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Finite basis problem for 2-testable monoids

Research Article

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Abstract: A monoid S^1 obtained by adjoining a unit element to a 2-testable semigroup S is said to be 2-*testable*. It is shown that a 2-testable monoid S^1 is either inherently non-finitely based or hereditarily finitely based, depending on whether or not the variety generated by the semigroup S contains the Brandt semigroup of order five. Consequently, it is decidable in quadratic time if a finite 2-testable monoid is finitely based.

MSC: 20M07

Keywords: Semigroups • Monoids • Varieties • Finitely based • Hereditarily finitely based

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1. Introduction

In the present article, all varieties are varieties of semigroups, that is, classes of semigroups that are closed under the formation of homomorphic images, subsemigroups, and arbitrary direct products. A variety is *finitely based* if its identities are finitely axiomatizable. A semigroup is *finitely based* if it generates a finitely based variety. The finite basis problem asks when a given finite semigroup is finitely based. This problem has been intensely investigated as early as the 1960s and is still open. Refer to the surveys of Shevrin and Volkov [20] and Volkov [27] for more information on semigroup varieties and the finite basis problem.

It is well known that the idempotent-generated 0-simple semigroup

$$A_2 = \langle a, b \mid a^2 = aba = a,\ b^2 = 0,\ bab = b \rangle$$

of order five plays several important roles in the theory of semigroup varieties. For instance, Trahtman [25] showed that the variety $\mathbf{A}_2$ generated by the semigroup A_2 coincides with the class of 2-*testable semigroups*, that is, semigroups that

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satisfy any identity formed by a pair of words that begin with the same letter, end with the same letter, and share the same set of factors of length two. The variety $\mathbf{A}_2$ also coincides with the variety generated by all aperiodic 0-simple semigroups [5] and is essential in the recent discovery and description of a new infinite series of limit varieties [16].

The semigroup A_2 is also an important example that is related to semigroups with very extreme and contrasting equational properties. By the early 1980s, Trahtman [22, 24] had proven that the semigroup A_2 is finitely based by the identities

$$x^3 \approx x^2, \qquad xyxyx \approx xyx, \qquad xyxzx \approx xzxyx. \tag{1}$$

Recently, the semigroup A_2 was shown to satisfy the stronger property of being *hereditarily finitely based* [9], that is, every semigroup in the variety $\mathbf{A}_2$ is finitely based. On the other hand, the semigroup A_2 can be used to construct non-finitely based semigroups. Volkov [26] demonstrated that the direct product of the semigroup A_2 with any finite group is non-finitely based. Trahtman [23] proved that the monoid A_2^1 obtained from A_2 by adjoining a unit element is non-finitely based, and Sapir [19] even proved that A_2^1 is *inherently non-finitely based* in the sense that any locally finite variety containing it is non-finitely based.

Since the variety $\mathbf{A}_2$ coincides with the class of 2-testable semigroups [25], it is reasonable to refer to a monoid S^1 as a 2-*testable monoid* whenever S is a semigroup from $\mathbf{A}_2$. Motivated by the contrasting equational properties of the semigroups A_2 and A_2^1, the present article is an in-depth investigation of the finite basis problem for 2-testable monoids. The non-finitely based monoid A_2^1 is vacuously 2-testable. It is routine to verify that the Brandt semigroup

$$B_2 = \langle c, d \,|\, c^2 = d^2 = 0,\, cdc = c,\, dcd = d \rangle$$

of order five satisfies the identities (1) and so belongs to the variety $\mathbf{A}_2$. Therefore the monoid B_2^1 is also 2-testable; this monoid is not only non-finitely based [18] but is also inherently non-finitely based [19].

Remark 1.1.

Up to isomorphism and anti-isomorphism, A_2^1 and B_2^1 are the only monoids that are minimal with respect to being non-finitely based [14]. Consequently, the semigroups A_2 and B_2 are the smallest possible examples of a finitely based semigroup S for which the monoid S^1 is non-finitely based.

For any semigroup or monoid S, let $\textsc{Var}\,\{S\}$ denote the semigroup variety generated by S. It is easy to show that if S and T are semigroups such that $S \in \textsc{Var}\,\{T\}$, then $S^1 \in \textsc{Var}\,\{T^1\}$; see also Almeida [1, Lemma 7.1.1]. Since the monoid B_2^1 is inherently non-finitely based [19], a 2-testable monoid S^1 is also inherently non-finitely based whenever $B_2 \in \textsc{Var}\,\{S\}$. It is thus natural to examine the finite basis problem for 2-testable monoids S^1 for which $B_2 \notin \textsc{Var}\,\{S\}$. The main goal of the present article is to show that such monoids must be hereditarily finitely based, thereby establishing a dichotomy for 2-testable monoids with respect to the finite basis property.

Theorem 1.2.

Let S be any semigroup in the variety $\mathbf{A}_2$.

(i) *If $B_2 \in \textsc{Var}\,\{S\}$, then the monoid S^1 is inherently non-finitely based.*

(ii) *If $B_2 \notin \textsc{Var}\,\{S\}$, then the monoid S^1 is hereditarily finitely based.*

Consequently, any 2-testable monoid is either inherently non-finitely based or hereditarily finitely based.

Remark 1.3.

(i) For any semigroup S in the variety $\mathbf{A}_2$, the semigroup B_2 belongs to the variety $\textsc{Var}\,\{S\}$ if and only if S does not satisfy the identity $xy^2x \approx xyx$ [6]. Therefore, in the presence of Theorem 1.2, checking the finite basis property of a finite 2-testable monoid S^1 is a problem of complexity $O(|S|^2)$.

(ii) Varieties generated by 2-testable monoids will be identified in a sequel article [13].

There are six sections in the present article. Notation and background material are given in Section 2. Main arguments of the proof of Theorem 1.2(ii) are given in Section 3, while the finer details are deferred to Sections 4–6.

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2. Preliminaries

Most of the notation and background material of this article are given in this section. Refer to the monograph of Burris and Sankappanavar [2] for more information on universal algebra.

2.1. Letters and words

Let $\mathcal{X}$ be a fixed countably infinite alphabet throughout. Denote by $\mathcal{X}^+$ and $\mathcal{X}^*$ the free semigroup and the free monoid over $\mathcal{X}$ respectively. Elements of $\mathcal{X}$ and $\mathcal{X}^*$ are referred to as *letters* and *words* respectively.

Let x be any letter and $\mathbf{w}$ be any word. Then

- the *content* of $\mathbf{w}$, denoted by $\mathsf{con}(\mathbf{w})$, is the set of letters occurring in $\mathbf{w}$;
- the *head* of $\mathbf{w}$, denoted by $\mathsf{h}(\mathbf{w})$, is the first letter occurring in $\mathbf{w}$;
- the *tail* of $\mathbf{w}$, denoted by $\mathsf{t}(\mathbf{w})$, is the last letter occurring in $\mathbf{w}$;
- the *initial part* of $\mathbf{w}$, denoted by $\mathsf{ini}(\mathbf{w})$, is the word obtained from $\mathbf{w}$ by retaining the first occurrence of each letter;
- the *final part* of $\mathbf{w}$, denoted by $\mathsf{fin}(\mathbf{w})$, is the word obtained from $\mathbf{w}$ by retaining the last occurrence of each letter;
- x is *simple* in $\mathbf{w}$ if x occurs exactly once in $\mathbf{w}$;
- $\mathbf{w}$ is *simple* if any letter occurs at most once in $\mathbf{w}$;
- $\mathbf{w}$ is *quadratic* if any letter occurs at most twice in $\mathbf{w}$.

Note that by definition, the empty word is simple and any simple word is quadratic.

Let $\mathbf{w}$ be any quadratic word. If $\mathbf{w} = \mathbf{a}x\mathbf{b}x\mathbf{c}$ for some $x \in \mathcal{X}$ and $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathcal{X}^*$ with $x \notin \mathsf{con}(\mathbf{abc})$, then the *distance* between the two occurrences of x in $\mathbf{w}$ is the length of $\mathbf{b}$. If $x_1, \ldots, x_r$ are all the non-simple letters of $\mathbf{w}$, then the *separation degree* of $\mathbf{w}$ is the sum $d_1 + \cdots + d_r$, where d_i is the distance between the two occurrences of x_i in $\mathbf{w}$.

2.2. Identities and varieties

An identity is written as $\mathbf{u} \approx \mathbf{v}$ where $\mathbf{u}, \mathbf{v} \in \mathcal{X}^+$. A semigroup S *satisfies* an identity $\mathbf{u} \approx \mathbf{v}$ if for any substitution φ from $\mathcal{X}$ into S, the elements $\mathbf{u}\varphi$ and $\mathbf{v}\varphi$ of S coincide. A variety $\mathbf{V}$ *satisfies* an identity $\mathbf{u} \approx \mathbf{v}$ if every semigroup in $\mathbf{V}$ satisfies $\mathbf{u} \approx \mathbf{v}$; this is indicated by $\mathbf{V} \models \mathbf{u} \approx \mathbf{v}$.

Let Σ be any set of identities. An identity $\mathbf{u} \approx \mathbf{v}$ is *deducible* from Σ if any semigroup that satisfies the identities in Σ also satisfies $\mathbf{u} \approx \mathbf{v}$; this is indicated by $\Sigma \vdash \mathbf{u} \approx \mathbf{v}$ or $\mathbf{u} \overset{\Sigma}{\approx} \mathbf{v}$. The variety *defined* by Σ is the class of all semigroups that satisfy all identities in Σ; in this case, Σ is a *basis* for the variety. A variety is *finitely based* if it possesses a finite basis. The subvariety of a variety $\mathbf{V}$ defined by Σ is denoted by $\mathbf{V}\Sigma$.

An identity σ *deletes* to an identity σ' if, up to renaming of letters, σ' is obtained from σ by removing all occurrences of some letters in σ. For example, the identity $xyxyzx \approx zx^2y$ deletes to the identities $a^2x_7a \approx x_7a^2$ and $p^2y_2 \approx y_2p$.

For any varieties $\mathbf{U}$ and $\mathbf{V}$ such that $\mathbf{U} \subseteq \mathbf{V}$, the *interval* $[\mathbf{U}, \mathbf{V}]$ is the set of all subvarieties of $\mathbf{V}$ containing $\mathbf{U}$. The lattice of subvarieties of $\mathbf{V}$ is denoted by $\mathcal{L}(\mathbf{V})$. Note that $\mathcal{L}(\mathbf{V}) = [\mathbf{0}, \mathbf{V}]$ where $\mathbf{0}$ is the variety of trivial semigroups.

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2.3. Some 2-testable monoids

It is routine to check that the subsets

$$A_0 = \{0, b, ab, ba\}, \qquad L_2 = \{a, ba\}, \qquad R_2 = \{a, ab\}, \qquad \text{and} \qquad N_2 = \{0, b\}$$

of A_2 are subsemigroups of A_2, and that the subset

$$B_0 = \{0, d, cd, dc\}$$

of B_2 is a subsemigroup of B_2. Note that L_2 is a left-zero semigroup, R_2 is a right-zero semigroup, and N_2 is a null semigroup. The variety $\mathrm{VAR}\{B_0\}$ is the unique maximal subvariety of both the varieties $\mathrm{VAR}\{A_0\}$ and $\mathrm{VAR}\{B_2\}$ [6]; a description of all subvarieties of $\mathrm{VAR}\{B_0\}$ is given by Lee [7]. Let $\mathbf{A}_0^1$, $\mathbf{B}_0^1$, $\mathbf{L}_2^1$, $\mathbf{R}_2^1$, and $\mathbf{N}_2^1$ be the varieties generated by the 2-testable monoids A_0^1, B_0^1, L_2^1, R_2^1, and N_2^1, respectively.

An identity $\mathbf{u} \approx \mathbf{v}$ is *quadratic* if the words $\mathbf{u}$ and $\mathbf{v}$ are quadratic. Define a relation $\stackrel{\circ}{=}$ on $\mathcal{X}^*$ by $\mathbf{u} \stackrel{\circ}{=} \mathbf{v}$ if the words $\mathbf{u}$ and $\mathbf{v}$ can be obtained from one another by rearrangement of letters. It is easy to verify the following well-known results.

Lemma 2.1.
Let $\mathbf{u} \approx \mathbf{v}$ be any identity. Then

(i) $\mathbf{L}_2^1 \vDash \mathbf{u} \approx \mathbf{v}$ *if and only if* $\mathrm{ini}(\mathbf{u}) = \mathrm{ini}(\mathbf{v})$*;*

(ii) $\mathbf{R}_2^1 \vDash \mathbf{u} \approx \mathbf{v}$ *if and only if* $\mathrm{fin}(\mathbf{u}) = \mathrm{fin}(\mathbf{v})$.

Further, if the identity $\mathbf{u} \approx \mathbf{v}$ is quadratic, then

(iii) $\mathbf{N}_2^1 \vDash \mathbf{u} \approx \mathbf{v}$ *if and only if* $\mathbf{u} \stackrel{\circ}{=} \mathbf{v}$.

Lemma 2.2.
Let $\mathbf{u} \approx \mathbf{v}$ be any quadratic identity such that $\mathrm{con}(\mathbf{u}) = \mathrm{con}(\mathbf{v})$. *Then*

(i) $\mathbf{A}_0^1 \vDash \mathbf{u} \approx \mathbf{v}$ *if and only if $\mathbf{u} \approx \mathbf{v}$ does not delete to any of the following identities:*

$$x^2 \approx x, \qquad xy \approx yx, \qquad xyx \approx x^2y, \qquad xyx \approx yx^2, \qquad x^2y \approx yx^2, \tag{2}$$
$$xy^2x \approx x^2y^2, \qquad xy^2x \approx y^2x^2, \qquad xyxy \approx x^2y^2, \qquad xyxy \approx y^2x^2, \qquad x^2y^2 \approx y^2x^2; \tag{3}$$

(ii) $\mathbf{B}_0^1 \vDash \mathbf{u} \approx \mathbf{v}$ *if and only if $\mathbf{u} \approx \mathbf{v}$ does not delete to any identity in* (2).

Proof. Part (i) follows from Edmunds [3, Lemma 4.1 and the proof of Proposition 3.2(a)]. Part (ii) follows from Edmunds [3, proof of Proposition 3.1(i)]. □

Proposition 2.3.
(i) *The variety $\mathbf{A}_0^1 \vee \mathbf{L}_2^1 \vee \mathbf{R}_2^1$ is defined by the identities*

$$xyxzx \approx xyzx, \qquad x^2yx \approx xyx, \qquad xyx^2 \approx xyx, \qquad x^3 \approx x^2. \tag{$\bigstar$}$$

(ii) *The variety $\mathbf{A}_0^1 \vee \mathbf{L}_2^1$ is defined by the identities* ($\bigstar$) *and*

$$xyxy \approx xy^2x. \tag{$\blacktriangleright$}$$

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(iii) *The variety* $\mathbf{A}_0^1 \vee \mathbf{R}_2^1$ *is defined by the identities* (★) *and*

$$xyxy \approx yx^2y. \tag{◀}$$

(iv) *The variety* $\mathbf{A}_0^1$ *is defined by the identities* (★), (▶), *and* (◀).

Remark 2.4.

Note that if a letter x occurs three or more times in a word $\mathbf{w}$, then all except the first and last occurrences of x in $\mathbf{w}$ can be eliminated by the identities (★). Therefore any word can be converted by the identities (★) into a unique quadratic word.

Proof of Proposition 2.3. Parts (ii)–(iv) are consequences of Edmunds [3, Propositions 3.2(a)] and Lee [11, Proposition 3.3 and Corollary 3.4]. Therefore it remains to verify part (i).

It is routine to show that the monoids A_0^1, L_2^1, and R_2^1 satisfy the identities (★). Therefore, to complete the proof, it suffices to show that any identity $\mathbf{u} \approx \mathbf{v}$ of the variety $\mathbf{A}_0^1 \vee \mathbf{L}_2^1 \vee \mathbf{R}_2^1$ is deducible from the identities (★). Since the variety $\mathbf{A}_0^1 \vee \mathbf{L}_2^1 \vee \mathbf{R}_2^1$ satisfies the identities (★), it follows from Remark 2.4 that the words $\mathbf{u}$ and $\mathbf{v}$ can be assumed to be quadratic. Hence $\mathbf{u}$ and $\mathbf{v}$ can be written as $\mathbf{u} = \mathbf{s}_1\mathbf{u}_1 \cdots \mathbf{s}_m\mathbf{u}_m$ and $\mathbf{v} = \mathbf{t}_1\mathbf{v}_1 \cdots \mathbf{t}_n\mathbf{v}_n$, where

- the letters of $\mathbf{s}_1 \in \mathcal{X}^*$ and $\mathbf{s}_2, \ldots, \mathbf{s}_m \in \mathcal{X}^+$ are all simple in $\mathbf{u}$,
- the letters of $\mathbf{u}_1, \ldots, \mathbf{u}_{m-1} \in \mathcal{X}^+$ and $\mathbf{u}_m \in \mathcal{X}^*$ are all non-simple in $\mathbf{u}$,
- the letters of $\mathbf{t}_1 \in \mathcal{X}^*$ and $\mathbf{t}_2, \ldots, \mathbf{t}_n \in \mathcal{X}^+$ are all simple in $\mathbf{v}$,
- the letters of $\mathbf{v}_1, \ldots, \mathbf{v}_{n-1} \in \mathcal{X}^+$ and $\mathbf{v}_n \in \mathcal{X}^*$ are all non-simple in $\mathbf{v}$.

Since the identity $\mathbf{u} \approx \mathbf{v}$ is satisfied by the monoids A_0^1, L_2^1, and R_2^1, it follows from Lemmas 2.1 and 2.2(i) that

(a) $\mathrm{ini}(\mathbf{u}) = \mathrm{ini}(\mathbf{v})$ and $\mathrm{fin}(\mathbf{u}) = \mathrm{fin}(\mathbf{v})$,

(b) $\mathbf{u} \approx \mathbf{v}$ does not delete to any identity from {(2), (3)};

specifically,

(c) $\mathbf{u} \approx \mathbf{v}$ does not delete to the identity $xyxy \approx x^2y^2$.

Since $\mathbf{u}$ and $\mathbf{v}$ are quadratic words, it follows from (b) that $m = n$, $\mathbf{s}_i = \mathbf{t}_i$, and $\mathbf{u}_i \stackrel{\circ}{=} \mathbf{v}_i$ for all i. Therefore $\mathbf{v} = \mathbf{s}_1\mathbf{v}_1 \cdots \mathbf{s}_m\mathbf{v}_m$. It then follows from (a) and (c) that $\mathbf{u}_i = \mathbf{v}_i$ for all i, whence the identity $\mathbf{u} \approx \mathbf{v}$ is trivial. Consequently, the identity $\mathbf{u} \approx \mathbf{v}$ is deducible from the identities (★). □

3. Proof of Theorem 1.2(ii)

It is routine to show that the semigroup C_0 with the following multiplication table satisfies the identities (1) and so belongs to the variety $\mathbf{A}_2$:

C_0	0	a	b	c	d	e
0	0	0	0	0	0	0
a	0	0	0	0	a	a
b	0	a	b	c	0	a
c	0	a	b	c	a	a
d	0	0	0	0	d	d
e	0	0	0	0	e	e

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The semigroups A_0, B_0, L_2, R_2, and N_2 are isomorphic to the subsemigroups

$$\{0, a, b, e\}, \qquad \{0, a, b, d\}, \qquad \{b, c\}, \qquad \{d, e\}, \qquad \text{and} \qquad \{0, a\}$$

of C_0 respectively. Let $\mathbf{A}_0 = \mathrm{V{\scriptsize AR}}\{A_0\}$, $\mathbf{B}_0 = \mathrm{V{\scriptsize AR}}\{B_0\}$, $\mathbf{B}_2 = \mathrm{V{\scriptsize AR}}\{B_2\}$, $\mathbf{C}_0 = \mathrm{V{\scriptsize AR}}\{C_0\}$, and $\mathbf{C}_0^1 = \mathrm{V{\scriptsize AR}}\{C_0^1\}$.

Lemma 3.1.
The variety $\mathbf{C}_0$ is the subvariety of $\mathbf{A}_2$ that is largest with respect to not containing the semigroup B_2.

Proof. This follows from Lee [6, Theorem 3.6] and Lee and Volkov [15, Theorem 4.2(iii)]. □

A complete description of the interval $[\mathbf{B}_0, \mathbf{A}_2]$ can be found in Lee [12, Figure 5]. The varieties $\mathbf{C}_0$ (denoted in Lee [12] by $\overline{\mathbf{B}}_2$) and $\mathbf{A}_2$ are very close since the interval $[\mathbf{C}_0, \mathbf{A}_2]$ coincides with the chain $\mathbf{C}_0 \subset \mathbf{B}_2 \vee \mathbf{C}_0 \subset \mathbf{A}_2$.

Lemma 3.2.

$$\mathbf{C}_0^1 = \mathbf{A}_0^1 \vee \mathbf{L}_2^1 \vee \mathbf{R}_2^1.$$

Proof. Since the monoids A_0^1, L_2^1, and R_2^1 are embeddable in C_0^1, the inclusion $\mathbf{A}_0^1 \vee \mathbf{L}_2^1 \vee \mathbf{R}_2^1 \subseteq \mathbf{C}_0^1$ holds. It is routine to show that the monoid C_0^1 satisfies the identities (★) so that the inclusion $\mathbf{C}_0^1 \subseteq \mathbf{A}_0^1 \vee \mathbf{L}_2^1 \vee \mathbf{R}_2^1$ holds by Proposition 2.3(i). □

Theorem 3.3.
The variety $\mathbf{C}_0^1$ is hereditarily finitely based. Equivalently, any semigroup that satisfies the identities (★) is finitely based.

Theorem 3.3 will be established over the next three sections. Restrictions on identities that can be used to define varieties in the interval $[\mathbf{A}_0, \mathbf{C}_0^1]$ are given in Section 4. These restrictions are then used in Sections 5 and 6 to show that all subvarieties of $\mathbf{C}_0^1$ are finitely based. Specifically, all subvarieties of $\mathbf{C}_0^1$ that contain the variety $\mathbf{A}_0 \vee \mathbf{B}_0^1$ are shown to be finitely based in Proposition 5.1, while those that do not contain the variety $\mathbf{A}_0 \vee \mathbf{B}_0^1$ are shown to be finitely based in Proposition 6.1.

Let S^1 be any 2-testable monoid such that $B_2 \notin \mathrm{V{\scriptsize AR}}\{S\}$. By Lemma 3.1, the semigroup S belongs to the variety $\mathbf{C}_0$ so that the monoid S^1 belongs to the variety $\mathbf{C}_0^1$. By Theorem 3.3, the monoid S^1 is hereditarily finitely based.

4. Identities of varieties containing $\mathbf{A}_0$

Recall from Proposition 2.3(i) and Lemma 3.2 that the variety $\mathbf{C}_0^1$ is defined by the identities (★). Since most equational deductions in the remainder of this article are deductions within the equational theory of the semigroup C_0^1, it is convenient to write $\Sigma_1 \Vdash \Sigma_2$, where Σ_1 and Σ_2 are any sets of identities, to stand for the deduction $(★) \cup \Sigma_1 \vdash \Sigma_2$.

Two words are said to be *disjoint* if they do not share any common letter. A word of length at least two is said to be *connected* if it cannot be written as a product of two disjoint nonempty words. An identity $\mathbf{u} \approx \mathbf{v}$ is *connected* if the words $\mathbf{u}$ and $\mathbf{v}$ are connected.

Lemma 4.1.
Let $\mathbf{u}$ and $\mathbf{v}$ be any words.

(i) *Suppose that $\mathbf{u} = \mathbf{u}_1 \cdots \mathbf{u}_m$ (respectively, $\mathbf{v} = \mathbf{v}_1 \cdots \mathbf{v}_n$) where $\mathbf{u}_1, \ldots, \mathbf{u}_m$ (respectively, $\mathbf{v}_1, \ldots, \mathbf{v}_n$) are pairwise disjoint words each of which is either connected or a singleton. Then $\mathbf{A}_0 \models \mathbf{u} \approx \mathbf{v}$ if and only if $m = n$ and $\mathbf{A}_0 \models \mathbf{u}_i \approx \mathbf{v}_i$ for all i.*

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(ii) *Suppose that either* $\mathbf{u}$ *or* $\mathbf{v}$ *is a simple word. Then* $\mathsf{A}_0 \models \mathbf{u} \approx \mathbf{v}$ *if and only if* $\mathbf{u} = \mathbf{v}$.

(iii) *Suppose that* $\mathbf{u}$ *and* $\mathbf{v}$ *are connected words. Then* $\mathsf{A}_0 \models \mathbf{u} \approx \mathbf{v}$ *if and only if* $\mathrm{con}(\mathbf{u}) = \mathrm{con}(\mathbf{v})$.

Proof. These results follow from the proof of part 4 of the first proposition in Edmunds [4]. □

For any ℓ and r from the set

$$\mathbb{N}^\star = \{0, 1, 2, \ldots, \infty, \infty + 1\},$$

define the words

$$\mathbf{p}_{(\ell)} = \begin{cases} \emptyset & \text{if } \ell = 0, \\ p_1 \cdots p_\ell & \text{if } 1 \le \ell < \infty, \\ p_1^2 p_2 & \text{if } \ell = \infty, \\ p_1^2 & \text{if } \ell = \infty + 1, \end{cases} \quad \text{and} \quad \mathbf{q}_{(r)} = \begin{cases} \emptyset & \text{if } r = 0, \\ q_1 \cdots q_r & \text{if } 1 \le r < \infty, \\ q_1 q_2^2 & \text{if } r = \infty, \\ q_2^2 & \text{if } r = \infty + 1. \end{cases}$$

Lemma 4.2.
Each variety in the interval $[\mathsf{A}_0, \mathsf{C}_0^1]$ *is defined within the variety* C_0^1 *by quadratic identities of the form*

$$\mathbf{p}_{(\ell)} \mathbf{u} \mathbf{q}_{(r)} \approx \mathbf{p}_{(\ell)} \mathbf{v} \mathbf{q}_{(r)} \tag{4}$$

where

(1†) $\ell, r \in \mathbb{N}^\star$;

(2†) $\mathbf{u}, \mathbf{v} \in \mathcal{X}^+$ *are connected and quadratic;*

(3†) $\mathbf{p}_{(\ell)}, \mathbf{u}, \mathbf{q}_{(r)}$ *are pairwise disjoint and* $\mathbf{p}_{(\ell)}, \mathbf{v}, \mathbf{q}_{(r)}$ *are pairwise disjoint;*

(4†) $\mathsf{A}_0 \models \mathbf{u} \approx \mathbf{v}$ *so that* $\mathrm{con}(\mathbf{u}) = \mathrm{con}(\mathbf{v})$;

(5†) $\ell = 0$ *if* $\mathrm{h}(\mathbf{u}) = \mathrm{h}(\mathbf{v})$;

(6†) $r = 0$ *if* $\mathrm{t}(\mathbf{u}) = \mathrm{t}(\mathbf{v})$.

Proof. It suffices to consider a variety in the interval $[\mathsf{A}_0, \mathsf{C}_0^1]$ of the form $\mathsf{C}_0^1\{\xi\}$, where

$$\xi : \mathbf{a} \approx \mathbf{b}$$

is any nontrivial, quadratic identity of the variety A_0, and show that

$$\mathsf{C}_0^1\{\xi\} = \mathsf{C}_0^1\{\xi_1, \ldots, \xi_k\} \tag{5}$$

for some identities $\xi_1, \ldots, \xi_k$ of the form (4) that satisfy conditions (1†)–(6†). By Lemma 4.1(ii), the words $\mathbf{a}$ and $\mathbf{b}$ are non-simple. By Remark 2.4 and Lemma 4.1(i),

$$\mathbf{a} = \mathbf{s}_1 \mathbf{u}_1 \mathbf{s}_2 \mathbf{u}_2 \cdots \mathbf{s}_k \mathbf{u}_k \mathbf{s}_{k+1} \quad \text{and} \quad \mathbf{b} = \mathbf{s}_1 \mathbf{v}_1 \mathbf{s}_2 \mathbf{v}_2 \cdots \mathbf{s}_k \mathbf{v}_k \mathbf{s}_{k+1}$$

where

- $\mathbf{s}_1, \ldots, \mathbf{s}_{k+1} \in \mathcal{X}^*$ are simple;
- $\mathbf{u}_1, \ldots, \mathbf{u}_k, \mathbf{v}_1, \ldots, \mathbf{v}_k \in \mathcal{X}^+$ are quadratic and connected;

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- $\mathbf{s}_1, \mathbf{u}_1, \ldots, \mathbf{s}_k, \mathbf{u}_k, \mathbf{s}_{k+1}$ are pairwise disjoint and $\mathbf{s}_1, \mathbf{v}_1, \ldots, \mathbf{s}_k, \mathbf{v}_k, \mathbf{s}_{k+1}$ are pairwise disjoint;
- $\mathsf{A}_0 \vDash \mathbf{u}_i \approx \mathbf{v}_i$ (so that $\operatorname{con}(\mathbf{u}_i) = \operatorname{con}(\mathbf{v}_i)$) for all $i \in \{1, \ldots, k\}$.

Without loss of generality, assume that $\operatorname{con}\left(\mathbf{p}_{(\ell)}\mathbf{q}_{(r)}\right) \cap \operatorname{con}(\mathbf{ab}) = \emptyset$ for all $\ell, r \in \mathbb{N}^\star$.

First consider the case when $k = 1$. Then the identity ξ is $\mathbf{s}_1\mathbf{u}_1\mathbf{s}_2 \approx \mathbf{s}_1\mathbf{v}_1\mathbf{s}_2$, so that the equation (5) holds with the identity ξ_1 being $\mathbf{p}_{(\ell)}\mathbf{u}_1\mathbf{q}_{(r)} \approx \mathbf{p}_{(\ell)}\mathbf{v}_1\mathbf{q}_{(r)}$, where $\ell = |\mathbf{s}_1|$ and $r = |\mathbf{s}_2|$.

Now suppose that $k \geq 2$. Denote by γ_1 the substitution

$$x \mapsto \begin{cases} q_1 & \text{if } x = \mathsf{h}(\mathbf{s}_2), \\ q_2^2 & \text{if } x \text{ occurs after } \mathsf{h}(\mathbf{s}_2) \text{ in } \mathbf{a}. \end{cases}$$

Then

$$\xi \Vdash \mathbf{s}_1\mathbf{u}_1 q_1 q_2^2 \overset{(\star)}{\approx} \mathbf{a}\gamma_1 \overset{\xi}{\approx} \mathbf{b}\gamma_1 \overset{(\star)}{\approx} \mathbf{s}_1\mathbf{v}_1 q_1 q_2^2 \vdash \xi_1 : \mathbf{s}_1\mathbf{u}_1 q_1 q_2^2 \approx \mathbf{s}_1\mathbf{v}_1 q_1 q_2^2$$

where $q_1 = \emptyset$ if and only if $\mathbf{s}_2 = \emptyset$, whence the identity ξ_1 is equivalent to $\mathbf{p}_{(\ell)}\mathbf{u}_1\mathbf{q}_{(r)} \approx \mathbf{p}_{(\ell)}\mathbf{v}_1\mathbf{q}_{(r)}$ with $\ell = |\mathbf{s}_1|$ and $r \in \{\infty, \infty + 1\}$. For $1 < i < k$, denote by γ_i the substitution

$$x \mapsto \begin{cases} p_1^2 & \text{if } x \text{ occurs before } \mathsf{t}(\mathbf{s}_i) \text{ in } \mathbf{a}, \\ p_2 & \text{if } x = \mathsf{t}(\mathbf{s}_i), \\ q_1 & \text{if } x = \mathsf{h}(\mathbf{s}_{i+1}), \\ q_2^2 & \text{if } x \text{ occurs after } \mathsf{h}(\mathbf{s}_{i+1}) \text{ in } \mathbf{a}. \end{cases}$$

Then

$$\xi \Vdash p_1^2 p_2 \mathbf{u}_i q_1 q_2^2 \overset{(\star)}{\approx} \mathbf{a}\gamma_i \overset{\xi}{\approx} \mathbf{b}\gamma_i \overset{(\star)}{\approx} p_1^2 p_2 \mathbf{v}_i q_1 q_2^2 \vdash \xi_i : p_1^2 p_2 \mathbf{u}_i q_1 q_2^2 \approx p_1^2 p_2 \mathbf{v}_i q_1 q_2^2$$

where $p_2 = \emptyset$ if and only if $\mathbf{s}_i = \emptyset$, and $q_1 = \emptyset$ if and only if $\mathbf{s}_{i+1} = \emptyset$, whence the identity ξ_i is equivalent to $\mathbf{p}_{(\ell)}\mathbf{u}_i\mathbf{q}_{(r)} \approx \mathbf{p}_{(\ell)}\mathbf{v}_i\mathbf{q}_{(r)}$ with $\ell, r \in \{\infty, \infty + 1\}$. Finally, denote by γ_k the substitution

$$x \mapsto \begin{cases} p_1^2 & \text{if } x \text{ occurs before } \mathsf{t}(\mathbf{s}_k) \text{ in } \mathbf{a}, \\ p_2 & \text{if } x = \mathsf{t}(\mathbf{s}_k). \end{cases}$$

Then

$$\xi \Vdash p_1^2 p_2 \mathbf{u}_k \mathbf{s}_{k+1} \overset{(\star)}{\approx} \mathbf{a}\gamma_k \overset{\xi}{\approx} \mathbf{b}\gamma_k \overset{(\star)}{\approx} p_1^2 p_2 \mathbf{v}_k \mathbf{s}_{k+1} \vdash \xi_k : p_1^2 p_2 \mathbf{u}_k \mathbf{s}_{k+1} \approx p_1^2 p_2 \mathbf{v}_k \mathbf{s}_{k+1}$$

where $p_2 = \emptyset$ if and only if $\mathbf{s}_k = \emptyset$, whence the identity ξ_k is equivalent to $\mathbf{p}_{(\ell)}\mathbf{u}_k\mathbf{q}_{(r)} \approx \mathbf{p}_{(\ell)}\mathbf{v}_k\mathbf{q}_{(r)}$ with $\ell \in \{\infty, \infty + 1\}$ and $r = |\mathbf{s}_{k+1}|$. Hence the deduction $\xi \Vdash \{\xi_1, \ldots, \xi_k\}$ holds and the inclusion $\mathbf{C}_0^1\{\xi\} \subseteq \mathbf{C}_0^1\{\xi_1, \ldots, \xi_k\}$ is established.

It remains to verify the deduction $\{\xi_1, \ldots, \xi_k\} \Vdash \xi$ so that the equation (5) holds. For each i, the word $\mathbf{u}_i$ is connected so that the letter $h_i = \mathsf{h}(\mathbf{u}_i)$ occurs at least twice in $\mathbf{u}_i$. Therefore $\mathbf{u}_i \overset{(\star)}{\approx} h_i^2\mathbf{u}_i$. Similarly, the letter $t_i = \mathsf{t}(\mathbf{v}_i)$ occurs at least twice in the word $\mathbf{v}_i$ so that $\mathbf{v}_i \overset{(\star)}{\approx} \mathbf{v}_i t_i^2$. The deduction $\{\xi_1, \ldots, \xi_k\} \Vdash \xi$ then follows since

$$\begin{aligned}
\mathbf{a} = \mathbf{s}_1\mathbf{u}_1\mathbf{s}_2\mathbf{u}_2\mathbf{s}_3\mathbf{u}_3 \cdots \mathbf{s}_k\mathbf{u}_k\mathbf{s}_{k+1} &\overset{(\star)}{\approx} \left(\mathbf{s}_1\mathbf{u}_1\mathbf{s}_2 h_2^2\right)\mathbf{u}_2\mathbf{s}_3\mathbf{u}_3 \cdots \mathbf{s}_k\mathbf{u}_k\mathbf{s}_{k+1} \\
&\overset{\xi_1}{\approx} \mathbf{s}_1\mathbf{v}_1\mathbf{s}_2\left(h_2^2\mathbf{u}_2\right)\mathbf{s}_3\mathbf{u}_3 \cdots \mathbf{s}_k\mathbf{u}_k\mathbf{s}_{k+1} \overset{(\star)}{\approx} \mathbf{s}_1\mathbf{v}_1\left(t_1^2\mathbf{s}_2\mathbf{u}_2\mathbf{s}_3 h_3^2\right)\mathbf{u}_3 \cdots \mathbf{s}_k\mathbf{u}_k\mathbf{s}_{k+1} \\
&\overset{\xi_2}{\approx} \mathbf{s}_1\left(\mathbf{v}_1 t_1^2\right)\mathbf{s}_2\mathbf{v}_2\mathbf{s}_3\left(h_3^2\mathbf{u}_3\right) \cdots \mathbf{s}_k\mathbf{u}_k\mathbf{s}_{k+1} \overset{(\star)}{\approx} \mathbf{s}_1\mathbf{v}_1\mathbf{s}_2\mathbf{v}_2\left(t_2^2\mathbf{s}_3\mathbf{u}_3\mathbf{s}_4 h_4^2\right)\mathbf{u}_4 \cdots \mathbf{s}_k\mathbf{u}_k\mathbf{s}_{k+1} \\
&\overset{\xi_3}{\approx} \ldots \overset{(\star)}{\approx} \ldots \overset{\xi_4}{\approx} \ldots \overset{(\star)}{\approx} \mathbf{s}_1\mathbf{v}_1\mathbf{s}_2\mathbf{v}_2 \cdots \mathbf{s}_{k-1}\mathbf{v}_{k-1}\left(t_{k-1}^2\mathbf{s}_k\mathbf{u}_k\mathbf{s}_{k+1}\right) \\
&\overset{\xi_k}{\approx} \mathbf{s}_1\mathbf{v}_1\mathbf{s}_2\mathbf{v}_2 \cdots \mathbf{s}_{k-1}\left(\mathbf{v}_{k-1} t_{k-1}^2\right)\mathbf{s}_k\mathbf{v}_k\mathbf{s}_{k+1} \overset{(\star)}{\approx} \mathbf{s}_1\mathbf{v}_1\mathbf{s}_2\mathbf{v}_2 \cdots \mathbf{s}_{k-1}\mathbf{v}_{k-1}\mathbf{s}_k\mathbf{v}_k\mathbf{s}_{k+1} = \mathbf{b}.
\end{aligned}$$

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Now each identity ξ_i is of the form (4) and it is easy to see that it satisfies conditions (1†)–(4†). Suppose that $\mathsf{h}(\mathbf{u}) = \mathsf{h}(\mathbf{v}) = h$. Since $\mathbf{u}$ and $\mathbf{v}$ are connected words, the letter h occurs more than once in both $\mathbf{u}$ and $\mathbf{v}$. Therefore

$$\xi_i \Vdash \mathbf{u}\mathbf{q}_{(r)} \overset{(\star)}{\approx} h^{|\mathbf{p}_{(\ell)}|}\mathbf{u}\mathbf{q}_{(r)} \overset{\xi_i}{\approx} h^{|\mathbf{p}_{(\ell)}|}\mathbf{v}\mathbf{q}_{(r)} \overset{(\star)}{\approx} \mathbf{v}\mathbf{q}_{(r)} \vdash \mathbf{u}\mathbf{q}_{(r)} \approx \mathbf{v}\mathbf{q}_{(r)},$$

whence the identity ξ_i can be chosen to satisfy condition (5†). By a symmetrical argument, each identity ξ_i can also be chosen to satisfy condition (6†). □

5. Proof of Theorem 3.3: subvarieties containing $\mathsf{A}_0 \vee \mathsf{B}_0^1$

Proposition 5.1.
Any variety in the interval $[\mathsf{A}_0 \vee \mathsf{B}_0^1, \mathsf{C}_0^1]$ is finitely based.

Proof. Let $\mathbf{V}$ be any variety in the interval $[\mathsf{A}_0 \vee \mathsf{B}_0^1, \mathsf{C}_0^1]$.

Case 1: $\mathsf{A}_0^1 \subseteq \mathbf{V}$. Then $\mathbf{V} \in [\mathsf{A}_0^1, \mathsf{C}_0^1]$ and $\mathbf{V}$ is shown to be finitely based in Subsection 5.1.

Case 2: $\mathsf{A}_0^1 \nsubseteq \mathbf{V}$. Then $\mathbf{V}$ is shown to be finitely based in Subsection 5.2. □

5.1. Varieties in $[\mathsf{A}_0^1, \mathsf{C}_0^1]$

The main result of this subsection, Proposition 5.8, establishes the finite basis property of every variety in the interval $[\mathsf{A}_0^1, \mathsf{C}_0^1]$.

Lemma 5.2.
Let $\mathbf{V}$ be any subvariety of C_0^1.

(i) *If $R_2^1 \notin \mathbf{V}$, then the variety $\mathbf{V}$ satisfies the identity*

$$hxyxyh \approx hxy^2xh. \tag{6}$$

(ii) *If $L_2^1 \notin \mathbf{V}$, then the variety $\mathbf{V}$ satisfies the identity*

$$hxyxyh \approx hyx^2yh. \tag{7}$$

Proof. (i) It follows from Almeida [1, Proposition 10.10.2(b)] that the identity $xyxy \approx xy^2x$ defines the subvariety of C_0^1 that is largest with respect to not containing R_2. Suppose that $R_2^1 \notin \mathbf{V}$. Let φ be the substitution $z \mapsto h^2zh^2$ for all $z \in \mathcal{X}$. Then it follows from Lee [8, Theorem 2] that the variety $\mathbf{V}$ satisfies the identity $(xyxy)\varphi \approx (xy^2x)\varphi$. It is easy to deduce that the variety $\mathbf{V}$ also satisfies the identity (6).

(ii) This is symmetrical to part (i). □

The identities

$$\lambda_m : \mathbf{p}_{(m)}xyxy \approx \mathbf{p}_{(m)}yx^2y, \qquad \rho_m : xyxy\mathbf{q}_{(m)} \approx xy^2x\mathbf{q}_{(m)},$$

where $m \in \mathbb{N}^*$, are required in the present subsection. It is straightforward to verify the following lemma.

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Lemma 5.3.

The following deductions hold:

(i) $\lambda_0 \Vdash \lambda_1 \Vdash \cdots \Vdash \lambda_\infty \Vdash \lambda_{\infty+1} \Vdash$ (7);

(ii) $\rho_0 \Vdash \rho_1 \Vdash \cdots \Vdash \rho_\infty \Vdash \rho_{\infty+1} \Vdash$ (6).

Lemma 5.4.

Let $\mathbf{u} \approx \mathbf{v}$ *be any quadratic, connected identity of the variety* A_0^1.

(i) *If* $\mathrm{ini}(\mathbf{u}) = \mathrm{ini}(\mathbf{v})$ *and* $\mathrm{t}(\mathbf{u}) = \mathrm{t}(\mathbf{v})$, *then* (6) $\Vdash \mathbf{u} \approx \mathbf{v}$.

(ii) *If* $\mathrm{h}(\mathbf{u}) = \mathrm{h}(\mathbf{v})$ *and* $\mathrm{fin}(\mathbf{u}) = \mathrm{fin}(\mathbf{v})$, *then* (7) $\Vdash \mathbf{u} \approx \mathbf{v}$.

(iii) *If* $\mathrm{h}(\mathbf{u}) = \mathrm{h}(\mathbf{v})$ *and* $\mathrm{t}(\mathbf{u}) = \mathrm{t}(\mathbf{v})$, *then* $\{(6), (7)\} \Vdash \mathbf{u} \approx \mathbf{v}$.

Proof. (i) Suppose that $\mathrm{ini}(\mathbf{u}) = \mathrm{ini}(\mathbf{v})$ and $\mathrm{t}(\mathbf{u}) = \mathrm{t}(\mathbf{v})$. Then by Lemma 2.1(i), the identity $\mathbf{u} \approx \mathbf{v}$ is satisfied by the variety $\mathsf{A}_0^1 \vee \mathsf{L}_2^1$. Recall from Proposition 2.3(ii) that the variety $\mathsf{A}_0^1 \vee \mathsf{L}_2^1$ is defined by the identities $\{(\bigstar), (\blacktriangleright)\}$. It follows that there exists a deduction sequence

$$\mathbf{u} = \mathbf{w}_0 \;\Rightarrow\; \mathbf{w}_1 \;\Rightarrow\; \ldots \;\Rightarrow\; \mathbf{w}_r = \mathbf{v} \tag{8a}$$

where each deduction $\mathbf{w}_j \Rightarrow \mathbf{w}_{j+1}$ involves an identity from $\{(\bigstar), (\blacktriangleright)\}$. (Recall that a deduction $\mathbf{w} \Rightarrow \mathbf{w}'$ involves an identity $\mathbf{z} \approx \mathbf{z}'$ if there exist words $\mathbf{h}, \mathbf{t} \in \mathcal{X}^*$ and an endomorphism φ of $\mathcal{X}^+$ such that $\mathbf{w} = \mathbf{h}(\mathbf{z}\varphi)\mathbf{t}$ and $\mathbf{w}' = \mathbf{h}(\mathbf{z}'\varphi)\mathbf{t}$.) For each j, since the identity $\mathbf{u} \approx \mathbf{w}_j$ is satisfied by the variety A_0 and the word $\mathbf{u}$ is connected, it follows from Lemma 4.1 that $\mathbf{w}_j$ is a connected word with $\mathrm{con}(\mathbf{w}_j) = \mathrm{con}(\mathbf{u})$. Let $t = \mathrm{t}(\mathbf{u}) = \mathrm{t}(\mathbf{v})$. Since the deductions $\mathbf{u} \overset{(\bigstar)}{\approx} \mathbf{u}t$ and $\mathbf{v} \overset{(\bigstar)}{\approx} \mathbf{v}t$ follow from the connectedness of the words $\mathbf{u}$ and $\mathbf{v}$, multiplying every word in the sequence (8a) on the right by the letter t results in the deduction sequence

$$\mathbf{u} \;\Rightarrow\; \mathbf{w}_0 t \;\Rightarrow\; \mathbf{w}_1 t \;\Rightarrow\; \ldots \;\Rightarrow\; \mathbf{w}_r t \;\Rightarrow\; \mathbf{v}$$

(where the first and last deductions involve identities from $(\bigstar)$). Hence there is no loss in generality to assume that every word in the sequence (8a) ends with the letter t.

Suppose that the deduction $\mathbf{w}_j \Rightarrow \mathbf{w}_{j+1}$ in (8a) involves the identity $(\blacktriangleright)$. Then $\mathbf{w}_j = \mathbf{abxyc}$ and $\mathbf{w}_{j+1} = \mathbf{abyxc}$ for some $\mathbf{x}, \mathbf{y} \in \mathcal{X}^+$, $\mathbf{a}, \mathbf{c} \in \mathcal{X}^*$, and $\mathbf{b} \in \{\mathbf{xy}, \mathbf{yx}\}$.

Case 1: $\mathbf{c} = \emptyset$. Then $t = \mathrm{t}(\mathbf{x}) = \mathrm{t}(\mathbf{y})$ by assumption. Since $\mathbf{b} \in \{\mathbf{xy}, \mathbf{yx}\}$, the letter t also occurs in $\mathbf{b}$. Therefore

$$\mathbf{w}_j = \mathbf{abxy} \overset{(\bigstar)}{\approx} \mathbf{ab}t\mathbf{yx}^2\mathbf{y}t \overset{(6)}{\approx} \mathbf{ab}t\mathbf{yxyx}t \overset{(\bigstar)}{\approx} \mathbf{abyx} = \mathbf{w}_{j+1}.$$

Case 2: $\mathbf{c} \neq \emptyset$. Since the word $\mathbf{w}_j$ is connected, there exists a letter z that is common to both its factors $\mathbf{abxy}$ and $\mathbf{c}$. Since $\mathbf{b} \in \{\mathbf{xy}, \mathbf{yx}\}$, the letter z is common to $\mathbf{ab}$ and $\mathbf{c}$. Therefore

$$\mathbf{w}_j = \mathbf{abxyc} \overset{(\bigstar)}{\approx} \mathbf{ab}z\mathbf{yx}^2\mathbf{y}z\mathbf{c} \overset{(6)}{\approx} \mathbf{ab}z\mathbf{yxyx}z\mathbf{c} \overset{(\bigstar)}{\approx} \mathbf{abyxc} = \mathbf{w}_{j+1}.$$

It follows from Cases 1 and 2 that any deduction $\mathbf{w}_j \Rightarrow \mathbf{w}_{j+1}$ in (8a) that involves the identity $(\blacktriangleright)$ can be replaced by a deduction sequence that involves identities from $\{(\bigstar), (6)\}$. Consequently, the deduction (6) $\Vdash \mathbf{u} \approx \mathbf{v}$ holds.

(ii) This is symmetrical to part (i).

(iii) Suppose that $\mathrm{h}(\mathbf{u}) = \mathrm{h}(\mathbf{v})$ and $\mathrm{t}(\mathbf{u}) = \mathrm{t}(\mathbf{v})$. By Proposition 2.3(iv), the variety A_0^1 is defined by the identities $\{(\bigstar), (\blacktriangleright), (\blacktriangleleft)\}$. Hence there exists a deduction sequence

$$\mathbf{u} = \mathbf{w}_0 \;\Rightarrow\; \mathbf{w}_1 \;\Rightarrow\; \ldots \;\Rightarrow\; \mathbf{w}_r = \mathbf{v} \tag{8b}$$

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where each deduction $\mathbf{w}_j \Rightarrow \mathbf{w}_{j+1}$ involves an identity from $\{(\bigstar), (\blacktriangleright), (\blacktriangleleft)\}$. For each j, since the identity $\mathbf{w}_j \approx \mathbf{u}$ is satisfied by the variety $\mathbf{A}_0$ and the word $\mathbf{u}$ is connected, it follows from Lemma 4.1 that $\mathbf{w}_j$ is a connected word with $\mathrm{con}(\mathbf{w}_j) = \mathrm{con}(\mathbf{u})$. Let $h = \mathrm{h}(\mathbf{u}) = \mathrm{h}(\mathbf{v})$ and $t = \mathrm{t}(\mathbf{u}) = \mathrm{t}(\mathbf{v})$. Since the deductions $\mathbf{u} \overset{(\bigstar)}{\approx} h\mathbf{u}t$ and $\mathbf{v} \overset{(\bigstar)}{\approx} h\mathbf{v}t$ follow from the connectedness of the words $\mathbf{u}$ and $\mathbf{v}$, multiplying every word in the sequence (8b) on the left by the letter h and on the right by the letter t results in the deduction sequence

$$\mathbf{u} \;\Rightarrow\; h\mathbf{w}_0t \;\Rightarrow\; h\mathbf{w}_1t \;\Rightarrow\; \ldots \;\Rightarrow\; h\mathbf{w}_rt \;\Rightarrow\; \mathbf{v}$$

(where the first and last deductions involve identities from $(\bigstar)$). Hence there is no loss in generality to assume that every word in the sequence (8b) begins and ends with the letters h and t respectively.

By arguments in the proof of part (i), any deduction $\mathbf{w}_j \Rightarrow \mathbf{w}_{j+1}$ in (8b) that involves the identity $(\blacktriangleright)$ can be replaced by a deduction sequence that involves identities from $\{(\bigstar), (6)\}$. By symmetry, any deduction $\mathbf{w}_j \Rightarrow \mathbf{w}_{j+1}$ in (8b) that involves the identity $(\blacktriangleleft)$ can be replaced by a deduction sequence that involves identities from $\{(\bigstar), (7)\}$. Consequently, the deduction $\{(6), (7)\} \Vdash \mathbf{u} \approx \mathbf{v}$ holds. □

Lemma 5.5.

Let $\mathbf{u} \approx \mathbf{v}$ be any quadratic, connected identity of the variety $\mathbf{A}_0^1$ such that $\mathrm{con}(\mathbf{uv}) \cap \mathrm{con}(\mathbf{p}_{(\ell)}\mathbf{q}_{(r)}) = \emptyset$ for some $\ell, r \in \mathbb{N}^*$.

(i) *If* $\mathrm{ini}(\mathbf{u}) = \mathrm{ini}(\mathbf{v})$ *and* $\mathrm{t}(\mathbf{u}) \neq \mathrm{t}(\mathbf{v})$, *then* $\mathbf{C}_0^1\{(6),\ \mathbf{uq}_{(r)} \approx \mathbf{vq}_{(r)}\} = \mathbf{C}_0^1\{\rho_r\}$.

(ii) *If* $\mathrm{h}(\mathbf{u}) \neq \mathrm{h}(\mathbf{v})$ *and* $\mathrm{fin}(\mathbf{u}) = \mathrm{fin}(\mathbf{v})$, *then* $\mathbf{C}_0^1\{(7),\ \mathbf{p}_{(\ell)}\mathbf{u} \approx \mathbf{p}_{(\ell)}\mathbf{v}\} = \mathbf{C}_0^1\{\lambda_\ell\}$.

Proof. (i) Suppose that $\mathrm{ini}(\mathbf{u}) = \mathrm{ini}(\mathbf{v})$ and $\mathrm{t}(\mathbf{u}) = x \neq y = \mathrm{t}(\mathbf{v})$. Since $\mathrm{con}(\mathbf{u}) = \mathrm{con}(\mathbf{v})$ by Lemma 4.1(iii), it follows from Lemma 2.2(i) that the identity $\mathbf{u} \approx \mathbf{v}$ does not delete to any identity in $\{(2), (3)\}$. Hence

$$\mathbf{u} = \mathbf{a}x\mathbf{b}y\mathbf{c}x \qquad \text{and} \qquad \mathbf{v} = \mathbf{d}y\mathbf{e}x\mathbf{f}y$$

for some $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}, \mathbf{f} \in \mathcal{X}^*$ with $y \in \mathrm{con}(\mathbf{ab}) \setminus \mathrm{con}(\mathbf{c})$ and $x \in \mathrm{con}(\mathbf{de}) \setminus \mathrm{con}(\mathbf{f})$. Let φ be the substitution $z \mapsto x$ for all $z \in \mathrm{con}(\mathbf{u}) \setminus \{x, y\}$. Then $xy\big((\mathbf{uq}_{(r)})\varphi\big) \overset{(\bigstar)}{\approx} xy^2x\mathbf{q}_{(r)}$ and $xy\big((\mathbf{vq}_{(r)})\varphi\big) \overset{(\bigstar)}{\approx} xyxy\mathbf{q}_{(r)}$ imply the deduction $\mathbf{uq}_{(r)} \approx \mathbf{vq}_{(r)} \Vdash \rho_r$. Hence

$$\mathbf{C}_0^1\{(6),\ \mathbf{uq}_{(r)} \approx \mathbf{vq}_{(r)}\} = \mathbf{C}_0^1\{(6),\ \mathbf{uq}_{(r)} \approx \mathbf{vq}_{(r)},\ \rho_r\}. \tag{9a}$$

Note that

(a) if some simple letter z of the word $\mathbf{u}$ belongs to the factor $\mathbf{c}$, then the identity $\mathbf{u} \approx \mathbf{v}$ deletes to the identity $y^2z \approx \mathbf{w}$ for some $\mathbf{w} \in \{zy^2, yzy\}$;

(b) if both occurrences of some non-simple letter z of the word $\mathbf{u}$ belong to the factor $\mathbf{c}$, then the identity $\mathbf{u} \approx \mathbf{v}$ deletes to the identity $y^2z^2 \approx \mathbf{w}$ for some $\mathbf{w} \in \{z^2y^2, zyzy, yz^2y\}$.

In both (a) and (b), the identity $\mathbf{u} \approx \mathbf{v}$ deletes to some identity from $\{(2), (3)\}$, contradicting an earlier observation. Thus neither (a) nor (b) is possible, whence each letter in the factor $\mathbf{c}$ is the last occurrence of some non-simple letter of the quadratic word $\mathbf{u}$, that is, each letter in $\mathbf{c}$ has a first occurrence somewhere in $\mathbf{a}$ or $\mathbf{b}$. Therefore

$$\mathbf{uq}_{(r)} \overset{(\bigstar)}{\approx} \mathbf{a}x\mathbf{b}\big(x\mathbf{c}y^2\mathbf{c}x\big)\mathbf{q}_{(r)} \overset{(6)}{\approx} \mathbf{a}x\mathbf{b}x\mathbf{c}y\mathbf{c}yx\mathbf{q}_{(r)} \overset{(\bigstar)}{\approx} \mathbf{a}x\mathbf{b}\mathbf{c}\big(xy^2x\mathbf{q}_{(r)}\big) \overset{\rho_r}{\approx} \mathbf{a}x\mathbf{b}\mathbf{c}xyxy\mathbf{q}_{(r)} \overset{(\bigstar)}{\approx} \mathbf{a}x\mathbf{b}\mathbf{c}xy\mathbf{q}_{(r)} = \mathbf{u}'\mathbf{q}_{(r)},$$

that is, the deduction $\{(6), \rho_r\} \Vdash \mathbf{uq}_{(r)} \approx \mathbf{u}'\mathbf{q}_{(r)}$ holds where $\mathbf{u}' = \mathbf{a}x\mathbf{bc}xy$ is a connected word. Hence

$$\mathbf{C}_0^1\{(6),\ \mathbf{uq}_{(r)} \approx \mathbf{vq}_{(r)},\ \rho_r\} = \mathbf{C}_0^1\{(6),\ \mathbf{u}'\mathbf{q}_{(r)} \approx \mathbf{vq}_{(r)},\ \rho_r\}. \tag{9b}$$

Observe that each identity in $\{(\bigstar), (6), \rho_r\}$ is formed by a pair of words with the same initial part. Therefore, since the identity $\mathbf{uq}_{(r)} \approx \mathbf{u}'\mathbf{q}_{(r)}$ is deduced from the identities $\{(\bigstar), (6), \rho_r\}$, it follows that

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(c) $\mathsf{ini}(\mathbf{u}') = \mathsf{ini}(\mathbf{u}) = \mathsf{ini}(\mathbf{v})$.

By Lemma 2.2(i), the variety $\mathbb{A}_0^1$ satisfies the identities $\{(\bigstar), (6), \rho_r\}$ so that

(d) $\mathbb{A}_0^1 \models \mathbf{u}' \approx \mathbf{u} \approx \mathbf{v}$.

Now since $\mathsf{t}(\mathbf{u}') = y = \mathsf{t}(\mathbf{v})$, it follows from (c), (d), and Lemma 5.4(i) that the deduction $(6) \Vdash \mathbf{u}' \approx \mathbf{v}$ holds; this deduction, together with the deduction $\rho_r \Vdash (6)$ in Lemma 5.3(ii), imply that

$$\mathbb{C}_0^1\left\{(6),\ \mathbf{u}'\mathbf{q}_{(r)} \approx \mathbf{v}\mathbf{q}_{(r)},\ \rho_r\right\} = \mathbb{C}_0^1\left\{\rho_r\right\}. \tag{9c}$$

The result is now obtained by combining (9a), (9b), and (9c).

(ii) This is symmetrical to part (i). □

Lemma 5.6.

Let $\mathbf{u} \approx \mathbf{v}$ be any quadratic, connected identity of the variety $\mathbb{A}_0^1$ such that $\mathsf{con}(\mathbf{uv}) \cap \mathsf{con}\left(\mathbf{p}_{(\ell)}\mathbf{q}_{(r)}\right) = \emptyset$ for some $\ell, r \in \mathbb{N}^*$.

(i) *If* $\mathsf{h}(\mathbf{u}) = \mathsf{h}(\mathbf{v})$ *and* $\mathsf{t}(\mathbf{u}) \neq \mathsf{t}(\mathbf{v})$, *then* $\mathbb{C}_0^1\left\{(6), (7),\ \mathbf{u}\mathbf{q}_{(r)} \approx \mathbf{v}\mathbf{q}_{(r)}\right\} = \mathbb{C}_0^1\left\{(7), \rho_r\right\}$.

(ii) *If* $\mathsf{h}(\mathbf{u}) \neq \mathsf{h}(\mathbf{v})$ *and* $\mathsf{t}(\mathbf{u}) = \mathsf{t}(\mathbf{v})$, *then* $\mathbb{C}_0^1\left\{(6), (7),\ \mathbf{p}_{(\ell)}\mathbf{u} \approx \mathbf{p}_{(\ell)}\mathbf{v}\right\} = \mathbb{C}_0^1\left\{(6), \lambda_\ell\right\}$.

(iii) *If* $\mathsf{h}(\mathbf{u}) \neq \mathsf{h}(\mathbf{v})$ *and* $\mathsf{t}(\mathbf{u}) \neq \mathsf{t}(\mathbf{v})$, *then* $\mathbb{C}_0^1\left\{(6), (7),\ \mathbf{p}_{(\ell)}\mathbf{u}\mathbf{q}_{(r)} \approx \mathbf{p}_{(\ell)}\mathbf{v}\mathbf{q}_{(r)}\right\} = \mathbb{C}_0^1\left\{\lambda_\ell, \rho_r\right\}$.

Proof. (i) Suppose that $\mathsf{h}(\mathbf{u}) = \mathsf{h}(\mathbf{v})$ and $\mathsf{t}(\mathbf{u}) = x \neq y = \mathsf{t}(\mathbf{v})$. Since $\mathsf{con}(\mathbf{u}) = \mathsf{con}(\mathbf{v})$ by Lemma 4.1(iii), it follows from Lemma 2.2(i) that the identity $\mathbf{u} \approx \mathbf{v}$ does not delete to any identity in $\{(2), (3)\}$. Hence

$$\mathbf{u} = \mathbf{a}x\mathbf{b}y\mathbf{c}x \qquad \text{and} \qquad \mathbf{v} = \mathbf{d}y\mathbf{e}x\mathbf{f}y$$

for some $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}, \mathbf{f} \in \mathscr{X}^*$ with $y \in \mathsf{con}(\mathbf{ab}) \setminus \mathsf{con}(\mathbf{c})$ and $x \in \mathsf{con}(\mathbf{de}) \setminus \mathsf{con}(\mathbf{f})$. The deduction $\mathbf{u}\mathbf{q}_{(r)} \approx \mathbf{v}\mathbf{q}_{(r)} \Vdash \rho_r$ can be obtained by following the proof of Lemma 5.5(i). Hence

$$\mathbb{C}_0^1\left\{(6), (7),\ \mathbf{u}\mathbf{q}_{(r)} \approx \mathbf{v}\mathbf{q}_{(r)}\right\} = \mathbb{C}_0^1\left\{(6), (7),\ \mathbf{u}\mathbf{q}_{(r)} \approx \mathbf{v}\mathbf{q}_{(r)},\ \rho_r\right\}. \tag{10a}$$

The deduction $\{(6), \rho_r\} \Vdash \mathbf{u}\mathbf{q}_{(r)} \approx \mathbf{u}'\mathbf{q}_{(r)}$, where $\mathbf{u}' = \mathbf{a}x\mathbf{bc}xy$, can also be obtained by following the proof of Lemma 5.5(i). Hence

$$\mathbb{C}_0^1\left\{(6), (7),\ \mathbf{u}\mathbf{q}_{(r)} \approx \mathbf{v}\mathbf{q}_{(r)},\ \rho_r\right\} = \mathbb{C}_0^1\left\{(6), (7),\ \mathbf{u}'\mathbf{q}_{(r)} \approx \mathbf{v}\mathbf{q}_{(r)},\ \rho_r\right\}. \tag{10b}$$

Now since $\mathbf{u}' \approx \mathbf{v}$ is a quadratic, connected identity of the variety $\mathbb{A}_0^1$ such that $\mathsf{h}(\mathbf{u}') = \mathsf{h}(\mathbf{v})$ and $\mathsf{t}(\mathbf{u}') = \mathsf{t}(\mathbf{v})$, the deduction $\{(6), (7)\} \Vdash \mathbf{u}' \approx \mathbf{v}$ holds by Lemma 5.4(iii). Further, the deduction $\rho_r \Vdash (6)$ holds by Lemma 5.3(ii). Therefore

$$\mathbb{C}_0^1\left\{(6), (7),\ \mathbf{u}'\mathbf{q}_{(r)} \approx \mathbf{v}\mathbf{q}_{(r)},\ \rho_r\right\} = \mathbb{C}_0^1\left\{(7), \rho_r\right\}. \tag{10c}$$

The result is now obtained by combining (10a), (10b), and (10c).

(ii) This is symmetrical to part (i).

(iii) This can be established with slight modifications to arguments in the proofs of parts (i) and (ii). But complete details are given here for the sake of clarity, since some details of the proof of part (i) were omitted due to similarities with the proof of Lemma 5.5(i).

Suppose that $\mathsf{h}(\mathbf{u}) = x \neq y = \mathsf{h}(\mathbf{v})$ and $\mathsf{t}(\mathbf{u}) = z \neq t = \mathsf{t}(\mathbf{v})$. Since $\mathsf{con}(\mathbf{u}) = \mathsf{con}(\mathbf{v})$ by Lemma 4.1(iii), it follows from Lemma 2.2(i) that the identity $\mathbf{u} \approx \mathbf{v}$ does not delete to any identity in $\{(2), (3)\}$. The assumption $\mathsf{h}(\mathbf{u}) = x \neq y = \mathsf{h}(\mathbf{v})$ implies that

$$\mathbf{u} = x\mathbf{a}_1 y\mathbf{b}_1 x\mathbf{c}_1 \qquad \text{and} \qquad \mathbf{v} = y\mathbf{a}_2 x\mathbf{b}_2 y\mathbf{c}_2$$

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for some $\mathbf{a}_i, \mathbf{b}_i, \mathbf{c}_i \in \mathscr{X}^*$ with $y \in \operatorname{con}(\mathbf{b}_1\mathbf{c}_1) \setminus \operatorname{con}(\mathbf{a}_1)$ and $x \in \operatorname{con}(\mathbf{b}_2\mathbf{c}_2) \setminus \operatorname{con}(\mathbf{a}_2)$. The assumption $\mathrm{t}(\mathbf{u}) = z \neq t = \mathrm{t}(\mathbf{v})$ implies that

$$\mathbf{u} = \mathbf{d}_1 z \mathbf{e}_1 t \mathbf{f}_1 z \qquad \text{and} \qquad \mathbf{v} = \mathbf{d}_2 t \mathbf{e}_2 z \mathbf{f}_2 t$$

for some $\mathbf{d}_i, \mathbf{e}_i, \mathbf{f}_i \in \mathscr{X}^*$ with $t \in \operatorname{con}(\mathbf{d}_1\mathbf{e}_1) \setminus \operatorname{con}(\mathbf{f}_1)$ and $z \in \operatorname{con}(\mathbf{d}_2\mathbf{e}_2) \setminus \operatorname{con}(\mathbf{f}_2)$. Let φ be the substitution $h \mapsto x$ for all $h \in \operatorname{con}(\mathbf{u}\mathbf{q}_{(r)}) \setminus \{x, y\}$. Then $\big((\mathbf{p}_{(\ell)}\mathbf{u}\mathbf{q}_{(r)})\varphi\big)yx \overset{(\star)}{\approx} \mathbf{p}_{(\ell)}xy^2x$ and $\big((\mathbf{p}_{(\ell)}\mathbf{v}\mathbf{q}_{(r)})\varphi\big)yx \overset{(\star)}{\approx} \mathbf{p}_{(\ell)}yxyx$ imply the deduction $\mathbf{p}_{(\ell)}\mathbf{u}\mathbf{q}_{(r)} \approx \mathbf{p}_{(\ell)}\mathbf{v}\mathbf{q}_{(r)} \Vdash \lambda_\ell$. Let χ be the substitution $h \mapsto z$ for all $h \in \operatorname{con}(\mathbf{p}_{(\ell)}\mathbf{u}) \setminus \{z, t\}$. Then $zt\big((\mathbf{p}_{(\ell)}\mathbf{u}\mathbf{q}_{(r)})\chi\big) \overset{(\star)}{\approx} zt^2z\,\mathbf{q}_{(r)}$ and $zt\big((\mathbf{p}_{(\ell)}\mathbf{v}\mathbf{q}_{(r)})\chi\big) \overset{(\star)}{\approx} ztzt\,\mathbf{q}_{(r)}$ imply the deduction $\mathbf{p}_{(\ell)}\mathbf{u}\mathbf{q}_{(r)} \approx \mathbf{p}_{(\ell)}\mathbf{v}\mathbf{q}_{(r)} \Vdash \rho_r$. Hence

$$\mathbf{C}_0^1\big\{(6), (7),\ \mathbf{p}_{(\ell)}\mathbf{u}\mathbf{q}_{(r)} \approx \mathbf{p}_{(\ell)}\mathbf{v}\mathbf{q}_{(r)}\big\} = \mathbf{C}_0^1\big\{(6), (7),\ \mathbf{p}_{(\ell)}\mathbf{u}\mathbf{q}_{(r)} \approx \mathbf{p}_{(\ell)}\mathbf{v}\mathbf{q}_{(r)},\ \lambda_\ell,\ \rho_r\big\}. \tag{11a}$$

Consider the word $\mathbf{u}$ written in the form $\mathbf{u} = x\mathbf{a}_1y\mathbf{b}_1x\mathbf{c}_1$ above. Note that

(a) if some simple letter h of the word $\mathbf{u}$ belongs to the factor $\mathbf{a}_1$, then the identity $\mathbf{u} \approx \mathbf{v}$ deletes to the identity $hy^2 \approx \mathbf{w}$ for some $\mathbf{w} \in \{yhy, y^2h\}$;

(b) if both occurrences of some non-simple letter h of the word $\mathbf{u}$ belong to the factor $\mathbf{a}_1$, then the identity $\mathbf{u} \approx \mathbf{v}$ deletes to the identity $h^2y^2 \approx \mathbf{w}$ for some $\mathbf{w} \in \{yh^2y, yhyh, y^2h^2\}$.

In both (a) and (b), the identity $\mathbf{u} \approx \mathbf{v}$ deletes to some identity from $\{(2), (3)\}$, contradicting an earlier observation. Thus neither (a) nor (b) is possible, whence each letter in the factor $\mathbf{a}_1$ is the first occurrence of some non-simple letter of the quadratic word $\mathbf{u}$, that is, each letter in $\mathbf{a}_1$ has a second occurrence somewhere in $\mathbf{b}_1$ or $\mathbf{c}_1$. Therefore

$$\begin{aligned}\mathbf{p}_{(\ell)}\mathbf{u}\mathbf{q}_{(r)} &\overset{(\star)}{\approx} \mathbf{p}_{(\ell)}(x\mathbf{a}_1y^2\mathbf{a}_1x)\mathbf{b}_1x\mathbf{c}_1\mathbf{q}_{(r)} \overset{(7)}{\approx} \mathbf{p}_{(\ell)}xy\mathbf{a}_1y\mathbf{a}_1x\mathbf{b}_1x\mathbf{c}_1\mathbf{q}_{(r)} \overset{(\star)}{\approx} (\mathbf{p}_{(\ell)}xy^2x)\mathbf{a}_1\mathbf{b}_1x\mathbf{c}_1\mathbf{q}_{(r)} \\ &\overset{\lambda_\ell}{\approx} \mathbf{p}_{(\ell)}yxyx\mathbf{a}_1\mathbf{b}_1x\mathbf{c}_1\mathbf{q}_{(r)} \overset{(\star)}{\approx} \mathbf{p}_{(\ell)}yx\mathbf{a}_1\mathbf{b}_1x\mathbf{c}_1\mathbf{q}_{(r)} = \mathbf{p}_{(\ell)}\mathbf{u}'\mathbf{q}_{(r)},\end{aligned}$$

that is, the deduction

$$\{(7), \lambda_\ell\} \Vdash \mathbf{p}_{(\ell)}\mathbf{u}\mathbf{q}_{(r)} \approx \mathbf{p}_{(\ell)}\mathbf{u}'\mathbf{q}_{(r)} \tag{11b}$$

holds where $\mathbf{u}' = yx\mathbf{a}_1\mathbf{b}_1x\mathbf{c}_1$ is a connected word such that $\mathrm{h}(\mathbf{u}') = \mathrm{h}(\mathbf{v})$ and $\mathrm{t}(\mathbf{u}') = \mathrm{t}(\mathbf{u})$. Now consider the word $\mathbf{v}$ written in the form $\mathbf{v} = \mathbf{d}_2t\mathbf{e}_2z\mathbf{f}_2t$ above. By a symmetrical argument, the deduction

$$\{(6), \lambda_\ell\} \Vdash \mathbf{p}_{(\ell)}\mathbf{v}\mathbf{q}_{(r)} \approx \mathbf{p}_{(\ell)}\mathbf{v}'\mathbf{q}_{(r)} \tag{11c}$$

is obtained where $\mathbf{v}' = \mathbf{d}_2t\mathbf{e}_2\mathbf{f}_2tz$ is a connected word such that $\mathrm{h}(\mathbf{v}') = \mathrm{h}(\mathbf{v})$ and $\mathrm{t}(\mathbf{v}') = \mathrm{t}(\mathbf{u})$. It follows from (11a), (11b) and (11c) that

$$\mathbf{C}_0^1\big\{(6), (7),\ \mathbf{p}_{(\ell)}\mathbf{u}\mathbf{q}_{(r)} \approx \mathbf{p}_{(\ell)}\mathbf{v}\mathbf{q}_{(r)}\big\} = \mathbf{C}_0^1\big\{(6), (7),\ \mathbf{p}_{(\ell)}\mathbf{u}'\mathbf{q}_{(r)} \approx \mathbf{p}_{(\ell)}\mathbf{v}'\mathbf{q}_{(r)},\ \lambda_\ell,\ \rho_r\big\}. \tag{11d}$$

Since $\mathbf{u}' \approx \mathbf{v}'$ is a quadratic, connected identity of $\mathbf{A}_0^1$ with $\mathrm{h}(\mathbf{u}') = \mathrm{h}(\mathbf{v}')$ and $\mathrm{t}(\mathbf{u}') = \mathrm{t}(\mathbf{v}')$, the deduction $\{(6), (7)\} \Vdash \mathbf{u}' \approx \mathbf{v}'$ holds by Lemma 5.4(iii); this deduction, together with the deduction $\{\lambda_\ell, \rho_r\} \Vdash \{(6), (7)\}$ in Lemma 5.3, imply that

$$\mathbf{C}_0^1\big\{(6), (7),\ \mathbf{p}_{(\ell)}\mathbf{u}'\mathbf{q}_{(r)} \approx \mathbf{p}_{(\ell)}\mathbf{v}'\mathbf{q}_{(r)},\ \lambda_\ell,\ \rho_r\big\} = \mathbf{C}_0^1\{\lambda_\ell, \rho_r\}. \tag{11e}$$

The result is now obtained by combining (11d) and (11e). □

Lemma 5.7.

Let $\sigma : \mathbf{p}_{(\ell)}\mathbf{u}\mathbf{q}_{(r)} \approx \mathbf{p}_{(\ell)}\mathbf{v}\mathbf{q}_{(r)}$ be the identity (4) that satisfies conditions (1†)–(6†) in Lemma 4.2. Suppose that the variety $\mathbf{A}_0^1$ satisfies the identity σ. Then $\mathbf{C}_0^1\{\sigma\} = \mathbf{C}_0^1(\Sigma_1 \cup \Sigma_2)$ for some $\Sigma_1 \subseteq \{(6), (7)\}$ and some $\Sigma_2 \subseteq \{\lambda_\ell, \rho_r\}$.

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Proof. Since A_0^1 is a monoid, it follows from condition (3†) of Lemma 4.2 that the variety $\mathbf{A}_0^1$ also satisfies the identity $\mathbf{u} \approx \mathbf{v}$. There are four cases determined by the conditions $\mathrm{ini}(\mathbf{u}) = \mathrm{ini}(\mathbf{v})$ and $\mathrm{fin}(\mathbf{u}) = \mathrm{fin}(\mathbf{v})$.

Case 1: $\mathrm{ini}(\mathbf{u}) = \mathrm{ini}(\mathbf{v})$ and $\mathrm{fin}(\mathbf{u}) = \mathrm{fin}(\mathbf{v})$. By Lemma 2.1, the identity σ is satisfied by the monoids L_2^1 and R_2^1. Since $\mathbf{C}_0^1 = \mathbf{A}_0^1 \vee \mathbf{L}_2^1 \vee \mathbf{R}_2^1$ by Lemma 3.2, the identity σ is also satisfied by the variety $\mathbf{C}_0^1$. Hence $\mathbf{C}_0^1\{\sigma\} = \mathbf{C}_0^1$.

Case 2: $\mathrm{ini}(\mathbf{u}) = \mathrm{ini}(\mathbf{v})$ and $\mathrm{fin}(\mathbf{u}) \neq \mathrm{fin}(\mathbf{v})$. Since $\mathrm{h}(\mathbf{u}) = \mathrm{h}(\mathbf{v})$, it follows from condition (5†) of Lemma 4.2 that the identity σ is $\mathbf{uq}_{(r)} \approx \mathbf{vq}_{(r)}$. By Lemma 2.1(ii), the identity σ is not satisfied by the monoid R_2^1. Hence

(a) $\mathbf{C}_0^1\{\sigma\} = \mathbf{C}_0^1\{(6),\ \mathbf{uq}_{(r)} \approx \mathbf{vq}_{(r)}\}$

by Lemma 5.2(i). If $\mathrm{t}(\mathbf{u}) \neq \mathrm{t}(\mathbf{v})$, then $\mathbf{C}_0^1\{\sigma\} \overset{\text{(a)}}{=} \mathbf{C}_0^1\{(6),\ \mathbf{uq}_{(r)} \approx \mathbf{vq}_{(r)}\} = \mathbf{C}_0^1\{\rho_r\}$ by Lemma 5.5(i). If $\mathrm{t}(\mathbf{u}) = \mathrm{t}(\mathbf{v})$, then the identity σ is $\mathbf{u} \approx \mathbf{v}$ by condition (6†) of Lemma 4.2, whence $\mathbf{C}_0^1\{\sigma\} \overset{\text{(a)}}{=} \mathbf{C}_0^1\{(6),\ \mathbf{u} \approx \mathbf{v}\} = \mathbf{C}_0^1(6)$ by Lemma 5.4(i).

Case 3: $\mathrm{ini}(\mathbf{u}) \neq \mathrm{ini}(\mathbf{v})$ and $\mathrm{fin}(\mathbf{u}) = \mathrm{fin}(\mathbf{v})$. By an argument that is symmetrical to Case 2, the variety $\mathbf{C}_0^1\{\sigma\}$ is either $\mathbf{C}_0^1\{\lambda_\ell\}$ or $\mathbf{C}_0^1(7)$.

Case 4: $\mathrm{ini}(\mathbf{u}) \neq \mathrm{ini}(\mathbf{v})$ and $\mathrm{fin}(\mathbf{u}) \neq \mathrm{fin}(\mathbf{v})$. Then

(b) $\mathbf{C}_0^1\{\sigma\} = \mathbf{C}_0^1\{(6), (7),\ \mathbf{p}_{(\ell)}\mathbf{uq}_{(r)} \approx \mathbf{p}_{(\ell)}\mathbf{vq}_{(r)}\}$

by Lemmas 2.1 and 5.2. There are four subcases determined by the conditions $\mathrm{h}(\mathbf{u}) = \mathrm{h}(\mathbf{v})$ and $\mathrm{t}(\mathbf{u}) = \mathrm{t}(\mathbf{v})$.

- **4.1:** $\mathrm{h}(\mathbf{u}) = \mathrm{h}(\mathbf{v})$ and $\mathrm{t}(\mathbf{u}) = \mathrm{t}(\mathbf{v})$. Then the identity σ is $\mathbf{u} \approx \mathbf{v}$ by conditions (5†) and (6†) in Lemma 4.2. Hence $\mathbf{C}_0^1\{\sigma\} \overset{\text{(b)}}{=} \mathbf{C}_0^1\{(6), (7),\ \mathbf{u} \approx \mathbf{v}\} = \mathbf{C}_0^1\{(6), (7)\}$ by Lemma 5.4(iii).
- **4.2:** $\mathrm{h}(\mathbf{u}) = \mathrm{h}(\mathbf{v})$ and $\mathrm{t}(\mathbf{u}) \neq \mathrm{t}(\mathbf{v})$. Then the identity σ is $\mathbf{uq}_{(r)} \approx \mathbf{vq}_{(r)}$ by conditions (5†) in Lemma 4.2. Hence $\mathbf{C}_0^1\{\sigma\} \overset{\text{(b)}}{=} \mathbf{C}_0^1\{(6), (7),\ \mathbf{uq}_{(r)} \approx \mathbf{vq}_{(r)}\} = \mathbf{C}_0^1\{(7), \rho_r\}$ by Lemma 5.6(i).
- **4.3:** $\mathrm{h}(\mathbf{u}) \neq \mathrm{h}(\mathbf{v})$ and $\mathrm{t}(\mathbf{u}) = \mathrm{t}(\mathbf{v})$. This is symmetrical to Subcase 4.2 so that $\mathbf{C}_0^1\{\sigma\} = \mathbf{C}_0^1\{(6), \lambda_\ell\}$.
- **4.4:** $\mathrm{h}(\mathbf{u}) \neq \mathrm{h}(\mathbf{v})$ and $\mathrm{t}(\mathbf{u}) \neq \mathrm{t}(\mathbf{v})$. Then $\mathbf{C}_0^1\{\sigma\} \overset{\text{(b)}}{=} \mathbf{C}_0^1\{(6), (7),\ \mathbf{p}_{(\ell)}\mathbf{uq}_{(r)} \approx \mathbf{p}_{(\ell)}\mathbf{vq}_{(r)}\} = \mathbf{C}_0^1\{\lambda_\ell, \rho_r\}$ by Lemma 5.6(iii). □

Proposition 5.8.
Any variety in the interval $[\mathbf{A}_0^1, \mathbf{C}_0^1]$ is finitely based.

Proof. Let $\mathbf{V}$ be any variety in the interval $[\mathbf{A}_0^1, \mathbf{C}_0^1]$. Then $\mathbf{V} = \mathbf{C}_0^1\Sigma$ for some set Σ of identities of the variety $\mathbf{A}_0^1$. By Lemma 4.2, the identities in the set Σ can be chosen to be of the form (4) that satisfy conditions (1†)–(6†).

Let $\sigma : \mathbf{p}_{(\ell)}\mathbf{uq}_{(r)} \approx \mathbf{p}_{(\ell)}\mathbf{vq}_{(r)}$ be any identity from the set Σ. By condition (3†) of Lemma 4.2, the variety $\mathbf{A}_0^1$ satisfies the identity $\mathbf{u} \approx \mathbf{v}$. Then it follows from Lemma 5.7 that $\mathbf{C}_0^1\{\sigma\} = \mathbf{C}_0^1(\Sigma_1^\sigma \cup \Sigma_2^\sigma)$ for some $\Sigma_1^\sigma \subseteq \{(6), (7)\}$ and some $\Sigma_2^\sigma \subseteq \{\lambda_\ell, \rho_r\}$. Since the identity σ is arbitrarily chosen from the set Σ, repeating the same argument on every identity in Σ yields $\mathbf{C}_0^1\Sigma = \mathbf{C}_0^1(\Sigma_1 \cup \Sigma_2)$ for some $\Sigma_1 \subseteq \{(6), (7)\}$ and some $\Sigma_2 \subseteq \{\lambda_n, \rho_n \mid n \in \mathbb{N}^*\}$. It follows from Lemma 5.3 that $\mathbf{V} = \mathbf{C}_0^1(\Sigma_1 \cup \Sigma_2')$ for some finite subset Σ_2' of Σ_2. □

5.2. Varieties in $[\mathbf{A}_0 \vee \mathbf{B}_0^1, \mathbf{C}_0^1]$ not containing $\mathbf{A}_0^1$

The main result of this subsection, Proposition 5.12, establishes the finite basis property of every variety in the interval $[\mathbf{A}_0 \vee \mathbf{B}_0^1, \mathbf{C}_0^1]$ that does not contain the variety $\mathbf{A}_0^1$.

Lemma 5.9.
Let $\mathbf{V}$ be any subvariety of $\mathbf{C}_0^1$ such that $A_0^1 \notin \mathbf{V}$. Then the variety $\mathbf{V}$ satisfies the identity

$$hxyxyh \approx hx^2y^2h. \tag{12}$$

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Proof. It follows from Torlopova [21] that the identity $xyxy \approx x^2y^2$ defines the subvariety of $\mathbf{C}_0^1$ that is largest with respect to not containing the semigroup A_0. Suppose that $A_0^1 \notin \mathbf{V}$. Let φ be the substitution $z \mapsto h^2zh^2$ for all $z \in \mathcal{X}$. Then it follows from Lee [8, Theorem 2] that the variety $\mathbf{V}$ satisfies the identity $(xyxy)\varphi \approx (x^2y^2)\varphi$. It is easy to deduce that the variety $\mathbf{V}$ also satisfies the identity (12). □

Let x and y be any distinct non-simple letters of a quadratic word $\mathbf{w}$. If $\mathbf{w} \in \mathcal{X}^*x\mathcal{X}^*xy\mathcal{X}^*y\mathcal{X}^*$, then x and y form a *standoff pair* in $\mathbf{w}$. A quadratic word with no standoff pairs is said to be *peaceful*.

Lemma 5.10.

Let $\mathbf{w}$ be any connected, quadratic word. Then the deduction (12) $\Vdash \mathbf{w} \approx \mathbf{w}^\pi$ holds for some peaceful, connected, quadratic word $\mathbf{w}^\pi$.

Proof. Suppose that the letters x and y form a standoff pair in $\mathbf{w}$. Then $\mathbf{w} = \mathbf{a}x\mathbf{b}xy\mathbf{c}y\mathbf{d}$ for some $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \in \mathcal{X}^*$. Since the word $\mathbf{w}$ is connected, its factors $\mathbf{a}x\mathbf{b}x$ and $y\mathbf{c}y\mathbf{d}$ share some common letter h. Further, the word $\mathbf{w}$ is quadratic so that the letter h is neither x nor y. Therefore

$$\mathbf{w} \overset{(\bigstar)}{\approx} \mathbf{a}x\mathbf{b}hx^2y^2h\mathbf{c}y\mathbf{d} \overset{(12)}{\approx} \mathbf{a}x\mathbf{b}hxyxyh\mathbf{c}y\mathbf{d} \overset{(\bigstar)}{\approx} \mathbf{a}x\mathbf{b}yx\mathbf{c}y\mathbf{d} = \mathbf{w}_{x,y},$$

that is, the identities $\{(\bigstar), (12)\}$ can be used to convert the word $\mathbf{w}$ into a word $\mathbf{w}_{x,y}$ in which the letters x and y no longer form a standoff pair. In other words, the identities $\{(\bigstar), (12)\}$ *reconciled* the standoff pair x and y. Observe that in the process of reconciliating a standoff pair x and y,

- the distance between the two occurrences of x is increased by one;
- the distance between the two occurrences of y is increased by one;
- the distance between the two occurrences of any other non-simple letter remains unchanged.

Therefore the separation degree of the resulting word $\mathbf{w}_{x,y}$ is at least two greater than the separation degree of $\mathbf{w}$. It is easy to see that the word $\mathbf{w}_{x,y}$ is connected and quadratic such that $\mathbf{w} \triangleq \mathbf{w}_{x,y}$.

There are at most $|\mathbf{w}|!$ words that are $\triangleq$-related to the word $\mathbf{w}$. Among all these $|\mathbf{w}|!$ words, one must possess the greatest possible separation degree, say d. By the above observations, the process of reconciliation can only be repeated on $\mathbf{w}$ at most $\lfloor d/2 \rfloor$ times. When no more reconciliation can be performed, the word is then the required peaceful, connected, quadratic word $\mathbf{w}^\pi$. □

Lemma 5.11.

Let $\mathbf{u}$ and $\mathbf{v}$ be peaceful, connected, quadratic words such that the identity $\mathbf{u} \approx \mathbf{v}$ does not delete to any identity in (2). Then the identity $\mathbf{u} \approx \mathbf{v}$ does not delete to any identity in (3).

Proof. Working toward a contradiction, suppose that the identity $\mathbf{u} \approx \mathbf{v}$ deletes to some identity in (3). By symmetry, it suffices to assume that the identity $\mathbf{u} \approx \mathbf{v}$ deletes to one of the following identities: $xy^2x \approx x^2y^2$, $xyxy \approx x^2y^2$, $yxyx \approx x^2y^2$, and $y^2x^2 \approx x^2y^2$. Hence the word $\mathbf{u}$ is either

$$\mathbf{a}x\mathbf{b}y\mathbf{c}y\mathbf{d}x\mathbf{e} \quad \text{or} \quad \mathbf{a}x\mathbf{b}y\mathbf{c}x\mathbf{d}y\mathbf{e} \quad \text{or} \quad \mathbf{a}y\mathbf{b}x\mathbf{c}y\mathbf{d}x\mathbf{e} \quad \text{or} \quad \mathbf{a}y\mathbf{b}y\mathbf{c}x\mathbf{d}x\mathbf{e}$$

for some $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e} \in \mathcal{X}^*$ with $x, y \notin \operatorname{con}(\mathbf{abcde})$, and

$$\mathbf{v} = \mathbf{a}'x\mathbf{b}'x\mathbf{c}'y\mathbf{d}'y\mathbf{e}'$$

for some $\mathbf{a}', \mathbf{b}', \mathbf{c}', \mathbf{d}', \mathbf{e}' \in \mathcal{X}^*$ with $x, y \notin \operatorname{con}(\mathbf{a}'\mathbf{b}'\mathbf{c}'\mathbf{d}'\mathbf{e}')$. If $\mathbf{c}' = \emptyset$, then the letters x and y form a standoff pair in the word $\mathbf{v}$. If $\mathbf{c}' \neq \emptyset$ and every letter in the factor $\mathbf{c}'$ is non-simple in the word $\mathbf{v}$, then $\mathbf{v}$ contains some standoff pair. If $\mathbf{c}' \neq \emptyset$ and some letter in the factor $\mathbf{c}'$ is simple in the word $\mathbf{v}$, then it is easy to show that the identity $\mathbf{u} \approx \mathbf{v}$ deletes to some identity in (2). □

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Proposition 5.12.
Any variety in the interval $[\mathbf{A}_0 \vee \mathbf{B}_0^1, \mathbf{C}_0^1]$ *that does not contain the variety* $\mathbf{A}_0^1$ *is finitely based.*

Proof. Let $\mathbf{V}$ be any variety in the interval $[\mathbf{A}_0 \vee \mathbf{B}_0^1, \mathbf{C}_0^1]$ such that $\mathbf{A}_0^1 \not\subseteq \mathbf{V}$. Then it follows from Lemma 5.9 that $\mathbf{V} = \mathbf{C}_0^1((12) \cup \Sigma)$ for some set Σ of identities of the variety $\mathbf{A}_0 \vee \mathbf{B}_0^1$. By Lemma 4.2, the identities in the set Σ can be chosen to be of the form (4) that satisfy conditions (1†)–(6†).

Let $\sigma : \mathbf{p}_{(\ell)}\mathbf{u}\mathbf{q}_{(r)} \approx \mathbf{p}_{(\ell)}\mathbf{v}\mathbf{q}_{(r)}$ be any identity from the set Σ. By condition (3†) of Lemma 4.2,

(a) $\mathbf{B}_0^1 \models \mathbf{u} \approx \mathbf{v}$.

Since the words $\mathbf{u}$ and $\mathbf{v}$ are connected and quadratic by condition (2†) of Lemma 4.2, it follows from Lemma 5.10 that the deduction

(b) $(12) \Vdash \{\mathbf{u} \approx \mathbf{u}^\pi, \mathbf{v} \approx \mathbf{v}^\pi\}$

holds for some peaceful, connected, quadratic words $\mathbf{u}^\pi$ and $\mathbf{v}^\pi$. Hence

(c) $\mathbf{C}_0^1\{(12), \sigma\} = \mathbf{C}_0^1\{(12), \sigma^\pi\}$

where σ^π is the identity $\mathbf{p}_{(\ell)}\mathbf{u}^\pi\mathbf{q}_{(r)} \approx \mathbf{p}_{(\ell)}\mathbf{v}^\pi\mathbf{q}_{(r)}$. By Lemma 2.2(ii), the variety $\mathbf{B}_0^1$ satisfies the identity (12). Therefore it follows from (a) and (b) that the variety $\mathbf{B}_0^1$ satisfies the identity $\mathbf{u}^\pi \approx \mathbf{v}^\pi$, whence by Lemma 2.2(ii), the identity $\mathbf{u}^\pi \approx \mathbf{v}^\pi$ does not delete to any identity in (2). By Lemma 5.11, the identity $\mathbf{u}^\pi \approx \mathbf{v}^\pi$ also does not delete to any identity in (3). Hence by Lemma 2.2(i), the variety $\mathbf{A}_0^1$ satisfies the identity $\mathbf{u}^\pi \approx \mathbf{v}^\pi$ so that

(d) $\mathbf{A}_0^1 \models \sigma^\pi$.

Since the identity σ is arbitrarily chosen from the set Σ, the construction of σ^π from σ can be repeated on every other identity in Σ to obtain the set $\Sigma^\pi = \{\sigma^\pi \mid \sigma \in \Sigma\}$ where each identity in Σ^π satisfies (c) and (d). Now the variety $\mathbf{C}_0^1\Sigma^\pi$ belongs to the interval $[\mathbf{A}_0^1, \mathbf{C}_0^1]$ and so is finitely based by Proposition 5.8. Therefore the variety $\mathbf{V} = \mathbf{C}_0^1((12) \cup \Sigma) \stackrel{(c)}{=} \mathbf{C}_0^1(12) \cap \mathbf{C}_0^1\Sigma^\pi$ is also finitely based. □

6. Proof of Theorem 3.3: subvarieties not containing $\mathbf{A}_0 \vee \mathbf{B}_0^1$

Proposition 6.1.
Any subvariety of $\mathbf{C}_0^1$ *that does not contain the variety* $\mathbf{A}_0 \vee \mathbf{B}_0^1$ *is finitely based.*

Proof. Let $\mathbf{V}$ be any subvariety of $\mathbf{C}_0^1$ that does not contain the variety $\mathbf{A}_0 \vee \mathbf{B}_0^1$. Then there are two cases depending on whether $\mathbf{A}_0 \not\subseteq \mathbf{V}$ or $\mathbf{B}_0^1 \not\subseteq \mathbf{V}$.

Case 1: $\mathbf{A}_0 \not\subseteq \mathbf{V}$. Then the variety $\mathbf{V}$ satisfies the identity $xyxy \approx x^2y^2$, since this identity defines the subvariety of $\mathbf{C}_0^1$ that is largest with respect to not containing the semigroup A_0 [21]. Any variety that satisfies the identities (★) and $xyxy \approx x^2y^2$ is finitely based [28, Chapter 3]; see also Luo and Zhang [17].

Case 2: $\mathbf{A}_0 \subseteq \mathbf{V}$ and $\mathbf{B}_0^1 \not\subseteq \mathbf{V}$. There are two subcases depending on whether or not the variety $\mathbf{V}$ contains the subvariety $\mathbf{N}_2^1$ of $\mathbf{B}_0^1$.

2.1: $\mathbf{N}_2^1 \subseteq \mathbf{V}$. Then $\mathbf{V} \in [\mathbf{A}_0 \vee \mathbf{N}_2^1, \mathbf{C}_0^1]$ and $\mathbf{B}_0^1 \not\subseteq \mathbf{V}$, and $\mathbf{V}$ is shown to be finitely based in Subsection 6.1.

2.2: $\mathbf{N}_2^1 \not\subseteq \mathbf{V}$. Then $\mathbf{V} \in [\mathbf{A}_0, \mathbf{C}_0^1]$ and $\mathbf{N}_2^1 \not\subseteq \mathbf{V}$, and $\mathbf{V}$ is shown to be finitely based in Subsection 6.2. □

Lemma 6.2.
Suppose that the identity $\mathbf{u} \approx \mathbf{v}$ *deletes to the identity* $\mathbf{u}' \approx \mathbf{v}'$. *Then the deduction* $\mathbf{u} \approx \mathbf{v} \Vdash h\mathbf{u}'h \approx h\mathbf{v}'h$ *holds.*

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Proof. Suppose that the identity $\mathbf{u}' \approx \mathbf{v}'$ is obtained from the identity $\mathbf{u} \approx \mathbf{v}$ by removing all occurrences of the letters $x_1, \ldots, x_r$. Denote by φ the substitution $x_i \mapsto h$ for all $i \in \{1, \ldots, r\}$. Then

$$\mathbf{u} \approx \mathbf{v} \Vdash h\mathbf{u}'h \overset{(\star)}{\approx} h(\mathbf{u}\varphi)h \approx h(\mathbf{v}\varphi)h \overset{(\star)}{\approx} h\mathbf{v}'h \vdash h\mathbf{u}'h \approx h\mathbf{v}'h. \qquad \square$$

6.1. Varieties in $[\mathsf{A}_0 \vee \mathsf{N}_2^1, \mathsf{C}_0^1]$ not containing B_0^1.

The main result of this subsection, Proposition 6.8, establishes the finite basis property of every variety in the interval $[\mathsf{A}_0 \vee \mathsf{N}_2^1, \mathsf{C}_0^1]$ that does not contain the variety B_0^1.

Lemma 6.3.
Let V *be any variety in the interval* $[\mathsf{A}_0 \vee \mathsf{N}_2^1, \mathsf{C}_0^1]$ *such that* $B_0^1 \notin \mathsf{V}$. *Then the variety* V *satisfies one of the following identities:*

$$hxyxh \approx hx^2yh, \qquad (13)$$
$$hxyxh \approx hyx^2h, \qquad (14)$$
$$hx^2yh \approx hyx^2h, \qquad (15)$$
$$hxyh \approx hyxh. \qquad (16)$$

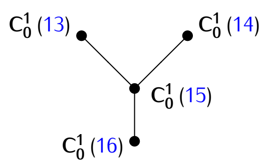
Proof. By assumption, some identity σ of the variety V is not satisfied by the monoid B_0^1; by Remark 2.4, this identity can be chosen to be quadratic. By Lemmas 2.1(iii) and 2.2, the identity σ deletes to one of the identities from (2) except $x^2 \approx x$. The result now follows from Lemma 6.2. $\square$

The following small semigroups are required in the proof of the next lemma:

$$I = \langle a, b \mid a^2 = ba = 0,\ ab = a,\ b^2 = b \rangle,$$
$$J = \langle a, b \mid a^2 = ab = 0,\ ba = a,\ b^2 = b \rangle,$$
$$K = \langle a, b \mid a^2 = b^2 = ba = 0 \rangle.$$

More information on the monoids I^1, J^1, and K^1 can be found in Lee [10, Section 1.3].

Lemma 6.4.
The varieties $\mathsf{C}_0^1\,(13)$, $\mathsf{C}_0^1\,(14)$, $\mathsf{C}_0^1\,(15)$, *and* $\mathsf{C}_0^1\,(16)$ *constitute the following subsemilattice of the lattice of all semigroup varieties:*



Proof. The inclusion $\mathsf{C}_0^1\,(15) \subseteq \mathsf{C}_0^1\,(13)$ holds since

$$(15) \Vdash hxyxh \overset{(\star)}{\approx} hx(hyx^2h) \overset{(15)}{\approx} hxhx^2yh \overset{(\star)}{\approx} hx^2yh \vdash (13).$$

By symmetry, the inclusion $\mathsf{C}_0^1\,(15) \subseteq \mathsf{C}_0^1\,(14)$ also holds. It is easy to verify that the inclusion $\mathsf{C}_0^1\,(16) \subseteq \mathsf{C}_0^1\,(15)$ and the intersection $\mathsf{C}_0^1\,(13) \cap \mathsf{C}_0^1\,(14) = \mathsf{C}_0^1\,(15)$ hold. The varieties $\mathsf{C}_0^1\,(13)$, $\mathsf{C}_0^1\,(14)$, $\mathsf{C}_0^1\,(15)$, and $\mathsf{C}_0^1\,(16)$ are distinct because $I^1 \in \mathsf{C}_0^1\,(13) \setminus \mathsf{C}_0^1\,(15)$, $J^1 \in \mathsf{C}_0^1\,(14) \setminus \mathsf{C}_0^1\,(15)$, and $K^1 \in \mathsf{C}_0^1\,(15) \setminus \mathsf{C}_0^1\,(16)$. $\square$

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Corollary 6.5.

Let V *be any variety in the interval* $[\mathsf{A}_0 \vee \mathsf{N}_2^1, \mathsf{C}_0^1]$ *such that* $B_0^1 \notin \mathsf{V}$. *Then precisely one of the following conditions holds:*

- $\mathsf{V} \subseteq \mathsf{C}_0^1\,(13)$ *and* $\mathsf{V} \nsubseteq \mathsf{C}_0^1\,(14)$;
- $\mathsf{V} \nsubseteq \mathsf{C}_0^1\,(13)$ *and* $\mathsf{V} \subseteq \mathsf{C}_0^1\,(14)$;
- $\mathsf{V} \subseteq \mathsf{C}_0^1\,(15)$ *and* $\mathsf{V} \nsubseteq \mathsf{C}_0^1\,(16)$;
- $\mathsf{V} \subseteq \mathsf{C}_0^1\,(16)$.

Proof. This follows from Lemmas 6.3 and 6.4. □

Let x be any non-simple letter in a quadratic word $\mathbf{w}$, say

$$\mathbf{w} = \mathbf{w}_1 x \mathbf{w}_2 x \mathbf{w}_3 \tag{17}$$

for some $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3 \in \mathcal{X}^*$ such that $x \notin \operatorname{con}(\mathbf{w}_1\mathbf{w}_2\mathbf{w}_3)$. Then the letter x is said to be *left-disciplined* in the word $\mathbf{w}$ if either $\mathbf{w}_1 = \emptyset$ or the letter $\mathrm{t}(\mathbf{w}_1)$ is the first occurrence of some non-simple letter in $\mathbf{w}$. A non-simple, quadratic word is said to be *left-disciplined* if all its non-simple letters are left-disciplined in it. Equivalently, a non-simple, quadratic word $\mathbf{w}$ is left-disciplined if and only if the first occurrences of non-simple letters in $\mathbf{w}$ form a prefix of $\mathbf{w}$.

Similarly, the letter x in (17) is said to be *right-disciplined* in the word $\mathbf{w}$ if either $\mathbf{w}_3 = \emptyset$ or the letter $\mathrm{h}(\mathbf{w}_3)$ is the second occurrence of some non-simple letter in $\mathbf{w}$. A non-simple, quadratic word is said to be *right-disciplined* if all its non-simple letters are right-disciplined in it. Equivalently, a non-simple, quadratic word $\mathbf{w}$ is right-disciplined if and only if the second occurrences of non-simple letters in $\mathbf{w}$ form a suffix of $\mathbf{w}$.

A word that is both left-disciplined and right-disciplined is said to be *disciplined*. Equivalently, a disciplined word $\mathbf{w}$ is non-simple, quadratic, and of the form

$$\mathbf{w} = \mathbf{h}\mathbf{w}'\mathbf{t}, \tag{18}$$

where the prefix $\mathbf{h}$ consists of all first occurrences of non-simple letters in $\mathbf{w}$, the suffix $\mathbf{t}$ consists of all second occurrences of non-simple letters in $\mathbf{w}$, and the factor $\mathbf{w}'$ consists of all simple letters of $\mathbf{w}$. Note that a disciplined word is necessarily connected and peaceful. The disciplined word $\mathbf{w}$ in (18) is said to be *well-disciplined* if the letters in the factor $\mathbf{w}'$ are in alphabetical order.

Lemma 6.6.

Let $\mathbf{w}$ *be any quadratic, connected word.*

(i) *The deduction* (13) $\Vdash \mathbf{w} \approx \mathbf{w}^\delta$ *holds for some right-disciplined, quadratic, connected word* $\mathbf{w}^\delta$.

(ii) *The deduction* (14) $\Vdash \mathbf{w} \approx \mathbf{w}^\delta$ *holds for some left-disciplined, quadratic, connected word* $\mathbf{w}^\delta$.

(iii) *The deduction* (15) $\Vdash \mathbf{w} \approx \mathbf{w}^\delta$ *holds for some disciplined, quadratic, connected word* $\mathbf{w}^\delta$.

(iv) *The deduction* (16) $\Vdash \mathbf{w} \approx \mathbf{w}^\delta$ *holds for some well-disciplined, quadratic, connected word* $\mathbf{w}^\delta$.

Further, the relation $\mathbf{w} \stackrel{\circ}{=} \mathbf{w}^\delta$ *holds in each of* (i)–(iv).

Proof. (i) Let $\mathbf{w}$ be any quadratic word and let x be any non-simple letter in $\mathbf{w}$ that is not right-disciplined. Then $\mathbf{w} = \mathbf{w}_1 x \mathbf{w}_2 x \mathbf{w}_3$ for some $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3 \in \mathcal{X}^*$ such that the letter $y = \mathrm{h}(\mathbf{w}_3)$ is either

(a) simple in $\mathbf{w}$, or

(b) the first occurrence of some non-simple letter in $\mathbf{w}$.

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Hence $\mathbf{w} = \mathbf{w}_1 x \mathbf{w}_2 x y \mathbf{w}_3'$ for some word $\mathbf{w}_3'$. In both cases (a) and (b), the letter y does not belong to the prefix $\mathbf{w}_1 x \mathbf{w}_2 x$. The connectedness of $\mathbf{w}$ implies the existence of a letter h that is common to both $\mathbf{w}_1 x \mathbf{w}_2 x$ and $y\mathbf{w}_3'$. Since the word $\mathbf{w}$ is quadratic, the letter h is distinct from x and y. Hence

$$\mathbf{w} \overset{(\bigstar)}{\approx} \mathbf{w}_1 x \mathbf{w}_2 (h x^2 y h) \mathbf{w}_3' \overset{(13)}{\approx} \mathbf{w}_1 x \mathbf{w}_2 h x y x h \mathbf{w}_3' \overset{(\bigstar)}{\approx} \mathbf{w}_1 x \mathbf{w}_2 y x \mathbf{w}_3',$$

that is, the identities $\{(\bigstar), (13)\}$ can be used to interchange the second occurrence of x with the first letter y of $\mathbf{w}_3$. Note that in this interchanging process,

- the distance between the two occurrences of x is increased by one;
- if y is non-simple, then the distance between the two occurrences of y is increased by one;
- the distance between the two occurrences of any other non-simple letter remains unchanged;
- the resulting word is connected and is $\mathrel{\hat{=}}$-related to $\mathbf{w}$.

It is easy to see that the aforementioned process can be repeated on any letter that is not right-disciplined until it is right-disciplined. There are at most $|\mathbf{w}|!$ words $\mathrel{\hat{=}}$-related to the word $\mathbf{w}$. Among all these $|\mathbf{w}|!$ words, one must possess the greatest possible separation degree, say d. Hence the process of interchanging can only be repeated on $\mathbf{w}$ at most d times. The resulting word $\mathbf{w}^\delta$ is then right-disciplined, connected, and quadratic such that $\mathbf{w} \mathrel{\hat{=}} \mathbf{w}^\delta$.

(ii) This is symmetrical to part (i).

(iii) By parts (i) and (ii), the deduction $\{(13), (14)\} \Vdash \mathbf{w} \approx \mathbf{w}^\delta$ holds for some quadratic, connected word $\mathbf{w}^\delta$ that is both left-disciplined and right-disciplined. The result now follows since $\mathbf{C}_0^1(15) = \mathbf{C}_0^1\{(13), (14)\}$ by Lemma 6.4.

(iv) Since the deduction $(16) \Vdash (15)$ holds by Lemma 6.4, it follows from part (iii) that the word $\mathbf{w}$ can be chosen to be disciplined, quadratic, and connected to begin with, say $\mathbf{w} = \mathbf{h}\mathbf{w}'\mathbf{t}$ is the word in (18). Since $\mathbf{w}$ is a connected word, it contains some non-simple letter h; this letter is clearly common to both the factors $\mathbf{h}$ and $\mathbf{t}$. Let x and y be any two consecutive letters in $\mathbf{w}'$, say $\mathbf{w}' = \mathbf{w}_1 x y \mathbf{w}_2$ for some $\mathbf{w}_1, \mathbf{w}_2 \in \mathcal{X}^*$. Since

$$\mathbf{w} \overset{(\bigstar)}{\approx} \mathbf{h}\mathbf{w}_1 (h x y h) \mathbf{w}_2 \mathbf{t} \overset{(16)}{\approx} \mathbf{h}\mathbf{w}_1 h y x h \mathbf{w}_2 \mathbf{t} \overset{(\bigstar)}{\approx} \mathbf{h}\mathbf{w}_1 y x \mathbf{w}_2 \mathbf{t},$$

the identities $\{(\bigstar), (16)\}$ can be used to interchange the letters x and y. It is easy to see how this can be repeated on the letters of the factor $\mathbf{w}'$ until a well-disciplined word is obtained. □

Lemma 6.7.

Let $\sigma : \mathbf{p}_{(\ell)}\mathbf{u}\mathbf{q}_{(r)} \approx \mathbf{p}_{(\ell)}\mathbf{v}\mathbf{q}_{(r)}$ be the identity (4) that satisfies conditions (1†)–(6†) in Lemma 4.2. Suppose that the identity σ is satisfied by the variety $\mathbf{A}_0 \vee \mathbf{N}_2^1$ but not by the variety $\mathbf{B}_0^1$. Then $\mathbf{C}_0^1\{\sigma\} = \mathbf{C}_0^1\{\theta, \sigma'\}$ for some identity θ from $\{(13), (14), (15), (16)\}$ and some identity σ' of the variety $\mathbf{A}_0 \vee \mathbf{B}_0^1$.

Proof. By condition (3†) of Lemma 4.2, the variety $\mathbf{N}_2^1$ satisfies the identity $\mathbf{u} \approx \mathbf{v}$. It follows from Lemma 2.1(iii) that

(a) the identity σ does not delete to the identity $x^2 \approx x$.

Let $\mathbf{V} = \mathbf{C}_0^1\{\sigma\}$. Then the variety $\mathbf{V}$ satisfies one of the four conditions in Corollary 6.5.

Case 1: $\mathbf{V} \subseteq \mathbf{C}_0^1(13)$ and $\mathbf{V} \nsubseteq \mathbf{C}_0^1(14)$. Then $\mathbf{V} = \mathbf{C}_0^1\{(13), \sigma\}$ and $\sigma \nVdash (14)$. By Lemma 6.6(i), the deduction $(13) \Vdash \{\mathbf{u} \approx \mathbf{u}^\delta, \mathbf{v} \approx \mathbf{v}^\delta\}$ holds for some right-disciplined, quadratic, connected words $\mathbf{u}^\delta$ and $\mathbf{v}^\delta$ such that $\mathbf{u} \mathrel{\hat{=}} \mathbf{u}^\delta$ and $\mathbf{v} \mathrel{\hat{=}} \mathbf{v}^\delta$. Hence $\mathbf{V} = \mathbf{C}_0^1\{(13), \sigma'\}$ where σ' is the identity $\mathbf{p}_{(\ell)}\mathbf{u}^\delta\mathbf{q}_{(r)} \approx \mathbf{p}_{(\ell)}\mathbf{v}^\delta\mathbf{q}_{(r)}$. By Lemma 4.1(iii), the identity σ' is satisfied by the variety $\mathbf{A}_0$. Since $\mathbf{u} \mathrel{\hat{=}} \mathbf{u}^\delta$ and $\mathbf{v} \mathrel{\hat{=}} \mathbf{v}^\delta$, it follows from (a) that

(b) the identity σ' does not delete to the identity $x^2 \approx x$.

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Recall that in a right-disciplined word $\mathbf{w}$, the second occurrences of non-simple letters of $\mathbf{w}$ form a suffix. Specifically, a right-disciplined word $\mathbf{w}$ cannot be of the form $\mathbf{a}x\mathbf{b}x\mathbf{c}y\mathbf{d}$ where y is a simple letter in $\mathbf{w}$. Therefore, since $\mathbf{u}^\delta$ and $\mathbf{v}^\delta$ are right-disciplined words,

(c) the identity σ' does not delete to any identity in $\{xyx \approx x^2y,\ x^2y \approx yx^2\}$.

Suppose that the identity σ' deletes to some identity in $\{xy \approx yx,\ xyx \approx yx^2\}$. Then by Lemma 6.2, either the deduction $\sigma' \Vdash$ (16) or the deduction $\sigma' \Vdash$ (14) holds. It then follows from Lemma 6.4 that the inclusion $\mathbf{V} \subseteq \mathbf{C}_0^1$ (14) holds, contradicting the assumption of the present case. Therefore the identity σ' does not delete to any identity in $\{xy \approx yx,\ xyx \approx yx^2\}$. In the presence of (b), (c), and Lemma 2.2(ii), the identity σ' is satisfied by the variety $\mathbf{B}_0^1$.

Case 2: $\mathbf{V} \nsubseteq \mathbf{C}_0^1$ (13) and $\mathbf{V} \subseteq \mathbf{C}_0^1$ (14). It follows from an argument symmetrical to Case 1 that $\mathbf{V} = \mathbf{C}_0^1\{(14), \sigma'\}$ for some identity σ' of the variety $\mathbf{B}_0^1$.

Case 3: $\mathbf{V} \subseteq \mathbf{C}_0^1$ (15) and $\mathbf{V} \nsubseteq \mathbf{C}_0^1$ (16). Then $\mathbf{V} = \mathbf{C}_0^1\{(15), \sigma\}$ and $\sigma \nVdash$ (16). By Lemma 6.6(iii), the deduction (15) $\Vdash \{\mathbf{u} \approx \mathbf{u}^\delta, \mathbf{v} \approx \mathbf{v}^\delta\}$ holds for some disciplined, quadratic, connected words $\mathbf{u}^\delta$ and $\mathbf{v}^\delta$ such that $\mathbf{u} \doteq \mathbf{u}^\delta$ and $\mathbf{v} \doteq \mathbf{v}^\delta$. Hence $\mathbf{V} = \mathbf{C}_0^1\{(15), \sigma'\}$ where σ' is the identity $\mathbf{p}_{(\ell)}\mathbf{u}^\delta\mathbf{q}_{(r)} \approx \mathbf{p}_{(\ell)}\mathbf{v}^\delta\mathbf{q}_{(r)}$. By Lemma 4.1(iii), the identity σ' is satisfied by the variety $\mathbf{A}_0$. Since $\mathbf{u} \doteq \mathbf{u}^\delta$ and $\mathbf{v} \doteq \mathbf{v}^\delta$, it follows from (a) that

(d) the identity σ' does not delete to the identity $x^2 \approx x$.

A disciplined word $\mathbf{w}$ cannot be of the form $\mathbf{a}x\mathbf{b}x\mathbf{c}y\mathbf{d}$ or $\mathbf{a}y\mathbf{b}x\mathbf{c}x\mathbf{d}$ where y is a simple letter in $\mathbf{w}$. Therefore, since $\mathbf{u}^\delta$ and $\mathbf{v}^\delta$ are disciplined words,

(e) the identity σ' does not delete to any identity in $\{xyx \approx x^2y,\ xyx \approx yx^2,\ x^2y \approx yx^2\}$.

Suppose that the identity σ' deletes to the identity $xy \approx yx$. Then the deduction $\sigma' \Vdash$ (16) holds by Lemma 6.2. It follows that the inclusion $\mathbf{V} \subseteq \mathbf{C}_0^1$ (16) holds, contradicting the assumption of the present case. Therefore the identity σ' does not delete to the identity $xy \approx yx$. In the presence of (d), (e), and Lemma 2.2(ii), the identity σ' is satisfied by the variety $\mathbf{B}_0^1$.

Case 4: $\mathbf{V} \subseteq \mathbf{C}_0^1$ (16). Then $\mathbf{V} = \mathbf{C}_0^1\{(16), \sigma\}$. By Lemma 6.6(iv), the deduction (16) $\Vdash \{\mathbf{u} \approx \mathbf{u}^\delta, \mathbf{v} \approx \mathbf{v}^\delta\}$ holds for some well-disciplined, quadratic, connected words $\mathbf{u}^\delta$ and $\mathbf{v}^\delta$ such that $\mathbf{u} \doteq \mathbf{u}^\delta$ and $\mathbf{v} \doteq \mathbf{v}^\delta$. Therefore $\mathbf{V} = \mathbf{C}_0^1\{(16), \sigma'\}$ where σ' is the identity $\mathbf{p}_{(\ell)}\mathbf{u}^\delta\mathbf{q}_{(r)} \approx \mathbf{p}_{(\ell)}\mathbf{v}^\delta\mathbf{q}_{(r)}$. By Lemma 4.1(iii), the identity σ' is satisfied by the variety $\mathbf{A}_0$. Since $\mathbf{u} \doteq \mathbf{u}^\delta$ and $\mathbf{v} \doteq \mathbf{v}^\delta$, it follows from (a) that the identity σ' does not delete to the identity $x^2 \approx x$. Since $\mathbf{u}^\delta$ and $\mathbf{v}^\delta$ are well-disciplined words, the identity does not delete to any identity from (2). Hence by Lemma 2.2(ii), the identity σ' is satisfied by the variety $\mathbf{B}_0^1$. □

Proposition 6.8.
Let $\mathbf{V}$ *be any variety in the interval* $[\mathbf{A}_0 \vee \mathbf{N}_2^1, \mathbf{C}_0^1]$ *such that* $B_0^1 \notin \mathbf{V}$. *Then the variety* $\mathbf{V}$ *is finitely based.*

Proof. By assumption, $\mathbf{V} = \mathbf{C}_0^1\Sigma$ for some set Σ of identities of the variety $\mathbf{A}_0 \vee \mathbf{N}_2^1$. By Lemma 4.2, the identities in the set Σ can be chosen to be of the form (4) that satisfy conditions (1†)–(6†). Let $\Sigma = \Sigma_0 \cup \Sigma_1$ where Σ_1 is the set of all identities from Σ that are satisfied by the variety $\mathbf{B}_0^1$ and $\Sigma_0 = \Sigma \setminus \Sigma_1$. Then it follows from Lemma 6.7 that $\mathbf{C}_0^1\Sigma_0 = \mathbf{C}_0^1(\Theta \cup \Sigma_0')$ for some set Θ of identities from $\{(13), (14), (15), (16)\}$ and some set Σ_0' of identities of the variety $\mathbf{A}_0 \vee \mathbf{B}_0^1$. Since the identities in Σ_0' and Σ_1 are satisfied by the variety $\mathbf{A}_0 \vee \mathbf{B}_0^1$, the variety $\mathbf{C}_0^1(\Sigma_0' \cup \Sigma_1)$ belongs to the interval $[\mathbf{A}_0 \vee \mathbf{B}_0^1, \mathbf{C}_0^1]$ and is finitely based by Proposition 5.1. Consequently, the variety $\mathbf{V} = \mathbf{C}_0^1\Theta \cap \mathbf{C}_0^1(\Sigma_0' \cup \Sigma_1)$ is finitely based. □

6.2. Varieties in $[\mathbf{A}_0, \mathbf{C}_0^1]$ not containing $\mathbf{N}_2^1$.

The main result of this subsection, Proposition 6.11, establishes the finite basis property of every variety in the interval $[\mathbf{A}_0, \mathbf{C}_0^1]$ that does not contain the variety $\mathbf{N}_2^1$.

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Lemma 6.9.

Let $\mathbf{V}$ *be any variety in the interval* $[\mathbf{A}_0, \mathbf{C}_0^1]$. *If* $N_2^1 \notin \mathbf{V}$, *then the variety* $\mathbf{V}$ *satisfies the identity*

$$hx^2h \approx hxh. \tag{19}$$

Proof. By Remark 2.4 and Lemmas 2.1(iii) and 4.1, the variety $\mathbf{V}$ satisfies some quadratic identity $\mathbf{u} \approx \mathbf{v}$ such that $\operatorname{con}(\mathbf{u}) = \operatorname{con}(\mathbf{v})$ and $\mathbf{u} \not\equiv \mathbf{v}$. Then the identity $\mathbf{u} \approx \mathbf{v}$ deletes to the identity $x^2 \approx x$. By Lemma 6.2, the variety $\mathbf{V}$ satisfies the identity (19). □

Lemma 6.10.

Let $\sigma : \mathbf{p}_{(\ell)}\mathbf{u}\mathbf{q}_{(r)} \approx \mathbf{p}_{(\ell)}\mathbf{v}\mathbf{q}_{(r)}$ *be the identity* (4) *that satisfies conditions* (1†)–(6†) *in Lemma 4.2. Suppose that the identity* σ *is not satisfied by the variety* $\mathbf{N}_2^1$. *Then* $\mathbf{C}_0^1\{\sigma\} = \mathbf{C}_0^1\{(19), \sigma'\}$ *for some identity* σ' *of the variety* $\mathbf{A}_0 \vee \mathbf{N}_2^1$.

Proof. It follows from Lemma 6.9 that $\mathbf{C}_0^1\{\sigma\} = \mathbf{C}_0^1\{(19), \sigma\}$. Note that if x is any simple letter in the connected word $\mathbf{u}$, say $\mathbf{u} = \mathbf{u}_1 x \mathbf{u}_2$ where the factors $\mathbf{u}_1$ and $\mathbf{u}_2$ share some common letter h, then

$$\mathbf{u} \overset{(\bigstar)}{\approx} \mathbf{u}_1(hxh)\mathbf{u}_2 \overset{(19)}{\approx} \mathbf{u}_1 hx^2h \mathbf{u}_2 \overset{(\bigstar)}{\approx} \mathbf{u}_1 x^2 \mathbf{u}_2.$$

Similarly, the identities $\{(\bigstar), (19)\}$ can be used to replace any simple letter x in $\mathbf{v}$ by x^2. Let $\mathbf{u}'$ and $\mathbf{v}'$ be the words obtained from the words $\mathbf{u}$ and $\mathbf{v}$ by replacing every simple letter x by x^2, and let σ' be the identity $\mathbf{p}_{(\ell)}\mathbf{u}'\mathbf{q}_{(r)} \approx \mathbf{p}_{(\ell)}\mathbf{v}'\mathbf{q}_{(r)}$. Then $\mathbf{C}_0^1\{\sigma\} = \mathbf{C}_0^1\{(19), \sigma\} = \mathbf{C}_0^1\{(19), \sigma'\}$. By conditions (2†) and (4†) of Lemma 4.2, the identity $\mathbf{u} \approx \mathbf{v}$ is satisfied by the variety $\mathbf{A}_0$ and the words $\mathbf{u}$ and $\mathbf{v}$ are connected. Clearly $\mathbf{u}'$ and $\mathbf{v}'$ are connected, quadratic words that contain no simple letters. By Lemmas 2.1(iii) and 4.1, the identity $\mathbf{u}' \approx \mathbf{v}'$ is satisfied by the variety $\mathbf{A}_0 \vee \mathbf{N}_2^1$. Therefore the identity σ' is also satisfied by the variety $\mathbf{A}_0 \vee \mathbf{N}_2^1$. □

Proposition 6.11.

Let $\mathbf{V}$ *be any variety in the interval* $[\mathbf{A}_0, \mathbf{C}_0^1]$ *such that* $N_2^1 \notin \mathbf{V}$. *Then the variety* $\mathbf{V}$ *is finitely based.*

Proof. It follows from Lemma 6.9 that $\mathbf{V} = \mathbf{C}_0^1((19) \cup \Sigma)$ for some set Σ of identities of the variety $\mathbf{A}_0$. By Lemma 4.2, the identities in the set Σ can be chosen to be of the form (4) with conditions (1†)–(6†). Let $\Sigma = \Sigma_0 \cup \Sigma_1$ where Σ_1 is the set of all identities from Σ that are satisfied by the variety $\mathbf{N}_2^1$ and $\Sigma_0 = \Sigma \setminus \Sigma_1$. Then it follows from Lemma 6.10 that $\mathbf{C}_0^1((19) \cup \Sigma_0) = \mathbf{C}_0^1((19) \cup \Sigma_0')$ for some set Σ_0' of identities of the variety $\mathbf{A}_0 \vee \mathbf{N}_2^1$. Since the identities in Σ_0' and Σ_1 are satisfied by the variety $\mathbf{A}_0 \vee \mathbf{N}_2^1$, the variety $\mathbf{V}' = \mathbf{C}_0^1(\Sigma_0' \cup \Sigma_1)$ belongs to the interval $[\mathbf{A}_0 \vee \mathbf{N}_2^1, \mathbf{C}_0^1]$. If $B_0^1 \notin \mathbf{V}'$, then the variety $\mathbf{V}'$ is finitely based by Proposition 6.8. If $B_0^1 \in \mathbf{V}'$, then the variety $\mathbf{V}'$ is finitely based by Proposition 5.1. In any case, the variety $\mathbf{V} = \mathbf{C}_0^1(19) \cap \mathbf{V}'$ is finitely based. □

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