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Aspects of multiple categories

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ASPECTS OF MULTIPLE CATEGORIES

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" In The Name Of Allah, Most Beneficent, Most Merciful"

To My Parents and Family

DECLARATION

The work of this thesis has carried out by the candidate and contains the results of his own investigations. The work has not already been accepted in substance for any degree, and is not being concurrently submitted in candidature for any degree. All sources of information have been acknowledged in the text.

DIRECTOR OF STUDIES CANDIDATE

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SUMMARY

The equivalence between the category of double categories with connections and the category of 2-categories was proved by C.P. Spencer and Y.L. Wong.

In this work we try to generalize this result i.e. to prove that there is an equivalence between the category of ω-categories with connections and the category of ∞-categories. This we have not done, though we have quite a lot of information on on the general case. We however managed to get a clear equivalence between triple categories with connection and 3-categories. In particular, we have

Theorem: The functors γ , λ form an adjoint equivalence

$\gamma : 3\text{-}\mathcal{G} \longrightarrow 3\text{-}\mathcal{C} : \lambda$

where $3\text{-}\mathcal{G}$ is the category of triple categories with connections and $3\text{-}\mathcal{C}$ is the category of 3-categories.

In chapter II we explore the equivalence between ω-categories and ∞-categories and get information as much as possible on this equivalence. In fact we define a functor

$\gamma : \omega\text{-}\mathcal{C}at \longrightarrow \infty\text{-}\mathcal{C}at$

where $\omega\text{-}\mathcal{C}at$ denotes the category of ω-categories and $\infty\text{-}\mathcal{C}at$ denotes the category of ∞-categories. Also we define an operation Ψ (we call it folding operation) in an ω-category G and prove that this operation transforms an element $x \in G$ into an element of the associated ∞-category γG.

The key problem which stands as an obstacle from establishing the equivalence in the general case is to find a good formula for the composition $\Psi(x \circ_i y)$ in G for n > 3.

In chapter III we give a full version of the equivalence between triple categories and 3-categories.

CONTENTS

CHAPTER I

INTRODUCTION

1.1 Overall aims and background.

This work develops some of the algebra of multiple categories, by relating notions of "ω-categories" and of "∞-categories".

Here an ω-category is an algebraic structure based on cubical sets with an extra structure introduced in [B-Hi-2], that of "connections". These are like extra degeneracies, in which some adjacent faces are equal, unlike the standard degeneracies of cubes in which some opposite faces are equal. Cubical sets with connections appear in many instances to combine the advantages of cubical and simplicial sets.

An ω-category is a cubical set with connections which in addition has n category structures in dimension n . An analogous concept of ω-groupoid was introduced by Brown-Higgins in [B-Hi-2]. The main result of that paper was an equivalence of categories between ω-groupoids and "crossed complexes" . This result has important implications for homotopy theory, which were exploited in [B-Hi-2]. Additionally, there are a number of other algebraic objects equivalent to ω-categories, for example simplicial T-complexes, cubical T-complexes, and ∞-groupoids (see [As-1], [B-Hi-3], [B-Hi-4]). Our aim is to extend some of these results from the groupoid to the category case.

This seems to be a difficult task. We focus attention on the relations between ω-categories and ∞-categories. There are several reasons for this. One is that it is not hard to give a definition of each of these objects, so that the question of their relationship arises immediately. Another is that the equivalence between ω-groupoids and ∞-groupoids given by Brown-Higgins is round about, going via crossed complexes. So is of interest to give, if possible, a direct proof.

A third reason is the importance of ∞-categories. They arise naturally in terms of homotopies and higher homotopies. They have been studied considerably by the Australian School (for example see [Jo-1], [K-st-1], [St-1]), for various reasons, including their occurrence in Computer Science.

However, manipulation with the elements of an ∞-category presents difficulties, because the compositions in different directions seem to have a different geometry. This leads to a number of "pasting problems" [K-St-1], which seem to be have been solved in principal in dimension 2.

By contrast, Spencer [S-1] has shown an equivalence between 2-categories and "double categories with connections", and he and Spencer-Wong [S-Wo-1] have shown the utility of this equivalence for homotopy theory. The basic idea is that complicated pasting in a 2-categories are replaced by a simpler manipulation with "thin elements" in a double category with connection.

The overall aim of this work is to provide a similar situation in all dimensions, that is to establish an equivalence between ω-categories with connection and ∞-categories. This we have not done, though we have quite a lot of information on the general case. We are however able to establish a result of this form in dimension 3. This give some evidence for the general case, and some idea of the kind of problems that have to be overcome in this approach to a verification of the general case.

The complications of this case are such that we have not been able to venture into potential applications. We hope this thesis will give some idea of the interest in the blend of algebra and geometry in this kind of "higher dimensional algebra", to use a phrase coined by R. Brown.

The notion of double category was first introduced by Ehresmann [Eh-1] and has occurred often in the literature (see for example [Gr-1], [Ma-1], [K-S-1], [[B-S-1], [S-W-1]).

Cubes in a double categories with connection were used by Spencer-Wong [S-W-1] to develop the abstract theory of homotopy pullbacks and pushouts introduced by Spencer in [S-1]. They have shown that there exists an equivalence between the category of 2-categories and the category of double categories with connections. Brown and Spencer in [B-S-1] have proved the equivalence between double groupoids and crossed modules, which was generalized by Brown and P. J. Higgins in [B-Hi-2] where they obtain an equivalence between the category of ω-groupoids and the category of crossed complexes (over groupoids). In [MO-1] G. Mosa has introduced the notion of ω-algebroids and develop a parallel theory in a more algebraic context. He proved an equivalence between the category of crossed modules (over algebroids) and the category of special double algebroids with connections. He also proved a similar result for the 3-dimensional case but with much less details. In [B-Hi-3], Brown and Higgins have proved a powerful result when they prove the equivalence between ω-groupoids and cubical T-complexes.

1.2 Structure and main results.

In chapter II we introduce the notion of ω-categories with connections via the cubical complexes. We established the relation of ∞-categories to ω-categories following a similar argument given by Brown-Higgins [B-Hi-3] in the relation of ∞-categories to ω-groupoids. This relation yields of the functor

$$\gamma : \omega\text{-}\mathcal{C}at \longrightarrow \infty\text{-}\mathcal{C}at ,$$

by the rule:

$$C_n = \{x \in G_n \mid \partial_j^\alpha x \in \varepsilon_1^{j-1} G_{n-j} \text{ for } 1 \le j \le n , \ \alpha = 0,1\} ,$$

where C is an ∞-category and G is an ω-category.

In section 5 we introduce an operation

$$\Psi : G_n \longrightarrow G_n$$

in an ω-category G_n. This operation is based on an operation ψ_j defined by G. Mosa [Mo-1], we define

$$\Psi_r = \psi_{r-1}\psi_{r-2}\cdots\psi_1 \text{ and } \Psi = \Psi_2\cdots\Psi_{n-1}\Psi_n ,$$

and in § 6 we prove that this operation transforms an element x in an ω-category to an element in the associated ∞-category.

In 2.6.5 we give an explanation why we could not give a formula for $\Psi(x \circ_i y)$ in the general case.

In § 7 we construct the coskeleton in terms of "shells" for an n-tuple category and we define ∂_i^α , ε_i , Γ_i , Γ'_i and the operations on $\Box\, G_n$ to prove the following ;

2.7.1 Proposition.

If $G = (G_n, \ldots, G_0)$ is an n-tuple category, then $G' = (\square\, G_n, G_{n-1}, \ldots, G_0)$ is an (n+1)-tuple category.

In the proof of this proposition we follow a similar argument to the corresponding case of algebroid given by Mosa [Mo-1].

The key point for defining the coskeleton of ω-category is shown in the following key proposition for the case where n = 3 ;

2.7.3 Proposition.

Let G be a triple category, and let $C = \gamma G$ be its associated 3-category. Let $\underline{x} \in \square\, G_2$ and $\xi \in C_3$. Then there exist $x \in G_3$ such that $\underline{\partial} x = \underline{x}$ and $\Psi x = \xi$ *if* and only if $d_1^{\alpha}\xi = \partial_1^{\alpha}\Psi\underline{x}$.

We also give the definition of the n-skeleton of an ω-category for $n \leq 3$ and the definition of a commuting shell:

2.7.5 Definition.

A shell $\underline{x} \in \square\, G_n$ is called a *commuting shell* if

$$\partial_1^0\Psi\underline{x} = \partial_1^1\Psi\underline{x} .$$

Chapter III is devoted to prove the equivalence

$$\gamma : 3\text{-}\mathcal{G} \rightleftarrows 3\text{-}\mathcal{C} : \lambda$$

between triple categories and 3-categories.

In § 1 and § 2 we explain the difficulty of finding a formula for the composition $\Psi(x \circ_i y)$, for x , $y \in G_n$, in the general case, but we were able to find a formula for that composition in dimension 3. First we give in (3.2.4) explicit formulae for the faces $\partial_1^0\Psi x$ and $\partial_1^1\Psi x$. These formulae play a key part in evaluating $\Psi(x \circ_i y)$ and in the proof of associative and interchange laws in $G = \lambda C$. Proposition 3.2.5 gives the evaluation of the composition $\Psi(x \circ_i y)$ for n = 3 .

In section 3 we define the functor $\lambda : 3\text{-}\mathcal{C} \longrightarrow 3\text{-}\mathcal{G}$ as follows:

given a triple category G with associated 3-category C = γG, and given $\underline{x} \in \Box G_2$, $\xi \in C_2$ with $d_1^\alpha \xi = d_1^\alpha \Psi \underline{x}$, we write $<\underline{x}, \xi>$ for the unique element $x \in G_3$ such that $\partial x = \underline{x}$ and $\Psi x = \xi$. Proposition 3.3.1 defines the compositions $x \circ_i y$ and shows that these compositions in G are also determined by γG . We define

$G_3 = \{<\underline{x} , \xi> : \underline{x} \in \Box G_2 , \xi \in C_3$ such that $\sigma_2 \underline{\partial} \Psi \underline{x} = \underline{\partial} \xi\}$,

where $\sigma_2 : \gamma G_2 \longleftrightarrow C_2$, and we define operations $\circ_i$ in G_3 .

In section 4 and 5 we prove the associative and interchange laws in G_3. The proof of these laws shows a great deal of complexity of algebra. By this we have a triple category $(G_3, \ldots, G_0)$ and isomorphism σ_3: $C_3 \longrightarrow \gamma G_3$ of 3-categories.

In the final section of this chapter we prove the main result in this work:

3.6.1 Theorem.

There is a functor λ from the category 3-$\mathfrak{G}$ of triple categories to the category 3-$\mathcal{C}$ of 3-categories such that λ : 3-$\mathfrak{G} \longrightarrow$ 3-$\mathcal{C}$ are inverse equivalencies.

CHAPTER II

ω-CATEGORIES, ∞-CATEGORIES AND FOLDING OPERATION

§ 2.0 Introduction

We begin this chapter by defining ω-categories and ∞-categories and establish the relations between them following an analogous to that between ω-categories and ∞-groupoids given by R.Brown and P.Higgins in [B-Hi-3]. By this we define a functor $\gamma : \omega\text{-}\mathcal{C}at \longrightarrow \infty\text{-}\mathcal{C}at$. For $x \in G$ (G is an ω-category) we define an operation $\Psi : G_n \longrightarrow G_n$ and prove that $\Psi x \in \gamma G$.

§ 2.1 Cubical complexes with connections.

2.1.1 Definition. [B-Hi-1]

A *cubical complex* K is a graded set $(K_n)_{n\geq 0}$ with face maps

$$\partial_i^\alpha : K_n \longrightarrow K_{n-1} \quad (i = 1,2,\dots,n \ ; \ \alpha = 0,1) ,$$

and degeneracy maps

$$\varepsilon_i : K_{n-1} \longrightarrow K_n \quad (i = 1,2,\dots,n) ,$$

satisfying the usual cubical relations namely

(i) $\partial_i^\alpha \partial_j^\beta = \partial_{j-1}^\beta \partial_i^\alpha \quad (i < j)$,

(ii) $\varepsilon_i \varepsilon_j = \varepsilon_{j+1} \varepsilon_i \quad (i \leq j)$,

(iii) $\partial_i^\alpha \varepsilon_j = \begin{cases} \varepsilon_{j-1} \partial_i^\alpha & (i \leq j) \\ \varepsilon_j \partial_{i-1}^\alpha & (i > j) \\ \mathrm{id} & (i = j) . \end{cases}$

2.1.2 Definition. [B-Hi-2]

Let K be a cubical complex. We say that K is a *cubical* *complex with connections,* if it has for $n \geq 2$ additional structures maps

$$\Gamma_i \, , \, \Gamma'_i \, : \, K_{n-1} \longrightarrow K_n \quad (i = 1,2,..,n-1) \, ,$$

satisfying the following relations:

(i) $$\Gamma_i\Gamma_j = \begin{cases} \Gamma_{j+1}\Gamma_i & (i \leq j) \, , \\ \Gamma_j\Gamma_{i-1} & (i > j) \, , \end{cases}$$

(ii) $$\Gamma'_i\Gamma'_j = \begin{cases} \Gamma'_{j+1}\Gamma'_i & (i \leq j) \, , \\ \Gamma'_j\Gamma'_{i-1} & (i > j) \, . \end{cases}$$

(iii) $$\Gamma_i\varepsilon_j = \begin{cases} \varepsilon_{j+1}\Gamma_i & (i < j) \, , \\ \varepsilon_j\Gamma_{i-1} & (i > j) \, , \\ \varepsilon_j^2 & (i = j) \, . \end{cases}$$

(iv) $$\Gamma'_i\varepsilon_j = \begin{cases} \varepsilon_{j+1}\Gamma'_i & (i < j) \, , \\ \varepsilon_j\Gamma'_{i-1} & (i > j) \, , \\ \varepsilon_j^2 & (i = j) \, . \end{cases}$$

(v) $$\partial_j^0\Gamma_j = \partial_{j+1}^0\Gamma_j = \mathrm{id} \, ,$$
$$\partial_j^1\Gamma_j = \partial_{j+1}^1\Gamma_j = \varepsilon_j\partial_j^1 \, .$$

(vi) $$\partial_j^1\Gamma'_j = \partial_{j+1}^1\Gamma'_j = \mathrm{id} \, ,$$
$$\partial_j^0\Gamma'_j = \partial_{j+1}^0\Gamma'_j = \varepsilon_j\partial_j^0 \, .$$

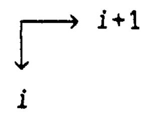
(vii) $$\partial_i^\alpha\Gamma_j = \begin{cases} \Gamma_{j-1}\partial_i^\alpha & (i < j) \, , \\ \Gamma_j \, \partial_{i-1}^\alpha & (i > j+1) \, . \end{cases}$$

(viii) $$\partial_i^\alpha\Gamma'_j = \begin{cases} \Gamma'_{j-1}\partial_i^\alpha & (i < j) \, , \\ \Gamma'_j\partial_{i-1}^\alpha & (i > j+1) \, , \end{cases}$$

(ix) $\Gamma_i \Gamma'_j = \begin{cases} \Gamma'_{j+1}\Gamma_i & (i < j)\ , \\ \Gamma'_j \Gamma_{i-1} & (i > j+1)\ . \end{cases}$

The functions Γ and Γ′ are first introduced by R. Brown [B-Hi-1] to deal with double groupoids. They are to be thought of as extra "degeneracies". A *degenerate cube* of type $\varepsilon_i x$ has a pair of opposite faces equal and all other faces degenerate. A cube of type $\Gamma_i x$ has a pair of *adjacent* faces equal and all other faces of type $\Gamma_j y$ or $\varepsilon_j y$. Those cubes can be represented by the following symbols which will be used frequently through this thesis

┘	┌	||	=	□
Γx	$\Gamma' x$	$\varepsilon_i x$	$\varepsilon_j x$	$\varepsilon_i \varepsilon_{i+1} x$



These elements are called *thin elements* and were initially introduced by R. Brown and P. J. Higgins in their discussion of double groupoids and other higher dimensional objects ([Br-1], [B-Hi-1], [B-Hi-2], [B-Hi-5], [B-S-1]).

2.1.3 Example.

Let X be a space. Then the *singular cubical complex* KX is a cubical complex where K_n is the set of continuous maps (singular n-cubes)

$$I^n \longrightarrow X$$

The connection $\Gamma_i : K_{n-1} \longrightarrow K_n$ is induced by the map

$$\gamma_i : I^n \longrightarrow I^{n-1}$$

defined by

$$\gamma_i(t_1, t_2, \ldots, t_n) = (t_1, t_2, \ldots, t_{i-1}, \max(t_i, t_{i+1}), t_{i+2}, \ldots, t_n)$$

§ 2.2 ω-Categories.

2.2.1 Definition.

An *ω-category* $G = (G_n ; \partial_i^\alpha , \varepsilon_i)$ is a cubical complex and for $n \geq 1$, G_n has n category structures $(G_n, \circ_i, \partial_j^0, \partial_j^1, \varepsilon_i)$ related appropriately to each other and to ∂_i^α , ε_i , with the following axioms:

(i) If x , $y \in G_n$, and $x \circ_j y$ is defined then for $\alpha = 0,1$

$$\partial_i^\alpha(x \circ_j y) = \begin{cases} \partial_i^\alpha x \circ_{j-1} \partial_i^\alpha y & (i < j) \\ \partial_i^\alpha x \circ_j \partial_i^\alpha y & i > j) \end{cases},$$

$$\partial_j^0(x \circ_j y) = \partial_j^0 x \quad , \quad \partial_j^1(x \circ_j y) = \partial_j^1 y ,$$

$$\varepsilon_i(x \circ_j y) = \begin{cases} \varepsilon_i x \circ_{j+1} \varepsilon_i y & (i \leq j) \\ \varepsilon_i x \circ_j \varepsilon_i y & (i > j) . \end{cases}$$

(ii) $\varepsilon_i \partial_j^0 \circ_j x = x = x \circ_j \varepsilon_i \partial_j^1 x$.

(iii) *(The interchange law).* If $i \neq j$, then

$$(x \circ_i y) \circ_j (z \circ_i w) = (x \circ_j z) \circ_i (y \circ_j w)$$

whenever both sides are defined.

(iv) If x , $y \in G_n$ and $x \circ_j y$ is defined, then

$$\Gamma_i(x \circ_j y) = \begin{cases} \Gamma_i x \circ_{j+1} \Gamma_i y (i < j), \\ \Gamma_i x \circ_j \Gamma_i y \quad (i > j), \end{cases}$$

$$\Gamma'_i(x \circ_j y) = \begin{cases} \Gamma'_i x \circ_{j+1} \Gamma'_i y & (i < j) , \\ \Gamma'_i x \circ_j \Gamma'_i y & (i > j) . \end{cases}$$

(v) $\Gamma'_j x \circ_{j+1} \Gamma_j x = \varepsilon_j x \quad , \quad \Gamma'_j x \circ_j \Gamma'_j x = \varepsilon_{j+1} x$,

(vi) *The transport laws.*

If x , $y \in G_n$ with $\partial_j^1 x = \partial_j^0 y$. Then

$$\Gamma_j(x \circ_j y) = (\Gamma_j x \circ_{j+1} \varepsilon_j y) \circ_j (\varepsilon_{j+1} y \circ_{j+1} \Gamma_j y)$$

$$\Gamma'_j(x \circ_j y) = (\Gamma'_j x \circ_{j+1} \varepsilon_{j+1} x) \circ_j (\varepsilon_j x \circ_{j+1} \Gamma'_j y) .$$

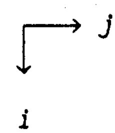
It is convenient to use a matrix notation for compositions of cubes. Thus, if $x_{sr} \in G_n$, $(1 \le s \le h\ ,\ 1 \le r \le k)$ are cubes in G_n satisfying

$$\partial_i^1 x_{s(r-1)} = \partial_i^0 x_{sr} \quad (1 \le s \le h\ ,\ 2 \le r \le k)\ ,$$

$$\partial_j^1 x_{(s-1)r} = \partial_j^0 x_{sr} \quad (2 \le s \le h\ ,\ 1 \le r \le k)\ ,$$

we write

$$\begin{bmatrix} x_{11} & x_{12} \cdots\cdots x_{1h} \\ x_{21} & x_{22} \cdots\cdots x_{2h} \\ \vdots & \vdots \qquad \vdots \\ x_{k1} & x_{k2} \cdots\cdots x_{kh} \end{bmatrix}$$



for

$$(x_{11} \circ_i \cdots\cdots \circ_i x_{k1}) \circ_j \cdots \circ_j (x_{1h} \circ_i \cdots \circ_i x_{kh})\ .$$

An *ω-subcategory* of G is a cubical subcomplex closed under all the connections and compositions $\circ_j$.

2.2.2 Definition.

A *morphism* between two ω-categories, $f : G \longrightarrow H$, is a family of category morphisms, $f_n : G_n \longrightarrow H_n$, such that $f_n : G_n \longrightarrow H_n$ commutes with all the structures. We denote the resulting category of ω-categories by *ω-Cat* .

2.2.3 Definition.

An ω-category G is called an *ω-category with connections* if the cubical complex G_n has connections.

2.2.4 Definition.

A *morphism* between ω-categories with connections, $f : G \longrightarrow D$ is a morphism of categories preserving the connections. The resulting category also will be denoted by *ω-Cat* .

For the rest of our thesis we will consider only ω-categories with connections. For shorthand we will call them just ω-categories unless stated otherwise.

It is clear that we can define finite dimensional versions of the above definitions.

2.2.5 Definition.

An *m-tuple category* is an m-truncated cubical complex $G = (G_m, G_{m-1}, \ldots, G_0)$ with connections, having n category structures in dimension n ($n \le m$), and satisfying all the laws for an ω-category in so far as they make sense. We denote by $\omega\text{-}\mathcal{C}at_m$ the category of m-tuple categories.

Note that for all $n \ge 2$ and $1 \le i \le n-1$, the pair (G_n, G_{n-1}) with the category structures in directions i and $i+1$ forms a double category.

§ 2.3 ∞-Categories.

2.3.1 Definitions.

An *n-fold category* is a class G together with n mutually compatible category structures $G^i = (G^i, \partial_i^0, \partial_i^1, \circ_i)$ where $0 \le i \le n-1$, each with G as its class of morphisms (and with ∂_i^0, ∂_i^1 giving the initial and final identities for $\circ_i$). The objects of the category structure G^i are here regarded as members of G, coinciding with the identity morphisms of G^i. The compatibility conditions are:

(i) $\partial_i^\alpha \partial_j^\beta = \partial_j^\beta \partial_i^\alpha$ for $i \ne j$ and $\alpha, \beta \in \{0,1\}$,

(ii) $\partial_i^\alpha (x \circ_j y) = \partial_i^\alpha x \circ_j \partial_i^\alpha y$ for $i \ne j$ and $\alpha = 0, 1$,

for all $x, y \in G$ and where $x \circ_j y$ is defined.

(iii) *(The interchange law)* If $i \ne j$, then

$$(x \circ_i y) \circ_j (z \circ_i w) = (x \circ_j z) \circ_i (y \circ_j w)$$

for all x , y , z , $w \in G$ such that both sides are defined.

We denote the two sides of (iii) by

$$\begin{bmatrix} x & y \\ z & w \end{bmatrix} \quad \begin{matrix} \longrightarrow i \\ \downarrow \\ j \end{matrix} \quad .$$

The category structure G^i on G is said to be *stronger* than the structure G^j if every object (identity morphism) of G^i is also an object of G^j . An *n*-fold category G is then called an *n-category* if the category structures G^0 , G^1 ,...., G^{n-1} can be arranged in a sequence of increasing (or decreasing) strength.

1.3.2 Definition

An ∞-*category* is a class G with mutually compatible category structures G^i for all integers $i \geq 0$ satisfying

$$\text{Ob } G^i \subset \text{Ob } G^{i+1} \quad \text{for all } i \geq 0 .$$

§ 2.4 The relation of ∞-categories to ω-categories.

In [B-Hi-4], R. Brown and P.Higgins have found a direct route from ω-groupoids to ∞-groupoids and used it to reformulate the definitions of ∞-groupoids and ∞-categories. They used this account to show how ∞-groupoids fit into the pattern of equivalencies established in [B-Hi-2] and [B-Hi-3]. They followed an elegant procedure for passing from an n-fold category G to an n-category induced on a certain subset C of G . This account and procedure are useful for our aim of establishing the equivalence between ω-categories with connections and ∞-categories. Below we have followed the same procedure to find the relationships between ω-categories and ∞-categories.

First, let G be an ω-category, we write

$$\eta_i^\alpha = \varepsilon_i \partial_i^\alpha : G_n \longrightarrow G_n$$

and

$$\mathrm{Ob}^i(G_n) = \varepsilon_i G_{n-1} = \{x \in G_n \mid \eta_i^\alpha x = x \text{ for } \alpha = 0,1\}$$

The axioms for ω-categories now ensure the category structures

$$(G_n,\ \eta_i^0,\ \eta_i^1,\ \circ_i)\ ,\ i = 1,2,\ldots,n$$

are mutually compatible. Thus for $n \geq 0$, G_n carries the structure of n-fold category and $\varepsilon_i : G_{n-1} \longrightarrow G_n$ embeds G_{n-1} as (n-1)-fold subcategory of the (n-1)-fold category obtained from G_n by omitting the j-th category structure.

Next we show how to pass from an n-fold category H to an n-category structures on a certain subset C of H. So let $H^i = (H,\ \partial_i^0,\ \partial_i^1,\ \circ_i)$, $i = 0,1,\ldots,n-1$, be the n-category structures on H . Write

$$B^i = \mathrm{Ob}(H^i) \cap \mathrm{Ob}(H^{i+1}) \cap \ldots \cap \mathrm{Ob}(H^{n-1})\ ,\ 0 \leq i \leq n-1\ ,$$

and define

$$C = \{\ x \in H \mid \partial_i^\alpha x \in B^i \text{ for } 0 \leq i \leq n-1\ ,\ \alpha = 0,1\ \}\ .$$

By the compatibility conditions, each B^i is an n-fold subcategory of H and hence C is also an n-fold subcategory of H , with category structures $C^i = (C,\ \partial_i^0,\ \partial_i^1,\ \circ_i)$. But, for $x \in C$, $\partial_i^\alpha x \in B^i \cap C$ so $\mathrm{Ob}(C^i) \subset B^i \cap C$; conversely, if $y \in B^i \cap C$ then $y \in B^i \subset \mathrm{Ob}(H^i)$, so $\partial_i^\alpha y = y$. Thus $\mathrm{Ob}(C^i) = B^i \cap C$. Since $B^0 \subset B^1 \subset \ldots \subset B^{n-1}$; it follows that C is an n-category.

Applying this procedure to the n-fold category G_n , we find that G_n is an n-fold category with respect to the structures H^i . Also

$$B^i = \mathrm{Ob}(G^i) \cap \mathrm{Ob}(G^{i+1}) \cap \ldots \cap \mathrm{Ob}(G^{n-1})$$

$$= \varepsilon_{n-i}G_{n-1} \cap \varepsilon_{n-i-1}G_{n-1} \cap \ldots \cap \varepsilon_1 G_{n-1} = \varepsilon_1^{n-i} G_i \ .$$

We therefore define

$$C_n = \{x \in G_n \mid \partial_j^\alpha x \in \varepsilon_1^{j-1} G_{n-j} \text{ for } 1 \le j \le n \ , \ \alpha = 0,1\} \ ,$$

and deduce that, for each $n \ge 0$, C_n is an n-fold category with respect to the structures $(C_n, \partial_i^0, \partial_i^1, \circ_i)$, $0 \le i \le n-1$. These structures are all categories. The family $(C_n)_{n\ge0}$ admits all the face operators ∂_i^β of G and also the first degeneracy operator ε_1 in each dimension. Since ε_1 embeds G_{n-1} in G_n as (n-1)-fold subcategory omitting $\circ_1$, it embeds C_{n-1} in C_n as (n-1)-subcategory omitting $\circ_{n-1}$. In other words, it preserves the operations $\circ_i$, $0 \le i \le n-2$ and its image is the set of identities of $\circ_{n-1}$. It follows that if we define

$$D = \varinjlim \left(C_0 \underset{\varepsilon_1}{\rightarrowtail} C_1 \underset{\varepsilon_1}{\rightarrowtail} C_2 \underset{\varepsilon_1}{\rightarrowtail} \ldots \right) ,$$

then the operations $\circ_i$ (for fixed i) in each dimension combine to give a category structure $D^i = (D, \partial_i^0, \partial_i^1, \circ_i)$ on D . Also $Ob(D^i)$ is D_i , the image of C_i in D . Thus if G is an ω-category, then G induces on D the structure of ∞-category.

Clearly, the structure on D can also be described in terms of the family $C = (C_n)_{n\ge0}$. The neatest way to do this is to use the operators

$$d_i^\alpha = (\partial_1^\alpha)^{n-i} = \partial_1^\alpha \partial_2^\alpha \ldots \partial_{n-i}^\alpha : G_n \longrightarrow G_i \ , \ 0 \le i \le n-1, \ \alpha = 0,1$$

$$\varepsilon_i = \varepsilon_1^{n-i} : G_i \longrightarrow G_n \ , \ 0 \le i \le n-1 \ .$$

Since C admits ε_1 and all ∂_i^α , there are induced operators

$$d_i^\alpha : C_n \longrightarrow C_i \ , \ s_i : C_i \longrightarrow C_n \ , \ 0 \le i \le n-1 \ .$$

If $x \in C_n$, we have $\partial_{n-i}^\alpha x = \varepsilon_1^{n-i-1} y$ for some $y \in G_i$ and this is unique, since ε_1 is an injection. The effect of d_i^α is to pick out this i-dimensional "essential face" y of x , because

$$d_i^\alpha x = \partial_1^\alpha \partial_2^\alpha \ldots \partial_{n-i-1}^\alpha (\partial_{n-i}^\alpha x) = (\partial_1^\alpha)^{n-i-1}(\varepsilon_1^{n-i-1} y) = y .$$

If we pass to $D = \varinjlim C_n$, the operators ε_i induce the inclusions $D_i \rightarrowtail D$ and the operators d_i^α induce the $d_i^\alpha : D \longrightarrow D$ since, for $x \in C_n$, we have $d_i^\alpha x = \varepsilon_1^{n-i} y$, where $y = d_i^\alpha x$.

Now we can give an equivalent definition of ∞-category to the definition (2.3) given previously.

2.4.1 Definition. [B-Hi-4]

An *∞-category* consists of

(i) A sequence $C = (C_n)_{n\geq 0}$ of sets.

(ii) Two families of functions

$$d_i^\alpha : C_n \longrightarrow C_i , \quad i = 0,1,2,\ldots.n-1 , \quad \alpha = 0,1 ,$$
$$s_i : C_i \longrightarrow C_n , \quad i = 0,1,2,\ldots.n-1 ,$$

satisfying the laws:

(ii)(a) $d_i^\alpha d_i^\beta = d_i^\alpha$ for $i < j$, $\alpha, \beta = 0,1,$

(ii)(b) $s_j s_i = s_i$ for $i < j$,

(ii)(c) $d_j^\beta s_i = \begin{cases} d_j^\beta & \text{for } j < i \\ 1 & \text{for } j = i \\ s_i & \text{for } j > i . \end{cases}$

(iii) Category structures $\circ_i$ on C_n $(0 \leq i \leq n-1)$ for each $n \geq 0$ such that $\circ_i$ has C_i as set of objects and d_i^0, d_i^1, s_i as its initial, final and identity maps. These category structures must satisfies the compatibility conditions:

(iii)(a) If $i > j$, $\alpha = 0,1$ and $x \circ_j y$ is defined, then

$$d_i^0(x \circ_j y) = d_i^0 x \circ_j d_i^0 y$$

(iii)(b) if $x \circ_j y$ is defined then

$$s_i(x \circ_j y) = s_i x \circ_j s_i y$$

(iv) *(The interchange law)* if $i \neq j$ then

$$(x \circ_i y) \circ_j (z \circ_i w) = (x \circ_j z) \circ_i (y \circ_j w)$$

The transition from an ∞-category C as defined in Section 2 to one of the above type is made by putting $C_n = Ob(A^n)$ and defining $s_i : C_i \longrightarrow C_n$ $(i < n)$ to be the inclusion map and $d_i^\alpha : C_n \longrightarrow C_i$ to be the restriction of $\partial_i^\alpha : A \longrightarrow A$.

In [S-1] it was shown that the category $\mathcal{G}_2$ of double categories with connections is equivalent to the category $\mathcal{C}_2$ of 2-categories. We prove in the next chapter that there is an equivalence between triple categories (3-$\mathcal{G}$) and 3-categories (3-$\mathcal{C}$) .

§ 2.5 **Folding operation.**

In this section we introduce an operation Ψ on cubes in an ω-category G (or in an m-tuple category). This operation has the effect of folding the *odd* faces $\partial_i^\alpha x$, where $i + \alpha$ is odd ,onto the face $\partial_1^0 \Psi x$and the *even* faces $\partial_i^\alpha x$, where $i + \alpha$ is even ,onto the face $\partial_1^1 \Psi x$ for $x \in G_n$ .This operation Ψ transforms x into an element of the associated ∞-category γG . It is important that Ψx is constructed from x and the "shell" of x consisting of all faces $\partial_j^\alpha x$ of x . This will imply that x itself can be reconstructed from Ψx and the shell of x .

In [B-Hi-2] R. Brown and P.J. Higgins have defined a similar folding operation Φ in an ω-groupoid which has the effect of folding all faces of $x \in G_n$ into the face $\partial_1^0 \Phi x$. This operation transforms an element x in an ω-groupoid to an element in the associated crossed complex.

In [Mo-1] G.H. Mosa also defined a folding operation in an ω-algebroid and proved that it transforms an element of ω-algebroid to an element of the associated crossed complex. We utilize some techniques in [Mo-1]

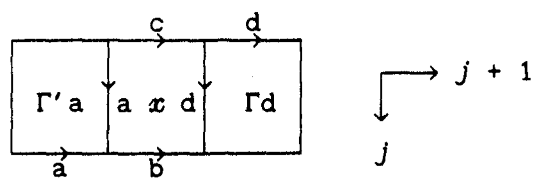
First, we define an operation

$$\psi_j : G_n \longrightarrow G_n \quad (1 \le j < n \le m) ,$$

by the formula

$$\psi_j x = \Gamma'_j \partial^0_{j+1} x \circ_{j+1} x \circ_{j+1} \Gamma_j \partial^1_{j+1} x , \qquad (2.5.1)$$

for $x \in G_n$ and $1 \le j \le n-1$, the effect of this operation can be seen from the diagram

$\psi_j x =$ 

in which unlabeled faces are appropriate degenerate cubes.

This operation was first introduced by G. Mosa in [Mo-s] and it is a generalization of the foldings in the case of dimension 2.

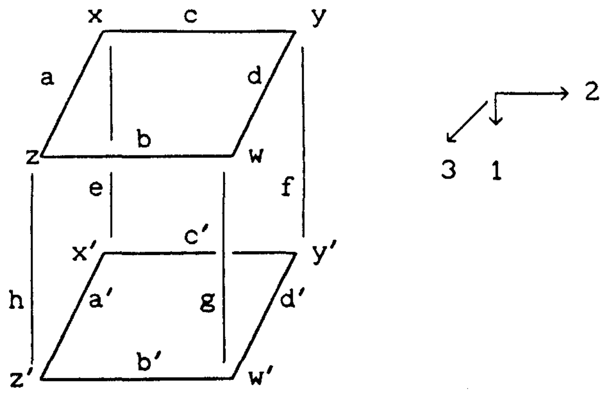
Second we define

$$\Psi_r = \psi_{r-1}\psi_{r-2} \cdot \cdots \psi_1 .$$

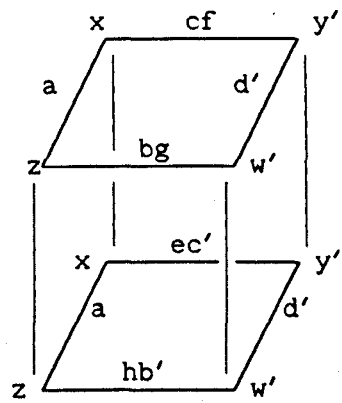
Finally, we define

$$\Psi = \Psi_2 \cdot \cdots \Psi_{n-1}\Psi_n .$$

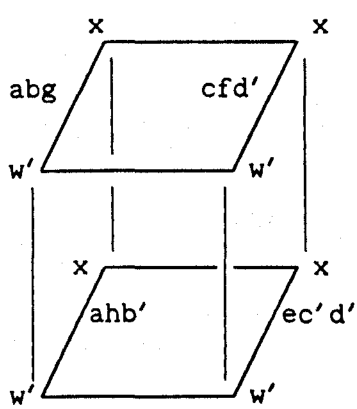
To give a clear picture for the above definitions, we shall use the cube in dimension 3 . So let $x \in G_3$ have edges and vertices given by:



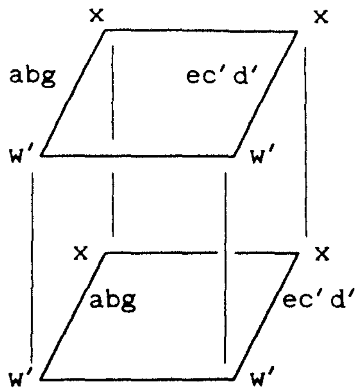
Then ψ_1 has edges and vertices in the form



and $\psi_2\psi_1$ has edges and vertices in the form



Thus $\Psi a = \Psi_2\Psi_3 = \psi_1\psi_2\psi_1$ has edges and vertices in the form



In chapter III we will see that the face $\partial_1^0 \Psi x$ is the "sum" of the faces $\partial_1^0 x$, $\partial_1^1 x$, $\partial_1^0 x$ while the face $\partial_1^1 x$ is the "sum" of the faces $\partial_1^1 x$, $\partial_2^0 x$, $\partial_3^1 x$.

This shows that the vertices and edges of Ψx are appropriate to an element of γC where C is a 3-category.

The operation ψ_j defined above satisfies several laws which will be stated and proved next. Those laws from 2.5.1 to 2.5.5 are taken entirely from [Mo-1].

2.5.2 Lemma.

(i) $$\partial_i^\alpha \psi_j = \begin{cases} \psi_{j-1}\partial_i^\alpha & (i < j) \\ \psi_j \partial_i^\alpha & (i > j+1) , \end{cases}$$

(ii) $$\partial_j^0 \psi_j x = \partial_j^0 x \circ_j \partial_{j+1}^1 x ,$$

(iii) $$\partial_j^1 \psi_j x = \partial_{j+1}^0 x \circ_j \partial_j^1 x ,$$

(iv) $$\partial_{j+1}^\alpha \psi_j = \varepsilon_j \partial_j^\alpha \partial_{j+1}^\alpha \ (= \varepsilon_j \partial_j^\alpha \partial_j^\alpha) ,$$

(v) $$\partial_j^1 \psi_j \psi_{j+1} \cdot\cdot \psi_{n-1} = \varepsilon_j \partial_j^1 \varepsilon_{j+1} \partial_{j+1}^1 \cdots \varepsilon_{n-1}^1 \partial_{n-1}^1 \partial_n^1 .$$

Proof.

(i) For $i < j$, let $x \in G_n$. Then

$$\partial_i^\alpha \psi_j x = \partial_i^\alpha(\Gamma'_j \partial_{j+1}^0 x \circ_{j+1} x \circ_{j+1} \Gamma_j \partial_{j+1}^1 x)$$

$$= \partial_i^\alpha \Gamma'_j \partial_{j+1}^0 x \circ_j \partial_i^\alpha x \circ_j \partial_i^\alpha \Gamma_j \partial_{j+1}^1 x \qquad \text{by (2.2.1)(i)}$$

$$= \Gamma'_{j-1} \partial_j^0 \partial_i^\alpha x \circ_j \partial_i^\alpha x \circ_j \Gamma_{j-1} \partial_j^1 \partial_i^\alpha x = \psi_{j-1} \partial_i^\alpha x ,$$

by (2.1.2)(vii, viii) and (2.2.1)(i).

For $i > j+1$, let $x \in G_n$. Then

$$\partial_i^\alpha \psi_j x = \partial_i^\alpha(\Gamma'_j \partial_{j+1}^0 x \circ_{j+1} x \circ_{j+1} \Gamma_j \partial_{j+1}^1 x)$$

$$= \partial_i^\alpha \Gamma'_j \partial_{j+1}^0 x \circ_{j+1} x \circ_{j+1} \Gamma_j \partial_{j+1}^1 x \qquad \text{by (2.2.1)(i) .}$$

$$= \Gamma'_j \partial_{j+1}^0 \partial_i^\alpha x \circ_{j+1} \partial_i^\alpha x \circ_{j+1} \Gamma_j \partial_{j+1}^1 \partial_i^\alpha x = \psi_j \partial_i^\alpha x$$

by (2.1.2)(vii, viii) and (2.2.1)(i).

(ii) Let $x \in G_n$. Then

$$\partial_{j+1}^\alpha \psi_j x = \partial_{j+1}^\alpha (\Gamma'_j \partial_{j+1}^0 x \circ_{j+1} x \circ_{j+1} \Gamma_j \partial_{j+1}^1 x) .$$

If $\alpha = 0$, then

$$\partial_{j+1}^0 \psi_j x = \partial_{j+1}^0 \Gamma'_j \partial_{j+1}^0 x$$

$$= \varepsilon_j \partial_j^0 \partial_{j+1}^0 x \qquad \text{by (2.1.2)(vi).}$$

If $\alpha = 1$, then

$$\partial_{j+1}^1 \psi_j x = \partial_{j+1}^1 \Gamma'_j \partial_{j+1}^1 x$$

$$= \varepsilon_j \partial_j^1 \partial_{j+1}^1 x \qquad \text{by (2.1.2)(vi).}$$

Thus $\partial_{j+1}^\alpha \ \psi_j = \varepsilon_j \partial_j^\alpha \partial_{j+1}^\alpha$.

(iv) $\partial_j^1 \psi_j x = \partial_j^1(\Gamma'_j \partial_{j+1}^0 x \circ_{j+1} x \circ_{j+1} \Gamma_j \partial_{j+1}^1 x)$,

$$= \partial_j^1 \Gamma'_j \partial_{j+1}^0 x \circ_j \partial_j^1 x \circ_j \partial_j^1 \Gamma_j \partial_{j+1}^1 x \qquad \text{by (2.2.1)(i) .}$$

$$= \partial_{j+1}^0 x \circ_j \partial_j^1 x \circ_j \varepsilon_j \partial_j^1 \partial_{j+1}^1 x \qquad \text{by (2.1.2)(v, vi).}$$

$= \partial^0_{j+1}x \circ_j \partial^1_j x$ (since $\varepsilon_j \partial^1_j \partial^1_{j+1} x$ is an identity for $\circ_j$).

(iii) Let $x \in G_n$. Then

$$\partial^0_j \psi_j x = \partial^0_j (\Gamma'_j \partial^0_{j+1} x \circ_{j+1} x \circ_{j+1} \Gamma_j \partial^1_{j+1} x)$$

$$= \partial^0_j \Gamma'_j \partial^0_{j+1} x \circ_j \partial^0_j x \circ_j \partial^0_j \Gamma_j \partial^1_{j+1} x \qquad \text{by (2.2.1)(i)} .$$

$$= \varepsilon_j \partial^0_j \partial^0_{j+1} x \circ_j \partial^0_j x \circ_j \partial^1_{j+1} x \qquad \text{by (2.1.2)(v, vi)}.$$

$= \partial^0_j x \circ_j \partial^1_{j+1} x$ (since $\varepsilon_j \partial^0_j \partial^0_{j+1} x$ is an identity for $\circ_j$).

(v) This follows from (iv) .

2.5.3 Lemma.

(i) $$\psi_j \varepsilon_i = \begin{cases} \varepsilon_i \psi_{j-1} & i < j) \\ \varepsilon_i \psi_j & (i > j+1) , \end{cases}$$

(ii) $\psi_j \varepsilon_j = \psi_j \varepsilon_{j+1} = \varepsilon_j$,

(iii) $\psi_j (\varepsilon_1)^j = (\varepsilon_1)^j$,

(iv) $$\psi_j \varepsilon_i \partial^\alpha_i = \begin{cases} \varepsilon_i \partial^\alpha_i \psi_j & (i < j) \\ \varepsilon_i \partial^\alpha_i \psi_j & (i > j+1) , \end{cases}$$

Proof.

(i) Let $x \in G_n$, then for $i < j$ we have

$$\psi_j \varepsilon_i = \Gamma'_j \partial^0_{j+1} \varepsilon_i x \circ_{j+1} \varepsilon_i x \circ_{j+1} \Gamma_j \partial^1_{j+1} \varepsilon_i x$$

$$= \Gamma'_j \varepsilon_i \partial^0_j x \circ_{j+1} \varepsilon_i x \circ_{j+1} \Gamma_j \varepsilon_i \partial^1_j x \qquad \text{by (2.1.1)(iii)}$$

$$= \varepsilon_i \Gamma'_{j-1} \partial^0_j x \circ_{j+1} \varepsilon_i x \circ_{j+1} \varepsilon_i \Gamma_{j-1} \partial^1_j x \qquad \text{by (2.1.2)(iii, iv)}$$

$$= \varepsilon_i (\Gamma'_{j-1} x \circ_j x \circ_j \Gamma_{j-1} x) \qquad \text{by (2.2.1)(i)}$$

$$= \varepsilon_i \psi_{j-1} .$$

For $i > j + 1$, let $x \in G_n$. Then

$$\psi_j \varepsilon_i = \Gamma'_j \partial^0_{j+1} \varepsilon_i x \circ_{j+1} \varepsilon_i x \circ_{j+1} \Gamma_j \partial^1_{j+1} \varepsilon_i x$$

$$= \Gamma'_j \varepsilon_{i-1} \partial^0_{j+1} x \circ_{j+1} \varepsilon_i x \circ_{j+1} \Gamma_j \varepsilon_{i-1} \partial^1_{j+1} x \qquad \text{by (2.1.1)(iii)}$$

$$= \varepsilon_i \Gamma'_j \partial^0_{j+1} x \circ_{j+1} \varepsilon_i x \circ_{j+1} \varepsilon_i \Gamma'_j \partial^1_{j+1} x \qquad \text{by (2.1.2)(iii, iv)}$$

$$= \varepsilon_i (\Gamma'_j \partial^0_{j+1} x \circ_{j+1} x \circ_{j+1} \Gamma'_j \partial^1_{j+1} x) \qquad \text{by (2.2.1)(i)}$$

$$= \varepsilon_i \psi_j \ .$$

(ii) $\psi_j \varepsilon_j x = \Gamma'_j \partial^0_{j+1} \varepsilon_j x \circ_{j+1} \varepsilon_j x \circ_{j+1} \Gamma_j \partial^1_{j+1} \varepsilon_j x$,

$$= \Gamma'_j \varepsilon_j \partial^0_j x \circ_{j+1} \varepsilon_j x \circ_{j+1} \Gamma_j \varepsilon_j \partial^1_j x \qquad \text{by (2.1.1)(iii)}$$

$$= \varepsilon_j \varepsilon_j \partial^0_j x \circ_{j+1} \varepsilon_j x \circ_{j+1} \varepsilon_j \varepsilon_j \partial^1_j x \qquad \text{by (2.1.2)(iii, iv)}$$

$= \varepsilon_j x$, (since $\varepsilon_j \varepsilon_j \partial^0_j x$ and $\varepsilon_j \varepsilon_j \partial^1_j x$ are identities).

(iii) Let $x \in G_n$, then

$$\psi_j (\varepsilon_1)^j x = \psi_j \varepsilon_1 (\varepsilon_1)^{j-1} = \varepsilon_1 \psi_{j-1} (\varepsilon_1)^{j-1} \qquad \text{by (2.5.3)(i)}$$

$$= \varepsilon_1 \varepsilon_1 \psi_{j-2} (\varepsilon_1)^{j-2} = (\varepsilon_1)^2 \varepsilon_1 \psi_{j-3} (\varepsilon_1)^{j-3}$$

Thus by induction we get

$$\psi_j (\varepsilon_1)^j x = (\varepsilon_1)^{j-1} \psi_1 \varepsilon_1 = (\varepsilon_1)^{j-1} \varepsilon_1 = (\varepsilon_1)^j \ .$$

(iv) For $i < j$, let $x \in G_n$, then

$$\psi_j \varepsilon_i \partial^\alpha_i x = \Gamma'_j \partial^0_{j+1} \varepsilon_i \partial^\alpha_i x \circ_{j+1} \varepsilon_i \partial^\alpha_i x \circ_{j+1} \Gamma_j \partial^1_{j+1} \varepsilon_i \partial^\alpha_i x$$

$$= \Gamma'_j \varepsilon_i \partial^0_j \partial^\alpha_i x \circ_{j+1} \varepsilon_i \partial^\alpha_i x \circ_{j+1} \Gamma_j \varepsilon_i \partial^1_j \partial^\alpha_i x$$

$$= \varepsilon_i \Gamma'_{j-1} \partial^\alpha_i \partial^0_{j+1} x \circ_{j+1} \varepsilon_i \partial^\alpha_i x \circ_{j+1} \varepsilon_i \Gamma_{j-1} \partial^\alpha_i \partial^1_{j+1} x$$

$$= \varepsilon_i \partial^\alpha_i \Gamma'_j \partial^0_{j+1} x \circ_{j+1} \varepsilon_i \partial^\alpha_i x \circ_{j+1} \varepsilon_i \partial^\alpha_i \Gamma_j \partial^1_{j+1} x$$

$$= \varepsilon_i (\partial^\alpha_i \Gamma'_j \partial^0_{j+1} x \circ_{j+2} x \circ_{j+2} \partial^\alpha_i \Gamma_j \partial^1_{j+1} x)$$

$$= \varepsilon_i \partial^\alpha_i (\Gamma'_j \partial^0_{j+1} x \circ_{j+1} x \circ_{j+1} \Gamma_j \partial^1_{j+1} x)$$

$$= \varepsilon_i \partial^\alpha_i \psi_j \ .$$

2.5.4 Lemma.

(i) $$\psi_j\Gamma_i = \begin{cases} \Gamma_i\psi_{j-1} & (i < j) \\ \Gamma_i\psi_j & (i > j+1) \end{cases}$$

(ii) $\psi_j\Gamma_j = \varepsilon_j$,

(iii) $\psi_j\Gamma_{j+1} = \Gamma'_j \circ_{j+1} \Gamma_{j+1} \circ_{j+1} \Gamma_j\varepsilon_{j+1}\partial^1_{j+1}$.

Proof.

(i) Let $x \in G_{n-1}$. Then for $i < j$, we get

$$\psi_j\Gamma_i x = \Gamma'_j\partial^0_{j+1}\Gamma_i x \circ_{j+1} \Gamma_i x \circ_{j+1} \Gamma_j\partial^1_{j+1}\Gamma_i x$$

$$= \Gamma'_j\Gamma_i\partial^0_j x \circ_{j+1} \Gamma_i x \circ_{j+1} \Gamma_j\Gamma_i\partial^1_j x \qquad \text{by (2.1.2)(vii, viii)}$$

$$= \Gamma_i\Gamma'_{j-1}\partial^0_j x \circ_{j+1} \Gamma_i x \circ_{j+1} \Gamma_i\Gamma_{j-1}\partial^1_j x \qquad \text{by (2.2.1)(ix)}$$

$$= \Gamma_i(\Gamma'_{j-1}\partial^0_j x \circ_j x \circ_j \Gamma_{j-1}\partial^1_j x) \qquad \text{by (2.2.1)(x)}$$

$$= \Gamma_i\psi_{j-1}x \ .$$

For $i > j + 1$, we have

$$\psi_j\Gamma_i x = \Gamma'_j\partial^0_{j+1}\Gamma_i x \circ_{j+1} \Gamma_i x \circ_{j+1} \Gamma_j\partial^1_{j+1}\Gamma_i x$$

$$= \Gamma'_j\Gamma_{i-1}\partial^0_{j+1}x \circ_{j+1} \Gamma_i x \circ_{j+1} \Gamma_j\Gamma_{i-1}\partial^1_{j+1}x$$

by (2.1.2)(vii, viii)

$$= \Gamma_i\Gamma'_j\partial^0_{j+1}x \circ_{j+1} \Gamma_i x \circ_{j+1} \Gamma_i\Gamma_j\partial^1_{j+1}x \qquad \text{by (2.1.2)(ix)}$$

$$= \Gamma_i(\Gamma'_j\partial^0_{j+1}x \circ_{j+1} x \circ_{j+1} \Gamma_j\partial^1_{j+1}x) \qquad \text{by (2.1.2)(x)}$$

$$= \Gamma_i\psi_j x \ .$$

(ii) Let $x \in G_{n-1}$. Then

$$\psi_j\Gamma_j x = \Gamma'_j\partial^0_{j+1}\Gamma_j x \circ_{j+1} \Gamma_j x \circ_{j+1} \Gamma_j\partial^1_{j+1}\Gamma_j x$$

$$= \Gamma'_j x \circ_{j+1} \Gamma_j x \circ_{j+1} \Gamma_j\varepsilon_j\partial^1_j x \qquad \text{by (2.1.2)(v, vi)}$$

$$= \varepsilon_j \circ_{j+1} \varepsilon_j \varepsilon_j \partial_j^1 x \qquad \text{by(2.1.2)(iii)}$$

$$= \varepsilon_j x \text{ , (since } \varepsilon_j \varepsilon_j \partial_j^1 x \text{ is an identity).}$$

(iii) Let $x \in G_{n-1}$. Then

$$\psi_j \Gamma_{j+1} x = \Gamma'_j \partial^0_{j+1} \Gamma_{j+1} x \circ_{j+1} \Gamma_j x \circ_{j+1} \Gamma_j \partial^1_{j+1} \Gamma_{j+1} x$$

$$= \Gamma'_j x \circ_{j+1} \Gamma_{j+1} x \circ_{j+1} \Gamma_j \varepsilon_{j+1} \partial^1_{j+1} x \quad \text{by (2.1.2)(v).}$$

2.5.5 Lemma.

(i) $$\psi_j \Gamma'_i = \begin{cases} \Gamma'_i \psi_{j-1} & (i < j) \\ \Gamma'_i \psi_j & (i > j+1) , \end{cases}$$

(ii) $\psi_j \Gamma'_j = \varepsilon_j$,

(iii) $\psi_j \Gamma'_{j+1} = \Gamma'_j \varepsilon_{j+1} \partial^0_{j+1} \circ_{j+1} \Gamma'_{j+1} \circ_{j+1} \Gamma_j$.

Proof.

(i) Let $x \in G_{n-1}$. Then for $i < j$, we get

$$\psi_j \Gamma'_i x = \Gamma'_j \partial^0_{j+1} \Gamma'_i x \circ_{j+1} \Gamma'_i x \circ_{j+1} \Gamma_j \partial^1_{j+1} \Gamma'_i x$$

$$= \Gamma'_j \Gamma'_i \partial^0_j x \circ_{j+1} \Gamma'_i x \circ_{j+1} \Gamma_j \Gamma'_i \partial^1_j x \qquad \text{by (2.1.2)(viii)}$$

$$= \Gamma'_i \Gamma'_{j-1} \partial^0_j x \circ_{j+1} \Gamma'_i x \circ_{j+1} \Gamma'_i \Gamma_{j-1} \partial^1_j x \quad \text{by (2.1.2)(ii)}$$

$$= \Gamma'_i (\Gamma'_{j-1} \partial^0_j x \circ_j x \circ_j \Gamma_{j-1} \partial^1_j x) \qquad \text{by (2.1.2)(x)}$$

$$= \Gamma'_i \psi_{j-1} x .$$

For $i > j + 1$, let $x \in G_{n-1}$ then

$$\psi_j \Gamma'_i x = \Gamma'_j \partial^0_{j+1} \Gamma'_i x \circ_{j+1} \Gamma'_i x \circ_{j+1} \Gamma_j \partial^1_{j+1} x \Gamma'_i$$

$$= \Gamma'_j \Gamma'_{i-1} \partial^0_{j+1} x \circ_{j+1} \Gamma'_i x \circ_{j+1} \Gamma_j \Gamma'_{i-1} \partial^1_{j+1} x$$

by (2.1.2)(viii)

$$= \Gamma'_i \Gamma'_j \partial^0_{j+1} x \circ_{j+1} \Gamma'_i x \circ_{j+1} \Gamma_i \Gamma'_j \partial^1_{j+1} x \quad \text{by (2.1.2)(ii)}$$

$$= \Gamma'_i(\Gamma'_j \partial^0_{j+1}x \circ_{j+1} x \circ_{j+1} \Gamma_j \partial^1_{j+1}x) \qquad \text{by (2.1.2)(x)}$$

$$= \Gamma'_i\psi_j \; .$$

(ii) Let $x \in G_{n-1}$. Then

$$\psi_j\Gamma'_jx = \Gamma'_j\partial^0_{j+1}\Gamma'_jx \circ_{j+1} \Gamma'_jx \circ_{j+1} \Gamma_j\partial^1_{j+1}\Gamma'_jx$$

$$= \Gamma'_j\varepsilon_j\partial^0_jx \circ_{j+1} \Gamma'_jx \circ_{j+1} \Gamma_jx \qquad \text{by (2.1.2)(vi)}$$

$$= \varepsilon_j\varepsilon_j\partial^0x \circ_{j+1} \varepsilon_jx \qquad \text{by (2.1.2)(iv)}$$

$$= \varepsilon_jx \quad \text{(since } \varepsilon_j\varepsilon_j\partial^0_jx \text{ is an identity).}$$

(iii) $\psi_j \Gamma'_{j+1}x = \Gamma'_j \partial^0_{j+1} \Gamma'_{j+1}x \circ_{j+1} \Gamma'_jx \circ_{j+1} \Gamma_j \partial^1_{j+1} \Gamma'_{j+1}x$

$$= \Gamma_j \varepsilon_{j+1} \partial^0_{j+1}x \circ_{j+1} \Gamma'_{j+1}x \circ_{j+1} \Gamma_jx \; .$$

by (2.1.2)(vi)

2.5.6 Proposition.

Let x , $y \in G_n$ *with* $\partial^\alpha_jx = \partial^\alpha_jy$, *where* $\alpha = 0,1$, *then*

$$\psi_i\ (x \circ_j y) = \begin{cases} \psi_ix \circ_j \psi_iy & \text{if } j \neq i, i+ \\ (\psi_ix \circ_{i+1} \varepsilon_i \partial^1_{i+1}y) \circ_i (\varepsilon_i \partial^0_{i+1}x \circ_{i+1} \psi_iy) & \text{if } j = i \\ (\varepsilon_i\partial^0_ix \circ_{i+1} \psi_iy) \circ_i (\psi_ix \circ_{i+1} \varepsilon_i\partial^1_iy) & \text{if } j = i+1 \; . \end{cases}$$

Proof.

Let $j < i$, then we have

$$\psi_i\ (x \circ_j y) = \Gamma'_i \partial^0_{i+1}(x \circ_j y) \circ_{i+1} (x \circ_j y) \circ_{i+1} \Gamma_i \partial^1_{i+1}(x \circ_j y)$$

$$= (\Gamma'_i \partial^0_{i+1}x \circ_j \Gamma'_i \partial^0_{i+1}y) \circ_{i+1} (x \circ_j y) \circ_{i+1} (\Gamma_i \partial^1_{i+1}x \circ_j \Gamma_i \partial^1_{i+1}y)$$

by (2.2.1)(i) and (2.1.2)(x)

$$= (\Gamma'_i \partial^0_{i+1}x \circ_{i+1} x \circ_{i+1} \Gamma_i\partial^1_{i+1}x) \circ_j (\Gamma'_i\partial^0_{i+1}y \circ_{i+1} y \circ_{i+1} \Gamma_i\partial^1_{i+1}y)$$

by (2.2.1)(iii)

$= \psi_i x \circ_j \psi_i y$.

For $j > i + 1$, we have

$$\psi_i (x \circ_j y) = \Gamma'_i \partial^0_{i+1}(x \circ_j y) \circ_{i+1} (x \circ_j y) \circ_{i+1} \Gamma_i \partial^1_{i+1}(x \circ_j y)$$

$$= \Gamma'_i (\partial^0_{i+1}x \circ_{j-1} \partial^0_{i+1}y) \circ_{i+1} (x \circ_j y) \circ_{i+1} \Gamma_i (\partial^1_{i+1}x \circ_{j-1} \partial^1_{i+1}y)$$

by (2.2.1)(i)

$$= (\Gamma'_i \partial^0_{i+1}x \circ_j \Gamma'_i \partial^0_{i+1}y) \circ_{i+1} (x \circ_j y) \circ_{i+1} (\Gamma_i \partial^1_{i+1}x \circ_j \Gamma_i\partial^1_{i+1}y)$$

by (2.1.2)(x)

$$= (\Gamma'_i \partial^0_{i+1}x \circ_{i+1} x \circ_{i+1} \Gamma_i \partial^1_{i+1}x) \circ_j (\Gamma'_i \partial^0_{i+1}y \circ_{i+1} y \circ_{i+1} \Gamma_i\partial^1_{i+1}y)$$

by (2.2.1)(iii)

$= \psi_i x \circ_j \psi_i y$.

The equalities for $j = i$, $i + 1$ follow from 2.1 in [S-1] since (G_n , G_{n-1}) is a double category for direction i , $i + 1$.□

§ 2.6 The associated ∞-category γG and Ψ .

In this section we state and prove some important results about the operation Ψ . These results prove that Ψx is an element of the associated ∞-category γG .

Before we give the following proposition we recall the following standard relations:

(2.6.a) $\varepsilon_n (\varepsilon_1)^{n-1}x = (\varepsilon_1)^n x$,

(2.6.b) $\partial^\alpha_{n+1} (\varepsilon_1)^n x = (\varepsilon_1)^n \partial^\alpha_1 x$,

2.6.1 Proposition.

Let $x \in G_n$, *then for* $n \ge 2$,

(*i*) $\Psi_n (\varepsilon_1)^n x = (\varepsilon_1)^n x$,

(*ii*) $\Psi_r (\varepsilon_1)^i y = (\varepsilon_1)^i x$, where $y \in G_{n-i}$ and $i > r$, *(iii) if $i > j+1$ then* $\partial_i^\alpha \Psi_{j+1} x = \Psi_{j+1} \partial_i^\alpha x$.

Proof.

We will use mathematical induction to proof this proposition and the next ones.

(i) For n = 2 we have

$\Psi_2 \ (\varepsilon_1)^2 x = \psi_1 \ (\varepsilon_1)^2 x = (\varepsilon_1)^2 x$. by (2.5.2)(ii)

Also

$\Psi_{n+1} \ (\varepsilon_1)^{n+1} x = \psi_n \ \Psi_n \ (\varepsilon_1)^n \ \varepsilon_1 x = \psi_n \ (\varepsilon_1)^n \ \varepsilon_1 x = (\varepsilon_1)^n \ \varepsilon_1 x$

by induction

$= (\varepsilon_1)^{n+1} x$.

Thus $\Psi_n \ (\varepsilon_1)^n x = (\varepsilon_1)^n x$ for all n .

(ii) $\Psi_r \ (\varepsilon_1)^i y = \Psi_r \ (\varepsilon_1)^r \ (\varepsilon_1)^{i-r} y = (\varepsilon_1)^r \ (\varepsilon_1)^{i-r} y = (\varepsilon_1)^i y$.

by (2.6.1)(i)

(iii) $\partial_i^\alpha \Psi_{j+1} x = \partial_i^\alpha \psi_j \psi_{j-1} \cdots \psi_1 x = \psi_j \partial_i^\alpha \psi_{j-1} \cdots \psi_1 x$ by (2.5.2)(i)

$= \psi_j \psi_{j-1} \partial_i^\alpha \psi_{j-2} \cdots \psi_1 x$ by (2.5.2)(i)

$= \psi_j \psi_{j-1} \cdots \psi_1 \partial_i^\alpha = \Psi_{j+1} \partial_i^\alpha$.

2.6.2 Proposition.

Let $x \in G_n$ *, then for* $n \geq 2$

(i) $\partial_n^\alpha \ \Psi_n x = (\varepsilon_1)^{n-1} \ (\partial_1^\alpha)^n x$,

(ii) $\partial_n^\alpha \ \Psi x = (\varepsilon_1)^{n-1} \ (\partial_n^\alpha)^n$,

(iii) $\partial_n^\alpha \ \Psi x = (\varepsilon_1)^{i-1} \ \partial_1^\alpha \ \partial_2^\alpha \ \cdots \partial_i^\alpha \ \Psi_{i+1} \ \Psi_{i+2} \ \cdots \Psi_n x$.

Proof.

(i) For n = 2 , we have

$\partial_2^\alpha \ \Psi_2 x = \partial_2^\alpha \ \psi_1 x = \varepsilon_1 \ \partial_1^\alpha \ \partial_2^\alpha x = \varepsilon_1 \ \partial_1^\alpha \ \partial_1^\alpha x = \varepsilon_1 \ (\partial_1^\alpha)^2 x$.

by (2.5.2)(iv) and (2.1.1)(i)

Also

$$\partial^\alpha_{n+1}\ \Psi_{n+1}x = \partial^\alpha_{n+1}\ (\psi_n\Psi_n)x$$

$$= \varepsilon_n\ \partial^\alpha_n\ \partial^\alpha_n\ \Psi_n x \qquad \text{by (2.5.2)(iv)}$$

$$= \varepsilon_n\ \partial^\alpha_n\ (\varepsilon_1)^{n-1}\ (\partial^\alpha_1)^n\ x$$

$$= \varepsilon_n\ (\varepsilon_1)^{n-1}\ \partial^\alpha_1\ (\partial^\alpha_1)^n x \qquad \text{by (2.6.b)}$$

$$= (\varepsilon_1)^n\ \partial^\alpha_1\ (\partial^\alpha_1)^n x \qquad \text{by (2.6.a)}$$

$$= (\varepsilon_1)^n\ (\partial^\alpha_1)^{n+1} x \quad \text{by induction ,}$$

Thus $\partial^\alpha_n\ \Psi_n x = (\varepsilon_1)^{n-1}\ (\partial^\alpha_1)^n x$ for all n .

(ii) For n = 2

$$\partial^\alpha_2\ \Psi x = \partial^\alpha_2\ \Psi_2 x = \partial^\alpha_2\ \psi_1 x$$

$$= \varepsilon_1\ \partial^\alpha_1\ \partial^\alpha_1 x = \varepsilon_1\ (\partial^\alpha_1 x)^2 \qquad \text{by (2.5.2)(iv)}$$

Also

$$\partial^\alpha_{n+1}\ \Psi x = \partial^\alpha_{n+1}\ \Psi_2\ \ldots\Psi_{n+1}x$$

$$= \Psi_2\ldots\Psi_n\ \partial^\alpha_{n+1}\ \Psi_{n+1}x \qquad \text{by (2.5.2)(i)}$$

$$= \Psi_2\ldots\Psi_n\ (\varepsilon_1)^n\ (\partial^\alpha_1)^{n+1}x \qquad \text{by (2.6.2)(i) and induction}$$

$$= \Psi_2\ldots\Psi_{n-1}\ (\varepsilon_1)^n\ (\partial^\alpha_1)^{n+1}x \qquad \text{by (2.6.1)(i)}$$

$$= \Psi_2\ldots\Psi_{n-2}\ (\varepsilon_1)^n\ (\partial^\alpha_1)^{n+1}x \qquad \text{by (2.6.1)(ii)}$$

$$= (\varepsilon_1)^n\ (\partial^\alpha_1)^{n+1}x\ . \qquad \text{by (2.6.1)(ii)}$$

Thus $\partial^\alpha_n\ \Psi x = (\varepsilon_1)^{n-1}\ (\partial^\alpha_n)^n x$ is true for all n .

(iii) $\partial^\alpha_i\ \Psi x = \partial^\alpha_i\ \Psi_2\ldots\Psi_n x$

$$= \Psi_2\ldots\Psi_{i-1}\ \partial^\alpha_i\ \Psi_i\ \Psi_{i+1}\ldots\Psi_n x \qquad \text{by (2.6.1)(iii)}$$

$$= \Psi_2\ldots\Psi_{i-1}\ (\varepsilon_1)^{i-1}\ (\partial^\alpha_1)^i\ \Psi_{i+1}\ldots\Psi_n x \qquad \text{by (2.6.2)(i)}$$

$$= \Psi_2\ldots\Psi_{i-2}(\varepsilon_1)^{i-1}\ (\partial^\alpha_1)^i\ \Psi_{i+1}\ldots\Psi_n x \qquad \text{by (2.6.1)(ii)}$$

$$= (\varepsilon_1)^{i-1} (\partial_1^{\alpha})^i \Psi_{i+1} \cdots \Psi_n x . \square$$

Thus Ψx is an element in the associated ∞-category γG .

It is clear that if $x \in C_n$, then the formula (2.5.1) becomes $\psi_j x = x$. This implies $\Psi x = x$, so we have:

2.6.3 Corollary.

$\Psi x = x$ if and only if x is an element in γG . In particular $\Psi^2 y = \Psi y$ for all $y \in G$.

2.6.4 Lemma.

Let $x \in G_{n-1}$ then ,

(i) $\Psi\varepsilon_i x = \varepsilon_i \Psi x$,

(ii) $\Psi\Gamma_i x = \varepsilon_1 \Psi x$ *and* $\Psi\Gamma'_i x = \varepsilon_1 \Psi x$, *for* $n = 2$, $i = 1,2$.

Proof.

(i) $\Psi\varepsilon_i x = \psi_{n-1} \cdots \psi_1 \varepsilon_i x$

$= \psi_{n-1} \cdots \psi_i \psi_{i-1} \varepsilon_i \psi_{1-2} \cdots \psi_1 x$ by (2.5.3)(i)

$= \psi_{n-1} \cdots \psi_i \varepsilon_{i-1} \psi_{1-2} \cdots \psi_1 x$ by (2.5.3)(ii)

$= \varepsilon_{i-1} \psi_{n-2} \cdots \psi_1 x$ by (2.5.3)(i)

$= \varepsilon_{i-1} \Psi_{n-1} x$.

(ii) for n = 2 we have

$\Psi\Gamma_i x = \psi_1 \Gamma_1 x = \varepsilon_1 x$. by (2.5.4)(ii)

For n = 3 and $i = 1$, we have

$\Psi\Gamma_i x = \psi_1 \psi_2 \psi_1 \Gamma_1 x$

$= \psi_1 \psi_2 \varepsilon_1 x$ by (2.5.4)(ii)

$= \psi_1 \varepsilon_1 \psi_1 x$ by (2.5.3)(i)

$= \varepsilon_1 \psi_1 x$ by (2.5.3)(ii)

$= \varepsilon_1 \Psi x$.

The case where $i = 2$ is proved in appendix II.

Thus $\Psi\Gamma_i x = \varepsilon_1 \Psi x$ and $\Psi\Gamma'_i x = \varepsilon_1 \Psi x$, for $1 < n \le 3$. $\square$

This lemma shows that $\varepsilon_i \Psi x$, $\Gamma_i \Psi x$ and $\Gamma'_i \Psi x$ are identities for $\circ_1$.

2.6.5 Remarks.

In the previous section and section 5 we investigate the folding operation Ψ in the general case except for finding an appropriate formula for Ψ on composite elements $x \circ_i y$. The key problem which stands as obstacle from finding this formula came from the fact that Ψx and Ψy lie in an ∞-category and so the faces of Ψx and Ψy contain much more information because they involve many faces which are not degenerate. In chapter III we give this formula for the case $n = 3$. It involves very complicated formulae which gives a clear indication that the formula $\Psi(x \circ_i y)$ for the general case looks extremely difficult with the available information. The same thing can be said about $\Psi\Gamma_i x$ and $\Psi\Gamma'_i x$ in Lemma 2.6.4 for the general case.

2.6.6 Lemma.

$$\Psi\psi_i = \Psi : G_3 \longrightarrow G_3 \quad (i = 1,2) .$$

The proof of this lemma and the next proposition will be given in III-2 since they require the compositions $\Psi(x \circ_i y)$, for x , $y \in G_3$ and $i = 1,2,3$, which will be determined by Proposition 3.2.5.

Recall from 2.1.2 that an element $x \in G_n$ $(n \geq 1)$ is thin if it can be written as a composite of $\varepsilon_i y$ or $\Gamma_i y$ or $\Gamma'_i y$ for $y \in G_{n-1}$.

The collection of all thin elements of G is closed under all the ω-category structures except the face operation. It is useful to think of the thin elements as the most general kind of degenerate cubes.

2.6.7 Proposition.

Let $x \in G_n$ $(1 \le n \le 3)$. *Then* x *is thin if and only if* $\Psi x = 1$.

The proof is given in III-2 .

§ 7 **Skeleton and coskeleton of ω-categories.**

If one ignores the elements of dimension higher than n in an ω-category, one obtains an n-tuple category G′. In [B-Hi-2] R. Brown and P. J. Higgins have constructed the skeleton and the coskeleton in an ω-groupoid. G. H. Mosa , in [Mo-1], has followed the same notations and terminology and constructed the coskeleton in an ω-algebroid. We will follow the same notations and terminology and construct the skeleton and coskeleton in an ω-category.

We start to construct the coskeleton in terms of "shells" as follows:

In any cubical complex K , an *r-shell* means a family $\underline{x} = (x_i^\alpha)$ of r-cubes ($i = 1, \dots, r+1$, $\alpha = 0, 1$) satisfying

$$\partial_j^\beta \underline{x}_i^\alpha = \partial_{i-1}^\alpha x_j^\alpha \text{ for } 1 \le j < i \le r+1 \text{ and } \alpha, \beta = 0, 1 .$$

In particular the faces $\partial_i^\alpha y$ of any (r+1)-cube form an r-shell ∂y . We denote by $\square K_r$, the set of all r-shells of K .

Let $K = (K_n, K_{n-1}, \dots, K_0)$ be an n-truncated cubical complex. Then $K' = (\square K_n, K_n, K_{n-1}, \dots, K_0)$ will denote the (n+1)-truncated cubical complex in which, for any $\underline{x} \in \square K_n$, $\partial_i^\alpha \underline{x}$ is defined to be x_i^α and for any $y \in K_n$, $\varepsilon_j y$ is defined to be the n-shell $\underline{z}$, where

$$(2.7.a)(i)\ \underline{z}_i^\alpha = \begin{cases} \varepsilon_{j-1}\, \partial_i^\alpha \underline{y} & (i < j) \\ \varepsilon_j\, \partial_{i-1}^\alpha \underline{y} & (i > j) \\ \underline{y} & (i = j) \end{cases} .$$

If K has connections, we can also define $\Gamma_j \underline{y} = \underline{w}$, $\Gamma'_j \underline{y} = \underline{e}$ where

$$(2.7.a)(ii)\ \underline{w}_i^\alpha = \begin{cases} \Gamma_{j-1}\, \partial_i^\alpha \underline{y} & (i < j) \quad \underline{w}_j^0 = \underline{w}_{j+1}^0 = \underline{y} \\ \Gamma_j\, \partial_{i-1}^\alpha \underline{y} & (i > j+1) \quad \underline{w}_j^1 = \underline{w}_{j+1}^1 = \varepsilon_j \partial_j^0 \underline{y} \end{cases}$$

$$(2.7.a)(iii)\ \underline{e}_i^\alpha = \begin{cases} \Gamma'_{j-1}\, \partial_i^\alpha \underline{y} & (i < j) \quad \underline{e}_j^0 = \underline{e}_{j+1}^0 = \varepsilon_j \partial_j^0 \underline{y} \\ \Gamma'_j\, \partial_{i-1}^\alpha \underline{y} & (i > j+1) \quad \underline{e}_j^1 = \underline{e}_{j+1}^1 = \underline{y} \end{cases} .$$

In this way K′ becomes an (n+1)-truncated cubical complex with connections.

Now we replace K by an n-tuple category G . We define $\circ_j$ in $\square\, G_n$ as follows:

Let $\underline{x}$, $\underline{y} \in \square\, G_n$ with $\partial_j^1 \underline{x} = \partial_j^0 \underline{y}$. Define $\underline{x} \circ_j \underline{y} = \underline{z}$ where

$$\underline{z}_j^0 = \underline{x}_j^0 \quad , \quad \underline{z}_j^1 = \underline{y}_j^1$$

$$(iv)\ \underline{z}_i^\alpha = \begin{cases} \underline{x}_i^\alpha \circ_j \underline{y}_i^\alpha & (i<j) \\ \underline{x}_i^\alpha \circ_j \underline{y}_i^\alpha & (i>j) \end{cases} .$$

2.7.1 Proposition.

The above structure $G' = (\square\, G_n, G_n, G_{n-1}, \ldots, G_0)$ *is an (n+1)-truncated* ω*-category.*

Proof.

Let $\underline{x}$, $\underline{y} \in \square\, G_n$ such that $\underline{x} \circ_j \underline{y}$ is defined. Then

$$(i)\ \partial_i^\alpha(\underline{x} \circ_j \underline{y}) = \partial_i^\alpha \underline{z} = \underline{z}_i^\alpha = \begin{cases} \underline{x}_i^\alpha \circ_j \underline{y}_i^\alpha & (i<j) \\ \underline{x}_i^\alpha \circ_j \underline{y}_i^\alpha & (i>j) \end{cases} .$$

$$= \begin{cases} \partial_i^\alpha \underline{x} \circ_j \partial_i^\alpha \underline{y} & (i<j) \\ \partial_i^\alpha \underline{x} \circ_j \partial_i^\alpha \underline{y} & (i>j) \end{cases} .$$

(ii) Let x , $y \in G_n$ such that $x \circ_j y$ is defined. Then for $k < i < j$ we get

$$\partial_k^\alpha[\varepsilon_i(x \circ_j y)] = \varepsilon_{i-1}\partial_k^\alpha(x \circ_j y)$$

$$= \varepsilon_{i-1}(\partial_k^\alpha x \circ_{j-1} \partial_k^\alpha y) \qquad \text{(since } \partial_k^\alpha x \text{ , } \partial_k^\alpha y \text{ are elements in } G_{n-1})$$

$$= \partial_k^\alpha\ \varepsilon_i x \circ_j \partial_k^\alpha\ \varepsilon_i y$$

$$= \partial_k^\alpha\ (\varepsilon_i x \circ_{j+1} \varepsilon_i y)$$

Thus $\varepsilon_i(x \circ_j y) = \varepsilon_i x \circ_j \varepsilon_i y$. Similarly we can prove that $\varepsilon_i(x \circ_j y) = \varepsilon_i x \circ_j \varepsilon_i y$ for $i > j$.

(iii) Let $\underline{x} \in \Box G_n$. Then for $k < j$, we get

$$\partial_k^\alpha(\varepsilon_j\ \partial_j^0 \underline{x} \circ_j \underline{x}) = \partial_k^\alpha\ \varepsilon_j\ \partial_j^0 \underline{x} \circ_{j-1} \partial_k^\alpha \underline{x}$$

$$= \varepsilon_{j-1}\ \partial_{j-1}^0\ \partial_k^\alpha \underline{x} \circ_{j-1} \partial_k^\alpha \underline{x}$$

$$= \partial_k^\alpha \underline{x} .$$

Thus $\varepsilon_j\ \partial_j^0 \underline{x} \circ_j \underline{x} = \underline{x}$. We can prove similarly that $\underline{x} = \underline{x} \circ_j \varepsilon_j\ \partial_j^1 \underline{x}$.

(iv) Let x , $y \in G_n$ such that $x \circ_j y$ is defined. Then for $k < i < j$ we have

$$\partial_k^\alpha\ [\Gamma_i(x \circ_j y)] = \Gamma_{i-1}\ \partial_k^\alpha(x \circ_j y)$$

$$= \Gamma_{i-1}(\partial_k^\alpha x \circ_{j-1} \partial_k^\alpha y)$$

$$= \Gamma_{i-1}\ \partial_k^\alpha x \circ_j \Gamma_{i-1}\ \partial_k^\alpha y$$

$$= \partial_k^\alpha\ (\Gamma_i x \circ_{j+1} \Gamma_i y) .$$

Thus $\Gamma_i(x \circ_j y) = \Gamma_i x \circ_{j+1} \Gamma_i y$. We can prove similarly that

$\Gamma_i(x \circ_j b) = \Gamma_i x \circ_j \Gamma_i y$ for $i < j$. We can follow the same routine to prove (iv) for Γ'_i .

(v) Let $x \in G_n$. Then for $k < j$, we get

$$\begin{aligned} \partial_k^\alpha (\Gamma'_j x \circ_{j+1} \Gamma_j x) &= \partial_k^\alpha \Gamma'_j x \circ_j \partial_k^\alpha \Gamma_j x \\ &= \Gamma'_{j-1} \partial_k^\alpha x \circ_j \Gamma_{j-1} \partial_k^\alpha x \\ &= \varepsilon_{j-1} \partial_k^\alpha x. \end{aligned}$$

Thus $\Gamma'_j x \circ_{j+1} \Gamma_j x = \varepsilon_j x$. Similarly we can prove that $\Gamma'_j x \circ_j \Gamma_j x = \varepsilon_{j+1} x$.

Thus $G' = (\square G_n, G_n, G_{n-1}, \ldots, G_0)$ is an (n+1)-truncated ω-category.

2.7.2 Proposition.

If $G = (G_n, G_{n-1}, \ldots, G_0)$ *is an n-tuple category, then the ω-category* $\bar{G}$ *with*

$$\bar{G}_m = \begin{cases} G_m & \text{for } m \le n \\ \square^{m-n} G_n & \text{for } m > n . \end{cases}$$

and operations defined as above, is the n-coskeleton of G .

Proof.

If H is any ω-category and $f_k : H_k \longrightarrow G_k$ are defined for $k = 0,1,2,\ldots,n$, that is

$$\begin{array}{ccccccccc} H_n & \longrightarrow & H_{n-1} & \longrightarrow & \cdots\cdots\cdots & \longrightarrow & H_1 & \longrightarrow & H_0 \\ \downarrow f_n & & \downarrow f_{n-1} & & & & \downarrow f_1 & & \downarrow f_0 \\ G_n & \longrightarrow & G_{n-1} & \longrightarrow & \cdots\cdots\cdots & \longrightarrow & G_1 & \longrightarrow & G_0 \end{array}$$

so as to form a morphism of n-tuple category from n-truncated H to G , then there is a unique extension to a morphism of

ω-categories $f : H \longrightarrow \bar{G}$ defined inductively by, for $\psi \in H$, $f_m \psi = \chi$, where $\chi_i^\alpha = f_{m-1}\partial_i^\alpha \psi$ $(m > n)$. This shows that $G \cong \mathrm{Cos}^n G$. □

We apply now the folding operation ψ_i and Ψ in the ω-category $\mathrm{Cosk}^n G$, where $G = (G_n, G_{n-1}, \ldots, G_0)$. Given an n-shell $\underline{\psi} = (\psi_i^\alpha) \in G_n$, we obtain n-shells $\psi_i\underline{\psi}$ and $\Psi\underline{\psi} = \Psi_2 \ldots \Psi_n \underline{\psi}$. By Proposition 2.6.2, all faces ∂_i^α of $\Psi\underline{\psi}$ are i-fold degenerate except for $i = 1$, where $\partial_1^0\underline{\psi}$ is a "kind of sum" of the odd faces $\partial_i^\alpha \underline{\psi}$ where $i + \alpha$ is odd, and $\partial_1^1 \underline{\psi}$ is a "kind of sum" of the *even* faces $\partial_i^\alpha x$, where $i + \alpha$ is even. If H is a given ω-category, then adjointness gives a canonical morphism

$$f : H \longrightarrow \mathrm{Cosk}^n H = \mathrm{cosk}^n(\mathrm{tr}^n H) ,$$

with $f_{n+1}x = \underline{\partial} x$ for $x \in H_{n+1}$. Since f preserves the folding operations we have

$$\Psi\underline{\partial} x = \underline{\partial}\Psi x \qquad (2.7.3)$$

for any element x of dimension at least two in an ω-category.

2.7.3 Proposition.

Let G *be a* ω*-category, and let* $C = \gamma G$ *be its associated* ∞*-category. Let* $\underline{x} \in \square G_{n-1}$ *and* $\xi \in C_n$. *Then there exist* $x \in G_n$ *such that* $\underline{\partial} x = \underline{x}$ *and* $\Psi x = \xi$ *if and only if* $d_1^\alpha \xi = \partial_1^\alpha \Psi \underline{x}$, $\alpha = 0, 1$.

Proof.

If $\underline{\partial} x = \underline{x}$ and $\Psi x = \xi$, then, by (2.7.3), $\underline{\partial}\Psi x = \Psi\underline{\partial} x = \Psi\underline{x}$, so $d_1^\alpha \xi = \partial_1^\alpha \Psi \underline{x}$. Suppose, conversely, that we are given $\underline{x}$ and ξ with $d_1^\alpha \xi = \partial_1^\alpha \Psi \underline{x}$. Then, since the faces $d_1^\alpha \xi$ and $\partial_1^\alpha \Psi \underline{x}$ determine the faces $d_i^\alpha \xi$ and $\partial_i^\alpha \Psi \underline{x}$ respectively, we have $\underline{\partial}\xi = \Psi\underline{x}$, an equation in $\square G_{n-1}$. We have to show that there is a unique $x \in G_n$ such that $\underline{\partial} x = \underline{x}$ and $\Psi x = \xi$. To prove this it is enough to show that if $\psi \in G_n$ and $\underline{\partial}\psi = \psi_i \underline{\chi}$, $\underline{\chi} \in \square G_{n-1}$, then there is a unique $\chi \in G_n$ with $\underline{\partial}\chi = \underline{\chi}$ and $\psi_i \chi = \psi$. This can be done by unwinding each $\psi_i \chi$

as follows;

$$\left[\begin{array}{c:c:c} \varepsilon_i\varepsilon_{i+1}\partial^0_{i+1}\partial^0_i\mathfrak{x} & \varepsilon_i\partial^0_i\mathfrak{x} & \Gamma'_i\partial^1_{i+1}\mathfrak{x} \\ \hdashline \Gamma'_i\partial^0_{i+1}\mathfrak{x} & \mathfrak{x} & \Gamma_i\partial^1_{i+1}\mathfrak{x} \\ \hdashline \Gamma_i\partial^0_{i+1}\mathfrak{x} & \varepsilon_i\partial^1_i\mathfrak{x} & \varepsilon_i\varepsilon_{i+1}\partial^1_{i+1}\partial^1_i\mathfrak{x} \end{array}\right] \qquad \begin{array}{l} \longrightarrow i+1 \\ \downarrow \\ i \end{array}$$

$$= \left[\begin{array}{c:c:c} \varepsilon_i\varepsilon_{i+1}\partial^0_{i+1}\partial^0_i\mathfrak{x} & \varepsilon_i\partial^0_i\mathfrak{x} & \varepsilon_i\varepsilon_{i+1}\partial^1_{i+1}\partial^1_i\mathfrak{x} \\ \hdashline \varepsilon_i\partial^0_{i+1}\mathfrak{x} & \mathfrak{x} & \varepsilon_i\partial^1_{i+1}\mathfrak{x} \\ \hdashline \varepsilon_i\varepsilon_{i+1}\partial^0_{i+1}\partial^0_i\mathfrak{x} & \varepsilon_i\partial^1_i\mathfrak{x} & \varepsilon_i\varepsilon_{i+1}\partial^1_{i+1}\partial^1_i\mathfrak{x} \end{array}\right]$$

$$= \mathfrak{x} \quad \left(\text{since } \left[\begin{array}{c} \Gamma'_i\partial^\alpha_{i+1}\mathfrak{x} \\ \Gamma_i\partial^\alpha_{i+1}\mathfrak{x} \end{array}\right] = \varepsilon_i\partial^\alpha_{i+1}\mathfrak{x}\ \right).$$

which shows how to recover $\mathfrak{x}$ from $\psi_i\mathfrak{x}$ and $\underline{\mathfrak{x}}$. This $\mathfrak{x}$ is unique and has boundary $\underline{\mathfrak{x}}$. □

By using the notations of thin elements Proposition 2.7.3 can be proved as follows:

$$\left[\begin{array}{c:c:c} \square & \| & \ulcorner \\ \hdashline \ulcorner & \mathfrak{x} & \lrcorner \\ \lrcorner & \| & \square \end{array}\right] = \left[\begin{array}{ccc} \square & \| & \square \\ = & \mathfrak{x} & = \\ \square & \| & \square \end{array}\right] \qquad \left(\text{since } \left[\begin{array}{c} \ulcorner \\ \lrcorner \end{array}\right] = [\ = \ ]\ \right),$$

$$= \mathfrak{x}\ . \square$$

2.7.4 Corollary.

A thin element of a triple category is determined by its faces. Given a shell $\underline{x} \in \square G_2$, *there is a thin element* t *with* $\underline{\partial}t = \underline{x}$ *if and only if* $\partial^0_1\Psi\underline{x} = \partial^1_1\Psi\underline{x}$.

Proof.

Put $\xi = 1$ in Proposition 2.7.3 and use the fact that t is thin if and only if $\Psi t = 1$ (see Proposition 2.6.6).

We can now describe the 3-skeleton construction of a triple category.

2.7.5 Definition.

A shell $\underline{x} \in \square G_n$ is called a *commuting shell* if

$$\partial_1^0 \Psi \underline{x} = \partial_1^1 \Psi \underline{x} .$$

This definition does not lead to a definition of skeleton in the general case because of the lack of good formulae for Ψ of a composition.

2.7.6 Proposition.

Given a double category $G = (G_2, G_1, G_0)$, *the 3-skeleton* S *of* G *is the triple subcategory of* $\bar{G} = \operatorname{cosk}^3 G$ *generated by* G. *For* $m \le 2$, $S_m = G_m$, *while* S_3 *consists entirely of thin elements, namely, the commuting shells in* $\square G_2$. Proof.

Let S_m be defined by

$$S_m = \begin{cases} G_m & \text{if } m \le 2 , \\ \{\underline{x} \in \square G_2;\ \partial_1^0 \Psi \underline{x} = \partial_1^1 \Psi \underline{x}\ \} & \text{if } m = 3 . \end{cases}$$

Then $G \subset S \subset \operatorname{cosk}^3 G$. By Corollary 2.7.4 applied to the triple category $\bar{G} = \operatorname{cosk}^3 G$, S_3 contains only thin elements. Clearly, S is closed under face maps, degeneracy maps and connections (since $\varepsilon_i x$ and $\Gamma_i y$ are always thin). Also, S_3 is closed under $\circ_i$ $(1 \le i \le 2)$; for if $\underline{x}$, $\underline{y} \in S_3$ and $\underline{x} \circ_i \underline{y}$ is defined, then $\underline{x} \circ_i \underline{y}$ has faces in S_2 and $\partial_1^0 \Psi(\underline{x} \circ_i \underline{y}) = \partial_1^1 \Psi(\underline{x} \circ_i \underline{y})$ because composites of thin elements in $\bar{G}$ are thin. Thus $\underline{x} \circ_i \underline{y} \in S_3$. Hence S is a triple subcategory of $\bar{G}$. Also, by Corollary 2.7.4, any triple subcategory of $\bar{G}$ containing G_2 must contain G_3 , so S is generated by G .

If H is any triple category and $\phi : G \longrightarrow tr^2 H$ is a morphism of triple categories, then ϕ extends uniquely to a morphism of triple categories $\phi : S \longrightarrow H$ by the inductive rule that, for any commuting shell $\underline{x} \in \Box S_2(m > n)$, $\phi_2(\underline{x})$ is the unique thin element t of H_3 such that $\partial_i^\alpha t = \phi_2 x_i^\alpha$ for $1 \le i \le m$ and $\alpha = 0,1$. The element t exists by Corollary 2.7.4 since the element $\phi_2 x_i^\alpha$ form a commuting shell in H. This shows that $S = sk^3 G$.□ Given a triple category G , we define $Sk^3 G = sk^3(tr^3 G)$ and call this the n-skeleton of G. There is a unique morphism $\sigma : Sk^3 G \longrightarrow G$ of triple categories (the adjunction) which is the identity in dimensions $0,\ldots,2$.

2.7.7 Proposition.

The adjunction $\sigma : Sk^3 G \longrightarrow G$ *is an injection and identifies* $Sk^3 G$ *with the triple subcategory category of* G *generated by* G_0, G_1, G_2 .

Proof.

It is clear that σ is the identity in dimensions 0,1,2. If $\underline{x} \in (sk^3 G)$, then $\sigma_3(\underline{x})$ is the unique element of G_3 with $\underline{\partial} t = \underline{x}$. So σ_3 is injective. Since G_0, G_1, G_2 generate $sk^3 G$ in $csk^3 G$, then it also generate $\sigma_3(sk^3 G)$ in G.□

CHAPTER III

THE EQUIVALENCE BETWEEN TRIPLE CATEGORIES WITH CONNECTIONS AND 3-CATEGORIES

§ 3.0 **Introduction**

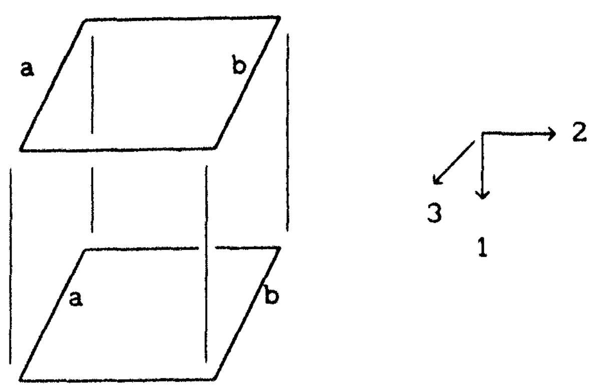
In this chapter we prove the equivalence between triple categories and 3-categories. One of the advantages of this proof is to highlight the key problems in the equivalence of the general case so one can concentrate the efforts to solve these problems. It seems that the key problem is to evaluate the composition $\Psi(x \circ_i y)$ because it involves many faces and edges.

§ 3.1 **The functor** γ : 3-$\mathcal{G}$ $\longrightarrow$ 3-$\mathcal{C}$.

In II-3 we have defined a functor γ : ω-$\mathcal{C}at$ $\longrightarrow$ ∞-$\mathcal{C}at$ by the rule

$$C_n = \{x \in G_n \mid \partial_j^\alpha x \in \varepsilon_1^{j-1} G_{n-j} \text{ for } 1 \le j \le n \text{ , } \alpha = 0,1\}$$

By this rule, C_3 is 3-fold category with respect to the structures $(C_3, \partial_i^0, \partial_i^1, \circ_i)$, for $0 \le i \le 2$. The elements of C are thus those cubes with boundaries partially represented by



The proof of the axioms of C are given in II-3 .

In chapter II we also constructed a "folding operation"

$$\Psi : G_n \longrightarrow G_n$$

for any ω-category G and proved that $\Psi G_n \subseteq (\gamma G)_n$.

The key difficulty in proving that $\gamma : \omega\text{-}\mathcal{C}at \longrightarrow \infty\ \mathcal{C}at$ is an equivalence of categories resides in finding appropriate formula for Ψ on composite elements $x \circ_i y$.

Recall that in the ω-groupoid case, Brown-Higgins [B-Hi-2] consider an analogous folding operation Φ and obtain a formula of the form

$$\Phi(x \circ_i y) = \begin{cases} \Phi y \circ (\Phi x)^{u_1 y} \text{if } n = 2 \text{ and} i = 1, \\ (\Phi x)^{u_1 y} \circ \Phi y \text{otherwise} , \end{cases}$$

where $u_1 y$ involves only one edge of y . In our more general case, the formula for $\Psi(x \circ_i y)$ should be expressed in terms of Ψx , Ψy and some "operations" involving the faces of x and y . The problem is that the folded form Ψx lies in an ∞-category and so the faces of Ψx contain much more information than in the case considered in [B-Hi-2] where Φx lies in a crossed complex, i.e. all faces but one of Φx are totally degenerate.

We are able to obtain a formula in dimension 3 . At present the general case looks difficult, and may need new ideas for codifying and applying the information contained in the faces of an element of an ∞-category.

3.2 The compositions $\Psi(x \circ_i y)$.

In II-5 we have defined an operation $\Psi : G_n \longrightarrow G_n$ and proved in I-6 that Ψ transfers an element $x \in G_n$ to an element Ψx in the associated n-category γG. Also we have seen from Propositions (2.6.1) and (2.6.2) that Ψx involves many faces which are not totally degenerate. In fact all the faces of Ψx are not degenerate except ∂_n^α . This makes the evaluation of $\Psi(x \circ_i y)$, for $i = 1,2,\ldots,n$, of great complexity. For this reason we will see the situation for the case n = 3 and evaluate $\Psi(x \circ_i y)$, for $i = 1,2,3$. This will give us a picture about the situation in the general case.

The best way to get this evaluation is to study the faces of Ψx , Ψy and $\Psi(x \circ_i y)$. We have seen that ∂_3^α are totally degenerate and ∂_2^α are partially degenerate i.e of the form of ε_1 . This suggests that the faces ∂_1^α are the key point to get the evaluation of the compositions $\Psi(x \circ_i y)$.

First we define operations in an n-category for n = 2,3 . These operations will help us in simplifying some of the complicated formulae.

3.2.1 Definition.

Let C be an ∞-category. If $\xi \in C_2$, θ , $\varphi \in C_1$, we define

$$\xi^\theta = \xi \circ_2 \varepsilon_1\theta \ , \quad {}^\theta\xi = \varepsilon_1\theta \circ_2 \xi \text{ and } {}^\theta\xi^\varphi = \varepsilon_1\theta \circ_2 \xi \circ_2 \varepsilon_1\varphi$$

This operation satisfies the following properties:

(i) $(\xi^\theta)^\varphi = \xi^{\theta\varphi}$ and ${}^\varphi(\xi^\theta) = {}^\varphi\xi^\theta$,

(ii) $(\xi \circ_1 \vartheta)^\theta = \xi^\theta \circ_1 \vartheta^\theta$ and ${}^\varphi(\xi \circ_1 \vartheta) = {}^\varphi\xi \circ_1 {}^\varphi\vartheta$,

(iii) $(\xi \circ_2 \vartheta)^\theta = \xi \circ_2 \vartheta^\theta$ and ${}^\varphi(\xi \circ_2 \vartheta) = {}^\varphi\xi \circ_2 \vartheta$, for ξ , $\vartheta \in C_2$, θ , $\varphi \in C_1$ and whenever the operations are defined.

Likewise we can define a similar operation in C_3 .

3.2.2 Definition.

Let C be a ∞-category, $\xi \in C_3$ and let θ , $\varphi \in C_i$ for $i = 1,2$, then we define

$$\xi^{\theta} = \xi \circ_3 \varepsilon_1^{3-i} \theta \text{ , } {}^{\theta}\xi = \varepsilon_1^{3-i} \theta \circ_3 \xi \text{ and } {}^{\theta}\xi^{\varphi} = \varepsilon_1^{3-i} \theta \circ_3 \xi \circ_3 \varepsilon_1^{3-i} \varphi$$

where the compositions are defined. This operation satisfies the following properties:

(i) $(\xi^{\theta})^{\varphi} = \xi^{\theta\varphi}$ and ${}^{\varphi}(\xi^{\theta}) = {}^{\varphi}\xi^{\theta}$

(ii) $(\xi \circ_2 \vartheta)^{\theta} = \xi^{\theta} \circ_2 \vartheta^{\theta}$ and ${}^{\varphi}(\xi \circ_2 \vartheta) = {}^{\varphi}\xi \circ_2 {}^{\varphi}\vartheta$,

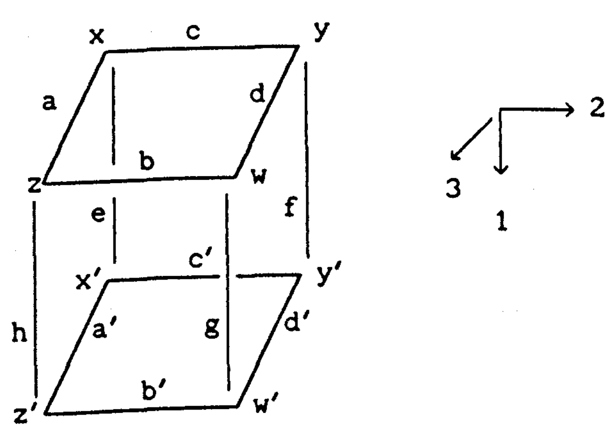
(iii) $(\xi \circ_3 \vartheta)^{\theta} = \xi \circ_3 \vartheta^{\theta}$ and ${}^{\varphi}(\xi \circ_3 \vartheta) = {}^{\varphi}\xi \circ_1 \vartheta$.

where $\vartheta \in C_3$ and $\xi \circ_2 \vartheta$ is defined.

3.2.3 Remark.

The analogous in higher dimensions of these operations has to be considerably more complicated than those dealt with above, because a line can be subdivided whereas a point cannot.

Now we want to see how the faces $\partial_1^{\alpha}\Psi x$ are composed, this will help us in evaluating the composition $\Psi(x \circ_i y)$ and in the proof of the associative and interchange laws. So let $x \in G_3$ have boundaries and vertices given by

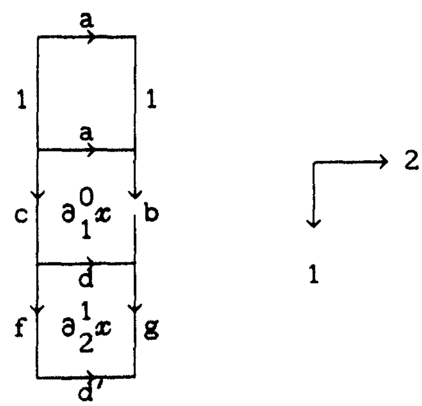


We want to see how the faces $\partial_1^\alpha \Psi x$ are formed from the faces $\partial_i^\alpha x$ when the operation Ψ is applied. This will support our claim in I-5 which asserts that $\partial_1^0 \Psi x$ is a kind of "sum" of the faces $\partial_i^\alpha x$ where $i + \alpha$ is odd, and $\partial_1^1 \Psi x$ is a kind of "sum" of the faces $\partial_i^\alpha x$ where $i + \alpha$ is even.

First we will prove this pictorially because that will help us to have a clear picture about what is going on, then we prove formally in the next proposition. We recall from I-5 that $\Psi_1 x = \psi_1 x$, $\Psi_2 x = \psi_2 \psi_1 x$ and $\Psi x = \Psi_2 \Psi_3 x = \psi_1 \psi_2 \psi_1 x$.

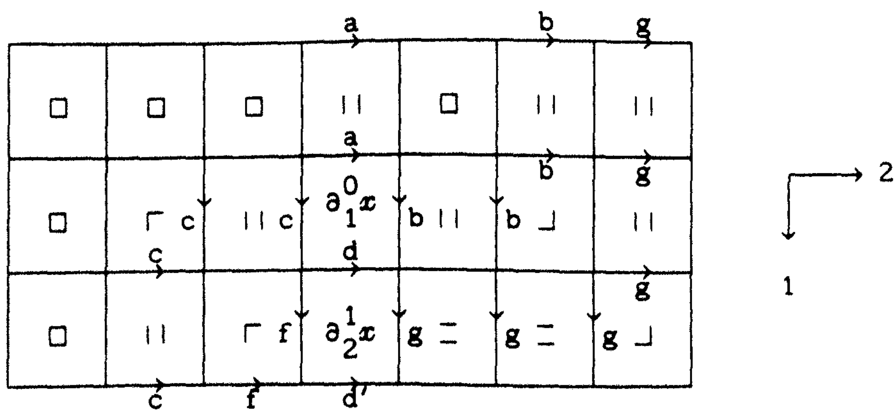
The following diagrams represent partially the faces $\partial_1^0 \psi_1$, $\partial_1^0 \psi_2 \psi_1$ and $\partial_1^0 \psi_1 \psi_2 \psi_1$ respectively,

$\partial_1^0 \Psi_1 x =$



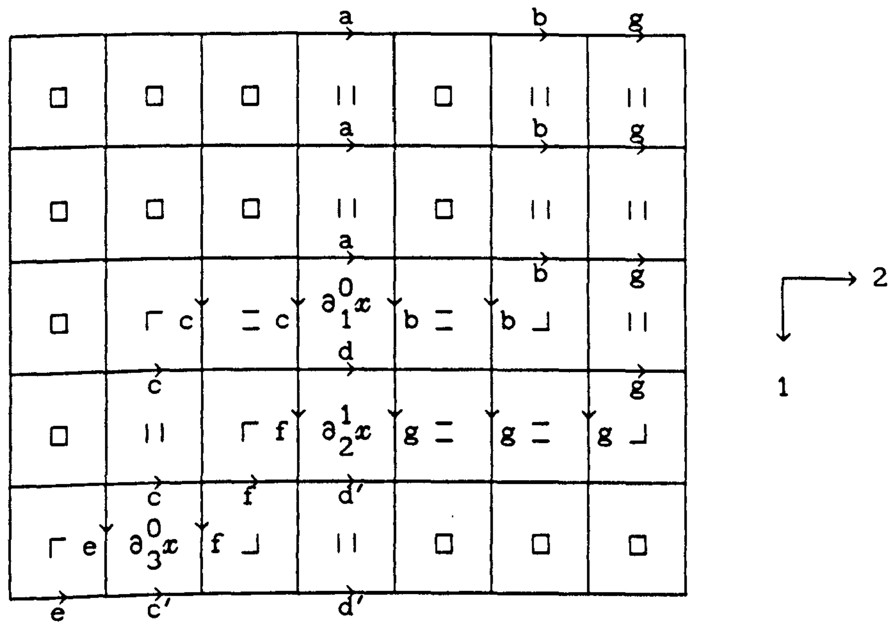
(Figure F-3.2.1)

$\partial_1^0 \Psi_2 x =$



(Figure F-3.2.2)

$\partial_1^0 \Psi x =$



(Figure F-3.2.3)

From this digram we notice that

$$(\text{row } 1 \circ_1 \text{row } 2 \circ_1 \text{row } 3) = \text{row } 3 \ ,$$

and since $\ulcorner_f \circ_1 \lrcorner_f = \, =_f$, then the above diagram can be reduced to

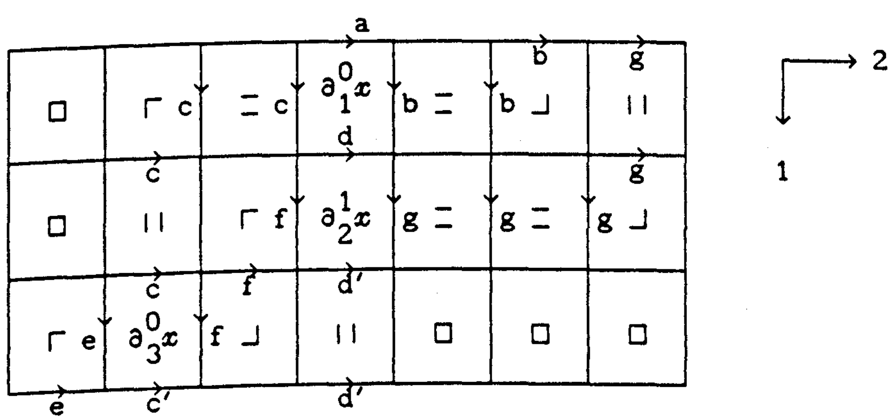


Figure (F-3.2.4)

Now we give the face $\partial_1^0 x$ in terms of elements of C_2 , i.e. the folded faces $\Psi\partial_i^\alpha$ for $i + \alpha$ is odd, and thin elements of the form ε_1 . To make it clear we can rearrange diagram F-3.2.4 to get

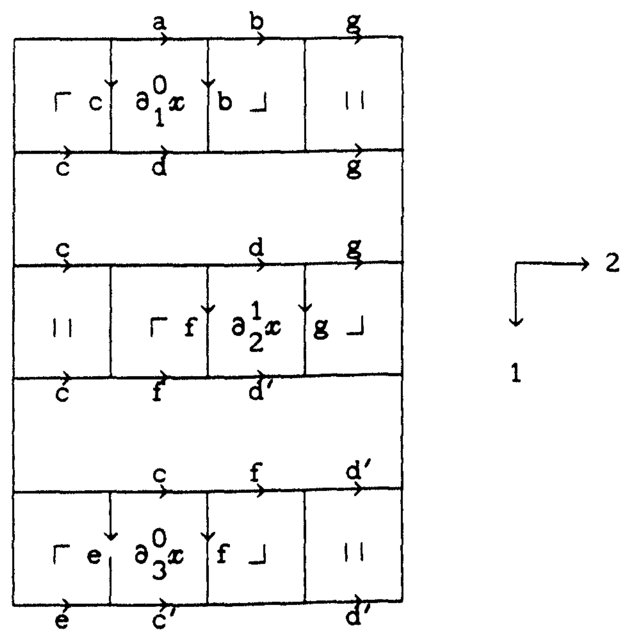


Figure (F-3.2.5)

and so $\partial_1^0 \Psi x$ can be given in the following formula

$$\partial_1^0 \Psi x = (\Psi\partial_1^0 x)^g \circ_1 {}^c(\Psi\partial_2^1 x) \circ_1 (\Psi\partial_3^0 x)^{d'} , \qquad (3.2.1)$$

Similarly we can follow the same steps and find that $\partial_1^1 \Psi x$ can be represented partially by

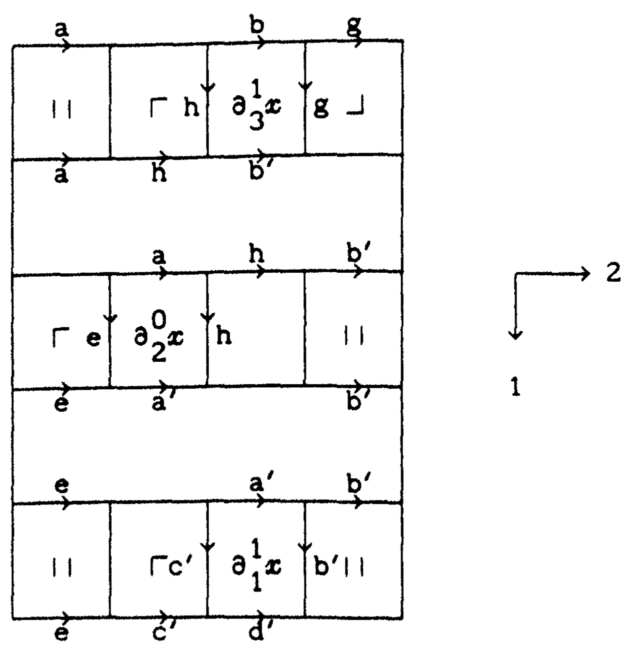


Figure (F-3.2.5)

and by formula $\partial_1^1\Psi x$ is given by

$$\partial_1^1\Psi x = {}^{a}(\Psi\partial_3^1 x) \circ_1 (\Psi\partial_2^0 x)^{d'} \circ_1 {}^{e}(\Psi\partial_1^1 x) \qquad (3.2.2)$$

The following diagrams represent the final reduced form of diagrams of $\partial_1^0\Psi x$ and $\partial_1^1\Psi x$ respectively.

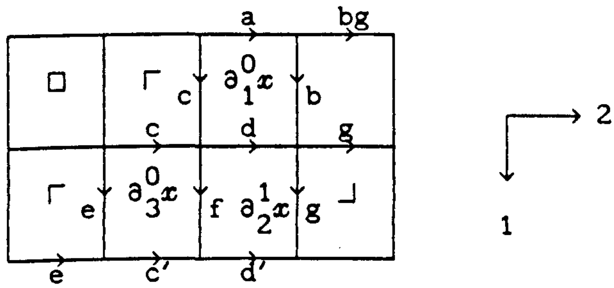


Figure (F-3.2.6)

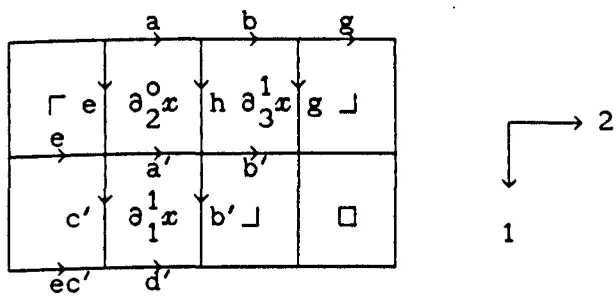


Figure (F-3.2.7)

and by formula $\partial_1^1\Psi x$ can be given as follows

$$\partial_1^1\Psi x = (\varepsilon_2 a \circ_2 \partial_1^0\Psi\varepsilon_1\partial_3^1 x) \circ_1 (\partial_1^0\Psi\varepsilon_1\partial_2^0 x \circ_2 \varepsilon_1 d') \circ_1 (\varepsilon_2 e \circ_2 \partial_1^0\Psi\varepsilon_1\partial_1^1 x) .$$

$$= {}^{a}(\Psi\partial_3^1 x) \circ_1 (\Psi\partial_2^0 x)^{b'} \circ_1 {}^{e}(\Psi\partial_1^1 x) . \qquad (3.2.2)$$

(we shall call the formulae (3.1.1) and (3.1.2) *the folded face formula* for $\partial_1^0\Psi x$ and $\partial_1^0\Psi x$ respectively).

In the following proposition we give formal proof for (3.2.1) and (3.2.2). This proof shows that the face $\partial_1^0\Psi x$ is a composite of the odd faces and $\partial_1^1\Psi x$ is a composite of the even faces of x . It also shows how complicated the situation of the general case will be.

3.2.4 Proposition.

Let $x \in G_3$, *then*

(i) $\partial_1^0 \Psi x = (\Psi\partial_1^0 x)^a \circ_1 {}^b(\Psi\partial_2^1 x) \circ_1 (\Psi\partial_3^0 x)^c$

(ii) $\partial_1^1 \Psi x = {}^d(\Psi\partial_3^1 x) \circ_1 (\Psi\partial_2^0 x)^e \circ_1 {}^f(\Psi\partial_1^1 x)$,

where $a = \partial_2^1\partial_2^1 x$, $b = \partial_2^0\partial_1^0 x$, $c = \partial_1^1\partial_1^1 x$, $d = \partial_1^0\partial_1^0 x$, $e = \partial_2^1\partial_1^1 x$ and $f = \partial_2^0\partial_2^0 x$.

Proof.

(i)

$$\partial_1^0\Psi x = \partial_1^0\Psi_2\Psi_3 x = \partial_1^0\psi_1\psi_2\psi_1 x = \partial_1^0\psi_2\psi_1 x \circ_1 \partial_2^1\psi_2\psi_1 x \quad \text{by (2.5.1)(ii)}$$

$$= \psi_1\partial_1^0\psi_1 x \circ_1 (\partial_3^0\psi_1 x \circ_2 \partial_2^1\psi_1 x) \quad \text{by (2.5.1)(i,iii)}$$

$$= \psi_1(\partial_1^0 x \circ_1 \partial_2^1 x) \circ_1 (\partial_3^0\psi_1 x \circ_2 \partial_2^1\psi_1 x) \quad \text{by (2.4.5)(i,ii,iv)}$$

$$= (\psi_1\partial_1^0 x \circ_2 \varepsilon_1\partial_2^1\partial_2^1 x) \circ_1 (\varepsilon_1\partial_2^0\partial_1^0 x \circ_2 \psi_1\partial_2^1 x) \circ_1 (\psi_1\partial_3^0 x \circ_2 \varepsilon_1\partial_1^1\partial_1^1 x)$$

$$= (\psi_1\partial_1^0 x \circ_2 \varepsilon_1 a) \circ_1 (\varepsilon_1 b \circ_2 \psi_1\partial_2^1 x) \circ_1 (\psi_1\partial_3^0 x \circ_2 \varepsilon_1 c) \quad \text{by (2.5.5)}$$

$$= (\Psi\partial_1^0 x)^a \circ_1 {}^b(\Psi\partial_2^1 x) \circ_1 (\Psi\partial_3^0 x)^c$$

(ii)

$$\partial_1^1\Psi x = \partial_1^1\Psi_2\Psi_3 x = \partial_1^1\psi_1\psi_2\psi_1 x = \partial_2^0\psi_2\psi_1 x \circ_1 \partial_1^1\psi_2\psi_1 x \quad \text{by (2.5.1)(iii)}$$

$$= (\partial_2^0\psi_1 x \circ_2 \partial_3^1\psi_1 x) \circ_1 \psi_1\partial_1^1\psi_1 x \quad \text{by (2.5.1)(i,ii)}$$

$$= (\varepsilon_1\partial_1^0\partial_1^0 x \circ_2 \psi_1\partial_3^1 x) \circ_1 \psi_1(\partial_2^0 x \circ_1 \partial_1^1 x) \quad \text{by (2.5.1)(i,iii,iv)}$$

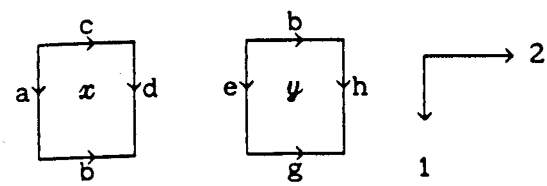
$$= (\varepsilon_1 d \circ_2 \psi_1\partial_3^1 x) \circ_1 (\psi_1\partial_2^0 x \circ_2 \varepsilon_1\partial_2^1\partial_1^1 x) \circ_1 (\varepsilon_1\partial_2^0\partial_2^0 x \circ_2 \psi_1\partial_1^1 x)$$

$$= {}^d(\Psi\partial_3^1 x) \circ_1 (\Psi\partial_2^0 x)^e \circ_1 {}^f(\Psi\partial_1^1 x) \ . \square$$

Now we move step further towards finding an evaluation of the compositions $\Psi(x \circ_i y)$ using the folded face formula of the faces ∂_1^0 and ∂_1^1 of Ψx and Ψy . We start by considering the case where $i = 1$ and study it in details to get intuition about the rest of the cases and possibly about the general case.

First we will have a quick look at the compositions $\Psi(x \circ_i y)$ in the case of dimension 2 , to see the analogy between the two cases and to get some light for our case and possibly for the general case. We first consider $\Psi(x \circ_1 y)$.

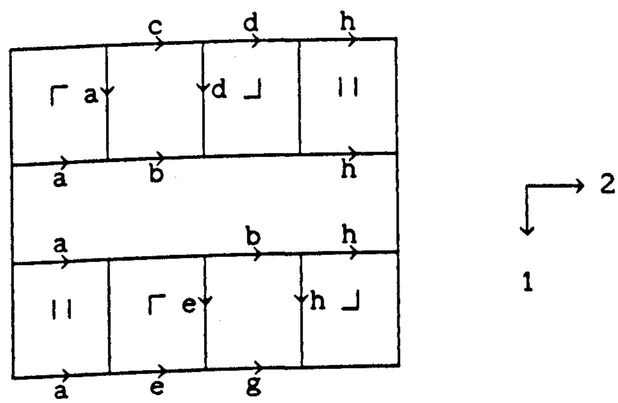
Let x , $y \in G_2$ with edges given by



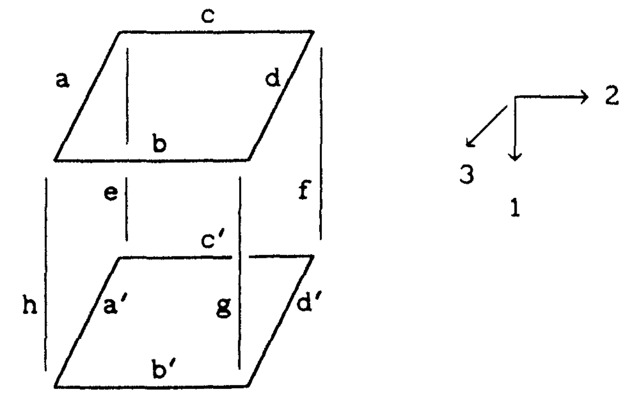
Then

$$\Psi(x \circ_1 y) = (\Psi x \circ_2 \varepsilon_1 h) \circ_1 (\varepsilon_1 a \circ_2 \Psi y)$$

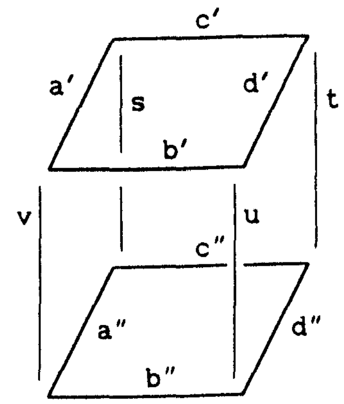
and pictorially $\Psi(x \circ_1 y)$ can be visualized by



In dimension 3 we have a similar situation in evaluating the compositions $\Psi(x \circ_i y)$ but in more complicated way. To explain the situation and make more clear let $x \in G_3$ be given with edges and boundaries given by



and $\psi \in G_3$ have edges and boundaries given by



then folded face formulae of ∂_1^α for Ψx , $\Psi\psi$ and $\Psi(x \circ_1 \psi)$ are:

$$\partial_1^0\Psi\psi = (\Psi\partial_1^0\psi)^u \circ_1 {}^{c'}(\Psi\partial_2^1\psi) \circ_1 (\Psi\partial_3^0\psi)^{d''} ,$$

$$\partial_1^1\Psi\psi = {}^{a'}(\Psi\partial_3^1\psi) \circ_1 (\Psi\partial_2^0\psi)^{b''} \circ_1 {}^{s}(\Psi\partial_1^1\psi) ,$$

$$\partial_1^0\Psi(x \circ_1 \psi) = (\Psi\partial_1^0x)^{gu} \circ_1 {}^{c}(\Psi\partial_2^1(x \circ_1 \psi)) \circ_1 (\Psi\partial_3^0(x \circ_1 \psi))^{d''}$$

$$\partial_1^1\Psi(x \circ_1 \psi) = {}^{a}(\Psi\partial_3^1(x \circ_1 \psi)) \circ_1 (\Psi\partial_2^0(x \circ_1 \psi))^{b''} \circ_1 {}^{es}(\Psi\partial_1^1\psi) .$$

First, since the faces $\partial_1^1\Psi x$ and $\partial_1^0\Psi\psi$ have one face in common, namely $\partial_3^0 x$ ($= \partial_3^1\psi$), then the order of the composition $\Psi(x \circ_1 \psi)$ starts with Ψx . Second, by examining the formulae of Ψx , $\Psi\psi$ and

$\Psi(x \circ_1 y)$, we notice the following:

(i) $\partial_1^0\Psi(x \circ_1 y)$ is composed of five squares, three are those which composed $\partial_1^0\Psi x$ and the remaining two are $\partial_2^1 y$ and $\partial_3^0 y$ of $\partial_1^0\Psi y$,

(ii) $\partial_1^0\partial_1^0\Psi(x \circ_1 y) = \partial_1^0\partial_1^1\Psi(x \circ_1 y) = abgu$, and

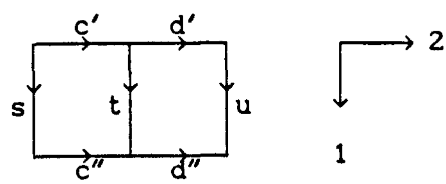
$\partial_1^1\partial_1^0\Psi(x \circ_1 y) = \partial_1^1\partial_1^1\Psi(x \circ_1 y) = es''c''d''$,

(iii) $\partial_1^1\Psi(x \circ_1 y)$ is composed of five squares, three are those which composed $\partial_1^1\Psi y$ and the remaining two are $\partial_2^0 x$ and $\partial_3^1 x$ of $\partial_1^1\Psi x$,

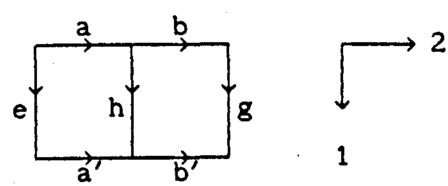
(iv) $\partial_1^0\partial_1^0\Psi x = \partial_1^0\partial_1^1\Psi x = abg$ and $\partial_1^1\partial_1^0\Psi x = \partial_1^1\partial_1^1\Psi x = ec'd'$,

(v) $\partial_1^0\partial_1^0\Psi y = \partial_1^0\partial_1^1\Psi y = a'b'u$, and $\partial_1^1\partial_1^0\Psi x = \partial_1^1\partial_1^1\Psi x = sc''d''$.

(vi) The composed face $(\partial_3^0 y \circ_2 \partial_2^1 y$) has boundaries given by



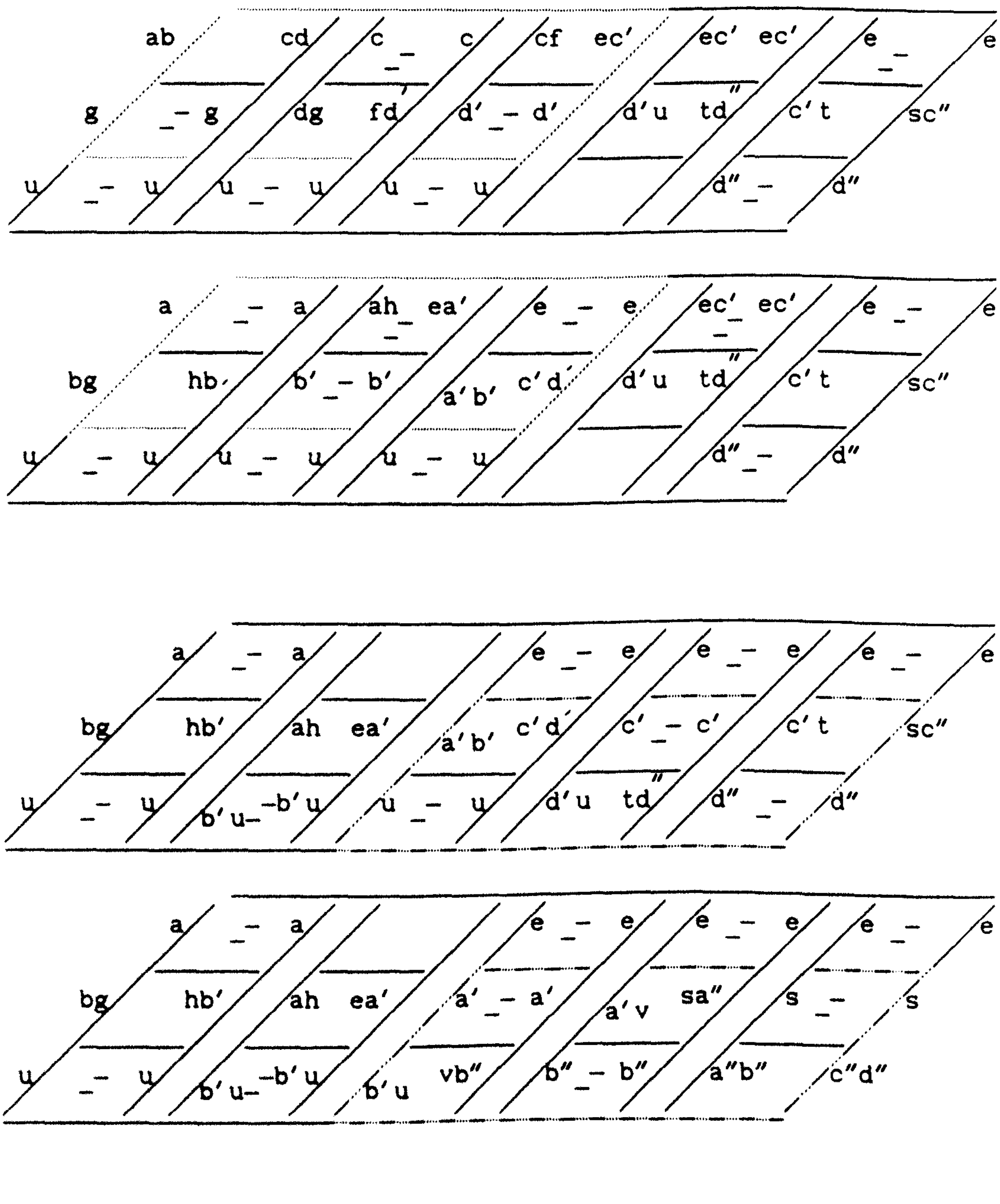
(vii) The composed face $(\partial_2^0 x \circ_2 \partial_3^1 x$) has boundaries given by



From the above discussion we can attach $\Psi\varepsilon_1\partial_2^0 x$ and $\Psi\varepsilon_1\partial_3^1 x$ to Ψx from the right to get $\partial_1^0\Psi x$ matching $\partial_1^0\Psi(x \circ_1 y)$, the resulting element is represented by $\Psi x \circ_2 \Psi\varepsilon_1\partial_2^1 y \circ_2 \Psi\varepsilon_1\partial_3^0 y$.

Similarly we attach $\Psi\varepsilon_1\partial_3^1 x \circ_2 \Psi\varepsilon_1\partial_2^0 x$ to $\Psi\psi$ from the left and get $\partial_1^0\Psi\psi$ matching $\partial_1^1\Psi(x \circ_1 \psi)$, the resulting element is represented by $\Psi\varepsilon_1\partial_3^1 x \circ_2 \Psi\varepsilon_1\partial_2^0 x \circ_2 \Psi\psi$.

The following digram makes it clearer

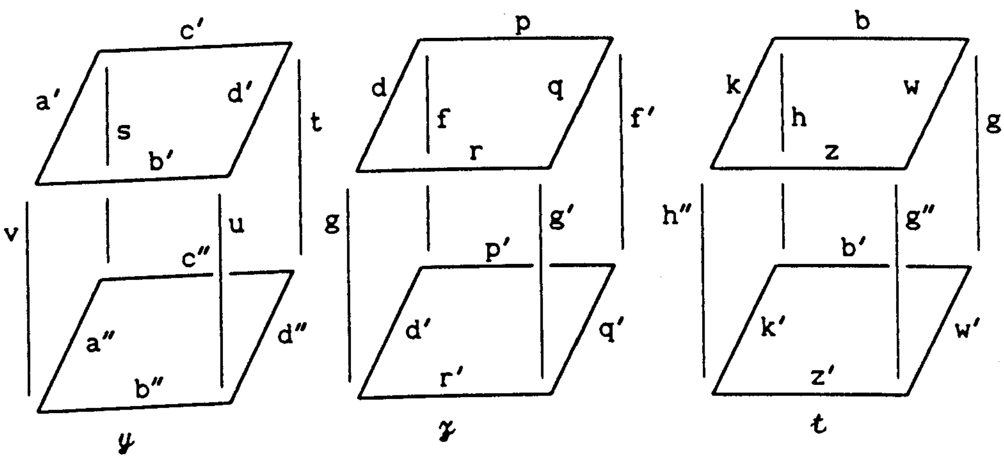


where the cube doted by ····· represents Ψx and the cube doted by – – represents $\Psi\psi$.

Unfortunately this evaluation gives a little light to the general case. Any how the following proposition evaluates the compositions $\Psi(x \circ_i y)$, for $i = 1,2,3$. The proof of this proposition is complicated and involves a lot of algebra, we put it in a separate appendix.

3.2.5 Proposition

Let $x \in G_3$ *with edges given as in* 3.2.4 *and let* y , z , $t \in G_3$ *with edges and boundaries given in the following diagrams*



such that $x \circ_1 y$, $x \circ_2 z$ and $x \circ_3 t$ *are well defined then*

$$(i)\ \Psi(x \circ_1 y) = [(\Psi x)^{u} \circ_2 {}^{ec'}(\Psi \varepsilon_1 \partial_2^1 y) \circ_2 {}^{e}(\Psi \varepsilon_1 \partial_3^0 y)^{d''}] \circ_1 [{}^{a}(\Psi \varepsilon_1 \partial_3^1 x)^{u} \circ_2 (\Psi \varepsilon_1 \partial_2^0 x)^{b'u} \circ_2 {}^{e}(\Psi y)] ,$$

$$(ii)\ \Psi(x \circ_2 z) = [(\Psi \varepsilon_1 \partial_1^0 x)^{rg'} \circ_2 {}^{c}(\Psi z) \circ_2 (\Psi \varepsilon_1 \partial_3^0 x)^{p'q'}] \circ_1 [{}^{ab}(\Psi \varepsilon_1 \partial_3^1 z) \circ_2 (\Psi x)^{r'} \circ_2 {}^{ec'}(\Psi \varepsilon_1 \partial_1^1 z)]$$

$$(iii)\ \Psi(x \circ_3 t) = [{}^{a}(\Psi \varepsilon_1 \partial_1^0 t)^{g''} \circ_2 {}^{ab}(\Psi \varepsilon_1 \partial_2^1 t) \circ_2 (\Psi x)^{w'}] \circ_1 [{}^{a}(\Psi t) \circ_2 (\Psi \varepsilon_1 \partial_2^0 x)^{b'w'} \circ_2 {}^{e}(\Psi \varepsilon_1 \partial_1^1 x)^{w'}] .$$

Proof.

The proof of this proposition is given in appendix II .

We now give proofs of Lemma 2.6.6 and Proposition 2.6.7 stated in II-6.

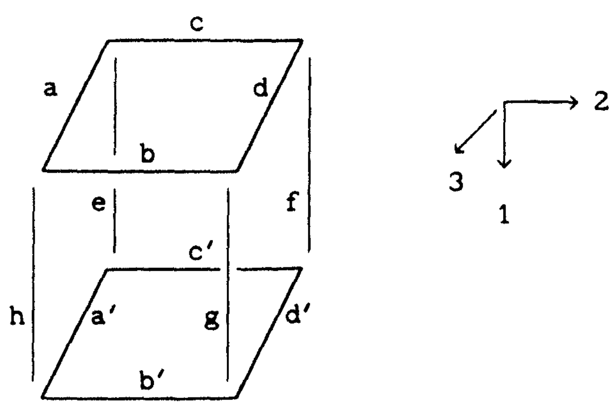
First we recall Lemma 2.6.6 ,

$$\Psi\psi_i : \Psi : G_3 \longrightarrow G_3 , \quad i = 1,2 .$$

Proof.

(i) the case for $i = 1$.

Let $x \in G_3$ have edges given by



then

$$\Psi\psi_1 x = \Psi(\Gamma'_1\partial_2^0 x \circ_2 x \circ_2 \Gamma_1\partial_2^1 x) \qquad \text{by definition of } \psi_1 x$$

$$= [(\Psi\varepsilon_1\partial_1^0(\Gamma'_1\partial_2^0 x \circ_2 x)^g \circ_2 {}^c(\Psi\Gamma_1\partial_2^1 x) \circ_2 (\Psi\varepsilon_1\partial_3^0(\Gamma'_1\partial_2^0 x \circ_2 x)^g)^{d'}] \circ_1$$

$$[{}^{ab}(\Psi\varepsilon_1\partial_3^1\Gamma_1\partial_2^1 x) \circ_2 (\Psi(\Gamma'_1\partial_2^0 x \circ_2 x)) \circ_2 {}^e(\Psi\varepsilon_1\partial_1^1\Gamma_1\partial_2^1 x)]$$

by (3.2.5)(ii)

$$= [(\Psi\varepsilon_1\partial_1^0\Gamma'_1\partial_2^0 x)^{bg} \circ_2 (\Psi\varepsilon_1\partial_1^0 x)^g \circ_2 {}^c(\Psi\Gamma_1\partial_2^1 x) \circ_2 (\Psi\varepsilon_1\partial_3^0 x)^{d'} \circ_2$$

$$(\Psi\varepsilon_1\partial_3^0\Gamma'_1\partial_2^0 x)^{c'd'}] \circ_1 [{}^{ab}(\Psi\varepsilon_1\partial_3^1\Gamma_1\partial_2^1 x) \circ_2 (\Psi\varepsilon_1\partial_1^0\Gamma'_1\partial_2^0 x)^{bg} \circ_2 (\Psi x) \circ_2$$

$$ {}^{ec'}(\Psi\varepsilon_1\partial_1^1\Gamma_1\partial_2^1 x) \circ_2 (\Psi\varepsilon_1\partial_3^0\Gamma_1'\partial_2^0 x)^{c'd'}] \circ_1 [{}^{a}(\Psi\varepsilon_1\partial_3^1(x \circ_2 \Gamma_1\partial_2^1 x) \circ_2 $$

$$ (\Psi\Gamma_1'\partial_2^0 x)^{b'} \circ_2 {}^{e}(\Psi\varepsilon_1\partial_1^1(x \circ_2 \Gamma_1\partial_2^1 x)] \qquad \text{by (2.5.6)} $$

$= [X] \circ_1 [Y] \circ_1 [Z]$, say.

We have to prove that $[Y] = \Psi x$ and $[X]$, $[Z]$ are identities for $\circ_1$. So

$$ [Y] = [{}^{ab}(\Psi\varepsilon_1\partial_3^1\Gamma_1\partial_2^1 x) \circ_2 (\Psi\varepsilon_1\partial_1^0\Gamma_1'\partial_2^0 x)^{bg} \circ_2 (\Psi x) \circ_2 $$

$$ {}^{ec'}(\Psi\varepsilon_1\partial_1^1\Gamma_1\partial_2^1 x) \circ_2 (\Psi\varepsilon_1\partial_3^0\Gamma_1'\partial_2^0 x)^{c'd'}] $$

$$ = [{}^{ab}(\Psi\varepsilon_1\Gamma_1\partial_2^1\partial_2^1 x) \circ_2 (\Psi\varepsilon_1\varepsilon_1\partial_1^0\partial_2^0 x)^{bg} \circ_2 (\Psi x) \circ_2 {}^{ec'}(\Psi\varepsilon_1\varepsilon_1\partial_1^1\partial_2^1 x) \circ_2 $$

$$ (\Psi\varepsilon_1\Gamma_1'\partial_2^0\partial_2^0 x)^{c'd'}] \qquad \text{by (2.1.1)(vii),(viii)} $$

$$ = [{}^{ab}(\varepsilon_1\varepsilon_1\partial_2^1\partial_2^1 x) \circ_2 (\varepsilon_1\varepsilon_1\partial_1^0\partial_2^0 x)^{bg} \circ_2 (\Psi x) \circ_2 {}^{ec'}(\varepsilon_1\varepsilon_1\partial_1^1\partial_2^1 x) \circ_2 $$

$$ (\varepsilon_1\varepsilon_1\partial_2^0\partial_2^0 x)^{c'd'}] \qquad \text{by (2.6.4)(i) and (2.6.4)} $$

$$ = [{}^{ab}(\varepsilon_1^2\partial_2^1\partial_2^1 x) \circ_2 (\varepsilon_1^2\partial_1^0\partial_2^0 x)^{bg} \circ_2 (\Psi x) \circ_2 {}^{ec'}(\varepsilon_1^2\partial_1^1\partial_2^1 x) \circ_2 $$

$$ (\varepsilon_1^2\partial_2^0\partial_2^0 x)^{c'd'}] $$

$$ = [{}^{ab}(\varepsilon_1^2 g) \circ_2 (\varepsilon_1^2 a)^{bg} \circ_2 (\Psi x) \circ_2 {}^{ec'}(\varepsilon_1^2 d') \circ_2 (\varepsilon_1^2 e)^{c'd'}] $$

$$ = [(\varepsilon_1^2 abg) \circ_2 (\varepsilon_1^2 abg) \circ_2 (\Psi x) \circ_2 (\varepsilon_1^2 ec'd') \circ_2 (\varepsilon_1^2 ec'd')] $$

$$ = [(\varepsilon_1^2 abg) \circ_2 (\Psi x) \circ_2 (\varepsilon_1^2 ec'd')] $$

$$ = \Psi x . $$

To prove that $[X]$ and $[Z]$ are identities for $\circ_1$ we have to show that $[X] = \varepsilon_1[\partial_1^0\Psi x]$ and $[Z] = \varepsilon_1[\partial_1^1\Psi x]$, which is true since

$$ [X] = [(\Psi\varepsilon_1\partial_1^0\Gamma_1'\partial_2^0 x)^{bg} \circ_2 (\Psi\varepsilon_1\partial_1^0 x)^{g} \circ_2 {}^{c}(\Psi\varepsilon_1\Gamma_1\partial_2^1 x) \circ_2 (\Psi\varepsilon_1\partial_3^0 x)^{d'} \circ_2 $$

$(\Psi\varepsilon_1\partial_3^0\Gamma_1'\partial_2^0 x)^{c'd'}]$

$= [(\varepsilon_1\Psi\varepsilon_1\partial_1^0\partial_2^1 x)^{bg} \circ_2 (\varepsilon_1\Psi\partial_1^0 x)^g \circ_2 {}^c(\varepsilon_1\Psi\Gamma_1\partial_2^1 x) \circ_2 (\varepsilon_1\Psi\partial_3^0 x)^{d'} \circ_2$

$(\varepsilon_1\Psi\Gamma_1'\partial_2^0\partial_2^0 x)^{c'd'}]$ by (2.6.4)(i) and (2.1.1)(vi)

$= \varepsilon_1[(\varepsilon_1\partial_1^0\partial_2^1 x)^{bg} \circ_1 (\Psi\partial_1^0 x)^g \circ_1 {}^c(\varepsilon_1\partial_2^1 x) \circ_1 (\Psi\partial_3^0 x)^{d'} \circ_1$

$(\varepsilon_1\partial_2^0\partial_2^0 x)^{c'd'}]$ by (2.5.3)(i) and (2.6.4)(i)

$= \varepsilon_1[(\varepsilon_1 a)^{bg} \circ_1 (\Psi\partial_1^0 x)^g \circ_1 {}^c(\varepsilon_1\partial_2^1 x) \circ_1 (\Psi\partial_3^0 x)^{d'} \circ_1 (\varepsilon_1 e)^{c'd'}]$,

$= \varepsilon_1[(\Psi\partial_1^0 x)^g \circ_1 {}^c(\varepsilon_1\partial_2^1 x) \circ_1 (\Psi\partial_3^0 x)^{d'}]$, by (2.2.1)(i)

(since

$\partial_1^1(\varepsilon_1 a)^{bg} = abg = \partial_1^0(\Psi\partial_1^0 x)^g$ and $\partial_1^1(\Psi\partial_3^0 x)^{d'} = ec'd' = \partial_1^0(\varepsilon_1 e)^{c'd'}$).

$= \varepsilon_1[\partial_1^0\Psi x] = \varepsilon_1[\partial_1^0 X]$, by (3.2.4)(i)

i.e. $[Y] \circ_1 [X] = [X]$.

Similarly we can prove that $[X] \circ_1 [Z] = [X]$. Thus $\Psi\psi_1 x = \Psi x$.

(ii) for $i = 2$, we apply a similar argument and get $\Psi\psi_2 x = \Psi x$.

Thus $\Psi\psi_i x = \Psi x$, for $i = 1,2$.□

Proposition 2.6.7.

Let $x \in G_n$ $(1 \le n \le 3)$. Then x is thin if and only if $\Psi x = 1$.

Proof.

By proposition (2.6.4), $\Psi\varepsilon_i y = 1$, $\Psi\Gamma_i y = 1$, and $\Psi\Gamma_i' y = 1$ for all $y \in G_2$ and $i = 1,2$. It follows from Proposition (3.2.5) that $\Psi x = 1$ whenever x is thin. To see the converse, we recall the definition

$$\psi_i x = [\Gamma_i'\partial_i^0 x \ , \ x \ , \ \Gamma_i\partial_i^1 x]_{i+1}$$

which can be rewritten as

$$x = \left[\begin{array}{c:c:c} \varepsilon_i\varepsilon_{i+1}\partial^0_{i+1}\partial^0_i z & \varepsilon_i\partial^0_i z & \Gamma'_i\partial^1_{i+1}z \\ \hline \Gamma'_i\partial^0_{i+1}z & z & \Gamma_i\partial^1_{i+1}z \\ \hline \Gamma_i\partial^0_{i+1}z & \varepsilon_i\partial^1_i z & \varepsilon_i\varepsilon_{i+1}\partial^1_{i+1}\partial^1_i z \end{array}\right]_{i+1}$$

These two equations show that $\psi_i x$ is thin if and only if x is thin. Hence Ψx is thin if and only if x is thin. In particular, if $\Psi x = 1$ then Ψx is thin, so x is also thin. □

3.3 The functor $\lambda : 3\text{-}\mathcal{C} \longrightarrow 3\text{-}\mathcal{G}$.

In this section we start to construct a triple category from a 3-category by using the folding operation.

In [Mo-1] G. Mosa has constructed a 3-tuple algebroid A_3 from a 3-truncated crossed complex $\underline{M}^3$. He defined the appropriate algebraic structure on A_3 but he did not prove that this structure is indeed a 3-tuple algebroid. He just refers to the proof for the case of dimension two which he proved earlier. In fact the proof for dimension 3 is much more complicated and involves a lot of information and complicated algebra.

Given a triple category G with associated 3-category $C = \gamma G$, and given $\underline{x} \in \Box G_2$, $\xi \in C_2$ with $d^\alpha_1\xi = d^\alpha_1\Psi\underline{x}$, we write $\langle\underline{x},\xi\rangle$ for the unique element $x \in G_3$ such that $\underline{\partial} x = \underline{x}$ and $\Psi x = \xi$. Proposition (3.3.1) shows that compositions in G are also determined by γG .

3.3.1 Proposition.

Let $x = \langle\underline{x} , \xi\rangle$, $y = \langle\underline{y} , \eta\rangle$, $z = \langle\underline{z} , \zeta\rangle$ *and* $t = \langle\underline{t} , \tau\rangle$ *in* G_3 *and let* $\underline{x}$, $\underline{y}$, $\underline{z}$ *and* $\underline{t}$ *have edges given as in* 3.2.5 *such that* *the compositions* $\underline{x} \circ_1 \underline{y}$, $\underline{x} \circ_2 \underline{z}$ *and* $\underline{x} \circ_3 \underline{t}$ *are well defined then*

$$(i)\ x \circ_1 y = \langle \underline{x} \circ_1 \underline{y} ; [(\xi)^{u} \circ_2 {}^{ec'}(s_1\sigma_2 y_2^1) \circ_2 {}^{e}(s_1\sigma_2 y_3^0)^{d''}] \circ_1 [{}^{a}(s_1\sigma_2 x_3^1)^{u} \circ_2 (s_1\sigma_2 x_2^0)^{b'u} \circ_2 {}^{e}(\eta)]\rangle$$

$$(ii)\ x \circ_2 z = \langle \underline{x} \circ_2 \underline{z} ; [(s_1\sigma_2 x_1^0)^{rg'} \circ_2 {}^{c}(\zeta) \circ_2 (s_1\sigma_2 x_3^0)^{p'q'}] \circ_1 [{}^{ab}(\Psi\varepsilon_1 z_3^1) \circ_2 (\xi)^{r'} \circ_2 {}^{ec'}(s_1\sigma_2 z_1^1)]\rangle$$

$$(iii)\ x \circ_3 t = \langle \underline{x} \circ_3 \underline{y} ; [{}^{a}(s_1\sigma_2 t_1^0)^{g''} \circ_2 {}^{ab}(s_1\sigma_2 t_2^1) \circ_2 (\xi)^{w'}] \circ_1 [{}^{a}(\tau) \circ (s_1\sigma_2 x_2^0)^{b'w'} \circ {}^{e}(s_1\sigma_2 x_1^1)^{w'}]\rangle .$$

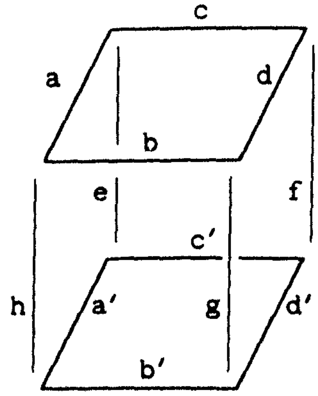
Proof.

This follows from Proposition 3.2.5 and the rule $\underline{\partial}(\underline{x} \circ_i \underline{y}) = \underline{\partial}\underline{x} \circ_i \underline{\partial}\underline{y}$.□

Now let $C = (C_3, C_2, C_1, C_0)$ be a 3-category and let $G_0 = C_0$, $G_1 = C_1$. In [S-1] C.B.Spencer has constructed a double category $G = \lambda C$ from a 2-category C and isomorphism $\sigma_2 : \gamma G_2 \longrightarrow C_2$. Then $(\square G_2, G_2, G_1, G_0)$ is a triple category and we define

$$G_3 = \{\langle \underline{x} , \xi\rangle : \underline{x} \in \square G_2 ,\ \xi \in C_3 \text{ such that } \sigma_2 \underline{\partial}\Psi\underline{x} = \underline{\partial}\xi\} ,$$

so by this definition if $\underline{x}$ have edges and boundaries given by



then the faces ∂_1^α of ξ are given by the following formulae

$$\partial_1^0\xi = (\sigma_2 x_1^0)^{g} \circ_1 {}^{c}(\sigma_2 x_2^1) \circ_1 (\sigma_2 x_3^0)^{d'}$$

$$\partial_1^1\xi = {}^{a}(\sigma_2 x_3^1) \circ_1 (\sigma_2 x_2^0)^{b'} \circ_1 {}^{e}(\sigma_2 x_1^1) .$$

For $y \in G_2$, let $\varepsilon_i y = (\varepsilon_i y, 1)$, where ε_i is defined by (2.7.a)(i). Then $\varepsilon_i y \in G_3$, since $\Psi\varepsilon_i y = 1$ by (2.6.4). The maps $\varepsilon_i : G_2 \longrightarrow G_3$, with the obvious face maps $\partial_i^\alpha : G_3 \longrightarrow G_2$ defined by $\partial_i^\alpha(\underline{x}, \xi) = x_i^\alpha$, give $(G_3, \ldots, G_0)$ the structure of an 3-cubical complex. Similarly one can defined connections $\Gamma_i, \Gamma_i' : G_2 \longrightarrow G_3$ by $\Gamma_i y = (\Gamma_i y, 1)$ and $\Gamma_i' y = (\Gamma_i' y, 1)$ It is clear by (2.6.4) that $\Gamma_i y$, $\Gamma_i' y \in G$.

We now define operations $\circ_i$, for $i = 1,2,3$, as follows. For $(\underline{x}, \xi)$, $(\underline{y}, \eta)$, $(\underline{z}, \zeta)$, $(\underline{t}, \tau) \in G_3$ with $\underline{x} \circ_1 \underline{y}$, $\underline{x} \circ_2 \underline{z}$ and $\underline{x} \circ_3 \underline{t}$ are well defined, let

$$(x \circ_1 y) = (\underline{x} \circ_1 \underline{y} ; [(\xi)^{u} \circ_2 {}^{ec'}(s_1\sigma_2 y_2^1) \circ_2 {}^{e}(s_1\sigma_2 y_3^0)^{d''}] \circ_1$$
$$[{}^{a}(s_1\sigma_2 x_3^1)^{u} \circ_2 (s_1\sigma_2 x_2^0)^{b'u} \circ_2 {}^{e}(\eta)])$$

$$(x \circ_2 z) = (\underline{x} \circ_2 \underline{z} ; [(s_1\sigma_2 x_1^0)^{rg'} \circ_2 {}^{c}(\zeta) \circ_2 (s_1\sigma_2 x_3^0)^{p'q'}] \circ_1$$
$$[{}^{ab}(\Psi\varepsilon_1 z_3^1) \circ_2 (\xi)^{r'} \circ_2 {}^{ec'}(s_1\sigma_2 z_1^1)])$$

$$(x \circ_3 t) = (\underline{x} \circ_3 \underline{t} ; [{}^{a}(s_1\sigma_2 t_1^0)^{g''} \circ_2 {}^{ab}(s_1\sigma_2 t_2^1) \circ_2 (\xi)^{w'}] \circ_1$$
$$[{}^{a}(\tau) \circ_2 (s_1\sigma_2 x_2^0)^{b'w'} \circ_2 {}^{e}(s_1\sigma_2 x_1^1)^{w'}]) .$$

We claim that $(G_3, \ldots, G_0)$ is now a triple category. Firstly, it is clear that, for $x \in G_2$, $\varepsilon_i x$ acts as an identity for $\circ_i$.

In the next two sections we prove the associative and interchange laws in G_3.

3.4 The associative law in λC_3.

In this section we prove the associative law. The key points in this proof are the next lemma and the interchange law in C_3 .

3.4.1 Lemma.

Let x , $y \in C_3$ *such that* $x \circ_3 y$ *is defined and let* $\partial_2^0 x = a$, $\partial_2^1 x = b$, $\partial_2^0 y = c$ *and* $\partial_2^0 y = d$, *then*

$$(x)^c \circ_2 {}^b(y) = {}^a(y) \circ_2 (x)^d .$$

Proof.

$$(x)^c \circ_2 {}^b(y) = (x \circ_3 \varepsilon_1^2 c) \circ_2 (\varepsilon_1^2 b \circ_3 y)$$

$$= (x \circ_2 \varepsilon_1^2 b) \circ_3 (\varepsilon_1^2 c \circ_2 y)$$

$$= x \circ_3 y \text{ , (since } \partial_2^1 x = b \text{ and } \partial_2^0 y = c \text{)}$$

$$= (\varepsilon_1^2 a \circ_2 x) \circ_3 (y \circ_2 \varepsilon_1^2 d)$$

$$\text{(since } \partial_2^0 x = a \text{ and } \partial_2^1 y = d)$$

$$= (\varepsilon_1^2 a \ {}_3\ y) \circ_2 (x \circ_3 \varepsilon_1^2 d)$$

$$= {}^a(y) \circ_2 (x)^d . \square$$

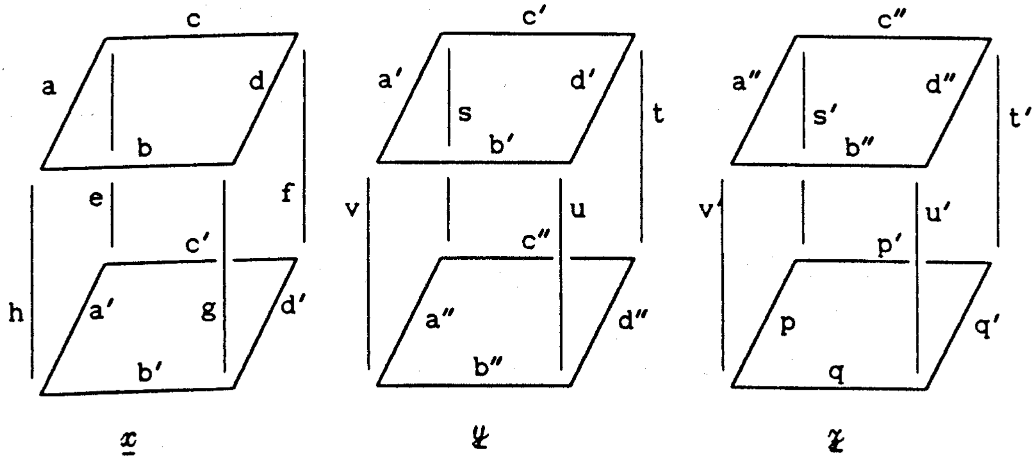
Now let $\mathbf{x} = \langle \underline{x} , \xi \rangle$, $\mathbf{y} = \langle \underline{y} , \eta \rangle$ and $\mathbf{z} = \langle \underline{z} , \zeta \rangle$ be elements of G_3 such that $\underline{x} \circ_i \underline{y}$, $\underline{y} \circ_i \underline{z}$ and $\underline{z} \circ_i \underline{t}$ are well defined (we will write $\mathbf{xy}$ for $\mathbf{x} \circ_i \mathbf{y}$ and $\xi\eta$ for $\xi \circ_i \eta$ in each case) then

$$(\mathbf{x} \circ_i (\mathbf{y} \circ_i \mathbf{z})) = (\underline{\boldsymbol{\omega}}, \boldsymbol{\omega}) , \qquad ((\mathbf{x} \circ_i \mathbf{y}) \circ_i \mathbf{z}) = (\underline{\boldsymbol{\omega}}, \boldsymbol{\omega}')$$

say, and we have to show that $\boldsymbol{\omega} = \boldsymbol{\omega}'$.

(i) The case for $i = 1$.

Let $\underline{x}$, $\underline{y}$ and $\underline{z}$ have edges given by



then

$$\omega = [(\xi\eta)^{u'} \circ_2 {}^{esc'}(s_1\sigma_2 z_2^1) \circ_2 {}^{es}(s_1\sigma_2 z_3^0)^{q'}] \circ_1$$

$$[{}^{a}(s_1\sigma_2(xy)_3^1)^{u'} \circ_2 (s_1\sigma_2(xy)_2^0)^{b''u'} \circ_2 {}^{es}(\zeta)] \qquad \text{by (3.2.5)(i)}$$

$$= [(\xi)^{uu'} \circ_2 {}^{ec'}(s_1\sigma_2 y_2^1)^{u'} \circ_2 {}^{e}(s_1\sigma_2 y_3^0)^{d''u'} \circ_2 {}^{esc'}(s_1\sigma_2 z_2^1) \circ_2$$

$${}^{es}(s_1\sigma_2 z_3^0)^{q'}] \circ_1 [{}^{a}(s_1\sigma_2 x_3^1)^{uu'} \circ_2 (s_1\sigma_2 x_2^0)^{b'uu'} \circ_2 {}^{e}(\eta)^{u'} \circ_2$$

$${}^{esc'}(s_1\sigma_2 z_2^1) \circ_2 {}^{es}(s_1\sigma_2 z_3^0)^{q'}] \qquad \text{by (3.2.5)(i)}$$

$$= [(\xi)^{uu'} \circ_2 {}^{e}(s_1\sigma_2(y_2^1 \circ_2 y_3^0))^{u'} \circ_2 {}^{es}(s_1\sigma_2(z_2^1 \circ_2 z_3^0))] \circ_1$$

$$[{}^{a}(s_1\sigma_2 x_3^1)^{uu'} \circ_2 (s_1\sigma_2 x_2^0)^{b'uu'} \circ_2 {}^{e}(\eta)^{u'} \circ_2 {}^{es}(s_1\sigma_2(z_2^1 \circ_2 z_3^0))] \circ_1$$

$$[{}^{a}(s_1\sigma_2(xy)_3^1)^{u'} \circ_2 (s_1\sigma_2(xy)_2^0)^{b''u'} \circ_2 {}^{es}(\zeta)] \qquad \text{by (2.5.6)}$$

$$= [(\xi)^{uu'} \circ_2 {}^{e}(s_1\sigma_2(y_2^1 \circ_2 y_3^0))^{u'} \circ_2 {}^{es}(s_1\sigma_2(z_2^1 \circ_2 z_3^0))] \circ_1$$

$$[{}^{a}(s_1\sigma_2 x_3^1)^{uu'} \circ_2 (s_1\sigma_2 x_2^0)^{b'uu'} \circ_2 {}^{e}(\eta)^{u'} \circ_2 {}^{es}(s_1\sigma_2(z_2^1 \circ_2 z_3^0))] \circ_1$$

$$[{}^{a}(s_1\sigma_2 x_3^1)^{uu'} \circ_2 (s_1\sigma_2 x_2^0)^{b'uu'} \circ_2 {}^{ea'}(s_1\sigma_2 y_3^1) \circ_2 {}^{e}(s_1\sigma_2 y_2^0)^{b''} \circ_2 {}^{es}(\zeta)]>$$

$= [(\xi)^{uu'} \circ_2 {}^{ec'}(s_1\sigma_2(yz)_2^1) \circ_2 {}^{e}(s_1\sigma_2(yz)_3^0)] \circ_1$

$[{}^{a}(s_1\sigma_2x_3^1)^{uu'} \circ_2 (s_1\sigma_2x_2^0)^{b'uu'} \circ_2 {}^{e}[\{(\eta)^{u'} \circ_2 {}^{s}(s_1\sigma_2(z_2^1 \circ_2 z_3^0)\} \circ_1$

$\{{}^{a'}(s_1\sigma_2y_3^1)^{u'} \circ_2 (s_1\sigma_2y_2^0)^{b''u'} \circ_2 {}^{s}(\zeta)\}]]$ by (3.2.1)(i)

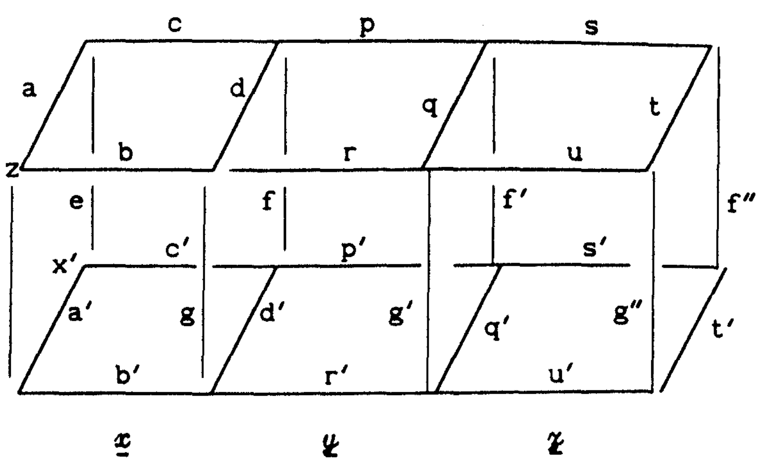
$= [(\xi)^{uu'} \circ_2 {}^{ec'}(s_1\sigma_2(yz)_2^1) \circ_2 {}^{e}(s_1\sigma_2(yz)_3^0)] \circ_1$

$[{}^{a}(s_1\sigma_2x_3^1)^{uu'} \circ_2 (s_1\sigma_2x_2^0)^{b'uu'} \circ_2 {}^{e}(\eta\zeta)]$ by (3.2.5)(i)

$= \omega' = (x \circ_1 (y \circ_1 z)).$

(ii) The case for $i = 2$.

Let $\underline{x}$, $\underline{y}$ and $\underline{z}$ have edges given by



$\omega = [(s_1\sigma_2 (xy)_1^0)^{ug''} \circ_2 {}^{cp}(\zeta) \circ_2 (s_1\sigma_2(xy)_3^0)^{s't'}] \circ_1$

$[{}^{abr}(s_1\sigma_2z_3^1) \circ_2 (\xi\eta)^{u'} \circ_2 {}^{ec'p'}(s_1\sigma_2z_1^1)]$ by (3.2.5)(ii)

$= [(s_1\sigma_2(xy)_1^0)^{ug''} \circ_2 {}^{cp}(\zeta) \circ_2 (s_1\sigma_2(xy)_3^0)^{s't'}] \circ_1$

$[{}^{abr}(s_1\sigma_2z_3^1) \circ_2 (s_1\sigma_2x_1^0)^{rg'u'} \circ_2 {}^{c}(\eta)^{u'} \circ_2 (s_1\sigma_2x_3^0)^{p'q'u'} \circ_2$

${}^{ec'p'}(s_1\sigma_2z_1^1)] \circ_1 [{}^{abr}(s_1\sigma_2z_3^1) \circ_2 {}^{ab}(s_1\sigma_2y_3^1)^{u'} \circ_2 (\xi)^{r'u'} \circ_2$

$${}^{ec'}(s_1\sigma_2 y_1^1)^{u'} \circ_2 {}^{ec'p'}(s_1\sigma_2 z_1^1)]$$ by (3.2.5)(ii) and (3.2.1)

$$= [(s_1\sigma_2(xy)_1^0)^{ug''} \circ_2 {}^{cp}(\zeta) \circ_2 (s_1\sigma_2(xy)_3^0)^{s't'}] \circ_1$$

$$[{}^{abr}(s_1\sigma_2 z_3^1) \circ_2 (s_1\sigma_2 x_1^0)^{rg'u'} \circ_2 {}^{c}(\eta)^{u'} \circ_2 (s_1\sigma_2 x_3^0)^{p'q'u'} \circ_2$$

$${}^{ec'p'}(s_1\sigma_2 z_1^1)] \circ_1 [{}^{ab}(s_1\sigma_2(yz)_3^1) \circ_2 (\xi)^{r'u'} \circ_2 {}^{ec'}(s_1\sigma_2(yz)_1^1)]$$

by (2.5.6) and (3.2.1)

$$= [(s_1\sigma_2 x_1^0)^{rug''} \circ_2 {}^{c}(s_1\sigma_2 y_1^0)^{ug''} \circ_2 {}^{cp}(\zeta) \circ_2 {}^{c}(s_1\sigma_2 y_3^0)^{s't'} \circ_2$$

$$(s_1\sigma_2 x_3^0)^{p's't'}] \circ_1 [{}^{abr}(s_1\sigma_2 z_3^1) \circ_2 (s_1\sigma_2 x_1^0)^{rg'u'} \circ_2 {}^{c}(\eta)^{u'} \circ_2$$

$${}^{ec'p'}(s_1\sigma_2 z_1^1) \circ_2 (s_1\sigma_2 x_3^0)^{p'q'u'}] \circ_1 [{}^{ab}(s_1\sigma_2(yz)_3^1) \circ_2 (\xi)^{r'u'} \circ_2$$

$${}^{ec'}(s_1\sigma_2(yz)_1^1)]$$

$$= [(s_1\sigma_2 x_1^0)^{rug''} \circ_2 {}^{c}(s_1\sigma_2 y_1^0)^{ug''} \circ_2 {}^{cp}(\zeta) \circ_2 {}^{c}(s_1\sigma_2 y_3^0)^{s't'} \circ_2$$

$$(s_1\sigma_2 x_3^0)^{p's't'}] \circ_1 [(s_1\sigma_2 x_1^0)^{rug''} \circ_2 {}^{cdr}(s_1\sigma_2 z_3^1) \circ_2 {}^{c}(\eta)^{u'} \circ_2$$

$${}^{cfp'}(s_1\sigma_2 z_1^1) \circ_2 (s_1\sigma_2 x_3^0)^{p's't'}] \circ_1 [{}^{ab}(s_1\sigma_2(yz)_3^1) \circ_2 (\xi)^{r'u'} \circ_2$$

$${}^{ec'}(s_1\sigma_2(yz)_1^1)]$$ by (3.4.1)

$$= [(s_1\sigma_2 x_1^0)^{rug''} \circ_2 {}^{c}\{(s_1\sigma_2 y_1^0)^{ug''} \circ_2 {}^{p}(\zeta) \circ_2 (s_1\sigma_2 y_3^0)^{s't'}\} \circ_1$$

$${}^{c}\{{}^{dr}(s_1\sigma_2 z_3^1) \circ_2 (\eta)^{u'} \circ_2 {}^{fp'}(s_1\sigma_2 z_1^1)\} \circ_2 (s_1\sigma_2 x_3^0)^{p's't'}] \circ_1$$

$$[{}^{ab}(s_1\sigma_2(yz)_3^1) \circ_2 (\xi)^{r'u'} \circ_2 {}^{ec'}(s_1\sigma_2(yz)_1^1)]$$ by (3.2.1)

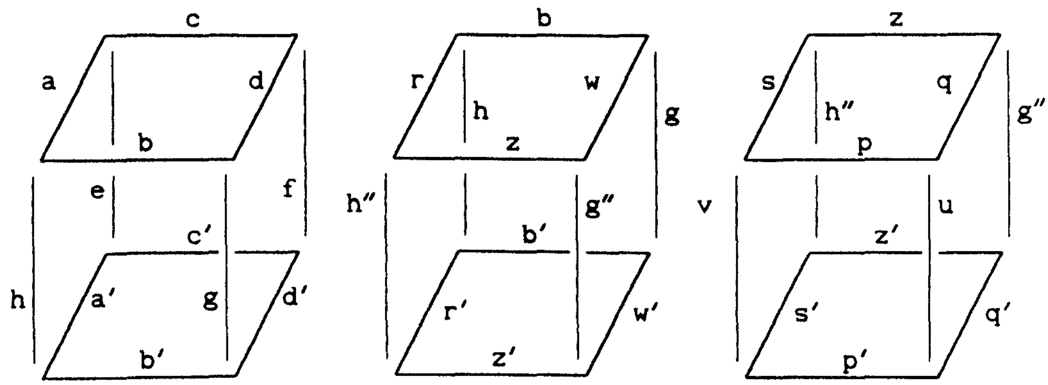
$$= [(s_1\sigma_2 x_1^0)^{rug''} \circ_2 {}^{c}(\eta\zeta) \circ_2 (s_1\sigma_2 x_3^0)^{p's't'}] \circ_1$$

$$[{}^{ab}(s_1\sigma_2(yz)_3^1) \circ_2 (\xi)^{r'u'} \circ_2 {}^{ec'}(s_1\sigma_2(yz)_1^1)]$$ by (3.2.5)(ii)

$= (x \circ_2 (y \circ_2 z)$ by (3.3.1).

(iii) The case of $\circ_3$.

Let $\underline{\boldsymbol{x}}$, $\boldsymbol{y}$ and $\boldsymbol{z}$ have edges given by



$\underline{\boldsymbol{x}}$ $\quad$ $\boldsymbol{y}$ $\quad$ $\boldsymbol{z}$

then

$$\boldsymbol{w} = [{}^{ar}(s_1\sigma_2\boldsymbol{z}_1^0)^u \circ_2 {}^{arz}(s_1\sigma_2\boldsymbol{z}_2^1) \circ_2 (\xi\eta)^{q'}] \circ_1$$

$$[{}^{ar}(\zeta) \circ_2 (s_1\sigma_2(\boldsymbol{xy})_2^0)^{z'q'} \circ_2 {}^{e}(s_1\sigma_2(\boldsymbol{xy})_1^1)^{q'}]$$ by (3.2.5)(iii)

$$= [{}^{ar}(s_1\sigma_2\boldsymbol{z}_1^0)^u \circ_2 {}^{arz}(s_1\sigma_2\boldsymbol{z}_2^1) \circ_2 {}^{a}(s_1\sigma_2\boldsymbol{y}_1^0)^{g''q'} \circ_2 {}^{ab}(s_1\sigma_2\boldsymbol{y}_2^1)^{q'} \circ_2$$

$$(\xi)^{w'q'}] \circ_1 [{}^{ar}(s_1\sigma_2\boldsymbol{z}_1^0)^u \circ_2 {}^{arz}(s_1\sigma_2\boldsymbol{z}_2^1) \circ_2 {}^{a}(\eta)^{q'} \circ_2$$

$$(s_1\sigma_2\boldsymbol{x}_2^0)^{b'w'q'} \circ_2 {}^{e}(s_1\sigma_2\boldsymbol{x}_1^1)^{w'q'}] \circ_1 [{}^{ar}(\zeta) \circ_2 (s_1\sigma_2(\boldsymbol{xy})_2^0 \circ_1 (\boldsymbol{xy})_1^1))^{q'}]$$

by (3.2.5)(iii) and (3.2.1)

$$= [{}^{a}({}^{r}(s_1\sigma_2(\boldsymbol{z}_1^0 \circ_1 \boldsymbol{z}_2^1) \circ_2 (s_1\sigma_2(\boldsymbol{y}_1^0 \circ_1 \boldsymbol{y}_2^1))^{q'}) \circ_2 (\xi)^{w'q'}] \circ_1$$

$$[{}^{ar}(s_1\sigma_2\boldsymbol{z}_1^0)^u \circ_2 {}^{arz}(s_1\sigma_2\boldsymbol{z}_2^1) \circ_2 {}^{a}(\eta)^{q'} \circ_2 (s_1\sigma_2\boldsymbol{x}_2^0)^{b'w'q'} \circ_2$$

$${}^{e}(s_1\sigma_2\boldsymbol{x}_1^1)^{w'q'}] \circ_1 [{}^{ar}(\zeta) \circ_2 \{s_1\sigma_2((\boldsymbol{x}_2^0 \circ_1 \boldsymbol{x}_1^1) \circ_2 (\boldsymbol{y}_2^0 \circ_1 \boldsymbol{y}_1^1))\}^{q'}])$$

by (2.5.6) and (2.2.1)(i)

$$= [{}^{a}\{s_1\sigma_2((\boldsymbol{y}_1^0 \circ_1 \boldsymbol{y}_2^1) \circ_2 (\boldsymbol{z}_1^0 \circ_1 \boldsymbol{z}_2^1))\} \circ_2 (\xi)^{w'q'}] \circ_1$$

$$[{}^{ar}(s_1\sigma_2\gamma_1^0)^u \circ_2 {}^{arz}(s_1\sigma_2\gamma_2^1) \circ_2 {}^{a}(\eta)^{q'} \circ_2 (s_1\sigma_2 x_2^0)^{b'w'q'} \circ_2$$

$${}^{e}(s_1\sigma_2 x_1^1)^{w'q'}] \circ_1 [{}^{ar}(\zeta) \circ_2 {}^{a}(s_1\sigma_2(y_2^0 \circ_1 y_1^1)^{q'} \circ_2$$

$$(s_1\sigma_2(x_2^0 \circ_1 x_1^1)^{w'q'}]) \qquad \text{by (2.5.6) and (2.2.1)(i)}$$

$$= [{}^{a}\{s_1\sigma_2((y_1^0 \circ_2 \gamma_1^0) \circ_1 (y_2^1 \circ_2 \gamma_2^1))\} \circ_2 (\xi)^{w'q'}] \circ_1$$

$$[{}^{ar}(s_1\sigma_2\gamma_1^0)^u \circ_2 {}^{arz}(s_1\sigma_2\gamma_2^1) \circ_2 {}^{a}(\eta)^{q'} \circ_2 (s_1\sigma_2 x_2^0)^{b'w'q'} \circ_2$$

$${}^{e}(s_1\sigma_2 x_1^1)^{w'q'}] \circ_1 [{}^{ar}(\zeta) \circ_2 {}^{a}(s_1\sigma_2 y_2^0)^{z'q'} \circ_2 {}^{ah}(y_1^1)^{q'} \circ_2$$

$$(s_1\sigma_2 x_2^0)^{b'w'q'} \circ_2 {}^{e}(s_1\sigma_2 x_1^1)^{w'q'}]) \qquad \text{by (2.5.6) and (2.2.1)(i)}$$

$$= [{}^{a}(s_1\sigma_2(y_1^0 \circ_2 \gamma_1^0))^u \circ_1 {}^{ab}(s_1\sigma_2(y_2^1 \circ_2 \gamma_2^1)) \circ_2 (\xi)^{w'q'}] \circ_1$$

$$[{}^{a}\{[{}^{r}(s_1\sigma_2\gamma_1^0)^u \circ_2 {}^{rz}(\gamma_2^1) \circ_2 (\eta)^{q'}] \circ_1 [{}^{r}(\zeta) \circ_2 (s_1\sigma_2 y_2^0)^{z'q'} \circ_2$$

$${}^{h}(s_1\sigma_2 y_1^1)^{q'}]\} \circ_2 (s_1\sigma_2 x_2^0)^{b'w'q'} \circ_2 {}^{e}(s_1\sigma_2 x_1^1)^{w'q'}])$$

by (2.2.1)(iii) , (2.5.6) and (3.2.1)

$$= [{}^{a}(s_1\sigma_2(y\gamma)_1^0)^u \circ_1 {}^{ab}(s_1\sigma_2(y\gamma)_2^1) \circ_2 (\xi)^{w'q'}] \circ_1$$

$$[{}^{a}(\eta\zeta) \circ_2 (s_1\sigma_2 x_2^0)^{b'w'q'} \circ_2 {}^{e}(s_1\sigma_2 x_1^1)^{w'q'}]) \qquad \text{by (3.2.5)(iii)}$$

$$= w' = (x \circ_3 (y \circ_3 \gamma)) \qquad \text{by (3.3.1).}\square$$

3.5 The interchange law in λC_3.

The proof of the interchange law will be more complicated because it involves four elements and two directions. The key points in this proof are Lemma 3.4.1 and the interchange law in C_3 .

Let $1 \le i < j \le 3$ and let $x = \langle \underline{x} , \xi \rangle$, $y = \langle \underline{y} , \eta \rangle$, $z = \langle \underline{z} , \zeta \rangle$, $t = \langle \underline{t} , \tau \rangle$ be elements of G_3 such that the composite shell

$$\underline{\varpi} = \begin{bmatrix} \underline{x} & \underline{y} \\ \underline{z} & \underline{t} \end{bmatrix} \qquad \begin{matrix} \ulcorner \!\longrightarrow j \\ \downarrow \\ i \end{matrix}$$

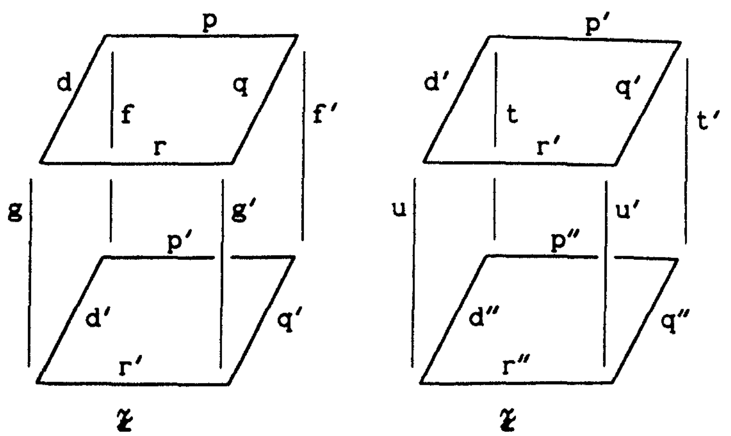
is defined. Then

$$(x \circ_i z) \circ_j (y \circ_i t) = (\underline{\varpi}, \omega), \qquad (x \circ_j y) \circ_i (z \circ_j t) = (\underline{\varpi}, \omega')$$

say, and we have to show that $\varpi = \varpi'$ in C_3 . We will prove each case individually.

(i) The case where $i = 1$ and $j = 2$.

let $\underline{x}$, $\underline{y}$ be given as in (3.4)(i) , $\underline{z}$ and $\underline{t}$ have boundaries and edges given by



then, by (3.2.5)

$$(\varpi) = [(s_1\sigma_2(xy)_1^0)^{rg'u'} \circ_2 {}^{c}(\zeta\tau) \circ_2 (s_1\sigma_2(xy)_3^0)^{p''q''}] \circ_1$$

$$[{}^{ab}(s_1\sigma_2(zt)_3^1) \circ_2 (\xi\eta)^{r''} \circ_2 {}^{esc''}(s_1\sigma_2(zt)_1^1)]$$

$$= [(s_1\sigma_2 x_1^0)^{rg'u'} \circ_2 {}^{c}(\zeta)^{u'} \circ_2 {}^{cfp'}(s_1\sigma_2 t_2^1) \circ_2 {}^{cf}(s_1\sigma_2 t_3^0)^{q''} \circ_2$$

$$(s_1\sigma_2(xy)^0_3)^{p''q''}] \circ_1 [(s_1\sigma_2 x^0_1)^{rg'u'} \circ_2 {}^{cd}(s_1\sigma_2 z^1_3)^{u} \circ_2 {}^{c}(s_1\sigma_2 z^0_2)^{r'u'} \circ_2$$

$${}^{cf}(\tau) \circ_2 (s_1\sigma_2(xy)^0_3)^{p''q''}] \circ_1 [{}^{ab}(s_1\sigma_2(zt)^1_3) \circ_2 (\xi)^{ur''} \circ_2 {}^{ec'}(s_1\sigma_2 y^1_2)^{r''} \circ_2$$

$${}^{e}(s_1\sigma_2 y^0_3)^{d''r''} \circ_2 {}^{esc''}(s_1\sigma_2 t^1_1)] \circ_1 [{}^{ab}(s_1\sigma_2(zt)^1_3) \circ_2 {}^{a}(s_1\sigma_2 x^1_3)^{ur''} \circ_2$$

$$(s_1\sigma_2 x^0_2)^{b'ur''} \circ_2 {}^{e}(\eta)^{r''} \circ_2 {}^{esc''}(s_1\sigma_2 t^1_1)]$$

$$= [(s_1\sigma_2 x^0_1)^{rg'u'} \circ_2 {}^{c}(\zeta)^{u'} \circ_2 {}^{cfp'}(s_1\sigma_2 t^1_2) \circ_2 {}^{cf}(s_1\sigma_2 t^0_3)^{q''} \circ_2$$

$$(s_1\sigma_2 x^0_3)^{tp''q''} \circ_2 {}^{e}(s_1\sigma_2 y^0_3)^{p''q''}] \circ_1 [(s_1\sigma_2 x^0_1)^{rg'u'} \circ_2 {}^{cd}(s_1\sigma_2 z^1_3)^{u} \circ_2$$

$${}^{c}(s_1\sigma_2 z^0_2)^{r'u'} \circ_2 {}^{cf}(\tau) \circ_2 (s_1\sigma_2 x^0_3)^{tp''q''} \circ_2 {}^{e}(s_1\sigma_2 y^0_3)^{p''q''}] \circ_1$$

$$[{}^{ab}(s_1\sigma_2 z^1_3)^{u'} \circ_2 {}^{abg}(s_1\sigma_2 t^1_3) \circ_2 (\xi)^{ur''} \circ_2 {}^{ec'}(s_1\sigma_2 y^1_2)^{r''} \circ_2 {}^{esc''}(s_1\sigma_2 t^1_1) \circ_2$$

$${}^{e}(s_1\sigma_2 y^0_3)^{d''r''}] \circ_1 [{}^{ab}(s_1\sigma_2 z^1_3)^{u'} \circ_2 {}^{abg}(s_1\sigma_2 t^1_3) \circ_2 {}^{a}(s_1\sigma_2 x^1_3)^{ur''} \circ_2$$

$$(s_1\sigma_2 x^0_2)^{b'ur''} \circ_2 {}^{e}(\eta)^{r''} \circ_2 {}^{esc''}(s_1\sigma_2 t^1_1)] ,$$

and

$$(w') = [(s_1\sigma_2 x^0_1)^{rg'u'} \circ_2 {}^{c}(\zeta)^{u'} \circ_2 (s_1\sigma_2 x^0_3)^{p'q'u'} \circ_2 {}^{ec'p'}(s_1\sigma_2 t^1_2) \circ_2$$

$${}^{ec'}(s_1\sigma_2 t^0_3)^{q''} \circ_2 {}^{e}(s_1\sigma_2 y^0_3)^{p''q''}] \circ_1 [{}^{ab}(s_1\sigma_2 z^1_3)^{u'} \circ_2 (\xi)^{r'u'} \circ_2$$

$${}^{ec'}(s_1\sigma_2 z^1_1)^{u'} \circ_2 {}^{ec'p'}(s_1\sigma_2 t^1_2) \circ_2 {}^{ec'}(s_1\sigma_2 t^0_3)^{q''} \circ_2 {}^{e}(s_1\sigma_2 y^0_3)^{p''q''}] \circ_1$$

$$[{}^{ab}(s_1\sigma_2 z^1_3)^{u'} \circ_2 {}^{a}(s_1\sigma_2 x^1_3)^{r'u'} \circ_2 (s_1\sigma_2 x^0_2)^{b'r'u'} \circ_2 {}^{e}(s_1\sigma_2 y^0_1)^{r'u'} \circ_2$$

$${}^{ec'}(\tau) \circ_2 {}^{e}(s_1\sigma_2 y^0_3)^{p''q''}] \circ_1 [{}^{ab}(s_1\sigma_2 z^1_3)^{u'} \circ_2 {}^{a}(s_1\sigma_2 x^1_3)^{r'u'} \circ_2$$

$$(s_1\sigma_2 x^0_2)^{b'r'u'} \circ_2 {}^{ea'b'}(s_1\sigma_2 t^1_3) \circ_2 {}^{e}(\eta)^{r''} \circ_2 {}^{esc''}(s_1\sigma_2 t^1_1)]$$

To simplify the situation, we write

$$(\varpi) = [Z] \circ_1 [T] \circ_1 [X] \circ_1 [Y]$$

$$(\varpi') = [Z'] \circ_1 [X'] \circ_1 [T'] \circ_1 [Y']$$

say, and we have to prove that $[Z] = [Z']$, $[Y] = [Y']$ and $[T] \circ_1 [X] = [X'] \circ_1 [T']$. So

$[Z] = [(s_1\sigma_2 x_1^0)^{rg'u'} \circ_2 {}^{c}(\zeta)^{u'} \circ_2 {}^{cfp'}(s_1\sigma_2 t_2^1) \circ_2 {}^{cf}(s_1\sigma_2 t_3^0)^{q''} \circ_2$

$(s_1\sigma_2 x_3^0)^{tp''q''} \circ_2 {}^{e}(s_1\sigma_2 y_3^0)^{p''q''}]$

$= [(s_1\sigma_2 x_1^0)^{rg'u'} \circ_2 {}^{c}(\zeta)^{u'} \circ_2 {}^{cfp'}(s_1\sigma_2 t_2^1) \circ_2 (s_1\sigma_2 x_3^0)^{p't'q''} \circ_2$

${}^{ec'}(s_1\sigma_2 t_3^0)^{q''} \circ_2 {}^{e}(s_1\sigma_2 y_3^0)^{p''q''}]$ by (3.4.1) and (2.4.1)(iv)

$= [(s_1\sigma_2 x_1^0)^{rg'u'} \circ_2 {}^{c}(\zeta)^{u'} \circ_2 (s_1\sigma_2 x_3^0)^{p'q'u'} \circ_2 {}^{ec'p'}(s_1\sigma_2 t_2^1) \circ_2$

${}^{ec'}(s_1\sigma_2 t_3^0)^{q''} \circ_2 {}^{e}(s_1\sigma_2 y_3^0)^{p''q''}]$ by (3.4.1) and (2.4.1)(iv)

$= [Z']$.

$[Y] = [{}^{ab}(s_1\sigma_2 y_3^1)^{u'} \circ_2 {}^{abg}(s_1\sigma_2 t_3^1) \circ_2 {}^{a}(s_1\sigma_2 x_3^1)^{ur''} \circ_2 (s_1\sigma_2 x_2^0)^{b'ur''} \circ_2$

${}^{e}(\eta)^{r''} \circ_2 {}^{esc''}(s_1\sigma_2 t_1^1)]$

$= [{}^{ab}(s_1\sigma_2 y_3^1)^{u'} \circ_2 {}^{a}(s_1\sigma_2 x_3^1)^{r'u'} \circ_2 {}^{ahb'}(s_1\sigma_2 t_3^1) \circ_2 (s_1\sigma_2 x_2^0)^{b'ur''} \circ_2$

${}^{e}(\eta)^{r''} \circ_2 {}^{esc''}(s_1\sigma_2 t_1^1)]$ by (3.4.1) and (2.4.1)(iv)

$= [{}^{ab}(s_1\sigma_2 y_3^1)^{u'} \circ_2 {}^{a}(s_1\sigma_2 x_3^1)^{r'u'} \circ_2 (s_1\sigma_2 x_2^0)^{b'r'u'} \circ_2 {}^{ea'b'}(s_1\sigma_2 t_3^1) \circ_2$

${}^{e}(\eta)^{r''} \circ_2 {}^{esc''}(s_1\sigma_2 t_1^1)]$ by (3.4.1) and (2.4.1)(iv)

$= [Y']$.

$[T] \circ_1 [X] = [(s_1\sigma_2 x_1^0)^{rg'u'} \circ_2 {}^{cd}(s_1\sigma_2 y_3^1)^{u'} \circ_2 {}^{c}(s_1\sigma_2 y_2^0)^{r'u'} \circ_2 {}^{cf}(\tau) \circ_2$

$$(s_1\sigma_2 x_3^0)^{tp''q''} \circ_2 {}^{e}(s_1\sigma_2 y_3^0)^{p''q''}] \circ_1 [{}^{ab}(s_1\sigma_2 z_3^1)^{u'} \circ_2 {}^{abg}(s_1\sigma_2 t_3^1) \circ_2$$

$$(\xi)^{ur''} \circ_2 {}^{ec'}(s_1\sigma_2 y_2^1)^{r''} \circ_2 {}^{e}(s_1\sigma_2 y_3^0)^{d''r''} \circ_2 {}^{esc''}(s_1\sigma_2 t_1^1)]$$

$$= [{}^{ab}(s_1\sigma_2 z_3^1)^{u'} \circ_2 (s_1\sigma_2 x_1^0)^{gr'u'} \circ_2 {}^{c}(s_1\sigma_2 x_2^0)^{r'u'} \circ_2 {}^{cf}(\tau) \circ_2$$

$$(s_1\sigma_2 x_3^0)^{tp''q''} \circ_2 {}^{e}(s_1\sigma_2 y_3^0)^{p''q''}] \circ_1 [{}^{ab}(s_1\sigma_2 z_3^1)^{u'} \circ_2 {}^{abg}(s_1\sigma_2 t_3^1) \circ_2$$

$$(\xi)^{ur''} \circ_2 {}^{ec'}(s_1\sigma_2 y_2^1)^{r''} \circ_2 {}^{ec't}(s_1\sigma_2 t_1^1) \circ_2 {}^{e}(s_1\sigma_2 y_3^0)^{p''q''}]$$

by (3.4.1) and (2.4.1)(iv)

$$= [{}^{ab}(s_1\sigma_2 z_3^1)^{u'} \circ_2 (s_1\sigma_2 x_1^0)^{gr'u'} \circ_2 {}^{c}(s_1\sigma_2 x_2^0)^{r'u'} \circ_2 {}^{cf}(\tau) \circ_2$$

$$(s_1\sigma_2 x_3^0)^{tp''q''} \circ_2 {}^{e}(s_1\sigma_2 y_3^0)^{p''q''}] \circ_1 [{}^{ab}(s_1\sigma_2 z_3^1)^{u'} \circ_2 {}^{abg}(s_1\sigma_2 t_3^1) \circ_2$$

$$(\xi)^{ur''} \circ_2 {}^{ec'}(s_1\sigma_2 y_2^1)^{r''} \circ_2 {}^{ec't}(s_1\sigma_2 t_1^1) \circ_2 {}^{e}(s_1\sigma_2 y_3^0)^{p''q''}]$$

$$= [{}^{ab}(s_1\sigma_2 z_3^1)^{u'} \circ_2 (s_1\sigma_2 x_1^0)^{gr'u'} \circ_2 {}^{c}(s_1\sigma_2 x_2^0)^{r'u'} \circ_2 (s_1\sigma_2 x_3^0)^{d'r'u'} \circ_2$$

$${}^{ec'}(\tau) \circ_2 {}^{e}(s_1\sigma_2 y_3^0)^{p''q''}] \circ_1 [{}^{ab}(s_1\sigma_2 z_3^1)^{u'} \circ_2 (\xi)^{r'u'} \circ_2$$

$${}^{ec'd'}(s_1\sigma_2 t_3^1) \circ_2 {}^{ec'}(s_1\sigma_2 y_2^1)^{r''} \circ_2 {}^{ec't}(s_1\sigma_2 t_1^1) \circ_2 {}^{e}(s_1\sigma_2 y_3^0)^{p''q''}]$$

by (3.4.1) and (2.4.1)(iv)

$$= {}^{ab}(s_1\sigma_2 z_3^1)^{u'} \circ_2 \{[(s_1\sigma_2 x_1^0)^{gr'u'} \circ_2 {}^{c}(s_1\sigma_2 x_2^0)^{r'u'} \circ_2 (s_1\sigma_2 x_3^0)^{d'r'u'} \circ_2$$

$${}^{ec'}(\tau)] \circ_1 [(\xi)^{r'u'} \circ_2 {}^{ec'd'}(s_1\sigma_2 t_3^1) \circ_2 {}^{ec'}(s_1\sigma_2 y_2^1)^{r''} \circ_2$$

$${}^{ec't}(s_1\sigma_2 t_1^1)]\} \circ_2 {}^{e}(s_1\sigma_2 y_3^0)^{p''q''}$$

$$= {}^{ab}(s_1\sigma_2 z_3^1)^{u'} \circ_2 \Big[\{[(s_1\sigma_2 x_1^0)^{g} \circ_2 {}^{c}(s_1\sigma_2 x_2^0) \circ_2 (s_1\sigma_2 x_3^0)^{d'}]^{r'u'} \circ_1$$

$$(\xi)^{r'u'}\} \circ_2 \{{}^{ec'}(\tau) \circ_1 {}^{ec'}[{}^{d'}(s_1\sigma_2 t_3^1) \circ_2 (s_1\sigma_2 y_2^1)^{r''} \circ_2$$

$${}^{t}(s_1\sigma_2 t_1^1)]\}\Big] \circ_2 {}^{e}(s_1\sigma_2 y_3^0)^{p''q''}$$

$$= {}^{ab}(s_1\sigma_2 z_3^1)^{u'} \circ_2 \left[(\xi)^{r'u'} \circ_2 {}^{ec'}(\tau) \right] \circ_2 {}^{e}(s_1\sigma_2 y_3^0)^{p''q''}$$

(by (3.2.4) and since $(\sigma_2 x_1^0)^{g} \circ_1 {}^{c}(\sigma_2 x_2^1) \circ_1 (\sigma_2 x_3^0)^{d'} = \partial_1^0 \xi$, ${}^{d'}(\sigma_2 t_3^1) \circ_1 (\sigma_2 t_2^1)^{r''} \circ_1 {}^{t}(\sigma_2 t_1^1) = \partial_1^1 \tau$)

$$= {}^{ab}(s_1\sigma_2 z_3^1)^{u'} \circ_2 \Big[\{(\xi)^{r'u'} \circ_1 [{}^{a}(s_1\sigma_2 x_3^1) \circ_2 (s_1\sigma_2 x_2^0)^{b'} \circ_2 {}^{e}(s_1\sigma_2 x_1^1)]^{r'u'}\} \{{}^{ec'}[(s_1\sigma_2 t_1^0)^{u'} \circ_2 {}^{p'}(s_1\sigma_2 t_2^0) \circ_2 (s_1\sigma_2 t_3^0)^{q''}] \circ_1 {}^{ec'}(\tau)\} \Big] \circ_2 {}^{e}(s_1\sigma_2 y_3^0)^{p''q''}$$

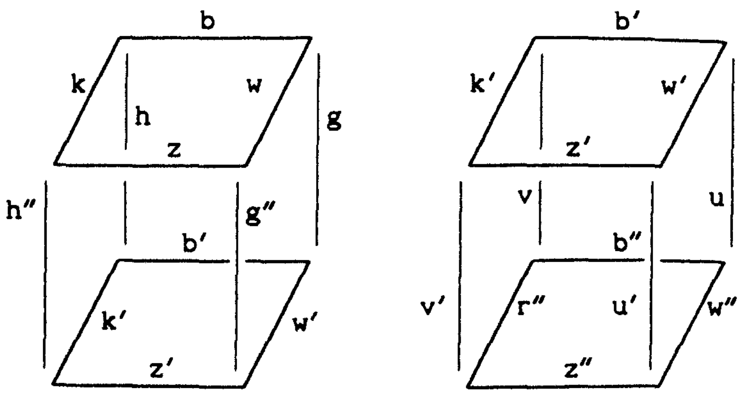
$$= [{}^{ab}(s_1\sigma_2 z_3^1)^{u'} \circ_2 \{(\xi)^{r'u'} \circ_2 {}^{ec'}(s_1\sigma_2 z_1^1)^{u'} \circ_2 {}^{ec'p}(s_1\sigma_2 t_2^0) \circ_2 {}^{ec'}(s_1\sigma_2 t_3^0)^{q''} \circ_2 {}^{e}(s_1\sigma_2 y_3^0)^{p''q''}] \circ_1 [{}^{a}(s_1\sigma_2 x_3^1)^{r'u'} \circ_2 {}^{ab}(s_1\sigma_2 z_3^1)^{u'} \circ_2 (s_1\sigma_2 x_2^0)^{b'r'u'} \circ_2 {}^{e}(s_1\sigma_2 x_1^1)^{r'u'} \circ_2 {}^{ec'}(\tau) \circ_2 {}^{e}(s_1\sigma_2 y_3^0)^{p''q''}]$$

$$= [X'] \circ_1 [T'] .$$

Thus $(\mathfrak{w}) = (\mathfrak{w}')$.

(ii) The case of $i = 1$ and $j = 3$.

Let x , y be given as in (3.4)(i) , z and t have edges and boundaries given by



z $\qquad$ t

then

$$(\omega) = [{}^{a}(s_1\sigma_2(zt)_1^0)^{g''u'} \circ_2 {}^{akz}(s_1\sigma_2(zt)_2^1) \circ_2 (\xi\eta)^{w''}] \circ_1$$

$$[{}^{a}(\zeta\tau) \circ_2 (s_1\sigma_2(xy)_2^0)^{b''w''} \circ_2 {}^{es}(s_1\sigma_2(xy)_1^1)^{w''}] .$$

$$= [{}^{a}(s_1\sigma_2(zt)_1^0)^{g''u'} \circ_2 {}^{ab}(s_1\sigma_2(zt)_2^1) \circ_2 (\xi)^{uw''} \circ_2 {}^{ec'}(s_1\sigma_2 y_2^1)^{w''} \circ_2$$

$${}^{e}(s_1\sigma_2 y_3^0)^{d''w''}] \circ_1 [{}^{a}(s_1\sigma_2(zt)_1^0)^{g''u'} \circ_2 {}^{ab}(s_1\sigma_2(zt)_2^1) \circ_2 {}^{a}(s_1\sigma_2 x_3^1)^{uw''} \circ_2$$

$$(s_1\sigma_2 x_2^0)^{b'uw''} \circ_2 {}^{e}(\eta)^{w''}] \circ_1 [{}^{a}(\zeta)^{u'} \circ_2 {}^{ahb'}(s_1\sigma_2 t_2^1) \circ_2$$

$${}^{ah}(s_1\sigma_2 t_3^0)^{w''} \circ_2 (s_1\sigma_2(xy)_2^0)^{b''w''} \circ_2 {}^{es}(s_1\sigma_2(xy)_1^1)^{w''}] \circ_1 [{}^{ak}(s_1\sigma_2 z_3^1)^{u'} \circ_2$$

$${}^{a}(s_1\sigma_2 z_2^0)^{z'u'} \circ_2 {}^{ah}(\tau) \circ_2 (s_1\sigma_2(xy)_2^0)^{b''w''} \circ_2 {}^{es}(s_1\sigma_2(xy)_1^1)^{w''}]$$

and

$$(\omega') = [(\xi\zeta)^{u'} \circ_2 {}^{ec'}(s_1\sigma_2(yt)_2^1) \circ_2 {}^{e}(s_1\sigma_2(yt)_3^0)^{d''w''}] \circ_1$$

$$[{}^{ak}(s_1\sigma_2(xz)_3^1)^{u'} \circ_2 (s_1\sigma_2(xz)_2^0)^{z'u'} \circ_2 {}^{e}(\eta\tau)] ,$$

$$= [{}^{a}(s_1\sigma_2 z_1^0)^{g''u'} \circ_2 {}^{ab}(s_1\sigma_2 z_2^1)^{u'} \circ_2 (\xi)^{w'u'} \circ_2 {}^{ec'}(s_1\sigma_2(yt)_2^1) \circ_2$$

$${}^{e}(s_1\sigma_2(yt)_3^0)^{d''w''}] \circ_1 [{}^{a}(\zeta)^{u'} \circ_2 (s_1\sigma_2 x_2^0)^{b'w'u'} \circ_2 {}^{e}(s_1\sigma_2 x_1^1)^{w'u'} \circ_2$$

$${}^{ec'}(s_1\sigma_2(yt)_2^1) \circ_2 {}^{e}(s_1\sigma_2(yt)_3^0)^{d''w''}] \circ_1 [{}^{ak}(s_1\sigma_2(xz)_3^1)^{u'} \circ_2$$

$$(s_1\sigma_2(xz)_2^0)^{z'u'} \circ_2 {}^{ea'}(s_1 t_1^0)^{u'} \circ_2 {}^{ea'b'}(s_1 t_2^1) \circ_2 {}^{e}(\eta)^{w''}] \circ_1$$

$$[{}^{ak}(s_1(xz)_3^1)^{u'} \circ_2 (s_1(xz)_2^0)^{z'u'} \circ_2 {}^{ea'}(\tau) \circ_2 {}^{e}(s_1 y_2^0)^{b''w''} \circ_2 {}^{es}(s_1 y_1^1)^{w''}] .$$

$$[X] = [{}^{a}(s_1\sigma_2(zt)_1^0)^{g''u'} \circ_2 {}^{ab}(s_1\sigma_2(zt)_2^1) \circ_2 (\xi)^{uw''} \circ_2 {}^{ec'}(s_1\sigma_2 y_2^1)^{w''} \circ_2$$

$${}^{e}(s_1 y_3^0)^{d''w''}]$$

$$= [{}^{a}(s_1\sigma_2 z_1^0)^{g''u'} \circ_2 {}^{ab}(s_1\sigma_2 z_2^1)^{u'} \circ_2 {}^{abg}(s_1\sigma_2 t_2^1) \circ_2 (\xi)^{uw''} \circ_2$$

$^{ec'}(s_1\sigma_2 y_2^1)^{w''} \circ_2 {}^{e}(s_1\sigma_2 y_3^0)^{d''w''}]$ by (2.2.1)(i), (2.5.6), (3.2.2)

$= [{}^{a}(s_1\sigma_2 z_1^0)^{g''u'} \circ_2 {}^{ab}(s_1\sigma_2 z_2^1)^{u'} \circ_2 (\xi)^{w'u'} \circ_2 {}^{ec'd'}(s_1\sigma_2 t_2^1) \circ_2$

$^{ec'}(s_1\sigma_2 y_2^1)^{w''}]$ by (3.4.1)

$= [{}^{a}(s_1\sigma_2 z_1^0)^{g''u'} \circ_2 {}^{ab}(s_1\sigma_2 z_2^1)^{u'} \circ_2 (\xi)^{w'u'} \circ_2 {}^{ec'}(s_1\sigma_2 \partial_2^1 yt)$

$= [X']$.

$[T] = [{}^{ak}(s_1\sigma_2 z_3^1)^{u'} \circ_2 {}^{a}(s_1\sigma_2 z_2^0)^{z'u'} \circ_2 {}^{ah}(\tau) \circ_2 (s_1\sigma_2 (xy)_2^0)^{b''w''} \circ_2$

$\quad {}^{es}(s_1\sigma_2 (xy)_1^1)^{w''}]$

$= [{}^{ak}(s_1\sigma_2 z_3^1)^{u'} \circ_2 {}^{a}(s_1\sigma_2 z_2^0)^{z'u'} \circ_2 {}^{ah}(\tau) \circ_2 (s_1\sigma_2 x_2^0)^{vb''w''} \circ_2$

$^{e}(s_1\sigma_2 y_2^0)^{b''w''} \circ_2 {}^{es}(s_1\sigma_2 y_1^1)^{w''}]$ by (2.2.1)(i), (2.5.6), (3.2.2)

$= [{}^{ak}(s_1\sigma_2 z_3^1)^{u'} \circ_2 {}^{a}(s_1\sigma_2 z_2^0)^{z'u'} \circ_2 (s_1\sigma_2 x_2^0)^{k'z'u'} \circ_2 {}^{eh'}(\tau) \circ_2$

$^{e}(s_1\sigma_2 y_2^0)^{b''w''} \circ_2 {}^{es}(s_1\sigma_2 y_1^1)^{w''}]$ by (3.4.1)

$= [{}^{ak}(s_1\sigma_2 (xz)_3^1)^{u'} \circ_2 (s_1\sigma_2 (xz)_2^0)^{z'u'} \circ_2 {}^{eh'}(\tau) \circ_2 {}^{e}(s_1\sigma_2 y_2^0)^{b''w''} \circ_2$

$^{es}(s_1\sigma_2 y_1^1)^{w''}]$ by (2.2.1)(i) and (2.5.6)

$= [T']$.

$[Y] \circ_1 [Z] = [{}^{a}(s_1\sigma_2 (zt)_1^0)^{g''u'} \circ_2 {}^{akz}(s_1\sigma_2 (zt)_2^1) \circ_2 {}^{a}(s_1\sigma_2 x_3^1)^{uw''} \circ_2$

$(s_1\sigma_2 x_2^0)^{b'uw''} \circ_2 {}^{e}(\eta)^{w''}] \circ_1 [{}^{a}(\zeta)^{u'} \circ_2 {}^{ahb'}(s_1\sigma_2 t_2^1) \circ_2 {}^{ah}(s_1\sigma_2 t_3^0)^{w''} \circ_2$

$(s_1\sigma_2 (xy)_2^0)^{b''w''} \circ_2 {}^{es}(s_1\sigma_2 (xy)_1^1)^{w''}]$

$= [{}^{a}(s_1\sigma_2 z_1^0)^{g''u'} \circ_2 {}^{ab}(s_1\sigma_2 z_2^1)^{u'} \circ_2 {}^{abg}(s_1\sigma_2 t_2^1) \circ_2 {}^{a}(s_1\sigma_2 z_3^0)^{uw''} \circ_2$

$(s_1\sigma_2 x_2^0)^{b'uw''} \circ_2 {}^{e}(\eta)^{w''}] \circ_1 [{}^{a}(\zeta)^{u'} \circ_2 {}^{ahb'}(s_1\sigma_2 t_2^1) \circ_2 {}^{ah}(s_1 y_3^1)^{w''} \circ_2$

$$(s_1\sigma_2 x_2^0)^{vb''w''} \circ_2 {}^{e}(s_1\sigma_2 y_2^0)^{b''w''} \circ_2 {}^{es}(s_1\sigma_2 y_1^1)^{w''}]$$

by (2.2.1)(i) and (2.5.6)

$$= [{}^{a}(s_1\sigma_2 z_1^0)^{g''u'} \circ_2 {}^{ab}(s_1\sigma_2 z_2^1)^{u'} \circ_2 {}^{a}(s_1\sigma_2 z_3^0)^{w'u'} \circ_2 {}^{ahb'}(s_1\sigma_2 t_2^1) \circ_2$$

$$(s_1\sigma_2 x_2^0)^{b'uw''} \circ_2 {}^{e}(\eta)^{w''}] \circ_1 [{}^{a}(\zeta)^{u'} \circ_2 {}^{ahb'}(s_1\sigma_2 t_2^1) \circ_2 (s_1\sigma_2 x_2^0)^{b'uw''} \circ_2$$

$${}^{ea'}(s_1\sigma_2 y_3^1)^{w''} \circ_2 {}^{e}(s_1\sigma_2 y_2^0)^{b''w''} \circ_2 {}^{es}(s_1\sigma_2 y_1^1)^{w''}]$$

$$= [{}^{a}\{(s z_1^0)^{g''} \circ_2 {}^{b}(s z_2^1) \circ_2 (s_1 z_3^0)^{w'}\}^{u'} \circ_1 {}^{a}(\zeta)^{u'}] \circ_2 {}^{ahb'}(s_1\sigma_2 t_2^1) \circ_2$$

$$(s_1 x_2^0)^{b'uw''} \circ_2 [{}^{e}(\eta)^{w''} \circ_1 {}^{e}\{{}^{a'}(s_1 y_3^1) \circ_2 (s y_2^0)^{b''} \circ_2 {}^{s}(s y_1^1)\}^{w''}]$$

by (2.4.1)(iv) and (3.2.2)

$$= {}^{a}(\zeta)^{u'} \circ_2 {}^{ahb'}(s_1\sigma_2 t_2^1) \circ_2 (s_1\sigma_2 x_2^0)^{b'uw''} \circ_2 {}^{e}(\eta)^{w''}$$

$$= {}^{a}(\zeta)^{u'} \circ_1 {}^{a}\{{}^{k}(s_1\sigma_2 z_3^1) \circ_2 (s_1\sigma_2 z_2^0)^{z'} \circ_2 {}^{h}(s_1\sigma_2 z_1^1)\}^{u'}] \circ_2$$

$${}^{ahb'}(s_1\sigma_2 t_2^1) \circ_2 (s_1\sigma_2 x_2^0)^{b'uw''} \circ_2 [{}^{e}\{(s_1\sigma_2 y_1^0)^{u} \circ_2 {}^{c'}(s_1\sigma_2 y_2^1) \circ_2$$

$$(s_1 y_3^0)^{d''}\}^{w''} \circ_1 {}^{e}(\eta)^{w''}]$$

$$= {}^{a}(\zeta)^{u'} \circ_2 {}^{ahb'}(s_1\sigma_2 t_2^1) \circ_2 (s_1\sigma_2 x_2^0)^{b'uw''} \circ_2 {}^{e}(s_1\sigma_2 y_1^0)^{uw''} \circ_2$$

$${}^{ec'}(s_1\sigma_2 y_2^1)^{w''} \circ_2 {}^{e}(s_1\sigma_2 y_3^0)^{d''w''}] \circ_1 [{}^{ak}(s_1\sigma_2 z_3^1)^{u'} \circ_2 {}^{a}(s_1\sigma_2 z_2^0)^{z'u'} \circ_2$$

$${}^{ah}(s_1\sigma_2 z_1^1)^{u'} \circ_2 {}^{ahb'}(s_1\sigma_2 t_2^1) \circ_2 (s_1\sigma_2 x_2^0)^{b'uw''} \circ_2 {}^{e}(\eta)^{w''}]$$

by (2.4.1)(iv)

$$= {}^{a}(\zeta)^{u'} \circ_2 (s_1\sigma_2 x_2^0)^{b'w'u'} \circ_2 {}^{ea'b'}(s_1\sigma_2 t_2^1) \circ_2 {}^{e}(s_1\sigma_2 y_1^0)^{uw''} \circ_2$$

$${}^{ec'}(s_1\sigma_2 y_2^1)^{w''} \circ_2 {}^{e}(s_1\sigma_2 y_3^0)^{d''w''}] \circ_1 [{}^{ak}(s_1\sigma_2 z_3^1)^{u'} \circ_2 {}^{a}(s_1\sigma_2 z_2^0)^{z'u'} \circ_2$$

$^{ah}(s_1\sigma_2 z_1^1)^{u'} \circ_2 (s_1\sigma_2 x_2^0)^{b'w'u'} \circ_2 {}^{ea'b'}(s_1\sigma_2 t_2^1) \circ_2 {}^{e}(\eta)^{w''}]$ by (3.4.1)

$= {}^{a}(\zeta)^{u'} \circ_2 (\sigma_2 s_1 x_2^0)^{b'w'u'} \circ_2 {}^{e}(s_1\sigma_2 y_1^0)^{w'u'} \circ_2 {}^{ec'd'}(s_1\sigma_2 t_2^1) \circ_2$

${}^{ec'}(s_1\sigma_2 y_2^1)^{w''} \circ_2 {}^{e}(s_1\sigma_2 y_3^0)^{d''w''}] \circ_1 [{}^{ak}(s_1\sigma_2 z_3^1)^{u'} \circ_2 {}^{a}(s_1\sigma_2 z_2^0)^{z'u'} \circ_2$

$(s_1\sigma_2 x_2^0)^{k'z'u'} \circ_2 {}^{ea'}(s_1\sigma_2 z_1^1)^{u'} \circ_2 {}^{ea'b'}(s_1\sigma_2 t_2^1) \circ_2 {}^{e}(\eta)^{w''}]$ by (3.4.1)

$= {}^{a}(\zeta)^{u'} \circ_2 (s_1\sigma_2 x_2^0)^{b'w'u'} \circ_2 {}^{e}(s_1\sigma_2 y_1^0)^{w'u'} \circ_2 {}^{ec'}(s_1\sigma_2 (yt)_2^1) \circ_2$

${}^{e}(s_1\sigma_2 (yt)_3^0)^{d''w''}] \circ_1 [{}^{ak}(s_1\sigma_2 (xz)_3^1)^{u'} \circ_2 (s_1\sigma_2 (xz)_2^0)^{z'u'} \circ_2$

${}^{ea'}(s\sigma_2 z_1^1)^{u'} \circ_2 {}^{ea'b'}(s_1\sigma_2 t_2^1) \circ_2 {}^{e}(\eta)^{w''}]$

by (2.5.6) , (3.2.1) and (2.2.1)(i)

$= [Z'] \circ_1 [Y']$.

Thus $(\omega) = (\omega')$.

(iii) The case where $i = 2$ and $j = 3$.

We follow a similar steps as (i) .

Thus

$$(x \circ_i z) \circ_j (y \circ_i t) = (x \circ_j y) \circ_i (z \circ_j t).$$

We now have a triple category $(G_3, \ldots, G_0)$, and we must identify γG_3 . For any $\xi \in C_3$, let $\underline{d}\xi$ denote the shell $\underline{x} \in \square G_2$ with $x_1^\alpha = \sigma_2 d_1^\alpha \xi$. Define

$$\sigma_3\xi = (\underline{d}\xi, \xi).$$

Clearly $\sigma_3\xi \in G_3$ and every element of γG_3 is of this form. The bijection $\sigma_3 : C_3 \longrightarrow \gamma G_3$ is compatible with the boundary maps since $d_1^\alpha \sigma_3 \xi = \partial_1^\alpha \sigma_3 \xi = \sigma_2 d_1^\alpha \xi$. To show that σ_3 preserves compositions let ξ , η , ζ , $\tau \in C_3$, and $\underline{d}\xi$, $\underline{d}\eta$, $\underline{d}\zeta$, $\underline{d}\tau$ be given with boundaries and edges as $\underline{x}$, $\underline{y}$, $\underline{z}$ and $\underline{t}$ in 3.3.1

respectively, then

$$(\underline{d}\xi,\xi) \circ_1 (\underline{d}\eta,\eta) =$$

$$(\underline{d}\xi \circ_1 \underline{d}\eta\ ,\ [(\xi)^{u} \circ_2 {}^{ec'}(s_1\sigma_2(\underline{d}\eta)_2^1) \circ_2 {}^{e}(s_1\sigma_2(\underline{d}\eta)_3^0)^{d''}] \circ_1$$
$$[{}^{a}(s_1\sigma_2(\underline{d}\xi)_3^1)^{u} \circ_2 (s_1\sigma_2(\underline{d}\xi)_2^0)^{b'u} \circ_2 {}^{e}(\eta)])$$

$$(\underline{d}\xi,\xi) \circ_2 (\underline{d}\zeta,\zeta) =$$

$$(\underline{d}\xi \circ_2 \underline{d}\zeta\ ,\ [(s_1\sigma_2(\underline{d}\xi)_1^0)^{rg'} \circ_2 {}^{c}(\zeta) \circ_2 (s_1\sigma_2(\underline{d}\xi)_3^0)^{p'q'}] \circ_1$$
$$[{}^{ab}(\Psi\varepsilon_1(\underline{d}\zeta)_3^1) \circ_2 (\xi)^{r'} \circ_2 {}^{ec'}(s_1\sigma_2(\underline{d}\zeta)_1^1)])$$

$$(\underline{d}\xi,\xi) \circ_3 (\underline{d}\tau,\tau) =$$

$$(\underline{d}\xi \circ_3 \underline{d}\tau\ ,\ [{}^{a}(s_1\sigma_2(\underline{d}\tau)_1^0)^{g''} \circ_2 {}^{ab}(s_1\sigma_2(\underline{d}\tau)_2^1) \circ_2 (\xi)^{w'}] \circ_1$$
$$[{}^{a}(\tau) \circ_2 (s_1\sigma_2(\underline{d}\xi)_2^0)^{b'w'} \circ_2 {}^{e}(s_1\sigma_2(\underline{d}\xi)_1^1)^{w'}]) .$$

Thus σ_3 is isomorphism of 3-categories.

By this we obtain a triple category G = λC and isomorphism of 3-categories.

In the following section we state the main result of this work, this result establishes the equivalence of triple categories with connections and 3-categories.

3.6 The equivalence between triple categories with connections and 3-categories.

3.6.1 Theorem. (The main result)

There is a functor λ from the category 3-𝔊 of triple categories to the category 3-𝔈 of 3-categories such that λ : 3-𝔊 ⟶ 3-𝔈 are inverse equivalencies.

Proof.

We have proved the existence of the functors

$\gamma : 3\text{-}\mathcal{G} \longrightarrow 3\text{-}\mathcal{C}$, $\lambda : 3\text{-}\mathcal{C} \longrightarrow 3\text{-}\mathcal{G}$ and isomorphism $\sigma_3 : C \longrightarrow \gamma G$. We now complete the proof of the equivalence. So let G' be a triple category and $\sigma' : C \longrightarrow \gamma G'$ be a morphism of 3-categories then there is a unique morphism $\theta : G \longrightarrow G'$ of triple categories such that the following diagram

$$\begin{array}{ccccc} C & \longrightarrow & \gamma\lambda C & \longrightarrow & \lambda C \\ & \searrow^{\sigma'} & \downarrow{\scriptstyle \gamma\theta} & & \downarrow{\scriptstyle \theta} \\ & & \gamma G' & \longrightarrow & G' \end{array}$$

commutes. We define θ by induction. For $n = 0,1$ it is clear that $G'_n = \gamma G'_n$. For $n = 2,3$, each $x' \in G'_n$ is uniquely determined by $(\underline{x}', \xi')$ where $\underline{x}' \in G'_{n-1}$, $\xi' \in \gamma G'_n$ and $d_1^{\alpha}\Psi\underline{x}' = d_1^{\alpha}\xi'$. This definition gives a morphism of triple categories. From this universal property, it follows that λ is a functor from $3\text{-}\mathcal{C}$ to $3\text{-}\mathcal{G}$ and is left adjoint to $\gamma : 3\text{-}\mathcal{G} \longrightarrow 3\text{-}\mathcal{C}$. The adjunction $\sigma_C : C \longrightarrow \gamma\lambda C$ is an isomorphism for all C , so $1_{3\text{-}\mathcal{C}} \cong \gamma\lambda$. Also, the adjunction $\lambda\gamma G' \longrightarrow G'$ is obtained by putting $G = \gamma G'$, $\sigma' =$ identity, in which case θ is an isomorphism $\lambda\gamma G' \longrightarrow G'$, as is clear from its definition. Hence $\lambda\gamma = 1_{3\text{-}\mathcal{G}}$ and we have inverse equivalencies λ and γ between $3\text{-}\mathcal{C}$ and $3\text{-}\mathcal{G}$. □

CHAPTER IV

COMMENTS AND POSSIBILITIES FOR FURTHER WORK

In this final brief chapter we make some remarks about the work of the thesis.

Technical work involved in this study is an indication of the difficulties underlying the use of multiple categories. However, the clear equivalence obtained in the 3-dimensional case is also an indication of the prospective power of this method.

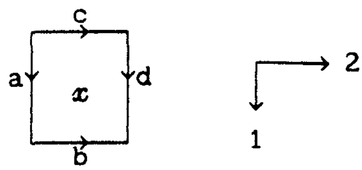
The Australian school on multiple categories are concentrating on the simplicial case, in order to define the simplicial nerve of an ∞-categories. This is achieved in an interesting and complex way (Street, Street-Walters). This is still a long way from obtaining an equivalence of categories, analogous to that between ∞-groupoids and simplicial T-complexes in the groupoid case.

The basic problem arising in this work is to find good formulae for $\Psi(x \circ_i y)$, $\Psi\Gamma_i$ and $\Psi\Gamma'$ for n > 3 .

APPENDIX I

Proof of Lemma (2.6.4) for n = 3 and $i = 2$.

This case is more complicated and will be proved using matrices. So let $x \in G_2$ have boundaries given by



then

$$\Psi\Gamma_2 x = \psi_1\psi_2\psi_1\Gamma_2 x$$

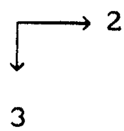
$$= \psi_1\psi_2(\Gamma'_1 x \circ_2 \Gamma_2 x \circ_2 \Gamma_1\varepsilon_2\partial_2^1 x) \,, \qquad \text{by (2.5.4)(iii)}$$

$$= \psi_1\psi_2(\Gamma'_1 x \circ_2 \Gamma_2 x \circ_2 \Gamma_1\varepsilon_2 d)$$

$$= \psi_1\psi_2(\Gamma'_1 x \circ_2 \Gamma_2 x \circ_2 \varepsilon_3\Gamma_1 d) \qquad \text{by (1.1.2)(iii)} .$$

By using matrices, $\psi_2(\Gamma'_1 x \circ_2 \Gamma_2 x \circ_2 \varepsilon_3\Gamma_1 d) =$

$$\begin{bmatrix} \Gamma'_2\Gamma'_1 a & \varepsilon_2\Gamma'_1 a & \varepsilon_2\Gamma'_1 a \\ \varepsilon_3\Gamma'_1 a & \Gamma'_2 x & \varepsilon_2 x \\ \varepsilon_3\Gamma'_1 a & \varepsilon_3 x & \Gamma'_2\Gamma_1 d \\ \Gamma'_1 x & \Gamma_2 x & \varepsilon_3\Gamma_1 d \\ \Gamma_2\Gamma'_1 d & \varepsilon_3\varepsilon_2 d & \varepsilon_3\Gamma_1 d \\ \varepsilon_2\varepsilon_2 d & \Gamma_2\varepsilon_2 d & \varepsilon_3\Gamma_1 d \\ \varepsilon_2\Gamma_1 d & \varepsilon_2\Gamma_1 d & \Gamma_2\Gamma_1 d \end{bmatrix}$$



$$
= \begin{bmatrix}
\Gamma_2'\Gamma_1'a & \varepsilon_2\Gamma_1'a & \varepsilon_2\Gamma_1'a \\
\varepsilon_3\Gamma_1'a & \Gamma_2'x & \varepsilon_2 x \\
\varepsilon_3\Gamma_1'a & \varepsilon_3 x & \Gamma_2'\Gamma_1 d \\
\Gamma_1'x & \Gamma_2 x & \varepsilon_3\Gamma_1 d \\
\Gamma_2\Gamma_1'd & \varepsilon_3\varepsilon_2 d & \varepsilon_3\Gamma_1 d \\
\varepsilon_3\varepsilon_2 d & \varepsilon_3\varepsilon_2 d & \varepsilon_3\Gamma_1 d \\
\varepsilon_2\Gamma_1 d & \varepsilon_2\Gamma_1 d & \Gamma_2\Gamma_1 d
\end{bmatrix}
\qquad
\begin{array}{l} \longrightarrow 2 \\ \downarrow \\ 3 \end{array}
$$

(since $\Gamma_2\varepsilon_2 d = \varepsilon_2^2 d = \varepsilon_3\varepsilon_2 d$).

Looking at row 6 and row 7 , we find that row 6 is an identity for row 7 , i.e. row 6 $\circ_3$ row 7 = row 7 . So

$\psi_2(\Gamma_1'x \circ_2 \Gamma_2 x \circ_2 \varepsilon_3\Gamma_1 d) =$

$$
\begin{bmatrix}
\Gamma_2'\Gamma_1'a & \varepsilon_2\Gamma_1'a & \varepsilon_2\Gamma_1'a \\
\varepsilon_3\Gamma_1'a & \Gamma_2'x & \varepsilon_2 x \\
\varepsilon_3\Gamma_1'a & \varepsilon_3 x & \Gamma_2'\Gamma_1 d \\
\Gamma_1'x & \Gamma_2 x & \varepsilon_3\Gamma_1 d \\
\Gamma_2\Gamma_1'd & \varepsilon_2\varepsilon_2 d & \varepsilon_3\Gamma_1 d \\
\varepsilon_2\Gamma_1 d & \varepsilon_2\Gamma_1 d & \Gamma_2\Gamma_1 d
\end{bmatrix}
\qquad
\begin{array}{l} \longrightarrow 2 \\ \downarrow \\ 3 \end{array}
$$

and $\psi_1\psi_2(\Gamma_1'x \circ_2 \Gamma_2 x \circ_2 \Gamma_1\varepsilon_2\partial_2^1 x) =$

$$\begin{bmatrix} \varepsilon_1^3\partial_1^0 c & \Gamma'_2\Gamma'_1 a & \varepsilon_2\Gamma'_1 a & \varepsilon_2\Gamma'_1 a & \Gamma_1\Gamma'_1 a \\ \varepsilon_1^3\partial_1^0 c & \varepsilon_3\Gamma'_1 a & \Gamma'_2 x & \varepsilon_2 x & \Gamma_1 x \\ \varepsilon_1^3\partial_1^0 c & \varepsilon_3\Gamma'_1 a & \varepsilon_3 x & \Gamma'_2\Gamma_1 d & \Gamma_1\Gamma_1 d \\ \varepsilon_1^2 c & \Gamma'_1 x & \Gamma_2 x & \varepsilon_3\Gamma_1 d & \varepsilon_1^3\partial_1^0 d \\ \Gamma'\Gamma'_1 d & \Gamma_2\Gamma'_1 d & \varepsilon_3\varepsilon_2 d & \varepsilon_3\Gamma_1 d & \varepsilon_1^3\partial_1^0 d \\ \Gamma'_1\Gamma_1 d & \varepsilon_2\Gamma_1 d & \varepsilon_2\Gamma_1 d & \Gamma_2\Gamma_1 d & \varepsilon_1^3\partial_1^0 d \end{bmatrix}$$

(for short hand we will use □ to denotes identities)

$$= \begin{bmatrix} \square & \Gamma'_2\Gamma'_1 a & \varepsilon_2\Gamma'_1 a & \varepsilon_2\Gamma'_1 a & \Gamma_1\Gamma'_1 a \\ \square & \varepsilon_3\Gamma'_1 a & \Gamma'_2 x & \varepsilon_2 x & \Gamma_1 x \\ \square & \varepsilon_3\Gamma'_1 a & \varepsilon_3 x & \Gamma'_2\Gamma_1 d & \Gamma_2\Gamma_1 d \\ \varepsilon_1^2 c & \Gamma'_1 x & \Gamma_2 x & \varepsilon_3\Gamma_1 d & \square \\ \Gamma'_2\Gamma'_1 d & \Gamma_2\Gamma_1 d & \varepsilon_3\varepsilon_2 d & \varepsilon_3\Gamma_1 d & \square \\ \Gamma'_1\Gamma_1 d & \varepsilon_2\Gamma_1 d & \varepsilon_2\Gamma_1 d & \Gamma_2\Gamma_1 d & \square \end{bmatrix}$$

(since $\Gamma_1\Gamma_1 = \Gamma_2\Gamma_1$ and $\Gamma'_1\Gamma_1 = \Gamma'_2\Gamma_1$ using (1.1.2)(i)(ii))

$$= \begin{bmatrix} \square & \Gamma'_1\Gamma'_1 a & \varepsilon_2\Gamma'_1 a & \varepsilon_2\Gamma'_1 a & \Gamma_1\Gamma'_1 a \\ \square & \varepsilon_3\Gamma'_1 a & \Gamma'_2 x & \varepsilon_2 x & \Gamma_1 x \\ \square & \varepsilon_3\Gamma'_1 a & \varepsilon_3 x & \Gamma'_2\Gamma_1 d & \Gamma_2\Gamma_1 d \\ \varepsilon_1^2 c & \Gamma'_1 x & \Gamma_2 x & \varepsilon_3\Gamma_1 d & \square \\ \Gamma'_2\Gamma'_1 d & \Gamma_2\Gamma_1 d & \varepsilon_3\varepsilon_2 d & \varepsilon_3\Gamma_1 d & \square \\ \Gamma'_1\Gamma_1 d & \varepsilon_2\Gamma_1 d & \varepsilon_2\Gamma_1 d & \Gamma_1\Gamma_1 d & \square \end{bmatrix}$$

(here the subdivision by the dotted line represents the entries which will be change and substitute by equivalent elements using the appropriate laws (see[Br-2]))

$$= \begin{bmatrix} \square & \Gamma_1'\Gamma_1'a & \varepsilon_2\Gamma_1'a & \Gamma_1\Gamma_1'a & \square \\ \square & \varepsilon_3\Gamma_1'a & \Gamma_2'x & \Gamma_1 x & \square \\ \square & \varepsilon_3\Gamma_1'a & \varepsilon_3 x & \varepsilon_3\Gamma_1 d & \square \\ \square & \Gamma_1'x & \Gamma_2 x & \varepsilon_3\Gamma_1 d & \square \\ \square & \varepsilon_3\Gamma_1 d & \varepsilon_3\varepsilon_2 d & \varepsilon_3\Gamma_1 d & \square \\ \square & \Gamma_1'\Gamma_1 d & \varepsilon_2\Gamma_1 d & \Gamma_1\Gamma_1 d & \square \end{bmatrix}$$

(since $\Gamma_2'\Gamma_1 d \circ_2 \Gamma_2\Gamma_1 d = \varepsilon_3\Gamma_1 d$ and $\Gamma_3'\Gamma_1' d \circ_2 \Gamma_3\Gamma_1' d = \varepsilon_3\Gamma_1' d$)

$$= \left[\begin{array}{c|c|c|c|c} \square & \Gamma_1'\Gamma_1'a & \varepsilon_2\Gamma_1'a & \Gamma_1\Gamma_1'a & \square \\ \hline \square & \varepsilon_3\Gamma_1'a & \Gamma_2'x & \Gamma_1 x & \square \\ \square & \varepsilon_3\Gamma_1'a & \varepsilon_3 x & \varepsilon_3\Gamma_1 d & \square \\ \hline \square & \Gamma_1'x & \Gamma_2 x & \varepsilon_3\Gamma_1 d & \square \\ \square & \varepsilon_3\Gamma_1 d & \varepsilon_3\varepsilon_2 d & \varepsilon_3\Gamma_1 d & \square \\ \hline \square & \Gamma_1'\Gamma_1 d & \varepsilon_2\Gamma_1 d & \Gamma_1\Gamma_1 d & \square \end{array}\right],$$

since row 3 and row 5 have entries either identities or of the form ε_3 , then

row 2 $\circ_3$ row 3 = row 2 and row 4 $\circ_3$ row 5 = row 4 ,

so, RHS =

$$\begin{bmatrix} \square & \square & \square & \square & \square \\ \square & \Gamma_1'\Gamma_1'a & \varepsilon_2\Gamma_1'a & \Gamma_1\Gamma_1'a & \square \\ \square & \varepsilon_3\Gamma_1'a & \Gamma_2'x & \Gamma_1 x & \square \\ \square & \Gamma_1'x & \Gamma_2 x & \varepsilon_3\Gamma_1 d & \square \\ \square & \Gamma_1'\Gamma_1 d & \varepsilon_2\Gamma_1 d & \Gamma_1\Gamma_1 d & \square \\ \square & \square & \square & \square & \square \end{bmatrix}$$

$$= \begin{bmatrix} \Gamma_1'\Gamma_1'a & \varepsilon_2\Gamma_1'a & \Gamma_1\Gamma_1'a \\ \varepsilon_3\Gamma_1'a & \Gamma_2'x & \Gamma_1 x \\ \Gamma_1'x & \Gamma_2 x & \varepsilon_3\Gamma_1 d \\ \Gamma_1'\Gamma_1 d & \varepsilon_2\Gamma_1 d & \Gamma_1\Gamma_1 d \end{bmatrix}$$

$$= \begin{bmatrix} \Gamma_1'\Gamma_1'a & \varepsilon_2\Gamma_1'a & \Gamma_1\Gamma_1'a \\ \Gamma_1'x & \varepsilon_2 x & \Gamma_1 x \\ \Gamma_1'\Gamma_1 d & \varepsilon_2\Gamma_1 d & \Gamma_1\Gamma_1 d \end{bmatrix}$$

(since $\varepsilon_3\Gamma_1'a \circ_3 \Gamma_1'x = \Gamma_1'x$, $\Gamma_2'x \circ_3 \Gamma_2 x = \varepsilon_2 x$ and $\Gamma_1 x \circ_3 \varepsilon_3\Gamma_1 d = \varepsilon_3\Gamma_1 d$)

$$= \left[\begin{array}{c|cc|} \Gamma_1'\Gamma_1'a & \varepsilon_2\Gamma_1'a & \Gamma_1\Gamma_1'a \\ \hline \Gamma_1'x & \varepsilon_2 x & \Gamma_1 x \\ \hline \Gamma_1'\Gamma_1 d & \varepsilon_2\Gamma_1 d & \Gamma_1\Gamma_1 d \end{array} \right] ,$$

since

column 2 = $\varepsilon_2(\Gamma_1'a \circ_2 x \circ_2 \Gamma_1 d)$ and column 3 = $\Gamma_1(\Gamma_1'a \circ_2 x \circ_2 \Gamma_1 d)$

then, column 2 $\circ_2$ column 3 = column 3 , and so

RHS =

$$= \begin{bmatrix} \Gamma_1'\Gamma_1'a & \Gamma_1\Gamma_1'a \\ \Gamma_1'x & \Gamma_1 x \\ \Gamma_1'\Gamma_1 d & \Gamma_1\Gamma_1 d \end{bmatrix}$$

since

column 1 = $\Gamma_1'(\Gamma_1'a \circ_2 x \circ_2 \Gamma_1 d)$ and column 2 = $\Gamma_1(\Gamma_1'a \circ_2 x \circ_2 \Gamma_1 d)$

then column 1 $\circ_2$ column 2 = $\varepsilon_1(\Gamma_1'a \circ_2 x \circ_2 \Gamma_1 d)$, and so

RHS =

$$= \begin{bmatrix} \varepsilon_1\Gamma_1'a \\ \varepsilon_1 x \\ \varepsilon_1\Gamma_1 d \end{bmatrix} = \varepsilon_1(\Gamma_1'a \circ_2 x \circ_2 \Gamma_1 d) = \varepsilon_1\psi_1 x = \Psi\varepsilon_1 x .$$

Similarly we can prove that $\Psi\Gamma_2'x = \varepsilon_1\Psi x$. Thus $\Psi\Gamma_i x = \varepsilon_1\Psi x$ and $\Psi\Gamma_i'x = \varepsilon_1\Psi x$, for $1 < n \le 3$. □

APPENDIX II

Proof of Proposition 3.2.5 :

(i) For $i = 1$, we have

$\Psi(x \circ_1 y) = \psi_1\psi_2\psi_1(x \circ_1 y)$

$= \psi_1\psi_2[(\psi_1 x \circ_2 \varepsilon_1\partial_2^1 y) \circ_1 (\varepsilon_1\partial_2^0 x \circ_2 \psi_1 y)]$ by (2.5.6)

$= \psi_1[\psi_2(\psi_1 x \circ_2 \varepsilon_1\partial_2^1 y) \circ_1 \psi_2(\varepsilon_1\partial_2^0 x \circ_2 \psi_1 y)]$ by (2.5.6)

$= \psi_1\Big\{[(\psi_2\psi_1 x \circ_3 \varepsilon_2\partial_3^1\varepsilon_1\partial_2^1 y) \circ_2 (\varepsilon_2\partial_3^0\psi_1 x \circ_3 \psi_2\varepsilon_1\partial_2^1 y)] \circ_1$

$[(\psi_2\varepsilon_1\partial_2^0 x \circ_3 \varepsilon_2\partial_3^1\psi_1 y) \circ_2 (\varepsilon_2\partial_3^0\varepsilon_1\partial_2^0 x \circ_3 \psi_2\psi_1 y)]\Big\}$ by (2.5.6)

To simplify the situation let

$A_1 = (\psi_2\psi_1 x \circ_3 \varepsilon_2\partial_3^1\varepsilon_1\partial_2^1 y)$, $A_2 = (\varepsilon_2\partial_3^0\psi_1 x \circ_3 \psi_2\varepsilon_1\partial_2^1 y)$,

$A_3 = (\psi_2\varepsilon_1\partial_2^0 x \circ_3 \varepsilon_2\partial_3^1\psi_1 y)$ and $A_4 = (\varepsilon_2\partial_3^0\varepsilon_1\partial_2^0 x \circ_3 \psi_2\psi_1 y)$, then

$\Psi(x \circ_1 y) = \psi_1[(A_1 \circ_2 A_2) \circ_1 (A_3 \circ_2 A_4)]$

$= [\psi_1(A_1 \circ_2 A_2) \circ_2 \varepsilon_1\partial_2^1(A_3 \circ_2 A_4)] \circ_1$

$[\varepsilon_1\partial_2^0(A_1 \circ_2 A_2) \circ_2 \psi_1(A_3 \circ_2 A_4)]$ by (2.5.6)

$= \Big\{[(\varepsilon_1\partial_1^0 A_1 \circ_2 \psi_1 A_2) \circ_1 (\psi_1 A_1 \circ_2 \varepsilon_1\partial_1^1 A_2)] \circ_2 \varepsilon_1\partial_2^1 A_4\Big\} \circ_1$

$\Big\{\varepsilon_1\partial_2^0 A_1 \circ_2 [(\varepsilon_1\partial_1^0 A_3 \circ_2 \psi_1 A_4) \circ_1 (\psi_1 A_3 \circ_2 \varepsilon_1\partial_1^1 A_4)]\Big\}$

by (2.5.6) and (1.2.1)(i)

We now compute each entry alone ;

$\varepsilon_1\partial_1^0 A_1 = \varepsilon_1\partial_1^0(\psi_2\psi_1 x \circ_3 \varepsilon_2\partial_3^1\varepsilon_1\partial_2^1 y)$

$= \varepsilon_1\partial_1^0\psi_2\psi_1 x \circ_3 \varepsilon_1\partial_1^0\varepsilon_2\partial_3^1\varepsilon_1\partial_2^1 y$ by (1.2.1)(i)

$= \varepsilon_1\psi_1\partial_1^0\psi_1 x \circ_3 \varepsilon_1\varepsilon_1\partial_1^0\partial_3^1\varepsilon_1\partial_2^1 y$ by (2.5.2)(i) and (1.1.1)(iii)

$$= \varepsilon_1\psi_1(\partial_1^0 x \circ_1 \partial_2^1 x) \circ_3 \varepsilon_1^2\partial_1^0\varepsilon_1\partial_2^1\partial_2^1 y$$

by (2.5.2)(ii) and (1.1.1)(iii)

$$= (\varepsilon_1\psi_1\partial_1^0 x)^g \circ_2 {}^c(\varepsilon_1\psi_1\partial_2^1 x) \circ_3 \varepsilon_1^2\partial_2^1\partial_2^1 y \qquad \text{by (2.5.6)}$$

$$= ((\varepsilon_1\psi_1\partial_1^0 x)^g \circ_2 {}^c(\varepsilon_1\psi_1\partial_2^1 x)) \circ_3 \varepsilon_1^2 u$$

$$= ((\varepsilon_1\psi_1\partial_1^0 x)^g \circ_2 {}^c(\varepsilon_1\psi_1\partial_2^1 x))^u$$

$$= (\varepsilon_1\psi_1\partial_1^0 x)^{gu} \circ_2 {}^c(\varepsilon_1\psi_1\partial_2^1 x)^u .$$

$$\psi_1 A_2 = \psi_1(\varepsilon_2\partial_3^0\psi_1 x \circ_3 \psi_2\varepsilon_1\partial_2^1 y) = \psi_1\varepsilon_2\partial_3^0\psi_1 x \circ_3 \psi_1\psi_2\varepsilon_1\partial_2^1 y$$

$$= \varepsilon_1\partial_3^0\psi_1 x \circ_3 \psi_1\varepsilon_1\psi_1\partial_2^1 y = \varepsilon_1\psi_1\partial_3^0 x \circ_3 \varepsilon_1\psi_1\partial_2^1 y ,$$

by (2.5.6) , (2.5.6)(ii)

since $\partial_1^1\varepsilon_1\psi_1\partial_3^0 = \varepsilon_1(ec')$ and $\partial_1^0\varepsilon_1\psi_1\partial_2^1 y = \varepsilon_1 d'u$, then by 3.4.1

$$\text{LHS} = \varepsilon_1\psi_1\partial_3^0 x \circ_3 \varepsilon_1\psi_1\partial_2^1 y = (\varepsilon_1\psi_1\partial_3^0 x)^{d'u} \circ_2 {}^{ec'}(\varepsilon_1\psi_1\partial_2^1 y) .$$

$$\psi_1 A_1 = \psi_1(\psi_2\psi_1 x \circ_3 \varepsilon_2\partial_3^1\varepsilon_1\partial_2^1 y) = \psi_1\psi_2\psi_1 x \circ_3 \psi_1\varepsilon_2\partial_3^1\varepsilon_1\partial_2^1 y \quad \text{by (2.5.6)}$$

$$= \Psi x \circ_3 \varepsilon_1\varepsilon_1\partial_2^1\partial_2^1 y = \Psi x \circ_3 \varepsilon_1^2 u = (\Psi x)^u . \quad \text{by (2.5.3)(ii)}$$

$$\varepsilon_1\partial_1^1 A_2 = \varepsilon_1\partial_1^1(\varepsilon_2\partial_3^0\psi_1 x \circ_3 \psi_2\varepsilon_1\partial_2^1 y) = \varepsilon_1\partial_1^1\varepsilon_2\partial_3^0\psi_1 x \circ_3 \varepsilon_1\partial_1^1\psi_2\varepsilon_1\partial_2^1 y$$

by (1.2.1)(i)

$$= \varepsilon_1\varepsilon_1\partial_1^1\psi_1\partial_3^0 x \circ_3 \varepsilon_1\psi_1\partial_1^1\varepsilon_1\partial_2^1 y \qquad \text{by (1.1.1)(iii)}$$

$$= \varepsilon_1^2(\partial_2^0\partial_3^0 x \circ_1 \partial_1^1\partial_3^0 x) \circ_3 \varepsilon_1\psi_1\partial_2^1 y \qquad \text{by (2.5.2)(iii)}$$

$$= \varepsilon_1^2 ec' \circ_3 \varepsilon_1\psi_1\partial_2^1 y = {}^{ec'}(\varepsilon_1\psi_1\partial_2^1 y) .$$

$$\varepsilon_1\partial_2^1 A_4 = \varepsilon_1\partial_2^1(\varepsilon_2\partial_3^0\varepsilon_1\partial_2^0 x \circ_3 \psi_2\psi_1 y) = \varepsilon_1\partial_2^1\varepsilon_2\partial_3^0\varepsilon_1\partial_2^0 x \circ_3 \varepsilon_1\partial_2^1\psi_2\psi_1 y$$

by (1.2.1)(i)

$$= \varepsilon_1\varepsilon_1\partial_1^1\varepsilon_1\partial_2^0\partial_2^0 x \circ_3 \varepsilon_1(\partial_3^0\psi_1 y \circ_1 \partial_2^1\psi_1 y)$$

by (1.1.1)(iii) and (2.5.2)(iii)

$$= \varepsilon_1^2\partial_2^0\partial_2^0 x \circ_3 \varepsilon_1(\psi_1\partial_3^0 y \circ_2 \varepsilon_1\partial_1^1\partial_1^1 y)$$

by (1.1.1)(iii) and (2.5.2)(i)

$$= \varepsilon_1^2 e \circ_3 \varepsilon_1\psi_1\partial_3^0 y \circ_3 \varepsilon_1\varepsilon_1 d'' = {}^{e}(\varepsilon_1\psi_1\partial_3^0 y)^{d''} .$$

$$\varepsilon_1\partial_2^0 A_1 = \varepsilon_1\partial_2^0(\psi_2\psi_1 x \circ_3 \varepsilon_2\partial_3^1\varepsilon_1\partial_2^1 y) = \varepsilon_1\partial_2^0\psi_2\psi_1 x \circ_3 \varepsilon_1\partial_2^0\varepsilon_2\partial_3^1\varepsilon_1\partial_2^1 y$$

by (1.2.1)(i)

$$= \varepsilon_1(\partial_2^0\psi_1 x \circ_2 \partial_3^1\psi_1 x) \circ_3 \varepsilon_1\varepsilon_1\partial_2^1\partial_2^1 y \qquad \text{by (2.5.2)(ii)}$$

$$= (\varepsilon_1(\varepsilon_1\partial_1^0\partial_1^0 x \circ_2 \psi_1\partial_3^1 x)) \circ_3 \varepsilon_1^2 u$$

$$= \varepsilon_1^2 a \circ_3 \psi_1\partial_3^1 x \circ_3 \varepsilon_1^2 u = {}^{a}(\psi_1\partial_3^1 x)^{u} .$$

$$\varepsilon_1\partial_1^0 A_3 = \varepsilon_1\partial_1^0(\psi_2\varepsilon_1\partial_2^0 x \circ_3 \varepsilon_2\partial_3^1\psi_1 y) = \varepsilon_1\partial_1^0\psi_2\varepsilon_1\partial_2^0 x \circ_3 \varepsilon_1\partial_1^0\varepsilon_2\partial_3^1\psi_1 y$$

by (1.2.1)(i)

$$= \varepsilon_1\psi_1\partial_1^0\varepsilon_1\partial_2^0 x \circ_3 \varepsilon_1\varepsilon_1\partial_1^0\psi_1\partial_3^1 y$$

by (2.5.2)(i) and (1.1.1)(iii)

$$= \varepsilon_1\psi_1\partial_2^0 x \circ_3 \varepsilon_1^2(\partial_1^0\partial_3^1 y \circ_1 \partial_2^1\partial_3^1 y) \qquad \text{by (2.5.2)(ii)}$$

$$= \varepsilon_1\psi_1\partial_2^0 x \circ_3 \varepsilon_1^2(b' \circ_1 u) = (\varepsilon_1\psi_1\partial_2^0 x)^{b'u} .$$

$$\psi_1 A_4 = \psi_1(\varepsilon_2\partial_3^0\varepsilon_1\partial_2^0 x \circ_3 \psi_2\psi_1 y) = \psi_1\varepsilon_2\partial_3^0\varepsilon_1\partial_2^0 x \circ_3 \psi_1\psi_2\psi_1 y \text{ by (2.5.6)}$$

$$= \varepsilon_1\varepsilon_1\partial_2^0\partial_2^0 x \circ_3 \Psi y = \varepsilon_1^2 e \circ_3 \Psi y = {}^{e}(\Psi y) \, .$$ by (2.5.3)(ii)

$$\psi_1 A_3 = \psi_1(\psi_2\varepsilon_1\partial_2^0 x \circ_3 \varepsilon_2\partial_3^1\psi_1 y) = \psi_1\psi_2\varepsilon_1\partial_2^0 x \circ_3 \psi_1\varepsilon_2\partial_3^1\psi_1 y$$ by (2.5.6)

$$= \psi_1\varepsilon_1\psi_1\partial_2^0 x \circ_3 \varepsilon_1\psi_1\partial_3^1 y = \varepsilon_1\psi_1\partial_2^0 x \circ_3 \varepsilon_1\psi_1\partial_3^1 y \, ,$$

since $\partial_1^1\varepsilon_1\psi_1\partial_2^0 x = \varepsilon_1(ea')$ and $\partial_1^0\varepsilon_1\psi_1\partial_3^1 y = b'u$, then by 3.4.1

$$\text{LHS} = \varepsilon_1\psi_1\partial_2^0 x \circ_3 \varepsilon_1\psi_1\partial_3^1 y = (\varepsilon_1\psi_1\partial_2^0 x)^{b'u} \circ_2 {}^{ea'}(\varepsilon_1\psi_1\partial_3^1 y) \, .$$

$$\varepsilon_1\partial_1^1 A_4 = \varepsilon_1\partial_1^1(\varepsilon_2\partial_3^0\varepsilon_1\partial_2^0 x \circ_3 \psi_2\psi_1 y) = \varepsilon_1\partial_1^1\varepsilon_2\partial_3^0\varepsilon_1\partial_2^0 x \circ_3 \varepsilon_1\partial_1^1\psi_2\psi_1 y$$

by (1.2.1)(i)

$$= \varepsilon_1\varepsilon_1\partial_1^1\varepsilon_1\partial_2^0\partial_2^0 x \circ_3 \varepsilon_1\psi_1\partial_1^1\psi_1 y$$ by (1.1.1)(iii)

$$= \varepsilon_1^2\partial_2^0\partial_2^0 x \circ_3 \varepsilon_1\psi_1(\partial_2^0 y \circ_1 \partial_1^1 y)$$

by (1.1.1)(iii) and (2.5.2)(iii)

$$= \varepsilon_1^2 e \circ_3 ((\varepsilon_1\psi_1\partial_2^0 y)^{b''} \circ_2 {}^{s}(\varepsilon_1\psi_1\partial_1^1 y))$$

$$= {}^{e}(\varepsilon_1\psi_1\partial_2^0 y)^{b''} \circ_2 {}^{es}(\varepsilon_1\psi_1\partial_1^1 y) \, .$$

We now come back to our original equation

$$\Psi(x \circ_1 y) = \left\{[(\varepsilon_1\partial_1^0 A_1 \circ_2 \psi_1 A_2) \circ_1 (\psi_1 A_1 \circ_2 \varepsilon_1\partial_1^1 A_2)] \circ_2 \varepsilon_1\partial_2^1 A_4\right\} \circ_1$$

$$\left\{\varepsilon_1\partial_2^0 A_1 \circ_2 [(\varepsilon_1\partial_1^0 A_3 \circ_2 \psi_1 A_4) \circ_1 (\psi_1 A_3 \circ_2 \varepsilon_1\partial_1^1 A_4)]\right\}$$

and compute the entries as follows

$$[(\varepsilon_1\partial_1^0 A_1 \circ_2 \psi_1 A_2) \circ_1 (\psi_1 A_1 \circ_2 \varepsilon_1\partial_1^1 A_2)] =$$

$$[(\varepsilon_1\psi_1\partial_1^0 x)^{gu} \circ_2 {}^{c}(\varepsilon_1\psi_1\partial_2^1 x)^{u} \circ_2 (\varepsilon_1\psi_1\partial_3^0 x)^{d'u} \circ_2 {}^{ec'}(\varepsilon_1\psi_1\partial_2^1 y)] \circ_1$$

$$[(\Psi x)^{u} \circ_2 {}^{ec'}(\varepsilon_1\psi_1\partial_2^1 y)]$$

$$= [((\varepsilon_1\psi_1\partial_1^0 x)^g \circ_2 {}^c(\varepsilon_1\psi_1\partial_2^1 x) \circ_2 (\varepsilon_1\psi_1\partial_3^0 x)^{d'})^u \circ_1 (\Psi x)^u] \circ_2$$

$$[{}^{ec'}(\varepsilon_1\psi_1\partial_2^1 y) \circ_1 {}^{ec'}(\varepsilon_1\psi_1\partial_2^1 y)]$$

$$= [((\varepsilon_1\psi_1\partial_1^0 x)^g \circ_2 {}^c(\varepsilon_1\psi_1\partial_2^1 x) \circ_2 (\varepsilon_1\psi_1\partial_3^0 x)^{d'}) \circ_1 (\Psi x)]^u \circ_2$$

$${}^{ec'}(\varepsilon_1\psi_1\partial_2^1 y)$$

$$= (\Psi x)^u \circ_2 {}^{ec'}(\varepsilon_1\psi_1\partial_2^1 y) \ .$$

and

$$[(\varepsilon_1\partial_1^0 A_3 \circ_2 \psi_1 A_4) \circ_1 (\psi_1 A_3 \circ_2 \varepsilon_1\partial_1^1 A_4)] =$$

$$[(\varepsilon_1\psi_1\partial_2^0 x)^{b'u} \circ_2 {}^e(\Psi y)] \circ_1 [(\varepsilon_1\psi_1\partial_2^0 x)^{b'u} \circ_2 {}^{ea'}(\varepsilon_1\psi_1\partial_3^1 y) \circ_2$$

$${}^e(\varepsilon_1\psi_1\partial_2^0 y)^{b''} \circ_2 {}^{es}(\varepsilon_1\psi_1\partial_1^1 y)$$

$$= [(\varepsilon_1\psi_1\partial_2^0 x)^{b'u} \circ_1 (\varepsilon_1\psi_1\partial_2^0 x)^{b'u}] \circ_2$$

$$[{}^e(\Psi y) \circ_1 {}^e(({}^{a'}(\varepsilon_1\psi_1\partial_3^1 y) \circ_2 (\varepsilon_1\psi_1\partial_2^0 y)^{b''} \circ_2 {}^s(\varepsilon_1\psi_1\partial_1^1 y))] \circ_2$$

$$= (\varepsilon_1\psi_1\partial_2^0 x)^{b'u} \circ_2$$

$${}^e[(\Psi y) \circ_1 (({}^{a'}(\varepsilon_1\psi_1\partial_3^1 y) \circ_2 (\varepsilon_1\psi_1\partial_2^0 y)^{b''} \circ_2 {}^s(\varepsilon_1\psi_1\partial_1^1 y))]$$

$= (\varepsilon_1\psi_1\partial_2^0 x)^{b'u} \circ_2 {}^e(\Psi y)$, by the folded face formula of (Ψy) .

Thus the final evaluation of our equation is

$$\Psi(x \circ_1 y) = \left\{[(\varepsilon_1\partial_1^0 A_1 \circ_2 \psi_1 A_2) \circ_1 (\psi_1 A_1 \circ_2 \varepsilon_1\partial_1^1 A_2)] \circ_2 \varepsilon_1\partial_2^1 A_4\right\} \circ_1$$

$$\left\{\varepsilon_1\partial_2^0 A_1 \circ_2 [(\varepsilon_1\partial_1^0 A_3 \circ_2 \psi_1 A_4) \circ_1 (\psi_1 A_3 \circ_2 \varepsilon_1\partial_1^1 A_4)]\right\}$$

$$= [(\Psi x)^u \circ_2 {}^{ec'}(\varepsilon_1\psi_1\partial_2^1 y) \circ_2 {}^e(\varepsilon_1\psi_1\partial_3^0 y)^{d''}] \circ_1$$

$$[{}^a(\psi_1\partial_3^1 x)^u \circ_2 (\varepsilon_1\psi_1\partial_2^0 x)^{b'u} \circ_2 {}^e(\Psi y)] \ .$$

For the other cases we will not mention the laws since they are the same as those used in (i)

(ii) for $i = 2$, we have

$$\Psi(x \circ_2 y) = \psi_1\psi_2\psi_1(x \circ_2 y)$$

$$= \psi_1\psi_2[(\varepsilon_1\partial_1^0 x \circ_2 \psi_1 y) \circ_1 (\psi_1 x \circ_2 \varepsilon_1\partial_1^1 y)]$$

$$= \psi_1[\psi_2(\varepsilon_1\partial_1^0 x \circ_2 \psi_1 y) \circ_1 \psi_2(\psi_1 x \circ_2 \varepsilon_1\partial_1^1 y)]$$

$$= \psi_1\{[(\psi_2\varepsilon_1\partial_1^0 x \circ_3 \varepsilon_2\partial_3^1\psi_1 y) \circ_2 (\varepsilon_2\partial_3^0\varepsilon_1\partial_1^0 x \circ_3 \psi_2\psi_1 y)] \circ_1$$
$$[(\psi_2\psi_1 x \circ_3 \varepsilon_2\partial_3^1\varepsilon_1\partial_1^1 y) \circ_2 (\varepsilon_2\partial_3^0\psi_1 x \circ_3 \psi_2\varepsilon_1\partial_1^1 y)]\} .$$

To simplify the situation we write

$A_1 = (\psi_2\varepsilon_1\partial_1^0 x \circ_3 \varepsilon_2\partial_3^1\psi_1 y)$, $A_2 = (\varepsilon_2\partial_3^0\varepsilon_1\partial_1^0 x \circ_3 \psi_2\psi_1 y)$,

$A_3 = (\psi_2\psi_1 x \circ_3 \varepsilon_2\partial_3^1\varepsilon_1\partial_1^1 y)$ and $A_4 = (\varepsilon_2\partial_3^0\psi_1 x \circ_3 \psi_2\varepsilon_1\partial_1^1 y)$, so our equation becomes

$$\Psi(x \circ_2 y) = \psi_1\{[A_1 \circ_2 A_2] \circ_1 [A_3 \circ_2 A_4]\}$$
$$= (\psi_1[A_1 \circ_2 A_2] \circ_2 \varepsilon_1\partial_2^1[A_3 \circ_2 A_4]) \circ_1$$
$$(\varepsilon_1\partial_2^0[A_1 \circ_2 A_2] \circ_2 \psi_1[A_3 \circ_2 A_4])$$
$$= \{[(\varepsilon_1\partial_1^0 A_1 \circ_2 \psi_1 A_2) \circ_1 (\psi_1 A_1 \circ_2 \varepsilon_1\partial_1^1 A_2)] \circ_2 \varepsilon_1\partial_2^1 A_4\} \circ_1$$
$$\{\varepsilon_1\partial_2^0 A_1 \circ_2 [(\varepsilon_1\partial_1^0 A_3 \circ_2 \psi_1 A_4) \circ_1 (\psi_1 A_3 \circ_2 \varepsilon_1\partial_1^1 A_4)]\} ,$$

we calculate each entry alone , so we have

$$\varepsilon_1\partial_1^0 A_1 = \varepsilon_1\partial_1^0(\psi_2\varepsilon_1\partial_1^0 x \circ_3 \varepsilon_2\partial_3^1\psi_1 y) = \varepsilon_1\partial_1^0\psi_2\varepsilon_1\partial_1^0 x \circ_3 \varepsilon_1\partial_1^0\varepsilon_2\partial_3^1\psi_1 y$$
$$= \varepsilon_1\psi_1\partial_1^0\varepsilon_1\partial_1^0 x \circ_3 \varepsilon_1\varepsilon_1\partial_1^0\psi_1\partial_3^1 y$$
$$= \varepsilon_1\psi_1\partial_1^0 x \circ_3 \varepsilon_1^2(\partial_1^0\partial_3^1 y \circ_1 \partial_2^1\partial_3^1 y)$$
$$= \varepsilon_1\psi_1\partial_1^0 x \circ_3 \varepsilon_1^2(rg') = (\varepsilon_1\psi_1\partial_1^0 x)^{rg'},$$

$$\psi_1 A_2 = \psi_1(\varepsilon_2\partial_3^0\varepsilon_1\partial_1^0 x \circ_3 \psi_2\psi_1 y) = \psi_1\varepsilon_2\partial_3^0\varepsilon_1\partial_1^0 x \circ_3 \psi_1\psi_2\psi_1 y$$

$$= \varepsilon_1\varepsilon_1\partial_2^0\partial_1^0 x \circ_3 \Psi y$$

$$= \varepsilon_1^2 c \circ_3 \Psi y$$

$$= {}^{c}(\Psi y) \ ,$$

$$\psi_1 A_1 = \psi_1(\psi_2\varepsilon_1\partial_1^0 x \circ_3 \varepsilon_2\partial_3^1\psi_1 y) = \psi_1\psi_2\varepsilon_1\partial_1^0 x \circ_3 \psi_1\varepsilon_2\partial_3^1\psi_1 y$$

$$= \psi_1\varepsilon_1\psi_1\partial_1^0 x \circ_3 \varepsilon_1\partial_3^1\psi_1 y$$

$$= \varepsilon_1\psi_1\partial_1^0 x \circ_3 \varepsilon_1\psi_1\partial_3^1 y$$

$$= \varepsilon_1\psi_1\partial_1^0 x \circ_3 \varepsilon_1\psi_1\partial_3^1 y \ ,$$

since $\partial_1^1\varepsilon_1\psi_1\partial_1^0 x = \text{cd}$ and $\partial_1^0\varepsilon_1\psi_1\partial_3^1 y$ then by 3.4.1

$$\text{LHS} = (\varepsilon_1\psi_1\partial_1^0 x)^{rg'} \circ_2 {}^{cd}(\varepsilon_1\psi_1\partial_3^1 y) \ .$$

$$\varepsilon_1\partial_1^1 A_2 = \varepsilon_1\partial_1^1(\varepsilon_2\partial_3^0\varepsilon_1\partial_1^0 x \circ_3 \psi_2\psi_1 y) = \varepsilon_1\partial_1^1\varepsilon_2\partial_3^0\varepsilon_1\partial_1^0 x \circ_3 \varepsilon_1\partial_1^1\psi_2\psi_1 y$$

$$= \varepsilon_1\varepsilon_1\partial_1^1\partial_3^0\varepsilon_1\partial_1^0 x \circ_3 \varepsilon_1\psi_1\partial_1^1\psi_1 y$$

$$= \varepsilon_1^2\partial_1^1\varepsilon_1\partial_2^0\partial_1^0 x \circ_3 \varepsilon_1\psi_1(\partial_2^0 y \circ_1 \partial_1^1 y)$$

$$= \varepsilon_2^2 c \circ_3 \varepsilon_1[(\psi_1\partial_2^0 y \circ_2 \varepsilon_1\partial_2^1\partial_1^1 y) \circ_1 (\varepsilon_1\partial_2^0\partial_2^0 y \circ_2 \psi_1\partial_1^1 y)]$$

$$= \varepsilon_2^2 c \circ_3 [(\varepsilon_1\psi_1\partial_2^0 y \circ_3 \varepsilon_1\varepsilon_1\partial_2^1\partial_1^1 y) \circ_2 (\varepsilon_1^2\partial_2^0\partial_2^0 y \circ_3 \varepsilon_1\psi_1\partial_1^1 y)]$$

$$= \varepsilon_2^2 c \circ_3 [(\psi_1\varepsilon_1\partial_2^0 y \circ_3 \varepsilon_1^2 r') \circ_2 (\varepsilon_1^2 f \circ_3 \psi_1\varepsilon_1\partial_1^1 y)]$$

$$= {}^{c}(\varepsilon_1\psi_1\partial_2^0 y \circ_3 \varepsilon_1^2 r') \circ_2 {}^{c}(\varepsilon_1^2 f \circ_3 \varepsilon_1\psi_1\partial_1^1 y)$$

$$= {}^{c}((\varepsilon_1\psi_1\partial_2^0 y)^{r'}) \circ_2 {}^{c}({}^{f}(\varepsilon_1\psi_1\partial_1^1 y)) = {}^{c}(\varepsilon_1\psi_1\partial_2^0 y)^{r'} \circ_2 {}^{cf}(\varepsilon_1\psi_1\partial_1^1 y) \ ,$$

$$\varepsilon_1\partial_2^1 A_4 = \varepsilon_1\partial_2^1(\varepsilon_2\partial_3^0\psi_1 x \circ_3 \psi_2\varepsilon_1\partial_1^1 y) = \varepsilon_1\partial_2^1\varepsilon_2\partial_3^0\psi_1 x \circ_3 \varepsilon_1\partial_2^1\psi_2\varepsilon_1\partial_1^1 y$$

$$= \varepsilon_1\psi_1\partial_3^0 x \circ_3 \varepsilon_1\partial_2^1\varepsilon_1\psi_1\partial_1^1 y$$

$$= \varepsilon_1\psi_1\partial_3^0 x \circ_3 \varepsilon_1\varepsilon_1\partial_1^1\psi_1\partial_1^1 y$$

$$= \varepsilon_1\psi_1\partial_3^0 x \circ_3 \varepsilon_1^2(\partial_2^0\partial_1^1 y \circ_1 \partial_1^1\partial_1^1 y)$$

$$= \varepsilon_1\psi_1\partial_3^0 x \circ_3 \varepsilon_1^2(p'q') = (\varepsilon_1\psi_1\partial_3^0 x)^{p'q'}$$

$$\varepsilon_1\partial_2^0 A_1 = \varepsilon_1\partial_2^0(\psi_2\varepsilon_1\partial_1^0 x \circ_3 \varepsilon_2\partial_3^1\psi_1 y) = \varepsilon_1\partial_2^0\psi_2\varepsilon_1\partial_1^0 x \circ_3 \varepsilon_1\partial_2^0\varepsilon_2\partial_3^1\psi_1 y$$

$$= \varepsilon_1(\partial_2^0\varepsilon_1\partial_1^0 x \circ_2 \partial_3^1\varepsilon_1\partial_1^0 x) \circ_3 \varepsilon_1\partial_3^1\psi_1 y$$

$$= \varepsilon_1(\varepsilon_1\partial_1^0\partial_1^0 x \circ_2 \varepsilon_1\partial_2^1\partial_1^0 x) \circ_3 \varepsilon_1\psi_1\partial_3^1 y$$

$$= \varepsilon_1^2(\partial_1^0\partial_1^0 x \circ_1 \partial_2^1\partial_1^0 x) \circ_3 \varepsilon_1\psi_1\partial_3^1 y$$

$$= \varepsilon_1^2(a \circ_1 b) \circ_3 \varepsilon_1\psi_1\partial_3^1 y$$

$$= {}^{ab}(\varepsilon_1\psi_1\partial_3^1 y)$$

$$\varepsilon_1\partial_1^0 A_3 = \varepsilon_1\partial_1^0(\psi_2\psi_1 x \circ_3 \varepsilon_2\partial_3^1\varepsilon_1\partial_1^1 y) = \varepsilon_1\partial_1^0\psi_2\psi_1 x \circ_3 \varepsilon_1\partial_1^0\varepsilon_2\partial_3^1\varepsilon_1\partial_1^1 y$$

$$= \varepsilon_1\psi_1\partial_1^0\psi_1 x \circ_3 \varepsilon_1\varepsilon_1\partial_1^0\varepsilon_1\partial_2^1\partial_1^1 y$$

$$= \varepsilon_1\psi_1(\partial_1^0 x \circ_1 \partial_2^1 x) \circ_3 \varepsilon_1^2\partial_2^1\partial_1^1 y$$

$$= \varepsilon_1[(\psi_1\partial_1^0 x \circ_2 \varepsilon_1\partial_2^1\partial_2^1 x) \circ_1 (\varepsilon_1\partial_2^0\partial_1^0 x \circ_2 \psi_1\partial_2^1 x)] \circ_3 \varepsilon_1^2 r'$$

$$= [(\varepsilon_1\psi_1\partial_1^0 x \circ_3 \varepsilon_1^2 g) \circ_2 (\varepsilon_1^2 c \circ_3 \varepsilon_1\psi_1\partial_2^1 x)] \circ_3 \varepsilon_1^2 r'$$

$$= [(\varepsilon_1\psi_1\partial_1^0 x)^g) \circ_2 {}^c(\varepsilon_1\psi_1\partial_2^1 x)]^{r'}$$

$$= (\varepsilon_1\psi_1\partial_1^0 x)^{gr'} \circ_2 {}^c(\varepsilon_1\psi_1\partial_2^1 x)^{r'} ,$$

$$\psi_1 A_4 = \psi_1(\varepsilon_2\partial_3^0\psi_1 x \circ_3 \psi_2\varepsilon_1\partial_1^1 y) = \psi_1\varepsilon_2\partial_3^0\psi_1 x \circ_3 \psi_1\psi_2\varepsilon_1\partial_1^1 y$$

$$= \varepsilon_1\psi_1\partial_3^0 x \circ_3 \psi_1\varepsilon_1\psi_1\partial_1^1 y$$

$$= \varepsilon_1\psi_1\partial_3^0 x \circ_3 \varepsilon_1\psi_1\partial_1^1 y = \varepsilon_1\psi_1\partial_3^0 x \circ_3 \varepsilon_1\psi_1\partial_1^1 y$$

since $\partial_1^1\psi_1\varepsilon_1\partial_3^0 x = ec'$ and $\partial_1^0\psi_1\varepsilon_1\partial_1^1 y = d'r'$, then by 3.4.1

$$\text{LHS} = (\varepsilon_1\psi_1\partial_3^0 x)^{d'r'} \circ_2 {}^{ec'}(\varepsilon_1\psi_1\partial_1^1 y)$$

$$\psi_1 A_3 = \psi_1(\psi_2\psi_1 x \circ_3 \varepsilon_2\partial_3^1\varepsilon_1\partial_1^1 y) = \psi_1\psi_2\psi_1 x \circ_3 \psi_1\varepsilon_2\partial_3^1\varepsilon_1\partial_1^1 y$$
$$= \Psi x \circ_3 \varepsilon_1\partial_3^1\varepsilon_1\partial_1^1 y = \Psi x \circ_3 \varepsilon_1\varepsilon_1\partial_2^1\partial_1^1 y$$
$$= \Psi x \circ_3 \varepsilon_1^2 r' = (\Psi x)^{r'}$$

$$\varepsilon_1\partial_1^1 A_4 = \varepsilon_1\partial_1^1(\varepsilon_2\partial_3^0\psi_1 x \circ_3 \psi_2\varepsilon_1\partial_1^1 y) = \varepsilon_1\partial_1^1\varepsilon_2\partial_3^0\psi_1 x \circ_3 \varepsilon_1\partial_1^1\psi_2\varepsilon_1\partial_1^1 y$$
$$= \varepsilon_1\varepsilon_1\partial_1^1\psi_1\partial_3^0 x \circ_3 \varepsilon_1\psi_1\partial_1^1\varepsilon_1\partial_1^1 y$$
$$= \varepsilon_1^2(\partial_2^0\partial_3^0 x \circ_1 \partial_1^1\partial_3^0 x) \circ_3 \varepsilon_1\psi_1\partial_1^1 y$$
$$= \varepsilon_1^2(e \circ_1 c') \circ_3 \psi_1\varepsilon_1\partial_1^1 y = {}^{ec'}(\psi_1\varepsilon_1\partial_1^1 y) .$$

We come back to our original equation

$$\Psi(x \circ_2 y) = \{[(\varepsilon_1\partial_1^0 A_1 \circ_2 \psi_1 A_2) \circ_1 (\psi_1 A_1 \circ_2 \varepsilon_1\partial_1^1 A_2)] \circ_2 \varepsilon_1\partial_2^1 A_4\} \circ_1$$
$$\{\varepsilon_1\partial_2^0 A_1 \circ_2 [(\varepsilon_1\partial_1^0 A_3 \circ_2 \psi_1 A_4) \circ_1 (\psi_1 A_3 \circ_2 \varepsilon_1\partial_1^1 A_4)]\} ,$$

and compute the entries as follows

$$[(\varepsilon_1\partial_1^0 A_1 \circ_2 \psi_1 A_2) \circ_1 (\psi_1 A_1 \circ_2 \varepsilon_1\partial_1^1 A_2)] =$$
$$[(\varepsilon_1\psi_1\partial_1^0 x)^{rg'} \circ_2 {}^{c}(\Psi y)] \circ_1 [(\varepsilon_1\psi_1\partial_1^0 x)^{rg'} \circ_2 {}^{cd}(\varepsilon_1\psi_1\partial_3^1 y)$$
$${}^{c}(\varepsilon_1\psi_1\partial_2^0 y)^{r'} \circ_2 {}^{cf}(\varepsilon_1\psi_1\partial_1^1 y)]$$
$$= [(\varepsilon_1\psi_1\partial_1^0 x)^{rg'} \circ_2 (\varepsilon_1\psi_1\partial_1^0 x)^{rg'}] \circ_2$$
$${}^{c}[(\Psi y) \circ_1 ({}^{d}(\varepsilon_1\psi_1\partial_3^1 y) \circ_2 (\varepsilon_1\psi_1\partial_2^0 y)^{r'} \circ_2 {}^{f}(\varepsilon_1\psi_1\partial_1^1 y))]$$
$$= (\varepsilon_1\psi_1\partial_1^0 x)^{rg'} \circ_2 {}^{c}(\Psi y)$$, by the folded face formula of Ψy .

and

$$[(\varepsilon_1\partial_1^0 A_3 \circ_2 \psi_1 A_4) \circ_1 (\psi_1 A_3 \circ_2 \varepsilon_1\partial_1^1 A_4)] =$$

$$[(\varepsilon_1\psi_1\partial_1^0 x)^{gr'} \circ_2 {}^{c}(\varepsilon_1\psi_1\partial_2^1 x)^{r'} \circ_2 (\varepsilon_1\psi_1\partial_3^0 x)^{d'r'} \circ_2 {}^{ec'}(\varepsilon_1\psi_1\partial_1^1 y)] \circ_1$$

$$[(\Psi x)^{r'} \circ_2 {}^{ec'}(\psi_1\varepsilon_1\partial_1^1 y)]$$

$$= [((\varepsilon_1\psi_1\partial_1^0 x)^{g} \circ_2 {}^{c}(\varepsilon_1\psi_1\partial_2^1 x) \circ_2 (\varepsilon_1\psi_1\partial_3^0 x)^{d'})^{r'} \circ_1 (\Psi x)^{r'}] \circ_2$$

$$[{}^{ec'}(\psi_1\varepsilon_1\partial_1^1 y) \circ_1 {}^{ec'}(\psi_1\varepsilon_1\partial_1^1 y)]$$

$$= [((\varepsilon_1\psi_1\partial_1^0 x)^{g} \circ_2 {}^{c}(\varepsilon_1\psi_1\partial_2^1 x) \circ_2 (\varepsilon_1\psi_1\partial_3^0 x)^{d'})^{r'} \circ_1 (\Psi x)]^{r'} \circ_2$$

$$[{}^{ec'}(\psi_1\varepsilon_1\partial_1^1 y)]$$

$$= (\Psi x)^{r'} \circ_2 {}^{ec'}(\psi_1\varepsilon_1\partial_1^1 y) .$$

Thus the final evaluation of our equation is

$$\Psi(x \circ_2 y) = \{[(\varepsilon_1\partial_1^0 A_1 \circ_2 \psi_1 A_2) \circ_1 (\psi_1 A_1 \circ_2 \varepsilon_1\partial_1^1 A_2)] \circ_2 \varepsilon_1\partial_2^1 A_4\} \circ_1$$

$$\{\varepsilon_1\partial_2^0 A_1 \circ_2 [(\varepsilon_1\partial_1^0 A_3 \circ_2 \psi_1 A_4) \circ_1 (\psi_1 A_3 \circ_2 \varepsilon_1\partial_1^1 A_4)]\} ,$$

$$= [(\varepsilon_1\psi_1\partial_1^0 x)^{rg'} \circ_2 {}^{c}(\Psi y) \circ_2 (\varepsilon_1\psi_1\partial_3^0 x)^{p'q'}] \circ_1$$

$$[{}^{ab}(\varepsilon_1\psi_1\partial_3^1 y) \circ_2 (\Psi x)^{r'} \circ_2 {}^{ec'}(\psi_1\varepsilon_1\partial_1^1 y)] .$$

(iii) for $i = 3$, we have

$$\Psi(x \circ_3 t) = \psi_1\psi_2\psi_1(x \circ_3 t)$$

$$\psi_1\psi_2(\psi_1 x \circ_3 \psi_1 t) = \psi_1[(\varepsilon_2\partial_2^0\psi_1 x \circ_3 \psi_2\psi_1 t) \circ_2 (\psi_2\psi_1 x \circ_3 \varepsilon_2\partial_2^1\psi_1 t)]$$

$$= [\varepsilon_1\partial_1^0(\varepsilon_2\partial_2^0\psi_1 x \circ_3 \psi_2\psi_1 t) \circ_2 \psi_1(\psi_2\psi_1 x \circ_3 \varepsilon_2\partial_2^1\psi_1 t)] \circ_1$$

$$[\psi_1(\varepsilon_2\partial_2^0\psi_1 x \circ_3 \psi_2\psi_1 t) \circ_2 \varepsilon_1\partial_1^1(\psi_2\psi_1 x \circ_3 \varepsilon_2\partial_2^1\psi_1 t)]$$

$$= [(\varepsilon_1\partial_1^0\varepsilon_2\partial_2^0\psi_1 x \circ_3 \varepsilon_1\partial_1^0\psi_2\psi_1 t) \circ_2 (\psi_1\psi_2\psi_1 x \circ_3 \psi_1\varepsilon_2\partial_2^1\psi_1 t)] \circ_1$$

$$[(\psi_1\varepsilon_2\partial_2^0\psi_1 x \circ_3 \psi_1\psi_2\psi_1 t) \circ_2 (\varepsilon_1\partial_1^1\psi_2\psi_1 x \circ_3 \varepsilon_1\partial_1^1\varepsilon_2\partial_2^1\psi_1 t)]$$

$$= [(\varepsilon_1\varepsilon_1\partial_1^0\varepsilon_1\partial_1^0\partial_1^0 x \circ_3 \varepsilon_1\psi_1\partial_1^0\psi_1 t) \circ_2 (\Psi x \circ_3 \varepsilon_1\varepsilon_1\partial_1^1\partial_1^1 t)] \circ_1$$

$$[(\varepsilon_1\varepsilon_1\partial_1^0\partial_1^0 x \circ_3 \Psi t) \circ_2 (\varepsilon_1\psi_1\partial_1^1\psi_1 x \circ_3 \varepsilon_1\varepsilon_1\partial_1^1\varepsilon_1\partial_1^1\partial_1^1 t)]$$

$$= [(\varepsilon_1^2\partial_1^0\partial_1^0 x \circ_3 \varepsilon_1\psi_1(\partial_1^0 t \circ_1 \partial_2^1 t)) \circ_2 (\Psi x \circ_3 \varepsilon_1^2 w')] \circ_1$$

$$[(\varepsilon_1^2 a \circ_3 \Psi t) \circ_2 (\varepsilon_1\psi_1(\partial_2^0 x \circ_1 \partial_1^1 x) \circ_3 \varepsilon_1^2 w')]$$

$$= [(\varepsilon_1^2 a \circ_3 \varepsilon_1\{(\psi_1\partial_1^0 t \circ_2 \varepsilon_1 g'') \circ_1 (\varepsilon_1 b \circ_2 \psi_1\partial_2^1 t)\}) \circ_2 (\Psi x)^{w'}] \circ_1$$

$$[({}^{a}(\Psi t) \circ_2 \varepsilon_1\{(\psi_1\partial_2^0 x \circ_2 \varepsilon_1 b') \circ_1 (\varepsilon_1 e \circ_2 \psi_1\partial_1^1 x)\}) \circ_3 \varepsilon_1^2 w')]$$

$$= [(\varepsilon_1^2 a \circ_3 \{(\varepsilon_1\psi_1\partial_1^0 t \circ_3 \varepsilon_1^2 g'') \circ_2 (\varepsilon_1^2 b \circ_3 \varepsilon_1\psi_1\partial_2^1 t)\}) \circ_2 (\Psi x)^{w'}] \circ_1$$

$$[({}^{a}(\Psi t) \circ_2 \{(\varepsilon_1\psi_1\partial_2^0 x \circ_3 \varepsilon_1^2 b') \circ_2 (\varepsilon_1^2 e \circ_3 \varepsilon_1\psi_1\partial_1^1 x)\}) \circ_3 \varepsilon_1^2 w')]$$

$$= [{}^{a}(\varepsilon_1\psi_1\partial_1^0 t \circ_3 \varepsilon_1^2 g'') \circ_2 {}^{a}(\varepsilon_1^2 b \circ_3 \varepsilon_1\psi_1\partial_2^1 t) \circ_2 (\Psi x)^{w'}] \circ_1$$

$$[{}^{a}(\Psi t) \circ_2 (\varepsilon_1\psi_1\partial_2^0 x \circ_3 \varepsilon_1^2 b')^{w'} \circ_2 (\varepsilon_1^2 e \circ_3 \varepsilon_1\psi_1\partial_1^1 x)^{w'}]$$

$$= [{}^{a}((\varepsilon_1\psi_1\partial_1^0 t)^{g''}) \circ_2 {}^{a}({}^{b}(\varepsilon_1\psi_1\partial_2^1 t)) \circ_2 (\Psi x)^{w'}] \circ_1$$

$$[{}^{a}(\Psi t) \circ_2 ((\varepsilon_1\psi_1\partial_2^0 x)^{b'})^{w'} \circ_2 {}^{e}((\varepsilon_1\psi_1\partial_1^1 x)^{w'})]$$

$$= [{}^{a}(\varepsilon_1\psi_1\partial_1^0 t)^{g''} \circ_2 {}^{ab}(\varepsilon_1\psi_1\partial_2^1 t) \circ_2 (\Psi x)^{w'}] \circ_1$$

$$[{}^{a}(\Psi t) \circ_2 (\varepsilon_1\psi_1\partial_2^0 x)^{b'w'} \circ_2 {}^{e}(\varepsilon_1\psi_1\partial_1^1 x)^{w'}]$$

$$= [{}^{a}(\Psi\varepsilon_1\partial_1^0 t)^{g''} \circ_2 {}^{ab}(\Psi\varepsilon_1\partial_2^1 t) \circ_2 (\Psi x)^{w'}] \circ_1$$

$$[{}^{a}(\Psi t) \circ_2 (\Psi\varepsilon_1\partial_2^0 x)^{b'w'} \circ_2 {}^{e}(\Psi\varepsilon_1\partial_1^1 x)^{w'}] .$$

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