the same observation:
'Much of what happens when we form and manipulate an image in the course of thinking may be in part a reflection of our underlying knowledge and beliefs about what would happen if we were dealing with real objects'

This view was also expressed by Battista (1994) who was interested in how the properties and behaviour of symbolic objects in a model simulate the properties and behaviour of the objects they represent. Battista (1994, page 93) explained how this is of importance to mathematics teaching:
'Thus research and theories on imagery may help inform theories dealing with construction of mental models in mathematics.'

Bishop (1989, page 11) believes that to draw valid inferences

[^7]regarding the visualisation process, the data must be derived from a broad range of situations, he explained:
'If we want to understand more about the visualisation process, we need to study it in a variety of task and stimulus contexts, and to move away from just 'problem-solving'.'

Two extremes of cognitive style have already been mentioned (as described by Sharma, 1979 and Presmeg, 1985) and we turn now to the contrast between the two in relation to the implications for learning mathematics.

## Verbalisers versus visualisers

The existence of two types of thinkers in relation to mathematical abilities (Krutetskii 1976), verbalisers (analytic thinkers), visualisers (geometric thinkers) and also a third type which is a combination of the two (harmonic thinkers) was described in Chapter 1. More recently Clements (1981) has suggested that rather than just three categories there may be a verbaliser/visualiser continuum.

Krutetskii's (1976) research, and latterly Presmeg's (1986), referred to a 'need' by thinkers at the geometric end of the scale (visualisers) to interpret visually an expression of an abstract mathematical relationship. In this sense, Krutetskii (1976) suggested that figurativeness often replaced logic for geometric thinkers. Krutetskii wrote of analytic thinkers having 'no need for visual supports' implying that this is a superior level of thought. However, Krutetskii (1976) also compared the thought processes of a geometric thinker and an analytic thinker in trying to explain the shape which is obtained when a right angled triangle is rotated about one of the shorter sides. The geometric thinker may consider the problem 'childish' since the solution is so obviously a cone, whereas the analytic thinker may describe in elaborate mathematical terms whereby a circular base would be obtained and that this is connected to a single point and may finally deduce that the shape is indeed a cone. This example demonstrates how switching to a visual strategy can be beneficial.

Lean and Clements' (1981) study of engineering students in Papua New Guinea supported Presmeg's (1986) results regarding cognitive style and mathematical abilities, reporting that there was a tendency for students who preferred to process mathematical information by verballogical means to out-perform more visual students on their mathematical tests.

Battista (1994) warned against drawing too many conclusions from these results. Battista (1994) applied Greeno's theory of conceptual domains to geometry and suggested that during the first 3 van Hiele levels, students reason in geometry by manipulating objects in mental models and observing the results, not by operating on propositions. Battista (1994, page 92) also asked:
'. . . if visual-spatial processing is important in mathematical thinking, what of the research suggesting that students who process mathematical information by verbal-logical means outperform students who process it visually (Presmeg 1986)?'
and responded:
'The answer may lie in how mathematics is viewed. I have suggested that the kinship between mathematical and spatial thinking applies to the learning of conceptual domains. But much current learning of school mathematics requires a different kind of learning - near-rote memorisation of procedures for manipulating symbols. Because such learning is not conceptual in nature, we should not expect it to involve spatial thinking or mental models.' 6

With regard to Lean and Clements' (1981) sample of subjects, who were all engineering students, there are some questions which could be asked, including 'How were the subjects taught mathematics?', and 'Were they taught algorithmically rather than conceptually?'. Skemp (1971) has described different levels of understanding in mathematics

[^8]and distinguished between what he terms 'instrumental' understanding which relies upon remembering techniques, and 'relational' understanding which relies upon understanding concepts.

Regarding the remembering of techniques, Wheatley (1977, page 37) stated:
'Our curricula stress rule-oriented sequential activities so extensively that children expect to apply a rule immediately.
Our society emphasises and rewards left hemisphere activities. This is particularly true of our schools. A premium is placed on being able to put ideas into words, to state them explicitly, and to operate with rules.'

Smith (1991, page 2) also commented upon the problem of emphasising verbal learning:
'It is not surprising that, in a system where academic ability is judged primarily in verbal terms, verbal methods tend to be emphasised even in spatial disciplines, so that 'able' children will find learning easier. . . . However, if spatial memory has an important role in many areas of the curriculum, then perhaps educators need to take a broader view of how children learn to think. . . it is possible that the intellectual potential of many children is being grossly underestimated and systematically underdeveloped.'

The left and right hemisphere activity of the brain supports the notion of two extremes of thinking, sequential and holistic thinking. According to Smith (1991), it is sequential thinking (left hemisphere) which receives more emphasis in our schools. In Chapter 1 it was mentioned that right hemispheric thinkers are better problem solvers, a skill considered important by mathematics educators, by the Cockcroft Committee (CITMS, 1982) and by the DFE (DFE, 1995). There are serious implications here for teaching; if the assessment and value judgements of pupils' mathematical ability are centred on verbal techniques, what should be done to bring about a more visual approach which would be
more effective in developing competence in problem solving? It may be the case that the curriculum is assessment driven, implying that items which appear on national tests will receive more emphasis in the classroom, and at the present time NFER are developing new mental tests for Key Stage 3 which will include more visual tasks ${ }^{7}$.

Although most learners will have a natural preference for either a visual or a verbal style, it seems desirable that they be able to be versatile. Krutetskii (1976) found that most gifted pupils can think in either the analytical way or the geometric way and can vary their approach according to the problem. Wheatley and Wheatley (1979) suggested that individual learners have preferred cognitive styles and are not necessarily equally proficient in both types of activity. Wheatley and Wheatley (1979) also suggested that low attainers find a spatial approach more accessible.

There seems to be a connection between spatial thinking and conceptual learning (conceptual learning being equivalent to Skemp's (1971) notion of relational understanding as distinct from instrumental). Bishop (1983) believes we must seek definitions of abilities and processes that help us to solve particular problems. McGee's (1979) two types of spatial ability already described include one (Spatial visualisation Vz ) which involves mental manipulation such as rotation, twisting or inverting .

Some researchers have hinted that imagery may have the limitation of forcing a single-case concreteness onto a mathematical (or other) situation thus preventing the opening up of a problem (Presmeg 1986; Lean \& Clements 1981). This may occur when the image is of a specific diagram and assumptions made about size, shape, geometry etc. It has already been noted that care should be taken that too much emphasis is not placed upon a static diagram at the expense of a dynamic mental image (Mason 1988). The distinction of different types of image described by Campbell et al. (1995) is useful here. They distinguished

[^9]between concrete, or 'rich' images, and image schemata. Campbell et al (1995, page 179) suggested that there may be a continuum stretching between such concrete images and image schemata and commented:
'the cut off point between one and the other is somewhat arbitrary but the distinction remains useful because abstraction is essential if visual imagery is to serve a useful function in mathematical thinking and the rigidity and inflexibility associated with the use of specific concrete visual images be transcended.'

Mason (1992, page 5) is convinced that with regard to the use of imagery in mathematics 'to get much educational benefit, students need to be active in processing images; they need to work on the images not just look at them.' Mason (1992) makes the distinction between looking at and looking through images; looking at is watching, passive observation, whereas looking through involves active knowledge of the image.

The results of Campbell et al. (1995) highlighted this distinction. Although the image is not a dynamic one, the sense of working on an image is conveyed in the painted cube task below which Campbell et al. (1995) asked subjects to work with:

A cube that is 3 cm by 3 cm by 3 cm was dipped in a bucket of red paint so that all of the outside was covered with paint. After the paint dried, the cube was cut into 27 smaller cubes, each measuring 1 cm on each edge. Some of the smaller cubes had paint on 3 faces, some on 2 faces, some only on 1 face and some had no paint on them at all. Find out how many of each kind of smaller cubes there were.

When solving this task, one student reported that the image was clear in his mind and he didn't need to draw a diagram. He could solve the problem completely by working on his mental image. In contrast, another student got extremely confused trying to look at her image and said (Campbell et al. 1995, page 185):
'The picture of the cube in my head is not clear and it's mucking me up because there are too many cubes . . . 27 smaller cubes.'

Thus, it is possible for some people to experience mathematics inwardly in the mind, although Mason (1988) reported that some individuals have 'fuzzier' images than others. Mason uses imagery to highlight and encourage the accurate use of language, as well as to create an environment which encourages formulation of conjectures and where participants can sharpen their images and modify them. Gates (1988) believes that imagery not only enhances understanding but also the reconstruction of ideas.

Mason (1995)8 has found that for a given sequence of 'imagine' tasks different people fail at different stages, finding the image control difficult. However, after a period of work, they find that they can manipulate much more visual information and for a longer time. The implication is that the practice of imagery is therefore desirable.

Visualisation is naturally employed in geometry but can also occur in arithmetic and algebra. Wheatley (1990) claimed that spatial sense can be useful in numerical as well as geometrical settings. Albert Einstein stated that he always thought about anything in terms of mental pictures. Indeed, some mathematicians have claimed that all mathematical tasks require spatial thinking (Fennema, 1979). The extent to which this is true may vary according to gender. Fennema (1977) had earlier reported no sex differences in spatial ability when background was controlled for. Battista (1990) found that spatial visualisation is an important factor in geometry learning but that it contributed differently to males' and females' performance. Battista (1990) found that males relied more upon visualisation strategies and females upon logical reasoning. Battista (1990, page 59) commented that this result had implications for teaching: 'However, from an equity in learning perspective, the ultimate goal of studying gender differences is the design of effective intervention strategies. From this perspective, the results of the

8 Personal communication September 1995
current study are important because they suggest factors that may relate to gender differences in geometry learning and indicate that certain instructional practices may create or exacerbate these differences.'

Bishop (1980, page 261), in setting psychological development research in context for mathematics learning, stated:
'The tasks still remain for the mathematics educator of planning curricular sequences and developing teaching approaches which will achieve the goals of mathematics education.'

### 2.6 Can visualisation skills be taught?

As early as 1973, Bishop found that the mere handling of 3-D shapes in a structured learning situation promoted growth in spatial ability. Lean (1981, in Bishop, 1983, page 186) suggested that the optimum time to develop spatial ability is age 7-12. Lean (1981) summarised the literature on spatial training and concluded that there was sufficient evidence in the literature to support the assertion that the interpretation of diagrams and spatial conventions could be taught, but that the situation for mental manipulation of images was far less clear. Bishop summed up the situation (1983, page 186),
'What are needed now are more training studies using clinical testing procedures, and involving retention and transfer tasks.
and (1983, page 199),
'As was indicated earlier, the research is not conclusive and it is likely that the difficulty lies with the assessment of VP (visual processing) as much as with its development. . . . What is clear is that figural and nonfigural stimuli need to be used.'

Other research findings (Smith, 1991) make it seemingly unacceptable to view spatial ability as an innate ability but one which can and should be taught. Bishop (1980, page 259) is also of the opinion that it is unacceptable to speak of 'mathematical ability' as innate, he stated:
'An 'ability' has the flavour of an individual difference, possibly
inherited, but certainly given with each child. 'Abilities' on the other hand, are described in more teachable terms, possibly capable of development within each child, but certainly there as objectives.'

Smith (1991) recommended that pupils who do experience spatial activities in the classroom should be encouraged to draw their observations rather than to write about what they see. What is so frequently the case is that pupils are asked to write about, not draw, what they see so as to somehow 'intellectualise' it.

Lean and Clements (1981) suggested that spatial skills are trainable but at a young age and warned that many studies seemed to prove that there may be an increase in performance in a particular test but that this may be due to retention or transfer rather than an increase in ability.

An alternative view is that there is an advantage in teaching spatial skills particularly since the skills are cross-curricular. A study which was carried out by Tuckey and Selvaratnam (1993) in relation to stereo-chemistry teaching concluded that spatial ability is not innate but has to be acquired and can be taught. They stated that a logical solution to a 3-D problem requires a stepwise approach (i) visualise making use of depth cues (ii) orientate (iii) understand the translation required and (iv) visualise how the positions (of atoms) change. They too found a correlation between spatial skills and achievement in chemistry.

Bishop (1978) and Mitchelmore (1976) have also concluded that there is value in creating spatial training programmes but suggest that a wider agenda is needed. However, in a later paper Mitchelmore (1980) warned that passive viewing of 3-D representations is unlikely to be effective in training spatial ability. This suggests a resonance with Mason's (1992) view expressed earlier about working on images.

Battista et al. (1982) were interested in whether instruction in geometry could improve spatial ability. They measured spatial ability
using the Purdue Spatial Visualisation Test ${ }^{9}$ and studied the effectiveness of a geometry course given to 82 pre-service teachers in improving their spatial ability score. They found that the activities used in the course, which included investigating the symmetry of polygons by manipulating concrete models, paper folding, tracing and using a Mira10, did improve spatial ability. Battista et al. (1982) also suggested that it is important to include imagery in early years of mathematics teaching (concrete operational stage of cognitive development) since pupils' thought is greatly dependent upon concrete and pictorial representations.

Clements and Battista (1992, page 426) supported the need for teaching programmes to improve spatial ability:
'Space intuitions . . . . . do not develop inevitably into increasing correspondence with pure logic or mathematics.'
and:
'Progress from one . . . (van Hiele). . level to the next is more dependent on instruction than on age or biological maturation.'

Earlier in this chapter it was noted that Mason (1991) acknowledged that the skill of visualisation may need awakening in pupils. Mason has various suggestions for enabling this, including animations, sequences of diagrams and the practice of imaging tasks.

In this context, Ben-Chaim et al. (1989, page 50) reported that research findings support the notion that visualisation skills are teachable. They believe the skill to be an important one for mathematics and stated that:
'Visualisation provides the learners with additional strategies potentially enriching their problem solving repertoire.'
and (Ben-Chaim et al., 1989, page 58)
'Spatial visualisation topics and activities should be explicitly taught throughout the mathematics curriculum. . . . . . Teachers need to purposely explore the relationships between the geometric

9 One example of an item on this test was shown earlier in Figure 2.2.
10 A 'Mira' is a two way mirror, it acts as a mirror but is also transparent.
topics and the different aspects of spatial analysis.'

School texts are now placing more emphasis upon developing spatial skills and providing tasks which involve imagery. One example, from Harper et al. (1988, page 38), is shown in Figure 2.4.
a) Three of these shapes can be posted through this slot.






b) Which of the remaining shapes can be posted through this slot?



Source: Harper et al. (1988)
Figure 2.4 Imagery task from a secondary school mathematics text book

Commenting on the work of Kent \& Hedger (1980), who
encouraged creativity through visualising and thinking in moving pictures, Bishop (1989, page 13) stated:
'Certainly the teaching shown in Kent and Hedger's study (1980) appeared to be very helpful, from the perspective of visualisation.' Kent \& Hedger (1980) believe that pupils work better with pictures in the mind than with abstract symbols and view visualisation and transforming pictures in the mind as a dormant and valuable skill.

### 2.7 Mathematical topics which make use of visual imagery

Reflecting on his own mathematics education and describing his need for a comprehensible mental image for the objects in the course he studied on topology, Dawson (1988. page 31) stated:
'Now number theory I could grasp, but topology was a different matter. There, again, I could not create images in my mind. Yes I could 'learn' the theorems, I could 'do' the tests, I even passed the topology course with an excellent mark, but I knew deep inside, that I did not have the foggiest notion as to what topology was all about, because I did not have the necessary images with which to work.'

There are topics in mathematics where the ability to visualise is essential and others where it is desirable. The most fundamental circumstance where visualisation skills are required is in the interpretation or drawing of 2-D representations of 3-D objects or situations. This can be extended to include predicting what an object might look like when viewed from a different direction. Questions at GCSE level now involve problems of this kind, such as imaging a cube and transformations of a cube with letters painted on each of the faces. The question which pupils may be asked to work out is what letters are on the hidden faces. Figure 2.5 shows a question from a recent GCSE paper (1998).


These diagrams show two views of a cube.
Its faces have the letters $A, B, C, D, E$ F on them
(a) Which letter is on the face opposite A?
(b) Which letter is on the face opposite B?

Source: WJEC (1998)

Figure 2.5 Imagery problem with a cube

Fielker (1993, page 23) has written an entire book on the topic of mental geometry and offered a similar 'imagine a cube' task:
'Start with a cube. Cut a little piece off each corner. What have you got now? How many faces has it? What shape are they? How many of each shape? How many corners?'

In secondary school mathematics, some common topics requiring imagery include, drawing or imagining cross sections of solid, or solids of revolution in the calculus. Topics from areas not related to geometry which would employ visualisation are directed numbers and fractions. When teaching pupils about directed numbers a common context which teachers use is that of a lift travelling to floors above or below ground
level (zero). The image is helpful in developing the concept - just the kind of image that Dawson (1999) was looking for in the quote at the start of this section.

In dealing with fractions, many pupils find it helpful to look at pictures of the situation. For example, Clements (1980) gave 12 year olds the following questions in a test. The calculation required was $1-1 / 4$ and it was given in two forms, symbolically as:
'Write in the answer $1-1 / 4=$ '
and verbally as:
'A cake is cut into four equal parts and Bill takes one of the parts. What fraction of the cake is left?'

About $50 \%$ of the children got the first version correct whereas over $75 \%$ got the second version correct. Clements (1980) found on interviewing the children that some of them had found imagery helpful in the second version. Pupils do need to learn how to do the former, formalised version, however, when pupils are moving away from the need to see a diagram, towards working with abstractions, the ability to visualise can be helpful.

Some less abstract thinkers could improve their ability to deal with problems by strengthening their powers of visualisation. A similar situation to the fractions of a cake task applies to basic operations with numbers where the use of imagery can be helpful. Wheatley (1991) suggested that rather than posing the abstract problem 36-29, it may be beneficial to some pupils to put the task in a meaningful setting. For example, ' 36 chairs are needed for a school party. 29 are already in the room. How many more are needed?' Furthermore, learning that combining 6 and 4 to make 10, leads on to the ability to deal with large numbers whose cardinalities have not been directly perceived, similarly, so can spatial abstractions become possible. Situations that merely point to the possibility of performing operations, can be manipulated mentally
without actually performing the operations ${ }^{11}$.
Another example where fractions occur is in intuitive proofs such as the sum of an infinite series. For example the series

$$
1 / 2+1 / 4+1 / 8+1 / 16+\ldots \quad \text { sums to } 1
$$

whereas the series

$$
1 / 4+1 / 16+1 / 64+\ldots \text { sums to } 1 / 3
$$

To be able to form a picture of the series and see 12 that it converges can be useful both in understanding and recalling by imagining the pictures representing the series shown in Figure 2.6.


Figure 2.6 The 'pictures' of two infinite series converging to 1 and to $1 / 3$

A well known puzzle involving jugs of water is also a suitable problem which can be helped by visualisation. The puzzle is to measure 4 litres of water using only jugs of capacity 3 litres and 5 litres. The solution requires several steps. Firstly, the 5 litre jug is filled and some of the water used to fill the 3 litre jug, thus leaving 2 litres in the 5 litre jug. The 3 litre jug is emptied and the remaining 2 litres is poured into the 3 litre jug. The 5 litre jug is refilled and one litre pour off into the 3 litre jug which is currently holding 2 litres. 4 litres now remain in the 5 litre jug! The possibility of seeing the jugs rather than just a string of number equations offers some help in understanding the problem.

Visualisation can also be helpful in forming a conjecture by generalising from a visual pattern. Figure 2.7 shows an ancient visual
11 For example, if two identical rectangles are placed one above and one of them rotated about the centre, the question 'what is the shape of the overlap' may be posed; it may not be necessary to perform the mental rotation in order to come to a decision.
12 Mason (1988) has discussed the use of the verb 'to see' in relation to awareness and understanding
proof that the sum of the sequence of odd numbers is equal to the sequence of square numbers. This type of visual proof can sometimes be beneficial in explaining proof to students.


Source: Nelsen (1993)
Figure 2.7 The geometrical representation of the sum of the odd numbers

Logic problems known as 'linear syllogisms' can also be solved effectively using imagery. For example:

Anne is taller than Jean
Mary is shorter than Jean
Who is tallest?
Kaufmann (1985) reported that subjects in research studies may solve this
problem by combining the premises into a unitary representation, that is standing the characters in a row in the form of a mental image and 'reading off' the answer.


Source: Clements and Wattanawaha (1978)
Figure 2.8 Dissection of a cube task

Clements and Wattanawaha (1978) devised a series of spatial tasks which they used to test visualisation abilities. These materials may have been devised to test visualisation abilities but they may also be used to develop these abilities. Bishop (1980, page 258) has remarked about such test materials:
'these are available to other workers . . . . which could be used . . . as tasks for training spatial abilities. They can also stimulate the
development of teaching material for classroom use.'
Some of the tasks require knowledge of the conventions in drawing 3-D objects, for example, in Figure 2.8 which shows a dissection of a cube task, the reader needs to understand the significance of the dotted lines. Other tasks, such as the paper folding task below in Figure 2.9 are more intuitive.


Source: Clements and Wattanawaha (1978)
Figure 2.9 Paper folding task

Visualisation skills and the use of imagery have become
recognised as an issue in the teaching of mathematics in recent years. Fielker's (1993) book entitled 'Mental Geometry' also includes the paper folding task which is shown in Figure 2.10.

## Fold a sheet of paper in half

Cut something out of the fold, so that when you unfold the paper you will have a square hole.
Try again, but this time end up with a triangular hole. What
about a circle? A shape with five sides?
Fold a sheet of paper in half, and then in half again
Cut off the corner, like this.
What shape will the hole be when
you unfold the paper? Think about
it before you unfold.
How can you cut to make
a square? A rhombus?


Fold a sheet of paper in half, in half again, and then like this, to make an angle of $45^{\circ}$

Cut off the corner.
What shape will you have when
you unfold?
What different shapes can you get,
always with one straight cut?


How can you fold a piece of paper so that you can make an 8 -pointed star with one straight cut?

Source: Fielker (1993)
Figure 2.10 Paper folding task

A Welsh Office funded project (McLeay 1988) resulted in four teaching packs, all related to geometry ${ }^{12}$. One focused on knots and

[^10]Chapter 2
incorporated mental imagery. Figure 2.11 shows an example of the tasks (McLeay 1991).


Source: McLeay (1991)
Figure 2.11 Knots mental imagery tasks

This focus on developing mental imagery has been confirmed by the fact that SMP, one of the major secondary school texts, has produced a book specifically on this topic (1994). This book not only includes tasks
on 2-D shapes and rigid object such as dice but also tasks on knots. Figure 2.12 is taken from SMP (1994).


For each drawing, if you pulled the two ends of the strings would you make a knot?

Source: SMP (1994)

Figure 2.12 Mental imagery tasks with ropes

### 2.8 Computers and visualisation

Some researchers in the field of imagery are enthusiastic about the possibilities offered to learners by computers. They (for example Ernest 1989b) believe that the computer can help in developing visualisation skills. Ernest (1989b, page 20) claimed that some computer programmes provide:
'the opportunity to utilise the following processes and strategies: visualisation skills and visual thinking in solving the problem . .' and (Ernest, 1989b, page 25)
'the microcomputer has tremendous classroom potential because it is: . . . . visual - it offers exciting displays which aid spatial visualisation abilities;'

Love and Tahta (1991) focussed on the possibility of exploring the structurally stable states of geometry utilising the 'underlying dynamic' of varying that which may change and noticing that which may not. They wrote (1991, page 259):
'If imagery can provide an entry into this 'underlying dynamic', then it is certainly true that computers now offer a very powerful method of working directly with images. As Gattegno emphasised, they also offer entry into controlled transformations of images, a dynamic which he characterised as algebraic.
With regard to our understanding of imagery, Love and Tahta (1991, page 259) continued:
'Certainly it is likely that increased experience of working with computers, as well as increased digestion of the findings from the field of artificial intelligence, will change our understanding of imagery drastically in the next few years.'

Bishop (1989) reported that the computer can be a powerful tool to develop visualisation although it does not help in getting at the process of visualisation.

Whilst recognising that it is important for mathematics teachers to employ the latest technology to enrich their pupils' experiences, Mason
(1988) urged teachers to try to develop the skill of visualising in their pupils rather than allowing the technology to do it all. Mason (1988, page 299) commented upon the importance of personal imagery as distinct from animations on a computer screen:
'Computer animations can display such dynamics beautifully, but they yield a different experience from doing it yourself in your head.'

Computer technology does offer teachers a great resource for developing imagery but it needs to be used wisely. Love (1995, page 125) urged:
'If we are to be clearer about how the computer might be used as a stimulus for learners' developing control of their imaginations, we need to consider the purposes of the earlier stimuli and how their features are utilised.'

Kaput (1992, page 515) discussed the impact which technology has had on the curriculum in mathematics as follows:
'These changes affect the decisions on what mathematics should be included in the school mathematics curricula. But the same technological forces that shape the mathematics also deeply affect the teachability and learnability of mathematics, both new and old.'

Love (1995, page 127) described how an activity which previously he had used with pencil and paper (the earlier stimuli) was adapted to the computer and where the opportunity for learning became much enhanced with the use of a dynamic geometry package such as CABRI14:
'Mark three points $A, B, C$ (at the vertices of a small acute angled triangle).
Mark a fourth point $P$. Reflect $P$ in $A$; reflect the image in $B$, and

[^11]reflect this new point in C. Continue reflecting $A, B, C$ in turn, joining each point to its reflection. What happens? Is it coincidence? Choose another point $P$ and repeat the process.'

The last request to 'repeat the process' is carried out in CABRI by simply dragging $P$ to various different positions. With pencil and paper perhaps two or three other points $P$ may have been explored by the pupil. In CABRI it is possible to explore many more and the fixed nature of the geometry and 'underlying dynamic' is immediately seen 'dynamically'. The student can then make conjectures about what might happen in special cases for A, B and C. Some personal imagery can take place at this stage in order to visualise what happens for, say, an equilateral triangle $A B C$ or a right-angled triangle and this may be quickly confirmed by the dynamic geometry on the screen. Figure 2.13 shows a general case from Love (1995).


Source: Love (1995)
Figure 2.13 Reflection in three points, scalene ABC

Love (1995) also warned that the computer must be used in conjunction with other visualisation activities or :
'It may be that the power of computers to present visualisations will cause working geometrically, in the imagination, to atrophy
just as surely as the advent of printing made the image-based 'art of memory' fall into decay.'

The use of computer animations or simulations has been shown in another study to aid learning. Kaput (1992) carried out a comparison of a teaching programme using Dienes blocks with another using a computer simulation of Dienes blocks. The findings suggested that the use of the computer simulation led to increased understanding of the number system and the algorithms built on it. Kaput (1992) used the term 'constraint-support' (CS) structure to describe interactive systems such as computer programmes. The successful students internalised the CS action to build their own knowledge structures. The computer system also reduced the cognitive strain by handling some of the translation activities with the blocks which otherwise would have to be carried out by the student whilst performing the calculations.

A study of the utility of computers in aiding visual thinking and geometry learning was carried out by Clements \& Battista (1990). They worked with children using LOGO to investigate whether LOGO does facilitate children's transition from the visual to the analytic level of thinking. Subjects were given LOGO programming training and, through interviews, their level of conceptual development was compared to a control group. The conclusion was that LOGO programming enriches children's geometric conceptualisations and develops the sophistication of their geometric thinking.

In a further study, Battista et al. (1991) described working with LOGO with young children (grade 2 and grade 5) to produce squares and tilted squares. Interviews confirmed that imagery and visual reasoning were practised. The grade 2 children were able to imagine that the square which they had drawn on screen 'in a tilt' could be turned to the standard orientation using spatial imagery. The grade 5 children were asked which of a range of quadrilaterals could be generated using a given rectangle procedure in LOGO. The reasoning practised by the grade 5 children included generating an image, inspecting an image and
transforming an image.

The use of imagery is also evident in the use of LOGO to construct turtle paths as described in Clements et al. (1996). A mental prediction first had to be made in order to decide upon the instruction which must be entered into the programme in order to create a path or to close a shape. Clements et al. (1996, page 332) explained that children in their study were less proficient in combining turn commands than forward and back commands. This could be due to the fact that forward and backward movements result in a line, whereas a turn does not. They explained; 'The turning motion itself usually does not leave a trace (and the line of sight must be constructed from memory)'. They added that the tasks were instructive and that the subjects, 'evinced a progressive building of imagery and concepts related to turns.'

The introduction of software, such as LOGO and CABRI, has opened up new possibilities in the teaching of geometry and spatial visualisation. LOGO encourages the practice of generating, inspecting and transforming images whilst CABRI can, to some extent, replace the concrete inflexible diagram of a particular figure, perhaps a circle drawn in an exercise book or pictured in the mind, with the more dynamic imagery which is desirable for a child to be able to operate with generalities. Thus a moving and deformable image may be viewed on the computer screen and not just imagined.

We not only need to offer these experiences at the computer to children, but also to consider personal image formation and also how these images can be controlled and developed. Knowledge of the genesis of imagery over time, its subsequent development in early life, and an acknowledgement of what is easy to visualise and what is not, are all important factors. The spatial skills which constitute the particular interests of this thesis are those which are needed to consider a viewed object, to speculate on how it might change and then to administer the changes mentally.

### 2.9 Concluding thoughts and questions

A matter of some interest to mathematics teachers is what might foster or inhibit the use of imagery and visualisation. Computers may offer support for this, given the right activities. The usefulness of computers in this regard may support the picture in the mind notion of imagery. Are visual images pictures of diagrams or interpretations of diagrams, or may they be both? If they can be both, how are the two linked and what are the consequences of these variants for mental processing of images? Some of these process aspects of visualisation are considered next in Chapter 3.

Researchers have put forward different reasons and benefits for how and why pupils use visualisation in problem solving. Can we identify more closely how imagery is used for reasoning in mathematics, which children are likely to use it and in which topics? Much research has been carried out to discover what are the 'abilities' which are needed in doing mathematics, is it possible to teach these abilities?

All of these questions are worthy of consideration and will be discussed further in Chapter 4.

Finally, included here are some instructions on how to tie a bow tie. It is possible that many people would find these sets of instructions shown in Figures 2.14 and 2.15 very difficult to comprehend. We have a duty as educators to improve the development of imagery and spatial awareness.


Source: Lanvin bow tie packaging
Figure 2.14 How to tie a bow tie


Source: Gombrich (1990)
Figure 2.15 How to tie a bow tie including verbal instructions

## Chapter 3

## The psychological perspective on mental imagery

### 3.0 Introduction

This chapter considers the psychologist's approach when studying imagery which is necessarily different from the approach of the mathematics educator's discussed in the previous chapter. Psychologists are more concerned with the nature of a mental image and the process of how images are formed than with the effect that imagery has on learning. Psychologists observe individual differences and wonder why they exist whereas the educator worries about what can be done about them.

This chapter examines the psychology research on imagery starting with the long-standing debate on the nature of a mental image and the early work of Roger Shepard (for example, 1971, 1973, 1990) and his colleagues (principally Cooper and Metzler). Following the major study of Shepard and Metzler (1971) on mental rotation, much interest was shown in fully understanding whether an image was visual or spatial or verbal (propositional). Many researchers have tried to resolve this 'imagery debate' by using a variety of different tasks and methods. These research studies will be discussed together with some of the ingenious experiments which have been devised ranging from mentally folding squares to form a cube (Shepard and Feng, 1972) to imagining letters rotated and then placing them one above another (Finke, Pinker and Farah, 1989).

### 3.1 Early work on imagery

In some manner, information about the world is represented in the brain. Following the philosophical contributions of Aristotle, Descartes and others who took the view that images in the brain were pictorial,
behaviourists working in the 1920s, such as Watson (see Skinner, 1953), put forward the theory that imagery was entirely verbal. This theory was challenged once more in the 1960s when faith in behaviourism declined and much empirical work on mental representations and imagery commenced. This debate has continued to engage the interest of many psychology researchers.

The early work on mental imagery was concerned mainly with the format of mental images, whether they are 'propositional' or 'pictorial'. Of particular note was the work of Paivio (1971), Pylyshyn (1973) and Kosslyn (1980). Paivio (1971) began by studying verbal learning and predicting how well a set of words could be memorised according to how easily one could visualise their referents. He found that the use of visual imagery led to better memory performance than verbal rehearsal. As a result, Paivio (1971) put forward a dual coding theory whereby he suggested that images are richly pictorial and that the memory stores information both verbally and pictorially. Paivio (1971) claimed that the pictorial system is better for spatial processing whereas the verbal system is more useful for serial or sequential processing (remembering the order of objects). He summarised his results thus (1971, page 242):
'The number of items correctly remembered in such tasks uniformly increases from abstract words, to concrete words to pictures. This does not occur in tasks such as immediate memory span and discrimination of recency, which principally involve memory for the order of items. These findings are generally consistent with the view that either images or words, or both can serve as effective memory codes for the retrieval of item information. Non verbal images may suffice in recognition memory, which does not require a verbal response, but the verbal code must be stored along with the image, or be retrievable from $i t$, in the case of tasks such as free verbal recall, which require a verbal response.'

Pylyshyn (1973) was an advocate of the structural descriptions (verbal) theory and was quick to attack the depictive mental images
theory asserting that all internal representations must be propositional. Pylyshyn's (1973) argument rested on the claim that information could not be stored in the form of mental photographs because there would be too many of them and the brain has no means of organising such a vast array.

Kosslyn, who took the alternative pictorial view, has continued to develop his pictorial theory of imagery, his first version being described in 'Image and Mind' in 1980 and developed later in 'Image and Brain' in 1994. Kosslyn's theory is described in more detail in section 3.3 of this chapter.

The debate between the two factions and their preferred theories was summed up neatly by Clements (1981, page 5):
'Somewhat ironically, Kosslyn and Pomerantz (1977) criticised the propositional representation theory for the same reason that Pylyshyn criticised picture in the mind theories. If Pylyshyn could ask how, according to Paivio's dual coding theory, verbal information and mental images could be transformed into each other, then Pylyshyn himself could be asked a similar question with respect to the propositional representation theory; how can information be transformed from an abstract representation into verbal information or a mental image?'

Much experimental evidence has now been produced but Shepard \& Metzler (1971) and Cooper \& Shepard (1973) were the first to show that the processes of imagery could be studied scientifically and that valid and reliable results could be obtained.

### 3.2 Shepard and Metzler's experiments

Early work by Shepard and Metzler (1971) concerned the mental rotation of visually observed polyhedral figures. These researchers presented pairs of 2-D polyhedral figures depicting 3-D structures comprised of cubes. Pairs consisted of identical or different structures presented at
different angular disparities. The question was asked 'Are these two objects the same?' with decision times to respond 'same' or 'different' recorded. A typical pair is shown in Figure 3.1


Source: Shepard \& Metzler (1971)

Figure 3.1 Example of Shepard \& Metzler polyhedral structures

Decision time was found to increase with degree of rotation, in fact a linear function described the relation between decision time and orientation difference whether or not the rotation was in the picture plane or in depth. The graph relating decision time to orientation difference is shown in Figure 3.2.

Shepard and Metzler (1971) described the process of decision making as one of mentally rotating one figure of the pair to attempt to match the other and hence more mental rotation was required for pairs with more angular disparity. Reflecting on these early studies, Cooper and Shepard (1990, page 125) observed that it was necessary to justify the proposition that the imagined rotation imitated the actual rotation and commented:
'It is tempting to view the imagined rotation as the internal simulation of an external rotation. Such a description, however,
would be justified only if we could demonstrate that the internal process passes through intermediate states corresponding to the intermediate orientations of a physical object rotating in the external world.'


Source: Shepard \& Metzler (1971)
Figure 3.2 Graph showing the linear relationship between decision time and orientation difference

Further research by Cooper and Shepard (1973) sought to justify their claim concerning the imagined rotations and details regarding this question are given in section 3.4 of this chapter. But whether or not the imagined rotations do emulate real objects, the results of the mental rotation experiments suggest that we do possess mental images, and that
they are somehow manipulable in mental space.

### 3.3 The nature of a mental image - the 'imagery debate'

Psychologists cannot agree on the nature of a mental image. Most theories fall into one of two groups - those that liken mental images to pictures and those that liken them to linguistic descriptions. The pictorial protagonists, such as Kosslyn and Shwartz (1977) do not insist that literally we have pictures in our heads but that our image representation is in a way something similar to the organisation of pictures and that pictures can be built up from information stored in the brain.


Source: Kosslyn and Schwartz (1977)
Figure 3.3 Computer generated surface representation image of a car
(Note that different letters indicate the recency of being 'refreshed' by the model.)

Kosslyn and Shwartz (1977), were able to simulate the process of visual imagery by computer which, they claimed, made the depictive theory plausible. They were able to model how people represent information in, and later retrieve information from, visual mental images. A distinction was made in their model between a 'surface' image which is quasi-pictorial, retrieved quickly and which fades in clarity
towards the edges, and a 'deep representation' which might contain additional facts about an image and could enable construction of different situations for the image. An example of one of their images in a surface representation is shown in Figure 3.3. Kosslyn and Shwartz' (1977) model embodied earlier empirical findings, including those of the first part of their own 1977 study, and the resulting theory was supported by the 'sufficiency proof' that their computer model was adequate to account for some range of data.

The focus of much research has been on how an image is produced and manipulated in the brain. Mental images are notoriously difficult to study since they cannot be put on public display but, as Kosslyn (1994) pointed out, neither can electrons, quarks and black holes. Psychologists have strived as enthusiastically as the physicists have done to work on the challenge to understand the seeming mysteries of imagery.

Logie (1991) considered mental imagery to be a difficult area to explore and commented that if asked to describe a scene from memory, many people would report experiencing a visual scene and a process of scanning their image. Regarding images in short term memory, Logie (1991, page 77) commented:
'One overriding impression is that people can retain such information over periods of several seconds, much longer than it could be retained in a purely sensory store. And yet, the information can be retained only as long as we make a conscious effort to keep it there. Thus the notion of a cognitive mechanism for temporary retention of visual images of scenes has an intuitive appeal.'

Kosslyn (1980) described such a mechanism, the 'visual buffer', together with the subsystems which are necessary to support his theory on imagery. Kosslyn (1980) believed that images are generated rather than retrieved. In his more recent work, Kosslyn (1994) elaborated on the subsystem of a 'visual buffer' for holding an image in short term memory and likened the sensation of image to a display on a cathode ray
screen. Another subsystem which Kosslyn (1994) called 'associative memory' is important in spatial relations and patterns. Information is stored which carries object details and can be retrieved so as to activate an image. Kosslyn (1987, page 149) asked us to consider how we might answer these questions:
'Which is darker green, a Christmas tree or a frozen pea?'
and
'Which is larger a tennis ball or an orange?'
The information we must retrieve from our memory is not direct factual knowledge, and presumably has not been explicitly considered previously (unless we have been asked the question before!). Imagery must be used when the sought information is not stored as a piece of existing knowledge (as when comparing, say, a mouse and an elephant). However, we do not experience images all the time. Kosslyn (1987 page 154) remarked:
'When we need an image, it is generated on the basis of stored information. For example, if you are asked to describe the shape of Snoopy's ears, you probably form an image of the dog's head; but you probably did not have the image until you tried to answer the question. The image comes to mind, is generated, only when you need it. The image is a transient representation in short term memory that is generated on the basis of information stored in long term memory.'
The experience of imaging Snoopy in order to find out about his ears is a very different experience to merely thinking about Snoopy without forming any image ${ }^{1}$.

A diagrammatic summary of Kosslyn's theory and sub-systems has been designed by Tye (1991) who put forward his own composite theory of mental imagery, favouring Kosslyn's pictorial theory but adapting it to include some linguistic additions (see Figure 3.4). Tye's (1991) view is that images are interpreted symbol-filled arrays.

[^12]

Source: Tye (1991)
Figure 3.4 Basic components of Kosslyn's theory

Kosslyn (1994) described the notion of a 'foundation part' of an image and stored concomitant spatial relations with other parts of the global image. Kosslyn (1994) believed that certain transformations, which access stored information, are particularly difficult to achieve when the foundation part of the image is changed or distorted.

Lowe (1987) described a property similar to Kosslyn's foundation part. Lowe worked on image matching in artificial intelligence and developed a model for a mental image with a computer vision system. He explained the use of 'trigger features' in his model for object recognition, which can be a part of the image focussed upon so as to reduce the amount of search that would otherwise be required in recognising the object. Lowe (1987, page 356) expressed it thus:
'While it is true that the appearance of a three-dimensional object

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can change completely as it is viewed from different viewpoints, it is also true that many aspects of an object's projection remain invariant over large ranges of viewpoints (examples include instances of connectivity, collinearity, parallelism, texture properties and certain symmetries).'
Perhaps Lowe's ideas offer some explanation for Shepard and Metzler's constant gradient graphs for picture plane and depth plane rotations.

Logie (1995), writing in the area of 'working memory', considered that there was some overlap with his own notion of working memory and Kosslyn's memory subsystems. Logie (1995) described the functional role of imagery in certain tasks and the picture superiority effect. This effect occurs when subjects are asked to perform tasks involving mental comparative judgements. The tasks are performed more quickly when subjects are shown pictures of the objects rather than given their names. Logie (1995), reflecting on all the experimental work carried out on imagery, posed a tongue in cheek question: 'What is imagery used for outside of cleverly designed laboratory experiments?' to which Logie (1995, page 35) himself offered the following practical example:
'Trying to find a building in a town can be a lot easier if we can generate in advance an image of the building from a verbal description of its size relative to neighbouring buildings. Likewise trying to find a four-centimetre nail among a pile of nails of varying sizes is much easier if we can create an image of the nail in its correct size before starting the search.'

The sub-title of Kosslyn's (1994) book 'Image and Brain' is 'The resolution of the imagery debate', which suggests that the issue is now closed. Not everyone would agree with that testimony but Tye (1991, page xiii) commented about Kosslyn's work of the period up until 1991:
'If Kosslyn is correct that the pictorial view provides the best explanation for a wide variety of experimental results, then it appears that the historical philosophers had the right conception of imagery after all (though for the wrong reasons)'

### 3.4 The evidence: Are images visual or spatial?

Cooper and Shepard (1990) provided a review of the evidence for imagery in mental rotation from the various studies carried out by each of them with their co-workers. Cooper and Shepard (1973) carried out experiments with subjects on mental rotation of rotated letter forms ( $F$, G, J, K and R) printed normally or as mirror images. They found that subjects needed to rotate the forms to the upright position before making judgements about the nature of the letters. This suggests that subjects are viewing some internal visual image in order to make a decision. Cooper and Shepard (1973) suggested that these results indicated that images do go through intermediate states in the process of being transformed, and hence are pictorial.


Source: Cooper (1975)
Figure 3.5 The 2-D polygons used in Cooper's experiment

Cooper's (1975) experiments required subjects to 'memorise' a set of 2-D polygonal shapes in both standard and reflected versions (see Figure 3.5). Subjects were then shown a single image with some rotation from the memorised version and asked whether it was the standard or reflected form. The results showed a linear increase of decision time with degree of rotation from the memorised position. Commenting upon this and further work on random polygon figures, Cooper and Shepard (1990, page 131) claimed:
'Taken together, our results amount to objective evidence of a mental process that models the rotation of objects in the physical world.'

Strong evidence that images are transformed in the visual buffer was found by Shwartz (1981, reported in Kosslyn 1994, page 355). Shwartz (1981) showed his subjects random angular polygon figures and asked them to mentally rotate the figure presented a specific number of degrees. When the subject had done this a second figure was presented. The subject had to make a decision as to whether or not the figures were identical. Shwartz (1981) found that subjects' decision times increased progressively with more rotation from the upright.

Some opposing evidence was put forward by Finke and Schmidt (1978) who claimed that their subjects, when attempting to 'image', experienced perceptual after-effects of imagined line orientation but not imagined colour. The authors contended that this suggests that images are spatial but not visual. On the other hand, a study carried out by Intons-Peterson (unpublished paper reported in Farah et al. 1988) showed that subjects required less time to form an image when the colour of the image matches the colour of the perceptual surface on which the image is projected, which implies that the imagery is visual.

Hinton (1979), adopted the structural description approach, that parts of an image together with their locations, lengths and orientations are stored, and argued that Shepard and Metzler's task must be done using spatial rather than purely visual imagery since picture plane and
depth rotations took the same time. Hinton (1979) considered that if visual imagery were used, the depth rotation tasks should take longer owing to the complex changes which would be needed to the diagram. Subjects would have to carry out additional foreshortening and hidden line removal operations in the depth rotations whereas these would not be required for the picture plane rotations. Extra time would therefore be needed if the imagery used were visual.

The work of Farah et al. (1988) added to the debate as to whether mental images are visual or spatial. They deliberated as to whether the images are in analogue or in array format, whether they are propositional or descriptive. Farah et al. (1988) claimed there was widespread agreement that images represent some of the spatial properties of visual stimuli in an analogue format (although they acknowledged that there were dissenters such as Pylyshyn). However, Farah et al. (1988) agreed with Hinton's (1979) argument, drawn from Shepard and Metzler's finding that subjects could mentally rotate objects in depth as quickly and accurately as in the picture plane, which suggested that the imagery used was spatial not visual. They stated (1988, page 442):
'Visual representations have also been distinguished from spatial representations based on their perspective properties, such as foreshortening and occlusion. . . . . Thus the finding that subjects can mentally rotate objects in depth as quickly and accurately as in the picture plane, even though the appearance of an object undergoing a depth rotation changes in more complex ways than the appearance of an object undergoing a picture plane rotation, is taken as evidence for the spatial nature of imagery.'

As a result, Farah et al. (1988) proposed that images may entail a combination of formats. They suggested that the broad meaning of the term 'imagery', used as a single term to cover such a diverse range of interpretations, has impeded research to understand it. There is more than one kind of mental imagery, each kind having properties worthy of study. Farah et al. (1988, page 443) explained:
'different mental imagery tasks call upon different kinds of imagery representations, some of which are visual and some of which are spatial. . . . In effect, imagery researchers have been misled by the use of a common term, 'imagery', for what are in fact two distinct types of representation.'
They also noted that early this century neurologists studying brain damaged patients discovered that the identification of visual stimuli occurred independently of, and in a different part of the brain to, spatial localisation. Visual impairments are associated with right temporal damage and spatial impairments with right parietal damage but each can continue to function without the other.

Farah et al. (1988) conducted two clinical case studies of brain damaged patients. They set the patients both visual imagery tasks and spatial imagery tasks. The visual imagery tasks involved colour, size comparisons and shape recognition. Those tasks involving colour typically asked the subject to name the colour of an object not verbally associated with its colour such as a football. The size comparison tasks were like those used by Kosslyn (1987), objects being quite similar in size (Kosslyn mentioned an orange and a tennis ball) and therefore requiring imagery rather than tacit knowledge. The shape recognition tasks required subjects to remember and visualise shapes of maps of various US states and then to name pairs which were most similar out of a number of triads. The spatial imagery tasks were of three types; the rotated letters used by Shepard and Cooper (1973), the Shepard and Metzler (1971) polyhedral shapes and mental image scanning tasks of the type used by Kosslyn, Ball and Reiser (1978). Farah et al. (1988) reported that the results from these patients provide neurological evidence that distinct visual and spatial imagery systems exist in different sections of the brain.

Logie (1995, page 96) conducted a study to distinguish the two memory stores, long term visual memory and the visual buffer, given in Tye's model shown in Figure 3.4. He described some of his findings with brain damaged patients thus:
'These patients could copy drawings and could reproduce patterns if they were asked to draw them immediately after they had been removed. However, they were unable to remember anything of visually presented patterns if there was a brief delay between removal of the pattern and the request to draw, or recognise the pattern.'

Logie (1995, page 96) remarked upon the experience of one of his memory impaired patients who seemed to use verbal codes to supplement poor temporary visual memory:
'if shown a pen for a few seconds, after a three minute delay, he could correctly identify a pen from distractor objects such as a comb, a pencil, or some chalk. However, he was not retaining information about the visual appearance of the pen because a few moments later he was unable to select the presented pen from among several other pens serving as distractors. . . . . . Verbal short term memory and long-term visual memory were unimpaired in (this) patient.'

Similar evidence distinguishing between short and long term visual memory was supplied by Riddoch (1990) in a case study of a left temporo-parietal brain damaged patient. Riddoch (1990) found that her patient could not perform mental rotations but could retain visually presented information for at least 10 seconds. She concluded that the system for generating and manipulating visual images might be distinct from that involved in short term visual storage.

There have been very few reports of brain damaged patients experiencing sudden imagery loss but the data that exist suggest that this loss of imagery is not contingent on damage to the right hemisphere (Richardson, 1991). It seems that posterior areas of the left hemisphere are more commonly involved, however, the clinical literature offers only sparse evidence.

Kosslyn's study (1987) found that the standard view that imagery
occurs only in the right hemisphere, is not correct. Kosslyn (1987) carried out studies of split-brain patients and developed a theory for some of the high level processing subsystems involved in imagery. As well as reporting his findings revealing which parts of the brain are used in imagery, he also reported on the variability in the neuro-psychological literature on lateralisation. Kosslyn (1987) found that the left hemisphere is used for imagery and in some cases is superior at some aspects of mental imagery to the right hemisphere. The right is deficient in devising alternative strategies and cannot generate multipart images. Both sides of the brain are involved in imagery and both can perform many of the tasks, generally the left for image generation and the right for all other sub tasks. One way of testing this theory, Kosslyn stated, would be to construct a simulation model and discover how it lateralises depending upon the parameter values.


Source: Kosslyn et al. (1989)
Figure 3.6 Task used to determine which side of the brain is used for visual judgements of spatial relations

Kosslyn et al. (1986) conducted experiments to discover which
kinds of tasks were better suited to the two hemispheres. The stimuli which they used were randomly shaped blobs with a dot placed nearby. The dot was either on the blob outline, very close to the outline or some distance away from the outline. An example is shown in Figure 3.6. One group of subjects were asked to judge whether the blob was on or off the line and another group to judge whether or not the blob was within 2 mm of the line. They found that the left hemisphere is better at saying if a dot is on or off the outline of a blob whereas the right is much better at judging how far off the line the blob lies. The results are shown in the graph in Figure 3.7


Source: Kosslyn (1987)

Figure 3.7 Graph showing response times for left and right hemispheres

Kosslyn (1987) also explained that transformations are especially difficult in the right hemisphere. He differentiated between so called 'blink transformations' and 'shift transformations'. Shift
transformations involve altering the existing image and tend to be quicker than blink transformations. A blink transformation is when the old image fades and the new one forms by accessing stored information. Kosslyn (1994) explained that blink transformations, which access stored information, are particularly difficult to achieve when the foundation part, or invariant part, of the image is changed or distorted.

Yet another complex aspect of imagery is that required when a subject imagines new situations for a viewed object. Kosslyn (1994, page 350) coined a term for this:
'one can visualise an object that was previously viewed in a static situation and imagine its appearance as it rotates, is stretched, breaks into pieces, and so forth. In this case one is not simply 'playing back' previously encoded memories. I call this a 'motionadded' transformation'
The notion of motion-added transformations will receive more attention in Chapter 9.

A final comment on the mental process which is generally described as the 'mental transformation of an image' comes from Freyd (1987). Freyd (1987) studied dynamic imagery using sequences of images with implicit dynamic information. Freyd (1987) found it helpful to look at the process in a different way and preferred to speak about the 'imagining of transformations' as opposed to the 'transforming of images'.

### 3.5 Mental rotation as a strategy

### 3.5.1 Effect of image size

After Shepard and Metzler's studies, subsequent research concentrated upon similar stimuli depicting essentially rigid objects. Some studies were carried out using objects and images of various sizes. The findings of Shwartz (1981, reported in Kosslyn 1994, page 355) showed that larger objects took longer to process, and hence supported the argument for
mental rotation in the visual buffer. Shwartz (1981) found that subjects required more time to rotate the larger stimuli progressively farther from the standard upright than the smaller stimuli.

Kosslyn (1994) examined Shwartz' (1981) findings and noted that they were not replicated in other studies (for example, Suzuki and Nakata 1988). Suzuki and Nakata (1988) investigated the effect of image size on Shepard and Metzler (1971) shapes ${ }^{2}$ and reported a seemingly conflicting finding to that of Shwartz (1981) who had used 2-D shapes. Suzuki and Nakata (1988) used small, medium and large images of the 3D shapes and found that for 'same' pairs the small images took longer than the large. Figures were presented at near, medium and far distances and it was found that it was the retinal size rather than actual size of the image which was the determining factor for decision time.

It should be noted that the different results of Suzuki and Nakata's (1988) tasks were possibly due to difference between their 3-D shapes and the random 2-D polygons of Shwartz, the complexity of the tasks was different and subjects possibly interpreted and processed the images differently. Kosslyn (1994) offered the explanation that Shwartz' figures were essentially very different from those of Suzuki and Nakata (1988) in that all the information required to judge same/different for Shwartz' figures was at the edges of these figures. Hence subjects would require longer times to scan larger perimeters. Kosslyn (1994) affirmed that this result supported the depictive image theory.

In contrast, Suzuki and Nakata's (1988) images carried more spatial information. Suzuki and Nakata (1988) found a linear relationship between angle and decision time only for same pairs. For different pairs no linear relationship was found. Suzuki and Nakata (1988) also found that mental rotation cannot be asserted to be a strategy for tasks involving 'different' pairs; they found no evidence in their results to support rotation as a solution strategy. Increasing the angular rotation did not result in an increase in reaction time where pairs were different.

[^13]Kosslyn (1994, page 355) in trying to explain Suzuki and Nakata's (1988) result for pairs which were different stated:
'This result suggests that the different stimuli were quite distinct, and could be discriminated purely on the basis of bottom-up matching in the pattern activation subsystem. If so, then the subjects presumably rotated the 'same' pairs because they feared that the members of at least some 'different' pairs were subtly different and hence allowed the imagery feedback process to be completed. . . . . In all of these circumstances, the subjects may scan local details to augment the global image during the rotation process.'

### 3.5.2 Other factors which affect image rotation

The presence in an image of a 'right way up' has been shown to affect how subjects process an image rotation. Reisberg and Chambers (1991) reported an interesting contrast between two ways of reorienting an image, either by the rotation of an image or by the reassigning of the top of an image. Their experiments required subjects to look at a line drawing and to form a mental image of it, after the line drawing was removed the subjects were asked either (i) to rotate the image through $90^{\circ}$ clockwise or anti clockwise or (ii) to reassign the top of the drawing to be the left or the right hand side of their image. Subjects were then asked to inspect their image mentally and to see if it resembled some familiar shape (which was a map of Texas in some experiments). Their results indicated that the method of reassigning the 'top' of an image for rotation tasks was more effective than actual rotation with regard to recalling and reinterpreting the image.

Chambers and Reisberg's earlier experiments (1985) described how subjects were able to recall mental images, but it was not until they drew these recalled images and inspected the drawing that an alternative interpretation was possible. In Reisberg and Chambers' (1991, page 341) later study, one of the experiments required subjects to draw their mental image. Only after the subjects had sketched their image did any of them

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recognise the familiar shape, the state of Texas (see Figure 3.8). Reisberg and Chambers (1991) reported that despite coaching, hints and prior practice no subjects discovered the alternative construal of the image before sketching it, although after sketching of the imaged stimulus 8 out of 15 of them were able to reconstrue their drawing.


Source: Reisberg and Chambers (1991)
Figure 3.8 The 8 drawings which could be reconstrued as a map of Texas

Commenting upon their subjects' ability (or inability) to interpret their images Reisberg and Chambers (1991, page 338) explained:
'Changing how a form is understood can drastically change phenomenal appearance' 'Images are saturated with our understanding of the form and are unambiguous with regard to what they represent.'
and
'learning from imagery will be determined by how the image is understood as much as by the geometry of the imaged form'

In contrast to this result, however, Finke, Pinker and Farah (1989) reported that their subjects were able to use imagery to discover novel interpretations for combinations of patterns. Their subjects were asked to imagine a capital letter D , mentally rotate it $90^{\circ}$ anti clockwise and then
put a capital letter J directly beneath it. The subjects were able to discover the umbrella thus depicted. Similarly, Brandimonte, Hitch and Bishop (1992) gave subjects 'subtraction' tasks involving drawings of familiar objects. Their subjects were able to mentally subtract parts of an image and discover a new interpretation.

Tarr and Pinker (1989) gave subjects extensive practice at recognising several shapes in left and right handed versions and in one specific ('canonical') orientation. They followed this with different orientations of the same shapes and found that, with practice, subjects recognised the shapes almost equally quickly at all the familiar (already seen) orientations. The standard versions of the shapes used by Tarr and Pinker (1989) are shown in Figure 3.93.


Source: Tarr and Pinker (1989)
Figure 3.9 Standard versions of letter-like asymmetrical characters

Tarr and Pinker (1989) claimed that these results indicated that subjects had stored representations of the different orientations of the shapes (not just the canonical form) which they could call upon to respond to recognition tasks, hence no mental rotation was required for

3 Note that the 'bottom' of each image is marked by the short line segment which acts as a foot or base and which is also the only terminating line segment. This was intended to give this 'upright' version 'canonical' status.
these 'known' orientations. The set of shapes were stored in memory including the canonical representation. When novel rotations were introduced, mental rotation was required and extra time was needed to rotate the new unfamiliar image to one of the familiar stored orientations.

Jolicoeur (1988) reported similar findings in experiments carried out on the naming of disoriented objects. Jolicoeur (1988) reported that the most likely explanation for his results was that representations of objects were stored in memory with codes relating to particular orientations and that the matching process became increasingly slower as the orientation of the displayed object deviated from the orientation stored in memory. He found that mental rotation was used to determine handedness but not to recognise shape.

### 3.5.3 Verbal coding

The role of verbal coding is an important one, but when used, does seem to restrict alternative visual construals of the image. A major consideration in the mental representation of any problem solving process refers to the roles of verbal (analytical) or spatial (analogue) thinking and the mixing of the two. Denis and Cocude (1989) showed that subjects could build up (visual) images using verbal descriptions of a map of an island and that the same mechanisms were involved in processing these images as apply to those which are constructed from perception. Analysis of response times for their mental scanning tasks showed that subjects produced the same time/distance relationship as is typically observed when memorising a real map rather than a verbally described one, that is, the longer the distance between two points of an imaged configuration the longer the scanning time.

Brandimonte, Hitch and Bishop (1992) reported on the efficiency of using verbal versus pictorial strategies on their tasks involving subtraction of parts of the image. They found that verbal strategies were often preferred and were commonly used even when these strategies
were ineffective. Brandimonte, Hitch and Bishop (1992) found that their subjects performed better if they were prevented from using verbal strategies. Kosslyn (1994, page 337) considered this to give additional evidence for visual imagery and explained the process as follows:
'In my terms, the subjects would encode the input image of the first object, which is organised into perceptual units by processing in the visual buffer and in the preprocessing subsystem. Given the nature of the task, the subjects would attend to individual parts, and store them along with the global shape in the exemplar pattern activation subsystem. When the second stimulus is presented, it is also encoded and stored. . . . .The resulting image is then 'inspected' in the usual way.'

### 3.6 Comparison of 2-D and 3-D tasks

Much of the research already described in this chapter has used stimuli chosen without specific attention to whether they were 2-D or 3-D. The comparative complexity of 2-D and 3-D object rotations has been explored by Jolicoeur, Regehr, Smith \& Smith (1985) who reported that rotation of 3-D objects takes longer than rotation of the same outlines when portrayed as 2-D images. Jolicoeur et al. (1985) noted that rotation of 2-D shapes takes less time than 3-D where rotation differences are greater than $60^{\circ}$. They inferred that for rotations of less than $60^{\circ}$ subjects were performing a holistic rotation of the entire image regardless of dimensionality but for rotations greater than $60^{\circ}$ this does not seem to be the case. They stated (1985, page 101):
'This aspect of the results raises an intriguing question: Are different processes at work in the mental rotation of twodimensional and three-dimensional rotations?'
It is possible that, for larger rotations, subjects construct a verbal description of the structure of the shapes and that these descriptions are compared rather than the surface images themselves (of the 2-D shapes).

Cooper (1991) suggested that 3-D imagery is 'obligatory' in
constructing representations of objects from two-dimensional information even when the task requires only 2-D information.

Jolicoeur (1988) reported results for identification of some standardised objects, such as a desk or a snail, after 'disorientation' Although many of the images were of 3-D objects the disorientations were all in the picture plane. Jolicoeur (1988) did not explore whether there were any effects due to 2-D or 3-D objects. He reported that disoriented patterns were usually identified more slowly than upright patterns and that recognition was also usually increasingly error prone with increasing disorientation.

### 3.7 The effect of practice

Wallace and Hofelich (1992) investigated whether improvement in one process due to practice of tasks involving that process is transferred to other tasks. They gave subjects the mental rotation tasks of 2-D random polygon shapes similar to those used by Cooper ${ }^{4}$ (1975) and found that both accuracy and speed improved with practice. Wallace and Hofelich (1992) also found that the practice of their tasks improved performance on geometric analogy tasks, where subjects had to perform two image transformations. This improvement also transferred in the other direction, i.e. from geometric analogy tasks to mental rotation tasks.

Kosslyn et al. (1989) found further evidence for practice improving performance. They found that during the course of their tasks involving distance encoding, with practice the left hemisphere develops a new categorical spatial relation.

With regard to the practice of imagery for motor processes, for example in the case of athletes or gymnasts, Kosslyn (1994, page 348) observed:
'there is a literature that imaging practicing can actually improve performance in some situations, provided that imaged practice is

[^14]intermixed with actual practice'

### 3.8 Deformable structures

Although our everyday experience is usually with 3-D objects and often rigid objects, frequently it is not. Clothes, for example, may assume many different shapes, yet remain topologically the same (i.e. homeomorphic). The mental representation of deformable structures, and hence the ability to manipulate them, has received relatively little attention, the closest perhaps being the paper folding studies of Shepard and Feng (1972).

The paper folding stimuli of Shepard and Feng (1972) consisted of single figures comprising six connected squares which when folded formed a cube i.e a net. Two arrows drawn on each figure pointed to the edges of two of the squares. An example is shown in Figure 3.10.


Source: Shepard \& Feng (1972)

Figure 3.10 Net for a cube used in Shepard \& Feng's experiment

The task set by Shepard and Feng (1972) was to mentally fold each

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figure in an attempt to bring the arrow tips into coincidence. In relating their study to the findings of the earlier one of Shepard and Metzler (1971), Shepard and Feng (1972, page 228), stated:
'The question naturally arises as to whether evidence can be found for a similar sort of isomorphism between physical operations and their purely mental analogues when the relevant physical operations are of a more complex nature and require, for example, a sequence of distinct spatial manipulations.'

Shepard and Feng (1972) believed that this type of folding task was most likely to be a test of spatial ability since it was very unlikely to be performed by verbal processes. They stated (1972, page 229):
'Our approach departs from earlier studies of spatial abilities using tasks of this kind, however, . . . . our concern is with the dependence of the time to solve individual problems upon identifiable parameters of those problems (or certain subtypes to which they belong), and with what this can tell us about the nature of the underlying mental process.'

Some of the nets used by Shepard and Feng (1972) involved more foldings than others and some required several squares to be carried along with each fold. It was expected that decision time would increase as a function of the number of foldings required. The results indicated that both the number of folds and the number of squares carried had an effect on reaction time. The results were essentially the same as those for Shepard and Metzler (1971) in that decision time increased linearly with an increase in the number of squares carried in foldings required to close the cube and attempt the matching. Shepard and Feng (1972, page 242) noted:
'Whether these Ss nevertheless did solve the problems by a process of mental folding . . . . or whether they they resorted to some more verbal process cannot be definitely settled.'

A combination of methods is probably the answer to their question.

### 3.9 Summary

Much has been done to examine and understand the processes of imagery. Kosslyn has developed his theory of mental imagery and how it is related to visual perception. Kosslyn's model provides a good explanation for many of the research findings.

The scientific investigation of imagery has clarified many of the processes in the workings of the brain and provided guidance to educators on how these workings may be enhanced. Much information is now available regarding the mental processing of many kinds of images, both 2-D and 3-D, and many kinds of transformations. It has also been shown that certain imagery skills may be enhanced through practice. However, the objects which have been largely overlooked are non-rigid objects where a change in their shape may come about in routine use. It is unclear as to why the investigation of non-rigid structures has not been continued following the Shepard and Feng (1972) experiment, perhaps due to the lack of stimuli having a legitimate systematic basis for variation. The area deserves more investigation, not least because the ability to deal with such objects may or may not denote another spatial skill. This question along with the other issues to be investigated will be considered in the next chapter.

## Chapter 4

## Knots and Knot Mathematics

> "In a knot of eight crossings, which is about the average size knot, there are 256 different 'over and under' arrangements possible . . . Make only one change in this 'over and under' sequence and either an entirely different knot is made or no knot at all may result."

(The Ashley Book of Knots, 1944, page 18)

### 4.1 The history of knot tying

### 4.1.1 Legends

The story begins with Aphrodite. According to the Greek myth, she would rise from the sea each morning and teach the women how to make fishing nets. These nets were surrounded by mystery, due to the fact that they utilised knots, and it is implicit in the legend that knowledge of how to tie knots was a very specialised skill; because of the complex nature of knots, only the divine or select few could understand their intricacies. The belief that knots have a mystical significance is manifest in many cultures. In Europe, sailors believed that the wind could be summoned using 'magic knots', whilst other people believed that knots could cause or cure certain illnesses (MacFarlan, 1983). According to the ancient Greeks a 'Hercules' knot ${ }^{1}$ tied in the girdle ${ }^{2}$ was said to have beneficial effects in preventing illness. Saint Francis of Assisi wore a triple overhand knot in his girdle to signify the three oaths of poverty, chastity and obedience. At the other extreme, 'witches' were

[^15]executed for a variety of evil doings, one example being bewitching others by utilising 'magic' knots.

The Greek legend of Gordius is well known and receives much empathy from those of us who know how difficult it can be to untie a complex knot! Perhaps the legendary knot was a 'mathematical' knot. Mathematical knots are always depicted as 'endless knots' with the ends spliced together and hence cannot be undone. An example of a mathematical or endless knot is shown in Figure 4.1. Alexander the Great's solution using his sword was effective if not subtle, and was in fact the only possible way to undo a mathematical knot!3 The mathematics of knots is discussed later in this chapter.


Figure 4.1 A trefoil knot

### 4.1.2 The earliest knot

Archaeologists would assert that knots existed long before this Greek legend and that they were used by archaic homo sapiens and

[^16]Neanderthals somewhere between 500,000 and 250,000 years ago for the purpose of fitting points onto spears as well as for securing decorative artifacts such as beads and pendants (Warner and Bednarik, 1996). Various materials, usually vegetable, would have been used and thus there is no surviving evidence to support the theory since these materials decompose rapidly. However, the evidence from the artifacts, such as needles and beads suggest that knots must have been in existence.

Tying things together was a necessary skill employed for a variety of tasks. It is known that Neanderthals were sea-going and boats, coracles and rafts required materials to be tied together. Unfortunately, due to decomposition of the evidence we do not know the precise knots which the Neanderthals used. The first concrete evidence of a precise knot dates from 9000 years ago where the knot used is the mesh knot, a knot which is commonly used to this day in making fishing nets. A diagram of this knot is shown in Figure 4.2.


Figure 4.2 The mesh knot (also known as the weaver's knot and the sheet bend)

### 4.1.3 Inventing the first knot

The most likely way that our ancestors discovered the utility of knots is probably by first of all wrapping some fibrous material such as plant fibres around a spear or other object, and then making random tucks in the
strands (Warner, 1996). It is possible that by chance the overhand knot or a hitch of some kind was produced. Fig 4.3 shows the simplest knot of all, the overhand knot ${ }^{4}$.


Figure 4.3 The overhand knot

In order to make the wrapping more secure, more wraps and tucks would be made resulting in some random composite knot. This second tying, where the ends would need to be bent back on themselves inwards, would have occurred only when a strong and flexible woven rope medium was available which could withstand such bends and twists. It seems likely that a date of 350,000-250,000 years ago for the first true knot can be estimated (Warner \& Bednarik, 1996).

### 4.1.4 Evidence of actual knots

Antrea net 7000 BC
A fragment of a Finnish fishing net, known as the Antrea net, was the first physical evidence of a knot to be discovered and this was dug out of a peat bog in 1913 (van de Kleij, 1996). The find consisted of the net itself together with floats and weights. Knots had been used so as to create a sizeable mesh out of fibres (the net measured about 30 metres by 1.5 metres), Carbon dating has verified the floats to be of the period $7280 \pm$ 210 BC. Figure 4.4 shows a sketch of the Antrea net (from van de Kleij

[^17]1996) made with two adjacent mesh knots ${ }^{5}$.


Source: van der Kleij (1996)

Figure 4.4 The Antrea net

The customary knot used for netting is still the mesh knot. Figure 4.5 shows a net in progress. The top section in the diagram shown is the part of the net already made, a netting shuttle is used to trace the path of the lower thread.


Source: van de Griend (1996a)
Figure 4.5 Making a net

[^18]Today, fishing nets are made by machine but the skill of knot tying is still needed to make the inevitable repairs which have to be made to the machine made nets.


Source: Wells (1991)
Figure 4.6 The bowline

Frequently, what is essentially the same knot has been used for a variety of different purposes and given several different names (Ashley, 1944). The mesh knot is one such knot. In its other forms it is also known as the sheet bend and the weaver's knot. When tied in a different way its structure is also in effect the same knot as the bowline shown in Figure 4.6, a knot used for many purposes including climbing and sailing. The only difference between the bowline and the mesh knot is that one continuous strand is used to tie a bowline whereas the mesh knot can be considered to be two distant sections being joined together. If the two loose ends of the mesh knot (or sheet bend) are connected, the result is the same as the bowline with the ends joined. These two possible variations are shown in the diagram in Figure 4.7.


Source: Wells (1991)
Figure 4.7 The bowline and the sheet bend with ends joined
"Ice man"
The Ice Man dates from 3400 BC and was found in 1991 in the South Tyrol (van de Kleij, 1996). This find is still being investigated but many artifacts verify the use of knots in their manufacture, these artifacts include a sewn leather quiver and shoes with laces of leather. Among the knots identified are the reef knot (See Figure 4.8), the overhand knot (Figure 4.2) and the strap knot (see Figure 4.9). A strap knot is similar to the reeving used in making a daisy chain where a slit is made in a strap end (or stalk) and straps (or stalks) are reeved together. However, a strap knot is different from a daisy chain in that the straps are tightened so that the slits are adjacent.

## Swiss neolithic lake dwellings

Swiss villages dating from 3000 BC have yielded important finds consisting of the remains of nets, carrying bags and straw hats which were found preserved in the mud of a lake (van de Kleij, 1996).


Figure 4.8 The reef knot


Figure 4.9 The strap knot

## Danish sites

Various knots have been found at Danish sites including a noose found around the neck of a skeleton dating from 3500 BC, a leister (pronged salmon spear) with lashings made from a row of half hitches and fish hooks attached to a line using the clove hitch dating from 5000 BC (van de Kleij, 1996).

## Egyptian knots

The earliest evidence of Egyptian knots is 1350 BC (Wendrich, 1996) and these were very simple unsophisticated knots such as overhand knots and reef knots. (Presumably the ancient Egyptians were more interested in numbers than in knots!) A typical use would be for making carrying
nets for amphora, commonly made from overhand knots.


Source: McLeay (1994)
Figure 4.10 Half hitch and clove hitch

All these finds suggest that human beings invented knots before numbers, the earliest evidence for a system of written numerals dates back to 3000 BC , compared to knots which date back to at least 7000 BC . So we might tentatively suggest that the topological development of man predates numerical development.

Ashley (1944) in his 'Book of Knots' claims to have been taught by an uncle how to tie a reef knot at the age of 3 years. He also says (1946 page 8),
'the simple act of tying a knot is an adventure in unlimited space. A
bit of string affords a dimensional latitude that is unique among entities.'
This statement offers a clear expression of the fascination which knots have generated over the centuries. As well as for their ubiquitous utility and fascination, knots have also been used as an early form of counting.

### 4.1.5 Knots and numbers

The word 'knots' has a maritime interpretation, it can also mean the unit of measure for the speed of a ship. The origin of the term derives from
the method used to estimate the speed of a ship whereby sailors counted knots on a mariner's logline. A logline consists of a log tied to a line with the line having knots tied in it at regular spacings. The log was cast out behind the ship and, as the line was dragged out over the side, sailors counted the number of knots disappearing over the stern in a fixed time. Thus the speed of the ship in 'knots' was obtained.

Various American Indians used knots as devices for counting. The Zuñi of New Mexico had an efficient method which avoided tying too many knots. They used different knots to signify the different numbers 1,5 and 10. The number 1 was represented by an overhand knot, 5 by a reef knot and 10 by a complex nine-crossing knot, (of which it is now known there are 49 different types!). The Zuñi operated a system similar to Roman numerals whereby the relative positions of these knot numerals was important.


Source: Christensen (1996)

Figure 4.11 Zuñi knots

The Incas used a still more sophisticated device, the Quipu, mainly for keeping records of population and food reserves, they were the early book-keepers. Their knots system required structure recognition, a system which was effective due to the distinctness of the knots. They
used the family of multiple overhand knots - (the trefoil family), whereby the number of lappings signified the number (see van de Griend 1996a). The knots were also tied and arranged such that cheats could not tamper with them without considerable difficulty and risk of being found out. Thus their bookkeeping was assured of consistency and security (van de Griend, 1996a).


Source: van de Griend (1996a)
Figure 4.12 Quipu knots


Source: Flegg (1975)
Figure 4.13 Millers' knots

Until the beginning of this century, German millers had a system of knot numerals for their transactions with bakers (Flegg, 1975). The
millers needed to record the amounts and the kinds of flour they delivered. They did this by using different knots tied in the drawstrings of the flour sacks. Different numbers were represented by special knots rather than by a series of the same knot. The forms of these special knots thus represent a form of knot numerals for $1,2,5,10,20$ and 60 . These are shown in Figure 4.13 as the knots 1, 2, 4, 5, 6 and 7 respectively' ${ }^{6}$ The knots 8-10 were used to signify different types of flour.

### 4.1.6 Knot trickery and knot mysticism

To primitive man, the incomprehensible and erratic workings of knots were attributable to divine intervention. Hence mystical and supernatural powers were attached to knots. Many cultures have revered all things to do with knots, magicians and mystics have frequently played upon this situation and families have used notions from geometry as symbolism for cultural identity.


Figure 4.14 Borromean rings

[^19]The Borromeo family have a family crest known as the Borromean rings (see Cromwell et al. ,1998). This crest can be seen in the palace dating from 1630 on the island of Isola Bella in Lake Maggiore in Northern Italy. The Borromean crest consists of three 'rings' or circles linked in such a way that they are bound together, they cannot be separated (see Figure 4.14). However, no single pair of rings is linked and if just one of the rings is cut then the whole is no longer linked at all and the rings fall apart. There are other examples of these interlaced rings in other parts of Italy, sometimes with varied interlacing and with different properties of connectedness (Cromwell et al., 1998).

The Borromean crest demonstrates an interesting symbolism, perhaps of family unity, expressed in the form of loops embedded in space (a definition of a knot is a loop or loops embedded in space ${ }^{7}$ ). This suggests an understanding of the geometry. The Celts too understood something of the geometry of knots. Cromwell (1993) analysed many intricate Celtic knotted patterns. Celtic knots consist of alternating knots, the endless repetitive and cyclical nature of an alternating knot being used to signify the changing rhythms of life. The artist's fascination with knots is still evident in art to this day. Figure 4.15 shows a sculpture of a trefoil knot made out of a Möbius band (Robinson, 1992).

Magicians too make use of the mathematical (topological) properties of knots and links and many illusions can be produced. One such 'trick' or illusion involves a large number of rings which are linked together to make a long 'chain'. One ring can appear to slip down the whole length of the chain. The illusion is created by the effect of the top ring falling one place (not all the way down the chain but just one position) and releasing the ring below it which falls one position and so on. The observer 'sees' one ring fall the whole length of the connected rings. The method of linkage is shown in Figure 4.16.

[^20]

Figure 4.15 'Immortality', John Robinson's sculpture of a trefoil knot8

8 This sculpture can be viewed in the foyer of the School of Mathematics, University of Wales Bangor and at: http://www.bangor.ac.uk/SculMath/image/immortal.htm


Figure 4.16 Linkages for the key ring trick

Sailors were the early experts in knot tying and would use their specialised knowledge in a more mundane but cunning way. They would tie a special variant of the reef knot to secure their kit bag so as to detect whether the bag had been tampered with whilst their back was turned. A thief would assume that the knot was the simple reef knot and retie the bag in this way. However, the thief knot must be tied in a totally different way, the shape of the knot is the same, the only difference between the reef knot and the 'thief' knot is the position of the loose ends. Hence a knowledgeable seafarer with a trained and careful eye could tell whether his bag had been tampered with.

Early seafarers showed a high level of skill and knowledge about the different forms of knots, famous families of the renaissance period understood something of their properties, Celtic and Islamic art showed an understanding of their topological features. In spite of this, the formal study and mathematical theory of knots did not commence until the end of the last century.


Source: van de Griend (1996a)

Figure 4.17 A reef knot and a thief knot

## 4. 2 Knot mathematics

### 4.2.1 The beginnings

Although knots and knot patterns have fascinated humans for centuries the mathematical study of knots is very recent. The most important aspects of intuitive knot theory are knot structure and the transformation properties of knots.

A fundamental breakthrough occurred in 1898 with the table of knots showing their structures. This table was drawn up by the Scottish physicist Peter Guthrie Tait. Tait's table of knots listed the diagrams of all the possible prime knots ${ }^{9}$ with up to 9 crossings. From that moment, mathematicians set about trying to classify and describe knots. Many properties have now been defined but the topic is very complex and still relatively unexplored. Most of the theoretical work has been done this century beginning with Alexander in the 1920s and continuing to the present day with new knot invariants being described as recently as 1993 by Birman and Lin.

[^21]
### 4.2.2 Chronology

A summary of the main events is shown in Table 4.1, starting from 1680 with Leibniz and the notion that geometrical figures could be considered as aggregates of smaller building blocks. Leibniz tried to formulate basic properties of geometrical figures by using symbols to represent them and tried to combine these symbols to produce other properties. Leibniz did not actually apply this study to knots, but geometry of this kind was new and was the precursor to the work carried out by Alexandre Theophile Vandermonde 90 years later placing knots in the field of geometry. Alexandre Theophile Vandermonde (1735-1796) was the author of the first scientific paper in which a mathematican discussed the problem of constructing a mathematical theory of knots and it contained the folowing quote (Turner, 1996, page 261):
"Whatever the twists and turns of a system of threads in space, one can always obtain an expression for the calculation of its dimensions, but this expression will be of little use in practice. The craftsman who fashions a braid, a net, or some knots will be concerned, not with questions of measurement, but with those of position; what he sees there is the manner in which the threads are interlaced.'

TABLE 4.1
Summary of events in the development of knot theory

| Date | Mathematician | Significant Work |
| :--- | :--- | :--- |
| 1680 | Leibniz | Symbols used as basic geometrical units |
| 1771 | Alexandre Vandermonde | Knots considered within the field of geometry |
| 1800 | Gauss | Sketches of knots as closed curves |
| 1847 | Listing | First attempt at classification of knots |
| 1898 | Peter Guthrie Tait | List of all the knots up to 9 crossings |
| 1928 | Alexander | First knot polynomial |
| 1935 | Reidemeister | Allowed moves for transforming knots |
| 1969 | Conway | New polynomial |
| 1985 | Jones, Homfly | New polynomials |
| 1990 | Kauffman | Bracket polynomial |

Some decades later, Gauss became interested in knots and in how
they could be described mathematically. In papers discovered after his death in 1855, Gauss had made sketches of thirteen different knots with English names written beside them (van de Griend, 1996a). Gauss' interest in electromagnetism and his work on the inductance in two linked circular wires and the concept of winding number, caused him to consider the knotting together of closed curves, a concept of fundamental importance in modern topology and knot theory.

Listing was a student of Gauss and a main founder of knot theory. In 1847 Listing described the concept of an oriented crossing and left and right handedness. He assigned a symbol to each kind of crossing and was able to represent a knot by a series of these symbols. Figure 4.18 shows the two kinds of crossings which Listing defined.


Figure 4. 18 Right and left handed crossings

Listing tried to classify knots of less than 7 crossings using knot diagrams or projections. He was the first person to persist in representing knots as knotted circles and obtained diagrams by projecting these onto the plane. He proposed the first invariant property for a knot, a form of polynomial which later proved not to be invariant, however, he set the scene for others to look for true invariants.

The principal aim of knot theory is to find a collection of knot invariants which is adequate for a distinction to be made between any two non-equivalent knots (No single invariant which can do this as yet
exists).

Otto Boeddicker was also a student of Gauss in Göttingen and had an interest in knots. One of his papers showed diagrams of a pentoil knot in its two forms with the crossings numbered (see Figure 4.18).

Peter Guthrie Tait was the Scot who in 1898 listed all the knots up to 9 crossings ( 82 knots) and also worked on 10 and 11 crossings. Appendix A shows all the knots with crossing number up to 8 (thirty five knots). Tait also developed the concept of unknotting number or Gordian number, that is the minimal number of crossing changes (or cuts) required to reduce the knot to the unknot. After Tait had come up with his table, interest turned away from enumeration.


Source: van de Griend (1996a)
Figure 4.19 Two forms of the pentoil knot as shown in a paper by Boeddicker

Another of the problems which interested Tait was how to tell when two knots are the same, or isotopic, that is when one of them can be deformed by continuous transformation into the other. This problem became known as the knot problem and would not be fully dealt with until the 1990s with the work on knot polynomials by Kauffman and
others. A related problem to isotopy is that of amphicheirality, or the property where a knot will transform into its own mirror image. As early as 1890 Tait made a conjecture that odd crossing number knots were not amphicheiral (would not transform into their mirror image) and hence were different knots. This conjecture has now been proved.

Powerful knot invariants were needed to distinguish between knots, particularly mirror images. It became clear that to study the space around a knot ${ }^{9}$ or (link) would be more productive than the study of the knot itself since the space around the knot (or the knot complement) is 3dimensional and carries much spatial or topological information. The notion of a knot group was born. The knot group manages to capture much of the characteristics of a knot and it is quite rare that two different knots have the same group. However, it was soon discovered that the granny and the reef knot, two simple knots, do have the same knot group. Thus the knot group did not contain all information about a knot and thus did not precisely define the knot. Eventually, knot theorists returned to the idea of using the knot itself for the search for invariant properties.

In 1914 Max Dehn showed that the left and right handed trefoil knots are indeed distinct. He was able to do this by thickening the knot's curve to a tube, removing the space inside the tube so formed and placing a co-ordinate system upon this exterior of the knot.

The next great step forward was made in 1928 by Alexander who obtained a polynomial for a knot. Crucially, differently deformed versions of the same knot yielded the same polynomial. For example, the polynomial for the trefoil is:

$$
t^{2}-t+1
$$

[^22]This is the case no matter how the trefoil knot is deformed. Its drawbacks are that it cannot distinguish between mirror images and also that it fails for some knots of more than 9 crossings.

In 1932 Kurt Reidemeister, who was working in Königsberg, produced a notation for Tait's list. He ordered it and sorted it and gave a numbering system to all the knots. He also produced his most famous 'Reidemeister' moves and theorem. Figure 4.20 illustrates these moves and Reidemeister's theorem of 1932 stated (van de Griend, 1996a, page 233);

If two knots are topologically equivalent their diagrams can be transformed one to the other by some finite sequence of Riedemeister moves.'

Another perplexing fact about knots is that they untie in the 4th dimension. Ashley (1944, page 8) expresses concern at this notion:
'Here is a Mr Klein who claims to have proved that knots cannot exist in space of four dimensions. This in itself is bad enough, but if someone else should come forward to prove that heaven does not exist in three dimensions, what future is there left for the confirmed knot tier?'

Some investigations were carried out in Germany in 1877 (reported in van de Griend 1996a) to test whether psychic mediums, who were supposedly in touch with the 4 th dimension, could untie knots without cutting. Needless to say this proved to be a fruitless quest.

In 1969 John Conway expanded Tait's table of knots and completed it up to 11 crossing knots. The expanded table is shown in Table 4.2.


Figure 4.20 Reidemeister moves
Note: The colouring of the strands illustrates the invariant property of 3-colouring described later in this chapter in section 4.2.3

## TABLE 4.2

Table of knots and crossing numbers

| Crossing number | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
| :--- | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: |
| How many knots? | 1 | 1 | 2 | 3 | 7 | 21 | 49 | 165 | 552 |

The table was further extended to 13 crossings by Thistlethwaite in 1982. The number of knots with 12 crossings is 2176 and with 13 crossings the number is 9988 . It is now estimated (van de Griend 1996a) that there are more than 150,000 alternating knots with 15 crossings.

Vaughan Jones developed a new knot polynomial in 1985 which sparked off an explosion of discoveries in the subsequent few years, notably the Kauffman polynomial and the Homfly ${ }^{11}$ polynomial. We turn now to these discoveries and the notion of knot invariants.

### 4.2.3 Knot Invariants

An invariant is a number or mathematical expression which carries information about a system and whose value does not change when the system is transformed in some defined way. If we compute the value of an invariant for two systems and find that they are different then we know that the two systems are different. However, the converse is not necessarily true, if two systems have the same value for some invariant they may or may not be the same. In the case of knots, two knots may have the same crossing number but they may be different knots.

## Crossing number

The crossing number of a knot is defined as the minimum number of crossings which remain after all unnecessary crossings have been removed (by applying the Reidemeister moves). Finding the crossing

[^23]number of a knot can be a practical task with rope or an exercise in mental manipulation, by unravelling mentally all the twists and extra loops in a knot diagram. Some of the puzzles in McLeay (1994) require this kind of visualisation. Figure 4.21 shows how the original knot diagram with 4 crossings has been reduced to 3 . The crossing number of the knot is 3 .


Source: McLeay (1994)
Figure 4.21 Knots puzzle


Figure 4.22 Rules for 3-colouring knots

## 3-colourability

Some knots can have all the strands coloured in such a way that only 3 colours need to be used and no two colours occur at any particular crossing. The rules for 3 -colouring are illustrated by the diagram shown in Figure 4.22

Some knots can be 3-coloured and others cannot. The property is invariant under the Riedemeister moves since if one diagram of a knot can be 3-coloured so can all other diagrams of any form of the knot. No matter what other arrangement we find of the knot then the diagram for that arrangement will be 3 -colourable. Figure 4.23 shows that the reef knot is 3 -colourable and the figure eight knot is not.


Figure 4.23 The reef knot and the figure eight knot with colouring

## Knot families and knot groups

Knots can be classified into families according to some similar properties of shape or structure. Two examples of knot families are the figure eight family of knots and the torus knots. Figure 4.24 shows the figure eight family and Figure 4.25 shows some exciting pictures of torus knots from the Knot Plot web site at:

[^24]

Source: McLeay (1994)
Figure 4.24 The 'figure eight' family

## Polynomials

The first of these invariants to be discovered was Alexander's knot polynomial. For a n-crossing knot he derived a $n \times n$ matrix encoding the nature of the crossings such that every element of the matrix was one of the following $0,-1, t$ or $1-\mathrm{t}$. Alexander then calculated the determinant of this matrix and arrived at his knot polynomial.

An interesting property of the Alexander polynomial is that when two prime knots are combined, the Alexander polynomial for their knot composition is given by the multiplication of the original Alexander polynomials. For example the polynomial for the trefoil is

$$
t^{2}-t+1
$$

and for the granny knot it is

$$
\left(t^{2}-t+1\right)^{2}
$$

The Alexander polynomial for the figure eight knot is

$$
t^{2}-3 t+1
$$

## Torus Knots



Source: www.cs.ubc.ca/nest/imager/contributions/scharein/knot-theory/torus.html
Figure 4.25 Some knots from the family of torus knots

Another pleasing pattern which is evident is that alternating knots have alternating signs in their polynomial. However, it cannot distinguish between mirror images, the Alexander polynomial for each trefoil is the same and hence the Alexander polynomial for the reef knot and granny knot are also the same.

Later, a neater method of recursion would be used to derive these polynomials. In 1969 Conway derived his polynomial recursively using a skein relation. Although related to Alexander's polynomial the method of derivation was totally different. Conway's polynomial was constructed without the use of determinants but was calculated directly from diagrams by a method of recursion. Unfortunately, Conway's polynomial too cannot distinguish between mirror images.

Vaughan Jones described a new polynomial in 1986 which was good at distinguishing mirror images. The Jones polynomials for the two forms of trefoil are

$$
t+t^{3}-t^{4}
$$

and

$$
t^{-1}+t^{-3}-t^{-4}
$$

Jones' work was developed in two different ways to result in the Homfly polynomial in 1985 and Kauffman's bracket polynomial in 1990 both being polynomials in two variables. The Kauffman polynomials for the trefoils are:
for the right handed trefoil

$$
-x^{-4}+x^{-2} y+x^{-2} y^{-1}
$$

and the left handed trefoil

$$
-x^{4}+x^{2} y+x^{2} y^{-1}
$$

and for the figure eight knot

$$
x^{-2}+x^{2}-y-y^{-1}+1
$$

### 4.3 Summary

This chapter has examined the history of knot tying, beginning with legends about knots and then tracing the developments in the use of knots from prehistory up to recent times. Examples have been cited where knowledge about the intricacies of knots can be used to symbolise, to mystify and to bewitch. The importance of knots to civilisation has been described and it has been pointed out that familiarity with knots predates number as a significant human skill.

The mathematical study of knots has been included to give some background to the nature of the tasks used in the study and to give an insight into this unusual field of spatial mathematics. Knot theory is an area of mathematics which is still growing. What began as a simple tale about fishing nets has today developed into a field which informs scientists studying DNA as well as those studying weather patterns. But the simple task of tying a knot is one which can be viewed as an exercise in spatial thinking.

Finally, the existence of a mathematical formalism supports the validity of their use as a variable in a controlled scientific experiment.

## Chapter 5

## The Research issues

### 5.0 Introduction

It has been demonstrated in Chapter 2 that imagery and spatial ability are helpful in the learning of mathematics. The debate regarding the nature of a mental image, whether it is pictorial or in some other format such as a spatial array of information, was discussed in Chapter 3. A study of the debate is helpful, since an understanding of the different processes of imagery may be important for the development and practice of the skill in the classroom.

It has been pointed out that imagery and spatial skills may be developed using the right activities but that not enough of these activities are taking place in many classrooms. A possible explanation for this dearth of visualisation activities is the lack of appropriate materials and sound knowledge of how to activate the teaching of imagery. A further possibility could be the fact that an exercise in imagery may have no concrete visible outcome in a pupil's exercise book and this may contribute to the teacher's unwillingness to spend time on such activities. Some important questions therefore are; what activities may be effective in developing imagery skills; and what can be done to encourage more practice of these skills in the classroom?

Activities which employ dynamic imagery have attracted some interest in research studies, and investigations of imagery have considered both 2-D and 3-D objects, but the perception of deformable 3-D structures and elucidation of strategies for their manipulation has received relatively little attention. These tasks are important since the issue of how imagery is used for reasoning in such manipulative tasks may be revealing and may enable pupils to develop their mathematical
skills by using spatial intuition. Much research has been carried out to discover what are the 'abilities' which are needed in doing mathematics. Is it possible that some intuitive abilities have been neglected?

Imagery ability is not only useful in geometry learning, but has also been shown to be productive in problem solving generally. The operations performed during the problem solving process and whilst working on mental models of situations in mathematics (perhaps involving algebraic structure) are often the same as those used in spatial tasks. Researchers have put forward different reasons and benefits for how and why pupils use visualisation in problem solving, but the inclusion of one more tool in the problem solving repertoire would seem to be sufficient reason to encourage its use. If this skill can be improved merely by providing the right activities then it would seem to be an appropriate quest to find some activities suitable for classroom use.

### 5.1 Pedagogy

As a secondary school mathematics teacher I have been aware of the problems experienced by some secondary pupils in carrying out spatial tasks and the relative ease with which other pupils may perform the same tasks. There is a need to present children with experiences and tasks which develop their spatial skills. As a member of the National Mathematics Project (Harper et al. 1991) writing team, working on a series of secondary school text books, both myself and the other authors felt it important to include activities involving mental transformation tasks in geometry ${ }^{1}$, not only to reinforce geometrical knowledge but also to practise more general spatial skills and visualisation.

In 1968, Pyshkalo (quoted in Hoffer 1983) noted that at that time geometry instruction was introduced in schools rather late in a child's development and the content was more concerned with measuring than with spatial operations. Pyshkalo proposed a less formal introduction to

[^25]geometry (qualitative rather than quantitative) involving familiarity with shapes and their transformations. Hoffer (1983) also noted that many elementary school teachers skip work on geometry because they do not consider it to be a 'basic skill'. In 1978, Clements and Wattanawaha (1978) suggested that very little systematic work on 'Shape and Space' specifically aimed at developing pupils' spatial and imagery abilities was being done in school classrooms and their work sought to provide suitable innovative spatial tasks.

To some extent, Hoffer's (1983) concern that geometry was not introduced to the child early enough has been addressed in the UK with the introduction of the National Curriculum (DFE, 1995). The Programmes of Study for the primary years, Key Stages 1 and 2, include work on 'Shape and Space' and visualisation is mentioned at all Key Stages 1-4. For the early years at Key Stage 1 the document (DFE, 1995, page 5) states :
'Pupils should be taught to: describe and discuss shapes and patterns that can be seen or visualised;'

Specified in the National Curriculum (DFE, 1995, page 16) in the Programmes of Study are two categories of teacher activities. One category of activities follows the instruction 'Pupils should be given opportunities to' and the other 'Pupils should be taught to'. Included in the list which pupils should be taught to do, are the following statements (DFE, 1995, page 9):
Key Stage 2
'Pupils should be taught to visualise and describe shapes and movements'
and for Key Stages 3 and 4 (DFE, 1995, page 16):
'Pupils should be taught to recognise and visualise the transformations of translation, reflection, rotation and enlargement,
Unfortunately, the scope of the document does not extend to offering suggestions on how pupils should be taught these things.

One of the original aims of the NMP (Harper et al. 1991) writing team had been to address this problem by incorporating 'mental geometry' tasks into the classroom materials. The NMP school texts which were produced contained many activities which were concerned with static rigid 3-D objects², but none concerned freely deformable nonrigid objects. Subsequently, in trying to find novel, interesting and challenging spatial problems which would engage pupils, I became interested in the mathematics of knots.

### 5.2 Knots as a context for spatial tasks

Knots provided a new context for exploring space and mathematics. Knot theory is a part of mathematics with its own theorems and mathematical structure and although knot theory can be a very abstract topic, many issues involved in knot theory can be demonstrated by referring to the actual knots and then mental imagery employed to replicate possible manoeuvres. Thus I have worked with Year 9 children (age 14) in mathematics masterclasses ${ }^{3}$, and with teachers in workshops and at conferences. The activities include many problems on various mathematical properties of knots, with ropes and the actual knots available, as well as with puzzles and problems using diagrams of knots. Denis (1991, page 110) advocated this approach and commented,
'. . . images can be manipulated in a way that realistically simulates transformations on physical objects, thus making the successive states resulting from these transformations cognitively available, manipulations of images have the advantage of being performed in a flexible and rapid manner enabling on-line inspection of current hypotheses. As a result the processing of a problem whose components can be readily implemented in a picture-like representation may be enhanced by a well-conducted imagery process that capitalises on the representational properties of these components.'

2 one example was shown earlier in Chapter 2, Figure 2.1
3 Groups exist throughout the country and are sponsored by the Royal Institution. Bangor is just one of the 'out of London groups' and is funded by Anglesey Aluminium

There is always variation in the ability of pupils (and their teachers) to do the tasks mentally. All of the pupils attending masterclasses are present because they enjoy doing mathematics, but like Krutetskii's pupils they think in different ways. Some seem to have a confidence and an automatic ease with the process of mental manipulation of the knots whilst others experience great difficulty. Nevertheless, it appears that with the presence of the knots themselves, and with the experience of working on the tasks, all pupils begin to model mentally the transformations which the actual knot can undergo and develop a conception of the way in which knots transform in space. The pupils begin to learn how to manipulate them in their mind - to imagine and to visualise how the knots can move and change shape in space.

Generally, most individuals have had experience of the topological aspects of items such as clothes. Clothes can change shape but remain topologically the same whether they are folded or being worn. We all know how to put on a jumper and some of us know how to take one off without first removing the jacket! Similarly, tying things together is an activity which most people have tried. It may be sensible to foster and extend these everyday topological experiences so that pupils are able to imagine how to do other tasks with deformable objects and hence develop their mathematical and spatial awareness.

Knots provide a useful medium for linking mental geometry to such ubiquitous human skills as weaving, knotting, ornamental knot work, tailoring etc, skills which have been well developed in many cultures since antiquity. It could be argued that the fact that these skills have been known to humans for so long supports the view, as Piaget, Inhelder and Szeminska (1960) claimed, that topological understanding is more fundamental and intuitive than the concepts of Euclidean geometry. However, Bell, Costello and Küchemann (1983) believe there is a distinction between simple appreciation of a shape and its topology, and a deeper more analytical understanding of spatial relationships, they
remarked (Bell et al., 1983, page 152):
'It seems clear that some topological notions are understood earlier than Euclidean ones, but the view that the child's first spatial concepts are exclusively topological is misleading and restrictive. It may be more appropriate to identify a progression from a reliance on simple qualitative cues to the recognition and application of increasingly complex spatial relationships. This progression involves a transition from global recognition of an entire shape to the analysis and appreciation of its component parts and properties.'

Global recognition of an entire shape has already been observed for 2-D shapes (for example, Tarr and Pinker, 1989), but what of 3-D deformable structures? Can evidence from mental knot tasks be found to support the view of Bell et al. (1983) expressed above? The idea of 'simple qualitative cues' leading to 'complex spatial relationships' is worthy of exploration in the context of knots.

### 5.3 Setting up a framework for the research

In setting up the research and formulating the research questions, it was considered useful to review Ernest's (1998) suggestions:

1. link with and build on existing knowledge in the relevant educational research literature;
2. use organised processes of enquiry, systematic methods of research, linked to existing methodology;
3. result in a systematically organised text, document or other public communicative form, so that others can assess the results of the educational research;
4. possibly engage in theory-building resulting in the construction of some systematically organised system of reflective knowledge.

The issue of relevant research literature raised by item 1 has been given due consideration in Chapters $1-3$ and forms the basis for the planning of this study. The organised processes of enquiry and
appropriate methodology raised in item 2 derive from the review of the literature. The planning will be mentioned briefly here but will be explained more fully in each of the Chapters 6,7 and 8 covering each phase of the study. With regard to item 3 and the evaluation of the outcomes of the research, the results of the pilot study which is described in Chapter 6 have already been published in a mathematics education journal (McLeay and Piggins, 1996) and a psychology journal (McLeay and Piggins, 1998). Item 4 will be addressed in Chapter 9 where an evaluation of the research framework and results will be given.

Much research has already been carried out to find out more about spatial ability and its development. This research seeks to extend this knowledge so as to assist creativity in problem solving and thus provide pupils with an additional skill in their 'mathematical tool box' An investigation will be carried out which offers subjects tasks requiring mental manipulation of deformable objects and which relates to previous research on rigid objects. The framework and methodology therefore will relate closely with previous research in the area. Further investigation will be carried out of a qualitative nature to enrich the interpretation of the quantitative study.

Knowledge about the cognitive process of imagery itself is relevant. Similarly, other aspects of mental imagery, such as levels of difficulty of certain tasks may be useful. This facility level of certain tasks can be ascertained using the psychometric approach and measured by how long it takes subjects to solve certain mental tasks. Different information, including personal strategies, can be gathered using other methodologies similar to clinical case study investigations.

Bishop (1992, page 712) made a plea for research to be made relevant to the classroom:
'If the object of research is the improvement of mathematics teaching then it clearly makes sense to examine the normal activities associated with mathematics teaching to see if they are creating the obstacles to improvement, that is to what extent are
they part of the solution rather than just part of the problem?'

Clearly it is desirable that research results in the provision of materials which fit in with the 'normal activities' of mathematics teaching so as to enhance pupils' imagery and spatial ability and improve pupils' learning. The development of visualisation skills requires practice so as to be able not only to construct an image but also to control it and manipulate it for some mathematical purpose. Previous research has shown that subjects can be reluctant to use imagery even when it would be useful to solve a problem. It is hoped that this research will help towards empowering pupils to be less reluctant imagers.

### 5.4 Research questions

With regard to the issues raised by previous research and which are of interest in this study, the questions for investigation fall into two groups. The two groups may be described as (i) pedagogic and (ii) cognitive. The questions are as follows:
(i) Pedagogic:

How can the skills of imagery and visualisation be developed?
Can visualisation be taught using knot tasks? and if so 'Which tasks enable/effect learning?'.
Can the tasks be sequenced ranging from easy to difficult?
Is it possible to devise a teaching programme (using knots) to improve spatial ability?
Do subjects improve their performance of the knot tasks? Do they learn simply by doing?
(ii) Cognitive:

What enables us to know how an object will move?
What factors influence the strategy employed by verbalisers and visualisers?
Are verbalisers forced to visualise for these knots tasks?
Do visualisers verbalise? Do visualisers form short cut methods similar to the verbally encoded propositions or procedures described by Battista (1994)

In order to address some of the questions in group (i), we wish to find out which are the spatial tasks that children are likely find easier or more intuitive. Tasks which are more topological in nature may be more intuitive and within the subject's everyday experiences rather than some of the very particular experimental spatial tasks explored in previous work.

One of the issues, 'can visualisation be taught?' was a concern explored by Lean (1981) and by Smith (1991), but there are still no clear guidelines on how this could be done or which experiences would be beneficial. Research in the psychology literature (see for example Wallace and Hofelich, 1992; Kosslyn et al., 1989) discussed in Chapter 3 has shown that practising mental acrobatics improves performance on spatial tasks and also that the improved performance transfers to other tasks. The sequencing of such tasks, however, has not been addressed directly or made explicit.

With regard to the related question 'what are the tasks which enable/effect learning?', Battista et al. (1982) found that teaching a geometry course to 82 pre-service teachers improved their spatial ability ${ }^{4}$. The activities they used in the course focused mainly on informal geometry with some logical deductive thinking but did not include topological questions.

The previous research described in Chapters 2 and 3 considered some of the questions above, but the nature of the objects studied in all of the previous studies was different, the objects were rigid. The mental manipulation of deformable objects has received little attention and could be useful in identifying different processes in visualisation, processes such as the 'imagining of transformations' as opposed to the 'transforming of images' defined by Freyd (1987). These different kinds of tasks may produce some additional strategies for spatial and visual thinking and enhance the development of mental skills.

[^26]Some of these questions can be explored using a similar methodology to that used by Shepard and Metzler (1971) in their experiments. The difference in this research will be that subjects will be shown diagrams of pairs of deformable objects with angular disparities rather than rigid objects.

During the tests to be administered, subjects may intuitively develop strategies or skills to deal with the tasks in much the same way as we learn by our experiences in everyday life. The subjects' reactions to the tasks will be sought and their ability to perform the tasks measured both quantitatively and by verbal feedback from subjects. Inevitably some tasks will be more difficult than others and this information will be of use if such tasks are to be used in planning a teaching programme for improving spatial ability and visualisation.

In order to answer some of the other questions classed above as (ii) cognitive, a more qualitative study will be required. These questions can be explored by in depth interviewing of subjects. This will be done by video case study 5 and will involve subjects carrying out the mental tasks and asking them to explain their thoughts and strategies. This data should give information not only on possible strategies but also any changes in strategy and learning of strategies as the tasks become more familiar.

In summary, the framework for this research will be to use a combination of two methodologies: psychometric testing to collect quantitative data using measurement of decision times as an indicator, together with case study interviews and the collection of qualitative data. These data may then be used to design curricular sequences to improve spatial ability and allow mental manipulation skills to be developed.

[^27]
### 5.5 Developing a teaching programme

How might the answers to the research questions be used to improve teaching? Hoffer (1983, page 225) was concerned with students' perception and acquisition of concepts and asked:
'Once we know how students perceive and reason with objects and relations, what learning experiences can we provide to help the students gain insight into the subject?'
The relationship between everyday experiences and mathematical thought was commented upon by Cooper \& Shepard (1990, page 121): 'The ability to represent objects or arrangements of objects and their transformations in space clearly is valuable in managing the concrete realities of everyday life, making it possible to plan actions and to anticipate outcomes. It may also play an important role in abstract thought.'

If we wish to develop pupils' ability in mathematics it is vital that we do not ignore the development of visualisation skills. It has been claimed (Lean, 1981) that visualisation is not an innate skill and can be developed in pupils (Clements and Battista, 1992; Tuckey and Selvaratnam, 1993) but that the results are more effective when the experience takes place early in a child's schooling (Bishop, 1983).

With regard to the question of what tasks enable the acquisition of visualisation skills, a range of possible activities have already been developed. Clements and Wattanawaha (1978) attempted to classify some spatial problems and they included some non-rigid objects in their space visualisation test. Figure 5.1 shows a topological ropes task and Figure 5.2 shows another paper folding task 6 .

Tasks which are more likely to succeed are those which build upon the young child's experiences. A question which arises is: How do we know when a rope is knotted? This knowledge draws upon our experiences of everyday items such as shoes, parcels and tangled string.

[^28]The topology of string may be a familiar concept to the child and may be used as a basis for teaching visualisation.


Source: Clements and Wattanawaha (1978)

Figure 5.1 Ropes knotting task

Most of the spatial tasks in Clements and Wattanawaha (1978) do not involve topology and a tentative criticism might be that neither do they all necessitate visualisation in order to be solved, some can be solved logically. For example, the task shown in Figure 5.2 which involves paper folding may not be a purely spatial task, it is possible to apply rules of geometry to solve it and the solution arrived at logically. The cut hole is decreasing in size from left to right. After unfolding, this situation is reversed (the cut is reflected in the fold line) and the correct picture must show the size of the hole decreasing from right to left (as it approaches the edge of the paper). Hence the answer must be be B or D. But it cannot be D since the the diamond shaped hole shown in $D$ does not have vertical line symmetry and so the answer is B. Krutetskii's (1976) analytic thinkers would probably have solved it in this way.


Source: Clements and Wattanawaha (1978)

Figure 5.2 Paper folding visualisation question

In the case of the ropes task, it is not easy to see how this can be solved without some visualisation taking place. We must mentally model a real situation of untangling or pulling the rope. Equally it is clear that, apart from the ropes A and B, this is a more complex task than unfolding a piece of paper. The reason why this is so is possibly due to the fact that there is more information to hold in the mind at any time. Or could it be that only visual imagery will suffice to solve the puzzle
and this is a high level skill?

A selection of different activities is always desirable in the classroom for any teaching programme so that all pupils have an opportunity to find something which interests them, something which they can understand and can do. A range of possible tasks have been developed and this study attempts to add more activities to this list so that all pupils may find an activity to suit them.

Some criticisms of the claim have been that the skills which can be taught are very task specific and not transferable. This research will attempt to explore subjects' thinking in how they are solving the tasks and determining whether or not they are developing general spatial strategies.

The main concern is with developing spatial abilities in an informed and precise manner and with investigating the skills which visualisers employ and thereby devise experiences to develop these skills. These skills can then be employed in the learning of, and attaching meaning to, mathematics as well as enabling the visualisers to improve and to make full use of their abilities.

### 5.6 Visualisation strategies

What can research tell us about the strategies used to solve mental tasks? Much of the previous research on mental manipulation has focussed on reorienting planar 2-D or rigid 3-D objects. Although some studies report that mental rotation was not always found to be a strategy (Suzuki and Nakata, 1988) many suggest that it is the main strategy (Shepard and Metzler, 1971; Shwartz, 1981).

What can this study tell us about the strategies used to solve mental tasks? This research will explore the effect of orientation differences but will also try to find out which tasks are performed by mental rotation and which are solved by other means, such as
reassigning the top of an image as identified by Reisberg and Chambers (1991). An attempt will be made to discover whether some of the reported alternative methods which have been employed for 2-D objects, such as reassigning the top of an image discussed in Chapter 3, are feasible for 3-D objects, and also to discover which other methods are used and how these other methods may vary according to the stimulus.

Where manipulation of an image is involved, the question as to whether the nature of a mental image is visual or propositional, becomes even more complex. If the subject is required to manipulate a currently viewed object and to transform it mentally, will a visual method or a verbal method be preferred? The viewed image is pictorial and exists in the visual perception processing buffer but if it needs to be altered in some way how will this alteration be achieved? Does the presence of the object influence the strategy employed? Are verbalisers forced to visualise for these knots tasks? Do visualisers verbalise? Do the subjects in this research learn to by-pass the visual process as Battista (1994) has suggested as familiarity with the tasks increases, do visualisers form short cut methods which may be verbal? These questions will be explored during the case study phase of the research.

Krutetskii (1976) described two kinds of thinkers (analytic and geometric) with a third intermediate category (harmonic). Lean \& Clements (1981) suggested that there is in fact not just an intermediate third category but a continuum along which all individuals may be situated. The latter notion seems an appealing one, allowing for flexible switching of strategies to suit the task involved. Whether subjects switch strategies in these tasks and what factors influence any switching of strategies is another interesting question to be considered.

In trying to improve children's acquisition of spatial skills we must know what the skills are. Individuals may solve the same problem in different ways (Clements, 1983) and indeed two responses to a problem may appear identical but may be produced using completely different solution paths (Lesh \& Landau, 1983). So we need to know about all the
possible strategies in order to offer a range of sequenced tasks which will appeal to the different modes of thinking of each individual. In this study, some attempt will be made to do this by studying a selection of subjects some of whom have a background in mathematics and others who are qualified in language. It is expected that the former will be good spatial thinkers, whereas it is presumed that the latter should show a different (propositional) approach to the tasks.

The 'need' by some pupils to use visual images to solve problems has been described by Presmeg (1986). The process of bypassing this phase as the problems become more familiar was considered by Battista (1994) to be enabling to pupils, but the visual strategies are still useful for new problem situations.

### 5.7 Summary

The exact method or strategy per se used by a subject on a particular item is not the main concern, but knowledge about strategies used by different individuals generally may be of help in devising activities for teaching programmes. How the imagery process copes with manipulation of deformable objects as distinct from rigid objects is not in itself the issue, but the different skills employed need to be recognised and opportunity given in the classroom for their practice and development. The extent to which visualisation methods are employed may reflect a subject's prior experiences but the extent to which the visualisation process increases or recedes during practice may be a function of the tasks themselves.

In the experiments described in this thesis, subjects will view the presented image before and during any mental rotation or unravelling, but must hold any intermediate mental images. By varying strategies, the subjects may learn to solve the tasks and this may entail interpreting an intermediate 'imaged' form of the knot diagram.

With regard to the question of how the tasks are solved, verbally or visually, there is a range of possibilities and perhaps there is room for
a combination of methods here. Although a knot is a 3 dimensional object, the knot images in these experiments could be considered to be 2$D$ in nature with a verbal addendum to describe crossings and/or deformability.

One of the questions which this research attempts to answer is, 'Which knot tasks are more difficult?'. The further question 'Why are some knots more difficult?' will be considered in the discussion of the results in Chapter 9. One aim of this research study is to investigate some mental capacities and to look for ways of developing or enabling these mental capacities. However, the main purpose of this research is thus to:

- Extend the ideas on imagery and visualisation which have been developed in the mental rotation of rigid structures to structures which can be deformed.
- Investigate the perception of deformable 3-D structures and elucidate strategies for their manipulation.
- Identify whether these activities can develop dynamic imagery and whether mental manipulation abilities can be improved with practice.

Bishop (1983, page 197) stated that he is:
'someone who sees empirical research as a way of shedding light on curricula and pedagogical issues by collecting data systematically, by interpreting those data, and by reflecting on their implications in relation to practice and to other research. ... and for ideas that will enable us to derive better tests, task materials, teaching materials and procedures, and educational practices to enable more children to feel successful and confident in mathematics.'
It is in the same spirit that this research has been carried out.

## Chapter 6

The Pilot study

### 6.0 Introduction

This experiment was devised as a result of the author having studied the undergraduate psychology courses on 'Perception' and 'Research Methods in Psychology' offered at the University of Wales, Bangor. A collaboration ensued with a visiting research fellow to that department which brought together a knowledge of mathematics and an expertise in experimental psychology. It had already been observed by the author that the knot puzzles previously referred to (McLeay, 1994) cause considerable difficulty to some people and none whatsoever to others. It had also been observed that some puzzles were harder to do than others and that a range of strategies could be employed in solving them. The question was, having recognised these facts how could they be investigated?

The research described in Chapter 2 highlighted the importance of imagery to mathematical thinking. Chapter 3 highlighted the scarcity of research data on mental manipulation of deformable objects such as knots. Chapter 4 explained the mathematics of knots including some of the mental geometry which can be involved. This chapter will explain the choice of stimuli and the setting up of an acceptable means of testing subjects with the various items. It was noted earlier that one of the most researched areas in mental manipulation was that of mental rotation and this seemed a suitable place to start the investigation.

A justification for the choice of stimuli and the procedure for carrying out the experiment follows and then the results of the experiment will be reported together with a discussion of what the results show.

### 6.1 Methodology

Some of the issues raised in Chapter 5 may be investigated using a similar methodology to that of the Shepard and Metzler (1971) experiments but using newer technologies. The tachistoscope which Shepard and Metzler (1971) used for presenting stimuli and the stop watch for recording response times are now outdated. The research questions raised can be investigated more effectively using a microcomputer both to present the stimuli and to record automatically accurate response times. This procedure is used frequently in experimental psychology and reduces the likelihood of recording errors which could be a function of experimenter expectancies or some type of observer bias. The microcomputer can be programmed to present as many different independent variables and to record as many different types of responses as creativity will allow. It was therefore an effective and appropriate method to explore the issues described.


Source: McLeay (1994)
Figure 6.1 Knots puzzle requiring mental manipulation

In this pilot study the mental manipulation of deformable structures was investigated. The approach taken was the experimental approach whereby the effect of systematically changing the variables was examined so as to identify causal relationships. Control of the variables was a prime concern. Although the knot puzzles in McLeay (1994) were appropriate tasks in terms of mental manipulation and the visualisation skills required (an example is shown in Figure 6.1), the experimental tasks needed to be rather more constrained so as to be able to investigate the effect of each of the variables.

### 6.1.1 The independent variables

The four independent variables under experimental control were:
(i) number of crossings, (ii) degree of rotation, (iii) topological status (knot or unknot1), and (iv) shape. The fourth variable, shape, is investigated but in a limited way. What cannot be considered here is the infinite variety of deformations of shape possible for any single knot. Any knot can be rearranged to give all manner of shapes and sizes of the loops but topologically it is still the same knot. We are not so much interested here in these mathematical intricacies, but whether the crossings in one rope diagram are the same as in another similarly drawn diagram. The research design selected offers a means of controlling the four variables listed and a way of seeking answers to the research questions. The method also links to previous research on mental manipulation with rigid objects.

As well as response times, the number of errors made by each subject was recorded. The particular knots and rotations which caused the errors were also noted. It was predicted that there would be more errors for pairs with rotation since this adds a further tier of complexity to the task.

Strategies were investigated by the use of a short interview at the end of each subject's performance of the experiment. The influence of
the utilisation of particular strategies and also instances of switching strategies were noted during the post experimental interviews in this pilot study. The post experimental interview is important in this type of research for various reasons. Christensen (1988, page 385) described the ethical function, of allowing subjects to comment freely on the experiment, and also made the following comment:
'the interview can provide information regarding the subjects' thinking or strategies used during the experiment . . . . . They (the subjects) relayed a specific strategy for having accomplished this task, which led to another study investigating strategies for learning.'

Verbal reports therefore can show where research is flawed or which direction it should take in the future. Such reports can also throw light on strategies and effects which the researcher had not expected.

In previous research studies, (for example, Cooper and Shepard, 1973; Suzuki and Nakata, 1988) specific strategies were required for the rigid objects used compared to a range of possible strategies for these objects with easy deformability. Fewer factors were needed in decision making for polyhedral figures than for knots and therefore significant differences in decision time between knot types was expected.

### 6.1.2 Individual subject variation

It has been noted by some researchers (for example, Logie, 1995) that when performing experimental tests, individual subjects do not necessarily respond in the same way to different variables in the tests. Individual subjects may perform differently on different parts of the test. With this in mind, as well as performing a general analysis of means of overall decision times for the groupings of pair types, the results were analysed by considering the variation in decision times for each individual subject across all knot and unknot pair types. For each item processed by a subject, the difference between the decision time and the subject's overall mean time was calculated and considered in relation to
the results as a whole. This aspect of the research was considered useful owing to the varied nature of the subject sample and the relatively novel nature of the stimuli.

### 6.2 Subjects

Twenty one subjects, 14 males and 7 females, two thirds of them students of the University of Wales Bangor, were chosen for their availability to take part in the experiment. This initial pilot study was carried out with a rather special 'purpose sample' and consequently there are limitations on the generalisability of the results. Their age range was 11-59 years most being late teens to early twenties. There was some variability in subjects' prior knowledge of knots. Viewing was binocular and all subjects admitted to normal visual acuity and wore correcting lenses for distance of viewing if this was usual. Eleven ( 8 male, 3 female) were undergraduates or postgraduates in mathematics, five ( 3 male, 2 female) were undergraduates or postgraduates in psychology, two (male) were school children age 11 and 12 years who were in the top set in mathematics and three others ( 1 male, 2 female) were postgraduates from the background of social sciences.

### 6.3 Stimuli

### 6.3.1 Motivation for choosing knots as the objects in the tasks

The stimuli used were knots, chosen for the fact that most people are familiar with their properties and for the existence of a developed mathematical formalism used to describe them ${ }^{2}$. The study adopted the spirit and methodology of Shepard and Metzler (1971) and related research experiments differing primarily in the employment of knots, deformable rather than rigid structures. Shepard and Metzler's ${ }^{3}$ (1971) 3D structures were always the same topologically although not necessarily

[^29]the same in shape. When Shepard and Metzler's shapes were not identical they were mirror images of each other. In this study the knot pairs chosen were not necessarily the same topologically but were always the same in outline shape and number of crossings.

### 6.3.2 The nature of the tasks

Stimuli were presented either as (i) a pair of knots; (ii) a pair of unknots; or (iii) a knot and an unknot.

A knot is defined as a closed loop configured such that no matter what transformations are applied to it, some interlacing remains; it is embedded in 3-D space. One of the characteristics by which knots can be described is by the minimum number of crossings which the knot can have. This is called the crossing number of a knot.

An unknot is defined as a closed loop configured such that it can be reduced to a simple closed loop with no crossings. In other words, crossings can be removed by sliding or twisting the rope. For practical purposes, knots or unknots in this study were restricted to 3,4 or 5 crossings and stimulus pairs always shared the same number of crossings. Knots were depicted as made of rope with the ends joined to form a closed loop. In the experiment the number of crossings (i.e. the number of times the rope passes over itself) either (i) cannot be reduced at all, that is the knot shows its true crossing number; or (ii) reduces to zero, if the figure is an unknot.

The two diagrams in Figure 6.2 look very similar but depict two different objects. The one on the left is a true knot with crossing number 4 whilst the one on the right can be reduced to a simple loop and is an unknot with crossing number zero.

In order for the pair to be different, that is not the same topologically, one of them has to be the unknot with all crossings removable whilst the other is the true knot. In order for the pair to be the same, either both of them must be unknots with all crossings
removable or both of them must be true knots.


Figure 6.2 A knot and an unknot

The knot diagrams were viewed on a computer monitor some 50 cms from the subjects and subtended an angle between $6^{\circ}$ and $6.5^{\circ}$ at the eye. A knot might be placed alongside an identical knot or a similar shaped unknot. To explore the effect of reorienting the stimuli some orientation difference was introduced, thus providing a useful connection to previous research. A typical pair is shown in Figure 6.3. The knot shapes used in the study were restricted to 3,4 or 5 crossings. The 3 and 4 crossing shapes are actually different representations (shapes) of the same knot, the trefoil knot in two forms and the figure-eight knot in two forms. The two 5 crossing knot shapes are actually different two different knots ${ }^{4}$. Figure 6.4 shows all the knot types used in this study.

[^30]

Figure 6.3 Example of a stimulus pair with rotation

Orientation differences within each pair were $0^{\circ}, 90^{\circ}, 180^{\circ}$ or $270^{\circ}$. Overall the same 126 pairs were presented in a different randomised order for each subject. The 126 items consisted of 21 items for each of the 6 knot shapes used. For each knot shape the 21 items were made up as follows:
knot/knot, four items with relative rotations $0^{\circ}, 90^{\circ}, 180^{\circ}$ or $270^{\circ}$; unknot/unknot, four items with relative rotations $0^{\circ}, 90^{\circ}, 180^{\circ}$ or $270^{\circ}$; knot/unknot thirteen out of the possible sixteen combinations of orientations for different pairs of knot with an unknot.

Three out of the four possibilities with $0^{\circ}$ relative rotation were removed so as to reduce the number of less complex items which required little or no mental manipulation. These orientation possibilities are explained below. Table 6.1 lists the pair combinations.


Figure 6.4 The six knot shapes used in the study

TABLE 6.1
Pair combinations of knots

| LEFT KNOT | 0 | RIGHT KNOT | 0 |
| :--- | :---: | :--- | ---: |
| LEFT KNOT | 0 | RIGHT KNOT | 90 |
| LEFT KNOT | 0 | RIGHT KNOT | 180 |
| LEFT KNOT | 0 | RIGHT KNOT | 270 |
| LEFT unKNOT | 0 | RIGHT unKNOT | 0 |
| LEFT unKNOT | 0 | RIGHT unKNOT | 90 |
| LEFT unKNOT | 0 | RIGHT unKNOT | 180 |
| LEFT unKNOT | 0 | RIGHT unKNOT | 270 |
| LEFT KNOT | 0 | RIGHT unKNOT | 0 |
| LEFT KNOT | 0 | RIGHT unKNOT | 90 |
| LEFT KNOT | 0 | RIGHT unKNOT | 180 |
| LEFT KNOT | 0 | RIGHT unKNOT | 270 |
| LEFT KNOT | 90 | RIGHT unKNOT | 0 |
| LEFT KNOT | 90 | RIGHT unKNOT | 180 |
| LEFT KNOT | 90 | RIGHT unKNOT | 270 |
| LEFT KNOT | 180 | RIGHT unKNOT | 0 |
| LEFT KNOT | 180 | RIGHT unKNOT | 90 |
| LEFT KNOT | 180 | RIGHT unKNOT | 270 |
| LEFT KNOT | 270 | RIGHT unKNOT | 0 |
| LEFT KNOT | 270 | RIGHT unKNOT | 90 |
| LEFT KNOT | 270 | RIGHT unKNOT | 180 |

This list of rotated and non rotated items was carefully planned so as to take account of all possible pair combinations. The combinations of 'same' objects, that is a knot with a knot or an unknot with an unknot, gave rise to fewer possibilities since for each rotation used, say a $90^{\circ}$ rotation, either the knot on the left or the knot on the right could be rotated but only one of these options would produce a distinct pair. For a knot with an unknot, either could be rotated through $90^{\circ}$ and paired with all other possibilities. This gives rise to sixteen possible oriented pairs four of which are pairs with $0^{\circ}$ relative rotation. It was decided that the inclusion of all of these pairs which had no relative orientation difference and were in essence posing the same task would be unnecessary and non-productive, as mentioned above. The number of these 'knot/unknot' pairs was reduced and three out of the four which had no relative rotation were excluded leaving just one 'different' pair
with $0^{\circ}$ rotation difference.

### 6.4 Hypothesis

The obvious tentative hypothesis which was posited was that the more complex the task the longer would be the time taken to solve it. Complexity was defined by (i) the number of crossings present - the higher the number of crossings the more complex the knot, and (ii) the amount of rotation - the more rotation the more complex the task.
Consequently, it was predicted (i) that the larger the number of crossings of the rope in a knot diagram the longer would be the decision time regarding its relationship to other knot diagrams;
and (ii) with regard to rotation, the hypothesis aligned with previous research findings in proposing that the more relative rotation within the pair then the longer would be the decision time.

The influence of strategies was investigated, however, the prediction was made that mental rotation would not be a prevalent strategy.

### 6.5 Procedure

Each subject was read a protocol (see Appendix II) explaining the experimental task and was shown a demonstration illustrating the manipulation of an actual closed loop knot (see Appendix (II). During the experiment, subjects were required to indicate if the stimulus pair were the same or different, the time taken to do this being recorded and termed 'decision time'. Each subject completed a practice session of 8 stimulus pairs which were checked for accuracy of response. When subjects indicated complete understanding of the tasks and of the experimental procedure then the data collection commenced.

Stimuli were presented on a monitor using a 'HyperCard' stack with automatic recording of decision times. Cards of stimulus pairs were presented alternately with a 'resting' card. The stimulus card contained

- pairs of knots, unknots or one of each,
- the question 'Are these knots the same?' and
- boxes labelled 'YES' and 'NO'.

The labelling of the boxes was randomised to avoid a laterality bias. An example of a stimulus card was shown earlier in Figure 6.3.

The resting card contained an empty box and the request 'Click here when you are ready to go on'. An arrow, the position of which was adjustable with movement of the mouse, was continuously present on every card. Succeeding stimulus pairs could only be initiated when the arrow was returned to the empty box in the middle of the resting card and the mouse button depressed. The resting card is shown in Figure 6.5. Stimulus presentation was thus ad libitum. The subject's decision time was measured by the time taken to move the arrow to the YES or NO box. The initial position of the arrow on the stimulus card was equidistant from each box.

## Click on the little button below when you are ready to proceed

## O

Figure 6.5 Resting card

Upon completion of all 126 stimulus pairs, subjects were given a
short debriefing interview during which their thinking was probed and they were encouraged to report the strategy or strategies which they considered they employed in making decisions.

### 6.6 Results

### 6.6.1 Strategies

Subjects reported 5 basic strategies for differentiating between the knots in a pair: rotation, unravelling, shape recognition, matching crossings and identifying sequences of crossings, the most frequently used strategy being unravelling. These strategies are defined as follows:

## Rotation

The mental rotation of the whole of the image of one of the pair to match the other. This is equivalent to the method used in the mental rotation of rigid objects.

## Unravelling

Unravelling the (un)knot systematically to remove crossings. For unknots, subjects notice 'superfluous' crossings and manipulate the image so as to remove crossings and eventually arrive at the simple loop.

## Shape recognition

Recognition of a knot or unknot by its global shape. Subjects become familiar with the knot shape and learn to recognise the knot and identify it generically.

## Matching crossings

Directly matching crossings according to their relative positions in each of the stimulus pair. Subjects may encode a verbal description or a perceptual organisation of information such as "The crossing at the 'base' has the rope on top as it goes down from right to left".

## Identifying sequences of crossings

Identifying sequences of crossings from the relative ordering of 'under' and 'over' elements in a configuration. Subjects may notice that the crossings in one figure have a sequence over, under, over, under, . . . , whereas the other of the pair has a different sequence.

All subjects reported reviewing their strategies during the course of the experiment and many considered that they changed strategies, the most common switch of strategy being to adopt the unravelling strategy. Although some subjects employed rotation initially if necessary, unravelling was reported as being more frequently employed as the experiment progressed. Three subjects reported searching for a 'foolproof strategy ${ }^{\prime}$, thus rejecting the rotation strategy as they considered it to be more error prone. It should be noted that the order of the tasks was randomised at the start of each experiment, thus no particular stimulus pairs were affected by the changes of strategy discussed here.

### 6.6.2 Data analysis

The distribution of decision times was positively skewed with a median of 7.7 seconds and a mean of 10.0 seconds ${ }^{5}$. Figure 6.6 shows the frequency distribution of the raw data and Figure 6.7 shows a box plot of these data.

For the purposes of the data analysis, a natural log transformation of decision time was used so as to achieve a good approximation to a normal distribution. A log transformation is effective in symmetrising a skewed distribution since it spreads out small values and compresses large ones. In order to carry out analysis of variance and employ an F test, several assumptions are made about the distribution of the data and the validity of these assumptions must be checked. When the necessary assumptions are upheld, the F test is is the most effective test in rejecting the null hypothesis when it is false. On testing these data for normality,

[^31]and for homogeneity of variance (homoscedasticity) across the groups, in both cases $p$ values of 0.001 were obtained for the null hypothesis. The other requirements: that the observations are independent, that they are measured on a number scale and that the effects of the variables are additive, are also met for the data analysed here. The mean for the normalised data is 2.11 with a median of 2.04 and standard deviation of 0.59. Figure 6.8 shows the frequency distribution of the transformed data superimposed onto the normal curve. The boxplot is shown beneath it.


Figure 6.6 Frequency distribution of decision times


Figure 6.7 Box plot showing distribution of decision times


Figure 6.8 Frequency distribution and box plot of log decision times

The responses were categorised by
(i) number of crossings $(3,4$ or 5$)$,
(ii) degree of relative rotation $\left(0^{\circ}, 90^{\circ}, 180^{\circ}\right.$ or $\left.270^{\circ}\right)$,
(iii) topological status of the pair combination (knot or unknot), and
(iv) shape (knot types shown in Figure 6.4).

The two dependent variables were decision time and error rate. An analysis of variance was carried out (using log decision time, so as to make the data appropriate for this procedure) and revealed several significant effects which are discussed below. The first part of the data analysis involved a one way analysis of variance to see which, if any, of the independent variables affected the decision time or error rate. The main aim of this first experiment was exploratory to see what effects may be occurring and what aspects are adding complexity to the tasks. A more in depth data analysis using two way analysis of variance to investigate interactions between variables will be carried out and reported on the larger data set in the second study.

## Knot or Unknot pairs?

Decision times were longer for knot pairs than for unknot pairs. In fact, unknot pairs had the shortest decision times of the three categories, shorter than the combination of a knot with an unknot. The difference in decision times between the three categories (unknot/unknot; unknot/knot and knot/knot) was significant $[\mathrm{F}(2,2643)=39.67, \mathrm{p}<0.001)]$. The mean times are shown in Table 6.2.

The mean of $\log t$ for all data was 2.11 with a standard deviation of 0.59 but particular values for standard deviations for groups of the data varied greatly and ranged from 0.45 to 0.9 . The tightest group was for unknot/unknot pairs with zero rotation. This group had a mean for log $t=1.89$ and standard deviation of 0.45 . The broadest ranging group was the group of incorrect responses for the unknot/unknot pairs where $\log t$ $=2.13$ with a standard deviation of 0.9 .

TABLE 6.2
Mean decision times

|  | Mean decision times (all values in seconds) |  |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | All data $n=2646$ | $\begin{gathered} \text { Correct } \\ \mathrm{n}=2446 \end{gathered}$ | Incorrect $n=180$ | $\begin{gathered} 0^{\circ} \\ n=378 \end{gathered}$ | $\begin{array}{r} 90^{\circ} \\ n=756 \end{array}$ | $\begin{gathered} 180^{\circ} \\ n=756 \end{gathered}$ | $\begin{gathered} 270^{\circ} \\ n=756 \\ \hline \end{gathered}$ |
| Knot/unknot | 9.7 | 9.6 | 10.3 | 8.9 | 9.8 | 9.8 | 10.0 |
| Unknot/unknot | 8.6 | 8.3 | 12.8 | 7.4 | 9.7 | 9.8 | 8.0 |
| Knot/knot | 12.2 | 11.7 | 21.2 | 9.3 | 11.6 | 16.0 | 12.1 |
| Overall | 10.0 | 9.8 | 12.5 | 8.6 | 10.0 | 10.7 | 10.0 |

Note: $\mathrm{n}=$ total number of trials in each category

## Orientation differences

Greater orientation differences produced longer decision times. There was no significant difference between $90^{\circ}$ and $270^{\circ}$ rotations and these were combined into one group. Significant differences in mean decision times between the three resulting categories were found $[F(2,2643)=6.93$, $\mathrm{p}<0.001$ ]. Decision times were fastest for $0^{\circ}$ rotation and slowest for $180^{\circ}$ and the overall means were $8.6 \operatorname{secs}\left(0^{\circ}\right), 10.0 \operatorname{secs}\left(90^{\circ}\right.$ and $\left.270^{\circ}\right)$ and 10.7 $\operatorname{secs}\left(180^{\circ}\right)$. See Table 6.3.

TABLE 6.3
Mean decision times by knot type and rotation

| Rotation | Mean decision times (all values in seconds) |  |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | All types $n=2646$ | Shape 1 $n=441$ | Shape 2 $n=441$ | Shape 3 $n=441$ | Shape 4 $n=441$ | Shape 5 $n=441$ | Shape 6 $n=441$ |
| $0^{\circ}$ | 8.6 | 9.6 | 8.1 | 6.6 | 9.4 | 8.9 | 8.8 |
| $90^{\circ}$ \& $270^{\circ}$ | 10.0 | 10.3 | 10.2 | 6.6 | 12.2 | 9.0 | 11.6 |
| $180^{\circ}$ | 10.7 | 11.1 | 13.2 | 7.1 | 10.9 | 10.0 | 11.8 |
| All rot's | 10.0 | 10.4 | 10.8 | 6.8 | 11.5 | 9.3 | 11.2 |

## Errors

There were 180 errors out of 2646 responses. The overall error rate ${ }^{6}$ was thus $6.8 \%$. The mean decision time for all errors was 12.5 seconds (see

[^32]Table 6.4) in comparison with 9.8 secs for correct responses (see Table 6.2). Incorrect responses were associated with longer decision times for pairs which were the same, increasing with rotation $[F(17,2628)=9.18, \mathrm{p}<$ 0.0001].

TABLE 6.4
Decision times for errors ( $6.8 \%$ of all data)

|  | Mean decision times (all values in seconds) |  |  |  |  |
| :--- | ---: | ---: | ---: | ---: | :---: |
|  | Rotation |  |  |  |  |
|  | All errors | $0^{\circ}$ | $90^{\circ} / 270^{\circ}$ | $180^{\circ}$ |  |
|  | $n=180$ | $n=11$ | $n=97$ | $n=72$ |  |
|  | 10.3 | 8.0 | 11.3 | 8.7 |  |
| Knot/unknot | 12.8 | 6.3 | 13.4 | 13.9 |  |
| Unknot/unknot | 21.2 | 4.7 | 10.5 | 24.6 |  |
| Knot/knot | 12.5 | 6.8 | 11.7 | 14.5 |  |
| Overall |  |  |  |  |  |

There was a higher error rate for $180^{\circ}$ rotation with $9.5 \%$ of items involving $180^{\circ}$ rotation being answered incorrectly compared to $6.8 \%$ overall (see Table 6.5). The variation in the occurrence of errors through all knot types is shown in Table 6.5.

TABLE 6.5
Percentage of errors by knot type and rotation

| Rotation | Error rate - incorrect responses as a \% of the total respo |  |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | Shape 1 $n=36$ | Shape 2 $n=46$ | Shape 3 $n=11$ | Shape 4 $n=35$ | Shape 5 $n=21$ | Shape 6 $\mathrm{n}=31$ | All types $n=180$ |
| $0^{\circ}$ | 3.2 | 4.8 | 1.6 | 1.6 | 3.2 | 3.2 | 2.9 |
| $90^{\circ}$ \& $270^{\circ}$ | 7.5 | 9.5 | 0.8 | 8.3 | 3.6 | 8.7 | 6.4 |
| $180^{\circ}$ | 11.9 | 15.0 | 6.3 | 10.3 | 7.9 | 5.6 | 9.5 |
| Overall | 8.1 | 10.4 | 2.5 | 7.9 | 4.8 | 7.0 | 6.8 |

## Crossing number and knot type

Knots with higher numbers of crossings were not found to have longer decision times. No significant difference was found between three crossings and five crossings $[F(1,1763)=1.66, p=0.198]$. The mean decision times were 10.6 seconds ( 3 crossings), 9.1 seconds ( 4 crossings) and 10.3 seconds ( 5 crossings).

There was a significant difference in decision time between knot types $[F(5,2640)=38.59, \mathrm{p}<0.0001]$ with one knot shape in particular requiring shorter decision times than others. This was the figure-eight knot (knot shape 3), identified by subjects as particularly easy to recognise, with a mean decision time of 6.8 seconds in comparison with the overall mean decision time of 10.0 seconds. This knot also produced the fewest errors, see Table 6.5. The longest mean decision time occurred for Shape 4, which took 11.5 seconds, see Table 6.3.

## Individual subject variation

The robustness of the results is demonstrated by considering the variation in decision times of individual subjects across knot and unknot pair types. Logie (1995) has made the point that most research reports upon the patterns of data over all subjects and he suggests that, when treated individually, the subjects' personal results may not necessarily demonstrate the found effects for the data overall. This is not the case here. For each item processed by a subject, the difference between the decision time and the subject's overall mean time was calculated. Meanadjusted decision times were then sorted by knot and unknot pair type and the mean deviation for each pairing plotted in rank order from fastest to slowest subject. The ranked deviations from mean decision times for different subjects for unknot pairs, knot pairs and knot/unknot pairs are illustrated in Figure 6.9.

It was found that the time taken to make a decision on a knot/unknot pair is close to the subject's overall mean decision time. Moreover, whilst the figure confirms that knot/knot pairs take longer than unknot/unknot pairs, it shows that the difference in log-times remains almost constant from the slowest subject to the fastest.


Fig 6.9 Graph showing ranked individual times for different pair types

Another of the effects, that Shape 3 is easier to process and results in shorter decision times than any other knot type, is also clearly demonstrated across subjects. For every subject the shortest mean decision time was for Shape 3.

The rotation effect is not so widespread across all individuals as only 13 out of the 21 subjects took longer for $180^{\circ}$ rotations than $0^{\circ}$ or $90^{\circ}$ and $270^{\circ}$.

The data for errors, when taken subject by subject, was not sufficient for analysis ${ }^{7}$ and no inference could be made.

[^33]
## Learning curve

The time taken for subjects to make a decision shortened on average as the experiment progressed. In other words the mere act of doing the tasks enabled them to learn how to solve them more easily. A main effect plot is shown in Figure 6. 10 which plots the mean times for each item in order from 1 to 126. The order was a different random order each time and so all knot types, pair types and rotations are randomly distributed throughout the item numbers. This has the effect of making a large spread but the overall trend is obvious. A linear regression was applied to the means and showed a negative time gradient of 0.00423 with a predicted shortening of decision time on average of 1.7 seconds.


Figure 6.10 Graph showing how decision times shorten as the experiment progresses

### 6.7 Summary of findings of the pilot study

The hypothesis that stimulus pairs of knots with higher numbers of crossings up to five would yield longer decision times was not supported. A linear relationship was not found between decision times and rotation. The increase in decision time brought about by a $90^{\circ}$ rotation was much greater than the increase in decision time brought about by increasing the rotation from $90^{\circ}$ to $180^{\circ}$, in some cases the latter increase in rotation had a zero effect. We may look to a number of factors to explain the findings reported here - the presence of unknots, the symmetry considerations, the shape of the knot, or the extent to which the tasks are performed as 3-D rotations. Indeed, perhaps no single factor is responsible. A consideration of these factors follows.

### 6.7.1 Knot or unknot pairs

Unknot pairs show the fastest decision times with knot pairs the slowest. Subjects reported attempting to unravel mentally a knot or unknot much of the time when making decisions. With unknots, the progressive mental removal of crossings would occur until only a simple loop remained. With knot pairs the processing would be expected to take longer since, after an unproductive attempt to unravel a knot mentally, an alternative strategy would have to be employed. This could possibly be mental rotation of the whole shape or alternatively a matching of crossings one by one would occur requiring subjects to process and remember a series of crossing comparisons sometimes involving changes in orientation. This would appear to be a more formidable task than the simpler outcome following the processing of an unknot.

### 6.7.2 Orientation differences

Lowest decision times and error rates were recorded for all stimulus pairs at $0^{\circ}$ of rotation. Decision times were essentially the same for $90^{\circ}$ and $270^{\circ}$ rotations. Knot pairs with rotation to $180^{\circ}$ took significantly longer for Shape 2 than those to the intermediate $90^{\circ}$ and $270^{\circ}$ positions suggesting perhaps that this was the most similar result to those obtained
by Shepard and Metzler (1971) and Shepard and Feng (1972).

Although mental rotation was one reported strategy used by subjects in decision making, no linear relationship between decision time and degree of rotation was found in this pilot study. The data show that for unknot/unknot pairs and for unknot/knot pairs there is no evidence to suggest mental rotation as a strategy. Other researchers, Suzuki and Nakata (1988), when investigating the effect of image size, have also found that mental rotation cannot be asserted to be a strategy for some tasks; they found no evidence in their results to support rotation as a solution strategy for 'different' pairs of figures like Shepard and Metzler's.

Rotation of knot figures does require additional processing time and perhaps the use of a range of strategies. Five strategies were reported as being used, only one of which involved mental rotation of the image. This is in contrast to Shepard and Metzler's (1971) results where mental rotation was claimed to be the only strategy used by their subjects.

### 6.7.3 Deformable compared to non-deformable

There are other possibilities which may be responsible for the lack of a uniform rotation gradient here. Firstly, the stimuli represented in the current study are deformable with the nature and relative position of crossings comprising the only constraints in confirming the knottedness, or otherwise, of the figure. The polyhedral figures, however, are completely rigid and non-deformable whilst the folding paper figures can only be deformed along adjoining sides of the squares; such figures are then only partially deformable or semi-rigid in nature. Specific strategies are required for rigid objects compared to a range of possible strategies for objects with easy deformability, mental rotation being only partially effective in the latter case. Fewer factors were needed in decision making for polyhedral figures and paper folding than for knots. In the first case only rotation within the plane or in 3-D was required for essentially the same type of figure. In the second case, only successive folding along
adjoining edges of the squares was required. In the case of knots and unknots a number of factors were present for consideration, differences in the number, nature and relative position of crossings, the overall shape of the stimulus and its figural symmetry. Hence it seems clearly a much more complex task.

### 6.7.4 Knot types

Shape 3 evoked faster decision times than other knots and was most easily recognised. This is shown both by anecdotal report and by shortest decision times, some subjects report that it had a pronounced 'right way up'.

In trying to explain the short decision times for this knot we may look to Kosslyn's (1994) theory of a 'foundation part' of an image and the stored concomitant spatial relations with other parts of the global image. This conception may be helpful to subjects in orienting the final image after mental rotation. Some knot types have obvious foundation parts and the presence of such a characteristic, as in Shape 3 with its 'top loop', may be a useful device in matching images.

Alternatively, Reisberg and Chambers (1991) report a strategy of (verbally) reassigning the 'top' of an image for rotation tasks and that this method is more effective than rotation for recalling and reinterpreting the image. It is also possible that the strong bilateral symmetry in the outline shape of Shape 3, a symmetry prevalent in our natural and manmade environment, may be a factor in the ease with which subjects can memorise and manipulate such shapes.

The stated hypothesis would have predicted Knots 3 and 4 to show similar decision times since they shared the same crossing numbers, however, this was not the case. In fact Shape 3 showed the shortest decision times of all the six types ( 6.8 seconds) and Shape 4 the longest (11.5 seconds).

### 6.7.5 Strategies and subjects

Overall, subjects in this study reported five major strategies which were employed to reach decisions. Personal means were found to vary considerably, the shortest being 4.8 seconds with standard deviation of 1.7 and the longest being 17.2 seconds with standard deviation 14.3. Logie's (1995) point that most research reports upon the patterns of data over all subjects and that, when treated individually, the subjects' personal results may not necessarily demonstrate the found effects for the data overall was investigated and found not to be the case here. As already noted our data show that the significant effects are present within subjects. Some subjects are slower than others, and some are more error prone, but they are all affected by the factors reported here.

### 6.7.6 Verbal or spatial?

A major consideration in the mental representation of any problem solving process refers to the roles of verbal (analytical) or spatial (analogue) thinking and the mixing of the two. Clearly the strategies reported above would support both verbal and spatial roles. It is reasonable to assume that the strategies of shape recognition and rotation would be primarily spatial in nature. Unravelling, could be viewed as an admixture of the two, whereas in viewing such local aspects of the figures as positions and sequences of crossings it seems likely that counting and labelling would more likely be verbal in nature.

The comparative complexity of 2-D and 3-D object rotations has been explored by Jolicoeur, Regehr, Smith \& Smith (1985) who report that rotation of 3-D objects takes longer than rotation of the same outlines when portrayed as 2-D images. Knot images could be considered to be 2-D in nature with a verbal addendum to describe crossings and/or deformability. Alternatively, the images can be said to have a surface representation (2-D image) and a deep representation (encoded structural information) as described by Kosslyn \& Shwartz (1977).

The relative efficacy of verbal versus pictorial strategies has been
studied by Brandimonte, Hitch and Bishop (1992). For their tasks, involving drawings of familiar objects, they found that verbal strategies were often preferred and were commonly used even when these strategies were ineffective. Jolicoeur et al. (1985, page 101) ask the question 'Are different processes at work in the mental rotation of twodimensional and three-dimensional representations?' They note that the complex transformations taking place when a 3-D shape is rotated only occur when the rotation is out of the image plane. When 2-D stimuli are involved rotations are usually in the image plane. For the knot tasks the rotations are certainly in the image plane but the images have both a surface representation and a deep (structural) representation. It is not confirmed that the Kosslyn and Shwartz (1977) model is used in any of the strategies described by the subjects in performing these tasks and neither is it established here which are the most efficient strategies.

### 6.8 Concluding remarks

What the research seems to show is that mental rotation is not the most efficient strategy for these tasks and that perhaps the non-rigid nature of the objects provides a route to a more efficient mental manipulation strategy. This alternative mental strategy might take the form of mentally untwisting, bending or sliding the rope. The fact that unravelling was reported to be used more frequently as the experiment progressed and also that unknot pairs had shortest decision times support this.

A number of major factors seem to be involved in these tasks: pattern recognition at a global and local level, prior knowledge of knots and the types of strategies employed, and the influence of verbal and spatial thinking. The next phase of the study was planned so as to find out more about these strategies.

An exploration of subjects' learning of strategies and changes of strategy as they carried out the tasks was needed as well as an inquiry into how the subjects learned to do the tasks. Also of interest was whether
subjects with expertise in language used different strategies from the mathematicians. This qualitative part of the research is described in Chapter 8.

In addition, one must ask the questions 'Can the results be replicated?' and 'Are the results reliable?'. These further questions are addressed in the second study by means of a modified set of tasks. The test items were combined and reduced and a larger sample size for the experiment was sought. This second experiment is described in Chapter 7.

## Chapter 7

## The second study

### 7.1 Methodology

The first experiment showed that valuable results could be collected and seemingly no major methodological problems had been found. All subjects had completed all the tasks and error rates were relatively low (7.8\%). The study uncovered some interesting results and one of the hypotheses, that relative rotation within a pair caused an increase in decision time, was confirmed. Furthermore the results showed that there was no significant difference between relative rotations of $90^{\circ}$ and $270^{\circ}$.

This similarity in answer times for $90^{\circ}$ and $270^{\circ} 1$ rotations was not a surprising result given that subjects could be expected to treat these as different directions of the same rotation, that is $90^{\circ}$ anti-clockwise or clockwise, rather than as increased rotation in a clockwise direction. It was likely that, where mental rotation was the employed strategy, subjects could process both directions of rotation equally and that a $270^{\circ}$ rotation was treated as a $90^{\circ}$ rotation in the opposite direction. Previous research such as Cooper and Shepard (1973), Tarr and Pinker (1989) assumed that transformations took the shortest way around to the upright.

Taking into consideration the results of the pilot study, a second experiment was devised which took into account the result for $90^{\circ}$ and

[^34]$270^{\circ}$ rotations by condensing these into one category $90^{\circ}$. As a result, a more balanced set of knot/unknot and same/different items was devised.

It was also useful to be able to reduce the demand made upon subjects carrying out the experiment. It was essential to maintain the full concentration of subjects throughout the performance of the tasks. By reducing the number of items and thus shortening the period of time required for subjects to complete the tasks it was also possible to secure the help of a larger number of postgraduate volunteers. It has been noted (Christensen (1988, page 133) that the degree of commitment of subjects has a great effect on the reliability of results. It was essential to recruit subjects who would take the experiment seriously and maintain their concentration and commitment throughout the experiment. In the pilot study the time taken for subjects to complete the experiment, not including the time taken for the follow up interview, ranged between 25 minutes and 45 minutes, a considerable commitment of time and concentration from subjects. Indeed all had shown total commitment to the importance of accurate data collection for the experiment. It was anticipated that the second study should take subjects no longer than half an hour.

Details of the items used in the second experiment are given in section 7.3. The four independent variables under experimental control were as before: the number of crossings, degree of rotation, topological status (knot or unknot), and knot shape. The design of this second experiment was essentially the same as for the pilot study in that stimuli were presented either as a pair of knots, a pair of unknots, or a knot and an unknot, the main difference being that a different selection of pairs was used. As explained above, orientation differences within each pair were now limited to $0^{\circ}, 90^{\circ}, 180^{\circ}$ and the number of items was reduced to 72 instead of the original 126.

The main aim of performing a second experiment was to replicate the results with a larger and different group and thus confirm the
reliability of this type of test in identifying complexity factors of spatial tasks. A further aim was to investigate different groups of subjects from different educational backgrounds and different aptitudes. As the effects of different kinds of thinking and/or past experience of spatial tasks was of interest, a further research question was 'Would the mathematicians necessarily be better spatial thinkers? 'Better' is measured here by speed and also by accuracy.

Individual subject variation had been noted in the pilot study and, whereas every subject can be expected to have their own individual abilities and weaknesses, the technical and educational background of subjects was expected to have an effect. Much research has been carried out to identify the differences in spatial skills between men and women and it was decided also to explore this aspect for these more unusual spatial tasks.

The strategies which had been identified in the pilot study seemed, in most cases, to be refined and improved during the course of the tasks to the extent that subjects developed a spatial dexterity. The graph in Chapter 6 which illustrated this learning effect was Figure 6.8. Another objective was to confirm this effect for a different and larger set of subjects. The question to be addressed was 'Could this learning be repeated with a different set of subjects and a shorter test with a reduced set of test items?'

### 7.2 Subjects

Forty-eight subjects ( 33 male and 15 female) who were in the age range 21 to 45 were tested and all had a background of higher education. Subjects were drawn from different sections of the University of Wales Bangor, most (40) were postgraduates studying to become secondary school teachers of mathematics or science. The modal age was 22 although five were mature students. Two smaller groups were tested, four students from the final year of the B. Ed. course with a literature emphasis and four computer scientists who were working on a multimedia research
project and who were trained in either programming or design. The latter group were predicted to have above average spatial skills. The sample thus covered a variety of aptitudes although it was not representative of the general population as it was restricted to individuals who were in, or had been through, higher education.

### 7.3 Stimuli

Overall, the same 72 pairs of knots or unknots were presented in a different randomised order for each subject. The 72 items consisted of 12 items for each of the 6 knot types used. These were made up as follows: knot/knot, three items with relative rotations $0^{\circ}, 90^{\circ}$, or $180^{\circ}$; unknot/unknot, three items with rotations $0^{\circ}, 90^{\circ}$, or $180^{\circ}$ (the latter groups making up six 'same' pairs); and knot/unknot orientations, six combinations of 'different' pairs.

These combinations gave a balanced number of items for each group and also provided a representative set of rotations. Table 7.1 lists all the pair combinations ${ }^{2}$.

In both groups of six pairs there are two pairs with zero relative rotation, two pairs with $90^{\circ}$ relative rotation and two with $180^{\circ}$ relative rotation, making 12 pairs in all for each knot shape. This is more balanced than in the pilot study which had 21 items in all for each knot shape, 3 items for each with $0^{\circ}$ rotation and 6 items each with $90^{\circ}, 180^{\circ}$ and $270^{\circ}$

[^35]TABLE 7.1
Pair combinations used in second experiment

| LEFT |  | RIGHT |  | RELATIVE <br> ROTATION | NUMBER of observation |
| :---: | :---: | :---: | :---: | :---: | :---: |
| LEFT KNOT | 0 | RIGHT KNOT | 0 | 0 | 276 |
| LEFT KNOT | 0 | RIGHT KNOT | 90 | 90 | 276 |
| LEFT KNOT | 0 | RIGHT KNOT | 180 | 180 | 276 |
| LEFT unKNOT | 0 | RIGHT unKNOT | 0 | 0 | 276 |
| LEFT unKNOT | 0 | RIGHT unKNOT | 90 | 90 | 276 |
| LEFT unKNOT | 0 | RIGHT unKNOT | 180 | 180 | 276 |
| LEFT KNOT | 0 | RIGHT unKNOT | 0 | 0 | 276 |
| LEFT KNOT | 0 | RIGHT unKNOT | 90 | 90 | 276 |
| LEFT KNOT | 0 | RIGHT unKNOT | 180 | 180 | 276 |
| LEFT KNOT | 90 | RIGHT unKNOT | 180 | 90 | 276 |
| LEFT KNOT | 180 | RIGHT unKNOT | 0 | 180 | 276 |
| LEFT KNOT | 270 | RIGHT unKNOT | 270 | 0 | 276 |

### 7.4 Hypotheses

Given the results from the pilot study, it was not expected that higher crossing numbers would result in longer decision times, but that the four crossing knot, shape 3 , would continue to show the shortest decision times. Knot/knot pairs were expected to require longer decision times as were pairs with some relative rotation.

A further hypothesis was that there would be different interactions between knot shapes and pair type or knot shapes and rotation in that some combinations would result in longer decision times than others, these differing rates thereby implying different levels of complexity or differing methods of solution. For example, if some knot pairs are more affected by rotation to varying degrees than others, this might suggest mental rotation being used as a strategy for those pairs. Deeper analysis of the interaction between pair type and knot shape and between rotation and knot shape was expected to show statistically significant differences and that these differences would show that mental rotation was not a
preferred strategy.

Finally, the different groups of subjects were expected to produce different decision times and it was predicted that the language specialists would require longer times. The overall effect of this would predict a longer mean decision time for the new sample than for the sample used in the pilot study.

### 7.5 Procedure

Each subject was read the protocol (see Appendix II) explaining the experimental task and was shown a demonstration illustrating the manipulation of an actual closed loop knot. Subjects were informed that they must make a decision about each of the stimulus pairs and that the time taken to do this would be recorded. Before the start of the experiment each subject completed a practice session of eight stimulus pairs which were checked for accuracy of response. As before, when subjects indicated complete understanding of the tasks and the experimental procedure, then the test commenced and data collection was recorded automatically.

### 7.6 Results ${ }^{3}$

As with the pilot study the distribution of decision times was positively skewed with a median of 8.6 seconds and a mean of 11.4 seconds (compared to median of 7.7 seconds and a mean of 10.0 seconds in the pilot study).

[^36]The subjects in this study had a different profile to the group in the pilot study. In the pilot study group 11 out of the 21 were final year undergraduates in mathematics or were studying for a PhD in mathematics, whereas less than half of the group in the second study had a background in mathematics (22 out of 48). This was one reason for predicting that the second experimental group would produce a longer mean decision time. In addition, the experiment consisted of fewer items and hence the opportunity to learn and improve was also reduced.

A frequency distribution for the data is shown in Figure 7.1. After applying a log transformation to the data, the distribution was tested for normality and was validated with a $p$ value of less than 0.01 for the null hypothesis (see Figure 7.2). The mean log decision time is 2.21 with a median of 2.14 and standard deviation 0.63 .


Figure 7.1 Frequency distribution of the raw data for decision times in seconds


Figure 7.2 Frequency distribution and boxplot of the transformed data, log time

As before, for the purposes of the data analysis, a natural log transformation of decision time was used and the responses were categorised by the original four variables, number of crossings ( 3,4 or 5 ), degree of relative rotation $\left(0^{\circ}, 90^{\circ}\right.$ or $\left.180^{\circ}\right)$, topological status (knot or unknot) of the pair combination, and shape.

An overview of all the data is illustrated in this introductory section. The first comparative graph, Figure 7.3, shows a box plot of log decision times for the three possibilities for crossing number. The stated hypothesis was that higher crossing numbers would not result in higher decision times, but that the four crossing knot, shape 3, would continue to show the shortest decision times. It can be seen immediately from Figure 7.3 that increase in crossing number is not a factor causing a linear increase in decision time, in fact the decision time overall for the 4
crossing knots is less than the decision times for either of the other two crossing numbers tested. This is mainly due to the effect of Shape 3 the figure eight knot, which it will be shown, continues to bring down the mean time for the 4 crossing knots as a grouping.


Figure 7.3 Box plot showing the variation of decision time with crossing number

Answer times for different knot types


Figure 7.4 Histograms showing the distribution of log decision times with knot shape

It can be seen easily from the histograms in Figure 7.44 that knot shape 3 has a smaller median decision time than Shape 6. Indeed with closer inspection it seems that Shape 3 has the smallest median of all the knot shapes 1 to 6 . These results indicate that the crossing number of the knot should not be considered as a characteristic grouping affecting decision times but that the knot shape can be used. What is not established as yet is the hierarchy of level of complexity for each of the knot shapes. What the results do suggest is that complexity does not depend upon the number of crossings in the knot shape.

From studying the illustration of the data in Figure 7.4 it seems that knot shape is an important factor in predicting decision time. The

4 In all the comparison histograms used in this chapter the frequencies are expressed as a percentage so as to aid comparison between groups
next variables to consider are rotation and pair type. Figure 7.5 shows the histograms for variation in log decision times according to rotation and pair type. It appears also that both of these variables are important factors in predicting decision time. The stated hypotheses that knot/knot pairs and that pairs with some relative rotation would require longer decision times seem to be upheld.

Answer times by relative rotation and pairtype


Figure 7.5. Histograms showing the variation of $\log$ decision times with rotation and pair type

There were also two new variables to be investigated in this second study, gender and educational background. Figure 7.6 shows the histogram for gender and Figure 7.7 shows the histogram for educational grouping. No obvious inferences can be drawn from Figure 7.6 but Figure 7.7 suggests that computer scientists are the fastest and linguists are the slowest at performing the tasks.

Answer times according to gender



Figure 7.6 Histogram showing the variation of log decision time according to gender


Figure 7.7 Histogram showing the variation of log decision time according to educational group

The two dependent variables were decision time and error rate. Graphs illustrating the broad view for the first of these, decision time, have been shown in Figures 7.1 to 7.7 and the next few graphs show broadly how error rate is affected by the same independent variables, rotation, pair type and knot shape. Figure 7.8 shows how the errors are distributed over the different relative rotations, for example it can be seen that almost $50 \%$ of the errors made were made on items involving $180^{\circ}$ rotation. It is interesting to note that although no linear relationship was found for increase in decision times with rotation, there seems to be a trend for error rate to increase with increased rotation. This again confirms the finding in the pilot study.

## Errors by rotation



Figure 7.8 Distribution of errors over rotations

Figure 7.9 shows the distribution of errors according to knot shape.

Shape 3 and 5 seem to show dramatically lower numbers of errors than the other knots, with Shape 1 having the highest number of errors.

## Errors by knot shape



Figure 7.9 Distribution of errors over knot shapes

There seems also to be a difference in number of errors ${ }^{5}$ made for different pair types with knot/unknot pairs causing more errors than the other pair types. This is illustrated in Figure 7.10.

[^37]

Figure 7.10 Error rates for different pair types

Error rates ${ }^{6}$ for the different groups are illustrated in Figure 7.11 and error rates for males and females in Figure 7.12.


Figure 7.11 Error rates for the different groups

[^38]

Figure 7.12 Error rates for males and females

Having looked at the characteristics of the data as a whole, an analysis of variance was carried out on the data and, as for the previous set of results, the assumptions required for such analysis outlined in section 6.6 .2 of Chapter 6 were tested for validity. Once again, p values of 0.001 were found regarding normality and homoscedasticity. Not only were several of the significant effects which had been revealed in the pilot study confirmed, but in addition some new influences and interactions emerged.

This time, as well as looking for main effects using a one-way analysis of variance, a two way analysis of variance was carried out to identify any interaction effects. To explain the kinds of effects which may be happening, if in looking for main effects, rotation is shown in the one way analysis of variance to have an effect of adding complexity to the tasks this can be investigated over all knot shapes to see if the effect is always the same. If, in a similar way, the shape of a knot is shown to have some effect on complexity, and it is already known that rotation affects complexity, what happens when the two are combined? Are the
two effects additive or will there be an interaction effect which alters this overall effect of rotation and knot type? What may happen is that one knot shape will be more affected by rotation than another, we may find that some knot shape 1, for example, is adversely affected by rotation to a greater extent than knot shape 2. These issues can be investigated using a two way analysis of variance. As with the first set of data in order for the assumptions for the F test to be upheld, a log transformation was applied to the data. The results of both analyses are described in the following subsections.

### 7.6.1 Knot or Unknot pairs?

Figure 7.5 illustrated the distribution of $\log$ decision times for different pair types. Decision times were significantly longer for knot pairs than for other pairs $[\mathrm{F}(1,3310)=22.72, \mathrm{p}<0.001)]$. There was no statistically significant difference in mean decision times between unknot pairs and the combination of a knot with an unknot $[F(1,2482)=0.69, \mathrm{p}=0.41)]$. The mean decision time for an unknot pair was 10.7 seconds compared with 11.1 seconds for a knot with an unknot. The mean time for a knot pair was 12.6 seconds. Figure 7.13 shows this graphically.


Figure 7.13 Mean decision times for different pair types

### 7.6.2 Orientation differences

Figure 7.5 illustrated the distribution of $\log$ decision times for different relative rotations. Once again pairs with no rotation offer the least difficulty to subjects in terms of producing shorter decision times. These items also have the smallest standard deviation of log time, standard deviations being 0.56 for $0^{\circ}$ compared to 0.64 and 0.67 for $90^{\circ}$ and $180^{\circ}$ respectively. Decision times were fastest for $0^{\circ}$ rotation and slowest for $180^{\circ}[\mathrm{F}(2,3309)=37.18, \mathrm{p}<0.001]$ and the overall means were 9.5 seconds $\left(0^{\circ}\right), 12.0$ seconds $\left(90^{\circ}\right)$ and 12.6 seconds $\left(180^{\circ}\right)$. Greater orientation differences were found to produce longer decision times overall, however, the differences in mean decision times between the $90^{\circ}$ and $180^{\circ}$ were not found to be significant. Mean times for all pair types and the three relative rotations are shown in Table 7.2 and in the graph in Figure 7.14.

TABLE 7.2
Mean decision times (all values in seconds)

|  | All data$n=3312$ | Correct$n=3044$ | Incorrect$n=268$ | Rotation |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  |  |  |  | $0^{\circ}$ | $90^{\circ}$ | $180^{\circ}$ |
|  |  |  |  | $\mathrm{n}=1104$ | $n=1104$ | $\mathrm{n}=1104$ |
| Knot/unknot | 11.1 | 10.9 | 13.4 | 10.7 | 11.6 | 11.1 |
| Unknot/unknot | 10.7 | 10.5 | 14.2 | 7.9 | 11.3 | 13.0 |
| Knot/knot | 12.6 | 12.3 | 16.8 | 8.8 | 13.6 | 15.4 |
| Overall | 11.4 | 11.1 | 14.2 | 9.5 | 12.0 | 12.6 |



Figure 7.14 Mean decision times for different rotations

### 7.6.3 Crossing number and knot type

Figure 7.3 illustrated in a box plot the distribution of log decision times for different crossing numbers and Figure 7.4 showed the histogram of distribution of $\log$ decision times for different knot shapes. The mean decision time for 4 crossings was 9.8 seconds compared to 12.0 seconds for

3 crossings and 12.3 seconds for 5 crossings (see Figure 7.15). Although crossing number did seem to be a factor which was giving rise to significantly different decision times $[\mathrm{F}(2,3309)=38.9, \mathrm{p}<0.001]$, the actual number of crossings could not be used as a predictor for decision times, some other effect was causing this difference. Knots with higher numbers of crossings were not found to take longer. The variable which was causing the significant difference was not so much the number of crossings as the knot shape.


Figure 7.15 Mean decision times for different crossing numbers

Within the group for crossing number 4, decision times for knot shape 3 (four crossings) were significantly different to decision times for knot shape 4 (also four crossings) $[F(1,1103)=168.21, \mathrm{p}<0.001]$. No significant difference was found between the knots with three crossings $[\mathrm{F}(1,1103)=1.1, \mathrm{p}=0.294]$ but there was a difference between the knots with five crossings $[\mathrm{F}(1,1103)=5.19, \mathrm{p}=0.023]$. When treated as two groups the three crossings knots were not significantly different to the
five crossing knots $[\mathrm{F}(1,2206)=0.01, \mathrm{p}=0.922]$.
There was a significant difference in decision time between knot shapes $[F(5,3306)=53,85, p<0.001]$ with shape 3 and shape 4 sharing the same crossing number but not the same decision times. Both in the pilot study and again here, the figure eight knot (shape 3) required shorter decision times than all the others. The mean decision time for this knot shape 3 was 7.5 seconds in comparison with 12.1 seconds for shape 4 , the overall mean being 11.4 seconds. Figure 7.16 shows the graph of mean decision times for different knot shapes.


Figure 7.16 Mean decision times for different knot shapes

The longest mean decision time for the various knot shapes was 13.1 seconds and occurred for Shape 6 . Table 7.3 shows all the data for the different shapes.

TABLE 7.3
Mean decision times by knot type and rotation

|  | Mean decision times (all values in seconds) |  |  |  |  |  |  |  |
| :--- | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: |
| Rotation |  All types Shape 1 Shape 2 Shape 3 Shape 4 Shape 5 Shape 6 |  |  |  |  |  |  |  |
|  | $n=3312$ | $n=552$ | $n=552$ | $n=552$ | $n=552$ | $n=552$ | $n=552$ |  |
| $0^{\circ}$ | 9.5 | 10.4 | 10.2 | 6.6 | 10.9 | 9.0 | 10.0 |  |
| $90^{\circ}$ | 12.0 | 13.4 | 12.4 | 7.4 | 12.4 | 12.1 | 14.5 |  |
| $180^{\circ}$ | 12.6 | 14.0 | 11.9 | 8.6 | 13.0 | 13.4 | 14.8 |  |
| All rot's | 11.4 | 12.6 | 11.5 | 7.5 | 12.1 | 11.5 | 13.1 |  |

### 7.6.4 Errors

The overall error rate was $8.1 \%$ ( 268 errors out of 3312 responses). The mean decision time for all errors was 14.2 seconds in comparison with 11.1 seconds for correct responses (see Table 7.2). This difference was significant $[F(1,3310)=29.87, \mathrm{p}<0.001]$, showing clearly that incorrect responses were associated with longer decision times. Table 7.4 shows the mean decision times for the errors data.

TABLE 7.4
Mean decision times for errors (8.1\% of all data)

| Mean decision times (seconds) |  |  |  |  |
| :--- | ---: | ---: | ---: | ---: |
|  | Rotation |  |  |  |
|  | All errors | $0^{\circ}$ | $90^{\circ}$ | $180^{\circ}$ |
|  | $n=268$ | $n=63$ | $n=76$ | $n=129$ |
|  | 13.4 | 13.6 | 13.3 | 13.2 |
| Knot/unknot | 14.4 | 6.9 | 10.1 | 16.3 |
| Unknot/unknot | 16.8 | 10.7 | 13.5 | 19.5 |
| Knot/knot | 14.2 | 12.9 | 12.8 | 15.7 |
| Overall |  |  |  |  |

The rate at which subjects made errors also varied with the different input variables. There was a higher error rate for $180^{\circ}$ rotation with $11.7 \%$ of items involving $180^{\circ}$ rotation being answered incorrectly $[F(2,265)=4.04, \mathrm{p}=0.019]$. Figure 7.17 shows the percentage error rates for different rotations.


Figure 7.17 Mean error rates for different rotations

The variation in the occurrence of errors through knot shapes is shown in Figure 7.18 and in Table 7.5 shape 1 being particularly error prone. Shapes 3 and 5 on the other hand produced significantly fewer errors $[F(5,3306)=13.68, \mathrm{p}<0.001]$. Where errors did occur for these knots, they occurred mostly for $180^{\circ}$. Table 7.5 shows the percentages of errors by knot type and by rotation. The data for knot shapes were combined to form two groups, knots 1,2 4 and 6 in one group and knots 3 and 5 in the other group, these formed two significantly different groups $[F(1,3310)=58.97, \mathrm{p}<0.001]$ with regard to error rate.


Figure 7.18 Mean error rates for different knots

Although there were fewer errors for shape 5, these items took significantly longer than other errors $[\mathrm{F}(5,262)=2.69, \mathrm{p}=0.02]$.

## TABLE 7.5

Percentage of errors (incorrect responses as a \% of the total responses) by knot shape and rotation

| Rotation | Shape 1 $n=74$ | Shape 2 $n=60$ | Shape 3 $n=15$ | Shape 4 $n=51$ | Shape 5 $n=18$ | Shape 6 $n=50$ | All types $n=268$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $0^{\circ}$ | 9.8 | 7.1 | 0.5 | 9.8 | 1.1 | 6.0 | 5.7 |
| $90^{\circ}$ | 12.0 | 10.9 | 1.6 | 8.2 | 1.6 | 7.1 | 6.9 |
| $180^{\circ}$ | 18.5 | 14.7 | 6.0 | 9.8 | 7.1 | 14.1 | 11.7 |
| Overall | 13.4 | 10.9 | 2.7 | 9.2 | 3.3 | 9.1 | 8.1 |

With regard to pair type, there was a significantly higher error rate for different pairs $[F(2,3301)=8.5, \mathrm{p}<0.001]$. Error rates were $10 \%$ for
knot/unknot pairs compared to $6.5 \%$ and $5.8 \%$ for unknot/unknot pairs and knot/knot pairs respectively. Of the errors which were made for these 'same' pairs, most occurred for $180^{\circ}$ rotation ( $72.2 \%$ and $64.6 \%$ for unknot/unknot pairs and knot/knot pairs respectively) whereas rotation appears not to affect the 'different' pairs. In other words subjects were saying erroneously 'yes', that a pair was the same $10 \%$ of the time when they were not. Similarly, the correct response for 'same' items is 'yes' but subjects were saying 'no' that the images were not the same.


Figure 7.19 Mean error rates for different pair types

### 7.6.5 Groups

Data for all 48 subjects are reported in this section. The footnote on page 177 at the start of this results section explained that not all of the data for all 48 subjects would be used in the analysis up to this point. For this part of the analysis the full data were analysed according to the four different
groups taking part in the study, mathematics students, science students, language students and IT staff researchers. Figure 7.7 showed the distributions of the log decision times for the four groups. There was a significant difference in decision times between the four groups with language group taking longer to perform the tasks than the other three groups $[F(3,3440)=44.67, \mathrm{p}<0.001]$ with a mean decision time of 15.4 seconds in comparison with the overall mean of 11.6 for the 48 subjects data. The computer scientists were the fastest group having a mean of 8.0 seconds.


Figure 7.20 Mean decision times according to group

No significant difference was found between the mathematics and science groups $[\mathrm{F}(1,2874)=0.02, \mathrm{p}=0.878]$ and their mean decision times
were 11.4 and 11.7 seconds respectively.


Figure 7.21 Error rates according to group

With regard to error rate the computer scientists did the worst with an error rate of $12.8 \%$. There was no significant difference between error rates in the other three groups ( $5.7 \% 7.3 \%$ and $8.0 \%$ linguists mathematicians, and scientists respectively)

### 7.6.6 Gender differences

In this second study it was decided to look for gender differences both in terms of decision times and error rates. An overview of the data for
gender was shown in the histograms in Figure 7.6.

It was described in the previous section that the four groups had significantly different decision times. The language group had significantly longer decision times whereas the computer scientists had significantly shorter decision times. The language group subjects were all female and the computer scientists were all male. It would not be possible to state which of the two variables, gender or educational group was responsible for any result involving these small groups. In fact it could be argued that the effect upon decision time is not generated by academic background at all but by gender.

In order to resolve this, the two large groups (the PGCE groups) which showed no inter-group difference were examined for gender difference. Analysis of variance on these groups show that the female subjects were faster than the male subjects $[F(1,2807)=8.43, p=0.004]$. It is interesting to note that if the data for the mathematicians and scientists had not been available a conclusion may have been made that the effect of gender is not significant. It may be the case that many studies which purport to find gender differences (usually the reverse of that found here!) may be identifying differences which are due to some other variable which has not been controlled for.

So with regard to the small group of linguists it is probably the fact that they are linguists and not that they are women that they take longer to perform the tasks. However, the result does appear even more surprising since females should take less time according to the finding on gender difference.

Error rate overall was $8.1 \%$. Error rates for gender show males with an error rate of $8.5 \%$ and females with $7.2 \%$. The error rate for the linguists was the lowest of all at $5.7 \%$, however, their sample size is too small and none of these differences is significant $[F(1,3311)=1.41, \mathrm{p}=$ $0.236]$.

### 7.6.7 Interactions

Initial analysis of the effects of the individual variables such as pair type showed that each of the variables under consideration had an effect on decision time. However, the effects were not always as predicted and, it was also possible that there may have been some interactions taking place between the variables, for example, different pair types may be affected differently by different rotations.


Figure 7. 22 Interaction plot for pair type and rotation acting on mean decision times

In fact greater orientation differences do not produce longer decision times for all the pair types. Interaction between rotation $\left(0^{\circ}, 90^{\circ}\right.$ and $180^{\circ}$ ) and pair type (knot/ knot, unknot/unknot or knot/unknot) can be observed in the data shown earlier in Table 7.2. Overall, rotation
has the effect of increasing decision time and overall, pair type has an effect on decision time with unknots being the fastest. However, the rotation effect is not additive for all pair types. An interaction plot demonstrates this and it is evident from the plot shown in Figure 7.22 that rotation has little effect on the decision times for knot/ unknot pairs. This suggests that mental rotation may not be the process by which these tasks are performed.

Whereas unknot/unknot pairings produced the shortest decision times overall, they did not produce shortest decision times for the $180^{\circ}$ rotation nor for the data which resulted from incorrect responses (see Table 7.2). The plot shows that the shortest decision times at $180^{\circ}$ rotation occur for knot/unknot pairs.


Figure 7.23 Interaction plot of pair type and rotation acting on error rate

A similar result can be seen in the occurrence of errors for different pair types under different rotations whereby error rate increases
with rotation for knot/knot pairs and for unknot/unknot pairs but remains fairly steady at around $9 \%$ for knot/unknot pairs under any rotation. See Figure 7.23. These results add weight to the proposal that knot/unknot pairs are not solved by mental rotation but by some different method(s).

With regard to the interaction between knot shape and rotation, one knot shape was more adversely affected by rotation than the others. Shape 6 was most affected perhaps suggesting that mental rotation was a strategy used for this knot shape. Decision times for all knot shapes were affected by rotation but Shape 3 was the least affected and Shape 6 the most.


Figure 7.24 Interaction plot of knot shape and rotation acting on mean log times

An interaction plot of knot shape and rotation is shown in Figure
7.24 and it can be seen that the steepest rotation gradients lie between $0^{\circ}$ and $90^{\circ}$ and the steepest gradient of all occurs for knot shape 6 between $0^{\circ}$ and $90^{\circ}$.

This interaction can be plotted in another way by switching the variables between axis and legend and this is shown in Figure 7.25. This plot shows clearly how shape 6 stands apart from the rest exhibiting the largest gap between the plotted points for $0^{\circ}$ and $90^{\circ}$. The knot shape which shows the largest gap between $90^{\circ}$ and $180^{\circ}$ is shape 5.


Figure 7.25 Interaction plot of rotation and knot shape acting on mean log times

The difference which rotation makes to decision times for Shape 6 and the other knots is illustrated in Figure 7.26 where knots 1 to 5 have been combined into one group and compared as a whole with Shape 6. Shape 6 shows the steepest rotation gradient between $0^{\circ}$ and $90^{\circ}$ as well as
showing longer decision times overall.


Figure 7. 26 Interaction comparing the decision times for knot shapes 1 to 5 with knot shape 6 with and without rotation

The effect of the interaction of knot shapes and rotation on error showed that Shape 4 error rates were little altered by rotation. All other shapes show the highest number of errors for $180^{\circ}$ rotation. It was reported earlier that Shapes 3 and 5 produced significantly fewer errors overall. Where errors did occur for these knot shapes they occurred mostly for $180^{\circ}$.

An interaction plot on error rate for all knot shapes under the effect of rotation is shown in Figure 7.27 where it can be seen that shapes 3 and 5 have the highest percentage of correct responses and yet have a steep rotation gradient (large gaps on the graph) from $90^{\circ}$ to $180^{\circ}$. This interaction plot also clearly shows how shape 6 has the steepest rotation gradient between $90^{\circ}$ and $180^{\circ}$. Shape 6 is most affected by rotation in error rates as well as decision times. The effect of rotation on knot shape

6 is clearly demonstrated in the errors data (see Table 7.5) where it can be seen that $14.1 \%$ of errors occurred on $180^{\circ}$ rotation compared to $6.0 \%$ and $7.1 \%$ for $0^{\circ}$ and $90^{\circ}$ respectively. It is also demonstrated in the interaction plot in Figure 7.27. The figure also shows that Shape 1 has the worst error rate of all for $180^{\circ}$ rotation.


Figure 7.27 Interaction plot of rotation and knot shape acting on error rate

The interaction between knot shape and pair type also shows some interesting properties. Knot/knot pairs times show a fairly steady increase in decision time through knots 3-6 (see Figure 7.28). Knots 3, 5 and 6 are fastest for knot/unknot pairs increasing through unknot/unknot to the longest decision time for knot/knot pairs. Knots 1,2 and 4 had their longest decision times for knot/unknot pairs.

The interaction between knot shape, pair type acting on error rate
shows mainly that shape 3 and 5 have unusually low error rates for unknot/knot pairs in comparison with other knot shapes (see Figure 7.29). Overall these two knots produced more accurate responses with little effect due to pair type whereas shapes 1 and 4 attract the worst error rate for unknot/knot pairs.


Figure 7.28 Interaction plot of knot shape and pair type acting on log decision time


Figure 7.29 Interaction plot of knot shape and pair type acting on error rate

When considering interactions between gender and the other variables the only noticeable effect is pair type. Females seem to show a steeper pair type gradient, taking longer than males for knot pairs. Females have more difficulty with knot/knot pairs than the other pair types. The interaction plot is shown in Figure 7.30.

No statistically significant interactions between group and other variables were found.


Figure 7.30 Interaction plot for gender and pair type acting on log decision time

### 7.6.8 Learning curve

Analysis of the decision times as the experiment progressed showed that there was an overall decrease. The order in the sequence of the 72 items has a significant effect on decision time $[\mathrm{F}(71,3240)=3.99, \mathrm{p}<0.001]$. The mean decision time for the first 12 items over all subjects was 14.3 seconds compared to the mean decision for the last 12 items of 8.8 seconds. One of the research questions to be addressed was, 'Can visualisation be taught using knot tasks?' This result suggests that the answer is 'yes'. What was of interest, was to find out if subjects could learn these strategies and improve their ability to solve these spatial tasks by practice. In fact subjects did learn how to do the tasks as they proceeded through the experiment.

The time taken to respond to any item decreased as the experiment
progressed and when decision times are plotted against the order of the item being measured, a steady decrease can be observed. This finding was explored both for the data as a whole and also for each individual subject. Figures 7.31 plots the mean decision times for the items against the order of execution from 1 to 72 of the data for all subjects. The order in the sequence and the decision time data was also plotted for each subject and a regression line fitted the data. In addition, an overall regression of decision time and order was carried out for the whole data set. These plots are shown in Figures 7.32 to 7.37 .


Figure 7.31 Graph showing how decision times shorten during the experiment

The regression plot for the complete set of data results in a negative gradient of 0.008 for log decision time over order of item. This amount to an average decrease in decision time due to learning of 1.8
seconds. Forty four out of forty six subjects show an individual negative gradient in the raw data and forty five subjects show an individual negative gradient in the data when corrected for the known affective variables7. This would suggest that 45 of the 46 subjects learn how to do the tasks during the course of the experiment. Table 7.6 gives the values for the learning gradient for each subject. The mean gradient for all subjects remains 0.008 even after adjustment due to the ordering of difficult items.

[^39]

Figure 7.32 Learning curves for subjects 1-8


Figure 7.33 Learning curves for subjects 9-16


Figure 7.34 Learning curves for subjects 17-24


Figure 7.35 Learning curves for subjects 25-32


Figure 7.36 Learning curves for subjects 33-40


Figure 7.37 Learning curves for subjects 41-46 and the overall learning curve page 216

TABLE 7.6
Learning curve gradients for all subjects

| Subject | Raw data gradients | Gradients after correction |
| :---: | :---: | :---: |
| 1 | -0.0122 | -0.0106 |
| 2 | -0.0041 | -0.0049 |
| 3 | -0.0136 | -0.0135 |
| 4 | -0.0006 | -0.0011 |
| 5 | -0.0165 | -0.0160 |
| 6 | -0.0074 | -0.0073 |
| 7 | -0.0075 | -0.0073 |
| 8 | -0.0017 | -0.0013 |
| 9 | 0.0032 | 0.0041 |
| 10 | -0.0049 | -0.0057 |
| 11 | -0.0078 | -0.0085 |
| 12 | -0.0152 | -0.0141 |
| 13 | -0.0083 | -0.0094 |
| 14 | -0.0038 | -0.0054 |
| 15 | -0.0111 | -0.0100 |
| 16 | -0.0063 | -0.0070 |
| 17 | -0.0069 | -0.0072 |
| 18 | -0.0079 | -0.0076 |
| 19 | -0.0082 | -0.0071 |
| 20 | -0.0002 | -0.0023 |
| 21 | -0.0115 | -0.0130 |
| 22 | 0.0016 | -0.0003 |
| 23 | -0.0081 | -0.0068 |
| 24 | -0.0119 | -0.0113 |
| 25 | -0.0133 | -0.0127 |
| 26 | -0.0086 | -0.0087 |
| 27 | -0.0105 | -0.0115 |
| 28 | -0.0053 | -0.0048 |
| 29 | -0.0150 | -0.0134 |
| 30 | -0.0145 | -0.0133 |
| 31 | -0.0048 | -0.0044 |
| 32 | -0.0098 | -0.0080 |
| 33 | -0.0056 | -0.0062 |
| 34 | -0.0124 | -0.0098 |
| 35 | -0.0051 | -0.0050 |
| 36 | -0.0125 | -0.0109 |
| 37 | -0.0070 | -0.0071 |
| 38 | -0.0092 | -0.0101 |
| 39 | -0.0098 | -0.0104 |
| 40 | -0.0083 | -0.0092 |
| 41 | -0.0109 | -0.0091 |
| 42 | -0.0011 | -0.0010 |
| 43 | -0.0076 | -0.0082 |
| 44 | -0.0060 | -0.0046 |
| 45 | -0.0055 | -0.0056 |
| 46 | -0.0084 | -0.0072 |

### 7.7 Summary

### 7.7.1 Limitations of the research

Subjects taking part in this study came from a variety of backgrounds, were of both genders and of various ages. They were, however, academically able with the majority being graduates in mathematics or the sciences. Although the significant effects present in the first study were confirmed here, it should not be assumed that they will apply across the ability range. The conclusions drawn regarding imagery and visualisation are specific to the sample.

The sample were, however, 'mixed ability' in relation to the range of individual means and error rates measured. No significant difference was found between the mathematics group and the science group. A difference was found, both in terms of decision times and accuracy, between the mathematics and scientists classified together, the graduates in social sciences and languages, and the group of IT specialists, but these were not balanced samples. A better research design would be to carry out the experiment with balanced subsets of these groups.

Although this is an unrepresentative sample, it is an interesting result that forty five out of the forty six subjects in the sample improved their performance of the tasks.

### 7.7.2 Main findings

The results of this second study confirm the main finding of the pilot study, that increasing rotation of these knot stimuli does not result in a corresponding proportional increase in decision time, the differences in mean decision times between the $90^{\circ}$ and $180^{\circ}$ were found not to be significant. Rotation does add to the complexity of the tasks but does not result in a requirement that mental rotation be used as a method of solution.

Increasing the number of crossings in a knot diagram (up to five) does not necessarily increase the time required to process the pairs involving that knot. Crossing number alone does not increase complexity, but different knot shapes do affect the ease with which subjects can solve the tasks and some shapes are affected more adversely than others by rotation. Different pair types gave different results for decision time and error rate. The comparisons of an unknot with another unknot, and an unknot with a knot are easier to resolve than the comparison of a knot with a knot. Errors were highest for comparison of knot/unknot pairs, subjects perhaps 'guessing' that the pair were the same when they were in fact different ${ }^{8}$.

Rotation was shown to adversely affect both decision time and accuracy overall but knot/unknot pairs, when considered separately, showed little increase in decision time with rotation and, whereas error rate increases with rotation for knot/knot pairs and for unknot/unknot pairs, it remains steady for knot/unknot pairs under any rotation.

Shape 6 was more adversely affected by rotation than any other knot shape. The percentage increase in mean decision time was $48.0 \%$ for Shape 6 whereas it was only $19.3 \%$ for other knots. Perhaps the asymmetrical nature of knot shape 6 together with the fact that it had the highest number of crossings in the experiment resulted in subjects being more confused when this shape was rotated.

A difference was found between males and females. These spatial tasks are solved more quickly by females than by males when other variables such as educational background are controlled for. Educational background does affect decision time and subjects with technical backgrounds performed the tasks more quickly than those from language studies backgrounds.

[^40]
### 7.7.3 Conclusions

All apart from one subject became more proficient at doing the tasks as the experiment progressed. The regression plots show that the subjects improved their spatial skills as a result of tackling the problems, 45 regressions showed a negative gradient indicating decreasing decision time during the course of the experiment.

The hierarchy of levels of complexity for each of the knot shapes which emerges from the results does not establish a clear explanation for the factors which affect their complexity. Bilateral symmetry and the existence of a foundation part of a shape do seem to assist mental processing, and Shape 3 incorporates both of these. There is some degree of symmetry in the other shapes ${ }^{9}$, knots 1 and 5 showing bilateral symmetry and knots 2 and 4 showing both a horizontal and vertical line of symmetry. However, none of these has a distinguishing local feature which could act as a foundation part. Knot shape 1 could be assigned a top and a bottom, as could Shape 5, but such designated features are not quite so easy to distinguish when rotated as the obvious loop in the figure eight knot. Furthermore, because knot shape 2 and 4 have an additional horizontal mirror line, it is difficult to process a $180^{\circ}$ rotation for these knots.

The other finding, that for comparable groups, females are faster at performing these spatial tasks than males, refutes the usual endorsement of a fundamental gender difference in spatial ability. Proponents of this theory may need to find other factors to explain any observed superiority in spatial ability of males.

The results reported thus far give some indication of the strategies which were used to solve certain items and also give an indication of the relative complexity of the tasks. These issues are explored in greater depth in the interviews with new subjects reported in the next chapter.

[^41]
## Chapter 8

The Case Studies

### 8.1 Issues

The previous two chapters have raised such issues as the efficacy of strategies and the acquisition of skills in performing the spatial tasks. This chapter describes how these matters were explored in more depth. The main reason for carrying out this further study was that qualitative data regarding spatial thinking and mental strategies would clarify the inferences made from the results of the quantitative data in the previous two studies. Furthermore, comments were recorded in Chapter 2 regarding research techniques for exploring imagery and it was noted that many researchers (such as Bishop, 1983) advocate a combination of quantitative and qualitative data collection. This view was accepted and appropriate procedures undertaken.

The opinions of subjects who have performed the tasks provide information regarding the level of difficulty of the tasks and confirmation or otherwise as to whether imagery skills are improved by thinking in this spatial way. If pupils' imagery skills are to be developed effectively then suitable sequenced tasks are needed. Interviews with subjects performing these mental manipulative tasks may provide the necessary information to do this. This information may then be used to plan activities for a spatial development curriculum.

In some ways the tasks are intuitive, individuals have experience of how ropes move, but mental tasks showing images of ropes ${ }^{1}$ will probably be unknown to subjects and how they react to the problems and to the mode of thinking required will be of interest. The mode of

[^42]thinking will be out of context rather than unfamiliar, since the subjects will be familiar with interwoven strands of rope or fibres of some kind. Any acknowledgement by the subjects of the usefulness of prior experience will also be of interest. If, as a result of working on the tasks, subjects do begin to feel 'successful and confident' in one kind of mental imagery, then the aim expressed by Bishop (1983) and reported at the end of Chapter 5 may have been met.

According to the data from both of the studies already described, the answer to the question 'Do subjects learn how to do the tasks?' seems to be 'Yes' (Figures 6.10 and 7.31). In following up on the quantitative study, this chapter explores such questions as 'Can this finding be confirmed verbally by the subjects themselves?', and 'What do subjects believe are the changes taking place in their performance of the tasks?' and also 'Do subjects report a dichotomy of approaches between verbal and spatial?'.

Subjects for the case studies were drawn from two backgrounds, mathematics and languages. Detailed verbal reports on the development of strategies were sought, whether they were verbal or visual, as well as comments on what the subjects found easy or difficult in the tasks and whether they became more able to solve them as the experiment progressed.

### 8.2 Methodology

The two approaches used in this research employ very different methodologies, each with a role in answering the research questions posed. A qualitative research technique was to be used in this phase of the study and the data to be collected would not be of measures such as decision times or error rates, but characteristic data regarding performance of the tasks.

The research method used was to record on video, the subjects performing a subset of the experimental tasks. Five subjects were studied
and, in seeking to determine various modes of thinking, it was considered beneficial to investigate subjects who were likely to be articulate and to be able to express their thoughts clearly. As in the quantitative studies, the subjects therefore represent an able and perhaps atypical sample. Two subjects were graduates in English (aged 22), one was a mathematics graduate (aged 23), one a Welsh graduate (aged 43) and one had a PhD in Celtic Studies (aged 43).

After completion of the tasks, extended interviewing took place whilst viewing the first recorded video tape. The purpose was to investigate in depth the subjects' personal perceptions of their thought processes during the performance of the tasks.

## TABLE 8.1

Knot pairs used for interviews with Subjects 1,2 and 3

| Knot | shape | Pair type |
| :---: | :--- | :---: | Rotation | 5 | unknot/unknot | 0 |
| :---: | :---: | :---: |
| 2 | unknot/unknot | 90 |
| 2 | knot/knot | 0 |
| 2 | unknot/knot | 180 |
| 6 | knot/knot | 180 |
| 6 | unknot/unknot | 0 |
| 6 | knot/knot | 90 |
| 5 | knot/knot | 0 |
| 2 | unknot/unknot | 0 |
| 3 | unknot/unknot | 0 |
| 2 | unknot/knot | 0 |
| 6 | unknot/knot | 0 |
| 1 | knot/knot | 90 |
| 3 | unknot/knot | 180 |
| 1 | unknot/knot | 0 |
| 4 | unknot/knot | 0 |
| 2 | unknot/knot | 90 |
| 3 | knot/knot | 180 |
| 6 | unknot/knot | 90 |
| 6 | knot/knot | 90 |

Twenty knot pairs were used so as to give a mixed sample of each
knot type, pair type and rotation. In all other respects the experimental setup was the same as in the pilot study. The interviews were carried out in two phases; the first group consisting of the first three subjects, were presented with the items listed in Table 8.1 and the second group consisting of subjects 4 and 5 were presented with the items shown in Table 8.2.

TABLE 8.2
Knot pairs used for interviews with Subjects 4 and 5

| Knot | shape | Pair type |
| :---: | :--- | :---: |
| 3 | unknot/unknot | 0 |
| 3 | unknot/knot | 0 |
| 5 | unknot/unknot | 90 |
| 6 | knot/knot | 180 |
| 1 | knot/knot | 180 |
| 5 | knot/knot | 90 |
| 4 | knot/knot | 90 |
| 4 | knot/knot | 0 |
| 2 | unknot/knot | 90 |
| 6 | unknot/unknot | 90 |
| 4 | unknot/knot | 0 |
| 6 | unknot/unknot | 0 |
| 4 | unknot/knot | 90 |
| 5 | unknot/knot | 90 |
| 3 | unknot/knot | 0 |
| 4 | knot/knot | 180 |
| 2 | unknot/unknot | 0 |
| 4 | unknot/knot | 90 |
| 2 | knot/knot | 90 |
| 1 | knot/knot | 90 |

Subjects were shown the initial demonstration as for the two previous studies and then viewed three random pairs to make sure that they had understood what the tasks involved. The process for filming was explained and when the subjects indicated that they had understood the procedure, the experiment commenced and filming began.

The main difference between performance of the tasks in the previous studies and in the case studies was that decision times were not
measured, this measurement was of no importance in this phase of the study. Instead, the subjects were requested to articulate their thoughts and to 'think aloud' as they progressed through the tasks and to try to explain the methods and strategies they employed in the decision making process.

The camera was placed behind and to the side of the subject so that the computer screen and the knot pair being worked on were visible on camera. Some discussion between subject and researcher did take place during this phase but the in depth questioning and analysis was left until the interview stage. After completion of the twenty trials the tape was removed from the camera and played back. During the viewing, the subject was invited to stop the tape as desired at any stage and to observe the process of their thinking. The subject was encouraged to expand upon the explanations being articulated on tape during the 'thinking aloud' process. This extended explanation was also recorded on videotape for analysis.

During this second part of the procedure, subjects were questioned as to which items they had found difficult and as to whether they had made any changes in strategy as the experiment progressed. A general discussion between subject and researcher occurred. For the five subjects in all who were interviewed, there resulted about 4 hours of videotape. The retrospective comments regarding reasoning processes were not only helpful for this research but were also beneficial to subjects as an introspective debriefing exercise. Transcripts of the interview part of the experiment are given in Appendix III.

### 8.3 The five domains to be reported

Details of the interviews are presented in this chapter according to five domains, namely; strategies, use of a foundation part, changes of strategy, learning of skills and finally, the relative difficulty or ease which subjects had experienced with the various items.

The previous studies had revealed some of the strategies used by subjects and these interviews were carried out so as to allow exploration of those strategies and also to identify any new strategies.

The presence of a foundation part seemed to be a factor affecting decision times in the earlier experiments, and the figure eight knot shape (shape 3), which exhibited a pronounced foundation part, showed shorter times. This effect was also reported by subjects in their debriefing interviews after the first experiments. Another comment emanating from these debriefing interviews was that subjects reported changing strategies as they became more familiar with the tasks. Deeper investigation of both of these issues is reported here.

The subjects' views on how successful they felt at the start and at the end of the experiment is discussed in the context of how much benefit, in relation to mental spatial skills, they had derived from doing the tasks. Likewise, their views on which tasks were harder to do and which were relatively easy is summarised.

In the following sections, details are given about each of the subjects, and their performance of the tasks is described. Strategies and modes of thinking used by the subjects are analysed and a detailed summary of the interviews is reported.

### 8.4 Subject 1

Subject 1 had graduated in English and was training to become a secondary school English teacher. She had a not uncommon fear of mathematics and considered her spatial ability to be weak. She used her hands a lot when doing the tasks and also when expressing herself. She was able to give quite detailed descriptions of her thoughts during the initial phase whilst performing the tasks as well as clear explanations later.

### 8.4.1 Strategies

Subject 1 started the first few items by tracing around each knot with one finger. She started the interview afterwards with:
'Initially it seemed easier to do it with my finger, but then it got confusing.'
This subject found for later pairs, with relative rotation, that she was unable to succeed with a tracing strategy and hence proposed in the above statement that the 'finger' method was confusing.

In the early stages of the experiment, this subject was tracing around the knot and looking for differences by using a 'matching' strategy (see Section 6.6.1 in Chapter 6) which she verbalised as follows :
'This one goes over and this one goes under'

She later employed a verbal technique corresponding to the 'identifying sequences of crossings' strategy described in Section 6.6.1 in Chapter 6, but when she noticed that one was the unknot, she adapted her verbal strategy. This is demonstrated in the following quote:
"They look the same initially - that's under, over, under . . . I'd say that one's a knot . . . . and that's a loop. Yes that's lying on top.'

The subject was compiling a sequence of words as a verbal coding of the knot by its various crossovers, under, over, over, over, under etc., but this descriptive coding was soon superseded by other strategies.

When explaining her thoughts about a shape 2 pair, presented as a knot/unknot pair with $90^{\circ}$ relative rotation, (see Figure 8.1) she said:
'Initially I was going like this (shows how she was following the rope around with her finger) but I think you can take that and pull it and then open the whole thing out.'


Figure 8.1 Shape 2 unknot/knot pair with $90^{\circ}$ relative rotation

Here the subject was articulating her new strategy for identifying unknots. In the statement 'open the whole thing out' she was starting to use a new strategy, 'unravelling', which may have resulted from her coding ${ }^{2}$. Talking about the figure 8 knot shape 3 , she said:
'The loop here and the loop there . . . one seemed to go through, the other seemed to be on top.
and also:
'This one is folded or balanced on top.'
Having realised, by whatever means verbal or visual, that some of the crossings can be removed the subject is articulating an essentially visual strategy.

A very quick response was forthcoming for the pair of unknots

[^43]shown in Figure 8.2. She said:
'I'd say they're identical and it's just a matter of untwisting that and pulling ' (she used her hands to demonstrate)


Figure 8.2 Shape 3 unknot/unknot pair with $0^{\circ}$ relative rotation

This subject did not readily use a strategy of mental rotation. Whenever there was a $90^{\circ}$ turn, rather than mentally rotating the image, the subject tracked the rope around to try to spot differences,

### 8.4.2 Foundation part

The figure 8 knot, shape 3, provoked the characteristic 'foundation part' response whereby this subject's attention was focused on the upper part of the shape which she, and other subjects, call the 'loop'. For the pair with $180^{\circ}$ relative rotation shown in Figure 8.3, this subject said,
'That one (pair) struck me straight away as being both knots because it seemed to go through the top loop in both of them.'
and also added 'It stood out.'


Figure 8.3 Shape $3 \mathrm{knot} / \mathrm{knot}$ pair with $180^{\circ}$ relative rotation

The statement 'it stood out' referred to the part of the rope passing through the 'top loop'. The subject was focussing on this, observing its route and then mentally judging the figure.

This subject spoke about a 'dominant' shape when describing a shape 2 pair with $180^{\circ}$ relative rotation. One of the pair drew her attention and she was inclined to start from a particular part of the knot diagram, (see Figure 8.4):
'This seems more dominant than this. It seems to lead the way more. This one (the one on the left) just seemed easier to follow.'

Later, when pressed to explain her use of the word 'dominant', she said:
'It was just that it went over and round and it seemed to have a strong shape to it.'


Figure 8.4 Shape 2 unknot/knot pair with $180^{\circ}$ relative rotation

The subject's attention was drawn to some part of the shape, possibly the lower half of the shape on the left, which seemed to either have more significance for her or was easier to recognise.

### 8.4.3 Changes of strategy

After only two or three pairs, subject 1 stopped tracing the path of the rope systematically with her 'under, over' coding and began to use a different strategy involving looking for unknots. She explained that the tracing strategy was relinquished but that the new strategy did not always work:
'That can also be misleading . . . . , because then I was trying to do a bit more, trying to see if they were a loop first of all or not.'

She found this method quite successful but when the knots wouldn't unravel she looked for an alternative. At this stage she tried the strategy which she liked the least - matching crossings:
'That one was harder. I was a bit lost there. It was more a case of following the rope and seeing if both fell in the same places'

She explained the situations in which she needed to switch to this matching strategy:
'I think that's how I went through with most of them but then it didn't work all the time. Usually when it didn't work it's because there was intertwining.' (she used her fingers interlocked to explain)

After the unravelling strategy failed the subject tried to explain the evolution of her other strategy. She said:
'I started working on one individually and then on the other individually and then matching them. But then sometimes it was easier to look at them both and to say 'oh that goes there' and then it goes there as well. And if you find a couple of things that do match up then just go through them both again until you find that they all either match up or are different.'
She added:
'The ones that had a lot of different loops on. I wasn't just looking at one for a long time on its own because that would be too confusing. . . ones with a lot of different loops and turns I was just looking at them both and without trying to go through one just seeing, do they both do this, do they both do this. It was easier to do it like that. If I was just to concentrate on one and just go into it, I would just confuse myself.'

She was beginning to consider a global shape, or at least consider a larger portion of the image as one element. Whether any mental rotation was involved in this global matching is unclear.

This subject's method in attempting a shape 2 unknot/knot pair
with $180^{\circ}$ relative rotation was unusual but topologically sound. She assumed that the $180^{\circ}$ rotation did not affect the topological properties of the knot, she seemed to be treating the two orientations as if there was no rotation but simply a distortion of the knot, she was accepting the objects as truly deformable. and she tried to compare the two global structures. Later she became uncertain about this as a strategy and began to check her assumptions by applying another method.

### 8.4.4 Learning

The subject recognised that she learned to use the unravelling strategy more effectively and also developed her matching strategy so that it was simpler to apply, she tried to remember the spatial properties as a whole and the global shape.
'After seeing a couple of the knots I got more into what I was looking at and so it was easy to actually break it down and see if there was a bit that was just twisted and folded on top which would . . pause . . If you folded it back and untwisted it it would just make a loop. It became easier to do that when my eye became a bit more tuned in.'
She repeated:
'I got more into what I was looking at.'
Thus emphasising that she was learning how to solve the tasks.

### 8.4.5 Difficulties

For those pairs which had no relative rotation, this subject commented that they were the easier ones to work on and it was easier to reach a decision. She also commented that when there was some rotation, it immediately had the effect of making the knots look different, and harder to compare. She commented:
'These look different straight away. That seems to make it a lot harder ( $90^{\circ}$ rotation), the fact that it's turned around I think I
should turn my head round to see them the same way.'
This statement implies that mental rotation was difficult for her and the
statement which followed: 'They are both knots', suggests that she found analysing the topology easier than performing a mental rotation.


Figure 8.5 Shape 6 knot/knot pair with $90^{\circ}$ relative rotation

In the following dialogue ${ }^{3}$ the subject was speaking about the pair with $90^{\circ}$ relative rotation shown in Figure 8.5. The idea of a journey around the knot refers to the matching process.

S I found it harder to try and imagine this one turned slightly although I tried to do that, I tried to visualise that it was straight and then look at it but I couldn't hold that for too long.
I So you went for this other strategy of looking at the journey round the knot?

S Yes.
I And what were you finding from that? What were you noticing?
S Just that they all went under at the same place or they all rested on

[^44]another part of the knot in the same place and that if they were all to undo that they would have the same picture.

Overall, the subject has summarised in this dialogue the utilisation and efficacy of her different visualisation strategies

### 8.5 Subject 2

Subject 2 had also graduated in English and was training to become a secondary school teacher. She was more apprehensive about her mathematical ability than subject 1 but was interested to find out about both the experiment and about how she would tackle the tasks.

### 8.5.1 Strategies

This subject began by trying to match crossings. Where there was rotation she tried to rotate and then match. One of her strategies for dealing with a rotated image was similar to the well documented one of reassigning the top of the image which was reported in Chapter 3. Speaking about the pair shown in Figure 8.6 she said:
'What I'm looking at is the top and the bottom of the knot on the right and then the left and the right of the other knot and whether they are going underneath or over the top. They look the same.'

Later during the interview she said:
'That's sideways on. There are two loops, one below the other so I called it . . . , I identified a top loop and the other one the bottom, so in this other one I'm looking at the sides, left and right.'


Figure 8.6 Shape 2 unknot/unknot pair with $90^{\circ}$ relative rotation

With a pair of knot shape 6 and no relative rotation, this subject just went through matching crossings one by one. With the same knot shape 6 and $180^{\circ}$ relative rotation, (See Figure 8.7) she attempted to deal with more than one crossing at a time and, although she was unable to focus on the whole image, she did deal with a larger portion of it at one time. She focussed on two regions of the shapes and said:
'These two bits are the same definitely and I'd say those two bits are the same.'


Figure 8.7 Shape 6 knot/knot pair with $180^{\circ}$ relative rotation

She explained further:
'As soon as I see them lying in a different way I look to see . . . . well that might be the same but it's lying on its side and this one's top . . . well the way I'm identifying them is this little loop here (left hand side) and this little loop here (right hand knot), nothing else. The big spaces confuse me so I have to find something small and start from there. I have to have one point to start from and I don't like looking at things that are in different ways. (rotated) So these two loops here are the same as these two loops here. Well that goes over the top and underneath ( RH ) and underneath there and over the top '( LH )

### 8.5.2 Foundation part

When working on the trefoil knot, shape 1 , with $90^{\circ}$ rotation this subject started to use a verbal description of the shape. She separated the whole
shape into two regions, the central 'egg shape' and the lower two crossings with outer loops:
'When I saw this egg shape to the right, I looked at the other one to see if I could see the egg shape then I checked, is it going underneath?'

In trying to clarify this notion, the following dialogue took place:
I When you had identified the 'egg shape' where did you head for then?
S The two other shapes - the other two shapes were similar and all I wanted to establish was when that egg shape comes up to the top like the left hand image are those two other things going to come down to the bottom. Are they going to be at the bottom?
I So you were looking generally for the two . . . or were you looking for the left part and then the right part?
S No, two.
I So you were trying to deal with two at once?
S Yes. The egg shape was a separate thing and then the other two shapes on each image together. Not one and then the other, not like that, the two together, I linked the two together and then I quickly looked at the ropy bit - is it lying on the top or underneath?

### 8.5.3 Changes of strategy

Although this subject talked a lot about unravelling in the introductory session, she did not use it very much as a strategy. She relied upon rotating and matching equivalent portions of the images. We may perhaps make the assumption that this subject was not proficient at mental manipulation and preferred a more verbal strategy. The following quote explains her thinking:
'What I'm trying to do here is identify which loop is which. If they are going to be the same there's going to be one that is shaped like that (pointing) but it may be in another place. It
looks like that one there. So it takes me longer (with rotation) because I've got to sort it out, switch this round and bring this loop up to the top. That's going under, that's going under so that's OK.'

### 8.5.4 Learning

This subject had difficulty with the knots at $180^{\circ}$ relative rotation. At one point she said:
'They're the same at the edge but the one (crossing) in the middle is the same in reverse.'
She assumed that the knots could still be the same when two crossings matched and a third was 'the same in reverse'. The topological properties were not being considered, or at least she was not calling upon this as a strategy.

Earlier in the experiment, having already found a difference at some point, this subject still continued comparing. This might have been because she was unconvinced by the first comparison rather than a lack of understanding of the fact that one difference was sufficient to respond 'No'. She eventually decided, incorrectly, that the pair were the same.

The subject learned that she did not always need to apply a mental rotation but used instead the relative positions of crossings within the shape:
'I didn't try to do that (rotate the image) with all the loops.
There were two loops and I thought, there they are up there and there they are down there.'

What the subject did was to key in to an equivalent point in both images. She identified a specific part in the first image and then looked for that part in the second image to see if the two crossings matched. The crossing or pair of crossings were in the same relative positions and she did not need to rotate these but knew which to compare by their relative
positions in the knot shape.

During the course of the experiment the subject became more confident and faster, but did not show any significant change in overall strategies. The strategies which she used remained the same and whilst they depended upon the pair type and relative rotation, the strategies which she employed in the beginning were still being used at the end, possibly with a more restricted use of rotation. Her strategies did not change due to any learned technique or one which she perceived to be more effective.

### 8.5.5 Difficulties

This subject had particular difficulty with knot shape 6. Referring to one of the items, the subject said:
'I didn't like this one because it had a lot of loops and I don't know where to start. Your eyes are jumping from one place to another.'

She had particular difficulty with this shape with $90^{\circ}$ rotation.

### 8.6 Subject 3

Subject 3 had graduated in mathematics and was training to be a secondary school mathematics teacher. She was confident both mathematically and spatially.

### 8.6.1 Strategies

Initially, Subject 3 used her fingers to trace around both knots simultaneously starting from two equivalent points. She always determined an equivalent starting point to begin with, for both the left and the right hand knot, and then traced the journey around. She matched over, under, over, under, sometimes using a sequence of crossings and other times matching crossings one by one. A typical
verbalisation as she worked was:
'Start from here, that's over, that's over (matching) under, under following right the way round that's under again they're the same.'

When faced with knot shape 2 and $180^{\circ}$ relative rotation she said:
'That one needs to be turned around, so starting from here (matching relative positions and tracing) ... they're not the same.'

With shape 3 at $180^{\circ}$ rotation and a knot/unknot pair she said:
'Those two are placed on top and that's one of each (talking about crossings) so they can't be the same.'
Once she had found one difference she was happy to answer ' $N o^{\prime}$ '. This was no doubt due to her mathematical training.

Subject 3 began to formulate another strategy with the two unknots pairing of shape 3 :
'I think this was one of the first points I actually realised that because they're both over the top (the top loops in the figure of eight) it struck me that they are not really knots. They are both the same unknots.'

### 8.6.2 Foundation Part

This subject did not appear to use a foundation part as a strategy, until, during the interview she said:
'Your eye automatically goes to the middle of this one, where they are actually knotted'
She was speaking about shape 4 (see Figure 8.8) and the section in the middle where the rope seems to link together. She felt that this was a focus for her attention in spite of the fact that the rope appeared linked at this point even when it was the unknot form. The only sense in which she may have been drawn to one part of a shape was in choosing where to begin her traced journey around the knot.


Figure 8.8 Shape 4 knot/unknot pair with $90^{\circ}$ relative rotation

At one point in the interview she commented that the choice of starting point may or may not have been conscious:
'I always started from the same point and I still started from here at that point by the look of things. . . But the one I started off at was where the difference was, whether that was that coincidence at that point or not I don't know.'

### 8.6.3 Changes of strategy

Eventually Subject 3 used the unravelling strategy more readily and for the pair shown in Figure 8.9, she said,
'This loop here lifts off and that's (the other knot) actually knotted.'


Figure 8.9 Shape 6 knot/unknot pair with $90^{\circ}$ relative rotation

She remarked in the following dialogue about the noticeable change in strategy:
S I didn't actually match on this one. I just looked and saw those two together and that leapt out and I saw they were different there. So this was actually different.
I You solved this differently?
S Yes. I just looked at that loop and saw it was on the top and it was unknotted, and I looked at that one and saw it was knotted (pointing at the loop). I didn't go through the same pattern of turning it the right way up and finding my starting point and tracing it round. It just seemed to be obvious - that was obviously a loop.' (an unknot)

Watching herself on film she explained how the realisation happened,
'I've noticed here that that one is a knot and that one isn't
knotted there. I still started from here (pointing) but where I started off at was where the difference was.'

### 8.5.4 Learning

The subject recognised that she had learned to use the unravelling strategy more frequently and noticed that in solving an early item she has not used the strategy which later she developed successfully during performance of the twenty pairs.
'Looking at that now I can see that that lifts up and that one doesn't'

In discussion she explained:
I You were doing a lot of tracing round and checking crossing by crossing weren't you?
S Yes. In the early stages, yes. Whereas now looking at this I can see that flips off there, it's not knotted there but it is there.
I So you probably wouldn't feel the need to use the same strategies that you were using.
S No, I wouldn't. I'd look at this and I might not even have to turn it round. I can identify that point as being that point and
. . . . you see, that's knotted and that isn't there.

This subject made some further comments regarding her initial strategy:
'At the start I seemed to spend longer getting it orientated and then going from there'
She explained that she thought the new strategy of unravelling was more effective:
'Before I would have had to go back and check it again. This time I didn't have to.'
She also commented that she learned that when a pair was different, one was the unknot, and hence 'undoable', whereas the other was the knot and fixed:
'I hadn't made that distinction that if they aren't the same, one
is going to be undoable and the other isn't. That has just come to me now.'

### 8.6.5 Difficulties

This subject also commented that pairs with rotation were more difficult, she said:
'Because they weren't the same way up I found it more difficult to picture in my mind so I started to tilt my head like that and follow it but I found it easier with my finger and followed them round. They were both at the same point at the same time. I could simply check that they were both doing the same thing. I think later on it became easier for me to visualise without having to use my (finger).'

The subject explained in more detail why she was slow at deciding upon a particular pair (shape 2 unknot/knot pair with $180^{\circ}$ relative rotation) she was watching on the video:
'I was checking on this one because I found this more difficult because they were upside down (a relative rotation of 180) and I had to actually think about my finger on. . . It seemed difficult. Once again I started from here, but it was difficult to match here, and follow round. I found that one much more difficult than the other ones and I had to check again at the end whether it was right. I was not that confident in myself like before, to say yes, they are the same. This one I just wanted to check again because it did not look immediately so right to me.'

### 8.7 Subject 4

Subject 4 has a first degree in Celtic Studies, a PhD in Breton and is an accomplished linguist who works as a researcher in bilingualism. Having found mathematics 'difficult' at school, she was not confident mathematically.

### 8.7.1 Strategies

This subject was very thoughtful and took a long time to reach a decision for the first few pairs. She found the tasks very difficult and was reluctant to make a decision until she had tried several times to confirm it. This was particularly noticeable when she was looking at a pair of the same knots.

While working on the pair of shape 5 unknots with $90^{\circ}$ relative rotation shown in Figure 8.10, she said:
'I'm trying to tilt them in my mind'

She found the mental rotation difficult and added:
'Every time I try to turn it round in my mind I lose my bearings spatially. I find it difficult spatially to make the correspondence. If I follow through both parts of the knot . . .'


Figure 8.10 Shape 5 unknot/unknot pair with $90^{\circ}$ relative rotation

When dealing with knot shape 6 and $180^{\circ}$ rotation (shown in Figure 8.7), she said:
'Now this looks as though the right hand one is upside down. So once again I'm trying to swivel it round in my mind and I find that hard.'

On several occasions this subject used the strategy of reassigning the top (or side) of an image. She remarked about shape 1 with $180^{\circ}$ rotation:
'If I tilt it so that the bottom becomes the top'
and:
'If you turn the left hand one so that the right hand side is at the top . . . '

With shape 6 and $180^{\circ}$ rotation, she said:
'This one $I$ can see there is a top and a bottom which are different. With the right hand one you can see that the top is at the bottom.'

In the following quote regarding shape 4 with $90^{\circ}$ rotation, she seemed to be combining the strategy of rotation ('tilting') with reassignment:
'The right hand one is as though it's on its side, if I can tilt it up in my mind so that the right hand bit is at the top . . .'

This subject also used the strategy of matching crossings within the pair as follows:
'I've checked the way the rope crosses itself and it seems to cross itself in exactly the same way.'

When speaking about another the pair, the subject took some time but noticed a difference after attempting to match crossings:
'I'm trying to follow with my eye the top and I'm going round the left one goes underneath no they're not the same because the intersection at the bottom is not the same.'

Reflecting upon this combination of rotation with matching, she said later:
'My strategy is trying to get the right hand one identical with the left by switching it round in my mind and following it round piece by piece and switching back from the left hand knot to the right hand and then when I've decided that bit is identical, then I go back to the left hand knot. But the problem is if they're not in identical positions it's not always easy to find your bearings. You get lost and I kept on getting lost. That was my problem.'

Sometimes this subject used a hitherto unarticulated strategy which can be termed topological equivalence ${ }^{4}$. She used this strategy with the shape 1 and shape 5, the trefoil knot and the pentoil knot. Mathematically, these knots belong to the family of torus knots5, which have a regular form and as such are more well-suited to this strategy than the other shapes. The subject did not rotate or try to match crossings but distorted the knot such that the relevant information held in the crossings remained but the 'loops' were manipulated until they resembled the loops in the other knot. She saw the stimuli as truly deformable objects and was able to mentally distort the rope. Speaking about the pair shown in Figure 8.11 she said:
'If you got hold of the right hand one and pulled it a little bit it would be exactly the same.'

[^45]

Figure 8.11 Shape $5 \mathrm{knot} / \mathrm{knot}$ pair with $90^{\circ}$ relative rotation

In the following quote the subject was continuing to use rotation as well as unravelling:
'Oh this is terrible. (after a long pause) . . . . This isn't a knot, I don't know if the others were knots but that strikes me immediately, that I can actually pick that up and straighten it out. I'm going to see if the rope is positioned in the same way. Once again I have to swivel the right hand knot around $90^{\circ}$. Yes they are the same, they aren't knots.'

She later emphasised the fact that she continued to use rotation as a strategy with shape 2 at $90^{\circ}$ rotation:
'I'm twisting (the subject means rotating) the right hand knot. I have to do this. I cannot do it any other way, I have to try and get them both - looking at them in the same position in my mind. I cannot do it as it stands.'

This comment ignores the fact that initially she was going round the knot as well as mentally rotating.

### 8.7.2 Foundation part

This subject not only commented about seeing a 'top' and a 'bottom' of some shapes, but also mentioned that she saw a 'star shape' and 'two triangles'. She said:
'Some of them look almost symmetrical. Like two circles where you have - like that - almost like two triangles.'

This would have constituted some sense of a foundation part for the pentoil knot, shape 5. However, this subject also commented on a particular part of shape 6:
'I think this is a bit easier than the one before (shape 5) because the one before looked the same no matter which angle you looked at it from, and this one, I can see that there's a top and a bottom which are different and then with the right hand one you can see that the top is at the bottom.'

In a similar way to other subjects, she spoke of a 'top loop in shape $2:$
'The top loop lies on top of the other loop and there's a twist in the middle.'

### 8.7.3 Changes of strategy

Once this subject noticed that she could undo one of the knots, she used this strategy more and more. In the first few cases she still used rotation as a back up, but eventually abandoned this strategy even as a checking mechanism. She summarised:
'I don't have to look any further I know they are not the same.'

With regard to the strategy of tracing around the path of the rope she said:
'What I've stopped doing is following it round. In the beginning I was following the whole piece of rope round, was tracing it like a path and I'm just looking at the . . . it seems like I'm just switching back and forth, getting the same positions, here yes that corresponds that corresponds - yes they're the same.'

She commented that processing a global picture was still difficult and not a preferred strategy:
'I still can't see the whole thing at once but I find it easier to keep my bearings as I go back to the left hand one and the right hand one and I compare each bit of it.'

At the end, this subject was unknotting without rotating although at the start she had felt the need to use both. When there were no unknots present and she couldn't use the unravelling strategy, she reverted to following the rope around or mentally rotating or, in the case of shape 5 , looking for topological equivalence.

### 8.7.4 Learning

This subject realised that the tasks which she had initially found very difficult she could now do with ease:
'It seems very obvious to me now that they are the same. That corresponds to that and that corresponds to that . .'
and:
'I can do it a lot quicker now. I can see immediately that that's not the same.'

Commenting upon a pair of unknots, she said:
'They look so easy now, I can just open them up and they're the same.'

For shape 3 and an unknot/knot pair she said:
'If I put my hand, in my imagination, on the piece of rope that
goes across at the top on the right hand knot I could just pick that up and straighten it out with the one on the left I couldn't do that because it goes in and out.'

This subject reflected that the practice of the spatial tasks had enabled her to deal with spatial problems which previously she would have avoided:
'I think I got used to it generally, I have more confidence now to look at it as a whole. It seems to be easier now to switch it and turn it upside down and say yes I think that is the same. I am more able now to look at the whole pattern.'
'I was a bit lost in the detail at the beginning because you feel you're looking at each separate little piece of knot'

When watching herself reasoning about one of the pairs she said: 'I wasn't sure - I'm sure now (about her answer), but I wasn't then.'
and she confirmed that although she found the thinking challenging at the start, she had now achieved a much better grasp of how to tackle the problems:
'It seems to be easier now to correspond. I still can't do it very well, hold the whole image in my mind, but I can check the various bits. I seem to have got my bearings.'
She also said:
'I think that what I'm doing now which I didn't think I could do before. . . . .
'I don't have to switch the whole thing round in my head, I can switch each little bit through $90^{\circ}$.'

This latter comment occurred because the subject was matching and rotating at the same time.

### 8.7.5 Difficulties

This subject articulated her difficulties very thoroughly:
'If they are in the same position it's quite easy. If they're not in the same position I get confused and I lose my bearings. It's very much like losing your way in a town when you turn a corner and you don't think you're in the same street because it looks different from the other way and I'm having this problem here. I'm pretty sure they're the same but . . . . (long pause) yes they are the same.'
and:
'I find it very hard to hold the whole picture in my mind.'

Occasionally, after taking some time on a pair she broke off to explain her general perplexity at spatial tasks:
'No it's still not the same . . . I find this very difficult, I have this problem reading maps I'm not a very visual person.'
'It's as though I'm losing my bearings . . . . (long pause) Every time I try and turn it round in my mind I lose my bearings.'
'I find it very difficult spatially to make them correspond'

This subject tried to compare the unfamiliarity of the tasks with other new learning experiences:
'It's like when you are learning a foreign language, at the beginning, everything when you are learning a foreign language is just a blur. It's just one thing of speech and you're trying to make sense of it all and then as you get your bearings you pick out words which are significant. It's almost as if with this it's like a foreign language and suddenly you see what is significant - you don't have to look at everything.'

The following dialogue illustrates her thinking further:
S You suddenly see what's significant, you don't have to look at everything. I can pick out the important bits with my eyes. Like a foreign language when you can pick out the important bits.
I So when it comes to solving it, is there any sense in which you page 253
are using language or pictures?
S 'I'm not using language - it's almost as if its familiar and I can see the whole thing and its immediately obvious that its identical. I don't do that any more.'

### 8.8 Subject 5

Subject 5 graduated in Welsh and works as a researcher preparing a Welsh and English dictionary of terminology. She considers herself weak mathematically and was a reserved and somewhat inhibited subject. She found the tasks difficult and spoke frequently about 'checking' in her strategies. She was able to visualise and hold a stationery image but found dynamic imagery very difficult.

### 8.8.1 Strategies

The first remarks were simply 'under, over, under, over' indicating that she was following the rope around. Later on, regarding crossings which were not alternately under and over in an unknot, she said:
'I'm picturing the rope just dropped down.6'

The novel way in which this subject seemed to deal with unknots was to think about the closed loop of rope, the unknot, as the starting point, and tried to imagine it folded onto itself. She said:
'If I thought it was twisted (not a knot) I'd visualise twisting it to see if it would work out.'
She started with an image of an unknot and tried to deform it into the image on the screen.

Ideally, she wanted to have access to the actual rope and became rather frustrated. She said:
'If I had that rope that's lying over there to play with I could see whether it matches.'

[^46]Since she could not manipulate the actual rope she explained:
'I had to visualise a circle of rope and twist it and drop it.'

This subject commented that she found the three crossing trefoil knot, shape 1 , with $180^{\circ}$ rotation relatively easy and that the mental rotation strategy was effective for this pair:
'That one was so simple in terms of . . . . , it only crosses over three times and you could see easily that it was . . . . , if you rotated the right hand one you would find that the images looked similar and you could see whether they were matching by again looking where they crossed over'

She further explained:
'First of all I changed the right hand image round so that it faced the same way as the left hand one. And then I checked the 3 crossovers points to make sure they weren't just lying on top of each other.'

It was not clear how subject 5 was 'changing round' the image, one can easily conclude that this was by mental rotation but it could be by some other form of mental processing. She frequently touched the screen with her finger whilst processing the pair but commented:
'Even though I was touching a point on the screen there, in my mind I was touching the rotated vision not what was actually on the screen, even though I was touching the screen.'
'The difficult part was holding the rotated vision in my mind and touching that not what was actually on screen.'

### 8.8.2 Foundation part

This subject did not seem to make use of a 'foundation part' for the manipulation of any of the shapes. Her statement about one of the knot pairs: 'I can't even work out which way to rotate it, to get it similar' perhaps indicates that this subject did not have access to the notion of a
foundation part at all. She did make one remark which shows some recognition of a whole shape and that was regarding her perception of the similarity of shape 5 to a Celtic knot design. She said:
'It's like a Celtic knot really. It's a common motif.'

### 8.8.3 Changes of strategy

At the start, rather than using rotation, this subject tried the method of tracing around the knot, noting 'unders' and 'overs'. She originally believed that her 'unders and overs' method would be a better method than mental rotation but then changed her mind:
'I think I decided after that (trying unders and overs) that it would be simpler just to turn the knot over (rotate it)and see if I could get it - the two pictures if they were aligned in the same direction to see if they corresponded. I decided I wanted a second method of proving whether they were knotted or not.'

The subject was reluctant to use rotation as a strategy since she doubted its effectiveness and returning again to the first method she said:
'I decided that I wanted a second method of proving whether they were knotted or not, as well as rotation I thought I'd investigate the way they crossed over each other.'

### 8.8.4 Learning

This subject made less progress with these spatial tasks than any of the other subjects. During the interview phase she reconsidered one of the pairs which she had answered incorrectly. This was the pair shown in Figure 8.1. The subject still could not see that the left hand image was an unknot. She said, 'I still can't see the difference.'

This was puzzling since earlier the subject has expressed ease with the other three crossing shape, shape 1 , giving the reason that it has only three crossings. After further study, she realised that two consecutive crossings go 'over' and deduced that the left hand image was in fact the
unknot.

When questioned about how she felt she had learned to solve the tasks, she explained:
'You had to use more than one criterion in a way. To look at them both, and try and turn them round to see if they match by looking at the crossover points, and then visualising if you think it's not a proper knot, but twisted over. Then visualising it as a rope being twisted, a circle of rope twisted and laid down in that pattern. Those were the three criteria that I did use.'

She continued:
'If I thought it was twisted, I'd visualise twisting it to see if it would work out whereas if I thought it was knotted, I wouldn't use that technique.'

### 8.8.5 Difficulties

When looking at shape 6 with $180^{\circ}$ rotation this subject said:
'It's more complicated. I can't make sense of it straight away, I can't even work out which way to rotate it.'

She also had difficulty with shape 4 and $180^{\circ}$ rotation. When there was no relative rotation the subject used matching for most cases and relied very little on unravelling, a strategy which did not spring readily into her mind.

This subject stated that she was able to recreate an image and could match and rotate the image, however, what she claimed to find extremely difficult was manipulating, twisting or distorting the image.

### 8.9 Summary

### 8.9.1 Common themes

Most subjects reported using a variety of strategies, the initial one being one to one matching of crossings, possibly using some kind of verbal under/over system. The other reported strategies of unravelling and recognition of a foundation part were present for four out of the five subjects. Mental rotation was reported as one of the strategies used, but less so towards the end of the experiment when more effective strategies had been discovered. However, for the knot/knot pairs most subjects continued to test their decision that the knots were the same by matching or rotating.

All except subject 5 actually reported that they felt more confident towards the end of the experiment, some saying that they felt they had improved their spatial skills. It was also evident that the mathematician in the group had modified her range of strategies. At the start she had been very methodical employing non-visual strategies which although effective were not the most efficient or the most suitable. As she became more familiar with the nature of the images, she adapted her initial strategy to a more visual one where imagery and mental transformations played a greater part.

The language specialists did not adhere to verbal strategies finding their spatial capacities improving as the experiment progressed. Although some verbal component could be identified at some point in each subject's technique, they all reported using visual processes in their principal strategy.

### 8.9.2 Differences

The subject who was a mathematics graduate tackled the tasks from the start in a confident and systematic manner, perhaps so effectively that she failed to search for alternative strategies, until the unravelling strategy presented itself in a particularly obvious way. Once she had
learned this strategy she used it very effectively. The subjects who were not used to solving problems of a spatial nature were more inclined to doubt the effectiveness of their initial strategies and try to seek alternatives.

One of the English graduates tried to deal with more than just one crossing at a time by grouping parts of the knot shape into some larger aggregated section, possibly so that she had fewer data points to remember.

The fourth subject, although she did not consider herself a capable mathematician, made the most remarkable and mathematically astute observation, that two knots (shape 5 pair shown in Figure 8.11) can be shown to be topologically equivalent without performing mental rotation, she explained how she was able to distort one of the knots to exactly match the other.

The strategies reported in this phase of the research were mostly the same as those reported in the short debriefing interviews which took place after the earlier quantitative experiments. There were two new strategies which emerged, one which could be termed 'topological equivalence', and the other which could be considered to be in the same genre as 'unravelling' but in reverse, was the strategy which 'reconstructed' the unknot diagram from the simple loop. Each of these were reported by one subject only and this latter strategy may have been adopted due to an association with the demonstration given at the start of the experiment rather than referral to any previous experience of ropes.

### 8.9.3 Generalisations

The tasks were seen to be quite difficult, mainly due to orientation differences and, in some cases, due to the presence of more crossovers. The items which subjects found the most difficult were those involving knot shapes 4 and 6, notable quotes regarding shape 6 being: 'I can't even
work out which way to rotate it.' and 'It had a lot of loops and I don't know where to start.'

In agreement with the findings from the first studies, shape 3 was considered to be relatively easier to deal with and items composed of two knots required the most checking.

The consensus seemed to be that visualisation was employed and that it improved during performance of the tasks. A fitting conclusion to this chapter would seem to be a collection of quotes from the subjects talking about their experience:

I think that what I'm doing now which I didn't think I could do before $\qquad$
'It became easier to do that when my eye became a bit more tuned in'
'My brain has just got used to finding the significant parts'
'You suddenly see what's significant'
'I got more into what I was looking at.
'You learn to know what to look for'
'I'd look at this and I might not even have to turn it round'
'Before I would have had to go back and check it again. This time I didn't have to.'
'It seems very obvious to me now'
'They look so easy now'
'I think I got used to it generally, I have more confidence now'
'I wasn't sure - I am sure now'

## Chapter 9

## Discussion and conclusion

### 9.0 Introduction

In the 1970s, Krutetskii's (1976) major study showed that visual imagery is a component of mathematical ability. More recently, many other researchers in the field of mathematics education, notably M.A. Clements, (1981, 1983), Clements \& Del Campo (1989), Clements \& Wattanawaha (1978), Clements \& Battista (1992), Battista (1990, 1994), Presmeg $(1986,1998)$ and Wheatley $(1977,1990,1991)$ have highlighted the importance of this ability in the teaching and learning of mathematics. Thus the ability to visualise and to manipulate images has become a concern for teachers, such that techniques to develop the ability in pupils are needed. During the same period, psychologists established certain facts concerning some of the processes involved in imagery and a theory of mental imagery was proposed by Kosslyn (1980, 1983). Much of the research in the psychology literature focussed upon the possibilities for image manipulation of 2-D or 3-D rigid objects. The research described in this thesis brings together these two separate academic fields and extends the investigations to the processes and skills involved in the mental manipulation of deformable objects.

The terms 'imagery', 'visualisation' and 'spatial ability' are used throughout this thesis, both in the contexts used in previous research and in the description of this research. A discussion and clarification of these terms was included in Chapter 2 and, in this chapter, refined interpretations which have evolved as a consequence of the research are proposed.

### 9.1 General comments on methodology

## Visualisation

One of the stated aims of this research was to investigate the perception of deformable 3-D structures and the strategies used for their manipulation. The choice of stimuli derived from the author's experiences using similar tasks with school pupils when it was observed that some pupils possess great skill in dealing with these images whilst others do not.

## The Mathematics

The stimuli used in the research reported in this thesis were knot diagrams and the research bears some similarity to the work of Strohecker (1991) who carried out her study using actual knots. Whilst the tasks in either study do not relate directly to knot theory, certain aspects are concerned with one of the major questions which some algebraists seek to answer, 'When are two knots the same?'.

The mathematical nature of the tasks is confirmed in the results of this study by the way in which the thinking of many of the subjects evolved whilst performing the tasks, from an initial haphazard strategy to a more logical and clearly applied strategy as the experiment progressed. The experience of these subjects may be compared to that of young children experiencing early mathematical activities such as in the comparison of a number of 3-D shapes, a typical challenge being to place a hexagonal prism into the correct shaped hole in a 'letter-box'. The way in which the child learns to perform the task is not yet known. Whether she tries to visualise the action, or learns to recognise the appropriate shape by some spatial information it is not possible to say, but the child is experiencing mathematical thinking in whichever way the task is performed ${ }^{1}$.

[^47]The notion of relationships of parts of objects is important in conceptualising 3-D solids and their cross sections On the subject of geometrical objects and spatial reasoning, Clements and Battista (1992, page 420) begin their comprehensive work by stating:
'School geometry is the study of those spatial objects, relationships, and transformations that have been formalized (or mathematized) and the axiomatic mathematical systems that have been constructed to represent them. . . . . . . . Usiskin (1987), for instance, has described four dimensions of geometry; (a) visualization, drawing, and construction of figures; (b) study of the spatial aspects of the physical world; (c) use as a vehicle for representing nonvisual mathematical concepts and relationships; and (d) representation as a formal mathematical system. The first three of these require spatial reasoning'

The 'mathematization' and constructed axiomatic mathematical systems as well as the notion of relationships are also important in the case of knots. Strohecker (1991, page 215), in describing her research on knots as follows:
'. . . . . a research project concerned with the development of understanding of topology, a branch of geometry concerned with properties of objects which are invariant when the object itself is distorted or deformed.
Strohecker (1991, page 215) continues by claiming that working with knots and studying the positions and interrelationships of crossings aids spatial awareness:
'Arriving at an understanding of such properties can involve a process of constructing ways of identifying relationships among parts of the object.'

Hence it can be argued that the study of knots can aid geometry learning, spatial reasoning and visualisation.

Relationships in mathematics
Battista (1994) suggested that cognitive development involves acquiring
facts about a new situation and establishing routes and relationships connecting these facts and assimilating them into a schema. With time, increasingly higher levels of interrelationships are established and these become integrated into the schema. The van Hiele levels also refer to the significance of relationships. At Level 2 (Descriptive/Analytic) there begins a development of awareness of parts of figures and their properties but relationships between classes of figures are not recognised. At Level 3 (Abstract/Relational) relationships and definitions are beginning to be recognised; for example, the relationship between a square, a rectangle and a parallelogram is understood.

Krutetskii (1976) also paid attention to how the learner deals with mathematical relationships when he characterised the geometric type by a tendency to interpret visually any mathematical relationship or situation. Brown and Wheatley (1997, page 69) emphasised the importance of this way of thinking and stated:
'We believe that forming images of mathematical relationships is essential for effective problem solving.'

Reynolds and Wheatley (1997, page 104) commented that imagery was the route by which relationships were explored by their subject, Elaine:
'Elaine's imaging activity was the result of her intention to make sense of relationships.'
According to Reynolds and Wheatley (1997), imagery had given this learner great mathematical power enabling her to construct, examine, and reconstruct complex mathematical relationships.'

It is abstract relationships which tend to be of prime importance in mathematics and, in order to establish mechanisms for coping with abstract relationships, concrete relationships concerning many kinds of objects must be constructed first of all.

## The Curriculum

The experiments were carried out using diagrams of knots presented on a
computer screen. At the start of each experiment, subjects were shown a demonstration using actual ropes but had no personal tactile experience. It was assumed that the samples of mature subjects would have had prior experience of string or rope and that they would have some tacit knowledge about the workings of topology. Before similar activities can be contemplated in the classroom, it would be desirable to begin by offering pupils experiences with actual ropes which they could touch and manipulate as they wished.

Mitchelmore (1980) warned that passive viewing of 3-D representations is unlikely to be effective in training spatial ability. Strohecker (1995, page 228), describing topological thinking in children and adults, remarked:
'The microworlds of knots make a similar offer (to the LOGO turtle). By twisting and turning with a piece of string, and by seeing and feeling the relationships of different parts of a knot, a child can construct vivid understandings of what is crucial about those relationships.'

The relationships were described by Strohecker's (1995) subjects in interesting and varied ways, for example, when describing the difference between a granny knot and a reef knot, one of her subjects explained:
'The granny is a kind of bridge between the Pretzel (trefoil) and the Square (reef knot) . . . It's like a Pretzel, double, but it's not quite a Square. 'cause it's a Square if you keep on going instead of if you do it right. Some people say that a Square is you do it one way and then you do it the other way, but since the string strings change places, its really doing it twice.'

Indeed the emphasis in Strohecker's (1995) research was as much concerned with the language and verbal descriptions of knots as with spatial conceptions and diagrams.

If activities involving knots are to be introduced into the curriculum, either by viewing on a computer screen or on the printed
page, pupils should engage in introductory experiences with real ropes first of all, perhaps in the way that Strohecker worked with her pupils, so as to ensure that all pupils have knowledge of the objects before they attempt to work on them mentally. Later activities would include the requirement that pupils work on their images in the way that Mason (1992) recommends.

### 9.2 Review of the aims of the research

The key aims of this research are related to improving the learning of mathematics, in particular through the development of imagery. The main questions which the research has sought to answer are whether the performance of the tasks used here can enhance visualisation ability and how these kinds of tasks may be sequenced so as to develop a teaching programme. Hence it may be possible to improve not just geometry learning but problem solving generally. The term 'geometry' is used in this thesis in much the same way as Freudenthal (NCTM, 1989, page 48) uses the term:
'Geometry is grasping space . . . that space in which the child lives, breathes and moves. The space that the child must learn to know, explore, conquer, in order to live, breathe and move better in it.'

This study aimed to examine novel kinds of structures occupying space and to extend the ideas concerning imagery and visualisation of mental rotation of rigid structures to structures which can be deformed.

The question regarding how to classify the tasks in terms of complexity was explored by measuring two fundamental indicators, the time taken to perform each of the tasks and the accuracy of response. These measures enabled a comparison to be made between the control variables. If a significant decrease in decision times occurred during the performance of the experiment, it may be concluded that subjects were discovering better strategies for solving the tasks, probably by improving their visual imagery skills.

The different strategies used by subjects were explored so as to provide insight into different modes of thinking and also to aid understanding on the part of educators of the range of cognitive styles which may come into play in dealing with deformable structures. The strategies applied in solving the spatial tasks used in this study may be such that they provide a route for teaching visualisation.

One of the research questions posed was concerned with what prior knowledge and experience enables us to predict the possible deformations of a non-rigid object, and specifically how easy it is to apply this knowledge to the tasks. This question was explored by questioning subjects about the strategies which they used in order to bring about the manipulations The methods of comparison of the deformable objects varied, and were sometimes adapted during the course of the experiment, and this response was investigated.

The question as to what type of thinkers prefer visual over verbal strategies and whether these strategies may be altered according to the tasks was dealt with by testing subjects from different backgrounds and with different knowledge. Facility with the tasks was measured and the strategies used by a variety of subjects at different stages in the performance of a series of the tasks were investigated.

### 9.3 Characteristics of this research

The literature makes some distinction between 'visual' and 'spatial' when describing tasks and strategies for solving them, but there are discrepancies and ambiguities in the meanings attached to the various terms. As a result of carrying out this study, some clearer definitions have emerged. In trying to make some distinction between the terms discussed throughout this thesis: 'visualisation', 'imagery', and 'spatial ability', it may now be useful to suggest the following more precise definitions of these terms:
(1) visualisation may be described as the actual process by which an
internal image is created,
(2) imagery may be used in the sense of the possession of an internal representation or mental image and the ability to perform manipulations on it, and crucially:
(3) spatial ability may be considered to be the awareness which is developed as a result of the use of both visualisation and imagery.

Wheatley's (1990) three components of imagery which derived from Kosslyn's (1983) theory were described earlier in Chapter 2:
(i) the construction of images from directly viewing the objects,
(ii) re-presentation of the image at some time after its original construction; and
(iii) transformation of an image

The first two of Wheatley's components relate to 'visualisation' (1) and the third relates to 'imagery' (2) in the sense defined here. Guay and McDaniel (1977) noted that high spatial ability (3) was characterised as requiring the visualisation of 3-D configurations and the mental manipulation of these visual images.

Bishop (1983) proposed two types of ability, one of which he named visual processing (VP). Bishop (1983) described VP ability as involving visualisation and the translation of abstract relationships and nonfigural information into visual terms and also the manipulation and transformation of visual representations and visual imagery. In considering the definitions (1) to (3) above, Bishop's VP ability encompasses (1) and (2) and it is suggested here that VP ability be separated into two parts, the second part, 'manipulation and transformation of visual representations', being defined as 'imagery'. Furthermore, what McGee (1979) describes as the two components of spatial ability ${ }^{2}$, may also be associated with 2 and 3 above.

Where manipulation of an image is involved, the question as to

[^48]whether the nature of a mental image is visual or propositional becomes even more complex. In the situations described here, the subject is required to manipulate a currently viewed image and to transform it mentally. The question arises as to whether this favours a visual method over a verbal method and, if so, is it therefore to be recommended as an educational practice.

The tasks used in this research were expected to be solved by different methods depending upon the prior experience of the subjects and upon the efficacy of any one method for a particular task. The tasks utilised deformable objects and hence allowed different methods of solution to the rigid shapes used in previous research. The strategies for solution may be more intuitive, since most individuals already have experience of ropes and knots, thus untutored skills may be called upon as a means of developing some other related mathematical abilities.

The extent to which subjects held a picture of the knot in their mind's eye and manipulated it cannot be verified, but it appears that some pictorial imagery did take place in the sense of Mason's (1988) visual experiments rather than some other form of internal representation.

The findings from the interviews have been analysed in some detail but Presmeg (1998, page 62) has indicated that this can be rather subjective and she made the plea for the reader to be able to make up her own mind:
> '. . . . I have to admit that I experience annoyance when a researcher is so explicit about his or her own subjective interpretations that the reporting does not enable me as a reader to get close to the people in the research, . . . Again a balance is required, in this case, between reporting of interpretations of the phenomena investigated, and of evidence for these interpretations, for instance in actual transcript data.'

One characteristic of this research has been to collect a large
amount of data, both quantitative and qualitative. These data have received detailed analysis here, but the opportunity for further study of the data from the video case studies is offered by the inclusion of the transcripts in Appendix III.

### 9.4 Characteristics of previous research

Much research has been carried out to discover what are the 'abilities' which are needed in doing mathematics, and many studies have drawn conclusions on the efficacy of different thinking styles. The consensus is that imagery is a valuable skill in mathematics, especially in problem solving, and must receive attention in the classroom. This has been acknowledged formally by inclusion in the National Curriculum for Mathematics (DFE, 1995).

Some of the previous research (Gattegno, 1965; Mason, 1991) has involved asking subjects to close their eyes in order to form pictorial images and manipulate them. Other research has considered how individuals implement imagery manipulations. (Kosslyn, 1987) has shown that different parts of the brain may be used for different purposes and that the left hemisphere may develop a categorical spatial relation. Whereas the right hemisphere is more concerned with the detail of spatial inter-relationships and possibly therefore with pictorial representations, the left can be employed for decision making regarding explicit spatial properties such as whether a shape is inside or outside another shape (Kosslyn et al., 1986) and hence with some other less pictorial form of internal representation.

Wallace and Hofelich (1992) considered the effect of practice and found that for mental rotation tasks of 2-D random polygon shapes both accuracy and speed improved with practice. They also found that the exercise of a mental process due to practice of tasks involving that process can be transferred to the performance of other tasks, for example, from mental rotation tasks to geometric analogy tasks.

### 9.5 Main findings of this research

With regard to the question of how the tasks used in this study were solved, either verbally or visually, it may be the case that for some subjects a combination of methods was used. It has already been shown that the ability to switch strategies is effective in mathematics (Smith 1991). Clearly the strategies reported in Chapter 8 would support both verbal and spatial roles and the most successful subjects were those who were able to use a range of strategies.

Shape recognition and rotation would be primarily spatial in nature. Unravelling is a novel strategy in imagery research and could be viewed as either verbal or visual depending upon whether sequences of crossings are labelled and analysed logically, or if immediate mental manipulation is performed.

In the analysis of the results, the question 'Why are some knots more difficult than others?' received attention both in terms of identifying high error rates and longer decision times as well as verbal reports from subjects in the case studies. The factors affecting the level of difficulty were rotation, knot shape and pair type, and there were some interactions between these factors.

The hypothesis that stimulus pairs of knots with higher numbers of crossings would yield longer decision times was not supported. Neither was a linear relationship found between decision times and rotation. Although subjects in the case studies mentioned the number of crossings present in the diagram as being a component affecting difficulty, the quantitative results did not confirm this. The number of crossings was not found to impart complexity to the tasks. The overall shape seemed to be the main contributing factor with shape 3 being the least difficult of all the knot shapes. The search for a hierarchy according to number of crossings failed, but the notion that 'families' of knots having similar shapes such as the 'figure eight family' shown in Figure 4.24, could form similarly complex groups, is one which is worthy of investigation.

The effect of rotation interacting with certain complex shapes served to compound the complexity for those shapes. Lowest decision times and error rates were recorded for all stimulus pairs at $0^{\circ}$ of rotation. Decision times for all knot shapes were affected by rotation but knot 3 was the least affected and knot 6 the most.

Pair type had an effect on the level difficulty, with unknot/unknot pairs being the simplest and knot/knot pairs the most difficult. Rotation had noticeably little effect upon unknot/knot pairs suggesting that mental rotation was not a strategy for these pair types. Not only was there no evidence to support rotation as a solution strategy for 'different' unknot/knot pairs, but neither was there for 'same' unknot/unknot pairs. However, knot/knot pairs with rotation did require additional processing time and in some cases (knot shape 2) a range of strategies including rotation may have been utilised.

In the pilot study, there was no significant difference between relative rotations of $90^{\circ}$ and $270^{\circ}$, and the differences in mean decision times between the $90^{\circ}$ and $180^{\circ}$ in the second study were found not to be significant. This provides further evidence that mental rotation was not a preferred strategy. Although mental rotation was one reported strategy used by subjects in decision making, no relationship between decision time and degree of rotation was found in the quantitative studies. Rotation clearly caused an increase in decision times overall, but this does not imply that mental rotation was the strategy used, rotation merely added complexity to the tasks. This is confirmed by the fact that the highest error rates occurred at $180^{\circ}$ rotation.

The interaction between rotation and knot shape showed that knot pairs with rotation to $180^{\circ}$ took significantly longer for Knot 2 than those to the intermediate $90^{\circ}$, suggesting perhaps that this was the closest result to those obtained by Shepard and Metzler (1971) and Shepard and Feng (1972). This may have been the only knot shape where mental
rotation was a plausible strategy.

When the data for errors were considered, apart from shape 3 which presented the least difficulty as measured by lowest error rate as well as shortest decision times, no statistically significant conclusions could be made regarding complexity for the remaining five knot shapes. There were fewer errors for shape 5 but this shape did not have significantly shorter decision times than the others.

With regard to gender, females were significantly faster than males when controlled for educational background. All subjects taking part in the research improved their performance during the course of the experiment. An important finding in this study has been the effect of the order of the item on the decision time recorded. Not only did subjects report that they felt that their ability improved as the experiment progressed, but the results clearly show that this is the case. This was true for both the quantitative studies and in the case studies, where subjects gave emphatic reports of noticing that they had learned how to do the tasks. The range of knots shapes used in the tests would suggest that a general ability to process these images is being developed.

### 9.6 Relationship to previous research

Although mental rotation has been shown to be a major strategy for some tasks involving rigid objects, the data from this research show that there is no evidence to support such a conclusion here. This conclusion has been reached in the past by other researchers, Suzuki and Nakata (1988), also found no evidence to support rotation as a solution strategy for 'different' pairs of Shepard and Metzler (1971) type figures.

The stimuli presented in this study were deformable, with the nature and relative attributes of crossings confirming the knottedness, or otherwise, of the figure. The figures were portrayed as comprised of rope which could be moved, twisted or turned in any manner. The figures used in most previous research were completely rigid and non-
deformable. Even the folding paper figures of Shepard and Feng (1972), whilst being deformable, could only be deformed along adjoining sides of the squares. In solving the Shepard and Feng (1972) tasks, only successive folding along adjoining edges of the squares was required. In the case of knots and unknots, a number of factors were present for consideration; differences in the number, nature and relative position of crossings, the overall shape of the stimulus, its topology and its figural symmetry.

The relative efficacy of verbal versus pictorial strategies was studied by Brandimonte, Hitch and Bishop (1992). For their tasks, involving drawings of familiar objects, they found that verbal strategies were often preferred and were commonly used even when these strategies were ineffective. This oddity was also observed by Denis (1991) who found that the most obvious strategy was not necessarily the one used. Denis (1991, page 124) believed that imagery is important for many types of problem and commented:
'situations where imagery is most readily available are not necessarily those where imagery is truly needed. For instance, while imagery is easily available in relation to concrete or spatial stimulus materials, imagery may be needed the most in abstract highly unstructured tasks.'

Reisberg and Chambers (1991) reported a strategy which can be considered to be verbal, involving the reassigning of the 'top' of an image. They found that this method was more effective than mental rotation for recalling and reinterpreting an image.

It has not been established here which are the most efficient strategies for performing these tasks but, in the case study interviews, subjects reported being more at ease with certain strategies and, as a result, found the later tasks easier to complete. The most notable of these strategies was learning how to undo the figure. This strategy highlights another complex aspect of imagery identified by Kosslyn (1994), that when a subject imagines new situations for a viewed object, novel.
previously unseen representations of an image are required. The notion, termed 'motion-added transformations' was proposed by Kosslyn (1994) whereby the transformed image assumes novel configurations or structures. In the case of knots, the subjects must use tacit knowledge of the possibilities for the movement of the ropes in the images so as to create the imagined novel representation.

For the tasks used in this study, the presented images involved transformations which were rotations in the image plane and the images had both a surface representation and a deep (structural, encoded shape information) representation as described by Kosslyn \& Shwartz (1977). The strategies described by the subjects in performing these tasks do not unequivocally confirm the use of the Kosslyn \& Shwartz (1977) model, but do suggest that the conceptions of surface and deep representation play a part.

In trying to explain the short decision times for one of the knots, shape 3, Kosslyn's (1994) theory of a 'foundation part' of an image and the theory proposed by Reisberg and Chambers (1991) of reassigning the 'top' of the image (some subjects report that it had a pronounced 'right way up' ) may provide an explanation. Some knot shapes had more obvious foundation parts than others and the presence of such a characteristic may be a factor in determining the complexity of the image. Another possibility is that the strong bilateral symmetry present in the outline shape of Knot 3, a symmetry prevalent in our natural and man-made environment, may be a contributory factor to the ease with which subjects can memorise and manipulate such shapes. A further study using diagrams of the family of figure eight knots of crossing number 4 upwards would clarify this.

Tarr and Pinker (1989) found that, with practice, subjects could recognise objects almost equally quickly at all familiar (already seen) orientations. This indicates that subjects could memorise the shapes and that they stored representations of different orientations of the shapes which later they called upon to respond to recognition tasks. This would
explain why no mental rotation was required in performing their tasks. The same may be true here, and the ease with which subjects could memorise the shapes and orientations of some of the knots might explain why some of them are performed more quickly than others.

Where subjects had to perform two image transformations Wallace and Hofelich (1992) found that practice improved performance on geometric analogy tasks. Kosslyn et al. (1989) found further evidence for practice improving performance. They found that, with practice, the left hemisphere of the brain develops a new function related to spatial processing. In the studies reported here, practice did result in an improved facility with the tasks, the ability of the subjects to perform the tasks in a shorter time improved as the experiments progressed. This suggests that certain spatial skills are improved as a result of tackling these problems.

### 9.7 Implications for the curriculum and for teaching

The results show that there is a difference between the method which subjects use for mental tasks involving rigid objects and that which seems to be preferred for some of these knot tasks. The existence of a topological strategy which can be brought into play where necessary, offers subjects increased scope for creativity and helps them to use a variety of strategies with this novel range of spatial tasks. The possibility of switching strategy is much more apparent in these tasks than other spatial tasks involving rigid structures. This would imply that these tasks would serve as a useful tool in teaching problem solving which has been shown to be improved by good spatial ability and a readiness to switch strategy when appropriate.

It has been mentioned earlier that memory plays a leading role in spatial ability (Smith, 1991). Other mathematical thinking also relies heavily on short term memory. The following remarks appear in the introduction to the SMP book (1994, page 6) 'Developing Mathematical Imagery':
'Much mathematical thinking can be analysed in terms of working with a pair of mental boxes (rather like a calculator with two memory cells and a display); we have to store information in one box while processing that held in the other box. Whenever this process is needed, it is crucial that the contents of the first box are remembered accurately, and that they are not 'forgotten' or changed as a result of thinking about, or processing, the contents of the second box. Such mental gymnastics can be taught and learned.'

The last sentence of the above quote makes a rather bold statement but the truth of it has been confirmed in this research.

Teachers have a duty to help pupils develop their mental strategies and mathematical thinking. The findings have implications for the teaching of visualisation with suggestions for incorporation into the curriculum of certain aspects of an area of mathematical theory (knot theory) into the school curriculum as a teaching aid. Indeed, the interim results of this study have been adopted by NFER and items have been incorporated into some new tests under development in 1999

It seems clear from this research that children can be given activities which improve their visualisation skills. The research has also identified some suitable activities, involving knots, for achieving this goal. Some relatively simple tasks have been distinguished from others which are more complex. The skills needed to perform these knot tasks are latent in most individuals and the potential for development of spatial ability by offering such tasks is a powerful one.

Just as language is a genetic potentiality in us all, it has been argued that perhaps the same may be true for the mental manipulation skills required to deal with knots and unknots. It is certainly a skill which has been available to humans throughout history. This raises another question worthy of investigation: 'Does a deficiency in the left parietal cortex of the brain, the region which is responsible for number
and which causes dyscalculia, affect ability in the area of mathematics known as topology.' It is possible that dyscalculics have no disability in this area. The part of the brain which may be required for topology is the right parietal cortex which is traditionally regarded as being responsible for spatial thinking. Perhaps dyscalculics could be helped by developing this other part of the brain. Keith Devlin (1988, page 230) has explained:
'Though topology is an extremely difficult subject to pursue properly, a facility to visualise geometric objects is all that is required . . . . to grasp the general principles.'

The tasks in this research were presented to subjects on a computer screen and as such may be considered to be intrinsically different to a picture on a page or to actual ropes. The ideal situation for the classroom would be to have a variety of means of presenting tasks to pupils, beginning in most cases with the opportunity to work with actual knots and ropes.

Other activities which may be used alongside knot tasks in the mathematics classroom include visualisation exercises such as those recommended by Mason ${ }^{3}$ (1991), where the entire process is carried out inside the visualiser's mind and may require the mental drawing of a shape followed by the manipulation of that shape. Other tasks could involve a starting point given by the teacher in the form of a picture on a card. The pupils could then be asked to imagine applying some transformation to the picture. A transformation such as reflection in the vertical could be carried out and the pupils may then be asked to describe or draw their new version or, more simply to pick out the correct one from a list, either on a printed page or on a computer screen. With practice the pupils may be asked to imagine two transformations, such as a reflection followed by a rotation.

What is clear is that as more emphasis is placed upon developing 'personal' mathematics through the use of imagery, all pupils should benefit, but particularly the average or less able child, not only from the

[^49]point of view of making respectable the visual approach, but also by allowing the child to do their own mathematics in their mind rather than always in an exercise book.

The tasks developed by Clements and Wattanawaha (1978) are also interesting and varied, and deserve to be included in classroom activities. Clements and Wattanawaha's (1978) tasks covered many aspects of imagery at a range of levels of difficulty. Teachers tend to be offered only a limited selection of tasks of this kind in current texts, but this deficiency may soon be rectified with the adoption of visual imagery tasks in the new tests being developed by NFER. Logic and imagery may be utilised to solve these tasks efficiently, and a rich variety of different imagery tasks, including knots, will assist pupils in becoming better problem solvers.

### 9.8 Concluding remarks

There are some questions which this research has raised and which are not fully answered here. One intriguing question is: 'What is it that makes knot shape 3 so much easier than the rest?'. An explanation has been offered regarding the presence of a strong 'foundation part' together with its bilateral symmetry, but further experiments need to be conducted in order to substantiate these ideas.

A more general question is related to the specificity or generality of the ability to perform these spatial tasks. Although the subjects in the study had different strengths and weaknesses academically, they were all highly educated in some field. It is not known whether the same effects would be observed with a more mixed sample and whether individual differences in pupils' abilities may interact with the training. It is possible that greater mathematical success may be evident for lower ability pupils who find a spatial approach more accessible (Wheatley and Wheatley, 1979).

Subjects in this study reported that they were unsure of the
effectiveness of some strategies for performing these tasks and that they needed to check by utilising a second alternative strategy. The inadequate strategies were those of mental rotation and matching of crossings. When subjects learned to use the unravelling strategy they were quite confident that it was effective.

What the research seems to show is that mental rotation is not the most efficient strategy for these tasks and that perhaps the non-rigid nature of the objects provides a route to a more efficient mental manipulation strategy via untwisting mentally and sliding the rope. The fact that unravelling was reported to be used more frequently as the experiment progressed and also that unknot pairs had shortest decision times support this.

There has been some debate in the literature as to which strategies are useful for individuals in performing visual tasks, but if the outcome of any strategy is the successful solution of a problem then it is a valid strategy. As has been pointed out by Bryant (1982), a child recognises that a certain strategy is effective when it consistently produces the same answers as another strategy.

The fact remains that many pupils need help to develop their thinking strategies and skills. Some skills may arise out of everyday experiences as well as classroom activities and we should be aware of this. Some learners will find certain strategies more successful than others and it is the role of the teacher to offer practice and support for the child's preferred strategy so long as that preferred strategy is effective.

This research has attempted to clarify some issues surrounding the teaching of visualisation and imagery. Some progress has been made and suggestions for suitable activities have been formulated. A small contribution to the literature has been made and with the continuing efforts of mathematics educators to produce better teaching materials, procedures and educational practices, then more children will become successful and confident in mathematics.

## Appendix 1

Table of knots up to 9 crossings
B. Q, D, P8.
8. 85. (8). 8

8, \& , \& , 8,


R2, (28) (6).
(8). B. B E B

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$4-7 t+11 t^{2}-7 t^{3}+4 t^{4}$

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## Knots Experiment Protocol - Pilot and second study

This is an experiment on knots and how you see them and manipulate them. If you normally wear corrective lenses for distance/near you should do so during the experiment.

This is a knot and this is an unknot or loop:

## <Show actual knot and unknot>

The difference between them is whatever you do to the knot it will always remain knotted whilst the unknot or loop can always be manipulated into a circular loop.

## <Demonstrate>

We say that two knots are the same if one can be manipulated into the other without cutting.

You will be shown pictures of

- pairs of knots
- pairs of unknots
- pairs containing one of each
and asked to answer the question 'Are these two knots the same? For example, the drawings may be like this:
<Show unknot in shape of Knot 1 next to knotted shape of Knot 1.
Demonstrate undoing the unknot and show that it is not possible to do this for the knot>

These two are not the same.
Or like this:
Show two trefoils where one has been rotated
These two are the same.
During the experiment you should not ask any questions, however you may wish to do so during the short practice session which we shall begin in a moment.

Your decision times will be recorded. You may take as long as you wish to respond, your criteria should be to aim for a correct answer. After you press the mouse button to indicate that you are ready a picture will be presented, this is called a trial. There will be (126) (72) such trials.

Do you have any questions before we begin?

## Knots Experiment Protocol - Case studies

This is an experiment on knots and how you see them and manipulate them. If you normally wear corrective lenses for distance/near you should do so during the experiment.

This is a knot and this is an unknot or loop:
<Show actual knot and unknot>
The difference between them is whatever you do to the knot it will always remain knotted whilst the unknot or loop can always be manipulated into a circular loop.

## <Demonstrate this>

We say that two knots are the same if one can be manipulated into the other without cutting.

You will be shown pictures of

- pairs of knots
- pairs of unknots
- pairs containing one of each
and asked to answer the question "Are these two knots the same?" For example the diagrams may be like this:
<Show unknot in shape of Knot 1 next to knotted shape of Knot 1. Demonstrate undoing the unknot and show that it is not possible to do this for the knot>

These two are not the same.
Or the diagrams may be like this:
<Show two knotted versions of Knot 1 where one has been rotated>
These two are the same.
During the experiment you should try to explain your thinking in deciding how to respond to the item on the screen. There will be 20 items. You may ask questions at any time.

## Appendix III

## Transcripts of Interviews

## Subject 1

Subj. 1
Initially, it seemed easier to do it with your finger, but then it got confusing.

Int.
What made it confusing?

Subj. 1
I think after seeing a couple of the knots I got more in tune with what I was looking at so it was easy to actually break it down if there was a little bit twisting or folding on top which would . . . . . if you folded it back and untwisted it, it would just make a loop. It became easier to do that when my eye became a bit more tuned in.

Int.
And you didn't feel that you needed to . . . . .

Subj. 1
Yes. I didn't feel I needed to do that. Funnily enough, I kept going to the left first. I don't know if that's because I'm left handed. . . But that can also be misleading because then I would try to do a bit more - trying to see if they were a loop first of all or not.

Int.
That was your first line of attack? To see if it would undo? And you thought 'Ah, I've found a strategy '. . . .

Subj. 1
Yes. Then I went through with most of them but then it didn't work all the time.

Int.
So when it didn't work, you had to try another strategy?

Subj. 1
Yes. Usually when it didn't work it was because there was intertwining and so then it was . . . .

That again was when I was looking for this . . . would they just open out, is one just folded or balanced on top of the other, looped on the other rope or is it intertwined?

Int.
That was the conversation about the dominant shape and you were lead to a dominant part of the shape.

Subj. 1
Yes. On the left hand side it was just because it went underneath and then it went over and round. It just seemed to have a strong shape to it.

Int.
So when you were trying to do some that you were not sure whether they would undo. . . You were then going to this strategy? Is that what was happening?

Subj. 1
If the matching was harder.

Int.
If you were not able to see a place that would undo with your first strategy, you were then looking at that strategy?

## Subj. 1

Yes, and then matching, seeing each one, seeing if this bit goes over there, this bit here goes over there. Does this bit go under as well. Or does it go under, or then, possibly working through it to see was there any . . . . . I started working on one individually and then on the other individually and then matching them and then sometimes it was easier to look at them both and to say 'that goes there' and 'that goes there' and if you find a couple of things that do match up then just go through them both again until you find that they all either match up or are different.

Int.
So sometimes you spent a little bit of time on one and then you looked at the other and spent quite a bit of time on that, but at other times you just had a quick glance at one bit of that one and then a quick glance at the other. I wonder why it was different?

Subj. 1
The ones that had a lot of different loops on, but I was not looking at one for a long time on its own. That was too confusing. Ones with a lot of different loops and turns - I was just looking at them both and without trying to go through one just to see if they both do this. Do they both do this? It was easier to just do that. If I was to just concentrate on the one and just go into it I would just confuse myself.

Int.
You seemed to get quite a number of that type! They keep coming up don't they?

They are a bit blurred on the screen but that was an unusual shaped one.
Do you remember the one I mean?

Subj. 1
Yes. That was a hard one. I did find that hard. It was turned anyway. The other one was on sideways and that made it harder. You don't see
just straight away to them - they looked like the same - go back to the usual procedure. It was more a case of I was a bit lost there, actually, following the rope and seeing if they were both balanced on. . .

Int.
When there was some amount of turn on one of the pictures, you started to follow the rope around.

Subj. 1
I found it hard to try and imagine this one turning slightly, although I tried to visualise that it was straight and then look at it, but I couldn't hold that for too long.

Int.
So you went for this other strategy of looking at the journey round the knot and what were you finding from that? What were you noticing?

Subj. 1
Just that they all went under at the same place or they all rested on another part of the knot in the same place and that if they were all to undo that they would have the same picture.

Int.
Could you rewind that one a second. I just saw something. Do you remember doing that one?

Subj. 1
That one initially struck me straight away as being knots because it seemed to go through the top loop in both of them.

Int.
One of the strands?

Subj. 1
Yes, it stood out.

Int.
And that made your mind up for you?

Subj. 1
I think that was a quick one - quite a basic shape. The one from the demo.

## Subject 2

Int.
You got a bit louder as time went on. Came to a conclusion - that's what you were saying. I liked this description, listen to what you said here.
'Taking the top and the bottom of one and going to the right and the left of the other.

Subj. 2
That's sideways on. Two loops, one below the other so I called it identified a top loop so the other one . . . . bottom . . . I'm looking at the sides, left and right.

Int.
'I called it top . . . and bottom', do you think you 'called' it in your mind? Do you know what you meant by that? If it happens again, let me know.

When you said 'the same in reverse' . . .

Subj. 2
Yes, I remember saying it thinking 'why did I say reverse?' Maybe I should have said 'turned over'.

Int.
The same but turned over? That's what you felt is it? So you were trying
to imagine it turned over?

## Subj. 2

I should look at it again just to see why I said reverse.

Int.
That's what you said then. Do you remember what you meant by that now? Something about that middle part.

Subj. 2
The middle part because they're the same at the edge. One was going under and one was going over the top. So maybe I'm using an incorrect term. I am just trying to say it's not the same at this point.

Int.
That was the difference, you found.

Subj. 2
One was going under and one was going over the top so I said 'in reverse'. Meaning . . . .

Int.
So you were explaining the difference? I see.

That particular shape - you didn't like that shape?

Subj. 2
No.

Int.
Because it had a lot of loops?

Subj. 2
So I have a lot of points to look at and at first I don't know where to start.
I am trying to find two points the same. I know these two are the same.

I'm working to where the next two points that are similar. And your eyes are jumping from one place to another and then when you think you have identified it, and looked at something else, you go back and think 'what did I say about that?'

Int.
You have forgotten by the time you get there. So, there was too much information.

Subj. 2
Too much detail. Yes.

Int.
You were actually very clear in your explanation when I was sitting there listening to this. You went round it very systematically. Do you
remember that?

Subj. 2
Yes. I found it more difficult and this one as well.

Int.
Do you remember that? - 'I have to have one point to start from'

Subj. 2
Yes. One point that is in each one. So I have kept to look at two - that is the middle loop. I just thought it's small enough for me to focus on to look at. Find the small loop in the other one. Not the other two. You can see the two loops at the side and the two at the bottom. I didn't want to look at that - that is too . . . . .

Int.
So you found that one identifiable point?

Subj. 2
Loop, yes. So on its side there, and then there. I thought 'well it's turned
up'. You know - go up - rotated so it's at the top on the left hand side.

Int.
So you were actually trying to picture that one on its side put back the other way. That's what you were trying to do?

Subj. 2
Yes. But I don't think I tried to do that with all loops. There were another two loops and I thought there they are up there and there they are down there so that must be the same, so as soon as I started seeing three things that looked the same, I just want to carry on and think 'well, it's got to be the same.'

Int.
So basically, you are taking one little bit and imagining that that bit has been rotated - checking that, and then you have moved somewhere else.

Subj. 2
Yes. I then saw the other two pieces. The other two loops. When I saw they were at a certain place, I just thought 'well, if that is rotated, they would come underneath'. But I was also assuming things.

Int.
What are you assuming? What do you mean 'I am assuming things'?
That the rest comes with it?

Subj. 2
Yes. Maybe if I went back and looked at some of those things and spent a little bit more time . . . . .

Int.
You felt as though you were having to rush?

Subj. 2
I feel I wanted to get through all of them.

Int.
But when you were dealing with the processing you said you identified the one point and you imagined that bit rotating and then you had to go to another part of it. You spoke about the two then. You spoke as though there were two pieces - that then had to be dealt with. Now, did you try to do these both at the same time?

Subj. 2
Yes. I just looked at the two loops in the one position. The two loops in the other position - looked at the shape - the shape was the same and I thought 'well, that's probably the same so, therefore, I am assuming that they are both . . . I could be wrong there.

Int.
But, basically, you were trying to deal with the whole part. What shall we say? The bottom in the left hand one with the two loops and you try to deal with both of those at the same time in the rotated one, rather than one at a time.

Subj. 2
Yes. Both together.

Int.
That's interesting.

This is the same strategy isn't it? This one doesn't have as many . . . .

Subj. 2
No, so that was easier .

Int.
And you were basically trying to get hold of the part of it and transfer it to the other piece. Can you remember how you were actually trying to do that transfer?

Subj. 2
I looked at it straight away and saw that they were obviously turned around and they could possibly be the same. But they were lying in different positions. So when I saw the egg shape on the right I looked at the other one to see if I could see a shape - egg shaped - where was it, where did it occur in the image? And then when I saw the egg shape then I checked is it going underneath.

Int.
Were you thinking 'egg-shape'?

Subj. 2
Egg-shape. Yes.

Int.
You were looking for an egg-shape?

Subj. 2
Yes. The other one was a fatter shape. So I thought if you turned that down the two other shapes are going to be lying at the bottom just like the image on the right hand side. So I checked the loop to make sure that it either went underneath or over the top. Were they both going the same? And they were the same. I may not have verbalised that - I just looked quickly and I thought 'that's the same'.

Int.
But when you were trying to identify the point of contact for the left hand with the right hand knot, one of them having been turned, you were thinking the word almost - egg-shaped. Do you think that was in your mind at the time? And then, when you identified the egg-shape, where did you head for next?

Subj. 2
The other two shapes were similar. They were similar shapes and all I
wanted to establish was that when that egg-shape comes up to the top like the left hand image, are those two other things going to come down at the bottom. Are they going to be at the bottom?

Int.
So, you were looking generally again for two - or were you looking for the left part and then the right part?

Subj. 2
No, two.

Int.
You were trying to deal with two at once?

Subj. 2
The egg-shape as a separate thing and then the other two shapes on each image - together - not one and then the other - not like that, but two together. I linked the two together. And then I quickly looked at the ropy bit to see if it's lying on the top or underneath. I tend to be quite impatient.

Int.
You were doing it like that rather than any of those other strategies, that perhaps when we did the demonstration about undoing the thing, you know the idea that maybe it would be an unknot and maybe you could lift it up. Then you were trying to identify the bits of the diagram. I wonder if you felt that you were using the unknotting strategy in any of the others. Did you think you were using that strategy at all?

Subj. 2
Yes I did. At times when you look at them there were one or two where I could see it was just a twist and lifted. It was not actually a knot.

## Subject 3

Subj. 3
Well this one, right from the start, I was checking that all these sizes were actually the same as it had occurred to me that maybe they could be different, whereas by the end of it I was not spending so long checking that they are actually the same size. I thought maybe you could have had a bigger one here and a smaller one there, but they were actually - you rotated them. There were not any situations where they were not the same because maybe that bit was bigger and that bit there.

Int.
These loops sticking out at the side?

Subj. 3
These loops were all the same size. I thought at one point maybe it could be they were different knots. I classified them as different if the loops were different sizes. And, of course, it would make a difference. As to which way I put the knots to check them. At the start I spent longer getting them orientated.

Int.
You were actually pointing at that point to both of them at the same time. Do you want to tell us what was going on when you were doing that.

Subj. 3
It was because they weren't the same way up I found it more difficult to picture in my mind. I tilted my head but I found it easier with my fingers and followed them round. They were both at the same point at the same time. I could check they were both doing the same thing. I think later on it became easier for me to visualise without having to use my .... . They were both over and then followed the knot itself.

Int.
Trying to keep your finger in the relatively same position?

Subj. 3
So I was actually following so it would be quite easy to go off in the wrong direction and end up going under rather than following it over.

Int.
How did you decide on which direction?

Subj. 3
Some of them I actually decided to turn that way - I think later on I changed my mind. Whichever way seemed easier at the time. For some reason, maybe because I went over - it's easier starting off going over. It does start off going over.

Int.
You preferred to start at an 'over'?

Subj. 3
Yes, and usually the point at the bottom. I seem to be going for a point there, rather than turn that round. Actually turning that round that way rather than that way.

Int.
Any feelings as to why that might have been?

Subj. 3
I don't know really. It just seemed to be the natural thing to do.

Int.
With that particular shape?

Subj. 3
With that shape. . . . I think that with others I did things differently. But right from the start I turned my head that way. I seem to try to turn this one round rather than this one up. Actually, first and then second rather
than second and then first.

They seemed to be much easier - when I was familiar with the shape and they were up the right way. I didn't really have to use my hands much. I just followed my eyes. It was much easier.

Int.
You seem to sometimes also have an immediate response and say something like 'They are up the right way.'

Subj. 3
In the first place I am always trying to check they are the same shape. That is bigger and so I turn it round the other way up.

Int.
So that was your first thought?

Subj. 3
Are they the right way up with each other and then I pick a point to start with. In this case it was just a case of following my eyes. Once again I start from the side. This side. Following it round. I was familiar with the shape after the last one.

Int.
That one was a relatively easy one because you had just done one like it.

Subj. 3
Yes.

I was checking on this because I found this more difficult because they were upside down and I had to actually think about my finger on . . . . It seemed difficult. Once again I start from here but it was difficult to match here and follow round. I found that much more difficult than the other ones. I had to check again at the end whether it was right. I was not that confident in myself like before, to say yes, they are the same. This one I
just wanted to check again because it did not look immediately so right to me.

Int.
So it was upside down? You recognised that - that was the first bit. One of them was upside down - and then you set about trying to deal with that by following your . . . . .

Subj. 3
Following the shape again - under over, under over.

Int.
Starting from the same point in each one.

Subj. 3
Yes, but strangely enough, I did start from this side. Which was . . . . . but looking at it now this comes out more to me that it is over that side. If I look at this now that goes under but it is not actually knotted here. On this side they are both . . . . . I did not think on those lines at the time, but towards the end I was thinking more on those ways. There seem to be more that would just sort of flip over and they were not knotted. I had not come across many like that.

Int.
At that stage, when you were trying to solve this one, you started here and what did you do with the one on this side?

Subj. 3
I started there.

Int.
Now did you do that because you turned it in your mind?

Subj. 3
Yes. I turned it, I visualised it that way. I just followed it round and
decided that matched with that.

Int.
How did you do that? Will you tell me again?

Subj. 3
Well, this one here went over and this one here went under. I matched them like that. They both went under.

I was just turning it round, checking that I had got things the right way up, and that I said this before that the sizes of the loops were the right way so these two could not have been maybe turned round the different shape. Only one chance of it being one way round.

Int.
You were saying where were these two.

Subj. 3
I would turn it round and check that that one corresponded with that one and that one corresponded with that one and they are different sizes.
Rather than having them . . . . both the same, they are different sizes. It is quite easy to match up.

Int.
So, how then did you end up dealing with this one?

Subj. 3
I turned that one upside down. I think I started from this point again and I traced it exactly the same way. I was just following the same pattern really.

Int.
You were doing a lot of that tracing round and checking crossing by crossing weren't you?

## Subj. 3

In the early stages, yes. Whereas now looking at this I can see that flips off there, it's not knotted there but it is there.

## Int.

So you probably would not feel the need to use those same strategies that you were using?

Subj. 3
No, I would not. I would look at this and I might not even have to turn it round. I can identify that point as being that point. You see? That's knotted and this isn't there. I think towards the end I was thinking more on those lines but initially I was still checking that if I turned it round they were exactly the same knots.

Int.
Also on one or two of the items I think you said 'that one's different' and then I don't think you were finished at that point. I think you were carrying on.

Subj. 3
Yes, just checking. I was not yet convinced that maybe this was all different. I could not maybe turn it round again.

Int.
By turning it in a different orientation, it might have ?

Subj. 3
If there was an odd one different - but there was more than two of them. At the back of my mind maybe I could have turned it round a different way. I didn't actually come to that so I didn't have to worry about it.

With these like that it seemed much easier to check using both hands. If there was one I wasn't sure about I went back and checked it. I was just showing the way my eyes were going really. Much easier - I could follow
it round.

Int.
Much easier that way up.

Subj. 3
Well, they are both the same. The same way up rather than one being upside down. It was much easier to visualise and just look right to left and right to left.

Int.
The way you are doing that one there . . . . You are actually keeping your finger at some point on both of these, on the same point.

Subj. 3
Actually on the point where it crosses. I was just going to where they crossed.

Int.
You were starting there?

Subj. 3
To find out the spots I turned that round - always turning the first to match the second it seems. I started off here and those points were exactly the same but I have just followed it round the same way. It went underneath there and underneath there too. I followed it around again. It goes over the top of that point and over the top of that one and it goes underneath there. Back over.

Int.
You were very methodical. You never got lost. That was very clear thinking.

That was too easy for you, that one!

Subj.
They looked the same right away. It was just a matter of following with your eyes - as soon as it came up that looks the same.

I think this was one of the first points I actually realised it goes . . . because those are both over the top. They are not really knots and that's the first one I have seen where they have been both the same - unknots. I hadn't really twigged that they were unknots the other times. This is one of the first times I have actually thought in my mind, yes, these are . . . . . they flip up so they're both unknotted, it's just a matter of lifting them up.

Int.
Although you had some in the past before this one, that would undo, they were unknots. . . .

Subj. 3
Have I had some that were the same? I cannot remember actually seeing any. I remember seeing the difference, one with a knot. I mean two that were not so obviously unknots and the same. This one was the first that struck me in my mind.

Int.
I have got a feeling there might have been another one that might have been an unknot. So there it jumped out at you. . that they were both unknots?

Subj. 3
I noticed here, that one there isn't a knot. It's not knotted there. I think that held the line. I think it jumped out more. I wasn't going through the whole tracing. I started off and then I saw that one was under and that one was over.

Int.
So you changed your strategy after you started?

Subj. 3
I always started from the same point and I still started from here at that point by the look of things. But the one I started off at was where the difference was, whether that was that coincidence at that point or not I don't know.

Int.
So having started off at that one . . . . .

Subj. 3
I think I may have been checking in my mind. Later on I was confident to say they are not the same now. But at this point I think I was still a bit tentative.

Int.
But you have already said at the beginning they are the right way up . That might have been the reason for you to have said 'because they are both the right way up' ..... you looked at these two maybe that was helpful?

Subj. 3
Yes. You can see instantly they are the right way up and you can see that these two are different.

I have just noticed that I started saying that this is different because this loop would come undone and I am saying that this one is tied up. These two together both over the top - the loop could undo itself. So I was trying to say 'that flips over' - 'that one's knotted', so that one's different.

Int.
So, that was really how you made up your mind.

Subj. 3
Yes, and that was one quite near the start. I think I was still going
through the same process, but as soon as I have seen . . . that . . . .

Int.
You had some of these earlier on. I don't think you were happy to say anything of the sort there because you were still checking one by one, the crossings, but by the time you got to this one, you started to find particular places to look.

Subj. 3
I was still looking in the same places. I have always been looking at where they crossed. I still started off again. I have not been looking at the whole picture - even this one I am starting off here and following them around. I do not look at that and think 'this is going to leap off' and 'that's not'. It was only by looking at the two individually again that one and then that one I thought that's going to leap off. That's going to come undone. Rather than looking at the picture as a whole.

Int.
You weren't actually looking at a single crossing though. You were looking at two there.

Subj. 3
I still went about it singly. Check that - Check that. Check that and say those two ..... I didn't look at this picture on its own and think 'that is going to come off.'

Int.
You were just collecting the information as you went round and using the up-to-date information.

Subj. 3
Yes, just the last few there, the last couple have been knots and loops that came undone. That is obviously why they are different then. Whereas up until that point I had just been tracing and had not really thought about it. Once I'd thought of that it seemed to be much easier.

That was a different one. I didn't actually match on this one. I just looked and saw those two together and that leapt out and I saw they were different there. So this was actually different.

Int.
You solved this differently?

Subj. 3
Yes. I just looked at that loop and saw it was on the top and it was unknotted, and I looked at that one and saw it was knotted (points at the loop). I didn't go through the same pattern of turning it the right way up and finding my starting point and tracing it round. It just seemed to be obvious - that was obviously a loop . . untied and that one there obviously was not.'

Int.
So, you didn't need to actually do any of this turn with that one.

Subj. 3
I think I may have gone like that originally, just to sort of . . . . I did at the start, can we just rewind it a bit?

Int.
'These are just reversed round the other way'. What were you thinking then?

Subj. 3
That one goes over and that one goes under. That one goes under and that one goes over. It should be the other way round. That should be over and that one under. I expected them to be the same. I would expect that to be on top and that underneath and they weren't.

Int.
And that made you think there were different?

Subj. 3
Yes. I was just checking that I could not turn it round. That they would be that way round - it would be symmetrical. I had to check if it was symmetrical.

Int.
You were doing this to . . . ?

Subj. 3
To see if that and that were not the same. So I thought one is under and one is over. It was quite nice - and there were two of them that were different again. I was just checking that there was a rotation that I had not missed. And by rotating it round . . . . opposite . . . I could accommodate the fact that they were opposites.

Int.
So, you were trying to see if the shape looked the same?

Subj. 3
If these two had been the same as these, then all three of them would be the same shape and then I turned it round.

Int.
So the fact that it was almost a symmetrical shape with all three made it a little bit tricky for you?

Subj. 3
It just made me check. At first glance that seems to me the way everything should be. I just wanted to check again because they both . . . . and did look quite symmetrical.

This is the same again. They looked the right way up. In the middle they looked the same but I was actually going right back to where I started from before, under and over, there's an obvious difference.

Int.
Yes, you said one was under and one was over.

## Subj. 3

I didn't even try to turn this one upside down because I think that was the right way up and if one is right and the other is wrong then if you put one the other way up it won't make any difference.

Int.
So on this one you have gone back to your old strategy.

Subj. 3
Just looking now, it has not leapt out at me this one, that that's not a knot. I find that more difficult. I couldn't . . . Your eye automatically goes to the middle of this one, where they are actually knotted

Int.
So, now looking at it you can see something new again?

Subj. 3
Yes. I did not spot this before, that that's not a knot.
Looking at it now I can see that that lifts up but I was going back to my old way. It didn't seem to me much slower.

Int.
You had that system working very effectively. If you were now try to do that one... .

Subj. 3
I would automatically turn it round. The same way. I just see the two overs. See it as a loop. I'm going to check that that loop. I don't know whether in the long run that would be quite inefficient.

Int.
Just tell me what you mean by 'checking that loop'.

Subj. 3
This one down here, is like a loop that would come undone.

Int.
O.K. You have identified this part here - this undoable part here . . . .

Subj. 3
I checked it. That loops up and I am not imagining it completely undone.
I am only imagining it lifting up.

Int.
But you seem to be saying also that you are not sure which would be the easiest way. You are doing the finger movements again.

Subj. 3
It all looks like completely knotted. I am just checking the knots are the same. I had not thought that if they are all knotted - if they are all knotted - I am just checking that they are all the same way. I haven't thought . . . . it is going to be an undoable knot. I haven't made any distinction . . . if they are not the same, one is going to be undoable and one isn't. I'm just checking that the two knots are going the same way. I hadn't thought of, at this point, that if they are all knotted . . . . I hadn't thought that they can either be all knotted or . . . I hadn't made that distinction that if they aren't the same, one is going to be undoable and the other isn't. That has just come to me now.

Int.
In effect, that's how you were doing this with the strategy that you seem to have been putting into practice effectively throughout.

Subj. 3
I was getting quite confident there. Saying 'Yes. That's knotted and that's
not. They are not the same.' Before, I would probably have checked it again. This time I didn't have to.

Int.
The first place you actually pointed to on the screen was up here.

## Subj. 3

I was matching up the same way again. To check the shapes were the same. I started here and here - always started on the right, it seems.

Int.
So, your mind had gone to here and your finger had gone to the part that you had identified as being equivalent to there.

## Subj. 3

I think I started the way I was doing them before, getting my mind to see if they were the same way up. I find it easy to picture them together and then, as soon as I had done that, I could see this one was knotted and this one wasn't. It seemed quite obvious. Because once again - it is right by where I started from. I don't know how it would have been if it had been that bit over there, the middle loop. If that had been the one that was undoable.

Subj. 3
I am checking this because although these two are the same it could still be that later on - that may be this had been turned round and . . . . so it was quite easy to start off with the same shape and it was the same way round as the one before. I still followed it round to check.

Int.
Obviously we had a random sample there, you did get quite a lot of those that seemed to be complicated ones. Do you remember when we first looked at them? As we got to the end there, you were actually managing to do them quite easily.

Subj. 3
Yes, they did get easier

Int.
So when the first was the picture of the rotation what made it look so difficult? Because you groaned when you saw some of them didn't you?

Subj. 3
It was one that had lots of bits to see, different sizes and they all are round the wrong way. Lots of loops, lots of sections, obviously different shapes. Some of them had little loops that were . . . . You had to check that they were actually different if they were very similar in size to other loops. Maybe double check again..

## Subject 4

Subj. 4
I suddenly realised that's not a knot and I don't know why I didn't know before. I think I kept looking . . . . .

Int.
I think this is one you actually spent a lot of time on.

Subj. 4
And the fact that I didn't realise, when I looked at it first, that it was not a knot. I don't know if I was looking for that. I had forgotten that you had said that some of them would not be knots. One thing I think I realise now, having done about twenty of them, I have gained more confidence to look at them as a whole. I think I was getting lost in the details at the beginning because you feel you are just looking at each separate little piece of knot.

I do find this very hard and I think I have decided now that it's the same, but it's only after looking at it for a very, very long time and I think I can
now switch it round in my mind more easily. But it has not been easy. My strategy is trying to get the right hand one identical with the left by switching it round in my mind and then following it round piece by piece and switching back from the left hand knot to the right hand knot and then when I have decided that bit is identical, then I go back to the left hand knot, but the problem is if they are not in an identical position, it's not always easy to find your bearings. You get lost and I kept on getting lost. That was my problem.

Int.
So, you were doing it by zooming in to a specific part and then you said eventually you got to looking at it more globally

Subj. 4
Yes, because I think I got a bit used to it. I realised you didn't have to look at every part of it to establish that they were not identical. Once you found one thing that was not identical, you could say 'well those are not the same.'

Int.
In the first place, you were not happy with just one thing not being identical.

Subj. 4
I am still not sure. I kept on thinking 'well, perhaps they are not identical'. If I switched it round, perhaps one is upside down because some of them look almost symmetrical. Like two circles where you have - like that - almost like two triangles and you wonder whether $\qquad$ you switched it back the other way round they would come back the same. So, I thought I had to check to try and look at it both ways.

Int.
So, you found that one quite hard? You were interested to know whether you finally got it right. You were saying to me a minute ago 'did I say yes'.

Subj. 4
I was not sure. I am sure now.

Int.
We have moved on to the next.

Subj. 4
Oh yes. This was a killer.

Int.
You didn't like this one?

Subj. 4
No. I think now I can see that one is upside down. I still get confused, but it's a little bit easier.

Int.
It's easier because you know how to do it now, or because the knot itself is easier?

Subj. 4
I don't know. I think it's because the knot itself is easier. I think I have got used to it generally and I think I have got more confidence now and look at it as a whole. It seems to be easier now to switch it and turn it upside down and say 'yes, I think that is the same'. It's as though I have got used to it. I'm more able to look at the whole pattern.

Int.
So, you are taking the whole thing and turning it upside down.

Subj. 4
I am trying to. Yes.

Int.
And just holding one picture and comparing that picture with the other one.

Subj. 4
Yes it's easier now and I would say . . . . I don't know what I said last time but I'd say those are the same. I don't think I would remember the individual knots because so many of them were similar. If they were all totally different, I would say 'yes'. I remember this one but some of them look quite alike, if not almost identical. I will just see what I said last time. I'm not very sure, am I? I'm saying 'yes' but I think I am sure now.

Int.
You were like this with all the ones that were turned.

Subj. 4
I found that very, very difficult.

Int.
I think some of them were found worse than others. I am not sure which ones you found worse than others. Certainly in the very beginning . . . .

Subj. 4
I think this is a bit easier than the one before because the one before looked the same whichever angle you looked at it.

Int.
The star shape?

Subj. 4
Yes, the star shape. This one I can see there is a top and a bottom which are different. With the right hand one you can see that the top is at the bottom.

Int.
You are picking the left hand one at this stage. Later on you switch.

## Subj. 4

I do not know why. I just thought it might be something to do with the way you read. You start from the left and you work over. And certainly I was looking at the left hand one first. I don't know why I switched to the right. It was just one of them I suddenly noticed that the right hand side was not a knot.

Int.
So you kind of focussed in . . . ? You must have scanned the two and seen something.

Subj. 4
Seen something in the right hand one. I am looking at that one now. I cannot remember what I said last time. I can see now that is not identical. I am turning the right hand one upside down so it corresponds with the left hand one and I can see that on the left hand side of the left hand one the top part goes underneath and the bottom half does that.

Int.
You are just comparing one little bit and finding a difference.

Subj. 4
Finding a difference and then there is no point in going over the rest. I find it so hard to switch them round and you are wondering whether there is something you have missed. Whether you should be looking at them from different angles. Because in some of the cases, where the knots appear identical the rope was not arranged exactly the same way.

Int.
Your strategy on that particular one . . . sounds similar to the one that you did in the first place. Did you find that one easy or hard? That shape.

Subj. 4
Relatively easy because I could get the different parts of it. A top and.... this one is hard again. These star shaped ones I found much harder.

Int.
You were talking about two triangles and I presume that that is one of the triangles. Is the other one going to there?

Subj. 4
Yes. It's not quite two triangles but that is one triangle.

Int.
But that is how you've pictured it or noted it ?

Subj. 4
That is why it seemed as though any way you turned it, it looked fairly similar - the same sort of shape. I will see if I agree with it now. Once again I'm turning the right . . .

Int.
You are worried that you can't find parts of it. This is what you mean by 'it's difficult to orient it' because that bit there you are not sure which that bit occurs to over here.

Subj. 4
I do know if I could actually get hold of it on the screen and turn it round - I find I get lost spatially. I just cannot correspond the different bits. I am just turning that bit round to the top in my mind.

Int.
They are not actually identical but they are close enough to make it difficult for you.

Subj. 4
It's as though all the bits look the same.

Int.
So, if I said I want to find that little bit there in the left hand knot?

Subj. 4
I think what I am doing now, which I didn't think I could do then - if I swivelled that round 90 degrees to there, then I compare those two bits then that would be obviously corresponding to that - swivel that round 90 degrees and I compare them all. But I didn't think of doing that at the time.

Int.
You were doing it a bit at a time.

Subj. 4
I am still doing a bit at a time, but I found that I don't have to swivel the whole thing round in my head. I can swivel each little bit through 90 degrees. So, I started with that. I swivelled that through 90 degrees and tried to hold the whole picture in my mind. But now, I have realised that what I can do - that goes through 90 degrees - that corresponds to that. If they go through 90 degrees that will correspond to that - which it does - and if that goes through 90 degrees, that will correspond to that which it does. I find it very hard to hold the whole picture in my mind. The only way I can solve it if they not in the same position is by swivelling it round in my mind and I have now discovered which I have just thought of, I hadn't thought of it before. I can take every little piece and swivel it in the same direction, which always seemed to be anti clockwise - I don't know why - and compare each little bit individually. I am still not finding it easy but I'm finding it easier.

Int.
Before, you felt very daunted by having to hold the image.

Subj. 4
I could not hold it in my mind. I could not do it. If I had them on a piece of paper I could just swivel them round, but you can't do that on the computer screen.

Int.
You were saying something there about rearranging it a little bit. I had a feeling that what you were saying - I remember now - was you thought that one of these loops had been enlarged on the right hand side - that the rope had somehow been moved.

Subj. 4
Now I look at it, it hasn't.

Int.
Do you remember feeling that at the time? I don't know whether - I think it is that knot. I certainly recall you saying something that I thought....

Subj. 4
What I think is that if I turned it - I am wondering whether I meant - if you turned it so that is at the top then it still corresponds - but not quite the same. The loops are not the same size. If you turn it that way, I can see now it's going to be almost identical. This is what I still have not worked out with these knots. If it all corresponds when you put that at the top, would it still correspond if you turned it another way? I don't know whether it would correspond.

Int.
It would. You mean if you actually just turned it anti clockwise rather than clockwise, or do you mean pick it up and turn it over?

Subj. 4
I have always been turning on anti clockwise so I presume that that corresponds to that - so that is what I have been working on. But I
suddenly thought of it now that if I turned this - if I tilted it . . . . I'm not sure if I did that.

Int.
And made this the top?

Subj. 4
Yes. It would still correspond.

Int.
It would in that knot.

Subj. 4
I don't know whether it would in all of them. I just don't know. This is what I have not worked out. This is why I find it very hard. If I am taking it for granted that that on the right hand knot corresponds to that on the left hand knot. I may not be right. It may not work that way. I still have not worked that one out with the knot. Would it always be the same whichever piece you started from?

Int.
Well there are some knots where there is a symmetry and it doesn't matter which you call the top. There are five possible 'tops' for that particular knot. It doesn't matter which one.

Subj. 4
It would always work out?

Int.
Yes.

Subj. 4
I cannot see that but I will accept it.

Int.
But it's not the same with all of them.

Subj. 4
So, it is important to work out.

Int.
Well, it is a possible strategy if you are a mathematician and you know something about knots, but the strategy for somebody who doesn't, would have to be more intuitive. The sort of strategies that you have been using.

Subj. 4
That's interesting that you used the word 'intuitive' because I wouldn't say that I am an intuitive person. I would not just look and think 'that looks the same'. I have to check. That is very much the kind of person I am. I have to check - almost like logically - the correspondence between each part. I would not just look at it and say 'they are the same'. I would not be able to do that. I would have to look and work out the correspondence. I wouldn't just get the general impression.

Int.
You checked more than once sometimes.

Subj. 4
Yes.

Int.
You started with this one as your key.

Subj. 4
Yes. I was always starting with the left hand one and then trying to turn the right hand one up and now it looks quite easy. It looks to me . . . . Actually, I didn't spend much time on it this time.

Int.
There were a number of this shape - more of this shape as it happens than some of the others. And at the beginning I think you thought
'Aarrgh!' when you saw it.

Subj. 4
I sort of became used to it.
It seemed to be easier now to correspond. I still can't do it very well, holding the whole image in my mind, but I can check the various bits. I have got my bearings. It's like looking at a map.

Int.
Bits became more familiar?

Subj. 4
Yes, and it seems very obvious now they're the same.

Int.
Which bits are the bits you are homing in on?

Subj. 4
Intersections - that corresponds to that, and that corresponds to that, and that corresponds to that.

Int.
But you just check them all. There is no place that you think 'I will go there first.'

Subj. 4
No. I always start from the left. I don't know why.

Int.
I don't know if it was that item. I think we may have gone on to the next item on the video there. Do you want to pause it there. There's one of these later ones, the same shape but a different pair and you were very
much describing this bit in the middle, I think. I think that came later.

Subj. 4
I think perhaps because I noticed that wasn't the same. The thing is, if you notice immediately that one thing is not the same, you can just say 'yes'. But with those - because it all seemed to correspond - I had to check each part.

Int.
This was the one and you were talking about crossing at the bottom weren't you?

Subj. 4
Yes, I got to the stage when I could turn them and then once again check. Of course, I can see now the one on the left is not a knot.

Int.
I don't think you did then. Can we rerun that one and see if you notice that?

Subj. 4
I don't think I noticed - only about halfway through - that some of them were knots. It's a pity these are not numbered actually. It would be a help.

Int.
This camera that's running now puts the time in the bottom corner.

Subj. 4
I can do it a lot quicker now. I can see immediately that is not the same because that part is not the same. I don't know - my eyes seem to be accustomed to it - and I can home in on a certain part of it. I can see that the rest is the same, but that goes underneath there, and it goes on top there and that is not a knot and that is. But at the beginning I was very unsure.

Int.
At the beginning, I didn't think you had found that strategy for unknotting at that point - it's quite interesting really how many into the experiment before you started noticing and I wonder if it was triggered by a particular knot.

Subj. 4
Yes. I think I suddenly noticed and I think I remembered that you had said.....

Is this that one?

Int.
This is this one there. This is not a knot.

Subj. 4
I think, yes. I think I saw that one. I could pick that one up and undo it.

Int.
But the previous one you could also now see.

Subj. 4
I can see it now. But I don't think I was looking for it. As I went on, I became more confident. I think at the beginning I was saying 'Aarrgh!' and immediately trying to home in on the details. Whereas now, I am just looking and taking the whole picture - looking at it for a minute as one whole thing. And that is when I think that is the first time I have looked at it as a whole thing and thinking that is not a knot. I can pick that up as an actual picture. I can put my hand into the computer screen and pick up the loop and it would just all unravel.

Int.
And that seemed to occur at this point in the experiment?

Subj. 4
Yes. I think it did. I just said on there that I didn't think it was a knot.
And then I seemed to be able to switch and swivel the right hand one in my head and it seems to be quite obvious now that they are the same. It seems to be easier to be able to switch back and forth between them. I still can't see the whole thing at once but I can find it easier to keep my bearings as I go back to the left hand one and the right hand one and I compare each bit of it.

Int.
So, even though you have noticed this part of the knot that will undo, you are still going to it an rotating it in some way.

Subj. 4
Because even if the left hand one is not a true knot, the right hand one might be. So I am still checking each part of it.

Int.
I see. You have noticed that this one will come undone and then

Subj. 4
I can see that that one will come undone as well. But I am still not sure whether it's arranged in the same way. It could have been that that one might have been crossed in another way. I don't know whether it would be possible. So, I am just checking through them.

Int.
So, you could see that this part of it would undo but had you actually thought it would all undo?

Subj. 4
Yes. I can see that. It is as though I can imagine it in my mind. Grabbing hold of the top loop and I can see there is nothing to stop it just opening out into one whole circle.

## Int.

And the same on that side?

Subj. 4
Yes. I can see the same - that that one would open as well. But I wasn't sure whether they were the same. I was just checking.

Int.
What did you say there?

Subj. 4
I said 'I am getting a bit tired of this.'
I can see immediately the one on the right hand side is not a knot and it seems to me that I know the arrangement that - I can see that that is on the top and there is just a twist in the middle and that is not a knot. I can see immediately that one part of it is underneath, the other one is on top which is the knot-like shape. So, it has taken me about two or three seconds to see that because I have got more confident. I can just look. At the beginning, I was getting lost in the details.

Int.
There is no turning involved in that one.

Subj. 4
No, it makes it easier. See how long it take me on this one.

Int.
You are describing these two points. They're different there, aren't they?

Subj. 4
I am still worrying whether I could turn them round upside down and whether they would be the same.

Int.
Yes. I know what you mean. Because they are so similar it's just a slight
difference in size of the loops.

Subj. 4
I just wondered whether I could turn them round.

Int.
Well, they are in identical positions - you started at the top and going round in your normal way.

Subj. 4
Whereas now I am just looking at it and saying they are the same and they are not knots. It is immediate now. I don't think it's something I could remember because there are so many of the same type - I could almost say that that is identical with the one that was about two goes ago.

Int.
But they were probably with a different orientation. So they're getting easier?

Subj. 4
They are getting easier. Yes.

Int.
And you are using more of this unknotting strategy. Do you remember?

Subj. 4
I can see now that the one on the left is a knot and the one on the right isn't.

Int.
I think this is where you decided to go for the one on the right first.
Do you think it is because of the orientation itself or the shape?

Subj. 4
Yes. It could be that right hand one is . . . . .

Int.
More the right way up? Although you are not familiar with these objects that seems the right way up and that one doesn't?

Subj. 4
Yes. I think that might be it.

Int.
I wonder if it looks as though it would sit without falling over more than the other one.

Subj. 4
I don't know. I have not noticed it. I am looking at the right hand one once again. I'm getting a bit more - I have to check every little piece. I think that the fact that I remember that some of them were not knots was a help. Because once you see that one is a knot and one isn't, you don't have to look any further.

Int.
And you got more strategies to rely on and more experience.

Subj. 4
I think it is experience. I seem to be able to find my bearings. That's the only way I can describe it. In the beginning I was totally lost. I could not see where I was supposed to be starting from. And now, I can almost see the whole thing. Once you have seen that there is one basic difference between them - in the beginning I was checking everything.

Int.
So, in that one now. Do you think you would do, any sense in which you would turn your head or trying to turn one of the pictures round.

Subj. 4
No, because I can see that the one on the right is not a knot and the one
on the left is. Again, I can see immediately the one on the left is not a knot. It's almost as if, when you are learning a foreign language, if you have ever experienced - like in Welsh or another foreign language. At the beginning when you are learning a foreign language everything is just a blur It is just one thing of speech and you are trying to make sense of it all. And then, as you get your bearings, you can pick out words that are significant. It is almost the same with this. Like a foreign language and then suddenly you see what is significant. You don't have to look at everything. I think that is the only way I can describe it. That suddenly it looks familiar and I realise that I don't have to look at it all and it's as though I can pick out the important bits with my eyes. Like a foreign language. When you are learning a foreign language you realise what the important bits are - that you can get the gist of it. You don't have to have it all washing over you like a blur of sound.

## Int.

You know which bits to look out for. So, when it comes to solving it and undoing it, or checking it or whatever, is there any sense in which you are using either language or pictures?

Subj. 4
I am not using language. It's almost as if it is familiar - like that - I know that's fairly simple because they are sort of looking in roughly the same position, but I am just ..... Once again, I started reading from right to left. I don't know why.

Int.
Even with this one?

Subj. 4
Yes. I have started on the right now. That crosses over and then that goes over that and I can see the whole thing and it's immediately obvious it's identical.

Int.
You seem to be on the left hand one actually.

Subj. 4
But there I looked at the right hand one but I think what is happening I am still following the whole rope round with my eye. Whereas now I am not bothering to do that. I'm just looking at the shape and I'm looking at where it crosses over and that's what I'm doing. I'm not following the whole thing round - that looks so easy now. I can just open them up.
I don't know whether I am slower because I'm trying to explain what I'm thinking.

Int.
Yes. You were being very careful at explaining.

Subj. 4
Yes. I think it probably that made me do it a little bit slower. But I still have the impression that now I can - I can see immediately there that they are not.

Int.
Your first line of attack here is the left hand side knot.

Subj. 4
I think that's because the left hand side is upright. Perhaps I have taken that to be - it looks like a more - an easier shape. It looks the right way up. Although I don't know why - on what basis I thought that out. It's as though I go for the one which is easier to decipher.

Int.
What you seem to be doing is looking and concentrating and describing the left hand without even looking at the right hand one at this stage. Because you started to look at the left hand one and said 'Oh yes! That is knotted.' rather than say 'let me just look at this part of the left hand
knot. . . . .

Subj. 4
Once I've discovered that some of them aren't knots - I'll check that first It's as though that's my first line of attack.

Int.
Then, if that failed, you had got somewhere to go back to.

Subj. 4
It does seem very easy now to see that they are not the same, but it did seem a lot more difficult. I realise now that one is not a knot. It is obvious now.

Int.
You were concentrating on the difference here. You were describing -

Subj. 4
I certainly homed in on one thing. I could have said, when I think about it, I'm still not looking globally all the time. I am getting lost in the details. Instead of having a look at both of them.

Int.
We'll see what the next picture is. We have gone on to this one.

Subj. 4
Obviously they are identical.

Int.
You say that with such conviction.

Subj. 4
It seems that I can seem to look at both of them at the same time, whereas before I think I was looking at one of them and then sort of following that right round - it seems to be easier now just to look at them both at
the same time and say 'Oh yes' as though my brain has just got used to finding the significant parts and corresponding them. I think it's a bit like reading a map. You learn to know what to look for.

Int.
It's interesting that it happens as quickly as it does though, because in the course of twenty of these items - by half way through, you have learned the strategy about the unknotting and by this stage now in reflecting, you seemed to really develop the strategy - develop the technique.

Subj. 4
It's as though I have got used to looking at them. But I don't know.
There seemed to be a lot of knots or pseudo knots that are quite similar. I don't know how I would be if you showed me different ones. I might find them more difficult. But, I think, perhaps, I would find them quite easy now. I can see they are the same. I don't know if I do that any more. What I stopped doing - I stopped following it round. At the beginning I was following the whole piece of rope round. I was tracing it like a path. And, I'm just looking - switching back and forth. I'm just getting the same position here. That corresponds, that corresponds, that corresponds. They are the same.

Int.
This is what I would like to ask you a bit more about. Now I've found one that you would come back to - remember that you did trace it round. As you were going round and observing, what do you think was going on? Do you think you were remembering a picture or do you think you were coding it in some way?

Subj. 4
I don't know.

Int.
What was happening there as you did that from the top - I'm going to follow this round.

Subj. 4
I think I'm remembering the picture. I don't think I'm putting it into any kind of code. I'm not saying what would go ..... I'm trying to look at that part and following it round and then trying to make it correspond with the other part. It's fairly easy with that ..... the right hand one is only tilted 45 degrees so it's not very difficult to do that. In the beginning in the first time I was following the whole piece of rope around.

## Subject 5

Int.
Keep playing on until there is something you want to add

## Subj. 5

I started by just looking, especially where the rope was going under and over each other thinking that with the simpler knots you were trying to fool me so that I might have been taking more time with the ones that looked similar than the ones that looked different, in case it was a trick question. So I was careful there to make sure that they were actually going under and over in the same way.

Int.
Right, so you thought that it looked easy in the beginning but you took more time to just check?

Subj. 5
I think I might have been going a bit slow in the beginning to make sure you weren't trying to pull a fast one on me.

Int.
What did you decide here? That picture; what did you say?

Subj. 5
I decided they weren't the same.

Int.
'Under, over, under, over, that's not a knot'.

Subj. 5
I'm picturing the rope just dropped down when it folds over itself.

Int.
You started counting, 'one, two, three, four'. Do you remember what you were doing there?

Subj. 5
Yes. Was it . . . ? Did it have the same number of overs. Was it crossing each other the same number of times?

Int.
Yes. You were counting all of these points?

Subj. 5
I think I decided after that it would be simpler to turn it over, to see if I could get the two pictures . . . If they were aligned in the same direction to see if they corresponded.

Int.
So originally you weren't rotating and then you decided that perhaps it would be a good idea to rotate it? Is that what you're saying?

Subj. 5
Yes, that's correct.

Int.
So at what point were you doing the rotating? After you'd already made a decision? Just checking?

Subj. 5
Here I haven't started rotating. I'll tell you when I remember when I've started to do that.

I was looking to see if they were knotted or just folded.

Int.
I think what you say here is 'Yes, they're both knotted but they still look different'.

Subj. 5
Then I realised that if I just turned one around it would . . . I think this was the one that was so simple . . it was easy to see that it . . I turned one around.

Int.
What's interesting to me is that you suddenly saw that 'Oh yes, all I need to do is rotate it'.

Subj. 5
I had thought about it when you were explaining to me with the knots on the table, but I thought that's too simplistic, she might be trying to fool me. I must be a naturally suspicious sort of person and I decided I wanted a second method of proving whether they were knotted or not.

Int.
Rather than just taking the image and turning it in your mind, you thought that was not going to be a satisfactory method? You wanted a more theoretical method?

Subj. 5
I thought I'd investigate the way they crossed over each other, if that would give me an indication of whether they were knotted or not.

Int.
And you started to do things like looking for it lying on top?

Subj. 5
Yes, I think that was the first thing I did.

Int.
And then when you came to this one now, you decided you were going to rotate it rather than what? What was your alternative?

Subj. 5
I think I was going to use that as an additional means of verification.
Because this one is so simple.

Int.
Simple in respect of . . .?

Subj. 5
It only crosses over three times and you could see easily that if you rotated the right hand one, you would find that the images looked similar and then you could see whether they were matching by looking at the places where they crossed over.

Int.
So you started by turning that one upright?

Subj. 5
Yes.

Int.
And then what did you do next?

Subj. 5
Let me play it again.
I think I was looking at the three crossover points there.

Int.
So do you take them one at a time?

Can we pause it there? I'd like to press you on this one as I'd like to know what you were thinking if you can remember.

Subj. 5
First of all I changed the right hand image round to face the same way round as the left hand one and then I looked at the three crossover points to make sure they were not just lying on top of each other.

Int.
OK

Subj. 5
So in a way I was adding matching the image with . . . . .

Int.
Testing for unravelling?

Subj. 5
Yes.

Int.
You said 'yes' to this one . . .

Subj. 5
'Twist it round', that means I . . . I turned it. I'm just checking there.

Int,
Because you'd made your decision?

You're doing them quickly.

If we go back to this idea of rotating as a strategy . . . Is that how you did that one?

Subj. 5
Yes, I've rotated it first in my mind.

Int.
So when you'd got the second image rotated you went through matching did you?

Subj. 5
Yes. Even though I was touching a point on the screen there, in my mind I was touching then the rotated vision, not what was actually on the screen, , even though I was touching the screen.

Int.
So you found it simple really to do that, then say I'm going to check that crossing with its rotated version?

Subj. 5
Yes.

Int.
And was that easy for you to do?

Subj. 5
Yes, once I'd figured it out. The difficult part was holding the rotated vision in my mind and touching that and not what was actually on screen.

Int.
Right. Did you, what shall we say, re-produce that image during the checking process, or did you keep going back reminding yourself how . . . have I got the right image after rotation?

Subj. 5
No. I noticed that I had rotated them all anti clockwise.
And I was holding that quite firmly in my mind.

Int.
Do you think that particular image was easier or harder or the same as any of the others, to rotate and remember?

Subj. 5
That was an easy one to remember, yes.

Int.
Because you mention in the previous one, you said that was easy because there were only three . . .

Subj. 5
Yes.

Int.
So that one you didn't find . . . although it had four in that one, you still found that quite easy to rotate?

Subj. 5
Yes.

That one doesn't need to rotate. Again I'm suspicious that they look so similar - what are you hiding from me?

Int.
OK. Just pause that for me. You answered 'yes' to that one, and I'm wondering if you have a look at it again now . . actually they're . . well I've implied already that they are actually different. Can you see? I'm wondering if there's something to do with . . . You said you had difficulty seeing some of the crossings there. If you look at that one now .

Subj. 5
Was it this one I found difficult? It was this sort of shading that I found difficult to follow.

Int.
I don't think that was where you said it on tape but perhaps it was a factor?

Subj. 5
Yes. You're telling me that they're different?

Int.
I'm telling you that those two are different.

Subj. 5
Yes?

Int.
And you were quite certain that they were the same weren't you?

Subj. 5
Yes, and I still see them as the same.

Int.
How were you doing it then? If you tried to do it again now . . .

Subj. 5
Well first of all I rotated it that way and I visualised them as the same, and when you said they were different, I rotated it the other way.

Int.
You did this one?

Subj. 5
No, I rotated it this way, instead of turning it up, I turned it clockwise and I looked at the crossover points and I thought 'under, under', and that's . . . . Ah, that's where I've made a mistake, there in the middle. Because that goes over.

Int.
Which way are you rotating it now?

Subj. 5
I'm rotating it clockwise.

Int.
So the middle is where you think you've made a mistake?

Subj. 5
Yes.

Int.
Which way round is this one then?

Subj. 5
If I rotate it clockwise, that crossover point is the same as that crossover point, but the middle one goes under there . . . . and it goes under here as well. I still can't see the difference.

Int.
You're rotating that one clockwise, you've checked that and you've checked that and I agree with you there, but have a look at the third one.

Subj. 5
Oh yes, it goes under there and it goes over there.

Int.
Right, so you would agree with that then now?

Subj. 5
After long thought. But I . . . well, I mean looking at it, it still doesn't jump out at me.

## Int.

At this point then, what would be your best strategy, from what you've done already and what you've picked up as you were doing it, what's your best shot at that one? How would you try to do that one now?

Subj. 5
That's the one that has just been twisted isn't it? Am I right? I mean it's not actually knotted so again I have to visualise a circle of rope and I twist it and drop it.

Int.
And is that one . . ?

Subj. 5
Well if you try to do that it doesn't work because that's over and that's under. They both fold up. (weave)

Int.
So in fact you weren't trying that strategy when you did it on screen?

Subj. 5
No.

Int.
You probably decided they were the same and didn't . . . . You did use that strategy in others didn't you?

Subj. 5
A complete circle, yes.

Int.
Shall we move on?

Subj. 5
It's more complicated. Because I can't make sense of it straight away. I can't even work out which way to rotate it, to get it similar. . .

Int.
You said 'They look as if they might not be proper knots'. Following on from that previous one In this one you started to see that there was a sense in which you could lift some of it up. Do you remember that?

Subj. 5
I remember thinking that if I had the actual circle of rope to fiddle around with and see whether I could reproduce that sort of image, by folding it.

Int.
So you did it by constructing it from the rope rather than deconstructing it from that image? That's interesting.

Subj. 5
Because it's not familiar to me I think. Like the simpler shapes were familiar. I'd come across those sort of shapes before.

Int.
And that one was just a mess really!

Subj. 5
Yes.

Int.
So you wanted to start from the simple loop of rope and make it look like that? That was the way you were doing it?

Subj. 5
Yes.

Int.
You said 'yes' again to that one.

Subj. 5
Yes, they look similar. Did I get it wrong?

Int.
Now have a look at it.

Subj. 5
It goes under and over, and under and under.

Int.
So the difference is there . .

Subj. 5
So that one's knotted. Have I been fooled by the way that middle bit has been twisted?

Int.
A minute ago you were looking at this one (crossing) and this one, the one at the bottom and this one at the top and then again in this diagram. And you seemed to find a difference there didn't you?

Subj. 5
Let's see.

Int.
I think you said 'yes'.

Subj. 5
What did 'yes' mean? 'Yes they're different' or 'Yes I'm satisfied with
the answer'? Yes, I think I said those are similar . . . . .

Int.
'Yes they look similar' and 'yes'. In fact I think you clicked on the 'yes' there but a moment ago you were seeming to find a difference.

Subj. 5
Yes. Both those crossover points, this passes over, and that's under and that's over.

Int.
O.K. So you've found a difference now.

Subj. 5
Yes. I didn't stop long enough did I? I was fooled by the fact that . . .

Int.
They were the same shape and the orientation was the same? . . . . You were happy that they were the same?

Subj. 5
I'm checking that all the crossover points to make sure that they all match. Again I've rotated this one. I can't see that one. .

Int.
You're looking at that crossing there and you're not sure which way it is? Well if I say that the way it's meant to look is that comes over and where that dark shadow is there, is meant to be under, is that clear to you now?

Subj. 5
Yes.

Int
I think you finally did interpret it like that.

Subj. 5
Yes, I worked out that the shadowy part was meant to be the shadow of being underneath. I'm not sure what foxed me.

Int.
So if that's that way . . . . if that particular crossing is over . . . . if you were to try and do that now . . . ?

Subj. 5
The crossovers are different ways. I'm not sure if I'm looking to see if they crossover . . . with this one on top there, would it pass . . . It depends on which way it turned doesn't it?

Int.
Whether you end up with that one matching with that, or that one matching with that?

Subj. 5
Yes.

Int.
Which direction do you want to try?

Subj. 5
Well if I turn it clockwise I can see that that matches with that but that doesn't match.

Int.
Now, what if you were trying to o that one from what you might have discovered during the 20 , what would be your best attempt, how would you . . . what would be your best strategy to deal with that one? Would you do the rotation. Is that what you would be trying to do?

Subj. 5
Instead of rotating anti clockwise, I would look at it now and find a
matching point and then rotate it either clockwise or anti clockwise according to whether - which point I'd fixed upon as being the same and then it would be easier to compare other points. Because if I'd rotated this anti clockwise . . . comparing these two well they're different, it's easier to start with something that's the same.

Int.
I'm with you.

Subj. 5
Whereas if I'd tried what I did now the other way I'd find 'yes' I can match those two points and then look at them.

## Int.

Just out of interest, you've come to the conclusion that you think they are different. So which is the knotted one and which is the unknotted one?

Subj. 5
That's the knotted one because it passes over one and under the other. This one's not knotted because the rope lies on top of the lower one and the two sides . . . if we look again using the trick of just visualising a circle of rope and twisting it and laying it down . . .

Subj. 5
So you're again doing that starting with a circle of rope. You're constructing again rather than deconstructing. You're starting with your circle and making it look like that shape? That's interesting, that's unusual that.

Subj. 5
Perhaps that starts from being frustrated at not having this actual rope here to play with which is where I started from. Thinking 'oh if I had that rope that's lying over there to play with I could see if I could match it.

## I found that one easy to visualise as a twist of rope lying down

Int.
It's interesting to me that the previous one . . . it didn't come out at you somehow, that fact, and you didn't use that strategy at all in that item and yet the next item along, suddenly your strategy is ' ah yes, I can see . . .

Subj. 5
Because I'm familiar with that shape.

Int.
Is that why you think it is?

Subj. 5
Yes.

Int
Why do you think that is a familiar shape. What does it remind you of?

Subj. 5
It's like a Celtic knot really.

Int.
Right, so that's what you mean by a familiar shape, you've seen it represented elsewhere?

Subj. 5
It's a common motif.

Int.
And because of that, what happened when you were looking?

Subj. 5
Because this was turned this way, those two points sort of come out at me
quite strongly.

Int.
They were not how you would normally have seen them, is that what you mean? When you've seen this familiar icon?

Subj. 5
Yes.

Int.
So you were immediately able to do that one, and spot that it was the wrong . . . .

Subj. 5
Yes.

Int.
O.K. Well you quickly dealt with that one. You were looking up here weren't you? And you can see I think the same thing that you were doing earlier.

So you dealt with that one quickly too. You're moving on rapidly with these. I didn't actually hear what you responded for this one here. The previous one . . y you'd had a lot of difficulty with it earlier and yet somehow that one seemed easier to do.

Subj. 5
Was it learning from experience or was I still getting it wrong?

Int.
No. I think you said they were both the same.

What do you say for these two?

Subj. 5
I'm saying 'here and here and here'

Int.
So you were just checking one to one? You weren't using any other strategy? Other than the one at a time going through each of the crossings?

Subj. 5
Because I didn't need to turn them round.

Int.
Right.

That's another one of that shape. Can you remember what you were doing?

Subj. 5
This isn't the one you've got up now? It looks the same to me.

Int.
That's the one we've got on screen now. Oh, you've done it already. I think you were finding that one easy by this stage.

You were checking each crossing there.
You're doing this matching again, one to one with the three . . .

Subj. 5
I must have decided that it was easier to match them by turning the left one because I've been quite consistently turning the right one.

Int.
I don't know whether you feel at the end of those that there is any kind of learning of strategies, that you feel as if now if you were to go and do another 20 if you are any better prepared having done a few of them?

Subj. 5
You had to use more than one criterion in a way. To look at them both, and try and turn them round to see if they match by looking at the crossover points, and then visualising if you think it's not a proper knot, but twisted over. Then visualising it as a rope being twisted, a circle of rope twisted and laid down in that pattern. Those were the three criteria that I did use.

Int.
When it was the situation that you were trying to make the loop of rope, construct it from the loop of rope, some of the shapes you seemed to immediately use that strategy and others you didn't. We talked about the pentoil as a sort of Celtic shape as being a familiar one, and it looked different. Any of the others?

Subj. 5
I think I used it to back up my theory that it was a piece of rope just twisted rather than knotted. If I thought it was twisted, I'd visualise twisting it to see if it would work out whereas if I thought it was knotted, I wouldn't use that technique.

## References

Adams, C. C. (1994) The Knot Book, New York, Freeman

APU (1980) Mathematical Development: Secondary Survey Report No. 1. London, HMSO

Ashley, C. W. (1944) The Ashley Book of Knots, New York, Doubleday

Bagnara, S., Simion, F. \& Tagliabue, M. E. (1988) 'Comparison processes on visual mental images', Memory \& Cognition 16, 138-46

Battista, M. T. (1990) 'Spatial visualisation and gender differences in high school geometry', Journal for Research in Mathematics Education 21 (1), 47-60

Battista, M. T. (1994) 'On Greeno's environmental/model view of conceptual domains: a spatial/geometric perspective', Journal for Research in Mathematics Education 25 (1), 86-94

Battista, M. T., Clements, D. H. \& Wheatley, G. H. (1991) 'Using spatial imagery in geometric reasoning' Arithmetic Teacher Nov 18-21

Battista, M. T., Talsma, G. \& Wheatley G. H. (1982) 'The importance of spatial visualisation and cognitive development for geometry learning in preservice elementary teachers', Journal for Research in Mathematics Education 13 (5), 332-340

Beeney, R. (1982) Geometric Images, Derby, Association of Teachers of Mathematics

Begle, E. G., (1969) 'The role of research in the improvement of mathematics education', Educational studies in Mathematics 2, 232-244 11, 23-41

Begle, E. G., (1969) 'The role of research in the improvement of mathematics education', Proceedings of the First International Congress of Mathematical Education 100-112 Dordrecht, The Netherlands

Bell, A. W., Costello, John and Küchemann, D. (1983) Research on Learning and Teaching, Part A, Windsor, NFER Nelson

Bell, A. (1994) 'What is Research in Mathematical Education and what are its results?' Paper presented at ICMI conference University of Maryland, Washington DC May 1994

Ben-Chaim, D., Lappan, G. \& Houang R.T. (1988) 'The effect of instruction on spatial visualisation skills of middle school boys and girls', American Education Research Journal 25, 51-71

Ben-Chaim, D., Lappan, G. \& Houang, R. T. (1989) 'The role of visualisation in the middle school curriculum', Focus on Learning problems in mathematics, 11 (1), 49-60

Birman, J. S. \& Lin X-S, (1993) 'Knot polynomials and Vassiliev's invariant' Inventiones Mathematicae 111 (2), 225-270

Bishop, A. J. (1973) 'Use of structural apparatus and spatial ability; a possible relationship', Research in Education, 9, 43-49

Bishop, A. J. (1980) 'Spatial abilities and mathematics education - a review', Educational Studies in Mathematics, 11, 257-269

Bishop, A. J. (1983) 'Space and geometry' in R. Lesh and M. Landau (eds) Acquisition of Mathematics Concepts and Processes 175-203. New York, Academic Press

Bishop, A. J. (1989) 'Review of research on visualisation in mathematics education', Focus on Learning Problems in Mathematics 11 (1), 7-16

Bishop, A. J. (1992) 'International perspectives on research in mathematics education' in D. A. Grouws, (ed), Handbook of Research on Mathematics Teaching and Learning, New York, Macmillan and National Council of Teachers of Mathematics.

Brandimonte, M. A., Hitch, G. J. \& Bishop, D. V. M. (1992) 'Manipulation of visual mental images in children and adults', Journal of experimental child psychology 53 (3), 300-31

Brown, D. L. \& Presmeg, N. C. (1993) 'Types of imagery used by elementary and secondary school students in mathematical reasoning', in I. Hirabayashi, N. Nohda, K. Shigematsu \& Fou-Lai Lin (eds), Proceedings of the Seventeenth International Conference for the Psychology of Mathematics Education, Japan Vol 2, 137-144

Brown, D. L. \& Wheatley G. H. (1997) 'Components of Imagery and mathematical understanding', Focus on Learning Problems in Mathematics 19 (1), 45-70

Bruner, J. (1963) The Process of Education, Cambridge Mass., Harvard University Press

Bruner, J. (1986) Actual Minds, Possible Worlds, Cambridge Mass., Harvard University Press

Bryant P. E. (1982 ) 'The role of conflict and agreement between intellectual strategies in children's ideas about measurement', British Journal of Psychology 73, 243-52

Campbell K. J., Collis, K. F. \& Watson, J. M. (1995) 'Visual processing during mathematical problem solving', Educational Studies in Mathematics 28, 177-194

Cheung Yiu Lin (1979) 'Imagery in mathematical thinking and learning',

International Journal for Mathematics Education Science and Technology 10 (1), 107-111

Christensen, L. B. (1988) Experimental Methodology Newton, Mass., Allyn and Bacon

CITMS (Cockcroft Committee of Inquiry into the Teaching of Mathematics in Schools) (1982) Mathematics Counts, London, HMSO

Clements, D. H. and Battista, M. T. (1990) 'The effects of LOGO on children's conceptualisations of angle and polygons', Journal for Research in Mathematics Education 21 (5), 356-371

Clements, D. H. and Battista, M. T. (1992) 'Geometry and Spatial Reasoning', in D. A. Grouws, (ed), Handbook of Research on Mathematics Teaching and Learning, New York, Macmillan and National Council of Teachers of Mathematics.

Clements, M. A. (1980) 'Analysing children's errors on written mathematical tasks', Educational Studies in Mathematics 11 (1), 1-21

Clements, M. A. (1981) 'Visual Imagery and School Mathematics', For the Learning of Mathematics 2 (2), 2-9 and 2 (3), 33-39

Clements, M. A. (1983) 'The question of how spatial ability is defined and its relevance to mathematics education Zentralblatt fur Didaktik der Mathematik 1 (10) 8-20

Clements, M. A. \& Del Campo, G. (1989) 'Linking verbal knowledge, visual images, and episodes for mathematical learning', Focus on Learning Problems in Mathematics 11 (1), 25-33

Clements, M. A. \& Wattanawaha (1978) 'The classification of spatial tasks suitable for the classroom', in D. Williams (ed) Learning and Applying Mathematics Melbourne, AAM p432-445

Cooper, L. A. (1975) 'Mental rotation of random two-dimensional shapes' Cognitive Psychology, 7, 20-43

Cooper, L. A. (1991) 'Dissociable aspects of the mental representations of visual objects' in R. H. Logie \& M Denis (eds), Mental Images in Human Cognition Amsterdam, North-Holland

Cooper, L. A. and Shepard, R. N. (1973) 'Chronometric studies of the rotation of mental images', in W. G. Chase (ed), Visual Information Processing. New York, Academic Press

Cooper, L. A. and Shepard, R. N. (1990) 'Turning something over in the Mind', in R. R. Llinas, The Workings of the Brain - Readings from Scientific American, Freeman.

Cromwell, P. (1993) 'Celtic knotwork - Mathematical art' The Mathematical Intelligencer 15 (1), 36-47

Cromwell, P., Beltrami, E. \& Rampichini, M. (1998) 'The Borromean Rings', The Mathematical Intelligencer 20 (1), 53-62

Dawe, L. (1993) 'Visual imagery and communication in the mathematics classroom' in M. Stephens, A.Waywood, D. Clarke and J. Izard (eds), Communicating Mathematics: Perspectives from Classroom Practice and Research, Hawthorn Victoria, Australian Council for Educational Research

Dawson, S. (1988) 'Words tiggered by images: images triggered by words', in J. Chatley, (ed) Mathematical Images, Derby, Association of Teachers of Mathematics

Denis, M. (1991) 'Imagery and thinking', in C. Cornoldo \& M. A. McDaniel, (eds) Imagery and Cognition, New York, Springer Verlag

Denis, M. \& Cocude, M. (1989) 'Scanning visual images generated from verbal descriptions', European Journal of Cognitive Psychology 1, 293-307

Descartes R. (1968) translation of 'Meditations on First Philosophy' in E. Muldane \& G. R. T. Ross, The philosophical works of Descartes, Cambridge, Cambridge University Press

Devlin, K. (1988) Mathematics: The new golden age London, Penguin

DFE (1995) Mathematics in the National Curriculum London, HMSO

Dickson, L., Brown, M. \& Gibson, O (1984) Children Learning Mathematics, London, Cassell

Donaldon, M. (1978) Children's Minds, London, Croom Helm

Dreyfus T. (1995) 'Imagery for diagrams' in R. Sutherland \& J. Mason (eds) Exploiting Mental Imagery with Computers, New York, Springer Verlag

Ernest, P. (1989a) 'The impact of beliefs on the teaching of mathematics' in Ernest, P. (ed) Mathematics Teaching - the State of the Art, London, Falmer Press

Ernest, P. (1989b) 'The role of the microcomputer in primary mathematics' in Ernest, P. (ed) Mathematics Teaching - the State of the Art, London, Falmer Press

Ernest, P. (1998) 'A post modern perspective on research in mathematics ducation', in A. Sierpinska \& J. Kilpatrick (eds), 'Mathematics Education as a Research Domain: A Search for Identity', ICMI Study Book 1, Dordrecht, Kluwer

Farah, M. J. , Hammond, K. M., Levine, D. N. \& Calvanio R. (1988)
'Visual and spatial mental imagery: dissociable systems of
representation', Cognitive Psychology 20, 439-462

Fennema, E. (1974) 'Mathematics learning and the sexes: a review', Journal for Research in Mathematics Education 5, 126-139

Fennema, E \& Sherman, J. (1977) 'Sex-related differences in mathematics achievement, spatial visualisation and affective factors' American Education Research Journal 14 (1), 51-71

Fielker, D. (1993) Starting from your Head - Mental Geometry, London, Beam

Finke, R., Pinker, S. \& Farah, M. J. (1989) 'Reinterpreting visual patterns in mental imagery.' Cognitive Science 13,51-78

Finke, R. A., \& Schmidt, M. J. (1978) 'The quantitative measure of pattern representation in images using orientation-specific aftereffects', Perception and Psychophysics 23,515-520

Flegg, G (1975) 'Art: Mathematics: An inter faculty 2nd level course' History of Mathematics, OU AM289 Milton Keynes Open University Press

Freyd, J. J. (1987) 'Dynamic mental representations', Psychological Review 94 (4) 427-438

Friedman, A. \& Harding, C. A. (1990) 'Seeing versus imagining movement in depth', Canadian Journal of Psychology, 44, 371-383

Gates, P. (1988) 'Working Mathematically on Circles and Imagery', Mathematics Teaching, 124, 21-23

Gates, P. Working Mathematically with Mental Imagery with Third Formers, PM64C Open University Video

Gattegno, C. (1971) What We Owe Children: the subordination of teaching to learning, London, Routledge and Kegan Paul

Gattegno, C. (1965) 'Mathematics and imagery', Mathematics Teaching 33, 22-24

Gilbert N. D., \& Porter, T. (1994) Knots and Surfaces, Oxford, Oxford University Press

Goldenberg, E. P. (1995) 'Ruminations about dynamic imagery (and a strong plea for research)', in R. Sutherland \& J. Mason (eds) Exploiting Mental Imagery with Computers, New York, Springer Verlag

Gombrich, E. (1990) 'Pictorial Instructions', in H. Barlow, C Blakemore \& M. Weston-Smith (eds) Images and Understanding, Cambridge, Cambridge University Press

Guay, R. (1977) Purdue spatial visualisation test: Rotations, West Lafayette, Ind:Purdue Research Foundation

Guay, R. B. \& McDaniel, E. D. (1977) 'The relationship between mathematics achievement and spatial abilitoes among elementary school children', Journal for Research in Mathematics Education 7, 211-215

Harper, E., McLeay, H., et al. (1988) NMP - Mathematics for Secondary Schools, Red Track Book 3, Harlow, Longman Group UK

Hinton, G. E. (1979) 'Imagery without arrays', Behavioural and Brain Sciences 2, 555-556

Hoffer, A. (1983) 'Van-Hiele based research', in R. Lesh and M. Landau (eds) Acquisition of Mathematics Concepts and Processes 205-227. New York, Academic Press

Hoz, R. (1981) 'The effects of rigidity on school geometry learning',

Jaworski, B. (1994) Investigating Mathematics Teaching, London, Falmer Press

Jolicoeur, P. (1988) 'Mental rotation and the identification of disoriented objects' Canadian Journal of Psychology 42 (4), 461-478

Jolicoeur, P., Regehr, S., Smith, L. B. J. P., \& Smith, G. N. (1985) 'Mental rotation of representations of two-dimensional and three-dimensional objects' Canadian Journal of Psychology 39 (1), 100-29

Kauffman, L. (1991) Knots and Physics, Singapore, World Scientific Kaufmann, G. (1985) 'A theory of symbolic representation in problem solving' Journal of Mental Imagery 9 (2), 51-70

Kent D. \& Hedger K. (1980) 'Growing tall', Educational Studies in Mathematics 11, 137-179

Kilpatrick, J (1992) 'A history of research in mathematics education', in D. A. Grouws, (ed), Handbook of Research on Mathematics Teaching and Learning, New York, Macmillan and National Council of Teachers of Mathematics.

Kosslyn, S. M. (1980) Image and Mind, Cambridge, Mass., Harvard University Press

Kosslyn, S. M. (1983) Ghosts in the Mind's Machine, New York, Norton

Kosslyn, S. M. (1987) 'Seeing and imagining in the cerebral hemispheres: a computational approach', Psychological Review 94 (2), 148-173

Kosslyn, S. M. (1994). Image and Brain, Cambridge, Mass, MIT Press.

Kosslyn, S. M., \& Shwartz, S. P. (1977) 'A simulation of visual imagery' Cognitive Science 1, 265-295.

Kosslyn, S. M., Koenig, O., Barrett, A., Cave, C. B., Tang, J. \& Gabrieli, J. D. E. (1989) 'Evidence for two types of spatial representation: Hemispheric specialisation for categorical and coordinate relations' Journal of Experimental Psychology, Human Perception and Performance 15, 723735.

Krutetskii, V. A. (1976) The Psychology of Mathematical Abilities in School Children, Chicago, University of Chicago Press.

Lacey, P. (1998) 'Using geometric images of number to teach mental addition and subtraction' Mathematics teaching 163, 31-35

Lean, G. \& Clements M. A. (1981) 'Spatial ability, visual imagery, and mathematical performance', Educational Studies in Mathematics 12, 267-299

Logie, R. H. (1991). 'Visuo-spatial short term memory: Visual working memory or visual buffer' in C. Cornoldo \& M. A. McDaniel (eds) Imagery and Cognition, New York, Springer Verlag

Logie, R. H. (1995) 'Visuo-Spatial Working Memory', Hove, UK, Lawrence Erlbaum

Love, E. (1995) 'The functions of visualisation in learning geometry' in R. Sutherland \& J. Mason (eds) Exploiting Mental Imagery with Computers, New York, Springer Verlag

Love, E. \& Tahta, D. 'Reflections on some words used in mathematics education' in D. Pimm \& E. Love (eds), Teaching and Learning School Mathematics, London, Hodder \& Stoughton

Mason, J. (1988) 'Imagery, Imagination and Mathematics Classrooms' in

Pimm, D. (ed) Mathematics, Teachers and Children, London: Hodder and Stoughton/ Open University

Mason, J. (1991) 'Questions about geometry' in D. Pimm \& E. Love (eds), Teaching and Learning School Mathematics, London, Hodder \& Stoughton

Mason, J. (1992) 'Doing and Construing Mathematics in Screenspace' in B. Southwell, B. Perry and K, Owens (eds), Space the First and Final Frontier, Proceedings of the Fifteenth Annual Conference of the Mathematics Education Research Group of Australia, University of Western Sydney

Macfarlan, A. \& Macfarlan, P. (1983) 'Knotcraft: The Practical and entertaining art of tying knots', New York, Dover

McGee, M.G. (1979) Human Spatial Abilities, New York, Praeger

McLeay, H. (1994) Strat-math: Strategies and Skills in Mathematics, Cardiff, Welsh Office

McLeay, H. (1991) 'Mathematics and Knots', Mathematics in School, 20 (1), 28-31

McLeay, H. (1994) The Knots Puzzle Book, Stradbroke England, Tarquin Publications.

McLeay, H. \& Piggins, D. (1996) 'The mental manipulation of 2-D representations of knots as deformable structures', Educational Studies in Mathematics 30, 399-414

McLeay, H. \& Piggins, D. (1998) 'Mental unknotting', Perceptual and Motor Skills 86, 707-719

Mitchelmore, M. C. (1980) 'Three-dimensional drawing in three
cultures', Educational Studies in Mathematics 11, 205-216

Moses, B. E. (1977) 'The nature of spatial ability and its relationship to mathematical problem solving' , PhD thesis Indiana University

Nelsen, R. B. (1993) Proofs without words - Exercises in visual thinking Washington, The Mathematical Association of America

Owens, K., Mitchelmore, M., Outhred, L. \& Pegg J. (1996) 'Space, geometry, measurement and visualisation' in Atweh, B. et al Research in Mathematics Education in Australasia 1992-1995

Paivio, A. (1971) Imagery and Verbal Processes New York, Holt, Rinehart \& Winston

Pascual-Leone, J. (1976) 'Metasubjective problems of constructive cognition: forms of knowing and their psychological mechanisms', Canadian Psychological Review 17, 110-25

Pesci A. (1995) ' Visualisation in mathematics and graphical mediators: an experience with 11-12 year old pupils' in R. Sutherland \& J. Mason (eds) Exploiting Mental Imagery with Computers, New York, Springer Verlag

Piaget J. \& Inhelder B. (1956) The Child's Conception of Space, London, Routledge and Paul

Piaget J. \& Inhelder B. (1971) Mental Imagery in the Child, London, Routledge and Paul

Piaget, J., Inhelder, B. and Szeminska, A. (1960) The Child's Conception of Geometry, London, Routledge and Kegan Paul

Polya, G. (1957) How to Solve it , Princeton NJ, Princeton University Press

Presmeg, N. (1985) The Role of Visually Mediated Processes in High School Mathematics: A Classroom Investigation, PhD Dissertation, University of Cambridge

Presmeg, N. (1986) 'Visualisation in High School Mathematics', For the Learning of Mathematics 6 (3), 42-46

Presmeg, N. (1998) 'Balancing complex human worlds: Mathematics education as an emergent discipline in its own right' in A. Sierpinska \& J. Kilpatrick (eds), 'Mathematics Education as a Research Domain: A Search for Identity', ICMI Study Book 1, Dordrecht, Kluwer

Pylyshyn Z. W. (1973) What the mind's eye tells the mind's brain: A critique of mental imagery', Psychological Bulletin 80, 1-24

Reisberg, D., \& Chambers, D. (1991) 'Neither pictures nor propositions: what can we learn from a mental image?' Canadian Journal of Psychology 45 (3), 336-52

Resnick L. (1988) 'Treating mathematics as an ill structured discipline' in R. Charles \& E. Silver (eds), The Teaching and Assessing of Mathematical Problem Solving, Reston, VA; National Council of Teachers of Mathematics

Reynolds, A. \& Wheatley, G. H. (1997) 'A student's imaging in solving a nonroutine task' Teaching Children Mathematics 4 (2), 100-104

Richardson, J. T. E. (1991) 'Imagery and the brain' in C. Cornoldo \& M. A. McDaniel (eds) Imagery and Cognition, New York, Springer Verlag Riddoch, M. J. (1990) 'Loss of visual imagery: A generation deficit' Cognitive Neuropsychology, 7, 249-273

Robinson, J. (1992) Symbolic Sculpture, Geneva, Edition Limitee

Schoenfeld, A. H. (1992) 'Learning to think mathematically: Problem solving, metacognition, and sense making in mathematics.' in D. A. Grouws, (ed), Handbook of Research on Mathematics Teaching and Learning, New York, Macmillan and National Council of Teachers of Mathematics.

Sharma, M. C. (1979) 'Children at risk for disabilities in mathematics', Focus on Learning Problems in Mathematics 1 (2) 63-64

Shepard, R. N. (1988) 'The role of transformations in spatial cognition' in J. Stiles-Davis, M. Kritchevsky, \& U. Bellugi (eds) Spatial Cognition Brain Bases and Development, Hillsdale NJ, Lawrence Erlbaum

Shepard, R. N., \& Feng, C. (1972) 'A chronometric study of mental paper folding' Cognitive Psychology 3, 228-243

Shepard, R. N., \& Metzler, J. (1971) 'Mental rotation of threedimensional objects' Science 171,701-703

Shwartz, S. P. (1981) 'The perception of disoriented complex objects' Manuscript, Yale University

Skemp, R. R. (1971) The Psychology of Learning Mathematics, Harmondsworth, Penguin Books

Skinner, B. F. (1953) Science and Human Behaviour, New York, Macmillan

Smith, I. (1964) Spatial Ability, San Diego, Knapp

Smith, P. (1991) 'Spatial ability' Topic 5, 1-6

SMP (1994) Developing Mathematical Imagery, Cambridge, Cambridge University Press

Strohecker, C. (1991) 'Elucidating styles of thinking about topology through thinking about knots' in I. Harel \& S. Papert (eds), Constructionism - research reports and essays, Norwood, N.J. : Ablex Pub. Corp

Strohecker, C. (1996) 'Understanding topological rlationships through comparisons of similar knots' AI and Society 10 (1), 1-58

Sutherland, P. (1992) Cognitive Development Today - Piaget and his Critics, London, Paul Chapman

Suwarsono, S. (1982) 'Visual imagery in the mathematical thinking of seventh grade students', PhD thesis Monash University

Suzuki, K., \& Nakata, Y. (1988) 'Does the size of figures affect the rate of mental rotation?' Perception and Psychophysics 44 (1), 76-80

Tarr, M. J. \& Pinker, S. (1989) 'Mental rotation and orientation dependence in shape recognition' Cognitive Psychology 21, 233-282

Thomas, N. \& Mulligan, J. (1995) 'Dynamic Imagery in children's representations of number' Mathematics Education Research Journal 7 (1), 5-25

Tuckey, H. \& Selvaratnam, M. (1993) ' Studies involving threedimensional visualisation skills in chemistry: a review' Studies in Science Education 21, 99-121

Turner, J. C. (1996) 'On theories of knots' in J. C. Turner and P. van de Griend (eds), History and Science of Knots, Singapore, World Scientific

Usiskin, Z. (1987) 'Resolving the continuuing dilemmas in school geometry' in M. M. Lindquist \& A. P. Shulte (eds), Learning and Teaching Geometry K-12 1987 Yearbook, Reston, VA; National Council of Teachers of Mathematics
van de Griend, P (1996a) 'A history of topological knot theory' in J. C. Turner and P. van de Griend (eds) History and Science of Knots, Singapore, World Scientific
van de Griend, P (1996b) 'On the true love knot' in J. C. Turner and P. van de Griend (eds) History and Science of Knots, Singapore, World Scientific
van de Kleij, G. (1996) 'On knots and swamps' in J. C. Turner and P. van de Griend (eds) History and Science of Knots, Singapore, World Scientific
van Hiele, P. M. (1986) Structure and Insight, Orlando, Academic Press
von Glasersfeld, E. (1987a) 'Learning as a constructive activity' in C. Janvier (ed) Problems of Representation in the Teaching and Learning of Mathematics , Hillsdale NJ, Lawrence Erlbaum
von Glasersfeld, E. (1987b) 'Preliminaries to any theory of representation' in C. Janvier (ed) Problems of Representation in the Teaching and Learning of Mathematics, Hillsdale NJ, Lawrence Erlbaum

Vygotsky, L. S. (1978) Mind in Society: The Development of the Higher Psychological Processes, Cambridge MA, Harvard University Press

Wallace, B. \& Hofelich, B. G. (1992) 'Process generalisation and the prediction of performance on mental imagery tasks', Memory and Cognition 20 (6), 695-704

Warner, C. (1996) 'Why knot? - Some speculations on the earliest knots' in J. C. Turner and P. van de Griend (eds) History and Science of Knots, Singapore, World Scientific

Warner, C. and Bednarik, R. G. (1996) 'Pleistocene knotting' in J. C.

Turner and P. van de Griend (eds) History and Science of Knots, Singapore, World Scientific

Wells, D. (1991) The Penguin Dictionary of Curious and Interesting Geometry, London, Penguin

Wendrich, W. (1996) 'Ancient Egyptian rope and knots' in J. C. Turner and P. van de Griend (eds) History and Science of Knots, Singapore, World Scientific

Wheatley, C. L. and Wheatley, G. H. (1979) 'Developing spatial ability', Mathematics in School 8 (1), 10-11

Wheatley, G. H. (1977) 'The right hemisphere's role in problem solving', Arithmetic Teacher 25 (2), 37-38

Wheatley, G. H. (1990) 'Spatial sense and mathematics learning' Arithmetic Teacher 37 (6), 10-11

Wheatley, G. H. (1991) 'Enhancing mathematics learning through imagery.' Arithmetic Teacher 39 (1), 34-36


[^0]:    2 The other data given in Table 1.1 are discussed later in this section

[^1]:    3 The constant ' $a$ ' refers to the amount of mental effort involved in attending to the specific instructions in the task.

[^2]:    4 Krutetskii emphasised that not all of these abilities are obligatory for mathematical success
    5 A third type, 'harmonic' is a combination of the other two
    6 Similar distinctions related to activity in right and left hemispheres in the brain were later described by Sharma (1979) and were mentioned earlier in this chapter

[^3]:    1 The 'propositional versus pictorial' debate is discussed in Chapter 3.

[^4]:    2 See also Chapter 3 section 3.3 for Kosslyn's ideas on retrieving an unfamiliar image

[^5]:    3 Krutetskii (1976), however, described spatial ability as a particular type of giftedness in mathematics rather than an essential skill or factor

[^6]:    4 An interesting observation which was noted by Smith (1991) is that high scores on such spatial tests often go to subjects who flexibly switch strategies between verbal and spatial in response to different test items.

[^7]:    5 See the tasks used by Clements \& Wattanawaha (1978) later in this chapter or Shepard and Feng's paper folding experiments in Chapter 3

[^8]:    6 There is an interesting quote from Goethe's Mephistopheles, translated in von Glasersfeld (1987, page 215) 'Just where we have no concepts, words come in very handy'

[^9]:    7 The most recent material, as yet unpublished, contains an item on knots

[^10]:    12 The four topics were Knots, Symmetry, Networks and Spherical Geometry.

[^11]:    14 CABRI enables the drawing and transformation of constrained and unconstrained geometric shapes on a computer screen thus allowing the demonstration of theorems and other geometric properties. Equivalent software which is also available is Geometry Inventor and Geometer's Sketchpad

[^12]:    1 Similar comparisons have been previously noted in Chapter 2 regarding a 'chiliagon' and a zebra

[^13]:    2 See Figure 3.1 for an example

[^14]:    4 These were shown in Figure 3.5

[^15]:    1 A Hercules knot has been shown to be a reef knot, see van de Griend (1996b)
    2 A belt made of rope or cord

[^16]:    3 However, the mathematician Felix Klein explained that a rope is knotted only in 3 dimensions and that in 4 dimensional space it can be undone. This notion is similar to the idea that a 2-D figure, say a letter ' $b$ ', and its mirror image ' $d$ ' cannot be superimposed without access to the third dimension. The ' $b$ ' must be lifted up out of the plane and turned over to become ' $d$ '.

[^17]:    4 Another form of which is the mathematical trefoil knot shown in Figure 4.1

[^18]:    5 A mesh knot was shown earlier in Figure 4.2.

[^19]:    6 Knot 3 in the Figure is an alternative form for the numeral 2

[^20]:    7 The mathematics of knots includes the study of links, links being knots made up of more than one strand.

[^21]:    9 A prime knot is one which cannot be made up of two or more simpler knots

[^22]:    9 The knot complement, $\mathrm{R}^{3}-\mathrm{L}$, carries much more topological information than the knot itself L

[^23]:    11 So called from the initials of the mathematicians who discovered it, Hoste, Ocneanu, Millet, Freyd, Lickorish and Yetter.

[^24]:    http://www.cs.ubc.ca/nest/imager/contributions/scharein/knot-theory/torus.html

[^25]:    1 Mental arithmetic is a common phenomenon in mathematics classrooms but 'mental geometry' is rather neglected.

[^26]:    4 One of their tasks was shown in Chapter 2 Figure 2.2

[^27]:    5 Further details of the methodology will be given in Chapter 8

[^28]:    6 One of Clements and Wattanawaha's (1978) paper folding tasks has already been shown earlier in Figure 2.9.

[^29]:    2 An introduction to the mathematical theory of knots and a description of some of their invariant properties is given in Chapter 5
    3 See Chapter 3 Figure 3.1 for an example

[^30]:    4 The fact that only one 3-crossing knot and only one 4 crossing knot exists whereas 2 knots with crossing number 5 exist was explained in Chapter 5

[^31]:    5 Since the data is skewed, standard deviation is not helpful here. Standard deviations for the natural logarithm of the answer times $(\log t)$ are reported over leaf.

[^32]:    6 In all tables the error rate is given as the percentage of incorrect responses out of the total number of responses (126)

[^33]:    7 These ranged from a minimum of 1 error ( 3 subjects)to a maximum of 31 errors ( 1 subject) out of the total of 126 items with a mean of 9 errors per subject.

[^34]:    ${ }^{1}$ The lack of a significant difference between $90^{\circ}$ and $270^{\circ}$ suggested either that a $270^{\circ}$ rotation was the same in terms of complexity as a $90^{\circ}$ rotation or that different directions were used mentally and that the direction of rotation was immaterial. It is likely that the latter is true.

[^35]:    2 Note: The items on the left hand side of the table did not always appear on the left hand side of the cards. Some randomising was introduced with regard to position of unknots and orientation.

[^36]:    3 For much of the analysis reported here, in order to preserve consistency in the cross tabulations, data from only 46 out of the 48 subjects ( 3312 observations) are used. The reason for this is that a computer system crash occurred during the course of the experiment for two of the subjects and resulted in 12 missing values on restart. However, one of these two subjects belonged to the smaller group of language specialists and in order to carry out the inter-group analysis, 3444 observations out of the total possible 3456 were used Missing values are noted in Minitab with an asterisk and the package is able to function and to carry out the usual statistical techniques whenever values are missing.

[^37]:    5 Note that there were balanced numbers of 'same' and 'different' pairs in the experiment, consequently there were twice as many unknot/knot pairs as knot/knot or unknot/unknot pairs. The data are shown as error rates overall, not as a percentage of the errors.

[^38]:    6 In the same way as for pair types, owing to the different numbers of subjects in each of the groups, errors are shown as error rates overall, not as a percentage of the number of errors made by the group

[^39]:    7 Some subjects may have had some of the more difficult items towards the end of the experiment and so the effects due to rotation, pair type and knot shape have been accounted for by taking the residuals after each analysis of variance and plotting the final set of data.

[^40]:    8 It should be noted that equal numbers of 'same' and 'different' pairs were presented so no 'probability estimates' were responsible here, see Table 7.1.

[^41]:    9 These symmetries are approximate and do not take into account the nature of the 'unders' or 'overs' in the crossings in knot diagram

[^42]:    1 See also the comment made in Chapter 2 regarding pictorial instructions on how to tie a bow tie shown in Figure 2.14

[^43]:    2 A sequence such as under, over, over over, under, suggests that the knot can be simplified, the part of the rope comprising the three 'overs' can be slid away from the 'under' strand and part of the 'knot' simplified

[^44]:    3 Throughout this chapter the abbreviations I and S are used for the Interviewer and the Subject respectively.

[^45]:    4 The strategy was evidently considered by Subject 1 in the initial part of her performance of the experiment whilst she was thinking aloud, but it was not developed. This was referred to in Section 8.4.3 of this chapter.

    5 A torus is a donut shape and a torus knot will wrap neatly around a donut without the rope touching or crossing itself on the torus. See Figure 4.25 in Chapter 4)

[^46]:    6 This effect of gravity has been commented upon in conference presentations of this research by delegates not involved in the current study

[^47]:    1 The child may learn that the 'green' shape goes into the 'square' hole, thus forming a concept of 'squareness' or, alternatively, may acquire the full spatial concept of, for example, a cuboid including the property that the 3-D solid will fit through a 2-D rectangular hole.

[^48]:    2 Vz - the ability to mentally manipulate a pictorially presented stimulus and SR-O - an aptitude to remain unconfused by the changing orientations in which a spatial configuration may be presented

[^49]:    3 See Chapter 2 section 2.1

