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Some Decision Problems for Extended Modular Groups

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Abstract. In this paper we investigate solvability of the word problem for Extended Modular groups, Extended Hecke groups and Picard groups in terms of complete rewriting systems. At the final part of the paper we examine the other important decision problem (conjugacy problem) for only Extended Modular groups.

Keywords: Conjugacy problem; Extended Modular groups; Rewriting systems; Word problem.

1. Introduction and Preliminaries

Algorithmic problems such as the *word, conjugacy* and *isomorphism problems* have played an important role in group theory since the work of M. Dehn in early 1900's. These problems are called *decision problems* which ask for a "yes" or "no" answer to a specific question. Among these decision problems especially the word problem has been studied widely in groups and semigroups (see [1]). It is well known that the word problem for finitely presented groups is not solvable in general; that is, given any two words obtained by generators of the group, there may be no algorithm to decide whether these words represent the same element in this group. Since a complete rewriting system for a group also gives a set of normal forms for elements of this group (i.e. for each group element there is a unique word representing it which cannot be rewritten), groups that are presented by *finite* and *complete* rewriting systems have solvable word problem ([2, 20]). Actually most of the idea of this paper will be constructed on this truth except that the final part. In this paper, it will be shown that each of *Extended Modular groups*, *Extended Hecke groups* and *Picard groups* has finite complete rewriting system (Theorem 2.1) and hence each of them has solvable word problem. Finally, we will show that the conjugacy problem is solvable for Extended Modular groups (Theorem 3.3).

As depicted above, since the main theory of this paper will be constructed over complete rewriting systems, let us recall some basic facts about these systems that will be needed in proofs. We note that the reader is referred to [2, 20, 23] for more detailed survey on (complete) rewriting sytems.

Let S be a set (called an alphabet) and let S^* be the free monoid consists of all words in the letters of S. The empty word in S^* will be represented by 1. A *rewriting system* on S^* is a subset $R \subseteq S^* \times S^*$ and an element $(u,v) \in R$, also written $u \to v$, is called a rule of R. The idea for a rewriting system is an algorithm for substituting the right-hand side of a rule whenever the left-hand side appears in a word. In general, for a given rewriting system R, we write $x \to y$ for $x, y \in S^*$ if $x = uv_1w$, $y = uv_2w$ and $(v_1, v_2) \in R$. Also we write $x \to^* y$ if $x = y$ or $x \to x_1 \to x_2 \to \cdots \to y$ for some finite chain of reductions. Furthermore an element x of S^* is called irreducible with respect to R if there is no possible rewriting (or reduction) $x \to y$; otherwise x is called *reducible*. The rewriting system R is

- *Noetherian* if there is no infinite chain of rewritings $x \to x_1 \to x_2 \to \cdots$ for any word $x \in S^*$,
- *Confluent* if whenever $x \to^* y_1$ and $x \to^* y_2$, there is a $z \in S^*$ such that $y_1 \to^* z$ and $y_2 \to^* z$,
- *Complete* if R is both Noetherian and confluent.

A complete rewriting system for a group is also known as a *complete presentation*. Finally a rewriting system is *finite* if both S and R are finite sets. Furthermore a *critical pair* of a rewriting system R is a pair of overlapping rules if one of the following forms:

(i) (r_1r_2, s), $(r_2r_3, t) \in R$ with $r_2 \neq 1$,

(ii) $(r_1r_2r_3, s)$, $(r_2, t) \in R$,

is satisfied. Also a critical pair is *resolved* in R if there is a word z such that $sr_3 \to^* z$ and $r_1t \to^* z$ in the first case or $s \to^* z$ and $r_1tr_3 \to^* z$ in the second. A Noetherian rewriting system is complete if and only if every critical pair is resolved [20]. Moreover the following lemma is important to get Noetherian condition.

Lemma 1.1. [15] *A rewriting system R on S is Noetherian if and only if there exists a reduction ordering on S^* which is compatible with R.*

A rewriting system for a group G is a *rewriting system for G as a monoid* if S generates G as a monoid. To get a simplier way, the monoid rewriting system can be written by $M = rws(S, R)$, where $R = \{r_1, r_2, \cdots, r_m\}$ is a set of pairs $r_i = (u_i, v_i)$ written $r_i = u_i \to v_i$. Knuth and Bendix [12] have developed an *algorithm* for creating a complete rewriting system M' for M (i.e. R is Noetherian and confluent), so that any word over S has a (unique) normal form with respect to M'. By considering overlaps of left-hand sides of rules, this algorithm basicly proceeds forming new rules when two reductions of an overlap word result in two distinct reduced forms.

Finite complete systems have been obtained for various types of groups, including the torus knot group and the Greendlinberger group [6], fundamental group of a closed orientable surface of genus g [9] and many Coxeter groups [9, 17]. Besides that since Extended Modular groups, Extended Hecke groups and Picard groups are really important for the people studying on both algebra and some part of analysis, it is therefore worth to examine whether these groups have complete rewriting systems or not. Hence let us present some introductory material about them as in the next two paragraphs.

In [8], Hecke introduced an infinite class of discrete groups $H(\lambda_q)$ of linear fractional transformations preserving the upper-half line. The *Hecke group* is the group generated by

$$x(z) = -\frac{1}{z} \qquad \text{and} \qquad u(z) = z + \lambda_q,$$

where $\lambda_q = 2cos\pi/q$ for the integer $q \geq 3$. Let $y = xu = -\frac{1}{z+\lambda_q}$. Then $H(\lambda_q)$ has a presentation $\mathcal{P}_{H(\lambda_q)} = \langle x, y; x^2, y^q \rangle$. For $q = 3$, the resulting Hecke group $H(\lambda_3) = \mathbf{M}$ is the *Modular group* $PSL(2, \mathbb{Z})$. By adding the reflection $r(z) = 1/\overline{z}$ to the generators of the modular group, the extended modular group $\overline{H}(\lambda_3) = \overline{\mathbf{M}}$ was defined in [11]. Then the *Extended Hecke group*, denoted by $\overline{H}(\lambda_q)$, was firstly defined in [10] by adding the reflection $r(z) = 1/\overline{z}$ to the generators of $H(\lambda_q)$ similar to the Extended Modular group $\overline{\mathbf{M}}$. The Hecke group $H(\lambda_q)$ is a subgroup of index 2 in $\overline{H}(\lambda_q)$. By [11], we know that the Extended Hecke group $\overline{H}(\lambda_q)$ is isomorphic to $D_2 *_{\mathbb{Z}_2} D_q$ (where D_q is the dihedral group having $2q$ elements) and has a presentation

$$\mathcal{P}_{\overline{H}(\lambda_q)} = \langle x, y, r; x^2, y^q, r^2, (xr)^2, (yr)^2 \rangle . \tag{1}$$

Again, for $q = 3$, it is obtained the *Extended Modular group* $\overline{\mathbf{M}}$. The Hecke groups $H(\lambda_q)$, Extended Hecke groups $\overline{H}(\lambda_q)$ and their normal subgroups have been extensively studied from many points of view in the literature (see, [13, 14] and [19]). The Hecke group $H(\lambda_3)$, the modular group $PSL(2, \mathbb{Z})$, and its normal subgroups have especially been of great interest in many fields of Mathematics, for example number theory, automorphic function theory and groups theory.

As a different view, in [5], the authors showed that the Extended Hecke group $\overline{H}(\lambda_q)$ is the semi-direct product (split extension) of the Hecke group $H(\lambda_q)$ by a cyclic group of order 2. Moreover, by considering the presentation (1), they gave the necessary and sufficient conditions of (1) to be *efficient* (which is an algebraic property) on the minimal number of generators. (We may refer [4] for the definition and some details of efficiency).

The *Picard group* $\mathbf{P}$ is $PSL(2,\mathbb{Z}[i])$, the group of linear fractional transformations with Gaussian integer coefficients. $\mathbf{P}$ is a free free product with amalgamation of the following form $\mathbf{P} = G_1 *_{\mathbf{M}} G_2$, where $G_1 \cong S_3 *_{\mathbb{Z}_3} A_4$, $G_2 \cong S_3 *_{\mathbb{Z}_2} D_2$ (we recall that D_2 is the Klein 4-group) and $\mathbf{M}$ is the Modular group $PSL(2,\mathbb{Z})$. By [3], it is known that a presentation for $\mathbf{P}$ is given by

$$\mathcal{P}_{\mathbf{P}} = \left\langle x,u,y,r; x^3, u^2, y^3, r^2, (xu)^2, (xy)^2, (ry)^2, (ru)^2 \right\rangle,$$

where $x(z) = \frac{i}{iz+1}$, $u(z) = -\frac{1}{z}$, $y(z) = \frac{z+1}{-z}$ and $r(z) = \frac{i}{iz}$.

2. Word Problem Part

In this section we state and prove that each of the Extended Modular, Extended Hecke and Picard groups has solvable word problem. In the light of the main aim of this paper, we should note that since Modular and Hecke groups are the free product of two cyclic groups, they have complete rewriting systems (due to having no overlap words) and so have solvable word problem.

Let us first suppose that

$$M_1 = rws(\{T,S,R,t,s,r\}, \{t^2 \to 1,\ s^3 \to 1,\ r^2 \to 1,\ (tr)^2 \to 1,\ (sr)^2 \to 1,\\ Tt \to 1,\ tT \to 1,\ Ss \to 1,\ sS \to 1,\ Rr \to 1,\ rR \to 1\})$$

is a monoid string rewriting system for the Extended Modular group $\overline{\mathbf{M}}$, where the ordering is *DegLex* related to

$$s < r < t < S < R < T. \tag{2}$$

(We note that DegLex is also known as LengthLex and ShortLex, and defines $w_1 < w_2$ if either $deg(w_1) < deg(w_2)$ or, in the case that the degrees (lengths) are equal, if the ith position is the first, working from left to right, in which w_1 and w_2 differ, then the ith letter of w_1 is less than that of w_2 in the ordering given to the alphabet). This system is obtained from the group presentation (1) by adding relations T,S and R to represent the inverses of t,s and r, respectively.

We also let

$$M_2 = rws(\{X,Y,R,x,y,r\}, \{x^2 \to 1,\ y^q \to 1,\ r^2 \to 1,\ (xr)^2 \to 1,\ (yr)^2 \to 1,\\ Xx \to 1,\ xX \to 1,\ Yy \to 1,\ yY \to 1,\ Rr \to 1,\ rR \to 1\})$$

be a monoid string rewriting system for the Extended Hecke group $\overline{H}(\lambda_q)$, where the ordering is *DegLex* related to

$$x < r < y < X < R < Y, \tag{3}$$

and

$$M_3 = rws(\{X, U, Y, R, x, u, y, r\}, \{x^3 \to 1,\ u^2 \to 1,\ y^3 \to 1,\ r^2 \to 1,\ (xu)^2 \to 1,\ (xy)^2 \to 1,\ (ry)^2 \to 1,\ (ru)^2 \to 1,\ Xx \to 1,\ xX \to 1,\ Uu \to 1,\ uU \to 1,\ Yy \to 1,\ yY \to 1,\ Rr \to 1,\ rR \to 1\})$$

be a monoid string rewriting system for the Picard group **P**, where the ordering is *DegLex* related to

$$x < y < u < r < X < Y < U < R. \tag{4}$$

We note that X, Y, R and U represent the inverses of x, y, r and u, respectively, as in M_1.

Now we have Table 1, Table 2 and Table 3 that show all overlap words and reduced forms of these words for groups which we studied on. In these tables the important point is the fourth column showing new rules such that some of these rules in each Table will be added to the related set M_1, M_2 and M_3 in details of the proof.

Table 1: The Extended Modular Group

	reducing from left	*reducing from right*	*new rules*
overlap words (uvw)	$uvw \to aw$ for $uv \to a$	$uvw \to ub$ for $vw \to b$	$aw \to ub$ or $ub \to aw$
t^2rtr	rtr	t	$rtr \to t$
$trtr^2$	r	trt	$trt \to r$
s^3rsr	rsr	s^2	$rsr \to s^2$
$srsr^2$	r	srs	$srs \to r$
Tt^2	t	T	$T \to t$
t^2T	T	t	$T \to t$
Ss^3	s^2	S	$s^2 \to S$
s^3S	S	s^2	$s^2 \to S$
Rr^2	r	R	$R \to r$
r^2R	R	r	$R \to r$
$trtrR$	R	trt	$trt \to R$
$Ttrtr$	rtr	T	$rtr \to T$
$Ssrsr$	rsr	S	$rsr \to S$
$srsrS$	R	srs	$srs \to R$

Now the first main theorem of this paper is the following.

Theorem 2.1. *There is a finite complete rewriting system for each Extended Modular, Extended Hecke and Picard groups.*

Proof. Since we have reduction orderings (2), (3) and (4), so by Lemma 1.1, it is easy to see that monoid rewriting systems M_1, M_2 and M_3 are Noetherian.

Table 2: The Extended Hecke Group

	reducing from left	*reducing from right*	*new rules*
overlap words (uvw)	$uvw \to aw$ for $uv \to a$	$uvw \to ub$ for $vw \to b$	$aw \to ub$ or $ub \to aw$
x^2rxr	rxr	x	$rxr \to x$
$xrxr^2$	r	xrx	$xrx \to r$
y^qryr	ryr	y^{q-1}	$y^{q-1} \to ryr$
$yryr^2$	r	yry	$yry \to r$
Xx^2	x	X	$X \to x$
x^2X	X	x	$X \to x$
Yy^q	y^{q-1}	Y	$y^{q-1} \to Y$
y^qY	Y	y^{q-1}	$y^{q-1} \to Y$
Rr^2	r	R	$R \to r$
r^2R	R	r	$R \to r$
$Xxrxr$	rxr	X	$rxr \to X$
$xrxrR$	R	xrx	$xrx \to R$
$Yyryr$	ryr	Y	$ryr \to Y$
$yryrR$	R	yry	$yry \to R$

Table 3: The Picard Group

	reducing from left	*reducing from right*	*new rules*
overlap words (uvw)	$uvw \to aw$ for $uv \to a$	$uvw \to ub$ for $vw \to b$	$aw \to ub$ or $ub \to aw$
x^3uxu	uxu	x^2	$uxu \to x^2$
$xuxu^2$	u	xux	$xux \to u$
x^3yxy	yxy	x^2	$yxy \to x^2$
$xyxy^3$	y^2	xyx	$xyx \to y^2$
r^2yry	yry	r	$yry \to r$
$ryry^3$	y^2	ryr	$ryr \to y^2$
r^2uru	uru	r	$uru \to r$
$ruru^2$	u	rur	$rur \to u$
Xx^3	x^2	X	$x^2 \to X$
x^3X	X	x^2	$x^2 \to X$
Uu^2	u	U	$U \to u$
u^2U	U	u	$U \to u$
Yy^3	y^2	Y	$y^2 \to Y$
y^3Y	Y	y^2	$y^2 \to Y$
$Xxuxu$	uxu	X	$uxu \to X$
$xuxuU$	U	xux	$xux \to U$
$Xxyxy$	yxy	X	$yxy \to X$
$xyxyY$	Y	xyx	$xyx \to Y$
$Rryry$	yry	R	$yry \to R$
$ryryY$	Y	ryr	$ryr \to Y$
$Rruru$	uru	R	$uru \to R$
$ruruU$	U	rur	$rur \to U$

Now let us examine the confluent property for each groups separately. To do that we will apply *Knuth-Bendix algorithm*.

I) For Extended Modular Groups. In Table 1, the rules $rtr \to t$, $trt \to r$, $trt \to R$ and $rtr \to T$ coincide with each other. This means that they are reduced to the same new rule $tr \to rt$. (Since we have the ordering $r < t$, the rule $tr \to rt$ must be choosen instead of $rt \to tr$). Also the rules $rsr \to s^2$, $srs \to r$, $rsr \to S$ and $srs \to R$ coincide and then we obtain $s^2r \to rs$ as a new rule. The other rules in Table 1 are trivial. Due to the new rules $tr \to rt$ and $s^2r \to rs$ are obtained by using the rules $(tr)^2 \to 1$ and $(sr)^2 \to 1$ in M_1, both of these rules obsolete the rules in M_1. So, by *algorithm*, we remove them from M_1 and then obtain the new rewriting system

$$M_1' = rws(\{T,S,R,t,s,r\},\{t^2 \to 1,\ s^3 \to 1,\ r^2 \to 1,\ tr \to rt,\ s^2r \to rs,\\ Tt \to 1,\ tT \to 1,\ Ss \to 1,\ sS \to 1,\ Rr \to 1,\ rR \to 1\}).$$

Checking all overlap words (which are t^2r, tr^2, s^3r, s^2r^2, Ttr, trR, Ss^2r and s^2rR) of rules in M_1', we find no potential failures of confluence. Thus M_1' is confluent and so the *algorithm* ends successfully. Therefore M_1' is a complete rewriting system, as required. (We note that the reduction steps of all overlap words in M_1' can be shown as in Figure 1).

II) For Extended Hecke Groups. We will apply the same steps as done in *I)*. Therefore let us consider Table 2. A simple calculation shows that the rules $rxr \to x$, $xrx \to r$, $rxr \to X$ and $xrx \to R$ coincide with each other and they are reduced to the same new rule $rx \to xr$. (Since we have the ordering $x < r$, the rule $rx \to xr$ must be choosen instead of $xr \to rx$). Moreover the rules $y^{q-1} \to ryr$, $yry \to r$, $ryr \to Y$ and $yry \to R$ coincide and so one can obtain $y^{q-1}r \to ry$ as a new rule. The remaining rules in Table 2 are trivial as in Table 1. On account of the new rules $rx \to xr$ and $y^{q-1}r \to ry$ are obtained by using the rules $(xr)^2 \to 1$ and $(yr)^2 \to 1$ in M_2, both of these rules obsolete the rules in M_2. Hence, by the *algorithm*, the new rewriting system

$$M_2' = rws(\{X,Y,R,x,y,r\},\{x^2 \to 1,\ y^q \to 1,\ r^2 \to 1,\ rx \to xr,\ y^{q-1}r \to ry,\\ Xx \to 1,\ xX \to 1,\ Yy \to 1,\ yY \to 1,\ Rr \to 1,\ rR \to 1\})$$

is obtained. Again checking all overlap words (which are r^2x, rx^2, y^qr, $y^{q-1}r^2$, Rrx, rxX, $Yy^{q-1}r$ and $y^{q-1}rR$) of rules in M_2', we see that each of these words is reduced to one different word. Hence we have a confluent rewriting system for the Extended Hecke group, and the *algorithm* ends successfully.

III) For Picard Groups. By considering Table 3, we see that

- $uxu \to x^2$, $xux \to u$, $uxu \to X$ and $xux \to U$ coincide and then they are reduced to the same new rule $x^2u \to ux$,
- $yxy \to x^2$, $xyx \to y^2$, $yxy \to X$ and $xyx \to y$ coincide and then they are reduced to the same new rule $x^2y^2 \to yx$,
- $yry \to r$, $ryr \to y^2$, $yry \to R$ and $ryr \to Y$ coincide and then they are reduced to the same new rule $y^2r \to ry$,

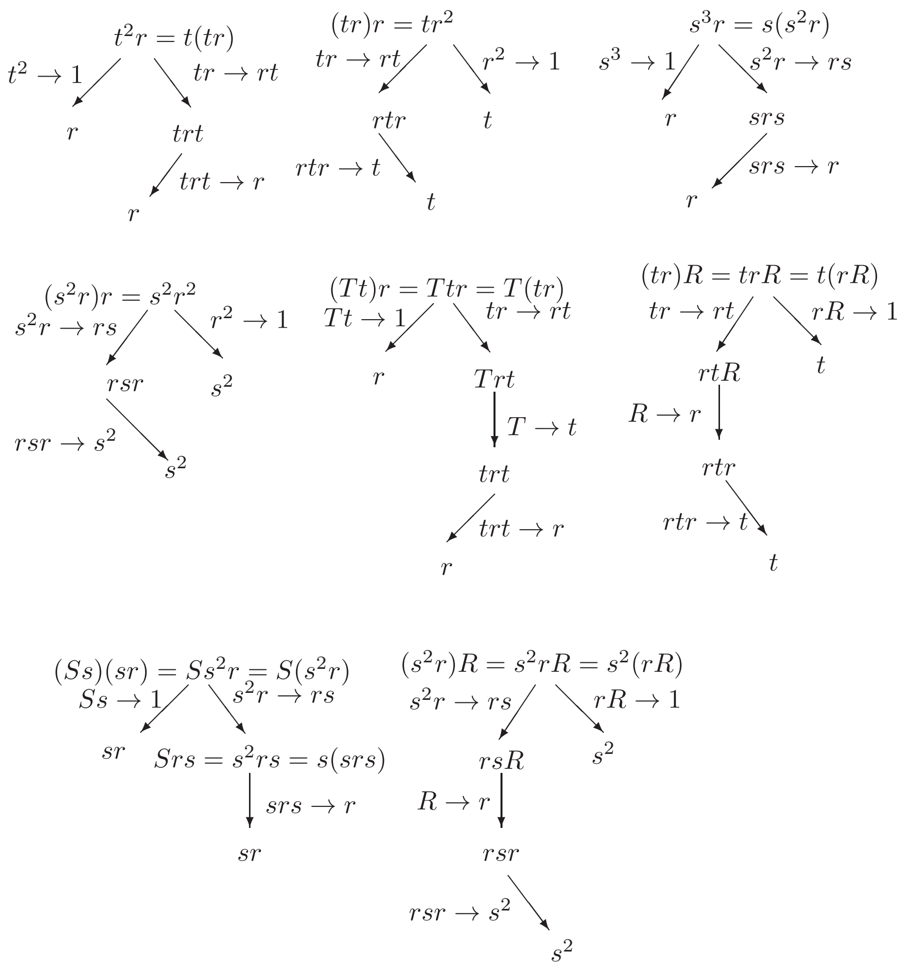

Figure 1:

- $uru \to r$, $rur \to u$, $uru \to R$ and $rur \to U$ coincide and then they are reduced to the same new rule $ru \to ur$ (since we have the ordering $u < r$).

As in other above cases the other rules in Table 3 are trivial. On account of the new rules $x^2u \to ux$, $x^2y^2 \to yx$, $y^2r \to ry$ and $ru \to ur$ are obtained by using the rules $(xu)^2 \to 1$, $(xy)^2 \to 1$, $(ry)^2 \to 1$ and $(ru)^2 \to 1$ in M_3, all of these rules obsolete the rules in M_3. So we can remove them. Hence we are left with the new rewriting system

$$
\begin{aligned}
M_3' = rws(\{X,U,Y,R,x,u,y,r\}, \{&x^3 \to 1,\ u^2 \to 1,\ y^3 \to 1,\ r^2 \to 1,\ x^2u \to ux,\\
&x^2y^2 \to yx,\ y^2r \to ry,\ ru \to ur,\\
&Xx \to 1,\ xX \to 1,\ Uu \to 1,\ uU \to 1,
\end{aligned}
$$

$$Yy \to 1,\ yY \to 1,\ Rr \to 1,\ rR \to 1\}).$$

If we check all overlap words (x^3u, x^2u^2, x^3y^2, x^2y^3, y^3r, y^2r^2, r^2u, ru^2, Xx^2u, x^2uU, Xx^2y^2, x^2y^2Y, Yy^2r, y^2rR, Rru and ruU) in M_3', then we see that each of them is reduced to one word separately. Thus we have a confluent rewriting system for the Picard group as well.

Hence the result. ■

Now we can state the whole aim of this section as in the following. Let us first recall that

> "*Let G be a group given by the finite presentation $\langle S; R\rangle$. Is there an algorithm that decides whether or not a given words is equivalent to the identity in G?*"

is the word problem for an arbitrary group G. As we noted in the first section, a complete rewriting system for G also gives a set of normal forms for elements of G; that is, for each group element there is a unique word representing it which cannot be rewritten. Therefore, since we have complete rewriting systems for the groups studied in here, by Theorem , we have the following result.

Corollary 2.2. *The word problem is solvable for Extended Modular groups, Extended Hecke groups and Picard groups.*

Remark 2.3. There is also another well known decision problem, namely *generalized word problem* or, equivalently, *membership problem* ([20]). Besides this problem was solved for Modular groups by Gurevich and Schupp (in a valuable paper [7]), we couldn't find any references in literature solving the membership problem for the groups studied in this paper and leave it as a future project.

3. Conjugacy Problem Part

In this section we consider another problem, namely conjugacy problem, for only Extended Modular groups and obtain a result (see Theorem 3.3). Actually there is no reference studying on the conjugacy problem for the other groups studied in the previous section. In general, the conjugacy problem can be expressed as in the following form.

> *Let G be a group given by the finite presentation $\langle S; R\rangle$. Is there an algorithm that decides whether or not any pair of words u and v are conjugate, i.e. there exist $w \in G$ such that $wu = vw$, in G?*

We can make a connection between conjugacy problem and conjugacy separability by Mostowski's following result.

Lemma 3.1. [18] *The conjugacy problem is solvable in finitely presented conjugacy separable groups.*

By (1), since $\overline{M}$ is finitely presented, to obtain the solvability of the conjugacy problem, we just need to prove that Extended Modular groups are conjugacy separable. Let us recall the definition of this important algebraic property. An element g of a group G is *conjugacy distinguished* if and only if given any element $h \in G$ *either* g is conjugate to h *or* there is a homomorphism γ of G onto a finite group such that $\gamma(g)$ is not conjugate to $\gamma(h)$. Then G is called *conjugacy separable* if every element of G are conjugacy distinguished.

Lemma 3.2. *The Extended modular group* $\overline{\mathbf{M}}$ *is conjugacy separable.*

Proof. By [11], we have $\overline{\mathbf{M}} = PGL(2,\mathbb{Z})$. According to [21], there is a free group F such that $[PGL(2,\mathbb{Z}) : F] < \infty$ and every infinite order element in $PGL(2,\mathbb{Z})$ are conjugacy distinguished. So to end up the proof, it remains to check whether the elements of finite order in $PGL(2,\mathbb{Z})$ are conjugacy distinduished. In fact, by [16], every such these elements in this group are conjugate to elements of dihedral groups D_2 and D_3 that are factors of $\overline{\mathbf{M}}$. Hence to show that $PGL(2,\mathbb{Z})$ is conjugacy separable, we need only prove that the conjugates of elements of each groups D_2 and D_3 are conjugacy distinguished.

Let $g \in PGL(2,\mathbb{Z})$ be finite order that conjugates to an element of D_2 or D_3. Also let h be any element of $PGL(2,\mathbb{Z})$ such that not conjugate to g. Then if h has infinite order in $PGL(2,\mathbb{Z})$, h is conjugacy distinguished in $PGL(2,\mathbb{Z})$ so there is a homomorphism φ of $PGL(2,\mathbb{Z})$ onto a finite group such that $\varphi(g)$ is not conjugate to $\varphi(h)$ in $\varphi(PGL(2,\mathbb{Z}))$. Thus we must consider h as a finite order in $PGL(2,\mathbb{Z})$. Hence we can obtain h conjugates to an element of D_2 or D_3. To show that there is a homomorphism φ of $PGL(2,\mathbb{Z})$ onto a finite group such that $\varphi(g)$ is not conjugate to $\varphi(h)$ in $PGL(2,\mathbb{Z})$, we can replace g and h by their conjugates in D_2 or D_3, and by representatives of their conjugacy classes in these subgroups.

Let $\langle a, b; a^2, b^2, ba = ab\rangle$ and $\langle c, d; c^3, d^2, dc = c^2 d\rangle$ be presentations for the groups D_2 and D_3, respectively. Then the elements $1, a, b, ab$ and $1, d, c, c^2, dc$ are the complete sets of conjugacy class representatives for the subgroups D_2 and D_3, respectively. Using the identifications $b = d$ and $bc = c^2 b$, we conclude that every element of finite order in $PGL(2,\mathbb{Z})$ is conjugate to one of the elements of the set $\{1, a, b, c, ab, bc\}$. It is clear that the orders of those elements in this set are $1, 2, 2, 3, 2, 2$, respectively.

If φ is a finite representation of $PGL(2,\mathbb{Z})$ faithful on the factors D_2 and D_3 of $PGL(2,\mathbb{Z})$, the images of two elements of different order will not be conjugate in $\varphi(PGL(2,\mathbb{Z}))$. According to [22], such a representation always exists. Therefore it must be considered that g and h are conjugate to different elements in the set $\{a, b, ab, bc\}$. Thus we obtain that the elements in the pairs $(\varphi(a), \varphi(b))$, $(\varphi(a), \varphi(ab))$, $(\varphi(a), \varphi(bc))$, $(\varphi(b), \varphi(ab))$, $(\varphi(b), \varphi(bc))$ and $(\varphi(ab), \varphi(bc))$ are not conjugate to each other.

Hence the result. ■

Then, by Lemmas 3.1 and 3.2, we have the following other main result of this paper.

Theorem 3.3. *The conjugacy problem is solvable for the Extended Modular group.*

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