Minimum number of distinct eigenvalues of graphs

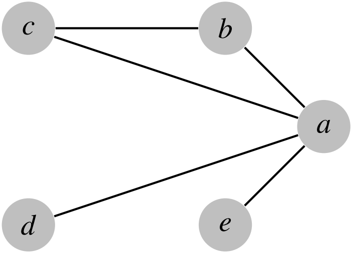
Shahla Nasserasr

Nova Southeastern University
November 15, 2016

joint work with Dr. Shaun Fallat and Dr. Karen Meagher (University of Regina)

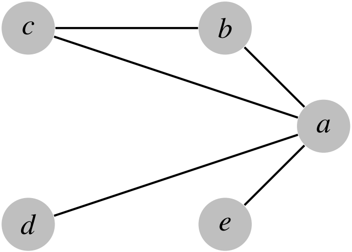
Introduction

- A *graph* is an ordered pair $G = (V, E)$, where
  $V = \{v_1, v_2, \ldots, v_n\}$ is the set of vertices,
  E is the set of edges, and
  every edge is a 2 element subset of V.
- *Example:*



$$V = \{a, b, c, d, e\}, E = \{\{a, b\}, \{a, c\}, \{a, d\}, \{a, e\}, \{b, c\}\}$$

- The *Adjacency matrix* of the graph G is an $n \times n$ matrix $A = [a_{ij}]$ where $a_{ij} = 1$ if and only if $\{i, j\}$ is an edge, and $a_{ij} = 0$ otherwise.
- *Example:*



G

	a	b	c	d	e
a	0	1	1	1	1
b	1	0	1	0	0
c	1	1	0	0	0
d	1	0	0	0	0
e	1	0	0	0	0

the adjacency matrix of G

- A is symmetric.

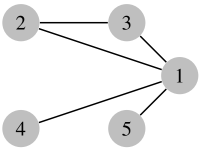
Compatible Matrices with G

An $n \times n$ symmetric matrix $A = [a_{ij}]$ is called *Compatible* with a graph G if

$$a_{ij} = \begin{cases} \text{nonzero}, & \text{if } ij \text{ is an edge} \\ \text{zero}, & \text{if } ij \text{ is not an edge} \end{cases}$$

- $S(G)$ is the set of all compatible matrices with G.

Example:



G

	1	2	3	4	5
1	a_{11}	a_{12}	a_{13}	a_{14}	a_{15}
2	a_{12}	a_{22}	a_{23}	0	0
3	a_{13}	a_{23}	a_{33}	0	0
4	a_{14}	0	0	a_{44}	0
5	a_{15}	0	0	0	a_{55}

a compatible matrix with G

Minimum Number of Distinct Eigenvalues of graphs

- If A is an $n \times n$ matrix, $q(A)$ denotes the number of distinct eigenvalues of A.
- *Example:*

$A=$

0	1	-1	1	1
1	1	1	0	0
-1	1	1	0	0
1	0	0	0	0
1	0	0	0	0

$\sigma(A) = \{-2, 0, 0, 2, 2\} \Rightarrow q(A) = 3$

- The *Minimum Number of Distinct Eigenvalues* of a graph G, denoted by $q(G)$, is

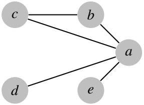
$$q(G) = \min\{q(A) \mid A \in S(G)\}.$$

The Minimum Rank Problem

- The *Minimum rank* of a graph G, denoted by $\mathrm{mr}(G)$, is

$$\mathrm{mr}(G) = \min\{\mathrm{rank}(A) \mid A \in S(G)\}.$$

- *Example:*



M =

a_{11}	a_{12}	a_{13}	a_{14}	a_{15}
a_{12}	a_{22}	a_{23}	0	0
a_{13}	a_{23}	a_{33}	0	0
a_{14}	0	0	a_{44}	0
a_{15}	0	0	0	a_{55}

$\Rightarrow \mathrm{rank}(G) \geq 3,$ A=

0	1	-1	1	1
1	1	1	0	0
-1	1	1	0	0
1	0	0	0	0
1	0	0	0	0

$\mathrm{rank}(A) = 3 \Rightarrow \mathrm{mr}(G) = 3.$

$q(G)$: A Lower Bound for $\mathrm{mr}(G) + 1$

Lemma

For any graph G,

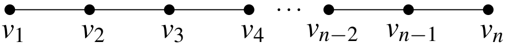
$$q(G) \leq \mathrm{mr}(G) + 1.$$

Proof. If $q(A) = q(G)$, then

$$A \sim \begin{bmatrix} \lambda_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_{\mathrm{mr}(G)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Paths

A *Path* on n vertices, denoted by P_n, is a graph with vertices $v_1, v_2, \ldots, v_n$ and edge set $\{\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{n-1}, v_n\}\}$.



Unique Paths

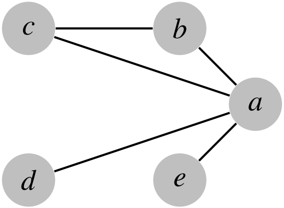
Theorem (Fonseca, 2010)

If there are vertices u, v in G at distance d and the path of length d from u to v is unique, then $q(G) \geq d+1$.

Proof.

- For $A \in S(G)$, the entry (u,v) of $A, A^2, \ldots, A^{d-1}$ equal 0
- The entry (u,v) of A^d is $\prod_{i=1}^{d} a_{v_i v_{i+1}} \neq 0$
- The matrices $I, A, A^2, \ldots, A^d$ are linearly independent
- The minimal polynomial of A has degree at least $d+1$
- $q(G) \geq d+1$.

Example:

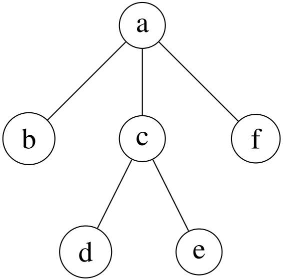


G

- $d(e, c) = 2,$
- the path from e to c of length 2 is unique
- $q(G) \geq 2 + 1 = 3$
- Note: In this case $q(G) = 3$

Corollary

For any tree T it is the case that $q(T) \geq \mathrm{diam}(T) + 1$.

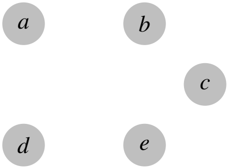


$$q(T) \geq diam(T) + 1 = 3 + 1 = 4$$

Graphs with q(G)=1

Lemma

For a graph G, $q(G) = 1$ if and only if G is an empty graph.



G

A=

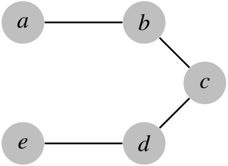
0	0	0	0	0
0	0	0	0	0
0	0	0	0	0
0	0	0	0	0
0	0	0	0	0

$$q(A) = q(G) = 1$$

Graphs with $q(G) = |V(G)|$

Lemma (Fiedler, 1969)

For a graph G on n vertices, $q(G) = n$ if and only if $G = P_n$, a path on n vertices.



P_5

A=

*	a_{12}	0	0	0
a_{12}	*	a_{23}	0	0
0	a_{23}	*	a_{34}	0
0	0	a_{34}	*	a_{45}
0	0	0	a_{45}	*

$$q(A) = q(G) = 5$$

Open Problem

Problem

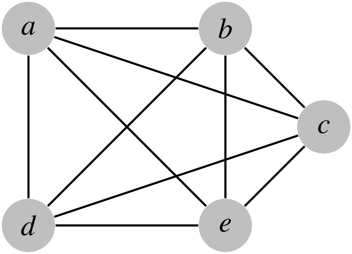
For a given k, characterize graphs G with $q(G) = k$.

The answer is known for $k = 1$ and $k = |V(G)|$. What about $k = 2$?

Complete Graphs

- A *Complete graph* is a graph, where there is an edge between every pairs of vertices.

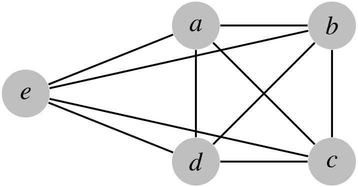
- *Example:* A complete graph on 5 vertices (K_5):



Graphs with $q(G) = 2$

Theorem

For any $n \geq 2$, we have $q(K_n) = 2$



K_5

0	1	1	1	1
1	0	1	1	1
1	1	0	1	1
1	1	1	0	1
1	1	1	1	0

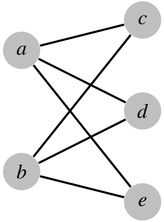
$q(K_5) = 2$

Theorem

If G is obtained from K_n by deleting a single edge, then $q(G) = 2$.

Graphs with $q(G) = 2$

- A *Bipartite graph* is a graph, where the vertices are partitioned into two sets X and Y, and the edges only connect a vertex from X to a vertex in Y (there are no edges from a vertex X to a vertex in X).
- *Example:* A complete bipartite graph $K_{2,3}$:



Lemma

For any $1 \leq m \leq n$ *we have*

$$q(K_{m,n}) = \begin{cases} 2, & \text{if } m = n; \\ 3, & \text{if } m < n. \end{cases}$$

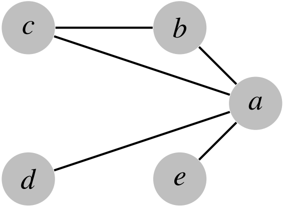
If $q(G) = 2, \ldots$

Theorem

If $q(G) = 2$, then for any independent set $\{v_1, v_2, \ldots, v_k\}$ of vertices

$$\left| \bigcup_{i \neq j} (N(v_i) \cap N(v_j)) \right| \geq k.$$

Example:



$\{e, c\}$ is an independent set

$|N(e) \cap N(c)| = |\{a\}| \not\geq 2$

$q(G) \neq 2.$

If $q(G) = 2, \ldots$

Corollary

In a graph G with $q(G) = 2$, any two non-adjacent vertices have at least two common neighbours.

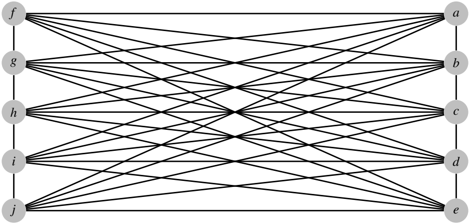
Corollary

In a graph G with $q(G) = 2$, every vertex lies in a cycle.

$q(G \vee G) = 2$

Theorem

Let G be a connected graph, then $q(G \vee G) = 2$.



$\mathrm{q}(P_5 \vee P_5) = 2$

Bipartition

Definition

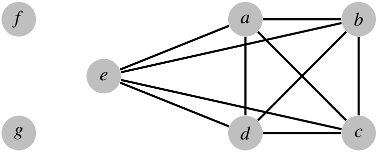
A graph G has *multiplicity bipartition* $[k, n-k]$ if it has two distinct eigenvalues where one eigenvalue has multiplicity k.

Subproblem: Given $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$, which graphs have multiplicity bipartition $[k, n-k]$.

Bipartition $[1, n-1]$

Lemma

A graph G has a multiplicity bipartition $[1, n-1]$ if and only if G is a complete graphs with isolated vertices.



$q(G) = 2$ with multiplicity bipartition $[1, n-1]$

Bipartition $[2, n-2]$

Theorem

Assume that G is a connected graph. The minimal multiplicity bipartition of G is $[2, n-2]$ if and only if

$$G = (K_{a_1} \cup K_{b_1}) \vee (K_{a_2} \cup K_{b_2}) \vee \ldots \vee (K_{a_k} \cup K_{b_k})$$

where $k > 1$, $a_i \geq 0, b_i \geq 0$, for $i = 1, 2, \ldots, k$, and G is not isomorphic to a complete graph, or to $(K_{a_1} \cup K_{b_1}) \vee K_1$.

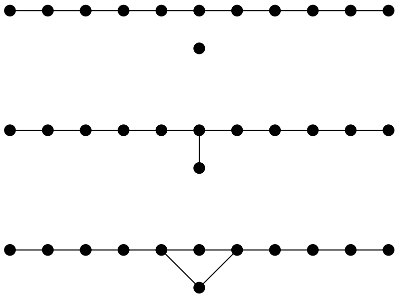
Graphs with multiplicity bipartition $[2, n-2]$ are characterized in several separate papers.

Graphs with $q(G) = n - 1$

Theorem (Barrett et al)

A graph G has $q(G) = |G| - 1$ *if and only if G is of the following form:*

- *the union of a path and an isolated vertex,*
- *a path with one leaf attached to an interior vertex,*
- *a path with an extra edge joining two vertices of distance 2.*



open problems

- Characterize graphs with $q(G) = k$ for $k = 2, 3, \ldots, n-2$.
- Find forbidden subgraphs for graphs with multiplicity bipartition $[k, n-k]$.
- For given graphs G and H, find $q(G \vee H)$.
- Find the relationship between $q(G)$ and other parameters of G such as independence number, degree of vertices, etc.

Thank You!