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ON QUADRATURE RULES ASSOCIATED WITH APPELL POLYNOMIALS

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Abstract: A quadrature rule using Appell polynomials and generalizing both the Euler-MacLaurin quadrature formula and a similar quadrature rule, obtained in Bretti et al [15], which makes use of Euler (instead of Bernoulli) numbers and even (instead of odd) derivatives of the given function at the extrema of the considered interval, is derived. An expression of the remainder term and a numerical example are also enclosed.

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1. Introduction

The Bernoulli polynomials $B_n(x)$ are usually defined (see e.g. Gradshteyn et al [1], p. xxix) starting from the generating function:

$$G(x,t) := \frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x)\,\frac{t^n}{n!}, \qquad |t| < 2\pi, \tag{1.1}$$

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and consequently, the Bernoulli numbers $B_n := B_n(0)$ can be obtained by the generating function:

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}. \tag{1.2}$$

The Bernoulli numbers (see Bernoulli [2] and Saalschuetz [3]) enter in many mathematical formulas, such as the Taylor expansion in a neighborhood of the origin of the trigonometric and hyperbolic tangent and cotangent functions, the sums of powers of natural numbers, the residual term of the Euler-MacLaurin quadrature formula (see Stoer [4]).

The Bernoulli polynomials, first studied by Euler (see Euler [5], Nörlund [6] and Bateman et al [7]), are employed in the integral representation of differentiable periodic functions, and play an important role in the approximation of such functions by means of polynomials.

The Euler polynomials $E_n(x)$ are defined by the generating function:

$$\frac{2e^{xt}}{e^t + 1} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}, \qquad |t| < \pi. \tag{1.3}$$

The Euler numbers $\mathcal{E}_n$ can be obtained by the generating function:

$$\frac{2}{e^t + e^{-t}} = \sum_{n=0}^{\infty} \frac{\mathcal{E}_n}{n!} t^n, \tag{1.4}$$

and the connection between Euler numbers and Euler polynomials is given by

$$E_n\left(\frac{1}{2}\right) = 2^{-n}\mathcal{E}_n, \qquad n = 0, 1, 2, \dots .$$

The first Euler numbers are given by

$$\mathcal{E}_0 = 1,\ \mathcal{E}_1 = 0,\ \mathcal{E}_2 = -1,\ \mathcal{E}_3 = 0,\ \mathcal{E}_4 = 5,\ \mathcal{E}_5 = 0,\ \mathcal{E}_6 = -61,\ \mathcal{E}_7 = 0,\ \dots .$$

For further values see Abramowitz et al [8], p. 810.

The Euler polynomials satisfy the symmetry relation

$$E_m(1-t) = (-1)^m E_m(t).$$

Since $E_m(0) = E_m(1) = 0$, for m even, we put, if m is odd:

$$e_m = E_m(0) = -E_m(1).$$

The connection of e_m with Euler numbers is given by the formula

$$e_m = -\frac{1}{2^m} \sum_{h=0}^{m} \binom{m}{h} \mathcal{E}_{m-h}. \tag{1.5}$$

The Euler polynomials are strictly connected with the Bernoulli ones, and enter in the Taylor expansion in a neighborhood of the origin of the trigonometric and hyperbolic secant functions.

The Appell polynomials (see Appell [9]) can be defined by considering the following generating function

$$G_R(x,t) = A(t)e^{xt} = \sum_{n=0}^{\infty} \frac{R_n(x)}{n!} t^n, \tag{1.6}$$

where

$$A(t) = \sum_{k=0}^{\infty} \frac{\mathcal{R}_k}{k!} t^k, \quad (A(0) \neq 0) \tag{1.7}$$

is analytic function at $t=0$, and $\mathcal{R}_k := R_k(0)$, $\mathcal{R}_0 = A(0) \neq 0$.

It is easy to see that

- If $A(t) = \frac{t}{e^t-1}$, then $R_n(x) = B_n(x)$,
- If $A(t) = \frac{2}{e^t+1}$, then $R_n(x) = E_n(x)$.
- If $A(t) = \alpha_1 \cdots \alpha_m t^m[(e^{\alpha_1 t}-1)\cdots(e^{\alpha_m t}-1)]^{-1}$, then $R_n(x)$ is the Bernoulli polynomials of order m (see Bateman et al [7]).
- If $A(t) = 2^m[(e^{\alpha_1 t}+1)\cdots(e^{\alpha_m t}+1)]^{-1}$, then $R_n(x)$ is the Euler polynomials of order m (see Bateman et al [7]).
- If $A(t) = e^{\xi_0+\xi_1 t+\cdots \xi_{d+1}t^{d+1}}$, $\xi_{d+1} \neq 0$, then $R_n(x)$ is the generalized Gould-Hopper polynomials (see Douak [10], Srivastava [11] and Gould et al [12]), including the Hermite polynomials when $d=1$ and classical $2-$orthogonal polynomials when $d=2$.

The differential equations satisfied by $B_n(x), E_n(x)$ and $R_n(x)$ have been presented in He et al [13].

We next assume that

$$\frac{A'(t)}{A(t)} = \sum_{n=0}^{\infty} \alpha_n \frac{t^n}{n!}. \tag{1.8}$$

The numerical coefficients $\alpha_k, k = 1,2,...,n-1$, are defined in (1.8), and are linked to the values $\mathcal{R}_k$ by the following relations:

$$\mathcal{R}_{k+1} = \sum_{h=0}^{k} \binom{k}{h} \mathcal{R}_h \alpha_{k-h}.$$

The $\mathcal{R}_k$ will be called Appell numbers, associated with $A(t)$.

It is useful to recall the following result, characterizing the Appell polynomials:

Theorem 1.1 *The only polynomials $P_n(x)$ satisfying the condition*

$$P_n'(x) = nP_{n-1}(x), \qquad n = 0, 1, 2, \ldots \tag{1.9}$$

are the Appell polynomials.

The proof is essentially contained in Specht [14], Sect. 1, althought in this article no reference is given to Appell polynomials.

In this note, starting from the Appell polynomials, and Appell numbers associated with $A(t)$, we construct a quadrature rule generalizing both the Euler-MacLaurin quadrature formula, and the formula presented in Bretti et al [15], which uses Euler numbers and even derivatives of the integrand at the extrema of the given interval.

2. The Euler-MacLaurin quadrature formula

In this section we recall briefly the classical Euler-MacLaurin quadrature formula (see Stoer [4]), and a similar formula derived by Bretti et al [15].

Given a function $f(x) \in C^{2m}[a,b]$ and an uniform partition of $[a,b]$ by means of points:

$$x_h := a + h\delta, \qquad \delta := \frac{b-a}{n}, \qquad h = 0, 1, 2, ..., n \tag{2.1}$$

where $x = a + t\delta$, putting $f(x) = f(a + t\delta) =: g(t)$, $t \in [0, n]$, it follows:

$$g^{(k)}(t) = \delta^k f^{(k)}(a + t\delta), \qquad k = 0, 1, ..., 2m. \tag{2.2}$$

Therefore

$$\int_a^b f(x)dx = \delta \int_0^n g(t)dt. \tag{2.3}$$

Denoting by $T(f)$ the quadrature of the integral (2.3) obtained by using the composite trapezoidal rule we can write:

$$\delta\left[\frac{1}{2}g(0) + g(1) + ... + g(n-1) + \frac{1}{2}g(n)\right] =$$

$$= \delta\left[\frac{1}{2}f(a) + f(x_1) + ... + f(x_{n-1}) + \frac{1}{2}f(b)\right] = T(f;\delta) = T(f) \tag{2.4}$$

Then the following proposition is valid:

Proposition 2.1 *If $f(x) \in C^{2m}[a,b]$, for the composite trapezoidal rule the asymptotic formula holds true*

$$T(f) = I_0 - I_1\delta^2 - I_2\delta^4 - ... - I_{m-1}\delta^{2m-2} + R_{2m}(f;\delta),$$

where:

$$I_0 = \int_a^b f(x)dx, \tag{2.5}$$

$$I_k = \frac{(-1)^k B_k}{(2k)!}\left[f^{(2k-1)}(b) - f^{(2k-1)}(a)\right], \qquad k = 1, 2, ..., m-1,$$

and

$$R_{2m}(f;\delta) = -\frac{\delta^{2m}}{(2m)!}\int_a^b \left[S_{2m}\left(\frac{x-a}{\delta}\right) - S_{2m}(0)\right] f^{(2m)}(x)dx = O(\delta^{2m}),$$

where the function $S_{2m}(x)$ *is the periodic continuation, of period* 1*, of the polynomial function* $B_{2m}(x)$ *starting from its values when* $x \in [0,1]$.

As a consequence the Euler-MacLaurin quadrature formula can be written in the form

$$\int_a^b f(x)dx = \delta\left[\frac{1}{2}f(a) + f(x_1) + ... + f(x_{n-1}) + \frac{1}{2}f(b)\right]$$

$$+\sum_{k=1}^{m-1}\frac{(-1)^k B_k}{(2k)!}\delta^k\left[f^{(2k-1)}(b) - f^{(2k-1)}(a)\right] + R_{2m}(f;\delta), \tag{2.6}$$

where:

$$R_{2m}(f;\delta) = \frac{\delta^{2m}}{(2m)!}\int_a^b \left[S_{2m}\left(\frac{x-a}{\delta}\right) - S_{2m}(0)\right] f^{(2m)}(x)dx = O(\delta^{2m}).$$

Remark 2.1 *In F.B. Hildebrand [16], p. 156, the particular case* $n = 1$ *is considered starting from the Euler-MacLaurin summation formula. In this case, the above formula (2.6) becomes:*

$$\int_a^b f(x)dx = \frac{b-a}{2}\left[f(a) + f(b)\right]$$

$$+\sum_{k=1}^{m-1}\frac{(-1)^k B_k}{(2k)!}(b-a)^{2k}\left[f^{(2k-1)}(b) - f^{(2k-1)}(a)\right] + R_{2m}(f;\delta), \tag{2.7}$$

and the remainder term can be written in the form

$$R_{2m}(f;\delta) = (-1)^m B_m \frac{(b-a)^{2m+1}}{(2m)!} f^{(2m)}(\xi), \qquad \xi \in [a,b]. \tag{2.8}$$

In Bretti et al [15], the use of Euler polynomials and Euler numbers allows to construct a quadrature rule similar to the previous one, but using Euler (instead of Bernoulli) numbers, and even (instead of odd) order derivatives of the given function at the extrema of the considered interval.

As a matter of fact, the following proposition holds true:

Proposition 2.2 *Consider a function $f(x) \in C^{2m}[a,b]$ and the corresponding integral over $[a,b]$. Let $x_i = a + i\delta$, $i = 0, 1, ..., n$, where $\delta := \frac{b-a}{n}$, and $f_i = f(x_i)$, $f_i^{(p)} = f^{(p)}(x_i)$, $p = 1, 2, ..., 2m$. Then the following composite trapezoidal rule holds true:*

$$\int_a^b f(x)dx = \delta\left(\frac{1}{2}f(a) + f_1 + ... + f_{n-1} + \frac{1}{2}f(b)\right)$$

$$-\sum_{k=2}^{m} \frac{e_{2k-1}}{(2k-1)!}\delta^{2k-1}\left[f^{(2k-2)}(b) + f^{(2k-2)}(a)\right] + R_E[f;\delta], \tag{2.9}$$

where the correction terms are expressed by means of the even derivatives of the given function $f(x)$ at the extrema, and the error term is given by

$$R_E[f;\delta] = \frac{\delta^{2m}}{(2m)!}\sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}} f^{(2m)}(x)E_{2m}\left(\frac{x-x_i}{\delta}\right)dx. \tag{2.10}$$

3. Quadrature rule associated with Appell polynomials

We recall that

- $R_n'(t) = nR_{n-1}(t)$
- $R_n(0) := \mathcal{R}_n$

where the $\mathcal{R}_n$ are the Appell numbers, associated with $A(t)$.
In the following we will assume for $A(t)$ the normalization $R_0(t) = \mathcal{R}_0 = 1$, so that the corresponding $R_k(t)$ polynomials are monic. By using the recursion relation, proved in He et al [13]:

$$R_{n+1}(t) = (t+\alpha_0)R_n(t) + \sum_{k=0}^{n-1}\binom{n}{k}\alpha_{n-k}R_k(t),$$

it is possible to construct, step by step, by recursion:

$$R_1(t) = t + \alpha_0,$$

$$R_2(t) = (t+\alpha_0)R_1(t) + \alpha_1 R_0(t) = (t+\alpha_0)^2 + \alpha_1,$$

$$R_3(t) = (t+\alpha_0)R_2(t) + \alpha_2 R_0(t) + 2\alpha_1 R_1(t) =$$

$$= (t+\alpha_0)^3 + 3\alpha_1(t+\alpha_0) + \alpha_2,$$

and so on.
Since Appell polynomials are determined by the sequence α_n, the recurrence relation:

$$\mathcal{R}_{n+1} = \sum_{k=0}^{n} \binom{n}{k} \mathcal{R}_k \alpha_{n-k}$$

is very useful.
In the following we will use the notation:

$$\mathcal{S}_k := R_k(1), \qquad k = 0, 1, 2, ... \tag{3.1}$$

and the coefficient

$$\alpha_0 = \frac{A'(0)}{A(0)} = \frac{\mathcal{R}_1}{\mathcal{R}_0} = \mathcal{R}_1 \tag{3.2}$$

will come into play.
Note that all the $\mathcal{S}_k$ can be written in terms of the Appell numbers $\mathcal{R}_h$, $h = 0, 1, ..., k$, associated with $A(t)$, since we can write:

$$A(t)e^t = \sum_{k=0}^{\infty} R_k(1)\frac{t^k}{k!} = \sum_{k=0}^{\infty} \frac{\mathcal{R}_k}{k!} t^k \sum_{k=0}^{\infty} \frac{t^k}{k!} = \sum_{k=0}^{\infty} \sum_{h=0}^{k} \binom{k}{h} \mathcal{R}_h \frac{t^k}{k!},$$

and consequently:

$$\mathcal{S}_k = \sum_{h=0}^{k} \binom{k}{h} \mathcal{R}_h, \qquad h = 0, 1, 2, ... \ .$$

The main result of this article is expressed by the following theorem:

Theorem 3.1 *Consider a function $f(x) \in C^{m+1}[a, b]$ and the corresponding integral over $[a, b]$. Let $x_i = a + i\delta$, $i = 0, 1, ..., n$, where $\delta := \frac{b-a}{n}$, and $f_i = f(x_i)$, $f_i^{(p)} = f^{(p)}(x_i)$, $p = 1, 2, ..., m + 1$. Then the following composite trapezoidal rule holds true:*

$$\int_a^b f(x)dx = \delta \left(f(b)\mathcal{S}_1 - f(a)\mathcal{R}_1 + \sum_{i=1}^{n-1} f_i \right)$$

$$+ \sum_{k=1}^{m} \frac{(-1)^k}{(k+1)!} \delta^{k+1} \left[f^{(k)}(b)\mathcal{S}_{k+1} - f^{(k)}(a)\mathcal{R}_{k+1} + \sum_{i=1}^{n-1} f_i^{(k)} \left(\mathcal{S}_{k+1} - \mathcal{R}_{k+1} \right) \right]$$

$$+ R_A[f], \tag{3.3}$$

where the correction terms are expressed by means of all derivatives of the given function $f(x)$, and the error term is given by

$$R_A[f; \delta] = \delta^{m+2} \frac{(-1)^{m+1}}{(m+1)!} \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} f^{(m+1)}(x) R_{m+1}\left(\frac{x - x_i}{\delta}\right) dx. \tag{3.4}$$

Proof. We consider first the trapezoidal rule extended to the whole interval.

Let be $f(x) \in C^{m+1}[a,b]$ and consider the integral of f over $[a,b]$. Putting $f(x) = f(a+\delta t) =: g(t)$, where $\delta = b-a$ denotes the length of the interval, and $t = \frac{x-a}{\delta}$, by using subsequent integration by parts we find:

$$\int_a^b f(x)dx = \delta \int_0^1 g(t)dt = \delta \int_0^1 g(t)dR_1(t) =$$

$$= \delta \left\{ [g(t)R_1(t)]_0^1 - \int_0^1 g'(t)R_1(t)dt \right\} =$$

$$= \delta \left\{ [g(1)R_1(1) - g(0)R_1(0)] - \frac{1}{2}\left[g'(t)R_2(t)\right]_0^1 + \frac{1}{2}\int_0^1 g''(t)R_2(t)dt \right\} =$$

$$= \delta \left\{ [g(1)(1+\alpha_0) - g(0)\alpha_0] - \frac{1}{2}\left[g'(1)R_2(1) - g'(0)R_2(0)\right] \right.$$

$$\left. + \frac{1}{2}\int_0^1 g''(t)R_2(t)dt \right\} =$$

$$= \delta \left\{ [g(1)(1+\alpha_0) - g(0)\alpha_0] - \frac{1}{2}\left[g'(1)R_2(1) - g'(0)R_2(0)\right] \right.$$

$$\left. + \frac{1}{6}\left[g''(t)R_3(t)\right]_0^1 - \frac{1}{6}\int_0^1 g'''(t)R_3(t)dt \right\} =$$

$$\cdots$$

$$= \delta \left\{ [g(1)(1+\alpha_0) - g(0)\alpha_0] + \sum_{k=1}^{m} \frac{(-1)^k}{(k+1)!}\left[g^{(k)}(1)R_{k+1}(1) - g^{(k)}(0)R_{k+1}(0)\right] \right.$$

$$\left. + \frac{(-1)^{m+1}}{(m+1)!}\int_0^1 g^{(m+1)}(t)R_{m+1}(t)dt \right\}.$$

Recalling the positions $f(x) = g(t)$, and (3.1)-(3.2), the preceding formula yields

$$\int_a^b f(x)dx = \delta \left\{ [f(b)\mathcal{S}_1 - f(a)\mathcal{R}_1] + \sum_{k=1}^{m} \frac{(-1)^k}{(k+1)!}\delta^k \left[f^{(k)}(b)\mathcal{S}_{k+1} - f^{(k)}(a)\mathcal{R}_{k+1}\right] \right.$$

$$\left. + \frac{(-1)^{m+1}}{(m+1)!}\delta^{m+1}\int_0^1 f^{(m+1)}(x)R_{m+1}\left(\frac{x-a}{\delta}\right)dx \right\}. \tag{3.5}$$

Consider now a partition of the interval $[a,b]$ into n partial intervals by means of equidistant knots $x_i = a + i\delta$, $i = 0, 1, ..., n$, where $\delta := \frac{b-a}{n}$, $f_i = f(x_i)$, $f_i^{(p)} = f^{(p)}(x_i)$, $p = 1, 2, ..., m+1$. From eq. (3.3) it follows:

$$\int_a^b f(x)dx = \sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}} f(x)dx = \delta\left(f(b)\mathcal{S}_1 - f(a)\mathcal{R}_1 + \sum_{i=1}^{n-1} f_i\right)$$

$$+\sum_{k=1}^{m}\frac{(-1)^k}{(k+1)!}\delta^{k+1}\left[f^{(k)}(b)\mathcal{S}_{k+1}-f^{(k)}(a)\mathcal{R}_{k+1}+\sum_{i=1}^{n-1}f_i^{(k)}\left(\mathcal{S}_{k+1}-\mathcal{R}_{k+1}\right)\right]$$

$$+\delta^{m+2}\frac{(-1)^{m+1}}{(m+1)!}\sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}}f^{(m+1)}(x)R_{m+1}\left(\frac{x-x_i}{\delta}\right)dx.$$

Note that the remainder term of the above procedure is expressed by

$$R_A[f;\delta]=\delta^{m+2}\frac{(-1)^{m+1}}{(m+1)!}\sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}}f^{(m+1)}(x)R_{m+1}\left(\frac{x-x_i}{\delta}\right)dx.$$

Remark 3.1 *In our opinion, althought general formulas of the same type of eq. (3.3) can be derived by using integration by parts, starting from the integral*

$$\frac{(-1)^{n+1}}{(n+1)!}\int_a^b f^{(n+1)}(x)\left(x^{n+1}-P_n(x)\right)dx,$$

where P_n *denotes an arbitrary polynomial of degree at most* n*, our approach represents the natural extension of the Euler-MacLaurin quadrature rule, a subject that, apparently, never appeared in literature before.*

4. Numerical example

In this section, we will show an application of the quadrature rule (3.3), associated with Appell polynomials. Of course the choice of the function $A(t)$ is arbitrary, but it is convenient to choose $A(t)$ in such a way that the corresponding Appell numbers $\mathcal{R}_k$ are not increasing.

Fix the integral N, and define:

$$A(t):=\frac{1}{e}\exp\left(\exp(-\frac{t}{N})\right),$$

so that the normalizing condition $A(0)=\mathcal{R}_0=1$ is satisfied.

Then, putting:

$$\frac{A'(t)}{A(t)}=\sum_{k=0}^{\infty}\alpha_k\frac{t^k}{k!}$$

and recalling the definition (1.7) of the function $A(t)$, the numerical values

$$\alpha_k=\frac{(-1)^{k+1}}{N^{k+1}}$$

are easily found. Furthermore Appell numbers can be computed by means of the recurrence relation:

$$\mathcal{R}_{h+1}=-\frac{1}{N}\sum_{k=0}^{h}\binom{h}{k}\frac{(-1)^{h-k}}{N^{h-k}}\mathcal{R}_k$$

The first values of the $\mathcal{R}_k$ are consequently:

$$\mathcal{R}_0 = A(0) = 1, \quad \mathcal{R}_1 = -\frac{1}{N}, \quad \mathcal{R}_2 = \frac{2}{N^2}, \quad \mathcal{R}_3 = -\frac{5}{N^3}, \quad \dots$$

and so on.

Remark 4.1 *Note that the Appell-type quadrature rule depends on the function $A(t)$ we start with, and consequently the obtained result could be worse than the corresponding obtained by using the Euler-MacLaurin one.*

It seems to be an interesting problem to investigate how to relate the integrand function $f(x)$ with the function $A(t)$ to be considered, in order to minimize the remainder term. However, this seems to be a difficult subject to work with.

In the following classical example, by using the function $A(t)$ of eq. (4.1), and assuming $N = 2$, we obtain better results with respect to other similar formulas.

Consider the integral

$$\mathcal{J} := \int_0^1 \frac{1}{x+1} dx = \int_0^{0.5} \frac{1}{1-x} dx \simeq 0.69314718246...,$$

where the "exact" value is obtained by using Mathematica©.

Computing $\mathcal{J}$ by the Euler-type quadrature rule, assuming $m = 6$ $n = 90$, and $A(t) = \frac{2}{e^t+1}$,

$$x(i) = a + (i-1)\delta = (i-1)\tfrac{1}{90}, \qquad (i = 1, ..., 90),$$

and recalling that, by eq. (1.5): $e_3 = \frac{1}{4}$, $e_5 = -\frac{1}{2}$, $e_7 = \frac{17}{8}$, $e_9 = -\frac{31}{2}$, $e_{11} = \frac{691}{4}$

The composite trapezoidal rule gives:

$$\mathcal{J}_{TRAP} = 0.6931548...$$

and, using the correction of the Euler-type quadrature rule, we found:

$$\mathcal{J}_{EUL} = 0.6931547...$$

and the following estimate for the remainder term:

$$|R_{12}^{(E)}| \leq 10^{-5}.$$

By using the Appell-type quadrature rule, with $m = 5$ e $n = 90$, $A(t) = e^{e^{-\frac{t}{2}}-1}$, (i.e. $N = 2$), we found the better approximation

$$\mathcal{J}_{APP} = 0.693147180...$$

and, for the remainder term, the better estimate:

$$|R_6^{(A)}| \leq 3 \cdot 10^{-9}.$$

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