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Differential equation of Appell polynomials via the factorization method

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Abstract

Let $\{P_n(x)\}_{n=0}^{\infty}$ be a sequence of polynomials of degree n. We define two sequences of differential operators Φ_n and Ψ_n satisfying the following properties:

$$\Phi_n(P_n(x)) = P_{n-1}(x),$$

$$\Psi_n(P_n(x)) = P_{n+1}(x).$$

By constructing these two operators for Appell polynomials, we determine their differential equations via the factorization method introduced by Infeld and Hull (Rev. Mod. Phys. 23 (1951) 21). The differential equations for both Bernoulli and Euler polynomials are given as special cases of the Appell polynomials. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

Let $\{P_n(x)\}_{n=0}^{\infty}$ be a sequence of polynomials of degree n. For $n = 0, 1, 2, \ldots,$ we define two sequences of differential operators Φ_n and Ψ_n, satisfying the following properties:

$$\Phi_n(P_n(x)) = P_{n-1}(x),$$

$$\Psi_n(P_n(x)) = P_{n+1}(x).$$

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Φ_n and Ψ_n play the role analogous to that of derivative and multiplicative operators, respectively, on monomials. The monomiality principle and the associated operational rules were used in [4] to explore new classes of isospectral problems leading to nontrivial generalizations of special functions. Most properties of the families of polynomials associated with these two operators can be deduced using operator rules with the Φ_n and Ψ_n operators. The operators we defined in this paper are varying with the degrees of polynomials n. The iterations of Φ_n and Ψ_n to $P_n(x)$ give us the following relations:

$$(\Phi_{n+1}\Psi_n)P_n(x) = P_n(x),$$

$$(\Psi_{n-1}\Phi_n)P_n(x) = P_n(x),$$

$$(\Phi_1\Phi_2\cdots\Phi_{n-1}\Phi_n)P_n(x) = P_0(x),$$

$$(\Psi_{n-1}\Psi_{n-2}\cdots\Psi_1\Psi_0)P_0(x) = P_n(x).$$

These operational relations allow us to derive a higher order differential equation satisfied by some special polynomials. The classical factorization method introduced in [8] was used to study the second-order differential equation.

In this paper, we construct Φ_n and Ψ_n for Appell polynomials $R_n(x)$. We then derive the corresponding differential equations by the factorization approach. As special cases of Appell polynomials, we also provide the differential equations for both Bernoulli polynomials $B_n(x)$ and Euler polynomials.

We briefly recall some of the properties of these polynomials. The Bernoulli polynomials $B_n(x)$ are usually defined (see e.g. [7, p. xxix]) starting from the generating function:

$$G(x,t):=\frac{te^{xt}}{e^t-1} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}, \quad |t| < 2\pi, \tag{1.1}$$

and consequently, the Bernoulli numbers $B_n:=B_n(0)$ can be obtained by the generating function:

$$\frac{t}{e^t-1} = \sum_{n=0}^{\infty} B_n\frac{t^n}{n!}.$$

The B_n are rational numbers. In particular, $B_0 = 1$, $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$, and $B_{2k+1} = 0$ for $k = 1,2,\ldots$.

$$B_0(x) = 1, \quad B_1(x) = x - \tfrac{1}{2}, \quad B_2(x) = x^2 - x + \tfrac{1}{6}.$$

The following properties are well known,

- $B_n(0) = B_n(1) = B_n,\ n \neq 1,$
- $B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k},$
- $B_n'(x) = nB_{n-1}(x).$

The Euler numbers E_n can be defined by the generating function

$$\frac{2}{e^t + e^{-t}} = \sum_{n=0}^{\infty} \frac{E_n}{n!}t^n.$$

The Euler polynomials $E_n(x)$ can be defined by the generating function

$$G_E(x,t) = \frac{2e^{xt}}{e^t + 1} = \sum_{n=0}^{\infty} \frac{E_n(x)}{n!} t^n. \tag{1.2}$$

The connection to the Euler numbers is given by

$$E_n\left(\frac{1}{2}\right) = 2^{-n} E_n.$$

The derivatives of $E_n(x)$ satisfy

$$E_n'(x) = nE_{n-1}(x).$$

The Bernoulli numbers (see [3–10]) enter in many mathematical formulas, such as the Taylor expansion in a neighborhood of the origin of the trigonometric and hyperbolic tangent and cotangent functions, the sums of powers of natural numbers, the residual term of the Euler–MacLaurin quadrature formula.

The Bernoulli polynomials, first studied by Euler (see [6–9,2]), are employed in the integral representation of differentiable periodic functions, and play an important role in the approximation of such functions by means of polynomials. They are also used in the remainder term of the composite Euler–MacLaurin quadrature formula (see [11]).

The Euler polynomials are strictly connected with the Bernoulli ones, and enter in the Taylor expansion in a neighborhood of the origin of the trigonometric and hyperbolic secant functions.

A recursive computation of the Bernoulli and Euler polynomials can be obtained by using the following formulas:

$$\sum_{k=0}^{n-1} \binom{n}{k} B_k(x) = nx^{n-1}, \quad n = 2, 3, \ldots,$$

$$E_n(x) + \sum_{k=0}^{n} \binom{n}{k} E_k(x) = 2x^n, \quad n = 1, 2, \ldots .$$

Some recurrent properties of the Bernoulli polynomials in terms of the Euler polynomials are also known, (see [7, p. xxix]).

The Appell polynomials [1] can be defined by considering the following generating function:

$$G_R(x,t) = A(t)e^{xt} = \sum_{n=0}^{\infty} \frac{R_n(x)}{n!} t^n, \tag{1.3}$$

where

$$A(t) = \sum_{k=0}^{\infty} \frac{R_k}{k!} t^k, \quad (A(0) \neq 0) \tag{1.4}$$

is analytic function at $t = 0$, and $R_k := R_k(0)$.

It is easy to see that

- If $A(t) = \frac{t}{e^t - 1}$, then $R_n(t) = B_n(t)$,
- If $A(t) = \frac{2}{e^t + 1}$, then $R_n(t) = E_n(t)$.

- If $A(t)=\alpha_1\cdots\alpha_m t^m[(e^{\alpha_1 t}-1)\cdots(e^{\alpha_m t}-1)]^{-1}$, then $R_n(t)$ is the Bernoulli polynomials of order m [2].
- If $A(t)=2^m[(e^{\alpha_1 t}+1)\cdots(e^{\alpha_m t}+1)]^{-1}$, then $R_n(t)$ is the Euler polynomials of order m [2].
- If $A(t)=e^{\xi_0+\xi_1 t+\cdots+\xi_{d+1}t^{d+1}}$, $\xi_{d+1}\neq 0$, then $R_n(t)$ is the generalized Gould–Hopper polynomials [5] including the Hermite polynomials when $d=1$ and d-orthogonal polynomials for each positive integer d.

These three polynomials have important applications in the theory of finite differences, in number theory and in classical analysis. These polynomials are closely related to corresponding sequences of numbers, namely the Bernoulli, Euler and generalized Bernoulli numbers. The differential equations satisfied by $R_n(x)$, $B_n(x)$ and $E_n(x)$ will be presented in the following section.

2. Differential equation for Appell polynomials $R_n(x)$

In this section, we derive a differential equation for the Appell polynomials $R_n(x)$ and give the recurrence relations and differential equations for the Bernoulli and Euler polynomials as special cases of the Appell polynomials.

Theorem 2.1. *The Appell polynomials $R_n(x)$ defined in Section* 1 *satisfy the differential equation*:

$$\frac{\alpha_{n-1}}{(n-1)!}y^{(n)}+\frac{\alpha_{n-2}}{(n-2)!}y^{(n-1)}+\cdots+\frac{\alpha_1}{1!}y''+(x+\alpha_0)y'-ny=0, \tag{2.1}$$

where the numerical coefficients α_k, $k=1,2,\ldots,n-1$ are defined in (2.3) *below, and are linked to the values R_k by the following relations*:

$$R_{k+1}=\sum_{h=0}^{k}\binom{k}{h}R_h\alpha_{k-h}.$$

Proof. Differentiating generation equation (1.3) with respect to x and equating coefficients of t^n, we obtain

$$R_n'(x)=nR_{n-1}(x).$$

As we have noticed in Section 1, the operator $\Phi_n=(1/n)D_x$ satisfies the following operational relation:

$$\Phi_n R_n(x)=R_{n-1}(x).$$

Next we find the operator Ψ_n such that

$$\Psi_n R_n(x)=R_{n+1}(x).$$

Differentiating generation equation (1.3) with respect to t,

$$\frac{\partial G_R(x,t)}{\partial t}=G_R(x,t)\left[\frac{A'(t)}{A(t)}+x\right]. \tag{2.2}$$

Next we assume that

$$\frac{A'(t)}{A(t)} = \sum_{n=0}^{\infty} \alpha_n \frac{t^n}{n!}. \tag{2.3}$$

Equating coefficients of t^n, in Eq. (2.2), we obtain

$$R_{n+1}(x) = (x + \alpha_0)R_n(x) + \sum_{k=0}^{n-1} \binom{n}{k} \alpha_{n-k} R_k(x). \tag{2.4}$$

This relation, starting from $n = 1$, and taking into account the initial value $R_0(x) = 1$, allows a recursive formula for the Generalized Bernoulli and Euler polynomials. We now use this recurrence relation to find the operator Ψ_n. It is easy to see that for $k = 0, 1, \ldots, n-1$,

$$R_k(x) = [\Phi_{k+1}\Phi_{k+2}\cdots\Phi_{n-1}\Phi_n]R_n(x) = \left(\prod_{j=1}^{n-k}\Phi_{k+j}\right) R_n(x).$$

The recurrence relation can be written as

$$\begin{aligned} R_{n+1}(x) &= (x + \alpha_0)R_n(x) + \sum_{k=0}^{n-1} \binom{n}{k} \alpha_{n-k} \left(\prod_{j=1}^{n-k}\Phi_{k+j}\right) R_n(x) \\ &= \left[(x + \alpha_0) + \sum_{k=0}^{n-1} \binom{n}{k} \alpha_{n-k} \left(\prod_{j=1}^{n-k}\Phi_{k+j}\right)\right] R_n(x) \\ &= \left[(x + \alpha_0) + \sum_{k=0}^{n-1} \frac{\alpha_{n-k}}{(n-k)!} D_x^{n-k}\right] R_n(x) \\ &= \Psi_n R_n(x), \end{aligned}$$

where

$$\Psi_n = (x + \alpha_0) + \sum_{k=0}^{n-1} \frac{\alpha_{n-k}}{(n-k)!} D_x^{n-k}$$

is the operator defined in Section 1. We now determine the differential equation for $R_n(x)$.

Applying both operators Φ_{n+1} and Ψ_n to $R_n(x)$, we have

$$(\Phi_{n+1}\Psi_n)R_n(x) = R_n(x).$$

That is,

$$\frac{1}{n+1} D_x \left[(x + \alpha_0) + \sum_{k=0}^{n-1} \frac{\alpha_{n-k}}{(n-k)!} D_x^{n-k}\right] R_n(x) = R_n(x).$$

This leads to the differential equation with $R_n(x)$ as a polynomial solution.

For Bernoulli polynomials, we have the following results.

Theorem 2.2. *For any integral $n \geqslant 1$, the following linear homogeneous recurrence relation for the Bernoulli polynomials defined in Section* 1 *holds true*:

$$B_n(x) = \left(x - \frac{1}{2}\right) B_{n-1}(x) - \frac{1}{n}\sum_{k=0}^{n-2} \binom{n}{k} B_{n-k}B_k(x). \tag{2.5}$$

Theorem 2.3. *The Bernoulli polynomials $B_n(x)$ defined in Section* 1 *satisfy the differential equation*

$$\frac{B_{n+1}}{n+1}y^{(n+1)} + \frac{B_n}{n}y^{(n)} + \frac{B_{n-1}}{n-1}y^{(n-1)} + \cdots + \frac{B_2}{2}y'' + \left(\frac{1}{2} - x\right) y' + ny = 0. \tag{2.6}$$

For Euler polynomials, we have the following results.

Theorem 2.4. *For any integral $n \geqslant 1$, the following linear homogeneous recurrence relation for the Euler polynomials defined in Section* 1 *holds true*:

$$E_{n+1}(x) = \left(x - \frac{1}{2}\right) E_n(x) + \sum_{k=0}^{n-1} \binom{n}{k} e_{n-k}E_k(x). \tag{2.7}$$

Theorem 2.5. *The Euler polynomials $E_n(x)$ defined in Section* 1 *satisfy the differential equation*

$$\frac{e_{n-1}}{(n-1)!}y^{(n)} + \frac{e_{n-2}}{(n-2)!}y^{(n-1)} + \cdots + \frac{e_1}{1!}y'' + \left(x - \frac{1}{2}\right) y' - ny = 0, \tag{2.8}$$

where the numerical coefficients e_k, $k = 1, 2, \ldots, n-1$, are linked to the Euler numbers E_k by the following relations:

$$e_k = -\frac{1}{2^k}\sum_{h=0}^{k} \binom{k}{h} E_{k-h}.$$

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