Applied Mathematics and Computation 229 (2014) 433–439

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

On Hessenberg and pentadiagonal determinants related with Fibonacci and Fibonacci-like numbers

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ARTICLE INFO

Keywords:
Fibonacci numbers
Fibonacci-like numbers
Hessenberg determinants
Pentadiagonal determinants

ABSTRACT

In this paper, we establish several new connections between the generalizations of Fibonacci and Lucas sequences and Hessenberg determinants. We also give an interesting conjecture related to the determinant of an infinite pentadiagonal matrix with the classical Fibonacci and Gaussian Fibonacci numbers.

1. Introduction

The Fibonacci sequence, say $\{f_n\}_{n\in\mathbb{N}}$ is the sequence of positive integers satisfying the recurrence relation $f_0 = 0, f_1 = 1$ and $f_n = f_{n-1} + f_{n-2}, n \geqslant 2$. The Lucas sequence, say $\{l_n\}_{n\in\mathbb{N}}$ is the sequence of positive integers satisfying the recurrence relation $l_0 = 2, l_1 = 1$ and $l_n = l_{n-1} + l_{n-2}, n \geqslant 2$.

In recent years, several connections between the Fibonacci and Lucas sequences with matrices have been given by researches. In [3], some classes of identities for some generalizations of Fibonacci numbers have been obtained. The relations between the Bell matrix and the Fibonacci matrix, which provide a unified approach to some lower triangular matrices, such as the Stirling matrices of both kinds, the Lah matrix, and the generalized Pascal matrix, were studied in [23]. In [16], İpek computed the spectral norms of circulant matrices with classical Fibonacci and Lucas numbers entries. In [4], Bozkurt first computed the spectral norms of the matrices related with integer sequences, and then he gave two examples related with Fibonacci, Lucas, Pell and Perrin numbers.

Some of traditional methods for calculation of the determinant of an $n \times n$ matrix are based on factorization in a product of certain matrices such as lower, upper, tridiagonal, pentadiagonal and Hessenberg matrices. A brief overview of the theory of determinants can be found, for example, in [14,21].

In some papers related with relationships between the Fibonacci and Lucas sequences with certain matrices, the results on relations between determinants of families of tridiagonal and pentadiagonal matrices with Fibonacci and Lucas numbers have been presented. Cahill and Narayan [7] showed how Fibonacci and Lucas numbers arise as determinants of some tridiagonal matrices. Strang [20] has introduced real tridiagonal matrices such that their determinants are Fibonacci numbers. Nallı and Civciv [19] gave a generalization of the presented in [7]. Also, Civciv [9] investigated the determinant of a special pentadiagonal matrix with the Fibonacci numbers. In [11], by the determinant of tridiagonal matrix, another proof of the Fibonacci identities is given. In [22], another proof of Pell identities is presented by the determinant of tridiagonal matrix.

In general we use the standard terminology and notation of Hessenberg matrix theory, see [13]. The determinant

$$H_n = |a_{ij}|_n,$$

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0096-3003/$ - see front matter © 2013 Elsevier Inc. All rights reserved.
http://dx.doi.org/10.1016/j.amc.2013.12.071

where $a_{ij} = 0$ when $i - j > 1$ or when $j - i > 1$ is known as a Hessenberg determinant or simply Hessenbergian. If $a_{ij} = 0$ when $i - j > 1$, the Hessenbergian takes the from

$$H_n = \begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1,n-1} & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2,n-1} & a_{2n} \\ & a_{32} & a_{33} & \cdots & \cdots & \cdots \\ & & a_{43} & \cdots & \cdots & \cdots \\ & & & \cdots & \cdots & \cdots \\ & & & & a_{n,n-1} & a_{nn} \end{vmatrix}_n.$$

If $a_{ij} = 0$ when $j - i > 1$, the triangular array of zero elements appears in top right-hand corner.

It is well known that several famous numbers may be represented as determinants of Hessenberg matrices. In many papers related with relationships between the Fibonacci and Lucas sequences with certain matrices, the results on relations between determinants of families of Hessenberg matrices with Fibonacci and Lucas numbers have been given [5, 6, 8, 20]. In [6], complex Hessenberg matrices such that their determinants are Fibonacci numbers have been introduced. It was showed in [8] that the maximum determinant achieved by $n \times n$ Hessenberg $(0,1)$-matrices is the nth Fibonacci number f_n. Esmaeili [10] gave several new classes of Fibonacci–Hessenberg matrices whose determinants are in the form $tf_{n-1} + f_{n-2}$ or $f_{n-1} + tf_{n-2}$ for some real or complex number t. In [18], by constructing new Fibonacci–Hessenberg matrices, another proofs of two results relative to the Pell and Perrin numbers is given.

Besides the usual Fibonacci and Lucas numbers many kinds of generalizations of these numbers have been presented in the literature. For any integer numbers $s > 0$ and $t \neq 0$ with $s^2 + 4t > 0$; the nth (s,t)-Fibonacci sequence, say $\{F_n(s,t)\}_{n\in\mathbb{N}}$ is defined recurrently by

$$F_{n+1}(s,t) = sF_n(s,t) + tF_{n-1}(s,t) \quad \text{for } n \geqslant 1, \tag{1}$$

with $F_0(s,t) = 0,\ F_1(s,t) = 1$.

For any integer numbers $s > 0$ and $t \neq 0$ with $s^2 + 4t > 0$; the nth (s,t)-Lucas sequence, say $\{L_n(s,t)\}_{n\in\mathbb{N}}$ is defined recurrently by

$$L_{n+1}(s,t) = sL_n(s,t) + tL_{n-1}(s,t) \quad \text{for } n \geqslant 1, \tag{2}$$

with $L_0(s,t) = 2,\ L_1(s,t) = s$.

In the rest of the paper, $F_n(s,t)$ and $L_n(s,t)$ would be written as F_n and L_n respectively.

The following table summarizes special cases of F_n and L_n:

(s,t)	F_n	L_n
$(1,1)$	Fibonacci numbers	Lucas numbers
$(2,1)$	Pell numbers	Pell–Lucas numbers
$(1,2)$	Jacobsthal numbers	Jacobsthal–Lucas numbers
$(3,-2)$	Mersenne numbers	Fermat numbers

Binet's formula are well known in the Fibonacci numbers theory [17]. Binet's formula allows us to express the (s,t)-Fibonacci and Lucas numbers in function of the roots α and β of the following characteristic equation, associated to the recurrence relation (1), or (2):

$$x^2 = sx + t. \tag{3}$$

Theorem 1 (Binet's formula). *The nth (s,t)-Fibonacci and Lucas numbers are given by*

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \text{ and } L_n = \alpha^n + \beta^n, \tag{4}$$

where α, β are the roots of the characteristic equation (3), and $\alpha > \beta$ (see [17]).

Note that, since $0 < s$, then

$$\beta < 0 < \alpha \quad \text{and} \quad |\beta| < |\alpha|,$$

$$\alpha + \beta = s \quad \text{and} \quad \alpha\beta = -t,$$

$$\alpha - \beta = \sqrt{s^2 + 4t}.$$

In this paper, we derive some relationships between the (s,t)-Fibonacci and Lucas numbers and determinants of some types of Hessenberg matrices, and we give a conjecture on the determinant of an infinite pentadiagonal matrix with the classical Fibonacci and Gaussian Fibonacci numbers.

The main contents of this paper are organized as follows: in Section 2, we introduce new classes of Hessenberg matrices whose determinants are the (s,t)-Fibonacci and Lucas numbers, where cofactor expansion is used to obtain these determinants. We also give the following interesting conjecture on the determinant of an infinite pentadiagonal matrix with the classical Fibonacci and Gaussian Fibonacci numbers in Section 3.

2. The determinants of Hessenberg matrices with the (s,t)-Fibonacci and Lucas sequences

Theorem 2. *For any integer numbers $s > 0$ and $t \neq 0$ with $s^2 + 4t > 0$, define the $(n+1) \times (n+1)$ matrix H_{n+1} as*

$$H_{n+1} = \begin{vmatrix} F_{2n} & 2tF_{2n-1} & (2t)^2F_{2n-2} & (2t)^3F_{2n-3} & \cdots & (2t)^{n-1}F_{n+1} & (2t)^nF_n \\ 1 & -s & s^2 & -s^3 & \cdots & \cdots & \binom{n}{0}(-s)^n \\ & 1 & -2s & 3s^2 & \cdots & \cdots & \binom{n}{1}(-s)^{n-1} \\ & & 1 & -3s & \cdots & \cdots & \binom{n}{2}(-s)^{n-2} \\ & & & \cdots & \cdots & \cdots & \cdots \\ & & & & & 1 & -sn \end{vmatrix}_{n+1}, \quad n \geqslant 0.$$

Then, the determinant H_{n+1} is given by

$$H_{n+1} = \begin{cases} \sqrt{(s^2+4t)^n}F_{2n}, & \text{if } n \text{ is zero or even}, \\ -\sqrt{(s^2+4t)^{n-1}}L_{2n}, & \text{if } n \text{ odd}. \end{cases}$$

Proof. If we expand the determinant H_{n+1} by the two elements in the last row, and repeat this operation on the determinants of lower order which appear, then we obtain

$$H_{n+1} = -\sum_{k=1}^{n}\binom{n}{k}s^kH_{n+1-k} + (-2t)^nF_n.$$

From where, the H_{n+1} term can be absorbed into the sum, thus we have

$$(-1)^n(2t)^nF_n = \sum_{k=0}^{n}\binom{n}{k}s^kH_{n+1-k}. \tag{5}$$

Since the polynomial in (5) is an Appell polynomial, using inverse relation of Appell polynomial (for more details see [1,2]) we obtain

$$H_{n+1} = (-1)^n\sum_{k=0}^{n}\binom{n}{k}(2t)^{n-k}s^kF_{n+k}. \tag{6}$$

From (4) and (6) we have

$$\begin{aligned} H_{n+1} &= \frac{(-1)^n}{\alpha-\beta}\sum_{k=0}^{n}\binom{n}{k}(2t)^{n-k}s^k\left(\alpha^{n+k}-\beta^{n+k}\right) \\ &= \frac{(-1)^n}{\alpha-\beta}\sum_{k=0}^{n}\binom{n}{k}(2t)^{n-k}\left[\alpha^n(\alpha s)^k-\beta^n(\beta s)^k\right] \qquad (7) \\ &= \frac{(-1)^n}{\alpha-\beta}\left[\alpha^n(2t+\alpha s)^n-\beta^n(2t+\beta s)^n\right] \\ &= \frac{(-1)^n}{\alpha-\beta}\left[\alpha^n\left(\alpha^2-\alpha\beta\right)^n-\beta^n\left(\beta^2-\alpha\beta\right)^n\right] \\ &= \frac{(-1)^n}{\alpha-\beta}\left[\alpha^{2n}(\alpha-\beta)^n-(-1)^n\beta^{2n}(\alpha-\beta)^n\right] \\ &= (\alpha-\beta)^n\left(\frac{(-1)^n\alpha^{2n}-\beta^{2n}}{\alpha-\beta}\right), \qquad (8) \end{aligned}$$

where α, β are the roots of the characteristic equation (3), and $\alpha > \beta$. Consequently, from (8) we get

$$H_{n+1}=\begin{cases}\sqrt{(s^2+4t)^n}F_{2n}, & \text{if } n \text{ is zero or even},\\ -\sqrt{(s^2+4t)^{n-1}}L_{2n}, & \text{if } n \text{ odd}.\end{cases}$$

This completes the proof. $\square$

For $\varphi_n(s)$, we have the following values:

(s,t)	H_2	H_3	H_4	H_5
$(1,1)$	$5F_4$	$-5L_6$	$25F_8$	$-25L_{10}$
$(1,2)$	$9F_4$	$-9L_6$	$81F_8$	$-81L_{10}$

Theorem 3. *For any integer numbers $s>0$ and $t\neq 0$ with $s^2+4t>0$, define the $(n+1)\times(n+1)$ matrix H^*_{n+1} as*

$$H^*_{n+1}=\frac{1}{n!}\begin{vmatrix} F_{2n} & 2tF_{2n-1} & (2t)^2F_{2n-2} & (2t)^3F_{2n-3} & \cdots & (2t)^{n-1}F_{n+1} & (2t)^nF_n \\ n & -s & & & & & \\ & n-1 & -2s & & & & \\ & & n-2 & -3s & & & \\ & & & \cdots & \cdots & \cdots & \cdots \\ & & & & & 1 & -ns \end{vmatrix}_{n+1}, \quad n\geqslant 0.$$

Then, the determinant H^*_{n+1} is given by

$$H^*_{n+1}=\begin{cases}\frac{1}{n!}\sqrt{(s^2+4t)^n}F_{2n}, & \text{if } n \text{ is zero or even},\\ -\frac{1}{n!}\sqrt{(s^2+4t)^{n-1}}L_{2n}, & \text{if } n \text{ odd}.\end{cases}$$

Proof. Since some of H^*_{n+1}'s elements are functions of n, the minor obtained by removing its last row and column is not equal to H^*_n . Hence, this implies that there is no obvious recurrence relation linking $H^*_{n+1}, H^*_n, H^*_{n-1}$, etc. Therefore, by a series of row operations which reduce some of its elements to zero, the determinant H^*_{n+1} can be obtained by transforming H_{n+1}.

By performing the row operations

$$R'_i=R_i-\left(\frac{i-1}{n+1-i}\right)(-s)R_{i+1},$$

with $2\leqslant i\leqslant n$, we get the determinant C_{n+1} with $(n-1)$ zero elements. Then, again by performing the row operations

$$R'_i=R_i-\left(\frac{i-1}{n+1-i}\right)(-s)R_{i+1},$$

with $2\leqslant i\leqslant n-1$, we get the determinant C_n with $(n-2)$ zero elements. Then, with $2\leqslant i\leqslant n-2$, etc., and, finally, with $i=2$, we get the determinant H^*_{n+1}. $\square$

For H^*_{n+1}, we have the following values:

(s,t)	H^*_2	H^*_3	H^*_4	H^*_5
$(1,1)$	$\frac{5}{2}F_4$	$-\frac{5}{6}L_6$	$\frac{25}{24}F_8$	$-\frac{5}{24}L_{10}$
$(1,2)$	$\frac{9}{2}F_4$	$-\frac{3}{2}L_6$	$\frac{81}{24}F_8$	$-\frac{27}{40}L_{10}$

Corollary 4. *For any integer numbers $s>0$ and $t\neq 0$ with $s^2+4t>0$, define the $(n+1)\times(n+1)$ matrix S_{n+1} and T_{n+1}, respectively, as*

$$S_{n+1}=\begin{vmatrix} \binom{n}{0} & \binom{n}{1} & \binom{n}{2} & \binom{n}{3} & \cdots & \binom{n}{n-1} & \binom{n}{n} \\ -\alpha s & 2t & & & & & \\ & -\alpha s & 2t & & & & \\ & & -\alpha s & 2t & & & \\ & & & \cdots & \cdots & \cdots & \cdots \\ & & & & & -\alpha s & 2t \end{vmatrix}_{n+1}$$

and

$$T_{n+1}=\begin{vmatrix} \binom{n}{0} & \binom{n}{1} & \binom{n}{2} & \binom{n}{3} & \cdots & \binom{n}{n-1} & \binom{n}{n} \\ -\beta s & 2t & & & & & \\ & -\beta s & 2t & & & & \\ & & -\beta s & 2t & & & \\ & & & \cdots & \cdots & \cdots & \cdots \\ & & & & & -\beta s & 2t \end{vmatrix}_{n+1},$$

with $n \geqslant 0$. *Then,*

$$\frac{(-1)^n}{\alpha-\beta}(\alpha^n S_{n+1}-\beta^n T_{n+1})=\begin{cases} \sqrt{(s^2+4t)^n}F_{2n}, & \text{if } n \text{ is zero or even}, \\ -\sqrt{(s^2+4t)^{n-1}}L_{2n}, & \text{if } n \text{ odd}. \end{cases}$$

Proof. Since

$$S_{n+1}=\sum_{k=0}^{n}\binom{n}{k}(2t)^{n-k}(\alpha s)^k$$

and

$$T_{n+1}=\sum_{k=0}^{n}\binom{n}{k}(2t)^{n-k}(\beta s)^k,$$

thus from the Eq. (8) we have $\frac{(-1)^n}{\alpha-\beta}(\alpha^n S_{n+1}-\beta^n T_{n+1})=(\alpha-\beta)^n\left(\frac{(-1)^n\alpha^{2n}-\beta^{2n}}{\alpha-\beta}\right)$. From the Binet's formulas of the nth (s,t)-Fibonacci and Lucas numbers, the result appears. $\square$

Theorem 5. *For any integer numbers* $s>0$ *and* $t\neq 0$ *with* $s^2+4t>0$, *define the* $(n+1)\times(n+1), n\geqslant 0$, *matrices* A_{n+1} *and* B_{n+1}, *respectively, as*

$$A_{n+1}=\begin{vmatrix} 1 & \alpha s & (\alpha s)^2 & (\alpha s)^3 & \cdots & (\alpha s)^{n-1} & (\alpha s)^n \\ -\frac{1}{t} & 1 & \alpha s & (\alpha s)^2 & \cdots & (\alpha s)^{n-2} & (\alpha s)^{n-1} \\ & -\frac{1}{t} & 1 & \alpha s & \cdots & (\alpha s)^{n-3} & (\alpha s)^{n-2} \\ & & \ddots & \ddots & & & \\ & & & \cdots & \cdots & \cdots & \cdots \\ & & & & & -\frac{1}{t} & 1 \end{vmatrix}_{n+1}$$

and

$$B_{n+1}=\begin{vmatrix} 1 & \beta s & (\beta s)^2 & (\beta s)^3 & \cdots & (\beta s)^{n-1} & (\beta s)^n \\ -\frac{1}{t} & 1 & \beta s & (\beta s)^2 & \cdots & (\beta s)^{n-2} & (\beta s)^{n-1} \\ & -\frac{1}{t} & 1 & \beta s & \cdots & (\beta s)^{n-3} & (\beta s)^{n-2} \\ & & \ddots & \ddots & & & \\ & & & \cdots & \cdots & \cdots & \cdots \\ & & & & & -\frac{1}{t} & 1 \end{vmatrix}_{n+1},$$

with $\alpha=\frac{s+\sqrt{s^2+4t}}{2}$ *and* $\beta=\frac{s-\sqrt{s^2+4t}}{2}$. *Then,*

$$A_{n+1} + B_{n+1} = \frac{1}{t^n} L_{2n}.$$

Proof. If we recall the properties of determinant and use the Binet's formulas of the nth (s,t)-Fibonacci and Lucas numbers, we obtain

$$\begin{aligned} A_{n+1} + B_{n+1} &= t^{-n}\left[(\alpha s + t)^n + (\beta s + t)^n\right] = t^{-n}\sum_{k=0}^{n}\binom{n}{k} t^{n-k}\left[(\alpha s)^k + (\beta s)^k\right] = \sum_{k=0}^{n}\binom{n}{k} t^{-k}\left[(\alpha s)^k + (\beta s)^k\right] \\ &= \sum_{k=0}^{n}\binom{n}{k}\left[\left(\frac{\alpha s}{t}\right)^k + \left(\frac{\beta s}{t}\right)^k\right] = \left(\frac{\alpha s}{t}+1\right)^n + \left(\frac{\beta s}{t}+1\right)^n = \left(\frac{\alpha s + t}{t}\right)^n + \left(\frac{\beta s + t}{t}\right)^n = \frac{1}{t^n}\left(\alpha^{2n} + \beta^{2n}\right) = \frac{1}{t^n} L_{2n}. \end{aligned}$$

Thus, the proof is completed.

For $n = 0$, we have

$$A_1 + B_1 = 2 = \frac{1}{t^0} L_0$$

and for $n = 1$ we have

$$A_2 + B_2 = \frac{1}{t}\left(s^2 + 2t\right) = \frac{1}{t} L_2.$$

□

3. The determinant of an infinite pentadiagonal matrix with Fibonacci and Gaussian Fibonacci numbers

Gaussian numbers were investigated in 1832 by Gauss [12]. A Gaussian integer is a complex number whose real and imaginary part are both integers. The Gaussian integers, with ordinary addition and multiplication of complex numbers, form an integral domain, usually written as $\mathbb{Z}[i]$. This domain does not have a total ordering that respects arithmetic, since it contains imaginary numbers. Gaussian integers are the set

$$\mathbb{Z}[i] = \left\{ a + ib : a, b \in \mathbb{Z} \text{ and } i^2 = -1 \right\}.$$

Horadam [15] examined Fibonacci numbers on the complex plane and established some interesting properties about them. Gaussian Fibonacci numbers (GFNS) $f_n^{(G)}$ are defined $f_n^{(G)} = f_{n-1}^{(G)} + f_{n-2}^{(G)}, n \geqslant 2$, where $f_0^{(G)} = i, f_1^{(G)} = 1$. The first six GFNs are $1, 1+i, 2+i, 3+2i, 5+3i$ and $8+5i$. Therefore, clearly, $f_n^{(G)} = f_n + if_{n-1}, n \geqslant 1$. Here, f_n is the nth classical Fibonacci number [17].

In the paper [9], the determinant of a pentadiagonal matrix with Fibonacci numbers such that

$$E_k = \begin{bmatrix} 1 - f_k f_{k-1} & f_{k+1} & f_k f_{k-1} & & & \\ -f_{k+1} & 1 - 2f_k f_{k-1} & \ddots & \ddots & & \\ f_k f_{k-1} & -f_{k+1} & \ddots & \ddots & \ddots & \\ & \ddots & \ddots & \ddots & f_{k+1} & f_k f_{k-1} \\ & & \ddots & \ddots & 1 - 2f_k f_{k-1} & f_{k+1} \\ & & & f_k f_{k-1} & -f_{k+1} & 1 - f_k f_{k-1} \end{bmatrix}_{k \times k}$$

was computed.

In here, we give the following interesting conjecture on the determinant of an infinite pentadiagonal matrix with the classical Fibonacci and Gaussian Fibonacci numbers:

Conjecture 6.

$$\begin{vmatrix} -f_1^{(G)} & i\sqrt{f_1} & 0 & 0 & 0 & 0 & 0 & \cdots \\ -if_2^{(G)}\sqrt{f_1} & -\overline{f_1^{(G)}}f_2^{(G)} & -if_3^{(G)}\sqrt{f_3} & i^2\sqrt{f_3 f_5} & 0 & 0 & 0 & \cdots \\ i^2\sqrt{f_1 f_3} & i\overline{f_1^{(G)}}\sqrt{f_3} & -\overline{f_2^{(G)}}f_3 & i\overline{f_2^{(G)}}\sqrt{f_5} & 0 & 0 & 0 & \cdots \\ 0 & 0 & -if_4^{(G)}\sqrt{f_5} & -\overline{f_3^{(G)}}f_4^{(G)} & -if_5^{(G)}\sqrt{f_7} & i^2\sqrt{f_7 f_9} & 0 & \cdots \\ 0 & 0 & i^2\sqrt{f_5 f_7} & i\overline{f_3^{(G)}}\sqrt{f_7} & -\overline{f_4^{(G)}}f_5^{(G)} & i\overline{f_4^{(G)}}\sqrt{f_9} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{vmatrix} = 0,$$

where f_n and $f_n^{(G)}$ are the nth classical Fibonacci and Gaussian Fibonacci numbers, respectively, $i^2 = -1$, and $\overline{f_n^{(G)}}$ is the conjugate of the nth Gaussian Fibonacci number.

4. Conclusions

We obtain formulas for the determinants of some Hessenberg matrices associated with the (s,t)-Fibonacci numbers and the roots of the characteristic equation (3) and they are computational feasible. Also, we give the following interesting conjecture on the determinant of an infinite pentadiagonal matrix with the classical Fibonacci and Gaussian Fibonacci numbers.

Acknowledgements

The authors wish to thank the editor and the referees for their thorough review and very useful suggestions.

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