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A Note on the Solutions of Some Linear Octonionic Equations

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Abstract: The main concerns of this paper are the linear equations with one term and one unknown of the forms: $\alpha(x\alpha)=\rho$, $\alpha(x\beta)=\rho$ and $(\alpha x)\beta=\rho$, and the linear equations with two terms and one unknown of the forms: $(\alpha x)\beta+(\gamma x)\delta=\rho$ and $\alpha(x\beta)+\gamma(x\delta)=\rho$ over the octonion field. Explicit general solutions of the equations in forms $\alpha(x\alpha)=\rho$, $\alpha(x\beta)=\rho$ and $(\alpha x)\beta=\rho$ are given, and solutions of the octonionic equations form $(\alpha x)\beta+(\gamma x)\delta=\rho$ and $\alpha(x\beta)+\gamma(x\delta)=\rho$ by matrix representation of octonions are derived using some particular cases. Examples of numerical equations are considered.

Keywords: The octonion field, The octonion equations.

1 Introduction

Even if are old, quaternions and octonions have at present many applications, as for example in physics, coding theory, computer vision, etc. For this reasons these algebras are intense studied. Research on solving the equations over the quaternion and octonion fields has attracted much interest. In [4] authors have described the set of solutions of the equation $x\alpha=x+\beta$ over an algebraic division ring. The author of the paper [5] has classified solutions of the quaternionic equation $ax+xb=c$. In [7] linear equations of the forms $ax=xb$ and $ax=\bar{x}b$ in the real Cayley–Dickson algebras (quaternions, octonions, sedenions) are solved and form for the roots of such equations is established. In [2] the solutions of the equations of the forms $ax=xb$ and $ax=\bar{x}b$ for some generalizations of quaternions and octonions are investigated. In [6], the $\alpha x\beta+\gamma x\delta=\rho$ linear quaternionic equation with one unknown, $\alpha x\beta+\gamma x\delta=\rho$, is solved. In [3], the quaternionic equation $ax+xb=c$ is studied. In [1], Bolat and İpek first have considered the linear octonionic equation with one unknown of the form $\alpha(x\alpha)=(\alpha x)\alpha=\alpha x\alpha=\rho$, with $0\neq\alpha\in\mathbf{O}$, secondly presented a method which is reduce this octonionic equation to an equation with the left and right coefficients to a real system of eight equations to find the solutions of this equation, and finally reached the solutions of the this linear octonionic equation from this real system.

In this study, we focus the linear equations with one term and one unknown of the forms: $\alpha(x\alpha)=\rho$, $\alpha(x\beta)=\rho$ and $(\alpha x)\beta=\rho$, and the linear equations with two terms and one unknown of the forms: $(\alpha x)\beta+(\gamma x)\delta=\rho$ and $\alpha(x\beta)+\gamma(x\delta)=\rho$ over the octonion field. Explicit general solutions of the equations in forms $\alpha(x\alpha)=\rho$, $\alpha(x\beta)=\rho$ and $(\alpha x)\beta=\rho$ are given, and solutions of the octonionic equations form $(\alpha x)\beta+(\gamma x)\delta=\rho$ and $\alpha(x\beta)+\gamma(x\delta)=\rho$ by matrix representation of octonions are derived using some particular cases. Our approach, to solve the problems in these types, is based on a new way of studying linear equations over the octonion field, which successfully overcomes the difficulty which arises from the noncommutative and nonassociative multiplication of octonions.

2 Some Preliminaries

In this section, we shortly review some definitions, notation and basic properties which we need to use in the presentations and proofs of our main results.

Let $\mathbf{O}$ be the octonion algebra over the real number field $\mathbb{R}$. In that case, $\mathbf{O}$ is an eight-dimensional

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Table 1: The multiplication table for the basis of **O**.

$\times$	1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
1	1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	e_1	-1	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6
e_2	e_2	$-e_3$	-1	e_1	e_6	e_7	$-e_4$	$-e_5$
e_3	e_3	e_2	$-e_1$	-1	e_7	$-e_6$	e_5	$-e_4$
e_4	e_4	$-e_5$	$-e_6$	$-e_7$	-1	e_1	e_2	e_3
e_5	e_5	e_4	$-e_7$	e_6	$-e_1$	-1	$-e_3$	e_2
e_6	e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	-1	$-e_1$
e_7	e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	-1

non-associative but alternative division algebra over its center field $\mathbb{R}$ and the canonical basis of **O** is

$$e_0=1,\ e_1=i,\ e_2=j,\ e_3=k,\ e_4=e,\ e_5=ie,\ e_6=je,\ e_7=ke. \tag{1}$$

The multiplication rules for the basis of **O** are listed in the table 1.

All elements of **O** take the form

$$\alpha=\alpha_0e_0+\alpha_1e_1+\alpha_2e_2+\alpha_3\,e_3+\alpha_4e_4+\alpha_5e_5+\alpha_6e_6+\alpha_7e_7,$$

with real coefficients $\{\alpha_i\}$. The conjugate of α is defined by

$$\overline{\alpha}=\alpha_0e_0-\alpha_1e_1-\alpha_2e_2-\alpha_3\,e_3-\alpha_4e_4-\alpha_5e_5-\alpha_6e_6-\alpha_7e_7$$

and the octonions α and β satisfy $\overline{(\alpha\beta)}=\overline{\beta}\,\overline{\alpha}$.

Let be $i^2=j^2=k^2=-1,\ ijk=-1$ and

$$\mathbf{H}=\{\alpha=\alpha_0+\alpha_1 i+\alpha_2 j+\alpha_3 k:\alpha_s\in\mathbb{R},\ s=0,1,2,3\}.$$

By the Cayley-Dickson process, any $\alpha\in\mathbf{O}$ can be written as

$$\alpha=\alpha'+\alpha''e$$

where $\alpha',\ \alpha''\in\mathbf{H}$.

The addition and multiplication for any $\alpha=\alpha'+\alpha''e,\ \beta=\beta'+\beta''e\in\mathbf{O}$ are defined respectively by

$$\begin{aligned}\alpha+\beta&=\left(\alpha'+\alpha''e\right)+\left(\beta'+\beta''e\right)\\&=\left(\alpha'+\beta'\right)+\left(\alpha''+\beta''\right)e\end{aligned}$$

and

$$\begin{aligned}\alpha\beta&=\left(\alpha'+\alpha''e\right)\left(\beta'+\beta''e\right)\\&=\left(\alpha'\beta'-\overline{\beta''}\alpha''\right)+\left(\beta''\alpha'+\alpha''\overline{\beta'}\right)e,\end{aligned} \tag{2}$$

where $\overline{\beta'}$ and $\overline{\beta''}$ denote the conjugates of the quaternions β' and β''.

The real and the imaginary parts of α are given by

$$\frac{\alpha+\overline{\alpha}}{2}=\alpha_0e_0$$

and

$$\frac{\alpha-\overline{\alpha}}{2}=\sum_{k=1}^{7}\alpha_ke_k$$

respectively.

The product of an octonion with its conjugate, $\overline{\alpha}\alpha=\alpha\overline{\alpha}$, is always a nonnegative real number:

$$\overline{\alpha}\alpha=\sum_{k=0}^{7}\alpha_k^2. \tag{3}$$

Using this, the norm of an octonion can be defined as

$$\|\alpha\|=\sqrt{\overline{\alpha}\alpha}.$$

This norm agrees with the standard Euclidean norm on $\mathbb{R}^8$ and the octonions α and β satisfy $\|\alpha\beta\|=\|\alpha\|\,\|\beta\|$.

The existence of a norm on **O** implies the existence of inverses for every nonzero element of **O**. The inverse of $\alpha\neq0$ is given by

$$\alpha^{-1}=\frac{\overline{\alpha}}{\|\alpha\|^2} \tag{4}$$

and it satisfies $\alpha^{-1}\alpha=\alpha\alpha^{-1}=1$.

For $k\in\mathbb{R}$, the octonion $k.\alpha$ is the octonion

$$k.\alpha=\sum_{i=0}^{7}(k\alpha_i)\,e_i. \tag{5}$$

Finally, the scalar product of the octonions $\alpha,\ \beta\in\mathbf{O}$ is

$$\langle\alpha,\beta\rangle=\sum_{i=0}^{7}\alpha_i\beta_i. \tag{6}$$

For all $\alpha,\beta\in\mathbf{O}$, the following equalities hold:

$$\alpha(\alpha\beta)=\alpha^2\beta,\ \ (\beta\alpha)\alpha=\beta\alpha^2,\ \ (\alpha\beta)\alpha=\alpha(\beta\alpha)=\alpha\beta\alpha. \tag{7}$$

Definition 1.*Let $x=\sum_{i=0}^{7}x_ie_i\in\mathbf{O}$. Then $\overrightarrow{x}=[x_0,x_1,x_2,x_3,x_4,x_5,x_6,x_7]^T$ is called the vector representation of x.*

Definition 2.[8] *Let $\alpha=\alpha'+\alpha''e\in\mathbf{O}$, where $\alpha'=\alpha_0+\alpha_1 i+\alpha_2 j+\alpha_3 k$, $\alpha''=\alpha_4+\alpha_5 i+\alpha_6 j+\alpha_7 k\in\mathbf{H}$. Then the 8×8 real matrix*

$$w(\alpha)=\begin{bmatrix}\alpha_0&-\alpha_1&-\alpha_2&-\alpha_3&-\alpha_4&-\alpha_5&-\alpha_6&-\alpha_7\\\alpha_1&\alpha_0&-\alpha_3&\alpha_2&-\alpha_5&\alpha_4&\alpha_7&-\alpha_6\\\alpha_2&\alpha_3&\alpha_0&-\alpha_1&-\alpha_6&-\alpha_7&\alpha_4&\alpha_5\\\alpha_3&-\alpha_2&\alpha_1&\alpha_0&-\alpha_7&\alpha_6&-\alpha_5&\alpha_4\\\alpha_4&\alpha_5&\alpha_6&\alpha_7&\alpha_0&-\alpha_1&-\alpha_2&-\alpha_3\\\alpha_5&-\alpha_4&\alpha_7&-\alpha_6&\alpha_1&\alpha_0&\alpha_3&-\alpha_2\\\alpha_6&-\alpha_7&-\alpha_4&\alpha_5&\alpha_2&-\alpha_3&\alpha_0&\alpha_1\\\alpha_7&\alpha_6&-\alpha_5&-\alpha_4&\alpha_3&\alpha_2&-\alpha_1&\alpha_0\end{bmatrix} \tag{8}$$

is called the left matrix representation of α over $\mathbb{R}$.

Let $c^1_{w(\alpha)}$ be the first column of the matris $w(\alpha)$.Then, it is obvious that $\overrightarrow{\alpha}=c^1_{w(\alpha)}$.

Theorem 1.[8] *Let $\alpha,\ x \in \mathbf{O}$ be given. Then*

$$\overrightarrow{\alpha x} = w(\alpha)\overrightarrow{x}. \tag{9}$$

Theorem 2.[8] *Let $\alpha = \alpha' + \alpha'' e \in \mathbf{O}$, where $\alpha' = \alpha_0 + \alpha_1 i + \alpha_2 j + \alpha_3 k$, $\alpha'' = \alpha_4 + \alpha_5 i + \alpha_6 j + \alpha_7 k \in \mathbf{H}$. Then the 8×8 real matrix*

$$v(\alpha) = \begin{bmatrix} \alpha_0 & -\alpha_1 & -\alpha_2 & -\alpha_3 & -\alpha_4 & -\alpha_5 & -\alpha_6 & -\alpha_7 \\ \alpha_1 & \alpha_0 & \alpha_3 & -\alpha_2 & \alpha_5 & -\alpha_4 & -\alpha_7 & \alpha_6 \\ \alpha_2 & -\alpha_3 & \alpha_0 & \alpha_1 & \alpha_6 & \alpha_7 & -\alpha_4 & -\alpha_5 \\ \alpha_3 & \alpha_2 & -\alpha_1 & \alpha_0 & \alpha_7 & -\alpha_6 & \alpha_5 & -\alpha_4 \\ \alpha_4 & -\alpha_5 & -\alpha_6 & -\alpha_7 & \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_5 & \alpha_4 & -\alpha_7 & \alpha_6 & -\alpha_1 & \alpha_0 & -\alpha_3 & \alpha_2 \\ \alpha_6 & \alpha_7 & \alpha_4 & -\alpha_5 & -\alpha_2 & \alpha_3 & \alpha_0 & -\alpha_1 \\ \alpha_7 & -\alpha_6 & \alpha_5 & \alpha_4 & -\alpha_3 & -\alpha_2 & \alpha_1 & \alpha_0 \end{bmatrix} \tag{10}$$

is called the right matrix representation of α over $\mathbb{R}$.

Let $c^1_{v(\alpha)}$ be the first column of the matris $v(\alpha)$. Then, it is obvious that $\overrightarrow{\alpha} = c^1_{v(\alpha)}$.

Theorem 3.[8] *Let $\alpha,\ x \in \mathbf{O}$ be given. Then*

$$\overrightarrow{x\alpha} = v(\alpha)\overrightarrow{x}. \tag{11}$$

Theorem 4.[8] *Let $\alpha,\ x \in \mathbf{O}$, $\lambda \in \mathbb{R}$ be given. Then*

1.$\alpha = \beta \Leftrightarrow w(\alpha) = w(\beta)$,
2.$w(\alpha+\beta) = w(\alpha) + w(\beta)$,
3.$w(\lambda\alpha) = \lambda w(\alpha)$, $w(1) = I_8$,
4.$w(\overline{\alpha}) = w^T(\alpha)$,
5.$\alpha = \beta \Leftrightarrow v(\alpha) = v(\beta)$,
6.$v(\alpha+\beta) = v(\alpha) + v(\beta)$,
7.$v(\lambda\alpha) = \lambda v(\alpha)$, $v(1) = I_8$,
8.$v(\overline{\alpha}) = v^T(\alpha)$.

Theorem 5.[8] *Let $\alpha \in \mathbf{O}$ be given with $\alpha \neq 0$. Then*

$$w^{-1}(\alpha) = w\left(\alpha^{-1}\right), \text{ and } v^{-1}(\alpha) = v\left(\alpha^{-1}\right). \tag{12}$$

Theorem 6.[8] *Let $\alpha,\ \beta \in \mathbf{O}$ be given. Then their matrix representations satisfy the following two identities*

$$w(\alpha\beta\alpha) = w(\alpha)\, w(\beta)\, w(\alpha) \tag{13}$$

and

$$v(\alpha\beta\alpha) = v(\alpha)\, v(\beta)\, v(\alpha). \tag{14}$$

3 Main Results and Examples

In this section, using the representation matrices $w(.)$ and $v(.)$ of octonions we give solutions of the some linear octonionic equations with one term and one unknown and then ones with two terms and one unknown over the octonion field.

3.1 Solutions of some linear octonionic equations with one term and one unknown

In this section, we deal with the linear octonionic equations with one term and one unknown of the forms: $\alpha(x\alpha) = \rho$, $\alpha(x\beta) = \rho$ and $(\alpha x)\beta = \rho$.

Proposition 1.*Let $\alpha = \sum_{i=0}^{7} \alpha_i e_i \in \mathbf{O} - \{0\}$ and $\rho = \sum_{i=0}^{7} \rho_i e_i \in \mathbf{O}$ be given. Then, the linear octonionic equation*

$$\alpha(x\alpha) = (\alpha x)\alpha = \alpha x \alpha = \rho, \tag{15}$$

has a unique solution $x \in \mathbf{O}$. The vector representation of x is

$$\overrightarrow{x} = \frac{1}{|\alpha|^4} w^T(\alpha)\, v^T(\alpha)\, \overrightarrow{\rho}, \tag{16}$$

or

$$\overrightarrow{x} = \frac{1}{|\alpha|^4} c^1_{w^T(\alpha) w(\rho) w^T(\alpha)}, \tag{17}$$

or

$$\overrightarrow{x} = \frac{1}{|\alpha|^4} c^1_{v^T(\alpha) v(\rho) v^T(\alpha)}, \tag{18}$$

and x octonion is

$$\begin{aligned} x = \varpi \Bigg[& \left(\sum_{k=0}^{7} \mu_k \rho_k\right) e_0 \\ & + (-2\alpha_0\alpha_1\rho_0 + \mu_1\rho_1 - 2\alpha_1\alpha_2\rho_2 - 2\alpha_1\alpha_3\rho_3 - 2\alpha_1\alpha_4\rho_4 - 2\alpha_1\alpha_5\rho_5 - 2\alpha_1\alpha_6\rho_6 - 2\alpha_1\alpha_7\rho_7) e_1 \\ & + (-2\alpha_0\alpha_2\rho_0 - 2\alpha_1\alpha_2\rho_1 + \mu_2\rho_2 - 2\alpha_2\alpha_3\rho_3 - 2\alpha_2\alpha_4\rho_4 - 2\alpha_2\alpha_5\rho_5 - 2\alpha_2\alpha_6\rho_6 - 2\alpha_2\alpha_7\rho_7) e_2 \\ & + (-2\alpha_0\alpha_3\rho_0 - 2\alpha_1\alpha_3\rho_1 - 2a_2a_3\rho_2 + \mu_3\rho_3 - 2a_3a_4\rho_4 - 2a_3a_5\rho_5 - 2a_3a_6\rho_6 - 2a_3a_7\rho_7) e_3 \\ & + (-2\alpha_0\alpha_4\rho_0 - 2\alpha_1\alpha_4\rho_1 - 2a_2a_4\rho_2 - 2a_3a_4\rho_3 + \mu_4\rho_4 - 2a_4a_5\rho_5 - 2a_4a_6\rho_6 - 2a_4a_7\rho_7) e_4 \\ & + (-2\alpha_0\alpha_5\rho_0 - 2\alpha_1\alpha_5\rho_1 - 2a_2a_5\rho_2 - 2a_3a_5\rho_3 - 2a_4a_5\rho_4 + \mu_5\rho_5 - 2\alpha_5\alpha_6\rho_6 - 2a_5a_7\rho_7) e_5 \\ & + (-2\alpha_0\alpha_6\rho_0 - 2\alpha_1\alpha_6\rho_1 - 2a_2a_6\rho_2 - 2a_3a_6\rho_3 - 2a_4a_6\rho_4 - 2\alpha_5\alpha_6\rho_5 + \mu_6\rho_6 - 2a_6a_7\rho_7) e_6 \\ & + (-2\alpha_0\alpha_7\rho_0 - 2\alpha_1\alpha_7\rho_1 - 2a_2a_7\rho_2 - 2a_3a_7\rho_3 - 2a_4a_7\rho_4 - 2a_5a_7\rho_5 - 2a_6a_7\rho_6 + \mu_7\rho_7) e_7 \Bigg] \end{aligned}$$

where $\varpi = \left(\sum_{i=0}^{7} \alpha_i^2\right)^{-\frac{1}{2}}$, $\mu_0 = -\sum_{i=0}^{7} (\alpha_i^2 + 2\alpha_0^2)$, $\mu_k = \sum_{i=0}^{7} (\alpha_i^2 - 2\alpha_k^2)$, $k = 1, \ldots, 7$.

Proof. From (15), we write that

$$\overrightarrow{\alpha(x\alpha)} = \overrightarrow{(\alpha x)\alpha} = \overrightarrow{\alpha x \alpha} = \overrightarrow{\rho}$$

and from here, considering first (9) and then (11) we obtain

$$v(\alpha)\, w(\alpha)\, \overrightarrow{x} = \overrightarrow{\rho}.$$

For $\alpha = \sum_{i=0}^{7} \alpha_i e_i \in \mathbf{O} - \{0\}$, since the matrices $v(\alpha)$ and $w(\alpha)$ are invertible, we obtain the vector representation of

x as

$$\begin{aligned}\overrightarrow{x} &= w^{-1}(\alpha)\, v^{-1}(\alpha)\, \overrightarrow{\rho} \\ &= w\left(\alpha^{-1}\right) v\left(\alpha^{-1}\right) \overrightarrow{\rho}, \\ &= w\left(\frac{\overline{\alpha}}{|\alpha|^2}\right) v\left(\frac{\overline{\alpha}}{|\alpha|^2}\right) \overrightarrow{\rho} \\ &= \frac{1}{|\alpha|^4} w(\overline{\alpha})\, v(\overline{\alpha})\, \overrightarrow{\rho} \\ &= \frac{1}{|\alpha|^4} w^T(\alpha)\, v^T(\alpha)\, \overrightarrow{\rho}.\end{aligned}$$

Now from (15) we write that

$$w(\alpha(x\alpha)) = w((\alpha x)\alpha) = w(\alpha x \alpha) = w(\rho)$$

and from here by using of the equality (13) we obtain

$$w(\alpha)\, w(x)\, w(\alpha) = w(\rho).$$

Therefore, we get

$$\begin{aligned}w(x) &= w^{-1}(\alpha)\, w(\rho)\, w^{-1}(\alpha) \\ &= w\left(\alpha^{-1}\right) w(\rho)\, w\left(\alpha^{-1}\right) \\ &= \frac{1}{|\alpha|^4} w^T(\alpha)\, w(\rho)\, w^T(\alpha),\end{aligned} \tag{19}$$

and thus considering the equality $\overrightarrow{x} = c^1_{w(x)}$, we obtain $\overrightarrow{x} = \frac{1}{|\alpha|^4} c^1_{w^T(\alpha) w(\rho) w^T(\alpha)}$. Similarly, from (15) we write

$$v(\alpha(x\alpha)) = v((\alpha x)\alpha) = v(\alpha x \alpha) = v(\rho)$$

and from here by using of the equality (14) we obtain

$$v(\alpha)\, v(x)\, v(\alpha) = v(\rho).$$

Therefore, we get

$$\begin{aligned}v(x) &= v^{-1}(\alpha)\, v(\rho)\, v^{-1}(\alpha) \\ &= v\left(\alpha^{-1}\right) v(\rho)\, v\left(\alpha^{-1}\right) \\ &= \frac{1}{|\alpha|^4} v^T(\alpha)\, v(\rho)\, v^T(\alpha),\end{aligned} \tag{20}$$

and thus considering the equality $\overrightarrow{x} = c^1_{v(x)}$, we find $\overrightarrow{x} = \frac{1}{|\alpha|^4} c^1_{v^T(\alpha) v(\rho) v^T(\alpha)}$.

Consequently, from (16), (17) or (18), the Eq.(19) is obtained.

*Example 1.*Consider the following equation:

$$(e_0+4e_1-e_2+3e_4+e_5-7e_6+5e_7)\, x\, (e_0+4e_1-e_2+3e_4+e_5-7e_6+5e_7) = -e_0+2e_1+3e_2+5e_3-3e_4-e_6$$

in $\mathbf{O}$. For $\alpha = e_0+4e_1-e_2+3e_4+e_5-7e_6+5e_7$ and $\rho = -e_0+2e_1+3e_2+5e_3-3e_4-e_6$, this equation is of the form

$$\alpha(x\alpha) = (\alpha x)\alpha = \alpha x \alpha = \rho.$$

For $\alpha_0 = 1,\ \alpha_1 = 4,\ \alpha_2 = -1,\ \alpha_3 = 0,\ \alpha_4 = 3,\ \alpha_5 = 1,\ \alpha_6 = -7,\ \alpha_7 = 5$ and $\rho_0 = -1,\ \rho_1 = 2,\ \rho_2 = 3,\ \rho_3 = 5,\ \rho_4 = -3,\ \rho_5 = 0,\ \rho_6 = -1,\ \rho_7 = 0$, from the formula (19) given for x in Proposition 1, we obtain the solution x as

$$x = \frac{53}{5202}e_0 + \frac{47}{2601}e_1 + \frac{155}{5202}e_2 + \frac{5}{102}e_3 - \frac{53}{1734}e_4 - \frac{1}{2601}e_5 - \frac{37}{5202}e_6 - \frac{5}{2601}e_7.$$

Proposition 2.*Let* $\alpha = \sum_{i=0}^{7} \alpha_i e_i,\ \beta = \sum_{i=0}^{7} \beta_i e_i \in \mathbf{O} - \{0\}$ *and* $\rho = \sum_{i=0}^{7} \rho_i e_i \in \mathbf{O}$ *be given. Then, the linear octonionic equation*

$$\alpha(x\beta) = \rho \tag{21}$$

has a unique solution $x \in \mathbf{O}$. *The vector representation of x and x octonion are respectively*

$$\overrightarrow{x} = \frac{1}{|\beta|^2 |\alpha|^2} v^T(\beta)\, w^T(\alpha)\, \overrightarrow{\rho} \tag{22}$$

and

$$\begin{aligned}x = \frac{1}{|\beta|^2 |\alpha|^2} \Bigg[&\left(\sum_{k=0}^{7} \beta_k \varpi_k\right) e_0 \\ &+ (-\beta_1\varpi_0+\beta_0\varpi_1-\beta_3\varpi_2+\beta_2\varpi_3-\beta_5\varpi_4+\beta_4\varpi_5+\beta_7\varpi_6-\beta_6\varpi_7)\, e_1 \\ &+ (-\beta_2\varpi_0+\beta_3\varpi_1+\beta_0\varpi_2-\beta_1\varpi_3-\beta_6\varpi_4-\beta_7\varpi_5+\beta_4\varpi_6+\beta_5\varpi_7)\, e_2 \\ &+ (-\beta_3\varpi_0-\beta_2\varpi_1+\beta_1\varpi_2+\beta_0\varpi_3-\beta_7\varpi_4+\beta_6\varpi_5-\beta_5\varpi_6+\beta_4\varpi_7)\, e_3 \\ &+ (-\beta_4\varpi_0+\beta_5\varpi_1+\beta_6\varpi_2+\beta_7\varpi_3+\beta_0\varpi_4-\beta_1\varpi_5-\beta_2\varpi_6-\beta_3\varpi_7)\, e_4 \\ &+ (-\beta_5\varpi_0-\beta_4\varpi_1+\beta_7\varpi_2-\beta_6\varpi_3+\beta_1\varpi_4+\beta_0\varpi_5+\beta_3\varpi_6-\beta_2\varpi_7)\, e_5 \\ &+ (-\beta_6\varpi_0-\beta_7\varpi_1-\beta_4\varpi_2+\beta_5\varpi_3+\beta_2\varpi_4-\beta_3\varpi_5+\beta_0\varpi_6+\beta_1\varpi_7)\, e_6 \\ &+ (-\beta_7\varpi_0+\beta_6\varpi_1-\beta_5\varpi_2-\beta_4\varpi_3+\beta\varpi_4+\beta_2\varpi_5-\beta_1\varpi_6+\beta_0\varpi_7)\, e_7 \Bigg],\end{aligned}$$

where

$$\left.\begin{aligned}\varpi_0 &= \alpha_0\rho_0+\alpha_1\rho_1+\alpha_2\rho_2+\alpha_3\rho_3+\alpha_4\rho_4+\alpha_5\rho_5+\alpha_6\rho_6+\alpha_7\rho_7, \\ \varpi_1 &= -\alpha_1\rho_0+\alpha_0\rho_1+\alpha_3\rho_2-\alpha_2\rho_3+\alpha_5\rho_4-\alpha_4\rho_5-\alpha_7\rho_6+\alpha_6\rho_7, \\ \varpi_2 &= -\alpha_2\rho_0-\alpha_3\rho_1+\alpha_0\rho_2+\alpha_1\rho_3+\alpha_6\rho_4+\alpha_7\rho_5-\alpha_4\rho_6-\alpha_5\rho_7, \\ \varpi_3 &= -\alpha_3\rho_0+\alpha_2\rho_1-\alpha_1\rho_2+\alpha_0\rho_3+\alpha_7\rho_4-\alpha_6\rho_5+\alpha_5\rho_6-\alpha_4\rho_7, \\ \varpi_4 &= -\alpha_4\rho_0-\alpha_5\rho_1-\alpha_6\rho_2-\alpha_7\rho_3+\alpha_0\rho_4+\alpha_1\rho_5+\alpha_2\rho_6+\alpha_3\rho_7, \\ \varpi_5 &= -\alpha_5\rho_0+\alpha_4\rho_1-\alpha_7\rho_2+\alpha_6\rho_3-\alpha_1\rho_4+\alpha_0\rho_5-\alpha_3\rho_6+\alpha_2\rho_7, \\ \varpi_6 &= -\alpha_6\rho_0+\alpha_7\rho_1+\alpha_4\rho_2-\alpha_5\rho_3-\alpha_2\rho_4+\alpha_3\rho_5+\alpha_0\rho_6-\alpha_1\rho_7, \\ \varpi_7 &= -\alpha_7\rho_0-\alpha_6\rho_1+\alpha_5\rho_2+\alpha_4\rho_3-\alpha_3\rho_4-\alpha_2\rho_5+\alpha_1\rho_6+\alpha_0\rho_7.\end{aligned}\right\} \tag{23}$$

*Proof.*From (21), we write that

$$\overrightarrow{\alpha(x\beta)} = \overrightarrow{\rho},$$

and from here, considering first (9) and then (11) we obtain

$$w(\alpha)\, v(\beta)\, \overrightarrow{x} = \overrightarrow{\rho}.$$

For $\alpha, \beta \in \mathbf{O} - \{0\}$, since the matrices $v(\beta)$ and $w(\alpha)$ are invertible, we get the vector representation of x as

$$\begin{aligned}\overrightarrow{x} &= v^{-1}(\beta)\, w^{-1}(\alpha)\, \overrightarrow{\rho} \\ &= v\left(\beta^{-1}\right) w\left(\alpha^{-1}\right) \overrightarrow{\rho} \\ &= \frac{1}{|\beta|^2 |\alpha|^2} v^T(\beta)\, w^T(\alpha)\, \overrightarrow{\rho}.\end{aligned} \tag{24}$$

Thus from (24), the Eq.(23) is obtained.

*Example 2.*Consider the following equation:

$$(e_0-2e_1+3e_3+5e_4-e_5-6e_6+e_7)\,(x\,(3e_0-e_1-2e_3+e_4-2e_6-4e_7))= e_0-2e_1+3e_2+6e_3+5e_4-2e_5+4e_6+8e_7$$

in $\mathbf{O}$. For $\alpha = e_0-2e_1+3e_3+5e_4-e_5-6e_6+e_7$, $\beta = 3e_0-e_1-2e_3+e_4-2e_6-4e_7$ and $\rho = e_0-2e_1+3e_2+6e_3+5e_4-2e_5+4e_6+8e_7$, this equation is of the form

$$\alpha(x\beta) = \rho.$$

Thus, from the formula (22) and (23) given for x, we obtain $\overrightarrow{x}$ and x such that

$$\overrightarrow{x} = \left[\tfrac{188}{2695}, -\tfrac{384}{2695}, -\tfrac{40}{539}, \tfrac{244}{2695}, \tfrac{288}{2695}, -\tfrac{26}{539}, -\tfrac{23}{2695}, \tfrac{214}{2695}\right]^T$$

and

$$x = \tfrac{188}{2695}e_0 - \tfrac{384}{2695}e_1 - \tfrac{40}{539}e_2 + \tfrac{244}{2695}e_3 + \tfrac{288}{2695}e_4 - \tfrac{26}{539}e_5 - \tfrac{23}{2695}e_6 + \tfrac{214}{2695}e_7,$$

respectively.

Proposition 3.*Let* $\alpha = \sum\limits_{i=0}^{7} \alpha_i e_i$, $\beta = \sum\limits_{i=0}^{7} \beta_i e_i \in \mathbf{O} - \{0\}$ *and* $\rho = \sum\limits_{i=0}^{7} \rho_i e_i \in \mathbf{O}$ *be given. Then, the linear octonionic equation*

$$(\alpha x)\,\beta = \rho \tag{25}$$

has a unique solution $x \in \mathbf{O}$. *The vector representation of* x *and* x *octonion are respectively*

$$\overrightarrow{x} = \frac{1}{|\alpha|^2\,|\beta|^2} w^T(\alpha)\, v^T(\beta)\, \overrightarrow{\rho} \tag{26}$$

and

$$\begin{aligned} x = {} & \frac{1}{|\alpha|^2\,|\beta|^2}\left[\left(\sum_{k=0}^{7}\alpha_k\eta_k\right)e_0\right. \\ & +(-\alpha_1\eta_0+\alpha_0\eta_1+\alpha_3\eta_2-\alpha_2\eta_3+\alpha_5\eta_4-\alpha_4\eta_5-\alpha_7\eta_6+\alpha_6\eta_7)\,e_1 \\ & +(-\alpha_2\eta_0-\alpha_3\eta_1+\alpha_0\eta_2+\alpha_1\eta_3+\alpha_6\eta_4+\alpha_7\eta_5-\alpha_4\eta_6-\alpha_5\eta_7)\,e_2 \\ & +(-\alpha_3\eta_0+\alpha_2\eta_1-\alpha_1\eta_2+\alpha_0\eta_3+\alpha_7\eta_4-\alpha_6\eta_5+\alpha_5\eta_6-\alpha_4\eta_7)\,e_3 \\ & +(-\alpha_4\eta_0-\alpha_5\eta_1-\alpha_6\eta_2-\alpha_7\eta_3+\alpha_0\eta_4+\alpha_1\eta_5+\alpha_2\eta_6+\alpha_3\eta_7)\,e_4 \\ & +(-\alpha_5\eta_0+\alpha_4\eta_1-\alpha_7\eta_2+\alpha_6\eta_3-\alpha_1\eta_4+\alpha_0\eta_5-\alpha_3\eta_6+\alpha_2\eta_7)\,e_5 \\ & +(-\alpha_6\eta_0+\alpha_7\eta_1+\alpha_4\eta_2-\alpha_5\eta_3-\alpha_2\eta_4+\alpha_3\eta_5+\alpha_0\eta_6-\alpha_1\eta_7)\,e_6 \\ & \left.+(-\alpha_7\eta_0-\alpha_6\eta_1+\alpha_5\eta_2+\alpha_4\eta_3-\alpha_3\eta_4-\alpha_2\eta_5+\alpha_1\eta_6+\alpha_0\eta_7)\,e_7\right], \end{aligned}$$

where

$$\left.\begin{aligned} \eta_0 &= \beta_0\rho_0+\beta_1\rho_1+\beta_2\rho_2+\beta_3\rho_3+\beta_4\rho_4+\beta_5\rho_5+\beta_6\rho_6+\beta_7\rho_7, \\ \eta_1 &= -\beta_1\rho_0+\beta_0\rho_1-\beta_3\rho_2+\beta_2\rho_3-\beta_5\rho_4+\beta_4\rho_5+\beta_7\rho_6-\beta_6\rho_7, \\ \eta_2 &= -\beta_2\rho_0+\beta_3\rho_1+\beta_0\rho_2-\beta_1\rho_3-\beta_6\rho_4-\beta_7\rho_5+\beta_4\rho_6+\beta_5\rho_7, \\ \eta_3 &= -\beta_3\rho_0-\beta_2\rho_1+\beta_1\rho_2+\beta_0\rho_3-\beta_7\rho_4+\beta_6\rho_5-\beta_5\rho_6+\beta_4\rho_7, \\ \eta_4 &= -\beta_4\rho_0+\beta_5\rho_1+\beta_6\rho_2+\beta_7\rho_3+\beta_0\rho_4-\beta_1\rho_5-\beta_2\rho_6-\beta_3\rho_7, \\ \eta_5 &= -\beta_5\rho_0-\beta_4\rho_1+\beta_7\rho_2-\beta_6\rho_3+\beta_1\rho_4+\beta_0\rho_5+\beta_3\rho_6-\beta_2\rho_7, \\ \eta_6 &= -\beta_6\rho_0-\beta_7\rho_1-\beta_4\rho_2+\beta_5\rho_3+\beta_2\rho_4-\beta_3\rho_5+\beta_0\rho_6+\beta_1\rho_7, \\ \eta_7 &= -\beta_7\rho_0+\beta_6\rho_1-\beta_5\rho-\beta_4\rho_3+\beta_3\rho_4+\beta_2\rho_5-\beta_1\rho_6+\beta_0\rho_7. \end{aligned}\right\} \tag{27}$$

*Proof.*From (25), we write that

$$\overrightarrow{(\alpha x)\,\beta} = \overrightarrow{\rho}$$

and from here, considering first (11) and then (9) we obtain

$$v(\beta)\,w(\alpha)\,\overrightarrow{x} = \overrightarrow{\rho}.$$

For $\alpha, \beta \in \mathbf{O} - \{0\}$, since the matrices $v(\beta)$ and $w(\alpha)$ are invertible, we obtain the vector representation of x as

$$\begin{aligned} \overrightarrow{x} &= w^{-1}(\alpha)\, v^{-1}(\beta)\, \overrightarrow{\rho} \\ &= w\left(\alpha^{-1}\right) v\left(\beta^{-1}\right) \overrightarrow{\rho} \\ &= \frac{1}{|\alpha|^2\,|\beta|^2} w^T(\alpha)\, v^T(\beta)\, \overrightarrow{\rho}. \end{aligned} \tag{28}$$

Thus from (28), the Eq.(27) is obtained.

*Example 3.*Consider the following equation:

$$((e_0-2e_1+3e_3+5e_4-e_5-6e_6+e_7)\,x)\,(3e_0-e_1-2e_3+e_4-2e_6-4e_7)= e_0-2e_1+3e_2+6e_3+5e_4-2e_5+4e_6+8e_7$$

in $\mathbf{O}$. For $\alpha = e_0-2e_1+3e_3+5e_4-e_5-6e_6+e_7$, $\beta = 3e_0-e_1-2e_3+e_4-2e_6-4e_7$ and $\rho = e_0-2e_1+3e_2+6e_3+5e_4-2e_5+4e_6+8e_7$, this equation is of the form

$$(\alpha x)\,\beta = \rho.$$

Thus, from the formulas (26) and (27) given for x, we obtain $\overrightarrow{x}$ and x such that

$$\overrightarrow{x} = \left[\tfrac{188}{2695}, -\tfrac{34}{2695}, -\tfrac{2}{539}, \tfrac{6}{539}, \tfrac{402}{2695}, -\tfrac{72}{539}, -\tfrac{9}{245}, \tfrac{60}{539}\right]^T,$$

and

$$x = \tfrac{188}{2695}e_0 - \tfrac{34}{2695}e_1 - \tfrac{2}{539}e_2 + \tfrac{6}{539}e_3 + \tfrac{402}{2695}e_4 - \tfrac{72}{539}e_5 - \tfrac{9}{245}e_6 + \tfrac{60}{539}e_7$$

respectively.

3.2 Solutions of some linear octonionic equations with two terms and one unknown

In this section, we deal with the linear octonionic equations with two terms and one unknown as x of the form

$$(\alpha x)\,\beta + (\gamma x)\,\delta = \rho \tag{29}$$

or equivalently

$$[v(\beta)\,w(\alpha) + v(\delta)\,w(\gamma)]\,\overrightarrow{x} = \overrightarrow{\rho}, \tag{30}$$

and of the form

$$\alpha\,(x\beta) + \gamma\,(x\delta) = \rho \tag{31}$$

or equivalently

$$[w(\alpha)\,v(\beta) + w(\gamma)\,v(\delta)]\,\overrightarrow{x} = \overrightarrow{\rho}, \tag{32}$$

where $\alpha, \beta, \gamma, \delta \in \mathbf{O} - \{0\}$ and $\rho = \sum\limits_{i=0}^{7} \rho_i e_i \in \mathbf{O}$.

Proposition 4. *Let* $\alpha=\sum_{i=0}^{7}\alpha_i e_i,\ \beta=\sum_{i=0}^{7}\beta_i e_i\in\mathbf{O}-\{0\}$ *and* $\rho=\sum_{i=0}^{7}\rho_i e_i\in\mathbf{O}$ *be given.*

1.Let be $\beta=\delta$ in (29) and (30).

(a)If $\alpha\neq-\gamma$, then the solutions of (30) and (29), respectively, are

$$\vec{x}=\frac{1}{|\beta|^2|\alpha+\gamma|^2}w^T(\alpha+\gamma)v^T(\beta)\vec{\rho}$$

and

$$\begin{aligned}x=&\frac{1}{|\beta|^2|\mu|^2}\left[\left(\sum_{k=0}^{7}\mu_k\eta_k\right)e_0\right.\\&+(-\mu_1\eta_0+\mu_0\eta_1+\mu_3\eta_2-\mu_2\eta_3+\mu_5\eta_4-\mu_4\eta_5-\mu_7\eta_6+\mu_6\eta_7)e_1\\&+(-\mu_2\eta_0-\mu_3\eta_1+\mu_0\eta_2+\mu_1\eta_3+\mu_6\eta_4+\mu_7\eta_5-\mu_4\eta_6-\mu_5\eta_7)e_2\\&+(-\mu_3\eta_0+\mu_2\eta_1-\mu_1\eta_2+\mu_0\eta_3+\mu_7\eta_4-\mu_6\eta_5+\mu_5\eta_6-\mu_4\eta_7)e_3\\&+(-\mu_4\eta_0-\mu_5\eta_1-\mu_6\eta_2-\mu_7\eta_3+\mu_0\eta_4+\mu_1\eta_5+\mu_2\eta_6+\mu_3\eta_7)e_4\\&+(-\mu_5\eta_0+\mu_4\eta_1-\mu_7\eta_2+\mu_6\eta_3-\mu_1\eta_4+\mu_0\eta_5-\mu_3\eta_6+\mu_2\eta_7)e_5\\&+(-\mu_6\eta_0+\mu_7\eta_1+\mu_4\eta_2-\mu_5\eta_3-\mu_2\eta_4+\mu_3\eta_5+\mu_0\eta_6-\mu_1\eta_7)e_6\\&\left.+(-\mu_7\eta_0-\mu_6\eta_1+\mu_5\eta_2+\mu_4\eta_3-\mu_3\eta_4-\mu_2\eta_5+\mu_1\eta_6+\mu_0\eta_7)e_7\right],\end{aligned}$$

where $\alpha+\gamma=\mu\in O-\{0\}$, $\mu=\sum_{i=0}^{7}\mu_i e_i$ and all η_i is as in (27).

(b)If $\alpha=\gamma$, then the solutions of (30) and (29), respectively, are

$$\vec{x}=\frac{1}{2|\beta|^2|\alpha|^2}w^T(\alpha)v^T(\beta)\vec{\rho}$$

and

$$\begin{aligned}x=&\frac{1}{2|\beta|^2|\alpha|^2}\left[\left(\sum_{k=0}^{7}\alpha_k\eta_k\right)e_0\right.\\&+(-\alpha_1\eta_0+\alpha_0\eta_1+\alpha_3\eta_2-\alpha_2\eta_3+\alpha_5\eta_4-\alpha_4\eta_5-\alpha_7\eta_6+\alpha_6\eta_7)e_1\\&+(-\alpha_2\eta_0-\alpha_3\eta_1+\alpha_0\eta_2+\alpha_1\eta_3+\alpha_6\eta_4+\alpha_7\eta_5-\alpha_4\eta_6-\alpha_5\eta_7)e_2\\&+(-\alpha_3\eta_0+\alpha_2\eta_1-\alpha_1\eta_2+\alpha_0\eta_3+\alpha_7\eta_4-\alpha_6\eta_5+\alpha_5\eta_6-\alpha_4\eta_7)e_3\\&+(-\alpha_4\eta_0-\alpha_5\eta_1-\alpha_6\eta_2-\alpha_7\eta_3+\alpha_0\eta_4+\alpha_1\eta_5+\alpha_2\eta_6+\alpha_3\eta_7)e_4\\&+(-\alpha_5\eta_0+\alpha_4\eta_1-\alpha_7\eta_2+\alpha_6\eta_3-\alpha_1\eta_4+\alpha_0\eta_5-\alpha_3\eta_6+\alpha_2\eta_7)e_5\\&+(-\alpha_6\eta_0+\alpha_7\eta_1+\alpha_4\eta_2-\alpha_5\eta_3-\alpha_2\eta_4+\alpha_3\eta_5+\alpha_0\eta_6-\alpha_1\eta_7)e_6\\&\left.+(-\alpha_7\eta_0-\alpha_6\eta_1+\alpha_5\eta_2+\alpha_4\eta_3-\alpha_3\eta_4-\alpha_2\eta_5+\alpha_1\eta_6+\alpha_0\eta_7)e_7\right],\end{aligned}$$

where all η_i is as in (27).

(c)If $\overline{\alpha}=\gamma$, then the solutions of (30) and (29), respectively, are

$$\vec{x}=\frac{1}{2\alpha_0|\beta|^2}v^T(\beta)\vec{\rho}$$

and

$$\begin{aligned}x=&\frac{1}{2\alpha_0|\beta|^2}\left[\left(\sum_{k=0}^{7}\beta_k\rho_k\right)e_0\right.\\&+(-\beta_1\rho_0+\beta_0\rho_1-\beta_3\rho_2+\beta_2\rho_3-\beta_5\rho_4+\beta_4\rho_5+\beta_7\rho_6-\beta_6\rho_7)e_1\\&+(-\beta_2\rho_0+\beta_3\rho_1+\beta_0\rho_2-\beta_1\rho_3-\beta_6\rho_4-\beta_7\rho_5+\beta_4\rho_6+\beta_5\rho_7)e_2\\&+(-\beta_3\rho_0-\beta_2\rho_1+\beta_1\rho_2+\beta_0\rho_3-\beta_7\rho_4+\beta_6\rho_5-\beta_5\rho_6+\beta_4\rho_7)e_3\\&+(-\beta_4\rho_0+\beta_5\rho_1+\beta_6\rho_2+\beta_7\rho_3+\beta_0\rho_4-\beta_1\rho_5-\beta_2\rho_6-\beta_3\rho_7)e_4\\&+(-\beta_5\rho_0-\beta_4\rho_1+\beta_7\rho_2-\beta_6\rho_3+\beta_1\rho_4+\beta_0\rho_5+\beta_3\rho_6-\beta_2\rho_7)e_5\\&+(-\beta_6\rho_0-\beta_7\rho_1-\beta_4\rho_2+\beta_5\rho_3+\beta_2\rho_4-\beta_3\rho_5+\beta_0\rho_6+\beta_1\rho_7)e_6\\&\left.+(-\beta_7\rho_0+\beta_6\rho_1-\beta_5\rho_2-\beta_4\rho_3+\beta_3\rho_4+\beta_2\rho_5-\beta_1\rho_6+\beta_0\rho_7)e_7\right].\end{aligned}$$

(d)If $\alpha=\overline{\gamma}$, then the solutions of (30) and (29), respectively, are

$$\vec{x}=\frac{1}{2\gamma_0|\beta|^2}v^T(\beta)\vec{\rho}$$

and

$$\begin{aligned}x=&\frac{1}{2\gamma_0|\beta|^2}\left[\left(\sum_{k=0}^{7}\beta_k\rho_k\right)e_0\right.\\&+(-\beta_1\rho_0+\beta_0\rho_1-\beta_3\rho_2+\beta_2\rho_3-\beta_5\rho_4+\beta_4\rho_5+\beta_7\rho_6-\beta_6\rho_7)e_1\\&+(-\beta_2\rho_0+\beta_3\rho_1+\beta_0\rho_2-\beta_1\rho_3-\beta_6\rho_4-\beta_7\rho_5+\beta_4\rho_6+\beta_5\rho_7)e_2\\&+(-\beta_3\rho_0-\beta_2\rho_1+\beta_1\rho_2+\beta_0\rho_3-\beta_7\rho_4+\beta_6\rho_5-\beta_5\rho_6+\beta_4\rho_7)e_3\\&+(-\beta_4\rho_0+\beta_5\rho_1+\beta_6\rho_2+\beta_7\rho_3+\beta_0\rho_4-\beta_1\rho_5-\beta_2\rho_6-\beta_3\rho_7)e_4\\&+(-\beta_5\rho_0-\beta_4\rho_1+\beta_7\rho_2-\beta_6\rho_3+\beta_1\rho_4+\beta_0\rho_5+\beta_3\rho_6-\beta_2\rho_7)e_5\\&+(-\beta_6\rho_0-\beta_7\rho_1-\beta_4\rho_2+\beta_5\rho_3+\beta_2\rho_4-\beta_3\rho_5+\beta_0\rho_6+\beta_1\rho_7)e_6\\&\left.+(-\beta_7\rho_0+\beta_6\rho_1-\beta_5\rho_2-\beta_4\rho_3+\beta_3\rho_4+\beta_2\rho_5-\beta_1\rho_6+\beta_0\rho_7)e_7\right].\end{aligned}$$

2.Let be $\alpha=\gamma$ in (29) and (30).

(a)If $\beta\neq-\delta$, then the solutions of (30) and (29), respectively, are

$$\vec{x}=\frac{1}{|\alpha|^2|\beta+\delta|^2}w^T(\alpha)v^T(\beta+\delta)\vec{\rho}$$

and

$$\begin{aligned}x=&\frac{1}{|\alpha|^2|\lambda|^2}\left[\left(\sum_{k=0}^{7}\alpha_k\varphi_k\right)e_0\right.\\&+(-\alpha_1\varphi_0+\alpha_0\varphi_1+\alpha_3\varphi_2-\alpha_2\varphi_3+\alpha_5\varphi_4-\alpha_4\varphi_5-\alpha_7\varphi_6+\alpha_6\varphi_7)e_1\\&+(-\alpha_2\varphi_0-\alpha_3\varphi_1+\alpha_0\varphi_2+\alpha_1\varphi_3+\alpha_6\varphi_4+\alpha_7\varphi_5-\alpha_4\varphi_6-\alpha_5\varphi_7)e_2\\&+(-\alpha_3\varphi_0+\alpha_2\varphi_1-\alpha_1\varphi_2+\alpha_0\varphi_3+\alpha_7\varphi_4-\alpha_6\varphi_5+\alpha_5\varphi_6-\alpha_4\varphi_7)e_3\\&+(-\alpha_4\varphi_0-\alpha_5\varphi_1-\alpha_6\varphi_2-\alpha_7\varphi_3+\alpha_0\varphi_4+\alpha_1\varphi_5+\alpha_2\varphi_6+\alpha_3\varphi_7)e_4\\&+(-\alpha_5\varphi_0+\alpha_4\varphi_1-\alpha_7\varphi_2+\alpha_6\varphi_3-\alpha_1\varphi_4+\alpha_0\varphi_5-\alpha_3\varphi_6+\alpha_2\varphi_7)e_5\\&+(-\alpha_6\varphi_0+\alpha_7\varphi_1+\alpha_4\varphi_2-\alpha_5\varphi_3-\alpha_2\varphi_4+\alpha_3\varphi_5+\alpha_0\varphi_6-\alpha_1\varphi_7)e_6\\&\left.+(-\alpha_7\varphi_0-\alpha_6\varphi_1+\alpha_5\varphi_2+\alpha_4\varphi_3-\alpha_3\varphi_4-\alpha_2\varphi_5+\alpha_1\varphi_6+\alpha_0\varphi_7)e_7\right],\end{aligned}$$

where $\lambda=\beta+\delta\in O-\{0\}$, $\lambda=\sum_{i=0}^{7}\lambda_i e_i$, and

$$\begin{aligned}\varphi_0=&\ \lambda_0\rho_0+\lambda_1\rho_1+\lambda_2\rho_2+\lambda_3\rho_3+\lambda_4\rho_4+\lambda_5\rho_5+\lambda_6\rho_6+\lambda_7\rho_7,\\\varphi_1=&-\lambda_1\rho_0+\lambda_0\rho_1-\lambda_3\rho_2+\lambda_2\rho_3-\lambda_5\rho_4+\lambda_4\rho_5+\lambda_7\rho_6-\lambda_6\rho_7,\\\varphi_2=&-\lambda_2\rho_0+\lambda_3\rho_1+\lambda_0\rho_2-\lambda_1\rho_3-\lambda_6\rho_4-\lambda_7\rho_5+\lambda_4\rho_6+\lambda_5\rho_7,\\\varphi_3=&-\lambda_3\rho_0-\lambda_2\rho_1+\lambda_1\rho_2+\lambda_0\rho_3-\lambda_7\rho_4+\lambda_6\rho_5-\lambda_5\rho_6+\lambda_4\rho_7,\\\varphi_4=&-\lambda_4\rho_0+\lambda_5\rho_1+\lambda_6\rho_2+\lambda_7\rho_3+\lambda_0\rho_4-\lambda_1\rho_5-\lambda_2\rho_6-\lambda_3\rho_7,\\\varphi_5=&-\lambda_5\rho_0-\lambda_4\rho_1+\lambda_7\rho_2-\lambda_6\rho_3+\lambda_1\rho_4+\lambda_0\rho_5+\lambda_3\rho_6-\lambda_2\rho_7,\\\varphi_6=&-\lambda_6\rho_0-\lambda_7\rho_1-\nu_4\rho_2+\nu_5\rho_3+\lambda_2\rho_4-\lambda_3\rho_5+\lambda_0\rho_6+\lambda_1\rho_7,\\\varphi_7=&-\lambda_7\rho_0+\lambda_6\rho_1-\lambda_5\rho-\lambda_4\rho_3+\lambda_3\rho_4+\lambda_2\rho_5-\lambda_1\rho_6+\lambda_0\rho_7.\end{aligned}$$

(b)If $\overline{\beta}=\delta$, then the solutions of (30) and (29), respectively, are

$$\vec{x}=\frac{1}{2\beta_0|\alpha|^2}w^T(\alpha)\vec{\rho}$$

and

$$\begin{aligned}x=&\frac{1}{2\beta_0|\alpha|^2}\left[\left(\sum_{k=0}^{7}\alpha_k\rho_k\right)e_0\right.\\&+(-\alpha_1\rho_0+\alpha_0\rho_1+\alpha_3\rho_2-\alpha_2\rho_3+\alpha_5\rho_4-\alpha_4\rho_5-\alpha_7\rho_6+\alpha_6\rho_7)e_1\\&+(-\alpha_2\rho_0-\alpha_3\rho_1+\alpha_0\rho_2+\alpha_1\rho_3+\alpha_6\rho_4+\alpha_7\rho_5-\alpha_4\rho_6-\alpha_5\rho_7)e_2\\&+(-\alpha_3\rho_0+\alpha_2\rho_1-\alpha_1\rho_2+\alpha_0\rho_3+\alpha_7\rho_4-\alpha_6\rho_5+\alpha_5\rho_6-\alpha_4\rho_7)e_3\\&+(-\alpha_4\rho_0-\alpha_5\rho_1-\alpha_6\rho_2-\alpha_7\rho_3+\alpha_0\rho_4+\alpha_1\rho_5+\alpha_2\rho_6+\alpha_3\rho_7)e_4\\&+(-\alpha_5\rho_0+\alpha_4\rho_1-\alpha_7\rho_2+\alpha_6\rho_3-\alpha_1\rho_4+\alpha_0\rho_5-\alpha_3\rho_6+\alpha_2\rho_7)e_5\\&+(-\alpha_6\rho_0+\alpha_7\rho_1+\alpha_4\rho_2-\alpha_5\rho_3-\alpha_2\rho_4+\alpha_3\rho_5+\alpha_0\rho_6-\alpha_1\rho_7)e_6\\&\left.+(-\alpha_7\rho_0-\alpha_6\rho_1+\alpha_5\rho_2+\alpha_4\rho_3-\alpha_3\rho_4-\alpha_2\rho_5+\alpha_1\rho_6+\alpha_0\rho_7)e_7\right].\end{aligned}$$

(c)If $\beta=\overline{\delta}$, then the solutions of (30) and (29), respectively, are

$$\vec{x}=\frac{1}{2\delta_0|\alpha|^2}w^T(\alpha)\vec{\rho}$$

and

$$x=\frac{1}{2\delta_0|\alpha|^2}\left[\left(\sum_{k=0}^{7}\alpha_k\rho_k\right)e_0\right.$$
$$+(-\alpha_1\rho_0+\alpha_0\rho_1+\alpha_3\rho_2-\alpha_2\rho_3+\alpha_5\rho_4-\alpha_4\rho_5-\alpha_7\rho_6+\alpha_6\rho_7)e_1$$
$$+(-\alpha_2\rho_0-\alpha_3\rho_1+\alpha_0\rho_2+\alpha_1\rho_3+\alpha_6\rho_4+\alpha_7\rho_5-\alpha_4\rho_6-\alpha_5\rho_7)e_2$$
$$+(-\alpha_3\rho_0+\alpha_2\rho_1-\alpha_1\rho_2+\alpha_0\rho_3+\alpha_7\rho_4-\alpha_6\rho_5+\alpha_5\rho_6-\alpha_4\rho_7)e_3$$
$$+(-\alpha_4\rho_0-\alpha_5\rho_1-\alpha_6\rho_2-\alpha_7\rho_3+\alpha_0\rho_4+\alpha_1\rho_5+\alpha_2\rho_6+\alpha_3\rho_7)e_4$$
$$+(-\alpha_5\rho_0+\alpha_4\rho_1-\alpha_7\rho_2+\alpha_6\rho_3-\alpha_1\rho_4+\alpha_0\rho_5-\alpha_3\rho_6+\alpha_2\rho_7)e_5$$
$$+(-\alpha_6\rho_0+\alpha_7\rho_1+\alpha_4\rho_2-\alpha_5\rho_3-\alpha_2\rho_4+\alpha_3\rho_5+\alpha_0\rho_6-\alpha_1\rho_7)e_6$$
$$\left.+(-\alpha_7\rho_0-\alpha_6\rho_1+\alpha_5\rho_2+\alpha_4\rho_3-\alpha_3\rho_4-\alpha_2\rho_5+\alpha_1\rho_6+\alpha_0\rho_7)e_7\right].$$

*Proof.*The equation (29) is equivalent to (30). According to some cases we now will find out $\overrightarrow{x}$ and x solutions of (30) and (29), respectively.

Case 1.Let $\beta=\delta$. Then from (30) we have

$$v(\beta)[w(\alpha)+w(\gamma)]\overrightarrow{x}=\overrightarrow{\rho}$$
$$v(\beta)[w(\alpha+\gamma)]\overrightarrow{x}=\overrightarrow{\rho},$$

and since $\beta\neq0$, $v(\beta)$ is invertible, and thus we write

$$[w(\alpha+\gamma)]\overrightarrow{x}=v^{-1}(\beta)\overrightarrow{\rho}. \quad (33)$$

Case (1-i).Let $\alpha\neq-\gamma$ in (33). Since $\alpha+\gamma\neq0$, $w(\alpha+\gamma)$ is invertible, and thus we obtain $\overrightarrow{x}$ such that

$$\overrightarrow{x}=w^{-1}(\alpha+\gamma)v^{-1}(\beta)\overrightarrow{\rho}$$
$$=\frac{1}{|\beta|^2|\alpha+\gamma|^2}w^T(\alpha+\gamma)v^T(\beta)\overrightarrow{\rho}. \quad (34)$$

Thus from (34) , the Eq.(33) is obtained.

Case (1-ii).Let $\alpha=\gamma$ in (33). Therefore, we have

$$[w(\alpha+\alpha)]\overrightarrow{x}=v^{-1}(\beta)\overrightarrow{\rho}$$
$$2w(\alpha)\overrightarrow{x}=v^{-1}(\beta)\overrightarrow{\rho},$$

and since $\alpha\neq0$, $w(\alpha)$ is invertible, and thus we obtain $\overrightarrow{x}$ such that

$$\overrightarrow{x}=\frac{1}{2}w^{-1}(\alpha)v^{-1}(\beta)\overrightarrow{\rho}$$
$$=\frac{1}{2|\beta|^2|\alpha|^2}w^T(\alpha)v^T(\beta)\overrightarrow{\rho}. \quad (35)$$

Thus from (35) , the Eq.(33) is obtained.

Case (1-iii).Let $\overline{\alpha}=\gamma$ in (33). Therefore, we have

$$w(\alpha+\overline{\alpha})\overrightarrow{x}=v^{-1}(\beta)\overrightarrow{\rho}$$
$$w(2\alpha_0)\overrightarrow{x}=v^{-1}(\beta)\overrightarrow{\rho}$$
$$2\alpha_0w(1)\overrightarrow{x}=v^{-1}(\beta)\overrightarrow{\rho},$$

and since $w(1)=I_8$, we obtain $\overrightarrow{x}$ such that

$$\overrightarrow{x}=\frac{1}{2\alpha_0|\beta|^2}v^T(\beta)\overrightarrow{\rho}. \quad (36)$$

Thus from (36) , the Eq.(33) is obtained.

Case (1-iv).Let $\alpha=\overline{\gamma}$ in (33). Therefore, we have

$$[w(\overline{\gamma}+\gamma)]\overrightarrow{x}=v^{-1}(\beta)\overrightarrow{\rho}$$
$$w(2\gamma_0)\overrightarrow{x}=v^{-1}(\beta)\overrightarrow{\rho}$$
$$2\gamma_0w(1)\overrightarrow{x}=v^{-1}(\beta)\overrightarrow{\rho},$$

and since $w(1)=I_8$, we obtain $\overrightarrow{x}$ such that

$$\overrightarrow{x}=\frac{1}{2\gamma_0|\beta|^2}v^T(\beta)\overrightarrow{\rho}. \quad (37)$$

Thus from (37) , the Eq.(33) is obtained.

Case 2.Let $\alpha=\gamma$. Then from (30) we have

$$[v(\beta)w(\alpha)+v(\delta)w(\alpha)]\overrightarrow{x}=\overrightarrow{\rho}$$
$$[v(\beta)+v(\delta)]w(\alpha)\overrightarrow{x}=\overrightarrow{\rho}$$
$$v(\beta+\delta)w(\alpha)\overrightarrow{x}=\overrightarrow{\rho}. \quad (38)$$

Case (2-i).Let $\beta\neq-\delta$ in (38). Since $\beta+\delta\neq0$ and $\alpha\neq0$, $v(\beta+\delta)$ and $w(\alpha)$ are invertible, and thus we obtain $\overrightarrow{x}$ such that

$$\overrightarrow{x}=w^{-1}(\alpha)v^{-1}(\beta+\delta)\overrightarrow{\rho}$$
$$=\frac{1}{|\alpha|^2|\beta+\delta|^2}w^T(\alpha)v^T(\beta+\delta)\overrightarrow{\rho}. \quad (39)$$

Thus from (39) , the Eq.(33) is obtained.

Case (2-ii).Let $\overline{\beta}=\delta$ in (38). Therefore, we have

$$v(\beta+\overline{\beta})w(\alpha)\overrightarrow{x}=\overrightarrow{\rho}$$
$$v(2\beta_0)w(\alpha)\overrightarrow{x}=\overrightarrow{\rho}$$
$$2\beta_0v(1)w(\alpha)\overrightarrow{x}=\overrightarrow{\rho},$$

and since $v(1)=I_8$ and $w(\alpha)$ is invertible, we obtain $\overrightarrow{x}$ such that

$$\overrightarrow{x}=\frac{1}{2\beta_0}w^{-1}(\alpha)\overrightarrow{\rho}$$
$$=\frac{1}{2\beta_0|\alpha|^2}w^T(\alpha)\overrightarrow{\rho}. \quad (40)$$

Thus from (40) , the Eq.(33) is obtained.

Case (2-iii).Let $\beta=\overline{\delta}$ in (38). Therefore, we have

$$v\left(\overline{\delta}+\delta\right)w(\alpha)\overrightarrow{x}=\overrightarrow{\rho}$$
$$v(2\delta_0)w(\alpha)\overrightarrow{x}=\overrightarrow{\rho}$$
$$2\delta_0v(1)w(\alpha)\overrightarrow{x}=\overrightarrow{\rho},$$

and since $v(1)=I_8$ and $w(\alpha)$ is invertible, we obtain $\overrightarrow{x}$ such that

$$\overrightarrow{x}=\frac{1}{2\delta_0}w^{-1}(\alpha)\overrightarrow{\rho}$$
$$\overrightarrow{x}=\frac{1}{2\delta_0|\alpha|^2}w^T(\alpha)\overrightarrow{\rho}. \quad (41)$$

Thus from (41) , the Eq.(33) is obtained.

Proposition 5.*Let $\alpha=\sum_{i=0}^{7}\alpha_ie_i$, $\beta=\sum_{i=0}^{7}\beta_ie_i\in\mathbf{O}-\{0\}$ and $\rho=\sum_{i=0}^{7}\rho_ie_i\in\mathbf{O}$ be given.*

1.Let be $\beta=\delta$ in (32) and (31).

(a)If $\alpha\neq-\gamma$, then the solutions of (32) and (31), respectively, are

$$\overrightarrow{x}=\frac{1}{|\beta|^2|\alpha+\gamma|^2}v^T(\beta)w^T(\alpha+\gamma)\overrightarrow{\rho}$$

and

$$x=\frac{1}{2|\beta|^2|\alpha|^2}\left[\left(\sum_{k=0}^{7}\beta_k\zeta_k\right)e_0\right.$$
$$+(-\beta_1\zeta_0+\beta_0\zeta_1-\beta_3\zeta_2+\beta_2\zeta_3-\beta_5\zeta_4+\beta_4\zeta_5+\beta_7\zeta_6-\beta_6\zeta_7)e_1$$
$$+(-\beta_2\zeta_0+\beta_3\zeta_1+\beta_0\zeta_2-\beta_1\zeta_3-\beta_6\zeta_4-\beta_7\zeta_5+\beta_4\zeta_6+\beta_5\zeta_7)e_2$$
$$+(-\beta_3\zeta_0-\beta_2\zeta_1+\beta_1\zeta_2+\beta_0\zeta_3-\beta_7\zeta_4+\beta_6\zeta_5-\beta_5\zeta_6+\beta_4\zeta_7)e_3$$
$$+(-\beta_4\zeta_0+\beta_5\zeta_1+\beta_6\zeta_2+\beta_7\zeta_3+\beta_0\zeta_4-\beta_1\zeta_5-\beta_2\zeta_6-\beta_3\zeta_7)e_4$$
$$+(-\beta_5\zeta_0-\beta_4\zeta_1+\beta_7\zeta_2-\beta_6\zeta_3+\beta_1\zeta_4+\beta_0\zeta_5+\beta_3\zeta_6-\beta_2\zeta_7)e_5$$
$$+(-\beta_6\zeta_0-\beta_7\zeta_1-\beta_4\zeta_2+\beta_5\zeta_3+\beta_2\zeta_4-\beta_3\zeta_5+\beta_0\zeta_6+\beta_1\zeta_7)e_6$$
$$\left.+(-\beta_7\zeta_0+\beta_6\zeta_1-\beta_5\zeta_2-\beta_4\zeta_3+\beta_3\zeta_4+\beta_2\zeta_5-\beta_1\zeta_6+\beta_0\zeta_7)e_7\right],$$

where $\alpha+\gamma=\mu\in\mathbf{O}-\{0\}$, $\mu=\sum_{i=0}^{7}\mu_ie_i$, and

$$\begin{aligned}
\zeta_0 &= \mu_0\rho_0+\mu_1\rho_1+\mu_2\rho_2+\mu_3\rho_3+\mu_4\rho_4+\mu_5\rho_5+\mu_6\rho_6+\mu_7\rho_7,\\
\zeta_1 &= -\mu_1\rho_0+\mu_0\rho_1+\mu_3\rho_2-\mu_2\rho_3+\mu_5\rho_4-\mu_4\rho_5-\mu_7\rho_6+\mu_6\rho_7,\\
\zeta_2 &= -\mu_2\rho_0-\mu_3\rho_1+\mu_0\rho_2+\mu_1\rho_3+\mu_6\rho_4+\mu_7\rho_5-\mu_4\rho_6-\mu_5\rho_7,\\
\zeta_3 &= -\mu_3\rho_0+\mu_2\rho_1-\mu_1\rho_2+\mu_0\rho_3+\mu_7\rho_4-\mu_6\rho_5+\mu_5\rho_6-\mu_4\rho_7,\\
\zeta_4 &= -\mu_4\rho_0-\mu_5\rho_1-\mu_6\rho_2-\mu_7\rho_3+\mu_0\rho_4+\mu_1\rho_5+\mu_2\rho_6+\mu_3\rho_7,\\
\zeta_5 &= -\mu_5\rho_0+\mu_4\rho_1-\mu_7\rho_2+\mu_6\rho_3-\mu_1\rho_4+\mu_0\rho_5-\mu_3\rho_6+\mu_2\rho_7,\\
\zeta_6 &= -\mu_6\rho_0+\mu_7\rho_1+\mu_4\rho_2-\mu_5\rho_3-\mu_2\rho_4+\mu_3\rho_5+\mu_0\rho_6-\mu_1\rho_7,\\
\zeta_7 &= -\mu_7\rho_0-\mu_6\rho_1+\mu_5\rho_2+\mu_4\rho_3-\mu_3\rho_4-\mu_2\rho_5+\mu_1\rho_6+\mu_0\rho_7.
\end{aligned}$$

(b)If $\alpha=\gamma$, then the solutions of (32) and (31), respectively, are

$$\overrightarrow{x}=\frac{1}{2|\beta|^2|\alpha|^2}v^T(\beta)\,w^T(\alpha)\overrightarrow{\rho}$$

and

$$x=\frac{1}{2|\beta|^2|\alpha|^2}\left[\left(\sum_{k=0}^{7}\beta_k\varpi_k\right)e_0\right.$$
$$+(-\beta_1\varpi_0+\beta_0\varpi_1-\beta_3\varpi_2+\beta_2\varpi_3-\beta_5\varpi_4+\beta_4\varpi_5+\beta_7\varpi_6-\beta_6\varpi_7)e_1$$
$$+(-\beta_2\varpi_0+\beta_3\varpi_1+\beta_0\varpi_2-\beta_1\varpi_3-\beta_6\varpi_4-\beta_7\varpi_5+\beta_4\varpi_6+\beta_5\varpi_7)e_2$$
$$+(-\beta_3\varpi_0-\beta_2\varpi_1+\beta_1\varpi_2+\beta_0\varpi_3-\beta_7\varpi_4+\beta_6\varpi_5-\beta_5\varpi_6+\beta_4\varpi_7)e_3$$
$$+(-\beta_4\varpi_0+\beta_5\varpi_1+\beta_6\varpi_2+\beta_7\varpi_3+\beta_0\varpi_4-\beta_1\varpi_5-\beta_2\varpi_6-\beta_3\varpi_7)e_4$$
$$+(-\beta_5\varpi_0-\beta_4\varpi_1+\beta_7\varpi_2-\beta_6\varpi_3+\beta_1\varpi_4+\beta_0\varpi_5+\beta_3\varpi_6-\beta_2\varpi_7)e_5$$
$$+(-\beta_6\varpi_0-\beta_7\varpi_1-\beta_4\varpi_2+\beta_5\varpi_3+\beta_2\varpi_4-\beta_3\varpi_5+\beta_0\varpi_6+\beta_1\varpi_7)e_6$$
$$\left.+(-\beta_7\varpi_0+\beta_6\varpi_1-\beta_5\varpi_2-\beta_4\varpi_3+\beta\varpi_4+\beta_2\varpi_5-\beta_1\varpi_6+\beta_0\varpi_7)e_7\right],$$

where all ϖ_i is as in (23).

(c)If $\overline{\alpha}=\gamma$, then the solutions of (32) and (31), respectively, are

$$\overrightarrow{x}=\frac{1}{2\alpha_0|\beta|^2}v^T(\beta)\overrightarrow{\rho}$$

and

$$x=\frac{1}{2\alpha_0|\beta|^2}\left[\left(\sum_{k=0}^{7}\beta_k\rho_k\right)e_0\right.$$
$$+(-\beta_1\rho_0+\beta_0\rho_1-\beta_3\rho_2+\beta_2\rho_3-\beta_5\rho_4+\beta_4\rho_5+\beta_7\rho_6-\beta_6\rho_7)e_1$$
$$+(-\beta_2\rho_0+\beta_3\rho_1+\beta_0\rho_2-\beta_1\rho_3-\beta_6\rho_4-\beta_7\rho_5+\beta_4\rho_6+\beta_5\rho_7)e_2$$
$$+(-\beta_3\rho_0-\beta_2\rho_1+\beta_1\rho_2+\beta_0\rho_3-\beta_7\rho_4+\beta_6\rho_5-\beta_5\rho_6+\beta_4\rho_7)e_3$$
$$+(-\beta_4\rho_0+\beta_5\rho_1+\beta_6\rho_2+\beta_7\rho_3+\beta_0\rho_4-\beta_1\rho_5-\beta_2\rho_6-\beta_3\rho_7)e_4$$
$$+(-\beta_5\rho_0-\beta_4\rho_1+\beta_7\rho_2-\beta_6\rho_3+\beta_1\rho_4+\beta_0\rho_5+\beta_3\rho_6-\beta_2\rho_7)e_5$$
$$+(-\beta_6\rho_0-\beta_7\rho_1-\beta_4\rho_2+\beta_5\rho_3+\beta_2\rho_4-\beta_3\rho_5+\beta_0\rho_6+\beta_1\rho_7)e_6$$
$$\left.+(-\beta_7\rho_0+\beta_6\rho_1-\beta_5\rho_2-\beta_4\rho_3+\beta_3\rho_4+\beta_2\rho_5-\beta_1\rho_6+\beta_0\rho_7)e_7\right].$$

(d)If $\alpha=\overline{\gamma}$, then the solutions of (32) and (31), respectively, are

$$\overrightarrow{x}=\frac{1}{2\gamma_0|\beta|^2}v^T(\beta)\overrightarrow{\rho}$$

and

$$x=\frac{1}{2\gamma_0|\beta|^2}\left[\left(\sum_{k=0}^{7}\beta_k\rho_k\right)e_0\right.$$
$$+(-\beta_1\rho_0+\beta_0\rho_1-\beta_3\rho_2+\beta_2\rho_3-\beta_5\rho_4+\beta_4\rho_5+\beta_7\rho_6-\beta_6\rho_7)e_1$$
$$+(-\beta_2\rho_0+\beta_3\rho_1+\beta_0\rho_2-\beta_1\rho_3-\beta_6\rho_4-\beta_7\rho_5+\beta_4\rho_6+\beta_5\rho_7)e_2$$
$$+(-\beta_3\rho_0-\beta_2\rho_1+\beta_1\rho_2+\beta_0\rho_3-\beta_7\rho_4+\beta_6\rho_5-\beta_5\rho_6+\beta_4\rho_7)e_3$$
$$+(-\beta_4\rho_0+\beta_5\rho_1+\beta_6\rho_2+\beta_7\rho_3+\beta_0\rho_4-\beta_1\rho_5-\beta_2\rho_6-\beta_3\rho_7)e_4$$
$$+(-\beta_5\rho_0-\beta_4\rho_1+\beta_7\rho_2-\beta_6\rho_3+\beta_1\rho_4+\beta_0\rho_5+\beta_3\rho_6-\beta_2\rho_7)e_5$$
$$+(-\beta_6\rho_0-\beta_7\rho_1-\beta_4\rho_2+\beta_5\rho_3+\beta_2\rho_4-\beta_3\rho_5+\beta_0\rho_6+\beta_1\rho_7)e_6$$
$$\left.+(-\beta_7\rho_0+\beta_6\rho_1-\beta_5\rho_2-\beta_4\rho_3+\beta_3\rho_4+\beta_2\rho_5-\beta_1\rho_6+\beta_0\rho_7)e_7\right].$$

2.Let be $\alpha=\gamma$ in (32) and (31).

(a)If $\beta\neq-\delta$, then the solutions of (32) and (31), respectively, are

$$\overrightarrow{x}=\frac{1}{|\alpha|^2|\beta+\delta|^2}v^T(\beta+\delta)\,w^T(\alpha)\overrightarrow{\rho}$$

and

$$x=\frac{1}{|\alpha|^2|\lambda|^2}\left[\left(\sum_{k=0}^{7}\lambda_k\varpi_k\right)e_0\right.$$
$$+(-\lambda_1\varpi_0+\lambda_0\varpi_1-\lambda_3\varpi_2+\lambda_2\varpi_3-\lambda_5\varpi_4+\lambda_4\varpi_5+\lambda_7\varpi_6-\lambda_6\varpi_7)e_1$$
$$+(-\lambda_2\varpi_0+\lambda_3\varpi_1+\lambda_0\varpi_2-\lambda_1\varpi_3-\lambda_6\varpi_4-\lambda_7\varpi_5+\lambda_4\varpi_6+\lambda_5\varpi_7)e_2$$
$$+(-\lambda_3\varpi_0-\lambda_2\varpi_1+\lambda_1\varpi_2+\lambda_0\varpi_3-\lambda_7\varpi_4+\lambda_6\varpi_5-\lambda_5\varpi_6+\lambda_4\varpi_7)e_3$$
$$+(-\lambda_4\varpi_0+\lambda_5\varpi_1+\lambda_6\varpi_2+\lambda_7\varpi_3+\lambda_0\varpi_4-\lambda_1\varpi_5-\lambda_2\varpi_6-\lambda_3\varpi_7)e_4$$
$$+(-\lambda_5\varpi_0-\lambda_4\varpi_1+\lambda_7\varpi_2-\lambda_6\varpi_3+\lambda_1\varpi_4+\lambda_0\varpi_5+\lambda_3\varpi_6-\lambda_2\varpi_7)e_5$$
$$+(-\lambda_6\varpi_0-\lambda_7\varpi_1-\lambda_4\varpi_2+\lambda_5\varpi_3+\lambda_2\varpi_4-\lambda_3\varpi_5+\lambda_0\varpi_6+\lambda_1\varpi_7)e_6$$
$$\left.+(-\lambda_7\varpi_0+\lambda_6\varpi_1-\lambda_5\varpi_2-\lambda_4\varpi_3+\lambda_3\varpi_4+\lambda_2\varpi_5-\lambda_1\varpi_6+\lambda_0\varpi_7)e_7\right],$$

where $\lambda=\beta+\delta\in\mathbf{O}-\{0\}$, $\lambda=\sum_{i=0}^{7}\lambda_ie_i$ and all ϖ_i is as in (23).

(b)If $\overline{\beta}=\delta$, then the solutions of (32) and (31), respectively, are

$$\overrightarrow{x}=\frac{1}{2\beta_0|\alpha|^2}w^T(\alpha)\overrightarrow{\rho}$$

and

$$x=\frac{1}{2\beta_0|\alpha|^2}\left[\left(\sum_{k=0}^{7}\alpha_k\rho_k\right)e_0\right.$$
$$+(-\alpha_1\rho_0+\alpha_0\rho_1+\alpha_3\rho_2-\alpha_2\rho_3+\alpha_5\rho_4-\alpha_4\rho_5-\alpha_7\rho_6+\alpha_6\rho_7)e_1$$
$$+(-\alpha_2\rho_0-\alpha_3\rho_1+\alpha_0\rho_2+\alpha_1\rho_3+\alpha_6\rho_4+\alpha_7\rho_5-\alpha_4\rho_6-\alpha_5\rho_7)e_2$$
$$+(-\alpha_3\rho_0+\alpha_2\rho_1-\alpha_1\rho_2+\alpha_0\rho_3+\alpha_7\rho_4-\alpha_6\rho_5+\alpha_5\rho_6-\alpha_4\rho_7)e_3$$
$$+(-\alpha_4\rho_0-\alpha_5\rho_1-\alpha_6\rho_2-\alpha_7\rho_3+\alpha_0\rho_4+\alpha_1\rho_5+\alpha_2\rho_6+\alpha_3\rho_7)e_4$$
$$+(-\alpha_5\rho_0+\alpha_4\rho_1-\alpha_7\rho_2+\alpha_6\rho_3-\alpha_1\rho_4+\alpha_0\rho_5-\alpha_3\rho_6+\alpha_2\rho_7)e_5$$
$$+(-\alpha_6\rho_0+\alpha_7\rho_1+\alpha_4\rho_2-\alpha_5\rho_3-\alpha_2\rho_4+\alpha_3\rho_5+\alpha_0\rho_6-\alpha_1\rho_7)e_6$$
$$\left.+(-\alpha_7\rho_0-\alpha_6\rho_1+\alpha_5\rho_2+\alpha_4\rho_3-\alpha_3\rho_4-\alpha_2\rho_5+\alpha_1\rho_6+\alpha_0\rho_7)e_7\right].$$

(c)If $\beta=\overline{\delta}$, then the solutions of (32) and (31), respectively, are

$$\overrightarrow{x}=\frac{1}{2\delta_0|\alpha|^2}w^T(\alpha)\overrightarrow{\rho}$$

and

$$x=\frac{1}{2\delta_0|\alpha|^2}\left[\left(\sum_{k=0}^{7}\alpha_k\rho_k\right)e_0\right.$$
$$+(-\alpha_1\rho_0+\alpha_0\rho_1+\alpha_3\rho_2-\alpha_2\rho_3+\alpha_5\rho_4-\alpha_4\rho_5-\alpha_7\rho_6+\alpha_6\rho_7)e_1$$
$$+(-\alpha_2\rho_0-\alpha_3\rho_1+\alpha_0\rho_2+\alpha_1\rho_3+\alpha_6\rho_4+\alpha_7\rho_5-\alpha_4\rho_6-\alpha_5\rho_7)e_2$$
$$+(-\alpha_3\rho_0+\alpha_2\rho_1-\alpha_1\rho_2+\alpha_0\rho_3+\alpha_7\rho_4-\alpha_6\rho_5+\alpha_5\rho_6-\alpha_4\rho_7)e_3$$
$$+(-\alpha_4\rho_0-\alpha_5\rho_1-\alpha_6\rho_2-\alpha_7\rho_3+\alpha_0\rho_4+\alpha_1\rho_5+\alpha_2\rho_6+\alpha_3\rho_7)e_4$$
$$+(-\alpha_5\rho_0+\alpha_4\rho_1-\alpha_7\rho_2+\alpha_6\rho_3-\alpha_1\rho_4+\alpha_0\rho_5-\alpha_3\rho_6+\alpha_2\rho_7)e_5$$
$$+(-\alpha_6\rho_0+\alpha_7\rho_1+\alpha_4\rho_2-\alpha_5\rho_3-\alpha_2\rho_4+\alpha_3\rho_5+\alpha_0\rho_6-\alpha_1\rho_7)e_6$$
$$\left.+(-\alpha_7\rho_0-\alpha_6\rho_1+\alpha_5\rho_2+\alpha_4\rho_3-\alpha_3\rho_4-\alpha_2\rho_5+\alpha_1\rho_6+\alpha_0\rho_7)e_7\right].$$

*Proof.*The equation (31) is equivalent to (32). According to some cases we now will find out $\overrightarrow{x}$ and x solutions of (32) and (31), respectively.

Case 1.Let $\beta=\delta$. Then from (32) we have

$$\begin{aligned}[w(\alpha)+w(\gamma)]\,v(\beta)\overrightarrow{x} &= \overrightarrow{\rho}\\ w(\alpha+\gamma)\,v(\beta)\overrightarrow{x} &= \overrightarrow{\rho}.\end{aligned} \tag{42}$$

Case (1-i).Let $\alpha\neq-\gamma$ in (42). Since $\alpha+\gamma\neq0$ and $\beta\neq0$, $w(\alpha+\gamma)$ and $v(\beta)$ are invertible, and thus we have $\overrightarrow{x}$ such that

$$w(\alpha+\gamma)v(\beta)\overrightarrow{x} = \overrightarrow{\rho}$$

$$\overrightarrow{x} = \frac{1}{|\beta|^2|\alpha+\gamma|^2}v^T(\beta)w^T(\alpha+\gamma)\overrightarrow{\rho}. \tag{43}$$

Thus from (43), the Eq.(42) is obtained.

Case (1-ii).Let $\alpha=\gamma$ in (42). Therefore, we have

$$\begin{aligned}w(\alpha+\alpha)v(\beta)\overrightarrow{x} &= \overrightarrow{\rho}\\ 2w(\alpha)v(\beta)\overrightarrow{x} &= \overrightarrow{\rho},\end{aligned}$$

and since $\alpha,\beta\neq0$, $w(\alpha)$ and $v(\beta)$ are invertible, and thus we obtain $\overrightarrow{x}$ such that

$$\begin{aligned}\overrightarrow{x} &= \frac{1}{2}v^{-1}(\beta)w^{-1}(\alpha)\overrightarrow{\rho}\\ &= \frac{1}{2|\beta|^2|\alpha|^2}v^T(\beta)w^T(\alpha)\overrightarrow{\rho}.\end{aligned} \tag{44}$$

Thus from (44), the Eq.(42) is obtained.

Case (1-iii).Let $\overline{\alpha}=\gamma$ in (42). Therefore, we have

$$\begin{aligned}w(\alpha+\overline{\alpha})v(\beta)\overrightarrow{x} &= \overrightarrow{\rho}\\ w(2\alpha_0)v(\beta)\overrightarrow{x} &= \overrightarrow{\rho}\\ 2\alpha_0 w(1)v(\beta)\overrightarrow{x} &= \overrightarrow{\rho},\end{aligned}$$

and since $w(1)=I_8$ and $\beta\neq0$, $v(\beta)$ is invertible, and thus we obtain $\overrightarrow{x}$ such that

$$\begin{aligned}\overrightarrow{x} &= \frac{1}{2\alpha_0}v^{-1}(\beta)\overrightarrow{\rho}\\ &= \frac{1}{2\alpha_0|\beta|^2}v^T(\beta)\overrightarrow{\rho}.\end{aligned} \tag{45}$$

Thus from (45), the Eq.(42) is obtained.

Case (1-iv).Let $\alpha=\overline{\gamma}$ in (42). Therefore, we have

$$\begin{aligned}w(\overline{\gamma}+\gamma)v(\beta)\overrightarrow{x} &= \overrightarrow{\rho}\\ w(2\gamma_0)v(\beta)\overrightarrow{x} &= \overrightarrow{\rho}\\ 2\gamma_0 w(1)v(\beta)\overrightarrow{x} &= \overrightarrow{\rho},\end{aligned}$$

and since $w(1)=I_8$ and $\beta\neq0$, $v(\beta)$ are invertible, and thus we obtain $\overrightarrow{x}$ such that

$$\begin{aligned}\overrightarrow{x} &= \frac{1}{2\gamma_0}v^{-1}(\beta)\overrightarrow{\rho}\\ &\frac{1}{2\gamma_0|\beta|^2}v^T(\beta)\overrightarrow{\rho}.\end{aligned} \tag{46}$$

Thus from (46), the Eq.(42) is obtained.

Case 2.Let $\alpha=\gamma$. Then from the Eq.(32) we have

$$\begin{aligned}w(\alpha)[v(\beta)+v(\delta)]\overrightarrow{x} &= \overrightarrow{\rho}\\ w(\alpha)[v(\beta+\delta)]\overrightarrow{x} &= \overrightarrow{\rho},\end{aligned}$$

and since $\alpha\neq0$, $w(\alpha)$ is invertible, we write

$$v(\beta+\delta)\overrightarrow{x} = w^{-1}(\alpha)\overrightarrow{\rho}. \tag{47}$$

Case (2-i).Let $\beta\neq-\delta$ in (47). Since $\beta+\delta\neq0$, $v(\beta+\delta)$ is invertible, and thus we obtain $\overrightarrow{x}$ such that

$$\begin{aligned}\overrightarrow{x} &= v^{-1}(\beta+\delta)w^{-1}(\alpha)\overrightarrow{\rho}\\ &= \frac{1}{|\alpha|^2|\beta+\delta|^2}v^T(\beta+\delta)w^T(\alpha)\overrightarrow{\rho}.\end{aligned} \tag{48}$$

Thus from (48), the Eq.(42) is obtained.

Case (2-ii).Let $\overline{\beta}=\delta$ in (47). Therefore, we have

$$\begin{aligned}v\left(\beta+\overline{\beta}\right)\overrightarrow{x} &= w^{-1}(\alpha)\overrightarrow{\rho}\\ v(2\beta_0)\overrightarrow{x} &= w^{-1}(\alpha)\overrightarrow{\rho}\\ 2\beta_0 v(1)\overrightarrow{x} &= w^{-1}(\alpha)\overrightarrow{\rho},\end{aligned}$$

and since $v(1)=I_8$, we obtain $\overrightarrow{x}$ such that

$$\overrightarrow{x} = \frac{1}{2\beta_0|\alpha|^2}w^T(\alpha)\overrightarrow{\rho}. \tag{49}$$

Thus from (49), the Eq.(42) is obtained.

Case (2-iii).Let $\beta=\overline{\delta}$ in (47). Therefore, we have

$$\begin{aligned}v\left(\overline{\delta}+\delta\right)\overrightarrow{x} &= w^{-1}(\alpha)\overrightarrow{\rho}\\ v(2\delta_0)\overrightarrow{x} &= w^{-1}(\alpha)\overrightarrow{\rho}\\ 2\delta_0 v(1)\overrightarrow{x} &= w^{-1}(\alpha)\overrightarrow{\rho},\end{aligned}$$

and since $v(1)=I_8$, we obtain $\overrightarrow{x}$ such that

$$\overrightarrow{x} = \frac{1}{2\delta_0|\alpha|^2}w^T(\alpha)\overrightarrow{\rho}. \tag{50}$$

Thus from (50), the Eq.(42) is obtained.

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