# The Nonlinear Time Lag Multivariable Gray Prediction Model Based on Interval Gray Numbers and Its Application[1]

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**Abstract**: The linear relationship of the original gray model does not consider the time lag effect of the current input parameters on the output parameters. In order to solve the problem, this paper takes an interval gray number sequence as the modeling sequence of the model. The nonlinear parameter $\gamma$ and the time lag parameter $\tau$ are introduced into the variable gray prediction color model, and a nonlinear time lag multi-variable gray prediction model ($\mathrm{MGM}(1, m|\tau, \gamma)$) for interval gray numbers is constructed. In view of the uncertain characteristics of the smog index data, this paper applies the improved model to the simulation and prediction of the smog index data. Compared with the original model, the results show that the prediction effect of the model proposed in this paper is superior to the original model in terms of its effectiveness and feasibility.

**Keywords**: gray system; interval gray number; nonlinear; time lag; smog

## 1 Introduction

Smog comes from a large amount of fine particulate emissions, calm weather and other factors. They have an adverse effect on the ecological environment and human physical and mental health in the world [1-2]. Many scholars have carried out relevant research on smog. Scholars usually use sampling analysis, factor analysis, regression analysis and other methods to predict the content of $PM_{10}$ and $PM_{2.5}$ related to smog in the air, and then combine meteorological elements to analyze and predict smog [3-6].

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These studies have made a certain contribution to the prediction of smog. However, these studies have not considered the uncertainty and hysteresis of the smog index data, and their prediction still has low accuracy. In view of uncertainty, hysteresis and non-linearity of the indicator data, the non-linear time-lag multi-variable gray forecasting model proposed in this paper can simulate and predict it more accurately.

The gray prediction model has been applied in many fields such as economy, production, and life. The gray prediction model theory has found the internal development law of the system by means of differential equations [7-11]. Among them, the GM (1, n) model can simulate data well. Many scholars have conducted in-depth research and improvement on the traditional model GM (1, n) model. These studies have demonstrated significant impacts in the related fields [12-14].

Among many research results of these scholars, the GM (1, m) model converts the "single variable model" to "multiple associated variable models". Relevant scholars have proposed $\mathrm{MGM}(1,\mathrm{m})$ model and verified that it can solve the issue of gray dynamic systems modeling with multiple interrelated factors [15]. The GM (1, m) is characterized by extending the gray prediction model that is only applicable to a single point to multi-point prediction. It has a wider range of applications. Other scholars have also confirmed through a sequence of examples that the accuracy of the GM (1, m) model is higher than that of the $\mathrm{GM}(1,1)$ alone [16]. Due to the high accuracy of the simulation and prediction of the GM (1, m), it has a wide range of applications and many scholars have performed further studies and improvements on their characteristics and optimization of background values [17-20]. These studies have driven the development of the traditional $\mathrm{MGM}(1,\mathrm{m})$, and laid a solid foundation for further innovation and improvement of the model. This paper will also study and improve the model.

Because the simulation and prediction of the smog index data sequence is the main content of the smog prediction, it is of great significance to the prevention and control of the smog. This paper will combine the characteristics of the smog index data sequence to improve the model. The process of forecasting the smog index data needs to consider three problems. Firstly, in actual application, the smog data and related prediction systems have a time lag. Secondly, the indicator data is nonlinear. Thirdly, the original model is suitable for ordinary real numbers. However, in many cases, the smog data involves gray numbers with uncertainty rather than simple real numbers.

In summary, this paper will make the following improvements to the original $\mathrm{MGM}(1,\mathrm{m})$: for the time lag issue of the indicator data, this paper introduces the time lag parameter $\tau$; for the original model with linear function relationship, this paper introduces the nonlinear parameter $\gamma$. With respect to the uncertainty, this paper will build a model based on interval gray numbers, so that the model can be applied to simulate and predict such uncertain data more efficiently. As a result, this paper constitutes a new model for interval gray numbers: a non-linear time lag multi-variable gray prediction model $MGM(1, m|\tau, \gamma)$. Subsequently, this paper applies the new model to the simulation and prediction of the smog indicator AQI and $PM_{2.5}$, and compares its results with the original model to verify the simulation and prediction accuracy of the new model.

## 2 Basic concepts of interval gray numbers

**Definition 1** [7] The gray number with both lower bound $a_k$ and upper bound $b_k$ is called the interval gray number and is written as $\otimes(k) \in [a_k, b_k]$, where, if $a_k = b_k$, the gray number $\otimes(k)$ degenerates to a real number $a_k$.

There is no perfect calculation rule for the interval gray number, so the kernel and degree of greyness are defined so that the real number algorithm can be used for related calculations. There are different definitions proposed. This paper uses the definition suitable for continuous interval gray numbers, taking the mean of the upper and lower bound as the kernel. The ratio of a gray number measurement to its universe is defined as its degree of greyness, as follows.

**Definition 2** [7] Suppose continuous gray number $\otimes(k) \in [a_k, b_k]$, $a_k \leq b_k$, in the absence of the distribution information of $\otimes(k)$, $\widetilde{\otimes}(k) = \frac{a_k+b_k}{2}$ is called the kernel of gray number $\otimes(k)$.

**Definition 3** [7]. Let the background of theorem $\Omega_k$ be generated by the gray number $\otimes(k)$, $\mu(\otimes_k)$ is the measure of number domain of the gray number $\otimes(k)$,then $g^{\circ}(\otimes_k) = \frac{\mu(\otimes_k)}{\mu(\Omega_k)}$ is the degree of greyness of the gray number $\otimes(k)$.

**Definition 4** [7] The sequence composed of interval gray numbers is called $X(\otimes) = \{\otimes(1), \otimes(2), \cdots, \otimes(n)\}$, where $\otimes(k) \in [a_k, b_k]$, $a_k \leq b_k$, $k = 1,2,\cdots n$. The sequence composed of the kernels and the sequence for the degree of greyness of all elements in the interval gray number sequence $X(\otimes)$ are called the kernel sequence $X(\widetilde{\otimes})$ and degree of greyness sequence $G^{\circ}(\otimes)$ of $X(\otimes)$ respectively,which can be written as

$$X(\widetilde{\otimes}) = \{\widetilde{\otimes}(1), \widetilde{\otimes}(2), \cdots, \widetilde{\otimes}(n)\}, \quad G^{\circ}(\otimes) = \{g^{\circ}(\otimes_1), g^{\circ}(\otimes_2), \cdots, g^{\circ}(\otimes_n)\}$$

## 3 Construction of a nonlinear multi-variable gray prediction model with time lag

### 3.1 Modeling mechanism of non-linear time lag multi-variable gray prediction model

The gray prediction model will encounter many problems in actual applications. The input value of the index data at a certain moment not only affects the simulated forecast output value of the index data at that moment, but also influences the output value after that moment. That is, it has a time lag. Considering that the time lag phenomenon is common in practical applications, this paper improves the original multi-variable gray prediction model by introducing the time lag parameter $\tau$. To deal with the nonlinear relationship of the index data for the original multi-variable gray prediction model, a nonlinear parameter $\gamma$ is also introduced to construct a nonlinear time lag prediction model.

**Definition 5** Let $x_1, x_2, \cdots, x_m$ represent a set of variables obtaining their observation values at time $-l, \cdots, -1,0,1,2, \cdots, n$. Here, $l, n$ are positive integers, and $X_j^{(0)}(k) = \left\{x_j^{(0)}(1), x_j^{(0)}(2), \cdots, x_j^{(0)}(n)\right\}$ is the sequence of observations at positive time, that is, $X_j^{(0)}(k)$ is the observation sequence of the j-th variable at $k = 1,2, \cdots, n$.

$X_j^{(0)}(k-\tau) = \left\{x_j^{(0)}(1-\tau), x_j^{(0)}(2-\tau), \cdots, x_j^{(0)}(n-\tau)\right\}$ is the $\tau$-phase lag sequence of sequence $X_j^{(0)}(k)$. That is, $X_j^{(0)}(k-\tau)$ is the sequence of observations of the variable at $= 1-\tau, 2-\tau, \cdots, n-\tau$, where $0 < \tau \leq l+1$.

$X_j^{(1)}(k-\tau) = \left\{x_j^{(1)}(1-\tau), x_j^{(1)}(2-\tau), \cdots, x_j^{(1)}(n-\tau)\right\}$ is the sequence of first-order accumulation of sequence $X_j^{(0)}(k-\tau)$, where $x_j^{(1)}(k-\tau) = \sum_{i=1}^{k} x_j^{(0)}(i-\tau)$, $k = 1,2, \cdots, n$

The $Z_j^{(1)}(k-\tau) = \left\{z_j^{(1)}(1-\tau), z_j^{(1)}(2-\tau), \cdots, z_j^{(1)}(n-\tau)\right\}$ is a sequence for the immediate mean of sequence $X_j^{(1)}(k-\tau)$, where$z_j^{(1)}(k-\tau) = 0.5(x_j^{(1)}(k-\tau-1) + x_j^{(1)}(k-\tau))$, $k = 2,3, \cdots, n$. Then a multivariate time lag nonlinear discrete $\mathrm{MGM}(1, \mathrm{m}|\tau, \gamma)$ model can be obtained:

$$\begin{cases} x_1^{(0)}(k) = a_{11}(z_1^{(1)}(k-\tau))^{\gamma_1} + a_{12}(z_2^{(1)}(k-\tau))^{\gamma_2} + \cdots + a_{1m}(z_m^{(1)}(k-\tau))^{\gamma_m} + b_1 \\ x_2^{(0)}(k) = a_{21}(z_1^{(1)}(k-\tau))^{\gamma_1} + a_{22}(z_2^{(1)}(k-\tau))^{\gamma_2} + \cdots + a_{2m}(z_m^{(1)}(k-\tau))^{\gamma_m} + b_2 \\ \vdots \\ x_m^{(0)}(k) = a_{m1}(z_1^{(1)}(k-\tau))^{\gamma_1} + a_{m2}(z_2^{(1)}(k-\tau))^{\gamma_2} + \cdots + a_{mm}(z_m^{(1)}(k-\tau))^{\gamma_m} + b_m \end{cases}$$

Among them, the power exponent is $\gamma_j$ ($j = 1,2, \cdots m$), the lag parameter is $\tau$, and the number of lag periods is $k = 2,3, \cdots, n$; when the multi-variable time lag nonlinear gray prediction model is used in the actual simulation prediction, the system development coefficient $a_{ij}$ and the gray effect $b_j$ are required.Where $\widehat{\boldsymbol{a}}_j = \left(\hat{a}_{j1}, \hat{a}_{j2}, \cdots, \hat{a}_{jm}, \hat{b}_j\right)^T$, $i, j = 1,2, \cdots, m$.

Obviously, when $\tau = 0$, $\gamma_j = 1$ ($j = 1,2,\cdots m$), the multivariate time lag nonlinear discrete model $\mathrm{MGM}(1,\mathrm{m}|\tau,\gamma)$ is the classic discrete model $\mathrm{MGM}(1,\mathrm{m})$. Therefore, the time lag nonlinear discrete model $\mathrm{MGM}(1,\mathrm{m}|\tau,\gamma)$ is a further extension of the model $\mathrm{MGM}(1,\mathrm{m})$.

**Theorem 1** Suppose that at time $-l,\cdots,-1,0,1,2,\cdots,n$, the observation values obtained by the variable $x_1,x_2,\cdots x_m$ are non-negative data, and the sequence $X_j^{(0)}(k)$, $X_j^{(0)}(k-\tau)$, $Z_j^{(1)}(k-\tau)$ are as described in Definition 5, $j=1,2,\cdots,m$, $k=1,2,\cdots,n$, then, the minimum quadratic estimation of m parameter vectors of the multivariate nonlinear discrete-time model $\mathrm{MGM}(1,\mathrm{m}|\tau,\gamma)$ is

$$\widehat{\boldsymbol{a}}_j = \left(\hat{a}_{j1},\hat{a}_{j2},\cdots,\hat{a}_{jm},\hat{b}_j\right)^T = (P^TP)^{-1}P^TY_j \tag{2}$$

Where $P = \begin{bmatrix} (z_1^{(1)}(2-\tau))^{\gamma_1} & (z_2^{(1)}(2-\tau))^{\gamma_2} & \cdots & (z_m^{(1)}(2-\tau))^{\gamma_m} & 1 \\ (z_1^{(1)}(3-\tau))^{\gamma_1} & (z_2^{(1)}(3-\tau))^{\gamma_2} & \cdots & (z_m^{(1)}(3-\tau))^{\gamma_m} & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ (z_1^{(1)}(n-\tau))^{\gamma_1} & (z_2^{(1)}(n-\tau))^{\gamma_2} & \cdots & (z_m^{(1)}(n-\tau))^{\gamma_m} & 1 \end{bmatrix}$, $Y_j = \begin{bmatrix} x_j^{(0)}(2) \\ x_j^{(0)}(3) \\ \vdots \\ x_j^{(0)}(n) \end{bmatrix}$, thus

The identification value $\hat{A} = (\hat{a}_{ij})_{m\times m}$, $\hat{B} = (\hat{b}_1,\hat{b}_2,\cdots,\hat{b}_m)^T$ of the parameter matrix A and B are

Proof : Substitute $X_j^{(0)}(k)$、$Z_j^{(1)}(k-\tau)$ into (1), that is $x_j^{(0)}(k) = a_{j1}(z_1^{(1)}(k-\tau))^{\gamma_1} + a_{j2}(z_2^{(1)}(k-\tau))^{\gamma_2} + \cdots + a_{jm}(z_m^{(1)}(k-\tau))^{\gamma_m} + b_j$, $j=1,2,\cdots,m$

$$\begin{cases} x_j^{(0)}(2) = a_{j1}(z_1^{(1)}(2-\tau))^{\gamma_1} + a_{j2}(z_2^{(1)}(2-\tau))^{\gamma_2} + \cdots + a_{jm}(z_m^{(1)}(2-\tau))^{\gamma_m} + b_j \\ x_j^{(0)}(3) = a_{j1}(z_1^{(1)}(3-\tau))^{\gamma_1} + a_{j2}(z_2^{(1)}(3-\tau))^{\gamma_2} + \cdots + a_{jm}(z_m^{(1)}(3-\tau))^{\gamma_m} + b_j \\ \vdots \\ x_j^{(0)}(n) = a_{j1}(z_1^{(1)}(n-\tau))^{\gamma_1} + a_{j2}(z_2^{(1)}(n-\tau))^{\gamma_2} + \cdots + a_{jm}(z_m^{(1)}(n-\tau))^{\gamma_m} + b_j \end{cases}$$

That is, $Y_j = P\boldsymbol{a}_j$, $\boldsymbol{a}_j = \left(a_{j1},a_{j2},\cdots,a_{jm},b_j\right)^T$, for the set of estimated values of $\hat{a}_{j1},\hat{a}_{j2},\cdots,\hat{a}_{jm},\hat{b}_j$, $\hat{a}_{j1}(z_1^{(1)}(k-\tau))^{\gamma_1} + \hat{a}_{j2}(z_2^{(1)}(k-\tau))^{\gamma_2} + \cdots + \hat{a}_{jm}(z_m^{(1)}(k-\tau))^{\gamma_m} + \hat{b}_j$ is used instead of $x_j^{(0)}(k)$, so that the error sequence $\varepsilon_j = Y_j - P\widehat{\boldsymbol{a}}_j$ can be obtained; if $s_j = \varepsilon_j^T\varepsilon_j = (Y_j - P\widehat{\boldsymbol{a}}_j)^T(Y_j - P\widehat{\boldsymbol{a}}_j) = \left[x_j^{(0)}(k) - \sum_{l=1}^{m}\hat{a}_{jl}(z_l^{(1)}(k-\tau))^{\gamma_l} - \hat{b}_j\right]^2$ is set, the

$\hat{a}_{j1}, \hat{a}_{j2}, \cdots, \hat{a}_{jm}, \hat{b}_j$ that minimizes $s_j$ should satisfy

$$\begin{cases} \dfrac{\partial s_j}{\partial \hat{a}_{j1}} = 2\left[\hat{b}_j - x_j^{(0)}(k) + \sum_{l=1}^{m} \hat{a}_{jl}(z_l^{(1)}(k-\tau))^{\gamma_l}\right](z_1^{(1)}(k-\tau))^{\gamma_1} = 0 \\ \dfrac{\partial s_j}{\partial \hat{a}_{j2}} = 2\left[\hat{b}_j - x_j^{(0)}(k) + \sum_{l=1}^{m} \hat{a}_{jl}(z_l^{(1)}(k-\tau))^{\gamma_l}\right](z_2^{(1)}(k-\tau))^{\gamma_2} = 0 \\ \vdots \\ \dfrac{\partial s_j}{\partial \hat{a}_{jm}} = 2\left[\hat{b}_j - x_j^{(0)}(k) + \sum_{l=1}^{m} \hat{a}_{jl}(z_l^{(1)}(k-\tau))^{\gamma_l}\right](z_m^{(1)}(k-\tau))^{\gamma_m} = 0 \\ \dfrac{\partial s_j}{\partial \hat{b}_j} = 2\left[\hat{b}_j - x_j^{(0)}(k) + \sum_{l=1}^{m} \hat{a}_{jl}(z_l^{(1)}(k-\tau))^{\gamma_l}\right] = 0 \end{cases}$$

$\hat{\boldsymbol{a}}_j = \left(\hat{a}_{j1}, \hat{a}_{j2}, \cdots, \hat{a}_{jm}, \hat{b}_j\right)^T = (P_j^T P_j)^{-1} P_j^T Y_j$; thus the identification value of parameter matrix A and parameter vector B can be obtained, as $\hat{A} = (\hat{a}_{ij})_{m\times m}$, $\hat{B} = (\hat{b}_1, \hat{b}_2, \cdots, \hat{b}_m)$ .

**Theorem 2** Suppose the estimated value of the structural parameters of the multi-variable time lag nonlinear $\mathrm{MGM}(1, \mathrm{m}|\tau, \gamma)$ discrete model is shown in Theorem 1. Take $\hat{x}_j^{(0)}(1) = x_j^{(0)}(1)$, $j = 1,2,\cdots,m$, and when $k \geq 2$, the discrete solution of the model is

$$\hat{X}^{(0)}(k) = \hat{A} Z^{(1)^{\gamma}}(k-\tau) + \hat{B} \tag{3}$$

Where $\hat{X}^{(0)}(k) = \begin{bmatrix} \hat{x}_1^{(0)}(k) \\ \hat{x}_2^{(0)}(k) \\ \vdots \\ \hat{x}_m^{(0)}(k) \end{bmatrix}$, $Z^{(1)^{\gamma}}(k-\tau) = \begin{bmatrix} (z_1^{(1)}(k-\tau))^{\gamma_1} \\ (z_2^{(1)}(k-\tau))^{\gamma_2} \\ \vdots \\ (z_m^{(1)}(k-\tau))^{\gamma_m} \end{bmatrix}$

**Proof** : When $k = 1$, take $\hat{x}_j^{(0)}(1) = x_j^{(0)}(1)$, $j = 1,2,\cdots,m$;

When $k \geq 2$, the $\left\{(z_1^{(1)}(k-\tau))^{\gamma_1}, (z_2^{(1)}(k-\tau))^{\gamma_2}, \cdots, (z_m^{(1)}(k-\tau))^{\gamma_m}\right\}^T$ is known, and substitute it to solve the identification values of the parameter matrix A and the parameter vector B, as $\hat{A} = (\hat{a}_{ij})_{m\times m}$, $B = (\hat{b}_1, \hat{b}_2, \cdots, \hat{b}_m)^T$, and directly simulate and predict the multi-variable

$$\begin{cases} \hat{x}_1^{(0)}(k) = \hat{a}_{11}(z_1^{(1)}(k-\tau))^{\gamma_1} + \hat{a}_{12}(z_2^{(1)}(k-\tau))^{\gamma_2} + \cdots + \hat{a}_{1m}(z_m^{(1)}(k-\tau))^{\gamma_m} + \hat{b}_1 \\ \hat{x}_2^{(0)}(k) = \hat{a}_{21}(z_1^{(1)}(k-\tau))^{\gamma_1} + \hat{a}_{22}(z_2^{(1)}(k-\tau))^{\gamma_2} + \cdots + \hat{a}_{2m}(z_m^{(1)}(k-\tau))^{\gamma_m} + \hat{b}_2 \\ \vdots \\ \hat{x}_m^{(0)}(k) = \hat{a}_{m1}(z_1^{(1)}(k-\tau))^{\gamma_1} + \hat{a}_{m2}(z_2^{(1)}(k-\tau))^{\gamma_2} + \cdots + \hat{a}_{mm}(z_m^{(1)}(k-\tau))^{\gamma_m} + \hat{b}_m \end{cases}$$

When $k = 2,3,\cdots,n$, we can get the analog value of the observation sequence of positive time, when $k \geq n+1$, we can get the prediction value of the observation sequence of positive time;

$\hat{X}^{(0)}(k) = \begin{bmatrix} \hat{x}_1^{(0)}(k) \\ \hat{x}_2^{(0)}(k) \\ \vdots \\ \hat{x}_m^{(0)}(k) \end{bmatrix}$, $Z^{(1)^\gamma}(k-\tau) = \begin{bmatrix} (z_1^{(1)}(k-\tau))^{\gamma_1} \\ (z_2^{(1)}(k-\tau))^{\gamma_2} \\ \vdots \\ (z_m^{(1)}(k-\tau))^{\gamma_m} \end{bmatrix}$, then the matrix form of the above multivariate nonlinear equations is $\hat{X}^{(0)}(k) = \hat{A}Z^{(1)^\gamma}(k-\tau) + \hat{B}$ t, as shown in formula (3).

### 3.2 Determination of time lag parameters

This section determines the time lag parameters of the nonlinear time lag multi-variable gray prediction model.

**Definition 6** Suppose the variables $x_1, x_2, \cdots, x_m$ obtain their observation values at the $-l, \cdots, -1,0,1,2, \cdots, n$ , let the current system characteristic sequence $X_j^{(0)}(k) = \left\{x_j^{(0)}(1), x_j^{(0)}(2), \cdots, x_j^{(0)}(n)\right\}$, that is the positive time observation value sequence, be the reference time sequence, and the $\tau$ period lag sequence $X_i^{(0)}(k-\tau) = \left\{x_i^{(0)}(1-\tau), x_i^{(0)}(2-\tau), \cdots, x_i^{(0)}(n-\tau)\right\}$ corresponding to the variable itself or another variable is the comparative time sequence, $i, j = 1,2, \cdots, m$ . Sequences $X'^{(0)}_i(k-\tau) = \left\{x'^{(0)}_i(1-\tau), x'^{(0)}_i(2-\tau), \cdots, x'^{(0)}_i(n-\tau)\right\}$ and $X'^{(0)}_j(k) = \left\{x'^{(0)}_j(1), x'^{(0)}_j(2), \cdots, x'^{(0)}_j(n)\right\}$ are the zero-zero image of sequence $X_i^{(0)}(k-\tau)$ and $X_j^{(0)}(k)$, respectively, where $x'^{(0)}_i(k-\tau) = x_i^{(0)}(k-\tau) - x_i^{(0)}(1-\tau)$, $x'^{(0)}_j(k) = x_j^{(0)}(k) - x_j^{(0)}(1)$, $k = 1,2, \cdots, n$. Then, time lag gray absolute correlation between the sequence $X_j^{(0)}(k)$ and $X_i^{(0)}(k-\tau)$ is:

$$\varepsilon_{ji} = \frac{1+|s_i|+|s_j|}{1+|s_i|+|s_j|+|s_i-s_j|} \tag{4}$$

The method determining the time lag parameter $\tau$: when the number of lag periods $\tau$ of the sequence $X_i^{(0)}(k-\tau)$ takes different values, the value of the gray absolute correlation $\varepsilon_{ji}$ of the time lag is different. The gray absolute correlation $\varepsilon_{ji}$ of $X_i^{(0)}(k-\tau)$ corresponding to the maximum $\varepsilon_{ji}$ is the time lag t between the sequence $X_i^{(0)}(k-\tau)$ compared with $X_j^{(0)}(k)$ . That is [23]:

$$\tau_{ji} = \tau_{max\,\varepsilon_{ji}} \tag{5}$$

$\tau_{ji}$ is also called the number of time lag of the variable $x_i$ relative to the variable $x_j$. Note that when the sequence of positive observations selected is $X_i^{(0)}(k)$ and the lag sequence is $X_j^{(0)}(k-\tau)$ ($i \neq j$), the time lag gray absolute correlation $\varepsilon_{ij}$ between the sequence $X_i^{(0)}(k)$ and $X_j^{(0)}(k-\tau)$ is different from that between $X_j^{(0)}(k)$ and $X_i^{(0)}(k-\tau)$. Therefore $\tau_{ji} \neq \tau_{ij}$, the time lag number of the variable $x_i$ relative to the variable $x_j$ is different from that of the variable $x_j$ relative to the variable $x_i$.

The arithmetic mean value of the time lag corresponding to all variables is determined as the final time lag parameter of the model,

$$\tau = \frac{1}{n^2}\sum_{j=1}^{n}\sum_{i=1}^{n}\tau_{ji} \tag{6}$$

This paper does not consider the case when the time lag parameter is a non-integer, and in which case it can be approximated with an integer value. That is, $\tau \in N^+$.

**3.3 Determination of nonlinear parameters**

The system variables in real life often have complex nonlinear relationships. The power exponent $\gamma_j$ ($j = 1,2,\cdots m$) introduced by the time lag nonlinear discrete $\mathrm{MGM}(1,\mathrm{m}|\tau,\gamma)$) reflects the nonlinear effect of the j-th lag variable on the current system variable. In the modeling process After the time-lag parameter $\tau$ is determined, the parameter $\gamma_j$, which reflects the nonlinear relationship between the variables, is unknown. The specific value of the nonlinear parameter $\gamma_j$ must be determined in advance to estimate the structural parameters, and then to solve the discrete solution of the model. In order to improve the accuracy of the model, the nonlinear parameter $\gamma_j$ can be solved by constructing a nonlinear optimization model that minimizes the average relative error and constrains the relationship between model parameters:

$$\min_{\gamma_j} avg(e(k)) = \frac{1}{m(n-1)}\sum_{j=1}^{m}\sum_{k=2}^{n}\left|\frac{x_j^{(0)}(k)-\hat{x}_j^{(0)}(k)}{x_j^{(0)}(k)}\right|, j = 1,2,\cdots,m \tag{7}$$

$$s.t.\begin{cases}\hat{x}_j^{(0)}(k) = \hat{a}_{j1}(z_1^{(1)}(k-\tau))^{\gamma_1} + \cdots + \hat{a}_{jm}(z_m^{(1)}(k-\tau))^{\gamma_m} + \hat{b}_j \\ \hat{\boldsymbol{a}}_j = (\hat{a}_{j1}, \hat{a}_{j2}, \cdots, \hat{a}_{jm}, \hat{b}_j)^T\end{cases}$$

$k = 2,3,\cdots,n$, $j = 1,2,\cdots,m$.

It can be seen that $\gamma_j$ determines the size of the average relative error $avg(e(k))$. In order to enable the model to predict the index data more accurately, $avg(e(k))$ needs to be minimized. The minimum value of $\gamma_j$ ($j = 1,2,\cdots m$) can be selected to determine the model, so that the model can be used to simulate the actual index data for predictions.

## 4 Construction of a non-linear time lag multi-variable gray prediction model for interval gray numbers

### 4.1 Calculation of kernel sequence and degree of greyness sequence of multivariate interval gray number sequence

For multi-variable gray number sequence $X_j^{(0)}(\otimes) = \left\{\otimes_j^{(0)}(1), \otimes_j^{(0)}(2), \cdots, \otimes_j^{(0)}(n)\right\}$, according to definition 2 and definition 3, this paper calculates the kernel sequence $X_j^{(0)}(\tilde{\otimes}) = \left\{\tilde{\otimes}_j^{(0)}(1), \tilde{\otimes}_j^{(0)}(2), \cdots, \tilde{\otimes}_j^{(0)}(n)\right\}$ and gray level sequence $G^{\circ(0)}(\otimes) = \left\{g_j^{\circ(0)}(\otimes_1), g_j^{\circ(0)}(\otimes_2), \cdots, g_j^{\circ(0)}(\otimes_n)\right\}$ of the interval gray number sequence corresponding to m variables, where

$$\begin{cases} \tilde{\otimes}_j^{(0)}(k) = \frac{a_{jk}+b_{jk}}{2} \\ g_j^{\circ(0)}(\otimes_k) = \frac{b_{jk}-a_{jk}}{\mu(\Omega_{jk})} \end{cases} \tag{8}$$

### 4.2 The establishment of a non-linear gray prediction model of kernel sequence

The model of multi-variable gray number kernel sequence $\mathrm{X}_{\mathrm{j}}^{(0)}(\tilde{\otimes}) = \left\{\tilde{\otimes}_{\mathrm{j}}^{(0)}(1), \tilde{\otimes}_{\mathrm{j}}^{(0)}(2), \cdots, \tilde{\otimes}_{\mathrm{j}}^{(0)}(\mathrm{n})\right\}(\mathrm{j} = 1,2,\cdots,\mathrm{m})$ is $\mathrm{MGM}(1,\mathrm{m}|\tau,\gamma)$

$$\begin{cases} \tilde{\otimes}_1^{(0)}(k) = a_{11}(z_1^{(1)}(\tilde{\otimes})(k-\tau))^{\gamma_1} + a_{12}(z_2^{(1)}(\tilde{\otimes})(k-\tau))^{\gamma_2} + \cdots + a_{1m}(z_m^{(1)}(\tilde{\otimes})(k-\tau))^{\gamma_m} + b_1 \\ \tilde{\otimes}_2^{(0)}(k) = a_{21}(z_1^{(1)}(\tilde{\otimes})(k-\tau))^{\gamma_1} + a_{22}(z_2^{(1)}(\tilde{\otimes})(k-\tau))^{\gamma_2} + \cdots + a_{2m}(z_m^{(1)}(\tilde{\otimes})(k-\tau))^{\gamma_m} + b_2 \\ \vdots \\ \tilde{\otimes}_m^{(0)}(k) = a_{m1}(z_1^{(1)}(\tilde{\otimes})(k-\tau))^{\gamma_1} + a_{m2}(z_2^{(1)}(\tilde{\otimes})(k-\tau))^{\gamma_2} + \cdots + a_{mm}(z_m^{(1)}(\tilde{\otimes})(k-\tau))^{\gamma_m} + b_m \end{cases}$$

Where $k = 2,3,\cdots,n$, $\boldsymbol{\tau}$ is the lag parameters of the system, reflecting the number of lag periods of the variables that affect the current system variables. .$\gamma_j$（$j = 1,2,\cdots m$） is the power index of the lag system, which can reflect the nonlinear effect of the j-th lag variable on the current system. $X_j^{(1)}(\tilde{\otimes}) = \left\{\tilde{\otimes}_j^{(1)}(1), \tilde{\otimes}_j^{(1)}(2), \cdots, \tilde{\otimes}_j^{(1)}(n)\right\}$ is the first-order accumulation sequence of kernel sequence $X_j^{(0)}(\tilde{\otimes})$. $\tilde{\otimes}_j^{(1)}(k) = \sum_{l=1}^{k}\tilde{\otimes}_j^{(0)}(l)$, $j = 1,2,\cdots,m$, $k = 1,2,\cdots,n$.

The estimated parameter value of the model obtained by the least square method is $(\hat{\boldsymbol{a}}_1, \hat{\boldsymbol{a}}_2, \cdots, \hat{\boldsymbol{a}}_m) = (P^T P)^{-1} P^T (Q_1, Q_2, \cdots, Q_m)$

Where $\hat{\boldsymbol{a}}_i = \begin{bmatrix} \hat{a}_{i1} \\ \hat{a}_{i2} \\ \vdots \\ \hat{a}_{im} \\ \hat{b}_i \end{bmatrix}$, $P = \begin{bmatrix} z_1^{(1)}(\widetilde{\otimes}_2) & z_2^{(1)}(\widetilde{\otimes}_2) & \cdots & z_m^{(1)}(\widetilde{\otimes}_2) & 1 \\ z_1^{(1)}(\widetilde{\otimes}_3) & z_2^{(1)}(\widetilde{\otimes}_3) & \cdots & z_m^{(1)}(\widetilde{\otimes}_3) & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ z_1^{(1)}(\widetilde{\otimes}_n) & z_2^{(1)}(\widetilde{\otimes}_n) & \cdots & z_m^{(1)}(\widetilde{\otimes}_n) & 1 \end{bmatrix}$, $Q_i = \begin{bmatrix} \widetilde{\otimes}_i^{(0)}(2) \\ \widetilde{\otimes}_i^{(0)}(3) \\ \vdots \\ \widetilde{\otimes}_i^{(0)}(n) \end{bmatrix}$,

$i = 1,2,\cdots,m. Z_j^{(1)}(\widetilde{\otimes}) = \left\{z_j^{(1)}(\widetilde{\otimes}_2), z_j^{(1)}(\widetilde{\otimes}_3), \cdots, z_j^{(1)}(\widetilde{\otimes}_n)\right\}$ is is the sequence of the immediate mean of $X_j^{(1)}(\widetilde{\otimes})$, $z_j^{(1)}(\widetilde{\otimes}_k) = 0.5(\widetilde{\otimes}_j^{(1)}(k-1) + \widetilde{\otimes}_j^{(1)}(k))$, $j = 1,2,\cdots,m$, $k = 2,3,\cdots,n$.And the identification value of parameter matrix A and parameter vector B are $\hat{A} = (\hat{a}_{ij})_{m\times m}$, $\hat{B} = (\hat{b}_1, \hat{b}_2, \cdots, \hat{b}_m)^T$,

Taking $\widehat{\widetilde{\otimes}}_j^{(0)}(1) = \widetilde{\otimes}_j^{(0)}(1)$, $j = 1,2,\cdots,m$, when $k \geq 2$, the discrete solution of the model is:

$$\hat{X}^{(0)}(\widetilde{\otimes}_k) = \hat{A}Z^{(1)^\gamma}(\widetilde{\otimes})(k-\tau) + \hat{B} \tag{9}$$

$$\text{Where} \quad \hat{X}^{(0)}(\widetilde{\otimes}_k) = \begin{bmatrix} \widehat{\widetilde{\otimes}}_1^{(0)}(k) \\ \widehat{\widetilde{\otimes}}_2^{(1)}(k) \\ \vdots \\ \widehat{\widetilde{\otimes}}_m^{(1)}(k) \end{bmatrix}, \quad Z^{(1)^\gamma}(\widetilde{\otimes}_k) = \begin{bmatrix} (z_1^{(1)}(\widetilde{\otimes})(k-\tau))^{\gamma_1} \\ (z_2^{(1)}(\widetilde{\otimes})(k-\tau))^{\gamma_2} \\ \vdots \\ (z_m^{(1)}(\widetilde{\otimes})(k-\tau))^{\gamma_m} \end{bmatrix} \tag{10}$$

**4.3 Establishment of multi-variable gray prediction model with nonlinear time lag for degree of greyness sequence**

The $\mathrm{MGM}(1,\mathrm{m}|\tau,\gamma)$ model for multi-variable gray number gray sequence $G^{\circ(0)}(\otimes) = \left\{g_j^{\circ(0)}(\otimes_1), g_j^{\circ(0)}(\otimes_2), \cdots, g_j^{\circ(0)}(\otimes_n)\right\}$ $(j = 1,2,\cdots,m)$ is as follows:

$$\begin{cases} g_j^{\circ(0)}(\otimes_1)(k) = c_{11}(z_1^{(1)}g^\circ(\otimes_{k-\tau}))^{\gamma_1} + c_{12}(z_2^{(1)}g^\circ(\otimes_{k-\tau}))^{\gamma_2} + \cdots + c_{1m}(z_m^{(1)}g^\circ(\otimes_{k-\tau}))^{\gamma_m} + d_1 \\ g_j^{\circ(0)}(\otimes_2)(k) = c_{21}(z_1^{(1)}g^\circ(\otimes_{k-\tau}))^{\gamma_1} + c_{22}(z_2^{(1)}g^\circ(\otimes_{k-\tau}))^{\gamma_2} + \cdots + c_{2m}(z_m^{(1)}g^\circ(\otimes_{k-\tau}))^{\gamma_m} + d_2 \\ \vdots \\ g_j^{\circ(0)}(\otimes_m)(k) = c_{m1}(z_1^{(1)}g^\circ(\otimes_{k-\tau}))^{\gamma_1} + c_{m2}(z_2^{(1)}g^\circ(\otimes_{k-\tau}))^{\gamma_2} + \cdots + c_{mm}(z_m^{(1)}g^\circ(\otimes_{k-\tau}))^{\gamma_m} + d_m \end{cases}$$

It is noted as

$$\hat{G}^{\circ(0)}(\otimes)(k) = \hat{C}\hat{G}^{\circ(0))^\gamma}(\otimes_{k-\tau}) + \hat{D} \tag{11}$$

The identification values of the parameter matrices C and D are $\hat{C} = (\hat{c}_{ij})_{m\times m}$, $\hat{D} = (\hat{d}_1, \hat{d}_2, \cdots, \hat{d}_m)^T$, $(\hat{\boldsymbol{c}}_1, \hat{\boldsymbol{c}}_2, \cdots, \hat{\boldsymbol{c}}_m) = (P^TP)^{-1}P^T(Q_1, Q_2, \cdots, Q_m)$ is given by the least squares estimate, where

$$\hat{C}_i = \begin{bmatrix} \hat{c}_{i1} \\ \hat{c}_{i2} \\ \vdots \\ \hat{c}_{im} \\ \hat{d}_i \end{bmatrix},\ P = \begin{bmatrix} \overset{\circ}{g}{}_1^{(1)}(\otimes_{2-\tau}) & \overset{\circ}{g}{}_2^{(1)}(\otimes_{2-\tau}) & \cdots & \overset{\circ}{g}{}_m^{(1)}(\otimes_{2-\tau}) & 1 \\ \overset{\circ}{g}{}_1^{(1)}(\otimes_{3-\tau}) & \overset{\circ}{g}{}_2^{(1)}(\otimes_{3-\tau}) & \cdots & \overset{\circ}{g}{}_2^{(1)}(\otimes_{3-\tau}) & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \overset{\circ}{g}{}_1^{(1)}(\otimes_{n-\tau}) & \overset{\circ}{g}{}_2^{(1)}(\otimes_{n-\tau}) & \cdots & \overset{\circ}{g}{}_m^{(1)}(\otimes_{n-\tau}) & 1 \end{bmatrix},\ Q_i = \begin{bmatrix} \overset{\circ}{g}{}_i^{(0)}(\otimes_2) \\ \overset{\circ}{g}{}_i^{(0)}(\otimes_3) \\ \vdots \\ \overset{\circ}{g}{}_i^{(0)}(\otimes_n) \end{bmatrix}$$

**5 Cases analysis**

In this section, the proposed model is used to simulate and predict the smog index data. he Beijing Air Quality Index (AQI) and $PM_{2.5}$ values from 5:00 to 16:00 in 2018 are taken as the smog data. $MGM(1,m)$ model (Referred to as model 1) and $MGM(1,m|\tau,\gamma)$ for interval gray numbers (hereinafter referred to as model 2) are separately applied to simulate and predict the AQI and $PM_{2.5}$, and their results are then compared.

The index data of AQI and $PM_{2.5}$ used in the experiment are gray numbers. Therefore, in this paper, the corresponding degree of greyness and kernels of AQI and $PM_{2.5}$ are calculated, and then the obtained data is substituted into MATLAB to calculate the time lag coefficient. The program performs calculations, and the time-lag coefficient is 3, and the nonlinear parameters are $\gamma_1 = 0.9$ and $\gamma_2 = 1.1$. Then, the coefficient is substituted into the program corresponding to the $MGM(1,m|\tau,\gamma)$ for interval gray numbers, and it is restored to the gray numbers. Finally, the simulated values, the predicted values and the corresponding simulated errors and prediction errors are obtained. According to the actual data from 8:00 to 16:00, the specific results are shown in Table 1 and Table 2. For a more intuitive expression, this paper makes a line chart to demonstrate the simulated prediction results of the two models about AQI and $PM_{2.5}$, as shown in Figures 1 to 4 .

Table 1 Simulation effect and prediction effect of model 1 and model 2 on AQI

| Time | Data | Model 1 | | | Model 2 | | |
|---|---|---|---|---|---|---|---|
| | | Analog value | Simulation error | | Analog value | Simulation error | |
| | | | Lower bound | Upper bound | | Lower bound | Upper bound |
| 8 | [198,379] | [198,379] | 0 | 0 | [198,379] | 0 | 0 |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 9 | [178,304] | [178.39,284.56] | 0.22 | 6.4 | [201.12,274.39] | 12.99 | 9.74 |
| 10 | [167,198] | [161.53,220.39] | 3.28 | 11.31 | [164.63,216.56] | 1.42 | 9.37 |
| 11 | [143,178] | [146.37,186.55] | 2.36 | 4.81 | [144.18,175.34] | 0.83 | 1.5 |
| 12 | [130,167] | [131.06,160.73] | 0.82 | 3.75 | [128.92,152.51] | 0.83 | 8.68 |
| 13 | [114,143] | [116.71,139.16] | 2.38 | 2.68 | [117.6,140.72] | 3.16 | 1.6 |
| 14 | [101,130] | [103.64,120.75] | 2.62 | 7.12 | [108.12,128.92] | 7.05 | 0.83 |
| | | Average simulation error | 1.67 | 5.15 | Average simulation error | 3.75 | 4.53 |
| | | Predictive value | Prediction error | | Predictive value | Prediction error | |
| | | | Lower bound | Upper bound | | Lowe r bound | Upper bound |
| 15 | [98,114] | [91.86,104.92] | 6.27 | 7.97 | [100.29,118.11] | 2.33 | 3.6 |
| 16 | [98,101] | [81.29,91.27] | 17.05 | 9.63 | [94.71,109.94] | 3.36 | 8.85 |
| | | Average prediction error | 11.66 | 8.8 | Average prediction error | 2.84 | 6.23 |

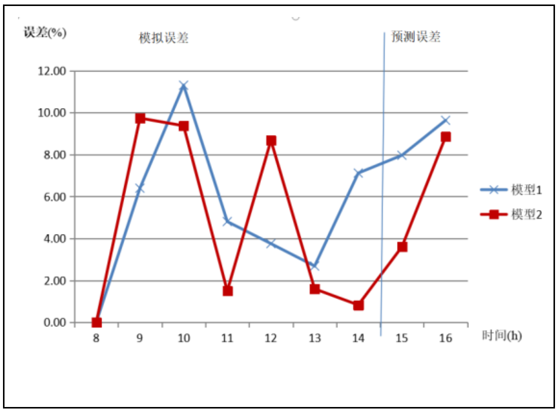


Fig.1 Simulation and prediction relative error of upper bound of AQI gray number

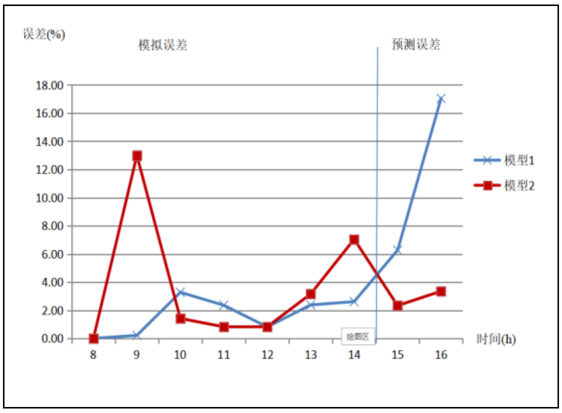


Fig. 2 Simulation and prediction relative error of AQI gray number lower bound

Table 2 Simulation effect and prediction effect of model 1 and model 2 on $PM_{2.5}$

| Time | Data | Model 1 | | | Model 2 | | |
|---|---|---|---|---|---|---|---|
| | | Analog value | Simulation error | | Analog value | Simulation error | |
| | | | Lower bound | Upper bound | | Lower bound | Upper bound |
| 8 | [61,92] | [61,92] | 0 | 0 | [61,92] | 0 | 0 |
| 9 | [53,77] | [52.01,74.85] | 1.87 | 2.79 | [55.57,73.99] | 4.86 | 3.91 |
| 10 | [46,61] | [46.16,62.74] | 0.34 | 2.85 | [47.3,60.84] | 2.82 | 0.25 |
| 11 | [40,53] | [41.41,53.89] | 3.52 | 1.68 | [41.95,50.73] | 4.87 | 4.28 |
| 12 | [37,46] | [37.03,46.53] | 0.08 | 1.14 | [37.47,44.31] | 1.27 | 3.67 |
| 13 | [33,40] | [33.01,40.26] | 0.04 | 0.65 | [33.75,40.2] | 2.27 | 0.51 |
| 14 | [30,37] | [29.36,34.90] | 2.13 | 5.69 | [30.56,36.32] | 1.88 | 1.85 |
| | | Average simulation error | 1.14 | 2.11 | Average simulation error | 2.57 | 2.07 |
| | | Predictive value | Prediction error | | Predictive value | Prediction error | |
| | | | Lower bound | Upper bound | | Lowe r bound | Upper bound |
| 15 | [28,33] | [26.06,30.29] | 6.93 | 8.21 | [27.82,32.8] | 0.63 | 0.61 |
| 16 | [28,30] | [23.09,26.32] | 17.53 | 12.26 | [25.58,29.88] | 8.65 | 0.41 |
| | | Average prediction error | 12.23 | 10.23 | Average prediction error | 4.64 | 0.51 |

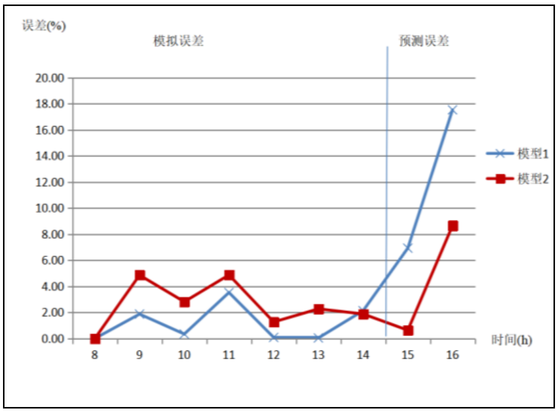


Fig.3 Simulation and prediction relative error of upper bound of $PM_{2.5}$ gray number

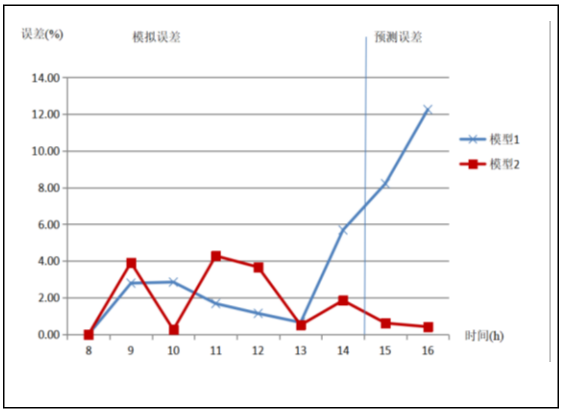


Fig.4 Simulation and prediction relative error of lower bound of $PM_{2.5}$ gray number

It can be seen from Tables 1 to 2 and Figures 1 to 4 that in the 8:00 to 14:00 period, the Model 1 and Model 2 are used to simulate the AQI and $PM_{2.5}$ data. At the time 15:00 to 16:00 period, AQI and $PM_{2.5}$ analog value are obtained

Among them, at 8:00 to 14:00 period, the simulated average errors of the lower bound of the gray number of AQI data by Model 1 and Model 2 are 1.67% and 3.75%, respectively, and the simulated average errors of the upper bound of the gray number of AQI data by Model 1 and Model 2 are 5.15%, 4.53%; the simulated average errors of the lower bound of the gray number of PM2.5 data by Model 1 and Model 2 are 1.14%, 2.57%; the simulated average errors of the upper bound of the gray number of PM2.5 by Model 1 and Model 2 are 2.11%, 2.07 %. At 15:00 ~ 16:00 period, the average prediction errors of the model 1 and model 2 for the lower bound of the gray number of AQI data are 11.66% and 2.84%, respectively, and the average simulation errors of the model 1 and model 2 for the upper bound of the gray number of AQI data are 8.8%, 6.23%; The average errors of the model 1 and model 2 for the lower bound of the gray number of PM2.5 data are 12.23% and 4.64%; the average errors of the model 1 and model 2 for the upper bound of the gray number of PM2.5 data are 10.23% and 0.51% respectively.

To sum up, the AQI and PM2.5 data can be simulated and predicted by applying the non-linear time lag multi-variable gray prediction model for interval gray numbers to obtain accurate prediction results. The average simulation error of the upper and lower bounds of Model 1 are 3.41% and 1.63%, the average simulation error of the upper and lower bounds of Model 2 are 4.41% and 2.58%, respectively. The simulation accuracy of Model 1 and Model 2 is high, and the simulation effect of Model 1 is better; the average prediction errors of the upper and lower bounds of Model 1 are 10.23% and 11.23%, respectively, and the upper and lower bound prediction errors of model 1 are 4.54% and 2.58% respectively. The comparison results show that the prediction effect of model 2 is better than that of the original model. Therefore, the improved model has a higher research and application value.

## 6 Conclusions

The gray system $MGM(1,m)$ model is suitable for the simulation and prediction of smog data, and its simulation accuracy is relatively high. Based on the original $MGM(1,m)$ model, for the uncertainty, lag effect, and non-linearity of the smog data, this paper introduces the interval gray number and time, lag parameters and nonlinear parameters. As a result, a nonlinear time lag model based on the interval gray number is constructed, and the model and the original model are applied to actual cases. This paper randomly selects the data from 5:00 to 16:00 on December 3, 2018 in Beijing, and compares the simulation accuracy and prediction accuracy. The results show that the prediction accuracy of the new model is better than that of the original model, and the model can be used to predict the uncertainty of the smog data. Therefore, the improved model is more accurate and its feasibility is stronger, it can be used for smog prediction, and it can obtain more accurate results.


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