The Efficiency of the Semi-Direct Products of Free Abelian Monoid with Rank n by the Infinite Cyclic Monoid

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Abstract. In this paper we give necessary and sufficient conditions for the efficiency of the semi-direct product of free abelian monoid with rank n by the infinite cyclic monoid.

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INTRODUCTION

Let $\mathscr{P} = [\mathbf{x};\mathbf{r}]$ be a finite presentation for a monoid M. Then the *Euler characteristic* of $\mathscr{P}$ is defined by $\chi(\mathscr{P}) = 1-|\mathbf{x}|+|\mathbf{r}|$ and an upper bound of M is defined by $\delta(M) = 1-rk_{\mathbf{Z}}(H_1(M))+d(H_2(M))$. In an unpublished work, S.J. Pride has shown that $\chi(\mathscr{P}) \geq \delta(M)$. With this background, one can define a monoid presentation $\mathscr{P}$ to be *efficient* if $\chi(\mathscr{P}) = \delta(M)$, and then M is called *efficient* if it has an efficient presentation.

It is well known that one of the effective way to show efficiency for the monoid M is to use spherical monoid pictures over $\mathscr{P}$. These geometric configurations are the representative elements of the Squier complex denoted by $\mathscr{D}(\mathscr{P})$ (see, for example [4], [5], [7]). Suppose $\mathbf{Y}$ is a collection of spherical monoid pictures over $\mathscr{P}$. Two monoid pictures $\mathbf{P}$ and $\mathbf{P}'$ are *equivalent relative to* $\mathbf{Y}$ if there is a finite sequence of monoid pictures $\mathbf{P} = \mathbf{P}_0, \mathbf{P}_1, \cdots, \mathbf{P}_m = \mathbf{P}'$ where, for $1 \leq i \leq m$, the monoid picture $\mathbf{P}_i$ is obtained from the picture $\mathbf{P}_{i-1}$ either by the insertion, deletion and replacement operations. By definition, a set $\mathbf{Y}$ of spherical monoid pictures over $\mathscr{P}$ is a *trivializer of* $\mathscr{D}(\mathscr{P})$ if every spherical monoid picture is equivalent to an empty picture relative to $\mathbf{Y}$. The trivializer is also called a set of generating pictures.

For any monoid picture $\mathbf{P}$ over $\mathscr{P}$ and for any $R \in \mathbf{r}$, $\exp_R(\mathbf{P})$ denotes the *exponent sum* of R in $\mathbf{P}$ which is the number of positive discs labelled by R_+, minus the number of negative discs labelled by R_-. For a non-negative integer n, $\mathscr{P}$ is said to be *n-Cockcroft* if $\exp_R(\mathbf{P}) \equiv 0 \,(\mathrm{mod}\; n)$, (where congruence (mod 0) is taken to be equality) for all $R \in \mathbf{r}$ and for all spherical pictures $\mathbf{P}$ over $\mathscr{P}$. Then a monoid M is said to be *n-Cockcroft* if it admits an n-Cockcroft presentation. In fact to verify the n-Cockcroft property, it is enough to check for pictures $\mathbf{P} \in \mathbf{Y}$, where $\mathbf{Y}$ is a trivializer (see [4], [5]). The 0-Cockcroft property is usually just called Cockcroft.

The following result is also an unpublished result by S.J. Pride.

Theorem 1 *Let $\mathscr{P}$ be a monoid presentation. Then $\mathscr{P}$ is efficient if and only if it is p-Cockcroft for some prime p.*

Let K be free abelian monoid of rank n with $\mathscr{P}_K = [y_1, y_2, \cdots y_n\,;\, y_iy_j = y_jy_i \;(1 \leq i < j \leq n)]$, and let A be the infinite cyclic monoid with $\mathscr{P}_A = [x\,;\,]$. Also let $\psi_{\mathscr{M}}$ be an endomorphism of K where $\mathscr{M}$ is the matrix on the positive integer

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set

$$\mathscr{M} = \begin{bmatrix} \alpha_{11} & \alpha_{12}\cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22}\cdots & \alpha_{2n} \\ \vdots & & \\ \alpha_{n1} & \alpha_{n2}\cdots & \alpha_{nn} \end{bmatrix}.$$

given by $y_i \mapsto y_1^{\alpha_{i1}} y_2^{\alpha_{i2}} \cdots y_n^{\alpha_{in}}$ for $1 \leq i \leq n$. Thus we have the presentation of $M = K \rtimes_\theta A$ where $\theta : A \longrightarrow End(K)$, $x \longmapsto \psi_{\mathscr{M}}$,

$$\mathscr{P}_M = [y_1, y_2, \cdots y_n, x\,;\, y_i y_j = y_j y_i \ (1 \leq i < j \leq n), T_{y_i x}(1 \leq i \leq n)], \tag{1}$$

for the monoid M where

$$T_{y_i x} : y_i x = x y_1^{\alpha_{i1}} y_2^{\alpha_{i2}} \cdots y_n^{\alpha_{in}}.$$

Now, by considering the presentation $\mathscr{P}_M$ in (1), we prove the following theorem as a main result in the present paper.

Theorem 2 *Let p be a prime or* 0*. Then the presentation $\mathscr{P}_M$ is p-Cockcroft if and only if, for all $1 \leq i < j \leq n$ and $1 \leq k, m \leq n$,*

$$\alpha_{jk}\alpha_{ik+m} - \alpha_{ik}\alpha_{jk+m} \equiv \begin{cases} 1\,(mod\ p), & if\ k = i\ and\ k+m = j, \\ 0\,(mod\ p), & otherwise \end{cases}$$

We may refer [1, 2, 3, 4, 5, 6, 7] to the reader for most of the fundamental material (for instance, *semidirect products of monoids, Squier complex, a trivializer set* of the Squier complex, *spherical* and *non-spherical monoid pictures*) which will be needed here.

TRIVIALIZER SET $\mathscr{D}(\mathscr{P}_M)$

Let us consider the relation $[(y_i y_j)\psi_x]_{\mathscr{P}_K} = [(y_j y_i)\psi_x]_{\mathscr{P}_K}$. Because of this, we get a non-spherical picture, say $\mathbf{B}_{S,x}$, over $\mathscr{P}_K$ where $S : y_i y_j = y_j y_i$ $(1 \leq i < j \leq n)$. Thus by using the subpicture $\mathbf{B}_{S,x}$, $T_{y_i x}$ discs and S disc, we have the generating pictures, say $\mathbf{P}_{S,x}$. Also, let $\mathbf{C}$ consists of the pictures $\mathbf{P}_{S,x}$. Let $\mathbf{X_K}$ be trivializer set of $\mathscr{D}(\mathscr{P}_K)$. We should note that since the monoid A is the infinite cyclic monoid, we don't have a trivializer set of $\mathscr{D}(\mathscr{P}_A)$. Let us consider the presentation $\mathscr{P}_M$, as in(1). Then, by [8], a trivializer set of $\mathscr{D}(\mathscr{P}_M)$ is

$$\mathbf{X_K} \cup \mathbf{C}.$$

The reason for us keeping work on the above monoid pictures is their usage in the important connection between *efficiency* and *p-Cockcroft property*. Therefore, in the present paper, we will use this connection to get the efficiency. To do that we will count the exponent sums of the discs in these above pictures to obtain p-Cockcroft property for the presentation $\mathscr{P}_M$ given in (1).

PROOF OF THE MAIN RESULT AND ITS APPLICATION

Let us consider the discs given in $\mathbf{P}_{S,x}$. To prove Theorem 2, we will count the exponent sums of the discs in these pictures. Here, when we calculate the number of S discs in $\mathbf{B}_{S,x}$ where $S : y_i y_j = y_j y_i$ such that $i < j$ and $i, j \in \{1, 2, \cdots, n\}$, we see that it is equal to $\alpha_{ji}\alpha_{ij} - \alpha_{ii}\alpha_{jj}$. On the other hand, we have also $y_1 y_2 = y_2 y_1$, $y_1 y_3 = y_3 y_1, \cdots, y_{n-1} y_n = y_n y_{n-1}$ discs different from $S : y_i y_j = y_j y_i$ in $\mathbf{B}_{S,x}$, say $\acute{S}$ discs. The number of $\acute{S}$ discs is $\alpha_{jk}\alpha_{ik+m} - \alpha_{ik}\alpha_{jk+m}$ where $1 \leq k, m \leq n$. At this point, it is easy to see that

$$\begin{aligned} exp_S(\mathbf{P}_{S,x}) &= 1 - exp_S(\mathbf{B}_{S,x}), \\ exp_{\acute{S}}(\mathbf{P}_{S,x}) &= exp_{\acute{S}}(\mathbf{B}_{S,x}) \end{aligned}$$

and to p-Cockcroft property be hold, we need to have

$$\begin{aligned} exp_S(\mathbf{P}_{S,x}) \equiv 0\,(mod\,p) \quad &\Leftrightarrow \quad exp_S(\mathbf{B}_{S,x}) \equiv 1\,(mod\,p), \\ exp_{\acute{S}}(\mathbf{P}_{S,x}) \equiv 0\,(mod\,p) \quad &\Leftrightarrow \quad exp_{\acute{S}}(\mathbf{B}_{S,x}) \equiv 0\,(mod\,p). \end{aligned}$$

By using this, if $k = i$ and $j = k+m$, we have $\alpha_{jk}\alpha_{ik+m} - \alpha_{ik}\alpha_{jk+m} \equiv 1(mod\, p)$. Otherwise, we get that $\alpha_{jk}\alpha_{ik+m} - \alpha_{ik}\alpha_{jk+m} \equiv 0(mod\, p)$. Moreover, in $\mathbf{P}_{S,x}$, we also have 2 times positive and 2 times negative $T_{y_i x}$ discs. That means

$$exp_{T_{y_i x}}(\mathbf{P}_{S,x}) = 0,$$

and so we say that p-Cockcroft property is hold for these discs. Hence the result.

We note that, by considering the trivializer set $\mathbf{X_K}$ of the Squier complex $\mathscr{D}(\mathscr{P}_K)$, it can be easily deduced that $\mathscr{P}_K$ are p-Cockcroft, in fact Cockcroft, presentations.

These all above procedure give us sufficient conditions to be the presentation $\mathscr{P}_M$ in (1) is p-Cockcroft for any prime p. In fact the converse part (necessary conditions) of the theorem is quite clear.

Let us suppose that the monoid K is presented by $\mathscr{P}_K = [y_1, y_2\,;\, y_1y_2 = y_2y_1]$. Hence we get the corresponding semi-direct product M with the presentation

$$\mathscr{P}_M = [y_1, y_2, x\,;\, y_1y_2 = y_2y_1, y_1x = xy_1^{\alpha_{11}}y_2^{\alpha_{12}}, y_2x = xy_1^{\alpha_{21}}y_2^{\alpha_{22}}\,]. \tag{2}$$

Let us consider presentation given in (2) .Then we can give the following corollary as a consequence of the main result.

Corollary 3 *Let p be a prime or* 0. *Then the presentation $\mathscr{P}_M$ is p-Cockcroft if and only if*

$$\alpha_{21}\alpha_{12} - \alpha_{11}\alpha_{22} \equiv 1(mod\, p).$$

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