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RIGHT FOCAL BOUNDARY VALUE PROBLEMS FOR DIFFERENCE EQUATIONS

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Abstract. An application is made of a new Avery *et al.* fixed point theorem of compression and expansion functional type in the spirit of the original fixed point work of Leggett and Williams, to obtain positive solutions of the second order right focal discrete boundary value problem. In the application of the fixed point theorem, neither the entire lower nor entire upper boundary is required to be mapped inward or outward. A nontrivial example is also provided.

Keywords: difference equation, boundary value problem, right focal, fixed point theorem, positive solution.

Mathematics Subject Classification: 39A10.

1. INTRODUCTION

For well over a decade, substantial results have been obtained for positive solutions and multiple positive solutions for boundary value problems for finite difference equations; see, for example [2, 5, 10, 11, 13, 15, 16, 20–23, 25–28].

Many of those results have been motivated by the applicability of a number of new fixed point theorems and multiple fixed point theorems as applied to certain discrete boundary value problems; such as the classical fixed point theorems of Guo and Krasnosel'skii [14, 17] or Leggett and Williams [19], along with several newer fixed point theorems by Avery *et al.* [1, 3, 6–9], and the fixed point theorem of Ge [12].

Recently, Avery, Anderson and Henderson [4] gave a topological proof in obtaining a Leggett-Williams type of fixed point theorem, which requires only that certain subsets of both boundaries of a subset of a cone for which $\|x\| > b$ and $\alpha(x) = a$, where α is a concave positive functional on the cone, be mapped inward and outward, respectively. This is an expansion result which is dramatically different from the Leggett-Williams fixed point theorem, which is in itself only a compression result. Moreover, this new fixed point theorem [4] is more general than those obtained by using Guo-Krasnosel'skii compression-expansion results which mapped at least one boundary inward or outward [1,8,14,19,24], or the topological generalizations of fixed point theorems introduced by Kwong [18] which require boundaries to be mapped inward or outward (invariance-like conditions). Moreover, conditions involving the norm in the original Leggett-Williams fixed point theorem were replaced in this recent fixed point theorem [4] by more general conditions on a convex functional.

In this paper, we give a first application of the Avery *et al.* fixed point theorem [4] to right focal boundary problems for finite difference equations, by demonstrating a technique that takes advantage of the flexibility of the new fixed point theorem in obtaining at least one positive solution for

$$\Delta^2 u(k) + f(u(k)) = 0, \quad k \in \{0, 1, \ldots, N\}, \tag{1.1}$$

$$u(0) = \Delta u(N+1) = 0, \tag{1.2}$$

where $f : [0, \infty) \to [0, \infty)$ is continuous. In Section 2, we provide some background definitions and we state the new fixed point theorem. In Section 3, we apply the fixed point theorem to obtain a positive solution to (1.1), (1.2), and in Section 3, we provide a nontrivial example of the existence result of Section 2.

2. BACKGROUND AND A FIXED POINT THEOREM

In this section, we present some definitions used for the remainder of the paper. In addition, we include a new fixed point theorem statement whose application, in the next section, will yield a solution of (1.1), (1.2).

Definition 2.1. Let E be a real Banach space. A nonempty closed convex set $P \subset E$ is called a *cone* if it satisfies the following two conditions:

(i) $x \in P, \lambda \geq 0$ implies $\lambda x \in P$;
(ii) $x \in P$, $-x \in P$ implies $x = 0$.

Definition 2.2. A map α is said to be a nonnegative continuous concave functional on a cone P of a real Banach space E if

$$\alpha : P \to [0, \infty)$$

is continuous and

$$\alpha(tx + (1-t)y) \geq t\alpha(x) + (1-t)\alpha(y)$$

for all $x, y \in P$ and $t \in [0, 1]$. Similarly we say the map β is a nonnegative continuous convex functional on a cone P of a real Banach space E if

$$\beta : P \to [0, \infty)$$

is continuous and

$$\beta(tx + (1-t)y) \leq t\beta(x) + (1-t)\beta(y)$$

for all $x, y \in P$ and $t \in [0, 1]$.

Let ψ and δ be nonnegative continuous functionals on a cone P; then, for positive real numbers a and b, we define the sets:

$$P(\psi, b) := \{x \in P : \psi(x) \leq b\}, \tag{2.1}$$

and

$$P(\psi, \delta, a, b) := \{x \in P : a \leq \psi(x) \text{ and } \delta(x) \leq b\}. \tag{2.2}$$

The following theorem [4] is the new fixed point theorem of compression-expansion and functional type.

Theorem 2.3. *Suppose P is a cone in a real Banach space E, α is a nonnegative continuous concave functional on P, β is a nonnegative continuous convex functional on P and $T : P \to P$ is a completely continuous operator. Assume there exist nonnegative numbers a, b, c and d such that:*

(A1) $\{x \in P : a < \alpha(x)$ *and* $\beta(x) < b\} \neq \emptyset$;
(A2) *if* $x \in P$ *with* $\beta(x) = b$ *and* $\alpha(x) \geq a$, *then* $\beta(Tx) < b$;
(A3) *if* $x \in P$ *with* $\beta(x) = b$ *and* $\alpha(Tx) < a$,*then* $\beta(Tx) < b$;
(A4) $\{x \in P : c < \alpha(x)$ *and* $\beta(x) < d\} \neq \emptyset$;
(A5) *if* $x \in P$ *with* $\alpha(x) = c$ *and* $\beta(x) \leq d$, *then* $\alpha(Tx) > c$;
(A6) *if* $x \in P$ *with* $\alpha(x) = c$ *and* $\beta(Tx) > d$, *then* $\alpha(Tx) > c$.

If

(H1) $a < c$, $b < d$, $\{x \in P : b < \beta(x)$ *and* $\alpha(x) < c\} \neq \emptyset$, $P(\beta, b) \subset P(\alpha, c)$, *and* $P(\alpha, c)$ *is bounded,*

then T *has a fixed point* x^* *in* $P(\beta, \alpha, b, c)$.
If

(H2) $c < a$, $d < b$, $\{x \in P : a < \alpha(x)$ *and* $\beta(x) < d\} \neq \emptyset$, $P(\alpha, a) \subset P(\beta, d)$, *and* $P(\beta, d)$ *is bounded,*

then T *has a fixed point* x^* *in* $P(\alpha, \beta, a, d)$.

3. SOLUTIONS OF (1.1), (1.2)

In this section, we impose growth conditions on f such that the right focal boundary value problem for the finite difference equation, (1.1), (1.2), has a solution as a consequence of Theorem 2.3. We note that from the nonnegativity of f, a solution u of (1.1), (1.2) is both nonnegative and concave on $\{0, 1, \ldots, N+2\}$. In our application of Theorem 2.3, we will deal with a completely continuous summation operator whose kernel is the Green's function, $H(k, \ell)$, for

$$-\Delta^2 v = 0 \tag{3.1}$$

and satisfying (1.2). In particular, for $(k, \ell) \in \{0, \ldots, N+2\} \times \{0, \ldots, N\}$,

$$H(k, \ell) = \frac{1}{N+2} \begin{cases} k, & k \in \{0, \ldots, \ell\}, \\ \ell + 1, & k \in \{\ell + 1, \ldots, N+2\}. \end{cases}$$

We observe that $H(k,\ell)$ is nonnegative, and for each fixed $\ell \in \{0,\ldots,N\}$, $H(k,\ell)$ is nondecreasing as a function of k. In addition, it is straightforward that, for $y,w \in \{0,\ldots,N+2\}$ with $y \le w$,

$$wH(y,\ell) \ge yH(w,\ell), \quad \ell \in \{0,\ldots,N\}. \tag{3.2}$$

Next, let $E = \{v : \{0,\ldots,N+2\} \to \mathbb{R}\}$ be endowed with the norm, $\|v\| = \max_{k\in\{0,\ldots,N+2\}} |v(k)|$. Choose

$$\tau \in \{1,\ldots,N-1\},$$

and define the cone $P \subset E$ by

$$P = \{v \in E : v \text{ is nondecreasing and nonnegative-valued on } \{0,\ldots,N+2\}, \\ \Delta^2 v(k) \le 0,\ k \in \{0,\ldots,N\}, \text{ and } (N+2)v(\tau) \ge \tau v(N+2)\}.$$

We note that, for any $u \in P$ and $y,w \in \{0,\ldots,N+2\}$ with $y \le w$,

$$wu(y) \ge yu(w). \tag{3.3}$$

For $v \in P$, we define a nonnegative concave functional α on P by

$$\alpha(v) := \min_{k\in\{\tau,\ldots,N+2\}} v(k) = v(\tau),$$

and a nonnegative, convex functional β on P by

$$\beta(v) := \max_{k\in\{0,\ldots,N+2\}} v(k) = v(N+2).$$

We note that for $v \in P$, in terms of the functionals,

$$(N+2)\alpha(v) \ge \tau\beta(v).$$

Now, we put growth conditions on f such that (1.1), (1.2) has at least one solution $u^* \in P(\beta,\alpha,b,c)$, as a consequence of Theorem 2.3 under the expansive condition (H1).

Theorem 3.1. *If $\tau \in \{1,\ldots,N-1\}$ is fixed, b and c are positive real numbers with $3b \le c$, and $f : [0,\infty) \to [0,\infty)$ is a continuous function such that:*

(i) $f(w) > \frac{c(N+2)}{\tau(N-\tau)}$, *for* $w \in [c, \frac{c(N+2)}{\tau}]$,
(ii) $f(w)$ *is decreasing, for* $w \in [0, \frac{b\tau}{N+2}]$, *with* $f(\frac{b\tau}{N+2}) \ge f(w)$, *for* $w \in [\frac{b\tau}{N+2}, b]$, *and*
(iii) $\sum_{\ell=0}^{\tau} \frac{(\ell+1)}{N+2} f(\frac{b\ell}{N+2}) < b - f(\frac{b\tau}{N+2})[\frac{(N+1)(N+2)-(\tau+1)(\tau+2)}{2(N+2)}]$,

then the discrete right-focal problem (1.1), (1.2) has at least one positive solutions $u^* \in P(\beta,\alpha,b,c)$.

Proof. First, we let
$$a = \frac{b\tau}{N+2} \text{ and } d = \frac{c(N+2)}{\tau}.$$
Then we have,
$$a = \frac{b\tau}{N+2} \le \frac{c\tau}{3(N+2)} < c$$
and
$$b \le \frac{c}{3} = \frac{d\tau}{3(N+2)} < d.$$

Next, we define the summation operator $T : E \to E$ by
$$Tu(k) = \sum_{\ell=0}^{N} H(k,\ell) f(u(\ell)), \quad u \in E, \quad k \in \{0, \dots, N+2\}.$$
It is immediate that T is completely continuous, and it is well known that $u \in E$ is a solution of (1.1), (1.2) if, and only if u is a fixed point of T. We now show that the conditions of Theorem 2.3 are satisfied with respect to T.

So, if we let $u \in P$, then $Tu(k) = \sum_{\ell=0}^{N} H(k,\ell) f(u(\ell)) \ge 0$ on $\{0, \dots, N+2\}$. Moreover, $\Delta^2(Tu)(k) = -f(u(k)) \le 0$, and so $\Delta(Tu)(k)$ is nonincreasing on $\{0, \dots, N+1\}$. From properties of $H(k,\ell)$, $\Delta(Tu)(N+1) = 0$, and so $\Delta(Tu)(k) \ge 0$ on $\{0, \dots, N+1\}$. Thus, $(Tu)(k)$ is nondecreasing on $\{0, \dots, N+2\}$. Moreover,
$$Tu(\tau) = \sum_{\ell=0}^{N} H(k,\ell) f(u(\ell)) \ge \frac{\tau}{N+2} \sum_{\ell=0}^{N} H(N+2,\ell) f(u(\ell)) = \frac{\tau}{N+2} Tu(N+2).$$
Therefore, we have $T : P \to P$.

We next proceed to verify properties (A1) and (A4) of Theorem 2.3 are satisfied. First, for any $L \in (\frac{2b}{2N+3-\tau}, \frac{2b}{N+1})$, the function u_L defined by
$$u_L(k) := \sum_{\ell=0}^{N} LH(k,\ell) = \frac{Lk}{2(N+2)}(2N+3-k) \in \{u \in P : a < \alpha(u) \text{ and } \beta(u) < b\},$$
since
$$\alpha(u_L) = u_L(\tau) = \frac{L\tau}{2(N+2)}(2N+3-\tau) > \frac{b\tau}{N+2} = a$$
and
$$\beta(u_L) = u_L(N+2) = \frac{L(N+2)}{2(N+2)}(2N+3-(N+2)) < b.$$

Similarly, for any $J \in (\frac{2c(N+2)}{\tau(2N+3-\tau)}, \frac{2c(N+2)}{\tau(N+1)})$, the function u_J defined by
$$u_J(k) := \sum_{\ell=0}^{N} JH(k,\ell) = \frac{Jk}{2(N+2)}(2N+3-k) \in \{u \in P : c < \alpha(u) \text{ and } \beta(u) < d\},$$

since

$$\alpha(u_J) = u_J(\tau) = \frac{J\tau}{2(N+2)}(2N+3-\tau) > c$$

and

$$\beta(u_J) = u_J(N+2) = \frac{J(N+2)}{2(N+2)}(2N+3-(N+2)) = \frac{J(N+1)}{2} < \frac{c(N+2)}{\tau} = d.$$

Hence we have

$$\{u \in P : a < \alpha(u) \text{ and} \beta(u) < b\} \neq \emptyset,$$

and

$$\{u \in P : c < \alpha(u) \text{ and} \beta(u) < d\} \neq \emptyset.$$

Therefore conditions (A1) and (A4) of Theorem 2.3 are satisfied.

Turning to (A2) of Theorem 2.3, let $u \in P$ with $\beta(u) = b$ and $\alpha(u) \geq a$. By the concavity of u, for $\ell \in \{0, \dots, \tau\}$, we have

$$u(\ell) \geq \left(\frac{u(\tau)}{\tau}\right)\ell \geq \frac{b\ell}{N+2}$$

and for all $\ell \in \{\tau, \dots, N+2\}$, we have

$$\frac{b\tau}{N+2} \leq u(\ell) \leq b.$$

Hence by (ii) and (iii), it follows that

$$\begin{aligned}\beta(Tu) &= \sum_{\ell=0}^{N} H(N+2,\ell) f(u(\ell)) = \sum_{\ell=0}^{N} \frac{(\ell+1)}{N+2} f(u(\ell)) \leq \\ &\leq \sum_{\ell=0}^{\tau} \frac{(\ell+1)}{N+2} f(\frac{b\ell}{N+2}) + \sum_{\ell=\tau+1}^{N} \frac{(\ell+1)}{N+2} f(\frac{b\tau}{N+2}) < \\ &< b - \frac{f(\frac{b\tau}{N+2})}{N+2}\left[\frac{(N+1)(N+2)-(\tau+1)(\tau+2)}{2}\right] + \\ &\quad + \frac{f(\frac{b\tau}{N+2})}{N+2}\left[\frac{(N+1)(N+2)-(\tau+1)(\tau+2)}{2}\right] = b,\end{aligned}$$

and so (A2) is satisfied.

Next, we establish (A3) of Theorem 2.3, and so we let $u \in P$ with $\beta(u) = b$ and $\alpha(Tu) < a$. By the properties of $H(k,\ell)$,

$$\begin{aligned}\beta(Tu) &= \sum_{\ell=0}^{N} H(N+2,\ell) f(u(\ell)) \leq \\ &\leq \frac{N+2}{\tau}\sum_{\ell=0}^{N} H(\tau,\ell) f(u(\ell)) = \frac{N+2}{\tau}\alpha(Tu) < \frac{a(N+2)}{\tau} = b,\end{aligned}$$

and (A3) also holds.

In dealing with (A5), let $u \in P$ with $\alpha(u) = c$ and $\beta(u) \le d$. Then for $\ell \in \{\tau, \dots, N+2\}$, we have

$$c \le u(\ell) \le d = \frac{c(N+2)}{\tau}.$$

By Property (i),

$$\begin{aligned}\alpha(Tu) &= \sum_{\ell=0}^{N} H(\tau,\ell) f(u(\ell)) \ge \sum_{\ell=\tau+1}^{N} H(\tau,\ell) f(u(\ell)) = \\ &= \sum_{\ell=\tau+1}^{N} \frac{\tau}{N+2} f(u(\ell)) > \sum_{\ell=\tau+1}^{N} \frac{c}{N-\tau} = c,\end{aligned}$$

and so (A5) is valid.

And now we address (A6). So, let $u \in P$ with $\alpha(u) = c$ and $\beta(Tu) > d$. Again by the properties of H,

$$\begin{aligned}\alpha(Tu) &= \sum_{\ell=0}^{N} H(\tau,\ell) f(u(\ell)) \ge \\ &\ge \frac{\tau}{N+2} \sum_{\ell=0}^{N} H(N+2,\ell) f(u(\ell)) = \\ &= \frac{\tau}{N+2} \beta(Tu) > \frac{\tau d}{N+2} = c,\end{aligned}$$

and so (A6) of Theorem 2.3 also holds.

Finally, we show that the conditions of (H1) are also in effect. To that end, if $u \in P(\alpha, c)$, then

$$\frac{\tau}{N+2} \beta(u) \le \alpha(u) \le c,$$

and hence

$$\|x\| = \beta(u) \le \frac{\alpha(u)(N+2)}{\tau} \le \frac{c(N+2)}{\tau}.$$

Thus $P(\alpha, c)$ is a bounded subset of P. Also, if $u \in P(\beta, b)$, then

$$\alpha(u) \le \beta(u) \le b < c,$$

and hence $P(\beta, b) \subset P(\alpha, c)$.

In addtion, for any $M \in (\frac{2b}{N+1}, \frac{c}{N+1})$, the function u_M defined by

$$u_M(k) := \sum_{\ell=0}^{N} M H(k,\ell) = \sum_{\ell=0}^{k-1} \frac{M(\ell+1)}{N+2} + \sum_{\ell=k}^{N} \frac{Mk}{N+2} = \frac{Mk}{2(N+2)}(2N+3-k)$$

belongs to the set $P(\beta, \alpha, b, c)$, since

$$\alpha(u_M) = u_M(\tau) = \frac{M\tau}{2(N+2)}(2N+3-\tau) < \frac{c\tau}{2(N+1)(N+2)}(2N+3-\tau) < c,$$

and

$$\beta(u_M) = u_M(N+2) = \frac{M(N+2)}{2(N+2)}(2N+3-(N+2)) =$$
$$= \frac{M}{2}(N+1) > \frac{2b}{2(N+1)}(N+1) = b.$$

Thus, we also have that $\{u \in P : b < \beta(u) \text{ and } \alpha(u) < c\} \neq \emptyset$. Hence the conditions of (H1) are met.

It follows from Theorem 2.3 that T has a fixed point $u^* \in P(\beta, \alpha, b, c)$, and as such u^* is a desired solution of (1.1), (1.2). The proof is complete. □

Example. Let $N = 8, \tau = 1, b = 1$, and $c = 3$. Notice that $\frac{c(N+2)}{\tau(N-\tau)} = \frac{30}{7}$, $\frac{c(N+2)}{\tau} = 30$, and $\frac{b\tau}{N+2} = \frac{1}{10}$. We define a continuous $f : [0, \infty) \to [0, \infty)$ by

$$f(w) = \begin{cases} -8w + 1, & 0 \leq w \leq \frac{1}{9}, \\ \frac{1}{9}, & \frac{1}{9} \leq w \leq 1, \\ \frac{22}{9}w - \frac{7}{3}, & w \geq 1. \end{cases}$$

Then:

(i) $f(w) > \frac{30}{7}$, for $w \in [3, 30]$,
(ii) $f(w)$ is decreasing on $[0, \frac{1}{10}]$, and $f\left(\frac{1}{10}\right) \geq f(w)$, for $w \in [\frac{1}{10}, 1]$, and
(iii) $\displaystyle\sum_{\ell=0}^{1} \frac{\ell+1}{10} f\left(\frac{\ell}{10}\right) = \frac{14}{100} < \frac{16}{100} = 1 - f\left(\frac{1}{10}\right)\left[\frac{9 \cdot 10 - 2 \cdot 3}{2 \cdot 10}\right].$

Therefore, by Theorem 3.1, the right focal boundary value problem,

$$\Delta^2 u(k) + f(u(k)) = 0, \ k \in \{0, \ldots, 8\},$$

$$u(0) = 0 = \Delta u(9),$$

has at least one positive solution, u^*, with

$$1 \leq u^*(10) \text{ and } u^*(1) \leq 3.$$

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