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On the Complete Join of Permutative Combinatorial Rees-Sushkevich Varieties

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Abstract

A semigroup variety is a Rees-Sushkevich variety if it is contained in a periodic variety generated by 0-simple semigroups. The collection of all permutative combinatorial Rees-Sushkevich varieties constitutes an incomplete lattice that does not contain the complete join $\mathbf{J}$ of all its varieties. The objective of this article is to investigate the subvarieties of $\mathbf{J}$. It is shown that $\mathbf{J}$ is locally finite, non-finitely generated, and contains only finitely based subvarieties. The subvarieties of $\mathbf{J}$ are precisely the combinatorial Rees-Sushkevich varieties that do not contain a certain semigroup of order four.

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1 Introduction

Recall that a semigroup is *0-simple* if it does not contain any nontrivial proper ideals. The class of 0-simple semigroups is one of the most important classes of semigroups. Indeed, as each finite semigroup can be obtained from finite 0-simple semigroups by a sequence of ideal extensions, the role that finite 0-simple semigroups play in semigroup theory is comparable to the role that finite simple groups play in group theory. Naturally, the varieties generated by 0-simple semigroups and their subvarieties deserve special attention.

Following Kublanovsky [4], any subvariety of a periodic variety generated by 0-simple semigroups is said to be a *Rees-Sushkevich variety*. Investigation of the lattice of Rees-Sushkevich varieties has recently been initiated by Reilly, Volkov, and the author (see [5]–[10], [12]–[14], and [19]). In particular, several aspects of the lattice $\mathfrak{C}$ of combinatorial Rees-Sushkevich varieties have

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been considered in [5]–[7], [10], and [19]. Recall that a semigroup variety is *combinatorial* if all groups in it are trivial.

A semigroup variety is *permutative* if it satisfies some permutation identity. Denote by $\mathfrak{P}$ the set of all permutative varieties in $\mathfrak{C}$. It is easy to show that $\mathfrak{P}$ is a lattice (Proposition 5). However, the complete join $\mathbf{J}$ of all varieties in $\mathfrak{P}$ is not permutative (Proposition 11) so that $\mathfrak{P}$ is an incomplete sublattice of the subvariety lattice $\mathcal{L}(\mathbf{J})$ of $\mathbf{J}$. The structure of $\mathcal{L}(\mathbf{J})$ is quite complex, for it follows from a result of Vernikov and Volkov [17] that every finite lattice is embeddable in $\mathcal{L}(\mathbf{J})$.

The objective of the present article is to investigate the variety $\mathbf{J}$ and its subvarieties. Specifically, it is shown that the variety $\mathbf{J}$ is locally finite, non-finitely generated, and contains only finitely based subvarieties. Consequently, $\mathcal{L}(\mathbf{J})$ is a countably infinite lattice. It is also shown that the subvarieties of $\mathbf{J}$ are precisely the varieties in $\mathfrak{C}$ that do not contain a certain semigroup of order four. The aforementioned properties of $\mathbf{J}$ are presented in Section 4.

2 Background

Let X^+ and X^* respectively be the free semigroup and free monoid over a countably infinite alphabet X. Elements of X are referred to as *letters*, and elements of X^+ and X^* are referred to as *words*.

The *head* and *tail* of a word $\mathbf{u}$ are respectively the first and last letters occurring in $\mathbf{u}$ and are denoted by $\mathsf{h}(\mathbf{u})$ and $\mathsf{t}(\mathbf{u})$. The *length* of $\mathbf{u}$ is the number $|\mathbf{u}|$ of letters occurring in $\mathbf{u}$ counting multiplicity. The *content* of $\mathbf{u}$ is the set of letters occurring in $\mathbf{u}$ and is denoted by $\mathsf{C}(\mathbf{u})$. The set of length-two factors of $\mathbf{u}$ is $\mathsf{C}_2(\mathbf{u}) = \{\mathbf{w} \in X^+ : |\mathbf{w}| = 2 \text{ and } \mathbf{u} \in X^*\mathbf{w}X^*\}$. It is easy to see that $\mathsf{C}_2(\mathbf{u}) = \mathsf{C}_2(\mathbf{v})$ implies that $\mathsf{C}(\mathbf{u}) = \mathsf{C}(\mathbf{v})$.

We write $\mathbf{u} = \mathbf{v}$ when $\mathbf{u}$ and $\mathbf{v}$ are identical words and write $\mathbf{u} \approx \mathbf{v}$ to stand for a semigroup identity. Let Σ be a set of identities. We write $\Sigma \vdash \mathbf{u} \approx \mathbf{v}$ or $\mathbf{u} \stackrel{\Sigma}{\approx} \mathbf{v}$ if the identity $\mathbf{u} \approx \mathbf{v}$ is derivable from the identities in Σ. The variety *defined* by Σ is the class of all semigroups that satisfy all identities in Σ and is denoted by $[\Sigma]$. If $\mathbf{V}$ is a variety with $\mathbf{V} = [\Sigma]$, then Σ is said to be a *basis* for $\mathbf{V}$. A variety is *finitely based* if it possesses a finite basis.

A *permutation identity* is an identity of the form $x_1 \cdots x_m \approx x_{1\alpha} \cdots x_{m\alpha}$ where $x_1, \ldots, x_m$ are distinct letters and α is a nontrivial permutation on $\{1, \ldots, m\}$. A *permutative variety* is a variety that satisfies some permutation identity.

Lemma 1 (Perkins [11]) *Each permutation identity implies the identity*

$$x_1 \cdots x_m yz w_1 \cdots w_m \approx x_1 \cdots x_m zy w_1 \cdots w_m \qquad (\pi_m)$$

for some $m \geq 1$. □

We refer the reader to [3] and [2] respectively for undefined terminology in semigroup theory and universal algebra.

3 The subvariety lattice $\mathcal{L}(\mathbf{A_2})$ of $\mathbf{A_2}$

Recall that $\mathfrak{C}$ denotes the lattice of all combinatorial Rees-Sushkevich varieties and $\mathfrak{P}$ denotes the set of all permutative varieties in $\mathfrak{C}$.

Denote by A_2 the idempotent-generated 0-simple semigroup of order five and by B_2 the Brandt semigroup of order five:

$$A_2 = \langle\, a, b : a^2 = aba = a,\ b^2 = 0,\ bab = b\,\rangle,$$
$$B_2 = \langle\, c, d : c^2 = d^2 = 0,\ cdc = c,\ dcd = d\,\rangle.$$

These two semigroups play very important roles in the theory of semigroup and especially in the theory of semigroup varieties. They appeared or were investigated in, for example, [1], [3]–[16], and [19]. Denote by $\mathbf{A_2}$ the variety generated by A_2.

Proposition 2 ([10, Proposition 1.2]) *The variety $\mathbf{A_2}$ is the largest combinatorial Rees-Sushkevich variety. Consequently, the lattice $\mathfrak{C}$ coincides with the subvariety lattice $\mathcal{L}(\mathbf{A_2})$ of $\mathbf{A_2}$.* □

In view of Proposition 2, any variety $\mathbf{V}$ in $\mathfrak{C}$ is defined within $\mathbf{A_2}$ by some set Σ of identities, that is, $\mathbf{V} = \mathbf{A_2} \cap [\Sigma]$.

Lemma 3 (Trahtman [15, 16]) *The identities*

$$x^3 \approx x^2, \quad xyxyx \approx xyx, \quad xyxzx \approx xzxyx \tag{1}$$

constitute a basis for $\mathbf{A_2}$. More generally, an identity $\mathbf{u} \approx \mathbf{v}$ holds in $\mathbf{A_2}$ if and only if $\mathsf{C}_2(\mathbf{u}) = \mathsf{C}_2(\mathbf{v})$, $\mathsf{h}(\mathbf{u}) = \mathsf{h}(\mathbf{v})$, and $\mathsf{t}(\mathbf{u}) = \mathsf{t}(\mathbf{v})$. □

A word of length at least two is said to be *connected* if it cannot be written as a product of two nonempty words with disjoint contents.

Lemma 4 *Let $\mathbf{u}, \mathbf{v}$ be connected words such that $\mathsf{C}(\mathbf{u}) = \mathsf{C}(\mathbf{v})$, $\mathsf{h}(\mathbf{u}) = \mathsf{h}(\mathbf{v})$, and $\mathsf{t}(\mathbf{u}) = \mathsf{t}(\mathbf{v})$. Suppose σ is an identity that does not hold in the semigroup B_2. Then the identity $\mathbf{u} \approx \mathbf{v}$ holds in the variety $\mathbf{A_2} \cap [\sigma]$. In particular, the following identities hold in any subvariety of $\mathbf{A_2}$ that does not contain B_2:*

$$xpyzqx \approx xpzyqx,\ xpyzx \approx xpzyx,\ xyzqx \approx xzyqx,\ xyzx \approx xzyx, \tag{2}$$
$$xpy^2qx \approx xpyqx,\ xpy^2x \approx xpyx,\ xy^2qx \approx xyqx,\ xy^2x \approx xyx, \tag{3}$$
$$x^2yx \approx xyx,\ xyx^2 \approx xyx. \tag{4}$$

Proof. Without loss of generality, suppose $\mathsf{C}(\mathbf{u}) = \mathsf{C}(\mathbf{v}) = \{x_1, \dots, x_n\}$. Denote by F_n the free object of $\mathbf{A_2} \cap [\sigma]$ over the generators $\{x_1, \dots, x_n\}$. Since $\mathbf{A_2}$ is locally finite, F_n is a finite semigroup. It follows from [1, Exercise 8.1.6] that the regular $\mathcal{D}$-classes of F_n are subsemigroups.

Now $\mathbf{u}$ and $\mathbf{v}$ are regular elements of F_n by [10, Proposition 2.2], and it follows from [1, Theorem 8.1.7] that they belong to the same $\mathcal{D}$-class D, which must be a rectangular band. Since $\mathsf{C}_2(\mathbf{uv}) = \mathsf{C}_2(\mathbf{vu})$, $\mathsf{h}(\mathbf{uv}) = \mathsf{h}(\mathbf{vu})$, and $\mathsf{t}(\mathbf{uv}) = \mathsf{t}(\mathbf{vu})$, the identity $\mathbf{uv} \approx \mathbf{vu}$ holds in $\mathbf{A_2}$ by Lemma 3. Therefore $\mathbf{u}$ and $\mathbf{v}$ are commuting elements in the rectangular band D and so must coincide in F_n. □

Proposition 5 *The set* $\mathfrak{P}$ *constitutes a sublattice of* $\mathcal{L}(\mathbf{A_2})$.

Proof. Suppose $\mathbf{U}, \mathbf{V} \in \mathfrak{P}$. Clearly $\mathbf{U} \cap \mathbf{V}$ is a variety in $\mathfrak{C}$ that satisfies all (permutation) identities of $\mathbf{U}$ and $\mathbf{V}$ so that $\mathbf{U} \cap \mathbf{V} \in \mathfrak{P}$. By Lemma 1, there exist $i, j \geq 1$ such that the permutation identities π_i and π_j hold in $\mathbf{U}$ and $\mathbf{V}$ respectively. Then $\mathbf{U} \vee \mathbf{V}$ is a variety in $\mathfrak{C}$ that satisfies the identity π_m where $m = \max\{i, j\}$. Therefore $\mathbf{U} \vee \mathbf{V} \in \mathfrak{P}$. □

Lemma 6 *The identities*

$$x^2yzw^2 \approx x^2zyw^2, \tag{5a}$$
$$x^2y^2z^2 \approx x^2yz^2, \tag{5b}$$

define the same subvariety in $\mathbf{A_2}$, *that is,* $\mathbf{A_2} \cap [(5a)] = \mathbf{A_2} \cap [(5b)]$.

Proof. It is straightforward to show that the identities (5a) and (5b) do not hold in B_2. Hence $\{(1), (5a)\} \vdash (3)$ and $\{(1), (5b)\} \vdash (2)$ by Lemma 4. It follows that the inclusion $\mathbf{A_2} \cap [(5a)] \subseteq \mathbf{A_2} \cap [(5b)]$ holds since

$$\begin{aligned}\{(1), (5a)\} \vdash x^2y^2z^2 &\overset{(1)}{\approx} x^2xy^2z^2 \overset{(5a)}{\approx} x^2y^2xz^2 \\ &\overset{(3)}{\approx} x^2yxz^2 \overset{(5a)}{\approx} x^2xyz^2 \\ &\overset{(1)}{\approx} x^2yz^2,\end{aligned}$$

and the inclusion $\mathbf{A_2} \cap [(5b)] \subseteq \mathbf{A_2} \cap [(5a)]$ holds since

$$\begin{aligned}\{(1), (5b)\} \vdash x^2yzw^2 &\overset{(1)}{\approx} x^2(xyzw)w^2 \overset{(5b)}{\approx} x^2(xyzw)^2w^2 \\ &\overset{(2)}{\approx} x^2(xzyw)^2w^2 \overset{(5b)}{\approx} x^2xzyww^2 \\ &\overset{(1)}{\approx} x^2zyw^2.\end{aligned}$$

□

Note that by Lemma 4, all subvarieties of $\mathbf{A_2} \cap [(5a)]$ satisfy the identities (2), (3), and (4). This result will be used in the remainder of this article without further reference.

A word is *simple* if all letters occurring in it have multiplicity one. Suppose X is (alphabetically) ordered by $<$. A word $\mathbf{u} = x_1 \cdots x_m$ is said to be an *ordered word* if $x_1 < \cdots < x_m$. Clearly an ordered word is necessarily simple. A word $\mathbf{u}$ is said to be in *canonical form* if any of the following conditions hold:

(A) $\mathbf{u} = x\mathbf{v}x$ for some ordered word $\mathbf{v} \in X^*$ with $x \notin \mathsf{C}(\mathbf{v})$;

(B) $\mathbf{u} = xy\mathbf{v}xy$ for some ordered word $\mathbf{v} \in X^*$ with $x, y \notin \mathsf{C}(\mathbf{v})$ and $x \neq y$.

Note that a word in canonical form is necessarily connected.

Lemma 7 *Let* $\mathbf{u}$ *be a connected word. Then there exists a unique word* $\mathbf{u}^\star$ *in canonical form such that* $\mathsf{C}(\mathbf{u}) = \mathsf{C}(\mathbf{u}^\star)$, $\mathsf{h}(\mathbf{u}) = \mathsf{h}(\mathbf{u}^\star)$, *and* $\mathsf{t}(\mathbf{u}) = \mathsf{t}(\mathbf{u}^\star)$. *Further, the identity* $\mathbf{u} \approx \mathbf{u}^\star$ *holds in the variety* $\mathbf{A_2} \cap [(5a)]$.

Proof. The existence and uniqueness of $\mathbf{u}^\star$ is easy to verify. Since any word in canonical form is connected, the identity $\mathbf{u} \approx \mathbf{u}^\star$ holds in $\mathbf{A_2} \cap [(5a)]$ by Lemma 4. □

Lemma 8 *A non-simple word* $\mathbf{u}$ *is equivalent within* $\mathbf{A_2} \cap [(5a)]$ *to a word* $\mathbf{p}\mathbf{w}^\star\mathbf{q}$ *where*

(i) $\mathbf{p}, \mathbf{q} \in X^*$ *are simple words*;

(ii) $\mathbf{w} \in X^+$ *is a connected word*;

(iii) $\mathsf{C}(\mathbf{p}), \mathsf{C}(\mathbf{w}), \mathsf{C}(\mathbf{q})$ *are pairwise disjoint sets.*

Proof. By assumption, we may write $\mathbf{u} = \mathbf{p}\mathbf{v}\mathbf{q}$ where $\mathbf{p}, \mathbf{q} \in X^*$ are simple, $\mathbf{v} \in X^+$ is non-simple with $h = \mathsf{h}(\mathbf{v})$ and $t = \mathsf{t}(\mathbf{v})$ each occurring at least twice in $\mathbf{v}$, and $\mathsf{C}(\mathbf{p}), \mathsf{C}(\mathbf{v}), \mathsf{C}(\mathbf{q})$ are pairwise disjoint sets. (Note that $h = t$ is possible). The words $\mathbf{v}$ and $\mathbf{v}^2$ are equivalent within $\mathbf{A_2} \cap [(5a)]$ since

$$\mathbf{v} \overset{(4)}{\approx} h^2\mathbf{v}t^2 \overset{(5b)}{\approx} h^2\mathbf{v}^2t^2 \overset{(4)}{\approx} \mathbf{v}^2.$$

Hence, by Lemma 7, the words $\mathbf{u}$ and $\mathbf{p}\mathbf{w}^\star\mathbf{q}$ are equivalent within $\mathbf{A_2} \cap [(5a)]$ where the word $\mathbf{w} = \mathbf{v}^2$ is connected. □

Proposition 9 *Every subvariety of* $\mathbf{A_2} \cap [(5a)]$ *is finitely based.*

Proof. The variety $\mathbf{A_2} \cap [(5a)]$ is clearly finitely based. Suppose $\mathbf{V}$ is a proper subvariety of $\mathbf{A_2} \cap [(5a)]$. Then $\mathbf{V}$ is defined within $\mathbf{A_2} \cap [(5a)]$ by some set Σ of identities. By Lemma 8, we may assume that all identities in Σ are formed by words that are either simple or of the form $\mathbf{p}\mathbf{w}^\star\mathbf{q}$. It follows from [18] that $\mathbf{V}$ is finitely based. □

4 Main results

Recall that $\mathbf{J}$ denotes the complete join of all varieties in $\mathfrak{P}$. This section presents several properties of $\mathbf{J}$ and its subvarieties.

Proposition 10 $\mathbf{J} = \mathbf{A_2} \cap [(5a)]$.

Proof. It is easy to see that within $\mathbf{A_2}$, the identity (5a) is a consequence of the identity π_m for any $m \geq 1$. Therefore $\mathbf{J} \subseteq \mathbf{A_2} \cap [(5a)]$.

Suppose $\mathbf{V}$ is a variety such that $\mathbf{J} \subset \mathbf{V} \subset \mathbf{A_2} \cap [(5a)]$. By Proposition 9, $\mathbf{V}$ is defined within $\mathbf{A_2} \cap [(5a)]$ by some finite set Σ of identities. We may assume that the identities in Σ do not hold in $\mathbf{A_2} \cap [(5a)]$. By Lemma 8, we may assume that all identities in Σ are formed by words that are either simple or of the form $\mathbf{pw}^\star\mathbf{q}$. Further, since $\mathbf{J}$ contains semilattices, each identity in Σ is formed by a pair of words with identical content. Let $\sigma : \mathbf{u} \approx \mathbf{v}$ be an identity in Σ.

Case 1: Suppose both $\mathbf{u}$ and $\mathbf{v}$ are simple. Then σ is a permutation identity. It follows that $\mathbf{V} \subseteq \mathbf{A_2} \cap [\sigma] \subseteq \mathbf{J}$, which is a contradiction.

Case 2: Suppose $\mathbf{u}$ is simple and $\mathbf{v}$ is non-simple (of the form $\mathbf{pw}^\star\mathbf{q}$). Then $\mathbf{u} = x_1 \cdots x_m$ and $\mathbf{v} = \mathbf{v}_1 x_k \mathbf{v}_2 x_k \mathbf{v}_3$ for some $k \in \{1, \ldots, m\}$ and some $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in X^*$ with $\mathsf{C}(\mathbf{v}_1\mathbf{v}_2\mathbf{v}_3) = \mathsf{C}(\mathbf{u}) \backslash \{x_k\}$. Denote by φ the substitution $x_k \mapsto xyzw$ and by χ the substitution $x_k \mapsto xzyw$. Then

$$\begin{aligned}\{(1), (2), \sigma\} \vdash \mathbf{u}\varphi &\overset{\sigma}{\approx} \mathbf{v}\varphi \\ &= \mathbf{v}_1\, xyzw\, \mathbf{v}_2\, xyzw\, \mathbf{v}_3 \\ &\overset{(2)}{\approx} \mathbf{v}_1\, xzyw\, \mathbf{v}_2\, xzyw\, \mathbf{v}_3 \\ &= \mathbf{v}\chi \overset{\sigma}{\approx} \mathbf{u}\chi.\end{aligned}$$

Hence $\mathbf{V}$ satisfies the permutation identity $\mathbf{u}\varphi \approx \mathbf{u}\chi$, and we arrive at the same contradiction in Case 1.

Therefore Cases 1 and 2 are both impossible, whence all identities in Σ are formed by non-simple words of the form $\mathbf{pw}^\star\mathbf{q}$. Suppose $\tau : \mathbf{u}_1 \approx \mathbf{u}_2$ is such an identity in Σ, say $\mathbf{u}_i = \mathbf{p}_i \mathbf{w}_i^\star \mathbf{q}_i$ for $i \in \{1, 2\}$. It is easy to show that if

$$\mathbf{p}_1 = \mathbf{p}_2, \quad \mathbf{q}_1 = \mathbf{q}_2, \quad \mathsf{h}(\mathbf{w}_1) = \mathsf{h}(\mathbf{w}_2), \quad \text{and} \quad \mathsf{t}(\mathbf{w}_1) = \mathsf{t}(\mathbf{w}_2), \tag{6}$$

then $\mathbf{u}_1 = \mathbf{u}_2$ so that the identity τ is contradictorily satisfied by $\mathbf{A_2} \cap [(5a)]$. Thus at least one of the four equalities in (6) do not hold, whence $\{(1), \pi_m\} \nvdash \tau$ for any $m \geq \max\{|\mathbf{u}_1|, |\mathbf{u}_2|\}$. But this contradicts the fact that $\mathbf{A_2} \cap [\pi_m] \subseteq \mathbf{J} \subset \mathbf{V} \subseteq \mathbf{A_2} \cap [\tau]$. Consequently, the identity τ, and hence $\mathbf{V}$, do not exist, whence $\mathbf{J} = \mathbf{A_2} \cap [(5a)]$. □

Proposition 11 *The variety* **J** *is not permutative. Consequently, the lattice* $\mathfrak{P}$ *is incomplete.*

Proof. By referring to the identity basis for **J** in Proposition 10, it is easy to show that **J** does not satisfy any of the identities π_m and hence cannot be permutative by Lemma 1. Therefore $\mathfrak{P}$ does not contain the complete join **J** of its varieties, whence it is a lattice (Proposition 5) that is incomplete. □

Proposition 12 *The variety* **J** *is locally finite and non-finitely generated.*

Proof. The variety **J** is locally finite since $\mathbf{A_2}$ is finitely generated. Let S be a semigroup in **J** with $|S| < m$. For $1 \leq i \leq m$, let $a, b, g_i, h_i \in S$. Since the list $g_1, \ldots, g_m$ contains an element (say g_i) that appears at least twice and that S satisfies the identities (4), we have

$$g_1 \cdots g_m \overset{(4)}{=} g_1 \cdots g_{i-1} g_i^2 g_i \cdots g_m.$$

By an identical argument, $h_1 \cdots h_m \overset{(4)}{=} h_1 \cdots h_j h_j^2 h_{j+1} \cdots h_m$ for some j. The identities (2) and (5b) also hold in S so that

$$\begin{aligned} & g_1 \cdots g_m \cdot ab \cdot h_1 \cdots h_m \\ \overset{(4)}{=}\; & g_1 \cdots g_{i-1}(g_i^2 g_i \cdots g_m \cdot ab \cdot h_1 \cdots h_j h_j^2) h_{j+1} \cdots h_m \\ \overset{(5b)}{=}\; & g_1 \cdots g_{i-1} g_i^2 (g_i \cdots g_m \cdot ab \cdot h_1 \cdots h_j)^2 h_j^2 h_{j+1} \cdots h_m \\ \overset{(2)}{=}\; & g_1 \cdots g_{i-1} g_i^2 (g_i \cdots g_m \cdot ba \cdot h_1 \cdots h_j)^2 h_j^2 h_{j+1} \cdots h_m \\ \overset{(5b)}{=}\; & g_1 \cdots g_{i-1} g_i^2 g_i \cdots g_m \cdot ba \cdot h_1 \cdots h_j h_j^2 h_{j+1} \cdots h_m \\ \overset{(4)}{=}\; & g_1 \cdots g_m \cdot ba \cdot h_1 \cdots h_m. \end{aligned}$$

Hence S satisfies the identity π_m. But **J** does not satisfy π_m (Proposition 11) and so cannot be generated by S. □

Proposition 13 *Every subvariety of* **J** *is finitely based. Consequently,* $\mathcal{L}(\mathbf{J})$ *is a countably infinite lattice.*

Proof. The first part follows from Propositions 9 and 10, while the second part holds since only countably many finite sets of identities exist up to relabelling of letters. □

The last result of this article involves the semigroup

$$Y = \langle\, e, f, s : e^2 = e,\ f^2 = f,\ ef = fe = 0,\ es = sf = s \,\rangle$$

of order four. It is easy to show that Y is isomorphic to a subsemigroup of the 0-simple semigroup B_2 and so belongs to $\mathbf{A_2}$ by Proposition 2.

Theorem 14 *The following statements on a variety* $\mathbf{V}$ *in* $\mathfrak{C}$ *are equivalent.*

(i) $\mathbf{V}$ *is contained in* $\mathbf{J}$;

(ii) $\mathbf{V}$ *does not contain* Y.

Consequently, $\mathbf{J}$ *is the largest variety in* $\mathfrak{C}$ *that does not contain* Y.

Proof. Since $e^2sef^2 = 0 \neq s = e^2esf^2$, the identity (5a) of $\mathbf{J}$ does not hold in the semigroup Y so that statement (i) implies statement (ii).

Conversely, suppose statement (ii) holds. Let S be any finite semigroup in $\mathbf{V}$. It follows from [1, Proposition 11.8.1] and the identities (1) of $\mathbf{A_2}$ that S satisfies at least one of the identities

$$(x^2yz^2)^2 \approx x^2yz^2, \qquad (\rho_1)$$
$$x^2yz^2x^2z^2 \approx x^2yz^2, \qquad (\rho_2)$$
$$x^2z^2x^2yz^2 \approx x^2yz^2. \qquad (\rho_3)$$

The variety generated by S does not contain B_2 so that S also satisfies the identities (3) by Lemma 4. Note that

$$\{(1),(3),\rho_1\} \vdash x^2yz^2 \overset{\rho_1}{\approx} (x^2yz^2)^2 \overset{(3)}{\approx} (x^2y^2z^2)^2 \overset{\rho_1}{\approx} x^2y^2z^2,$$
$$\{(1),(3),\rho_2\} \vdash x^2yz^2 \overset{\rho_2}{\approx} x^2yz^2x^2z^2 \overset{(3)}{\approx} x^2y^2z^2x^2z^2 \overset{\rho_2}{\approx} x^2y^2z^2,$$

and by a symmetrical argument, $\{(1),(3),\rho_3\} \vdash x^2yz^2 \approx x^2y^2z^2$. Therefore the identity (5b) holds in S.

We have thus shown that the identity (5b) holds in every finite semigroup of the locally finite variety $\mathbf{V}$ and hence must also hold in $\mathbf{V}$. Consequently, $\mathbf{V} \subseteq \mathbf{A_2} \cap [(5b)] = \mathbf{J}$ by Lemma 6 and Proposition 10. □

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