

On Modules for Which All Submodules Are Projection Invariant and the Lifting Condition

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Abstract. The notion of projection invariant subgroups was first introduced by Fuchs in [7]. We will define the module-theoretic version of the projection invariant subgroup. Let R be a ring and M a right R -module. We call a submodule N of M the projection invariant if every projection π of M onto a direct summand maps N into itself, i.e. N is invariant under any projection of M . In this note, we give several characterizations to these class of modules that generalize the recent results in [14]. We also define and study the PI-lifting modules which is a generalization of FI-lifting module. It is shown that if each M_i is a PI-lifting module for all $1 \leq i \leq n$, then $M = \bigoplus_{i=1}^n M_i$ is a PI-lifting module. In particular, we focus on rings satisfying the following condition:

(*) Every submodule of M is projection invariant.

We prove that if R has the (*) property, then $R \oplus R$ does not satisfy the (*) property.

Keywords: Fully invariant submodules; Projection invariant submodules; Duo modules and rings; Finite exchange property; Lifting modules.

1. Introduction

Throughout this paper, R is an associative ring with identity and all modules are unitary. For a right R -module M , we use $S = \text{End}_R(M_R)$ to denote the endomorphism ring of M . Obviously, the module M is an (S, R) -bimodule.

A submodule N of M is said to be a *fully invariant* if $f(N)$ is contained in N for every $f \in S$. Clearly, 0 and M are fully invariant submodules of M . The right R -module M is said to be *duo* provided every submodule of M is fully invariant. It is clear that every simple right R -module is a duo module. Moreover, if the right R -module R is a duo module, then the ring R is called *right duo*. Note that a ring R is a right duo ring if and only if every right ideal of R is a two-sided ideal, equivalently Ra is contained in aR for every element a in R .

Example 1.1. Let \mathbb{Z} be the ring of integers, n a positive integer and p a prime integer. Then, \mathbb{Z} and $\mathbb{Z}/\mathbb{Z}p^n$ are duo \mathbb{Z} -modules, but the field of rationals is not a duo \mathbb{Z} -module.

Example 1.2. (see [1, Example 6]) Let p be a prime integer. Then, we have the following properties:

- (1) The \mathbb{Z} -module $\mathbb{Z} \oplus A$ is not duo for any \mathbb{Z} -module A .
- (2) For any distinct prime integers p_i ($i = 1, 2, \dots, n$), the \mathbb{Z} -module $M = \bigoplus_{i=1}^n \mathbb{Z}/\mathbb{Z}p_i^{n_i}$ is a duo module for any positive integers n_i ($i = 1, 2, \dots, n$).
- (3) The \mathbb{Z} -module $\mathbb{Q} \oplus A$ is not a duo module for any \mathbb{Z} -module A .

An R -module M is said to have the *summand sum property* if the sum of any two direct summands of M is a direct summand of M (*SSP*).

M is said to have the *summand intersection property* if the intersection of any two direct summands of M is a direct summand of M (*SIP*) (see [8],[10],[19]).

Theorem 1.3. ([1, Theorem 5]) *Let M be a duo module. Then M has the SIP and the SSP.*

In Section 2, we obtain some new properties of fully invariant submodules and duo modules. In particular, it is shown that a direct summand complement of a direct summand of M is unique if and only if it is a fully invariant submodule of M .

Let M be a module and N be a submodule of M . We call N a *projection invariant submodule* of M if every projection π of M onto a direct summand maps N into itself, i.e. N is invariant under any projection of M . Clearly, each fully invariant submodule of a module is a projection invariant submodule.

In the first part of Section 3, we obtain some basic properties of projection invariant submodules and also we study the interrelation between these submodules and the finite exchange property.

In the last part of Section 3, we consider the condition (*).

We will prove that the condition (*) holds for every direct summand of $M = \bigoplus_{i \in I} M_i$ where each M_i ($i \in I$) is an indecomposable submodule of M if and only if

- (1) $\text{Hom}(M_i, M_j) = 0$ for all distinct $i, j \in I$, and
- (2) For every direct summand N of M , there exist a (finite) subset I' of I such that $N = \bigoplus_{i \in I'} (N \cap M_i)$.

Let M be an R -module and N be a submodule of M . N is called *small*, written $N \ll M$, if $M \neq N + L$ for every proper submodule L of M . Properties of small submodules are given in [13, Lemma 4.2] and [19, Proposition 19.3]. Let M be a module. M is said to be a *lifting module*, if for every submodule N of M , M has a decomposition $M = M_1 \oplus M_2$ with $M_1 \leq N$ and $M_2 \cap N$ small in M_2 , i.e. if for every submodule A of M there exists a direct summand B of M such that $B \leq A$ and A/B is small in M/B .

According to Koşan [12], the module M is called *FI-lifting* if for every fully invariant submodule A of M , there exists a direct summand B of M such that $B \subseteq A$ and A/B small in M/B as a generalization of lifting module. By [12], if X is a fully invariant submodule of a FI-lifting module M then M/X is FI-lifting. In this section, similar to FI-lifting modules, we define PI-lifting modules. M is called a *PI-lifting module* if for every projection invariant submodule A of M , there exists a direct summand B of M such that $B \subseteq A$ and A/B small in M/B . This definition is not meaningless, that is not every PI-lifting module is a lifting module. Let $M_{\mathbb{Z}} = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z}$. Then $M_{\mathbb{Z}}$ is a FI-lifting module by [12, Corollary 3.5]. Therefore, M is a PI-lifting module. We note that $M_{\mathbb{Z}}$ is not a lifting module by [11, Example 1]. On the other hand,

- (1) M is a PI-lifting module if and only if for every projection invariant submodule A of M there exist a decomposition $M = M_1 \oplus M_2$ such that $M_1 \leq A$ and $M_2 \cap A$ is small in M_2 .
- (2) By definitions, every lifting modules are FI-lifting and PI-lifting. One may suspect that if M is an FI-lifting module then it is also a PI-lifting module. But the following example eliminates this possibility: Let R be a simple domain that is not a division ring (e.g. the first Weyl Algebra over a field of characteristic 0). Then the only fully invariant right ideals of R are the trivial ones, so R_R is FI-lifting. Since the only idempotents of R are 0 and 1 any right ideal of R is projection invariant; but $J(R) = 0$, so that R_R is not PI-lifting.

In Section 4, we obtain some basic properties of projection invariant lifting modules. In particular, it is shown that if each M_i is a PI-lifting module, then $M = \bigoplus_{i=1}^n M_i$ is a PI-lifting module.

The texts by Anderson and Fuller [2] and Wisbauer [20] are the general references for notions of rings and modules not defined in this work.

2. Fully Invariant Submodules

The next results are well known facts proved for groups in Lemma 9.5, Theorem 9.6 and Corollary 9.7 of [7], respectively.

Lemma 2.1. *Let $M = M_1 \oplus M_2$ be a decomposition of M with associated projections $\pi_i : M \rightarrow M_i$ (for $i = 1, 2$). If we also have $M = M_1 \oplus M_3$ with projections $\pi'_1 : M \rightarrow M_1$ and $\pi_3 : M \rightarrow M_3$, then, for some endomorphism ϕ of M , we have*

$$\pi'_1 = \pi_1 + \pi_1\phi\pi_2 \quad \text{and} \quad \pi_3 = \pi_2 - \pi_1\phi\pi_2 \quad (2.1)$$

Conversely, if π'_1 and π_3 are endomorphisms of M satisfying (2.1) for some $\phi \in \text{End}(M)$, then $M = M_1 \oplus \pi_3(M)$.

Proof. Let $\phi = \pi_2 - \pi_3$. Then $M_1 \leq \text{Ker}(\phi)$. Since $\phi\pi_1 = 0$ and $1_M = \pi_1 + \pi_2$, $\phi = \phi\pi_1 + \phi\pi_2 = \phi\pi_2$. Let $m = m_1 + m_2 = m'_1 + m_3 \in M$ where $m_1, m'_1 \in M_1$, $m_2 \in M_2$, and $m_3 \in M_3$. Then $\phi(m) = (\pi_2 - \pi_3)m = m_2 - m_3 = m'_1 - m_1 \in M_1$. Hence $\pi_1\phi(m) = \phi(m)$ for all $m \in M$. Thus $\phi = \pi_1\phi$. Since $\phi = \pi_1\phi = \phi\pi_2$, we have $\phi = \pi_1\phi\pi_2$ and $\pi_3 = \pi_2 - \phi = \pi_2 - \pi_1\phi\pi_2$. Since $1_M = \pi'_1 + \pi_3$, then $\pi'_1 = 1_M - \pi_3 = \pi_1 + \pi_2 - \pi_3 = \pi_1 + \pi_1\phi\pi_2$.

Conversely, assume that π'_1 and π_3 are of the form (2.1). We add the equalities (2.1) side by side to get $\pi'_1 + \pi_3 = \pi_1 + \pi_2 = 1_M$. Also it is easy to check that π'_1 and π_3 are orthogonal idempotents in S . Then $M = \pi'_1(M) \oplus \pi_3(M)$. By (2.1), $\pi'_1(M) \leq \pi_1(M)$, and since $\pi_1(M) = M_1$ and $\pi_3(M) \cap M_1 = 0$, we have $M = M_1 \oplus \pi_3(M)$. ■

Theorem 2.2. *If M_1 is a direct summand of the module M , then the intersection of all direct summand complements of M_1 in M is the maximal fully invariant submodule of M that has the zero intersection with M_1 .*

Proof. Let K denote the intersection of all direct summand complements of M_1 in M . Let $M = M_1 \oplus M_2$ and both π_1 and π_2 be projections of M along M_1 and M_2 respectively, and let $\phi \in S = \text{End}(M)$. By Lemma 2.1, $M_3 = (\pi_2 - \pi_1\phi\pi_2)(M)$ is again a direct summand complement of M_1 in M . Let $x \in K$. Since $K \leq M_2 \cap M_3$, $(\pi_2 - \pi_1\phi\pi_2)(x) = x$ and $\pi_2(x) = x$. Hence $0 = (\pi_1\phi\pi_2)(x) = (\pi_1\phi)(x)$. Thus, $\phi(x) \in M_2$, for all direct summand complement of M_2 in M . It follows that $\phi(x) \in K$. Now clearly $M_1 \cap K = 0$. If L is any fully invariant submodule of M with $L \cap M_1 = 0$ and $M = M_1 \oplus M_2$, then, if $x \in L$ with $x = m_1 + m_2$ where $m_1 \in M_1$ and $m_2 \in M_2$, we have $m_1 = \pi_1(x) \in M_1 \cap L = 0$, and so $L = (L \cap M_1) \oplus (L \cap M_2) = L \cap M_2$. Hence $L \leq M_2$ for all direct summand complements of M_2 in M . Thus, $L \leq K$. This completes the proof. ■

Corollary 2.3. *Let M be a module. A direct summand complement of a direct summand of M is unique if and only if it is a fully invariant submodule of M .*

Following Warfield [18], we say that a ring R is *exchange* in case the regular right R -module R_R satisfies the (finite) exchange property, that is, for every R -module M and decompositions

$$M = X \oplus Y = \bigoplus_{i \in I} M_i$$

with $X \cong R_R$ (and I finite), there exist submodules $N_i \subseteq M_i$ such that

$$M = X \oplus (\oplus_{i \in I} N_i).$$

Some kinds of generalized exchange rings were studied by Chen Huayin in [5]. We remark here that GM-condition on a ring R was also stated by H.Chen and M.Chen which generalizes the known unit 1-stable range condition [6]. By using this GM-condition, they investigated the exchange rings with Artinian primitive factors satisfying the GM-condition.

It is well know that all continuous modules have the full exchange property (see [13]). The following theorem is a slight version of this result on quasi-injective modules. For some new results on injective module and quasi injective modules, the readers are referred to [16], [9] and [15].

Theorem 2.4. *Every quasi-injective duo module has the finite exchange property.*

Proof. Let M be a quasi-injective duo module with $S = \text{End}_R(M)$. Note that every module is a submodule of a quasi-injective module. Let N be a right R -module and $g : N \rightarrow M$ be a monomorphism. Then we may assume $g(N)$ is a fully invariant submodule of M . Since M is a quasi-injective module, for $\alpha, \beta \in S' = \text{End}_R(N)$ with $\alpha + \beta = 1_N$, there exist a $f \in S$ such that $g\alpha = fg$. It is easy to see that $g\beta = (1_M - \alpha)g$ and so $\alpha + (1_M - \alpha) = 1_M$. Now, since $S/J(S)$ is regular and self-injective, the ring S is an exchange ring by [17, Theorem 29.2]. By [17, Theorem 29.1], we have $e_1 \in \alpha S$ with $e_1^2 = e_1$ and $e_2 \in (1_M - \alpha)S$ with $e_2^2 = e_2$ such that $e_1 + e_2 = 1_M$. Let $e_1 = \alpha s_1$ and $e_2 = \alpha s_2$ for some $s_1, s_2 \in S$. Since $g(N)$ is a fully invariant submodule of M , there are unique $h_1, h_2, t_1, t_2 \in S'$ such that $gh_1 = e_1g$, $gh_2 = e_2g$, $gt_1 = s_1g$ and $gt_2 = s_2g$. Then h_1, h_2 are idempotents and $h_1 + h_2 = 1_N$. Since $g : N \rightarrow M$ is monomorphism, we have $h_1 = \alpha t_1$ and $h_2 = \beta t_2$. Now, by [17, Theorem 29.1], the ring S' is an exchange ring. This implies that N has the finite exchange property. ■

3. Projection Invariant Submodules

We list below some of the basic properties of projection invariant submodules that will be needed in this paper.

Proposition 3.1. *Let M be a module and N be a submodule of M . Then;*

(1) *N is a projection invariant submodule of M if and only if $\pi(N) = N \cap \pi(M)$ for every projection π of M .*

(2) *N is a projection invariant submodule of M if and only if N is an intersection of projection invariant submodules of M .*

(3) *Any sum and intersection of projection invariant submodules of M is again a projection invariant submodule of M .*

(4) A projection invariant direct summand of M is a fully invariant submodule of M .

(5) Let $M = M_1 \oplus M_2$ be a decomposition and N be any projection invariant submodule of M . Then $N = (N \cap M_1) \oplus (N \cap M_2)$.

(6) If $M = \bigoplus_{i \in I} M_i$ and N is a projection invariant submodule of M , then $N = \bigoplus_{i \in I} \pi_i(N) = \bigoplus_{i \in I} (M_i \cap N)$, where π_i is the i -th projection homomorphism of M along M_i .

Proof. (1) Assume that N is a projection invariant submodule of M . Let π be a projection of M . Then $\pi(N) \leq N \cap \pi(M) \leq \pi(N)$. Since $N \cap \pi(M) \leq \pi(N)$ always holds, then $\pi(N) = N \cap \pi(M)$. The converse is clear.

(2) Assume that N is a projection invariant submodule of M . Note that M is a projection invariant submodule of M . Since $N = N \cap M$, then N is the intersection of projection invariant submodules N and M . Conversely, let $N = \bigcap_{i \in I} N_i$ where N_i ($i \in I$) are projection invariant submodules of M and let π be a projection of M . Then $\pi(N) = \pi(\bigcap_{i \in I} N_i) \leq \bigcap_{i \in I} \pi(N_i) \leq \bigcap_{i \in I} N_i = N$. Hence $\pi(N) \leq N$.

(3) This is similar to [12, Lemma 3.2].

(4) Let M_1 be a projection invariant direct summand of M , $f \in S = \text{End}(M)$ and $M = M_1 \oplus M_2$. Let π_1 and π_2 be projections of M onto M_1 and M_2 , respectively. Let ϕ be any element in S . By Lemma 2.1, we obtain that $\pi_3 = \pi_1 - \pi_2 \phi \pi_1$ is a projection of M . By hypothesis, $\pi_3(M_1) \leq M_1$. Let $x \in M_1$. Then $\pi_3(x) = x - (\pi_2 \phi)(x) \in M_1$. Hence $(\pi_2 \phi)(x) = 0$. Thus $\phi(x) \in M_1$.

(5) Let π_1 and π_2 be projections of M along with M_1 and M_2 respectively. Then, for any $m = m_1 + m_2 \in M$ where $m_1 \in M_1$ and $m_2 \in M_2$, we have $\pi_1(m) = m_1$ and $\pi_2(m) = m_2$. Let $n = n_1 + n_2 \in N$ where $n_1 \in M_1$ and $n_2 \in M_2$. By hypothesis, we obtain that $\pi_1(n) = n_1 \in N$ and $\pi_2(n) = n_2 \in N$, and so $\pi_1(n) = n_1 \in N \cap M_1$ and $\pi_2(n) = n_2 \in N \cap M_2$. Then $n = n_1 + n_2 \in N \cap M_1 + N \cap M_2$. Hence $N \leq N \cap M_1 + N \cap M_2$. The rest is clear.

(6) This is similar to [12, Lemma 3.2]. ■

Let M and N be two submodules with $S = \text{End}_R(M_R)$ and $S' = \text{End}_R(N_R)$. For a right R -homomorphism $g : N \rightarrow M$, we consider the set $I = \{f \in S : gf = 0\}$. It is easy to see that I is a right ideal of S .

Proposition 3.2. *Let M be a quasi-projective module. With the above notation, if I is a projection invariant direct summand, then $S/I \cong S'$.*

Proof. Let M be a quasi-projective module and $I = \{f \in S : gf = 0\}$ for any right R -homomorphism $g : N \rightarrow M$. By Proposition 3.1, we may assume that I is fully invariant. Then we have the R -module homomorphism $h : M/\text{Rad}(M) \rightarrow S$ such that $hg = gf$. Now $\alpha : S \rightarrow S'$, defined by $\alpha(f) = h_f$ where h_f depends on any $f \in S$, is a homomorphism. Since M is a quasi-projective module, for any $\beta \in S'$, there exist a $f' \in S$ such that $\beta g = gf'$. It is easy to see that α is a surjective homomorphism and $\text{Ker}(\alpha) = I$. ■

Theorem 3.3. *Let M be a quasi-projective module and N be a module. With the above notations, let I be a projection invariant direct summand. If M has the finite exchange property, then*

- (1) N has the finite exchange property.
- (2) I is an exchange ring.
- (3) For any $f \in S$, if $\text{Im}(f - f^2) \subseteq I$ then there exists an idempotent $e \in S$ such that $\text{Im}(f - e) \subseteq I$.

Proof. (1) Assume that M has the finite exchange property. By [17, Theorem 28.7], the ring S is an exchange ring. Then, by Proposition 3.2, we have $S/I \cong S'$. By [3, Theorem 2.2], the ring S' is an exchange ring. Then N has the finite exchange property.

(2) Clear.

(3) For any $f \in S$, there exist an idempotent $e \in S$ such that $f - e = (f - f^2)f'$ for some $f' \in S$ by [17, Theorem 29.1]. This implies that $g(f - e) = g(f - f^2)f' = 0$, i.e., $\text{Im}(f - e) \subseteq I$. ■

We consider the condition (*) for an R -module M .

Clearly, duo modules satisfy the (*)-condition. If M is a right R -module, then M satisfies the (*)-condition because M is a duo module.

Proposition 3.4.

- (1) *If a module satisfies the (*)-condition, then any direct summand of it also satisfies the (*)-condition.*
- (2) *Let M be an R -module. Assume that the (*)-condition holds for every summands of M , i.e. all direct summands of M are projection invariant. Then M has the SIP and SSP properties.*

Proof. (1) Assume M satisfies the (*)-condition and $M = M' \oplus M''$ with M', M'' submodules of M . Let $\pi_{M'} : M \rightarrow M'$ be the canonical projection and let N be any submodule of M' . Suppose that π is a projection of M' , i.e. $\pi : M' = M' \oplus (0) \rightarrow M'$. Then $p = \pi_{\pi_{M'}}$ is a projection of M and $\pi(N) = p(N)$ which is contained in N because M satisfies the (*)-condition. It follows that M' satisfies the (*)-condition.

(2). Let M_1 and M_2 be direct summands of M . Note that, M_1 and M_2 are fully invariant submodules of M by Prop. 3.1(4). For some submodule M'_2 of M , let $M = M_2 \oplus M'_2$. By assumption and Prop. 3.1(6), we have $M_1 = (M_1 \cap M_2) \oplus (M_1 \cap M'_2)$. Clearly, $M_1 \cap M_2$ is a direct summand of M , i.e. M has the SIP property. Since $M_1 + M_2 = M_2 \oplus (M_1 \cap M'_2)$ and $M_1 \cap M'_2$ is a direct summand of M'_2 , then $M_1 \cap M'_2$ is a direct summand of M , i.e., M has the SSP property. ■

Remark 3.5. Note that Proposition 3.5(2) also follows from Proposition 3.1.(4) and Theorem 1.3.

In [14, Proposition 1.3], it is proved that any direct summand of a duo module is also a duo module.

Proposition 3.6. *Any direct summand of a duo module is also a duo module.*

Proof. The proof is clear from Props. 3.4 and 3.1. ■

Proposition 3.7. *Let M be an R -module.*

- (1) *Assume that M has a decomposition $M = M_1 \oplus M_2$ for some submodules M_1, M_2 of M . If M_1 is a projection invariant submodule of M , then $\text{Hom}(M_1, M_2) = 0$.*
- (2) *Assume that the $(*)$ -condition holds for every direct summand of M . If M has a decomposition $M = M_1 \oplus M_2$ for some submodules M_1, M_2 of M , then $\text{Hom}(M_1, M_2) = 0$.*

Proof. (1). By Prop. 3.1, we can suppose that M_1 is a fully invariant submodule of M . Let $f : M_1 \rightarrow M_2$ be any homomorphism. Let $p_1 : M \rightarrow M_1$ denote the canonical projection and let $i_2 : M_2 \rightarrow M$ denote inclusion. Then $f^* = i_2 f p_1$ is an endomorphism of M . By hypothesis, $f^*(M_1) \subseteq M_1$, so that $f(M_1) \subseteq M_1 \cap M_2 = 0$. It follows that $f = 0$.

(2). By (1) and Prop. 3.1(4). ■

Theorem 3.8. *Let a module $M = \bigoplus_{i \in I} M_i$ be a direct sum of submodules M_i ($i \in I$). Then, the $(*)$ -condition holds for every direct summand of M if and only if*

- (1) *The $(*)$ -condition holds for every direct summand of M_i for all $i \in I$,*
- (2) *$\text{Hom}(M_i, M_j) = 0$ for all distinct $i, j \in I$,*
- (3) *$N = \bigoplus_{i \in I} (N \cap M_i)$ for every direct summand N of M .*

Proof. Sufficiency. It is clear from by Props. 3.1 and 3.7.

(Necessity). Suppose that M satisfies (1), (2) and (3). Let K be any direct summand of M and let f be any endomorphism of M . For each j in I , let $p_j : M \rightarrow M_j$ denote the canonical projection and let $i_j : M_j \rightarrow M$ denote the inclusion. Then, by (1), we have $p_j f i_j (K \cap M_j) \subseteq K \cap M_j$ for all $j \in I$. Because every projection-invariant direct summand of M_j is a fully invariant submodule by Prop. 3.1(4). Moreover, we have $p_k f i_j (K \cap M_j) = 0$ for all distinct $j, k \in I$ by (2). Now, (3) gives $f(K) = \sum_{j \in I} f(K \cap M_j) \subseteq \sum_{j \in I} p_j f i_j (K \cap M_j) \subseteq \sum_{j \in I} (K \cap M_j) \subseteq K$. Thus, K is a fully invariant submodule of M and so a projection invariant submodule of M . ■

Corollary 3.9. *Let a module $M = \bigoplus_{i \in I} M_i$ be a direct sum of indecomposable submodules M_i ($i \in I$). Then the $(*)$ -condition holds for every direct summand of M if and only if*

- (1) *$\text{Hom}(M_i, M_j) = 0$ for all distinct $i, j \in I$.*

- (2) For every direct summand N of M , there exist a (finite) subset I' of I such that $N = \bigoplus_{i \in I'} (N \cap M_i)$.

Proof. (Sufficiency). Clear from by Thm. 3.8 and Prop. 3.4.

(Necessity). By Thm. 3.8. ■

Let R be a ring and let M be a right R -module. For any non-empty subset S of M , the *annihilator* of S (in R) will be denoted by $\text{ann}(S)$, i.e. $\text{ann}(S) = \{r \in R : sr = 0 \text{ for all } s \text{ in } S\}$. In case $S = \{m\}$, then we write $\text{ann}(m)$ for $\text{ann}(\{m\})$. We now prove another basic fact about direct sum decompositions.

Lemma 3.10. ([14, Lemma 2.4]) *Let a module $M = \bigoplus_{i \in I} M_i$ be a direct sum of submodules M_i ($i \in I$). Then the following statements are equivalent.*

- (1) $R = \text{ann}(m_i) + \text{ann}(m_j)$ for all $m_i \in M_i, m_j \in M_j$, for all $i \neq j$ in I .
- (2) $N = \bigoplus_{i \in I} (N \cap M_i)$ for every (cyclic) submodule N of M .

Moreover, in this case $\text{Hom}(M_i, M_j) = 0$ for all distinct i, j in I .

Theorem 3.11. *Let a module $M = \bigoplus_{i \in I} M_i$ be a direct sum of submodules M_i ($i \in I$). Then M satisfies the (*)-condition if and only if*

- (1) M_i satisfies the (*)-condition for all $i \in I$, and
- (2) $N = \bigoplus_{i \in I} (N \cap M_i)$ for every submodule N of M .

Proof. Using Lemma 3.10, the proof is similar to that of Cor. 3.9. ■

Corollary 3.12. *Let a module $M = \bigoplus_{i \in I} M_i$ be a direct sum of submodules M_i ($i \in I$). Then M satisfies the (*)-condition if and only if $M_i \oplus M_j$ satisfies the (*)-condition for all distinct i, j in I .*

Proof. (Sufficiency). By Prop. 3.4.

(Necessity). Let $M_i \oplus M_j$ satisfy the (*)-condition for all distinct $i \neq j$ in I . Then M_i satisfies the (*)-condition for all $i \in I$, by Prop. 3.4. Moreover, for all $i \neq j$ in I , $R = \text{ann}(m_i) + \text{ann}(m_j)$ for all $m_i \in M_i, m_j \in M_j$ by Prop. 3.1 and Lemma 3.10. Hence M satisfies the (*)-condition by Lemma 3.10 and Theorem 3.11. ■

If the right R -module R has the (*) property, we say R has the (*) property on the right side. Clearly commutative rings and division rings satisfy the (*) property on the right side.

Remark 3.13.

- (1) If R has the (*) property, then $R \oplus R$ does not satisfy the (*) property.
- (2) Any 2×2 matrix ring over division rings does not satisfy the (*) property.

Proof. Let A and B be right R -modules and $f : A \rightarrow B$ be an epimorphism. Then A is not projection invariant in $M = A \oplus B$, because if $p : M \rightarrow A$ denotes the canonical projection, then $fp : M \rightarrow B$ is a projection to the direct summand B , but $fp(A) = B$ is not contained in A . Hence the module M does not have property (*). In particular, for any non-zero module M , the module $M \oplus M$ does not have property (*), independent of M having property (*) or not. This shows that neither $R \oplus R$ nor the ring of 2×2 matrices over any ring R (R can be even a field) satisfies (*). ■

Question 3.14. Let R be a ring and R' be a proper subring of R . Does R'_R satisfy the (*)-property or not ?

4. The Lifting Condition

Following [12], the module M is called *FI-lifting* if for every fully invariant submodule A of M , there exists a direct summand B of M such that $B \subseteq A$ and A/B small in M/B .

Definition 4.1. A right R -module M is called *PI-lifting* if for every projection invariant submodule A of M , there exists a direct summand B of M such that $B \subseteq A$ and A/B small in M/B .

Lemma 4.2. The following statements are equivalent for a right R -module M .

- (1) M is a PI-lifting module.
- (2) For every projection invariant submodule A of M there is a decomposition $A = N \oplus S$ with N a direct summand of M and S small in M .
- (3) For every projection invariant submodule X of M , there exists an idempotent homomorphism $e : M \rightarrow X$ such that $(1 - e)(X) \leq (1 - e)(M)$.

Proof. (1) \Rightarrow (2). Let A be a projection invariant submodule of M . Since M is a PI-lifting module, there exist a decomposition $M = M_1 \oplus M_2$ such that $M_1 \leq A$ and $M_2 \cap A$ small in M_2 . Therefore $A = M_1 \oplus (A \cap M_2)$, as required.

(2) \Rightarrow (1). Assume that every projection invariant submodule has the stated decomposition. Let A be a projection invariant submodule of M . By hypothesis, there exist a direct summand N of M and a small submodule S of M such that $A = N \oplus S$. Now $M = N \oplus N'$ for some submodule N' of M . Consider the natural epimorphism $\pi : M \rightarrow M/N$. Then $\pi(S) = (S + N)/N = A/N$ small in M/N . Therefore M is a PI-lifting module. (1) \Leftrightarrow (3). Clear. ■

Theorem 4.3. Let $M = \bigoplus_{i=1}^n M_i$. If each M_i is a PI-lifting module, then M is a PI-lifting module

Proof. Let N be a projection invariant submodule of M . It is easy to see that

for every $1 \leq i \leq n$, $N \cap M_i$ is projection invariant in M_i by Lemma 3.1. Since M_i is a PI-lifting module for every i , there exist a direct summand K_i of M_i such that $K_i \leq N \cap M_i$ and $(N \cap M_i)/K_i$ is small in M_i/K_i for every i . Clearly, $K = \bigoplus_{i=1}^n K_i$ is a direct summand of M and $K \subseteq \bigoplus_{i=1}^n (N \cap M_i)$. We know that $\bigoplus_{i=1}^n (N \cap M_i) = M$ by Lemma 3.1. Now consider the homomorphism $\beta : \bigoplus_{i=1}^n (N_i/K_i) \rightarrow (\bigoplus_{i=1}^n M_i)/K$ with $(m_1 + K_1, \dots, m_n + K_n) \rightarrow (\sum_{i=1}^n m_i) + K$, where $m_i \in M_i$ for $1 \leq i \leq n$. Then $\beta(\bigoplus_{i=1}^n ((N \cap M_i)/K_i)) = (\bigoplus_{i=1}^n (N \cap M_i))/K$. Since any finite sum of small submodules again a small submodule, $\bigoplus_{i=1}^n ((N \cap M_i)/K_i)$ is small in $\bigoplus_{i=1}^n (M_i/K_i)$. Then by [13, Lemma 4.2], $(\bigoplus_{i=1}^n (N \cap M_i))/K$ is small in M/K . ■

We do not know if any direct sum of PI-lifting modules is a PI-lifting module.

Corollary 4.4. *If M is a finite direct sum of lifting (or hollow) modules, then M is a PI-lifting module.*

Example 4.5. Let R be a PID and M be any finitely generated R -module. We consider the torsion submodule $Tor(M)$ of M . Since $Tor(M)$ is a finite direct sum of hollow R -modules, then $Tor(M)$ is a PI-lifting module by Corollary 4.4.

Let M be a lifting module. By [12, Corollary 2.2], for every fully invariant submodule Y of M , M/Y is a lifting module. Let X be a fully invariant submodule of M . If M is an FI-lifting module then M/X is an FI-lifting module (see [12, Proposition 3.3]).

Proposition 4.6. *Let M be a module and X be a projection invariant submodule of M . Assume that X'/X is a projection invariant submodule of M/X where $X \leq X' \leq M$. Then X' is a projection invariant submodule of M . If M is a PI-lifting module then M/X is a PI-lifting module.*

Proof. Let Y be a submodule of M with $X \subseteq Y$ and let Y/X be a projection invariant submodule of M/X . By assumption, Y is a projection invariant submodule of M . Since M is a PI-lifting module, there exist a direct summand D of M such that $D \leq Y$ and Y/D is small in M/D . Assume $M = D \oplus D'$ for some submodule D' of M . Let π be the projection with the kernel D and $i : D' \rightarrow M$ the inclusion map. Now, $\alpha = i\pi : M \rightarrow M$ be a homomorphism of M . Since X and Y are projection invariant submodules of M , then $\alpha(X) \subseteq X$ and $\alpha(Y) \subseteq Y$. It is easy to see that $Y = \alpha^{-1}(Y)$. Now, $\alpha^{-1}(X) \subseteq Y = \alpha^{-1}(Y)$. Let K be a submodule of M with $\alpha^{-1}(X) \subseteq K$ and $M/\alpha^{-1}(X) = (Y/\alpha^{-1}(X)) + (K/\alpha^{-1}(X))$. Then $M = Y + K$ and since Y/D is small in M/D , $M = K$. Therefore $Y/\alpha^{-1}(X)$ is small in $M/\alpha^{-1}(X)$, namely $(Y/X)/(\alpha^{-1}(X)/X) \ll (M/X)/(\alpha^{-1}(X)/X)$. Now, we want to show that $\alpha^{-1}(X)/X$ is a direct summand of M/X . Since $M = D \oplus D'$, then $M = \alpha^{-1}(X) + D'$. Therefore $M/X = (\alpha^{-1}(X)/X) + (D' + X)/X$. Since

$\alpha^{-1}(X) \cap (D' + X) = X + (\alpha^{-1}(X) \cap D') = X$, then $\alpha^{-1}(X)/X$ is a direct summand of M/X . Hence M/X is a PI-lifting module. ■

Theorem 4.7. *Let $M = M_1 \oplus M_2$ be a module with the (*)-condition. Then M is a PI-lifting module if and only if each M_i is a PI-lifting module for $i = 1, 2$.*

Proof. By Theorem 4.3 and Proposition 4.6. ■

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