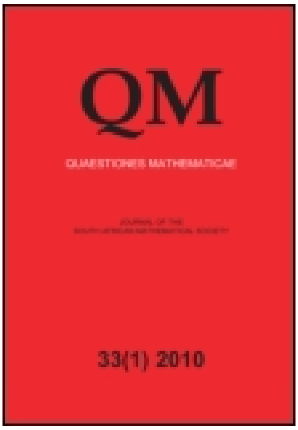

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On the support of general local cohomology modules and filter regular sequences

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ON THE SUPPORT OF GENERAL LOCAL COHOMOLOGY MODULES AND FILTER REGULAR SEQUENCES

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Abstract. Let R be a commutative Noetherian ring with non-zero identity and $\mathfrak{a}$ an ideal of R. In the present paper, we examine the question whether the support of $H^n_{\mathfrak{a}}(N, M)$ must be closed in Zariski topology, where $H^n_{\mathfrak{a}}(N, M)$ is the nth general local cohomology module of finitely generated R-modules M and N with respect to the ideal $\mathfrak{a}$.

Mathematics Subject Classification (2010): 13D45.

Key words: Local cohomology module, generalized local cohomology module, support of local cohomology module, filter regular sequence, Matlis duality functor.

1. Introduction.

Throughout this paper, we will assume that R is a commutative Noetherian ring with non-zero identity, $\mathfrak{a}$ is an ideal of R and M, N are two finitely generated R-modules. Also, we shall use $\mathbb{N}_0$ (respectively, $\mathbb{N}$) to denote the set of non-negative (respectively, positive) integers.

Local cohomology was first defined and studied by Grothendieck [3]. For each $n \in \mathbb{N}_0$, the nth local cohomology module of M with respect to an ideal $\mathfrak{a}$ is defined as

$$H^n_{\mathfrak{a}}(M) = \varinjlim_{m\in\mathbb{N}} \operatorname{Ext}^n_R(R/\mathfrak{a}^m, M).$$

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It is well known that, in general, the local cohomology modules $H^n_{\mathfrak{a}}(M)$ are not finitely generated for all $n \in \mathbb{N}$. One of the important problems concerning local cohomology is to find when the set of associated primes of $H^n_{\mathfrak{a}}(M)$ is finite (cf. [7, Problem 4]). There are several papers devoted to studying the associated prime ideals of local cohomology modules. We refer the reader to the papers of Hellus [4], Huneke and Sharp [9], Lyubeznik [15, 16], Singh [24], Katzman [10] and also Singh and Swanson [25]. So it is natural to ask whether the sets of primes minimal in the support of $H^n_{\mathfrak{a}}(M)$ are finite for all $n \in \mathbb{N}$. This is equivalent to asking the following question (see Lemma 2.1(i)).

QUESTION 1.1. Let R be a Noetherian ring, M a finitely generated R-module, $\mathfrak{a}$ an ideal of R, and n a non–negative integer. Is the support of $H^n_{\mathfrak{a}}(M)$ a Zariski-closed subset of $\operatorname{Spec}(R)$?

Recently, Huneke, Katz and Marly, in [8], provided some partial answers for Question 1.1 in the case when the ideal $\mathfrak{a}$ generated by n elements and the top local cohomology modules $H^n_{\mathfrak{a}}(M)$ are considered. For instance, they proved that:

- The support of $H^2_{(x,y)}(M)$ is closed for all $x, y \in R$.

Also, they showed that:

- If the support of $H^3_{\mathfrak{a}}(M)$ is closed for every three-generated ideal $\mathfrak{a}$ of R then, for all non-negative integers n, $\operatorname{Supp}_R H^n_{\mathfrak{b}}(M)$ is closed for every n-generated ideal $\mathfrak{b}$ of R.

Afterward, Khashyarmanesh, in [12], showed that:

- Over an arbitrary commutative ring R, the following conditions are equivalent:

(a) For all positive integers n, $\operatorname{Supp}_R H^n_{\mathfrak{a}}(M)$ is closed for every ideal $\mathfrak{a}$.

(b) For $i = 2, 3, 4$, $\operatorname{Supp}_R \operatorname{Hom}_R(R/(x_1, \ldots, x_{i+1}), H^i_{(x_1,\ldots,x_i)}(M))$ is closed, for every sequence $x_1, \ldots, x_{i+1}$ of elements of R such that $x_1, \ldots, x_i$ is an $(x_1, \ldots, x_{i+1})$-filter regular sequence on M.

(c) $\operatorname{Supp}_R H^2_{\mathfrak{a}}(M)$ is closed for every three-generated ideal $\mathfrak{a}$ of R, $\operatorname{Supp}_R H^3_{\mathfrak{a}}(M)$ is closed for every four-generated ideal $\mathfrak{a}$ of R, and $\operatorname{Supp}_R H^4_{\mathfrak{a}}(M)$ is closed for every five-generated ideal $\mathfrak{a}$ of R.

On the other hand, a generalization of the local cohomology functor has been given by Herzog in [5] (see also [27]). For each $n \in \mathbb{N}_0$, the ith generalized local cohomology module of the pair (N, M) with respect to an ideal $\mathfrak{a}$ is defined as

$$H^n_{\mathfrak{a}}(N, M) = \varinjlim_{m \in \mathbb{N}} \operatorname{Ext}^n_R(N/\mathfrak{a}^m N, M).$$

Clearly, $H^i_{\mathfrak{a}}(R, N) \cong H^i_{\mathfrak{a}}(N)$ for all $i \in \mathbb{N}_0$. So, we are led to the following natural question:

QUESTION 1.2. Let R be a Noetherian ring, M and N be finitely generated R-modules, $\mathfrak{a}$ an ideal of R, and n a non-negative integer. Is the support of $H^n_{\mathfrak{a}}(N, M)$ a Zariski-closed subset of $\mathrm{Spec}(R)$?

The finiteness properties of generalized local cohomology modules are not well understood (cf. [1], [6] and [14], [17]). In this paper we provide a partial answer to Question 1.2.

Now, let E be the injective hull of the direct sum of all simple R-modules and $D(-)$ be the functor $\mathrm{Hom}_R(-, E)$, which is a natural generalization of the Matlis duality functor to non-local rings (see [19, 20, 21, 22]). The co-support of an R-module L is defined as follows (cf. [22]):

$$\mathrm{co} - \mathrm{Supp}_R L = \mathrm{Supp}_R D(L).$$

So as a dual version "in some sense" of Questions 1.1 and 1.2, we have that:

QUESTION 1.3. Let R be a Noetherian ring, M and N finitely generated R-modules, $\mathfrak{a}$ an ideal of R, and n a non-negative integer. Is the co-support of $H^n_{\mathfrak{a}}(N, M)$ a Zariski-closed subset of $\mathrm{Spec}(R)$?

In Section 3, we provide a partial answer to Question 1.3.

Our terminology follows the textbook [2] on local cohomology. For basic properties of generalized local cohomology modules, we refer the reader to [1], [6] and [14].

2. Support of generalized local cohomology modules.

The concept of a filter regular sequence plays an important role in this paper. A sequence $x_1, \ldots, x_n$ of elements of the ideal $\mathfrak{a}$ of R is said to be an $\mathfrak{a}$-filter regular sequence on M, if

$$\mathrm{Supp}_R \left(\frac{(x_1, \ldots, x_{i-1})M :_M x_i}{(x_1, \ldots, x_{i-1})M} \right) \subseteq V(\mathfrak{a})$$

for all $i = 1, \ldots, n$, where $V(\mathfrak{a})$ denotes the set of prime ideals of R containing $\mathfrak{a}$. The concept of an $\mathfrak{a}$-filter regular sequence on M is a generalization of the one for a filter regular sequence which has been studied in [23], [26] and has led to some interesting results. Note that both concepts coincide if $\mathfrak{a}$ is the maximal ideal in a local ring. Also note that $x_1, \ldots, x_n$ is a weak M-sequence if and only if it is an R-filter regular sequence on M. It is easy to see that the analogue of [26, Appendix 2(ii)] holds true whenever R is Noetherian, M is finitely generated and $\mathfrak{m}$ is replaced by $\mathfrak{a}$; so that, if $x_1, \ldots, x_n$ is an $\mathfrak{a}$-filter regular sequence on M, then there is an element $y \in \mathfrak{a}$ such that $x_1, \ldots, x_n, y$ is an $\mathfrak{a}$-filter regular sequence on M. Thus, for a positive integer n, there exists an $\mathfrak{a}$-filter regular sequence on M of length n.

LEMMA 2.1. *Suppose that X is an R-module.*

(i) $\mathrm{Supp}_R X$ *is closed if and only if the number of the minimal elements in* $\mathrm{Supp}_R X$ *is finite.*

(ii) *Let* $0 \longrightarrow Y \longrightarrow X \longrightarrow Z \longrightarrow 0$ *be an exact sequence of* R*-modules. If the sets* $\mathrm{Supp}_R Y$ *and* $\mathrm{Supp}_R Z$ *are closed, then so is* $\mathrm{Supp}_R X$.

Proof. (i) Assume that the support of X is closed. Hence $\mathrm{Supp}_R X = V(\mathfrak{a})$ for some ideal $\mathfrak{a}$ of R. Let $\mathfrak{a} = \cap_{i=1}^t \mathfrak{q}_i$ be a minimal primary decomposition of $\mathfrak{a}$, where $\mathfrak{q}_i$ is a $\mathfrak{p}_i$-primary ideal for all i with $1 \leqslant i \leqslant t$. Then $V(\mathfrak{a}) = V(\cap_{i=1}^t \mathfrak{q}_i) = \cup_{i=1}^t V(\mathfrak{q}_i)$. Also, it is easy to see that $V(\mathfrak{q}_i) = V(\mathfrak{p}_i)$ for all $1 \leqslant i \leqslant t$. Therefore $V(\mathfrak{a}) = \cup_{i=1}^t V(\mathfrak{p}_i)$. So the number of the minimal elements in $\mathrm{Supp}_R X$ is finite. Conversely, if the number of the minimal elements in $\mathrm{Supp}_R X$ is finite, then clearly $\mathrm{Supp}_R X$ is closed.

(ii) It follows from (i). □

NOTATION 2.2. *For an* R*-module* X*, we denote the set of minimal elements in* $\mathrm{Supp}_R(X)$ *by* $\mathrm{minSupp}_R(X)$

In the following theorem, for a fixed integer n, we study the closeness of the support of the generalized local cohomology module $H^n_{(x_1,\ldots,x_n)}(N, M)$.

THEOREM 2.3. *Let* n *be a non-negative integer and* $x_1, \ldots, x_n$ *be an* $\mathfrak{a}$*-filter regular sequence on* M*, where* $\mathfrak{a} := (x_1, \ldots, x_n)$*. Assume that*

(i) $\mathrm{Supp}_R(\mathrm{Ext}_R^{n-i+1}(N, H^i_{\mathfrak{a}}(M))) \subseteq \mathrm{Supp}_R(\mathrm{Ext}_R^{n-i}(N, H^i_{\mathfrak{a}}(M)))$ *for all* $i = 0, 1, \ldots, n-1$,

(ii) $H_{\mathfrak{a}}^{n-i-2}(N, H^{i+1}_{(x_1,\ldots,x_{i+1})}(M)) = 0$*, for all* $i = 0, 1, \ldots, n-2$,

(iii) $\mathrm{Supp}_R(\mathrm{Ext}_R^{n-i}(N, H^i_{\mathfrak{a}}(M)))$ *is closed for all* $i = 1, \ldots, n-1$,

(iv) $\mathrm{minSupp}_R(H^n_{\mathfrak{a}}(M)) \subseteq \mathrm{Supp}_R(N)$*, and*

(v) *the set* $\mathrm{Supp}_R(H^n_{\mathfrak{a}}(M))$ *is closed.*

Then $\mathrm{Supp}_R(H^n_{\mathfrak{a}}(N, M))$ *is closed.*

Proof. Let x_{n+1} be an element in $\mathfrak{a}$ such that $x_1, \ldots, x_{n+1}$ is an $\mathfrak{a}$-filter regular sequence on M. (Note that the existence of such element is explained in the beginning of this section.) Put $S_0 := M$ and $S_i := H^i_{(x_1,\ldots,x_i)}(M)$ for $i = 1, \ldots, n+1$. Hence, by [11, Lemma 2.2], for each $i = 0, 1, \ldots, n$, we obtain the following exact sequence:

$$0 \longrightarrow H^i_{\mathfrak{a}}(M) \longrightarrow S_i \xrightarrow{f_i} (S_i)_{x_{i+1}} \longrightarrow S_{i+1} \longrightarrow 0.$$

Put $L_i := \mathrm{Im} f_i$ for $i = 0, 1, \ldots, n$. Since the multiplication by x_{i+1} provides an automorphism on $(S_i)_{x_{i+1}}$ and $H^j_{\mathfrak{a}}(N, (S_i)_{x_{i+1}})$ is an $\mathfrak{a}$-torsion module, for all $j \in \mathbb{N}_0$, it follows from the exact sequence $0 \longrightarrow L_i \longrightarrow (S_i)_{x_{i+1}} \longrightarrow S_{i+1} \longrightarrow 0$ that

$$H^0_{\mathfrak{a}}(N, L_i) = 0 \tag{1}$$

and

$$H^j_{\mathfrak{a}}(N, L_i) \cong H^{j-1}_{\mathfrak{a}}(N, S_{i+1}) \tag{2}$$

for all $i = 0, 1, \ldots, n$ and $j \in \mathbb{N}$. Hence, for $i = 0, 1, \ldots, n$, by applying the functor $H^j_{\mathfrak{a}}(N, -)$ on the exact sequence $0 \longrightarrow H^i_{\mathfrak{a}}(M) \longrightarrow S_i \longrightarrow L_i \longrightarrow 0$, in conjunction with (1), (2) and [14, Lemma 2.2], one can obtain an exact sequence:

$$\begin{aligned}
0 &\longrightarrow \operatorname{Ext}^1_R(N, H^i_{\mathfrak{a}}(M)) \longrightarrow H^1_{\mathfrak{a}}(N, S_i) \xrightarrow{g_1} H^0_{\mathfrak{a}}(N, S_{i+1}) \\
&\longrightarrow \operatorname{Ext}^2_R(N, H^i_{\mathfrak{a}}(M)) \longrightarrow H^2_{\mathfrak{a}}(N, S_i) \xrightarrow{g_2} H^1_{\mathfrak{a}}(N, S_{i+1}) \\
&\longrightarrow \ldots \\
&\longrightarrow \operatorname{Ext}^j_R(N, H^i_{\mathfrak{a}}(M)) \longrightarrow H^j_{\mathfrak{a}}(N, S_i) \xrightarrow{g_j} H^{j-1}_{\mathfrak{a}}(N, S_{i+1}) \\
&\longrightarrow \operatorname{Ext}^{j+1}_R(N, H^i_{\mathfrak{a}}(M)) \longrightarrow \ldots.
\end{aligned}$$

Now, let i be an arbitrary integer with $0 \leqslant i \leqslant n-1$. Then, by assumption (ii), there exists an exact sequence:

$$\begin{aligned}
0 \longrightarrow \operatorname{Ext}^{n-i}_R(N, H^i_{\mathfrak{a}}(M)) \longrightarrow H^{n-i}_{\mathfrak{a}}(N, S_i) &\xrightarrow{g_{n-i}} H^{n-i-1}_{\mathfrak{a}}(N, S_{i+1}) \\
&\longrightarrow \operatorname{Ext}^{n-i+1}_R(N, H^i_{\mathfrak{a}}(M)).
\end{aligned}$$

So, in view of the hypothesis in condition (i), it is routine to check that the minimal elements in $\operatorname{Supp}_R(H^{n-i}_{\mathfrak{a}}(N, S_i))$ are contained in the set

$$\operatorname{minSupp}_R(\operatorname{Ext}^{n-i}_R(N, H^i_{\mathfrak{a}}(M))) \cup \operatorname{minSupp}_R(H^{n-i-1}_{\mathfrak{a}}(N, S_{i+1})). \tag{3}$$

Thus, in view of assumption (iii) and (3), if $\operatorname{Supp}_R(H^{n-i-1}_{\mathfrak{a}}(N, S_{i+1}))$ is closed, then the support of $H^{n-i}_{\mathfrak{a}}(N, S_i)$ is also closed. So, by using the telescoping method, we need only to show that $\operatorname{Supp}_R(H^0_{\mathfrak{a}}(N, S_n))$ is closed. To achieve this, note that

$$H^0_{\mathfrak{a}}(N, S_n) \cong H^0_{\mathfrak{a}}(\operatorname{Hom}_R(N, S_n))$$

and $\operatorname{Hom}_R(N, S_n)$ is $\mathfrak{a}$-torsion. Hence

$$H^0_{\mathfrak{a}}(N, S_n) \cong \operatorname{Hom}_R(N, S_n) = \operatorname{Hom}_R(N, H^n_{\mathfrak{a}}(M)).$$

Since $\operatorname{minSupp}_R(H^n_{\mathfrak{a}}(M)) \subseteq \operatorname{Supp}_R N$, and the set $\operatorname{Supp}_R(H^n_{\mathfrak{a}}(M))$ is closed, the support of $H^0_{\mathfrak{a}}(N, S_n)$ is also closed by Lemma 2.1, as required. □

COROLLARY 2.4. *Let x_1, x_2 be an $\mathfrak{a}$-filter regular sequence on M, where $\mathfrak{a} := (x_1, x_2)$. Assume that*

(i) $\operatorname{Supp}_R(\operatorname{Ext}^{3-i}_R(N, H^i_{\mathfrak{a}}(M))) \subseteq \operatorname{Supp}_R(\operatorname{Ext}^{2-i}_R(N, H^i_{\mathfrak{a}}(M)))$ *for* $i = 0, 1$,

(ii) $H^0_{\mathfrak{a}}(N, H^1_{(x_1)}(M)) = 0$,

(iii) $\operatorname{Supp}_R(\operatorname{Ext}^1_R(N, H^1_{\mathfrak{a}}(M)))$ *is closed, and*

(iv) $\operatorname{minSupp}_R(H^2_{\mathfrak{a}}(M)) \subseteq \operatorname{Supp}_R N$.

Then $\operatorname{Supp}_R(H^2_{\mathfrak{a}}(N, M))$ *is closed.*

Proof. It immediately follows from [8, Theorem 1.2] and Theorem 2.3. □

Let $\mathcal{L}$ be a class of R-modules. We say that an R-module X is $\mathcal{L}$-*projective* if $\operatorname{Ext}^i_R(L, X) = 0$ for all $L \in \mathcal{L}$ and for all $i \in \mathbb{N}$ (see also [18]).

Similarly, we say that X is $\mathfrak{a}$-*projective* if $\operatorname{Ext}^i_R(T, X) = 0$ for every $\mathfrak{a}$-torsion module T and for all $i \in \mathbb{N}$. So we have the following corollary.

COROLLARY 2.5. *Let n be a non-negative integer and $x_1, \ldots, x_n$ be an $\mathfrak{a}$-filter regular sequence on M, where $\mathfrak{a} := (x_1, \ldots, x_n)$. Assume that*

(i) N *is* $\mathfrak{a}$*-projective,*

(ii) $H^{n-i-2}_{\mathfrak{a}}(N, H^{i+1}_{(x_1,\ldots,x_{i+1})}(M)) = 0$, *for all* $i = 0, 1, \ldots, n-2$,

(iii) $\operatorname{minSupp}_R(H^n_{\mathfrak{a}}(M)) \subseteq \operatorname{Supp}_R(N)$, *and*

(iv) *the set* $\operatorname{Supp}_R(H^n_{\mathfrak{a}}(M))$ *is closed.*

Then $\operatorname{Supp}_R(H^n_{\mathfrak{a}}(N, M))$ *is closed.*

3. Support of the Matlis dual of generalized local cohomology modules.

Let $\sum_R$ denote the direct sum

$$\bigoplus_{\mathfrak{m} \in \operatorname{MaxSpec}(R)} R/\mathfrak{m}$$

of all simple R-modules, E_R be the injective hull of $\sum_R$, and $D(-)$ be the functor $\operatorname{Hom}_R(-, E_R)$.

Note that $D(-)$ is a natural generalization of the Matlis duality functor to non-local rings.

Recall that the arithmetic rank of $\mathfrak{a}$, denoted by $\operatorname{ara}(\mathfrak{a})$, is the least number of elements of R required to generate an ideal which has the same radical as $\mathfrak{a}$.

PROPOSITION 3.1. *For any ideal $\mathfrak{a}$ of R,* $\operatorname{Hom}_R(R/\mathfrak{a}, D(H^n_{\mathfrak{a}}(M))) = 0$, *where* $n = \operatorname{ara}(\mathfrak{a})$.

Proof. Since $n = \operatorname{ara}(\mathfrak{a})$, there exists a sequence $y_1, \ldots, y_n$ of elements of R such that $\sqrt{\mathfrak{a}} = \sqrt{(y_1, \ldots, y_n)}$. Hence there exists $t \in \mathbb{N}$ such that $y^t_i \in \mathfrak{a}$ for all $1 \leqslant i \leqslant n$. Clearly $V(\mathfrak{a}) = V((y^t_1, \ldots, y^t_n))$. Also, by [28, Proposition 1.2], there exists an $(y^t_1, \ldots, y^t_n)$-filter regular sequence $x_1, \ldots, x_n$ on M such that $H^n_{(y^t_1,\ldots,y^t_n)}(M) \cong H^n_{(x_1,\ldots,x_n)}(M)$. It is easy to see that $x_1, \ldots, x_n$ is also an $\mathfrak{a}$-filter regular sequence on M. Thus $H^n_{\mathfrak{a}}(M) \cong H^n_{(x_1,\ldots,x_n)}(M)$. Now $\operatorname{Hom}_R(R/\mathfrak{a}, D(H^n_{\mathfrak{a}}(M))) = 0$ by [13, Lemma 3.2(i)]. □

In [12], it was shown that, for an $\mathfrak{a}$-filter regular sequence $x_1, \ldots, x_n$ on M,

$$\operatorname{Supp}_R(H^n_{\mathfrak{a}}(M)) = \operatorname{Supp}_R(\operatorname{Hom}_R(R/\mathfrak{a}, H^n_{(x_1,\ldots,x_n)}(M))).$$

Moreover, in view of Proposition 3.1, $\operatorname{Hom}_R(R/\mathfrak{a}, D(H^n_{\mathfrak{a}}(M))) = 0$, where $n = \operatorname{ara}(\mathfrak{a})$. In this section we study the support of $D(H^n_{\mathfrak{a}}(N, M))$ which is a dual of question 1.1 in [8] ‘in some sense’ in the context of the generalized local cohomology modules.

THEOREM 3.2. *Let n be a non-negative integer and $x_1, \ldots, x_n$ be an $\mathfrak{a}$-filter regular sequence on M, where $\mathfrak{a} := (x_1, \ldots, x_n)$. Assume that*

(i) $\operatorname{Supp}_R(D(H^{n-2-i}_{\mathfrak{a}}(N, H^i_{(x_1,\ldots,x_i)}(M))))$
$\subseteq \operatorname{Supp}_R(D(H^{n-1-i}_{\mathfrak{a}}(N, H^i_{(x_1,\ldots,x_i)}(M))))$ *for all* $i = 0, 1, \ldots, n-2$,

(ii) $\operatorname{Supp}_R(D(\operatorname{Ext}^{n-i}_R(N, H^i_{\mathfrak{a}}(M))))$ *is closed for all* $i = 0, 1, \ldots, n-1$,

(iii) $\operatorname{Ext}^{n+1-i}_R(N, H^i_{\mathfrak{a}}(M)) = 0$ *for all* $i = 0, 1, \ldots, n-1$, *and*

(iv) the set $\operatorname{Supp}_R(N \otimes_R D(H^n_{(x_1,\ldots,x_n)}(M)))$ *is closed.*

Then $\operatorname{Supp}_R(D(H^n_{\mathfrak{a}}(N, M)))$ *is closed.*

Proof. By using the method which we employed in the proof of Theorem 2.3 for $i = 0, 1, \ldots, n-1$, we have the following exact sequence

$$\begin{aligned}
\ldots &\longrightarrow D(H^{j-1}_{\mathfrak{a}}(N, S_{i+1})) \longrightarrow D(H^j_{\mathfrak{a}}(N, S_i)) \longrightarrow D(\operatorname{Ext}^j_R(N, H^i_{\mathfrak{a}}(M)))\\
&\longrightarrow D(H^{j-2}_{\mathfrak{a}}(N, S_{i+1})) \longrightarrow \ldots\\
&\longrightarrow D(H^1_{\mathfrak{a}}(N, S_{i+1})) \longrightarrow D(H^2_{\mathfrak{a}}(N, S_i)) \longrightarrow D(\operatorname{Ext}^2_R(N, H^i_{\mathfrak{a}}(M)))\\
&\longrightarrow D(H^0_{\mathfrak{a}}(N, S_{i+1})) \longrightarrow D(H^1_{\mathfrak{a}}(N, S_i)) \longrightarrow D(\operatorname{Ext}^1_R(N, H^i_{\mathfrak{a}}(M))) \longrightarrow 0.
\end{aligned}$$

Thus, in view of the hypothesis in conditions (i), (ii) and (iii), we have that the minimal element in $\operatorname{Supp}_R(D(H^{n-i}_{\mathfrak{a}}(N, S_i)))$ is a subset of

$$\operatorname{minSupp}_R(D(H^{n-i-1}_{\mathfrak{a}}(N, S_{i+1}))) \cup \operatorname{minSupp}_R(D(\operatorname{Ext}^{n-i}_R(N, H^i_{\mathfrak{a}}(M))))$$

for all $i = 0, 1, \ldots, n-1$. Hence we need only to show that $\operatorname{Supp}_R(D(H^0_{\mathfrak{a}}(N, S_n)))$ is closed. To do this, note that

$$\begin{aligned}
D(H^0_{\mathfrak{a}}(N, S_n)) &\cong D(H^0_{\mathfrak{a}}(\operatorname{Hom}_R(N, S_n)))\\
&\cong D(H^0_{\mathfrak{a}}(\operatorname{Hom}_R(N, H^n_{\mathfrak{a}}(M))))\\
&\cong D(\operatorname{Hom}_R(N, H^n_{\mathfrak{a}}(M)))\\
&\cong N \otimes_R D(H^n_{\mathfrak{a}}(M)).
\end{aligned}$$

The result now follows from (iv). □

COROLLARY 3.3. *Let x_1, x_2 be an $\mathfrak{a}$-filter regular sequence on M, where $\mathfrak{a} := (x_1, x_2)$. Assume that*

(i) $\mathrm{Supp}_R(D(H^0_{\mathfrak{a}}(N, M))) \subseteq \mathrm{Supp}_R(D(H^1_{\mathfrak{a}}(N, M)))$,

(ii) $\mathrm{Supp}_R(D(\mathrm{Ext}^1_R(N, H^1_{\mathfrak{a}}(M))))$ *is closed,*

(iii) $\mathrm{Ext}^{3-i}_R(N, H^i_{\mathfrak{a}}(M)) = 0$ *for all* $i = 0, 1$, *and*

(iv) the set $\mathrm{Supp}_R(N \otimes_R D(H^2_{(x_1,x_2)}(M)))$ *is closed.*

Then $\mathrm{Supp}_R(D(H^2_{\mathfrak{a}}(N, M)))$ *is closed.*

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REFERENCES

1. J. ASADOLLAHI, K. KHASHYARMANESH AND S. SALARIAN, On the finitenes properties of generalized local cohomology Modules, *Comm. Algebra* **30**(2) (2002), 859–867.
2. M. BRODMANN AND R.Y. SHARP, *Local cohomology – an algebraic introduction with geometric applications,* Cambridge studies in advanced mathematics No. 60, Cambridge University Press, 1998.
3. A. GROTHENDIECK, *Cohomologie locale des faisceaux cohérents et théorèmes de Leftschetz locayx et globaux (SGA 2)* North-Holland Publishing Co., Amsterdam, 1968.
4. M. HELLUS, On the set of associated primes of local cohomology modules, *J. Algebra* **237** (2001), 406–419.
5. J. HERZOG, *Komplex Auflösungen und dualiti in der lokalen Algebra,* Habilitationsschrift, Universität Regensburg, Germany, 1974.
6. J. HERZOG AND N. ZAMANI, Duality and vanishing of generalized local cohomology, *Arch. Math.* (Basel) **81**(5) (2003), 512–519.
7. C.L. HUNEKE, *Problems on local cohomology. Free resolutions in commutative algebra and algebraic geometry,* Res. Notes Math., Vol. 2, pp. 93–108, Jones and Bartlett, Boston, MA, 1992.
8. C. HUNEKE, D. KATZ AND T. MARLEY, On the support of local cohomology, *Journal of Algebra,* to appear.
9. C. HUNEKE AND R.Y. SHARP, Bass numbers of local cohomology modules, *Trans. Amer. Math. Soc.* **339**(2) (1993), 765–774.
10. M. KATZMAN, An example of infinite set of associated primes of local cohomology module, *J. Algebra* **252**(1) (2002), 161–166.
11. K. KHASHYARMANESH, On the Matlis duals of local cohomology modules, *Arch. Math.* (Basel) **88** (2007), 413–418.

12. ________________, On the support of local cohomology modules and filter regular sequences, *Journal of Commutative Algebra* **2**(2) (2010), 177–186.

13. K. KHASHYARMANESH AND F. KHOSH-AHANG On the finiteness properties of Matlis duals of local cohomology modules, *Proc. Indian. Sci. (Math. Sci.)* **118**(2) (2008), 197–206.

14. K. KHASHYARMANESH, M. YASSI AND A. ABBASI, Filter regular sequences and generalized local cohomology modules, *Comm. Algebra* **32**(1) (2004), 253–259.

15. G. LYUBEZNIK, Finiteness properties of local cohomology modules (an application of D-modules to commutative algebra), *Invent. Math.* **113**(1) (1993), 41–55.

16. ________________, Finiteness properties of local cohomology modules for regular local rings of mixed characteristic: the unramified case, *Comm. Algebra* **28**(12) (2000), 5867–5882.

17. A. MAFI, On the finitenes results of the generalized local cohomology Modules, *Algebra Colloq.* **16**(2) (2009), 325–332.

18. L. MAO AND N. DING, $\mathcal{L}$-injective hulls of modules, *Bull. Austral. Math. Soc.* **74** (2006), 37–44.

19. T. MARLEY, The associated primes of local cohomology modules over rings of small dimension, *Manuscripta Mathematica* **104** (2001), 519–525.

20. E. MATLIS, Injective nodules over Noetherian rings, *Pasific J. Math.* **8** (1958), 511–528.

21. A. OOISHI, Matlis duality and the width of a module, *Hiroshima Math. J.* **6**(3) (1976), 573–587.

22. A.S. RICHARDSON, Co-localization, co-support and local homology, *Rocky Mountain J. Math.* **36**(5) (2006), 1679–1703.

23. P. SCHENZEL, N.V. TRUNG AND N.T CUONG, Verallgemeinerte Cohen-Macaulay-Moduln, *Math. Nachr.* **85** (1978), 57–73.

24. A. SINGH, p-torsion elements in local cohomology modules, *Math. Res. Lett.* **7**(2–3) (2000), 165–176.

25. A. SINGH AND I. SWANSON, Associated primes of local cohomology modules and of Frobenius powers, *Int. Math. Res. Not.* **33** (2004), 1703–1733.

26. J. STÜCKRAD AND W. VOGEL, *Buchsbaum rings and applications,* VEB Deutscher Verlag der Wissenschaftan, Berlin, 1986.

27. N. SUZUKI, On the generalized local cohomology and its duality, *J. Math. Kyoto Univ.* **18** (1978), 71–85.

28. R. TAJARROD AND H. ZAKERI, Comparison of certain complexes of modules of generalized fractions and Čech complexes, *Comm. Algebra* **35**(12) (2007), 4032-4041.

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