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On the Cohomology of Spatial Polygons in Euclidean Spaces

Vehbi Emrah Paksoy
Nova Southeastern University, vp80@nova.edu

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Abstract

Space of spatial polygons in Euclidean spaces has been studied extensively in [3, 1, 2]. There is a beautiful description of cohomology in [1]. In this paper we introduce another, easy to compute method to obtain the cohomology without using toric variety arguments. We also give a criterion for a polygon space to be Fano, thus, having ample anticanonical class. [1]

1 Introduction and definitions

Let $\mathcal{P}_n$ be the space of all n-gons with distinguished vertices in Euclidean space $\mathbb{E}^3$. An n-gon P is determined by its vertices $v_1, \ldots, v_n$. These vertices are joined in cyclic order by edges $p_1, \ldots, p_n$ where p_i is the oriented line segment from v_i to v_{i+1}. Two polygons $P, Q \in \mathcal{P}_n$ are identified if and only if there exists an orientation preserving isometry g of $\mathbb{E}^3$ which sends the vertices of P to the vertices of Q.

Definition : Let $m = (m_1, \ldots, m_n)$ be an n-tuple of positive real numbers. Then $\mathcal{M}_n$(or just $\mathcal{M}$) is the space of n-gons with side lenghts $m_1, \ldots, m_n$ modulo isometries as above.

Note that for any $P \in \mathcal{P}_n$, the vector of lenghts of sides of P satisfies the following;

$$m_i < m_1 + m_2 + \cdots + \widehat{m_i} + \cdots + m_n; \qquad i = \overline{1, n} = 1, \ldots, n.$$

It is known that $\mathcal{M}$ has only isolated singularities corresponding to the degenerate polygons. It is non singular if all sums $\pm m_1 \pm m_2 \pm \cdots \pm m_n$ are non zero. The following theorem about the structure of $\mathcal{M}$ appears in [3].

Theorem : If lenghts of all sides $\|p_i\| = m_i$ are rational and $m_1 \pm m_2 \pm \cdots \pm m_n \neq 0$ then $\mathcal{M}$ is a non-singular projective variety.

We are going to use the relation between spatial polygons and stable weighted point configurations on the complex projective line. We have the following description;

Definition : A n-point configuration Σ is a collection of n-points $p_1, \ldots, p_n \in$

[1] I want to thank A. Klyachko for his ideas and the wonderful course "Geometric Invariant Theory" he taught at Bilkent University, Turkey.

$\mathbb{P}^1$. Assume there is a given positive weight m_i for each point. The configuration of weighted points is called semi-stable (resp. stable) if sum of the weights of equal points does not exceed(resp. less than) half the weight of all points.

Using Hilbert-Mumford stability criterion([4]), we can say that there exist a non-singular geometric factor of stable configurations with respect to a natural action of $PSL_2(\mathbb{C})$. It will be denoted by $\mathcal{C}_n(m)$ where $m = (m_1, \ldots, m_n)$ is the vector of weights. By definition, $\mathcal{C}_n(m)$ is non-empty iff the weights satisfy the following polygon inequality

$$m_i < m_1 + m_2 + \ldots + \widehat{m_i} + \ldots + m_n; \qquad i \in \{1, \ldots, n\} \qquad (*)$$

In a similar way, there exist a categorical factor of space of semi-stable configurations denoted by $\overline{\mathcal{C}}_n(m)$.

Under condition (*), the variety $\overline{\mathcal{C}}_n(m)$ is a projective compactification of $\mathcal{C}_n(m)$ by a finite number of points. Its ample sheaf $\mathcal{O}(1)$ and the corresponding line bundle $\mathcal{L}$ may be described as follows. Let $\mathcal{T}(p_i)$ be a tangent space at the point $p_i \in \mathbb{P}^1$. Then $\mathcal{L}$ is a line bundle on $\overline{\mathcal{C}}_n(m)$ with fiber

$$\mathcal{L}(\mathbf{p}) = \mathcal{T}(p_1)^{\otimes m_1} \otimes \mathcal{T}(p_2)^{\otimes m_2} \otimes \cdots \otimes \mathcal{T}(p_n)^{\otimes m_n}$$

at a point $\mathbf{p} = (p_1, \ldots, p_n) \in \overline{\mathcal{C}}_n(m)$. If all semi-stable configurations of weight m are stable then $\overline{\mathcal{C}}_n(m)$=$\mathcal{C}_n(m)$ is a non-singular projective variety of dimension $n-3$.

Example : Let all weights $m_i = 1$ i.e, $m = (1, \ldots, 1)$. Then $\overline{\mathcal{C}}_n(m) = \mathcal{C}_n(m)$ is a non-singular projective variety for odd n. In this case all sums $m_1 \pm m_2 \pm \cdots \pm m_n$ are non-zero.

Example : Let $\Sigma = (p_1, \ldots, p_n)$ be a configuration of n-points in $\mathbb{P}^1$ having one massive point, say m_1 i.e, $m_1 + m_i > \frac{m}{2}, m = m_1 + \cdots + m_n$ so that $p_1 \neq p_i$, $\forall i \neq 1$. Then we can interchange the coordinates in $\mathbb{P}^1$ such that $p_1 = \infty$, $p_2 = 0$ and $p_i = z_i, z_i \in \mathbb{C}$; z_i are defined uniquely up to multiplication by scalar multiplication $z \mapsto \lambda z$ which preserves $\infty, 0$. Then moduli of the configuration is eqivalent to

$$\{(z_3 : \ldots : z_n) | z_i \in \mathbb{C}, i = 3, \ldots, n; \text{ not all zero}\} = \mathbb{P}^{n-3}$$

Example : Let $\Sigma = (p_1, \ldots, p_n)$ be a configuration with three massive points

$$m_i + m_j > \frac{m}{2}, m_j + m_k > \frac{m}{2}, m_i + m_k > \frac{m}{2}$$

Then $p_i \neq p_j, p_j \neq p_k$ and $p_i \neq p_k$. By a suitable coordinate change we may take $p_i = 0, p_j = 1, p_k = \infty$ and hence the moduli of configuration is equivalent to

$$\prod_{\alpha \neq i,j,k} \mathbb{P}^1 = (\mathbb{P}^1)^{n-3}.$$

The following theorem reveals the relations between the variety of spatial polygons in $\mathbb{E}^3$ and stable configurations on $\mathbb{P}^1$.
Theorem : The algebraic variety of spatial polygons $\mathcal{M}$ is biregular equivalent to the variety $\overline{\mathcal{C}}_n(m)$ of semi-stable weighted configurations of points in the projective line.
Proof: See [**?**] $\square$

2 Cohomology

Algebraic variety of spatial polygons in Euclidean space $\mathbb{E}^3$ is equivalent to $\mathcal{C}_n(m)$, stable weighted configurations on complex projective line $\mathbb{P}^1 = S^2$ modulo Möbius group $PSL_2(\mathbb{C})$.
Let's define $\mathcal{L}_i$ to be the linear vector bundle over $\mathcal{C}_n(m)$ such that fiber at $\Sigma = (p_1, \dots, p_n)$ is equal to tangent space at $p_i \in \mathbb{P}^1, i = 1, \dots, n$. We call $\mathcal{L}_i$'s *natural bundles* on $\mathcal{C}_n(m)$.
We know that $\mathcal{C}_n(m)$ is non-empty if and only if weights $m = (m_1, \dots, m_n)$ satisfy;

$$m_i < m_1 + m_2 + \dots + \widehat{m_i} + \dots + m_n; \quad i = 1, \dots, n.$$

Under this condition variety of semi-stable configurations $\overline{\mathcal{C}}_n(m)$ is a projective compactification of $\mathcal{C}_n(m)$ by a finite number of points. Corresponding line bundle $\mathcal{L}$ of $\overline{\mathcal{C}}_n(m)$ can be written as

$$\mathcal{L}(\Sigma) = \mathcal{L}_1^{\otimes m_1} \otimes \dots \otimes \mathcal{L}_n^{\otimes m_n},$$

at a point $\Sigma = (p_1, \dots, p_n) \in \overline{\mathcal{C}}_n(m)$.
Setting $\mathcal{C}_n(m) = (\mathbb{P}_1^n)^s / PSL_2(\mathbb{C})$. We consider the map

$$\pi : (\mathbb{P}_1^n)^s = \underbrace{(\mathbb{P}^1 \times \dots \times \mathbb{P}^1)^s}_{n-copies} \longrightarrow \mathcal{C}_n(m).$$

With the fiber $\pi^{-1}(\Sigma) \simeq PSL_2(\mathbb{C})$. This is the structure of a principal $PSL_2(\mathbb{C})$ bundle.
Let ξ be the linear vector bundle such that for $\Sigma \in \mathcal{C}_n(m)$, $\xi(\Sigma)$ is the tangent space to fiber $\pi^{-1}(\Sigma)$ i.e, $\xi(\Sigma)$ is the tangent to $PSL_2(\mathbb{C}) = \mathfrak{sl}_2$ which acts on SL_2 by $adg : A \mapsto g^{-1}Ag$ and $PSL_2(\mathbb{C}) = SL_2 / \pm 1$.
Note that $det\ adg = 1$ since $A \in SL_2$. So determinant bundle $det\xi$ is trivial. Taking into consideration all above, we form the Euler sequence to be

$$0 \longrightarrow \xi \longrightarrow \bigoplus_{i=1}^{n} \mathcal{L}_i \longrightarrow \mathcal{T} \longrightarrow 0$$

where $\mathcal{T}$ is tangent bundle to $\mathcal{C}_n(m)$.

The *canonical* bundle of $\mathcal{C}_n(m)$ is defined to be the determinant bundle of 1-forms Ω on $\mathcal{C}_n(m)$. Namely,

$$\kappa = det\Omega.$$

We know that $\Omega = \mathcal{T}^*$, dual of tangent bundle $\mathcal{T}$. Then we say $-\kappa = det\mathcal{T}$ is the *anticanonical bundle*.

In an exact sequence

$$0 \longrightarrow E' \longrightarrow E \longrightarrow E'' \longrightarrow 0$$

of vector bundles we have $detE = (detE') \otimes (detE'')$ and $det \bigoplus E_i = \bigotimes E_i$, again E_i's are vector bundles. (see [**?**]). Then we arrive the following theorem

Theorem : $-\kappa = det\mathcal{T} = \bigotimes_{i=1}^n \mathcal{L}_i$ where $\mathcal{T}$ is the tangent bundle of $\mathcal{C}_n(m)$ and $\mathcal{L}_i$'s are natural bundles.

Proof: By above argument and Euler sequence

$$0 \longrightarrow \xi \longrightarrow \bigoplus_{i=1}^{n} \mathcal{L}_i \longrightarrow \mathcal{T} \longrightarrow 0$$

we have $det \bigoplus_{i=1}^n \mathcal{L}_i = det\mathcal{T} \otimes det\xi$ and $det\xi$ is trivial. So

$$det\mathcal{T} = \bigotimes_{i=1}^{n} \mathcal{L}_i \quad \square$$

.

Definition : A topological space X is called an even-cohomology space if its cohomology groups $H^*(X;\mathbb{Z})$ vanish for * odd.

The following lemma is a first step to determine the cohomology of spatial polygons. For the proof see [1],[**?**]

Lemma : $\mathcal{M}_n$ is an even-cohomology space. $\square$

As a consequence of the lemma, odd Betti numbers of $\mathcal{M}_n$ vanish. The following theorem is a useful tool for calculating Poincarè polynomials. Proof can be found in [**?**] and [1]

Theorem : Poincaré polynomial of the variety $\mathcal{M}_n$ is given by

$$P_q(\mathcal{M}_n) = \frac{1}{q(q-1)}((1+q)^{n-1} - \sum_{m_I \leq \frac{m}{2}} q^{|I|})$$

where $m = m_1 + \cdots + m_n; m_I = \sum_{i \in I} m_i$.

Let us go back to variety of weighted stable configurations. For any decomposition $I \amalg J \amalg K \ldots = \{1, \ldots, n\}$ let $D_{IJK\ldots}$ be the cycle of stable configurations $\Sigma = (p_1, \ldots, p_n)$ with $p_\alpha = p_\beta$ for α, β are in the same component $I, J, K, \ldots$. In particular, we define

$$D_{ij} = \text{ divisor of stable configurations with } p_i = p_j.$$

We would like to characterize all effective cycles in $\mathcal{C}_n(m)$ using degenerate configurations $D_{IJKL\cdots}$.
Theorem : Any effective cycle in $\mathcal{C}_n(m)$ is equivalent to positive combinations of degenerate configurations $D_{IJKL\cdots}$.
Proof: Theorem holds for special values for m_i's. For example, for one massive point or three massive points. In these cases $\mathcal{C}_n(m) \simeq \mathbb{P}^{n-3}$ and $\mathcal{C}_n(m) \simeq (\mathbb{P}^1)^{n-3}$ respectively.
It is possible to pass from one moduli space to another by a sequence of wall crossing $\mathcal{C}_n(m) \longrightarrow \mathcal{C}_n(\widetilde{m})$ such that only one inequality $m_I < \frac{m}{2}$ changes its direction to be $\widetilde{m_I} > \frac{m}{2}$ and all the other inequalities stay unchanged. In this case we may choose m and $\widetilde{m}$ to be arbitrary close the wall $m_I = \frac{m}{2}$.
Let $\{1, \ldots, n\} = I \amalg J,\ I$ is the special subset mentioned above. Assume $|I| = k,\ |J| = l$. Then

$$\mathcal{C}_n(m_I, m_i:\ i \in J) \simeq \mathbb{P}^{l-2} \subset \mathcal{C}_n(m),$$
$$\mathcal{C}_n(\widetilde{m}_J, m_i:\ i \in I) \simeq \mathbb{P}^{k-2} \subset \mathcal{C}_n(\widetilde{m})$$

and $\mathcal{C}_n(m)/\mathbb{P}^{l-2}$ is birationally equivalent to $\mathcal{C}_n(\widetilde{m})/\mathbb{P}^{k-2}$. Algebraic cycles in $\mathcal{C}_n(\widetilde{m})$ are those in $\mathcal{C}_n(m)$ and cycle in $\mathbb{P}^{l-2}$ are generated by degenerate configurations by the argument at the beginning of this proof.
□

Recall that $\mathcal{L}_i$'s are natural bundles on $\mathcal{M}$ such that the fiber at $\Sigma = (p_1, \ldots, p_n)$ is the tangent space at $p_i \in \mathbb{P}^1$. Set

$$l_i = [\mathcal{L}_i] = \{\text{ zeros of } s\} - \{\text{ poles of } s\}$$

where s is a rational section of $\mathcal{L}_i$.
Lemma : With the previous notations $l_i = D_{ij} + D_{ik} - D_{jk}$ which is independent of choice of j, k.
Proof: Let $t = \frac{p_i - p_j}{p_i - p_k} : \frac{z - p_j}{z - p_k}$ be local parameter at $z \in \mathbb{P}^1$ with $t(p_i) = 1$. Then

$$w_i = \frac{dt}{dz} = \frac{(p_k - p_j)dz}{(z - p_j)(z - p_k)}|_{z=p_i} = \frac{(p_k - p_j)dp_i}{(p_i - p_j)(p_i - p_k)}$$

is the rational section of dual bundle $\mathcal{L}_i^*$. So $[w_i] = D_{jk} - Dij - D_{ik}$. Therefore,

$$l_i = D_{ij} + D_{ik} - D_{jk}. \quad \square$$

Corollary : Some of the other relations between l_i and D_{ij} are as follows;

$$1) \qquad D_{ij} = \frac{1}{2}(l_i + l_j),$$
$$2) \qquad l_i - l_j = \frac{1}{2}(D_{ik} - D_{jk}).$$

Proof: We have $l_i = D_{ij} + D_{ik} - D_{jk}$. So
1)

$$\left.\begin{array}{l} l_i = D_{ij} + D_{ik} - D_{jk} \\ l_j = D_{ij} + D_{jk} - D_{ik} \end{array}\right\} \Rightarrow l_i + l_j = 2D_{ij}$$

so $D_{ij} = \frac{1}{2}(l_i + l_j)$.
2) Follows from above. □

The lemma gives an inductive procedure to evaluate any monomial in l_i in terms of "degenerate" cycles $D_{I,J,K,\ldots}$(in which all points $p_i \in I$ are glued together as well as for $J, K, L, \ldots$). The following corollary allows us to evaluate an arbitrary monomial in l_i. Note that non-zero cycles should contain at least three components and 3-component cycles represent a point provided m_I, m_J, m_K satisfy triangle inequality.
Corollary :$l_i \cdot D_{I,J,K,\ldots} = D_{(IJ),K,\ldots} + D_{(IK),J,\ldots} - D_{I,(JK),\ldots}$.
Proof: For any l_i and cycle $D_{I,J,K,\ldots}$, with $i \in I$ we may write

$$l_i \cdot D_{I,J,K,\ldots} = [\mathcal{L}_i | \mathcal{C}_n(m_I, m_J, m_K, \ldots)],$$

where right hand side of the above equation is the class of $\mathcal{L}_i$ in $\mathcal{C}_n(m_I, m_J, m_K, \ldots) \simeq D_{I,J,K,\ldots}$ and $\mathcal{C}_n(m_I, m_J, m_K, \ldots)$ is the moduli space of weighted stable configurations obtained from summing up weights of $\mathcal{C}_n(m)$ whose corresponding indices contained in $I, J, K, L, \ldots$
By lemma we may write

$$[\mathcal{L}_i | \mathcal{C}_n(m_I, m_J, m_K, \ldots)] = D_{IJ} + D_{IK} - D_{JK}$$

provided $i \in I$ and we obtain the equality in the statement of corollary. □
Example : For $i \neq j$ we can evaluate $l_j l_i$ as follows; we know that $l_i = D_{ij} + D_{ik} - D_{jk}$. So,

$$\begin{array}{c} l_j \cdot D_{ij} = D_{ijk} + D_{ijl} - D_{(ij)(kl)} \\ l_j \cdot D_{ik} = D_{ijk} + D_{(ik)(jl)} - D_{ikl} \\ l_j \cdot D_{jk} = D_{ijk} + D_{jkl} - D_{(il)(jk)} \end{array}$$

This implies

$$l_j l_i = D_{ijk} + D_{ijl} - D_{ikl} - D_{jkl} + D_{(ik)(jl)} + D_{(il)(jk)} - D_{(ij)(kl)}.$$

Example : For $p = l_i^2$, a similar calculation leads us to the formula

$$p = D_{ijk} + D_{ijl} + D_{ikl} + D_{jkl} - D_{(ij)(kl)} - D_{(ik)(jl)} - D_{(jk)(il)}.$$

This expression is independent of i, j, k, l.

By the equivalence of stable configurations and spatial polygons, we can relate the divisors D_{ij} by some kind of polygons. In other words, a divisor D_{ij} corresponds to a polgon in $\mathcal{M}$ with edges $(p_1, p_2, \ldots, p_n)$ and $p_i \upuparrows p_j$ i.e, p_i and p_j are parallel. For anti-parallel edges we write $p_i \uparrow\downarrow p_j$.
Using the following theorem we may calculate the cohomology rings of stable configurations, hence cohomology rings of spatial polygons. **Theorem :** The Chow(cohomology) ring of $\mathcal{C}_n(m)$ is generated by the class of divisors D_{ij} subject to the following relations;
1) $\forall$ quadruple (i, j, k, l) there are linear relations

$$D_{ij} + D_{km} = D_{ik} + D_{jm} = D_{im} + D_{kj} = \frac{1}{2}(l_i + l_j + l_k + l_m).$$

2) For any triple (i, j, k) there are quadratic relations

$$D_{ij}D_{jk} = D_{jk}D_{ki} = D_{ki}D_{ij} = D_{ijk}.$$

3) For any tree Γ with vertices in $I \subset \{1, \ldots, n\}$ such that $m_I > \frac{m}{2}$,

$$D_I = \prod_{(ij)\in\Gamma} D_{ij} = 0.$$

Proof: We know that the divisor D_{ij} generate the Chow ring. In the view of the formula $D_{ij} = \frac{1}{2}(l_i + l_j)$, relations in 1) becomes trivial. Using the same formula we also see that all products in quadratic relations 2) are equal to D_{ijk}. The product in 3) is a locus of configurations with equal points $p_i, i \in I$. Under the condition $m_I > \frac{m}{2}$, such configuration is unstable and hence $\prod_{(ij)\in\Gamma} D_{ij} = 0$. Observe that the quadratic relations ensure that the product D_I is independent of choice of tree Γ on vertices I.
To prove the completeness, we need to show that; for any disjoint subsets $I, J, K, M \subset \{1, \ldots, n\}$ we have

$$(*) \quad D_{(IJ),K,M} + D_{I,J,(KM)} = D_{(IK),J,M} + D_{I,K,(JM)} = D_{(IM),J,K} + D_{I,M,(JK)}.$$

In fact, if i, j, k, m are elements from I, J, K, M respectively then the above equation is equivalent to the following identities;

$$D_{I,J,K,M}(D_{ij} + D_{km}) = D_{I,J,K,M}(D_{ik} + D_{jm}) = D_{I,J,K,M}(D_{im} + D_{jk})$$

which follows from 1). So (*) holds.
Now, let us consider a puzzle; let's divide a heap of stones of masses m_i into three parts of masses m_I, m_J, m_K satisfying the triangle inequality. Then any other such division may be obtained from the initial one br removing a stone from one heap and putting it into another so that new heaps also satisfy the

triangle inequality.
Using the puzzle we can show that if $I \amalg J \amalg K = \{1, \dots, n\}$ is stable decomposition i.e, m_I, m_J, m_K satisfy the triangle inequality. Then $D_I D_J D_K = D_{IJK}$ is independent of stable decomposition.
Really, by the puzzle it is enough to check that

$$D_{I/\{i\},J\cup\{i\},K} = D_{IJK} \text{ if } I/\{i\}, J \cup \{i\}, K \text{ is stable.}$$

Applying (*) to the quadruple $\{i\}, I/\{i\}, J, K$ we get

$$D_{IJK} + D_{\{i\},I/\{i\},(JK)} = D_{(\{i\}J),K,I/\{i\}} + D_{\{i\},J,(KI/\{i\})}.$$

Using triangle inequalities $m_J + m_K > \frac{m}{2}$ and $m_K + M_{I/\{i\}} > \frac{m}{2}$ we obtain unstable decompositions (JK) and $(KI/\{i\})$ then

$$D_{\{i\},I/\{i\},(JK)} = D_{\{i\},J,(KI/\{i\})} = 0$$

.
So we obtained the desired result. It remains to check that
i) Linear relations between divisors are complete,
ii) Linear relation on divisor cycles $D_{IJK\dots}$ follows from the relation 1) in the statement of the theorem.
The second part *ii)* is the consent of (*). The first part *i)* necessarily says that the cross ratio $[p_i : p_j : p_k : p_l]$ generate the whole ring of non-vanishing regular functions on the divisors D_{ij}. Actually, if we fix $p_i = 0, p_j = \infty, p_k = 1$ then the complement of the divisor D_{ij} is a subset $X \subset \mathbb{C}^{n-3}$ with pairwise distinct components$\neq 0, 1$. Non-vanishing regular functions on X are generated by $(p_l - p_m)^{\pm 1}, p_l^{\pm 1}, (p_l - 1)^{\pm 1}$ and may be expressed by cross-ratio. Hence we are done. □
Corollary : The Chow(cohomology) ring $H^*(\mathcal{M})$ over $\mathbb{Z}$ is generated by the classes of natural bundles subject to relations

$$\begin{array}{ll} 1) & l_i^2 = p, \text{ independent of } i \\ 2) & \displaystyle\sum_{2k+r=|I|-1} p^k \sigma_r(l_I) = 0, \quad m_I > \frac{m}{2}, \end{array}$$

where σ_r is the r-th elementary symmetric polynomial and $m = m_1 + \cdots + m_n; m_I = \sum_{i \in I} m_i$.

Above corollary gives a handy method to determine the Chow ring of polygon spaces. Moreover, in what follows we will give explicit formulae to determine the monomials of the form $p^k l_J, J \subset I$ with $m_I > \frac{m}{2}$.

Non-Holomorphic Cycles. There are natural symplectic cycles in the polygon space $\mathcal{M}$, corresponding degenerate polygons with antiparallel sides. Tangent space to the moduli of such degenerate polygons is not closed under complex multiplication but as a topological cycle, it should be an integer linear combination of the holomorphic cycles $D_{I,J,K,\ldots}$. In a simplest case of non-holomorphic divisor, it may be expressed as follows,

Lemma : The cycle D^-_{ij}(see Fig.9b) of polygons with antiparallel vectors $p_i \uparrow\downarrow p_j$ is equivalent to $\frac{1}{2}(l_i - l_j)$ if D^-_{ij} is oriented by the vector p_j.

Proof: Let us compare the intersection indices of D^-_{ij} and $\frac{1}{2}(l_i - l_j)$ with holomorphic curves, that is with quadrangles D_{IJKL}. Both intersections $D^-_{ij} \cdot D_{IJKL}$ and $\frac{1}{2}(l_i - l_j) \cdot D_{IJKL}$ are zero if i, j belongs to the same set I, J, K, L. So we may suppose that $i \in I, j \in J$. Then the first intersection index is non-zero if and only if the quadrangle admits a degeneration into triangle with antiparallel vectors $p_I \uparrow\downarrow p_J$. Here $p_I = \sum_{i \in I} p_i$. In this case

$$||p_K - p_L|| < ||p_I - p_J|| < p_K + p_L \text{ i.e, } |m_K - m_L| < |m_I - m_J| < m_L + m_K.$$

Assume $m_I > m_J$ and $m_K > m_L$. Then previous inequality becomes

$$m_K + m_J < m_I + m_L, m_I < m_J + m_K + m_L. \quad (*)$$

In the previous section we saw that there are two types of quadrangles(shown in Fig.1) "triangle" type and "star" type which are represented by (ijk) and $\{(ij)(ik)(il)\}$ respectively. In our case we have

$$m_I + m_K > m_J + m_L \text{ and } m_K + m_J < m_I + m_L.$$

So if $m_I + m_J > m_K + m_L$ we obtain the star $\{(IJ)(IK)(IL)\}$, otherwise we find the triangle (IKL). By corollary(1.40) we have

$$l_i \cdot D_{IJKL} = D_{(IJ)KL} + D_{(IK)JL} - D_{I(JK)L}$$

.

Together with this formula and using the assumption $i \in I, j \in J$ we have

$$\begin{array}{lll} l_i \cdot D_{IJKL} = 0, & l_j \cdot D_{IJKL} = 2 & \text{for triangle } (IKL), \\ l_i \cdot D_{IJKL} = -1, & l_j \cdot D_{IJKL} = 1 & \text{for star } \{(IJ)(IK)(IL)\}. \end{array}$$

In both cases $\frac{1}{2}(l_i + l_j) \cdot D_{IJKL} = -1$.

Suppose $m_I < m_J$ then intersection indices change sign. As a result we get,

$$\frac{1}{2}(l_i - l_j) \cdot D_{IJKL} = \begin{cases} 1 & \text{if } m_I < m_J \\ -1 & \text{if } m_I > m_J. \end{cases}$$

To compare this to $D^-_{ij} \cdot D_{IJKL}$ we must choose an orientation of D^-_{ij}, which is in fact an orientation of the normal plane to p_i or p_j. Let us choose the orientation by the normal to p_i. Then

$$D^-_{ij} \cdot D_{IJKL} = \begin{cases} 1 & \text{if } m_I > m_J \\ -1 & \text{if } m_I < m_J \end{cases} = -\frac{1}{2}(l_i - l_j) \cdot D_{IJKL}.$$

So if we orient it by p_j we obtain the Lemma. $\square$

This lemma is quite useful in determining any product

$$D_I^\epsilon = 2^{1-|I|} \prod_{(ij)\in\Gamma} (l_i \pm l_j)$$

where $\epsilon(i,j) = -\epsilon(j,i) = \pm 1$ and $m_I > \frac{m}{2}$ with a tree Γ with vertices I. In a different point of view, we may consider Γ as partially oriented i.e, some edges have an orientation $i \to j$, yet the others may not be oriented. So we define

$$D_\Gamma = 2^{1-|I|} \prod_{(ij)\in\Gamma} (l_i \pm l_j) = 2^{1-|I|} \prod (l_i + l_j) \prod_{i\to j} (l_i - l_j).$$

Let us take Γ to be star $\{(i_0 i_1)(i_0 i_2) \cdots (i_0 i_q)\}$. Then

$$D_I^\epsilon = 2^{1-|I|} \prod_{i\in I/\{i_0\}} (l_{i_0} + \epsilon_i l_i)$$

with $\epsilon_i = \pm 1$ and $\epsilon_{i_0} = 1$.
Observe that

$$\prod_{(ij)\in\Gamma} (l_i + l_j) = \sum_{\substack{orientations\ of \\ \Gamma}} \prod_{i\in I} l_i^{deg^+ i} = \sum_{2k+|I_{odd}|=|I|-1} p^k l_{I_{odd}},$$

where $deg^+ i$ is the number of edges entering i and I_{odd} is a subset of vectors with odd deg^+. This leads to the equation

$$D_I^\epsilon = 2^{1-|I|} \sum_{\substack{J \subset I \\ |I/J| = 2k+1}} \epsilon_J l_J p^k \quad (*)$$

where $\epsilon_J = \prod_{i\in J} \epsilon_i, l_j = \prod_{j\in J} l_j$.
We may take the last formula as a system of $2^{1-|I|}$ equations with the same number of variables $l_J p^k, J \subset I$ and $|I/J| = 2k+1$. Note that the matrix of the system is invertible because its square is a scalar matrix.
The following theorem and its corollary are useful in calculating the cohomology.
Theorem : For any $J \subset I, m_I > \frac{m}{2}$ and $|I/J| = 2k+1$, the following formula holds

$$l_J \cdot p^k = \sum_\epsilon \epsilon_J D_I^\epsilon$$

where the product is taken over all combinations of signs $\epsilon_i = \pm 1, \epsilon_0 = 1$.
Proof: Using (*) we can write

$$\sum_\epsilon \epsilon_J D_I^\epsilon = 2^{1-|I|} \sum_{\substack{K \subset I \\ |I/K| = 2k+1}} (\sum_\epsilon \epsilon_J \epsilon_K) l_K p^k = l_J p^k$$

Since

$$\sum_{\epsilon} \epsilon_J \epsilon_K = \begin{cases} 0 & \text{if } J \neq K, \\ 2^{|I|-1} & \text{if } J = K. \end{cases} \qquad \square$$

Corollary : Let $|J| + 2k = n - 3 = dim\mathcal{M}$ and $I \supset J$ be any set of cardinality $n-2$, say $I = \{1, \ldots, n\}/\{\alpha, \beta\}, \alpha, \beta \notin I$. Then

$$l_J p^k = \sum_{|m_\alpha - m_\beta| < |(\epsilon_I, m_I)| < m_\alpha + m_\beta} sgn(\epsilon_I, m_I)\epsilon_{I/J}$$

where the sum is taken over all signs $\epsilon_i = \pm 1$, fixed on one element $\gamma \in I$ with $\epsilon_\gamma = 1$ and $(\epsilon_I, m_I) = \sum_{i \in I} \epsilon_i m_i$.
Proof: By the assumption of the corollary $|I| = n-2$ and hence $|I/J| = 2k+1$. Note that in the sum, we consider $\alpha, \beta \in \{1, \ldots, n\}/I$ such that the polygon degenerates into a triangle and satisfies the triangle inequality. So the previous theorem is applicable. $\square$

3 Fano Polygon Spaces

Proposition : Let the vertices of an element of the moduli space $\mathcal{M}$ be numbered as $1, 2, \ldots, n$. Then the first Chern class $c_1(\mathcal{M})$ is given by

$$c_1(\mathcal{M}) = \sum_{i=1}^{n} D_{ii+1}.$$

Here we set $D_{nn+1} = D_{n1}$
Proof: Using the natural bundles we write $c_1(\mathcal{M}) = \sum_i l_i$. Taking consecutive sum with the convention $l_{n+1} = l_1$ we have,

$$c_1(\mathcal{M}) = \sum_i l_i = \sum_{i=1}^{n} \frac{1}{2}(l_i + l_{i+1}) = \sum_{i=1}^{n} D_{ii+1}. \quad \square$$

Definition : A divisor D is ample if $D \cdot C > 0$, $\forall$ curve C contained in $\mathcal{M}_n$. Similarly, a vector bundle $\mathcal{L}$ is ample if $\mathcal{L}|_C$ is ample for every curve in $\mathcal{M}_n$.

Note that $D_{IJKL} \neq 0 \Leftrightarrow m_I, m_J, m_K, m_L$ satisfy $m_I < m_J + m_K + m_L$ for all I, J, K, L and in this case $D_{IJKL} \simeq \mathbb{P}^1$.
Theorem ;[Ampleness Criterion] $D = \sum_{i=1}^{n} a_i l_i$, where l_i's are characteristic classes of natural bundles $\mathcal{L}_i$ and a_i's are positive numbers, is ample if and only if for all quadruple D_{IJKL} we have

$$\begin{cases} a_I > 0 & \text{for "triangle"} \\ a_I < a_J + a_K + a_L & \text{for "star"} \end{cases}$$

Proof: Recall that there are two types of quadrangles;
1) Triangle type; for indices i, j, k, l

$$m_i + m_j > \frac{m}{2},\ m_j + m_k > \frac{m}{2},\ m_k + m_i > \frac{m}{2}$$

2) Star type; for indices i, j, k, l

$$m_i + m_j > \frac{m}{2},\ m_i + m_k > \frac{m}{2},\ m_i + m_l > \frac{m}{2}$$

For triangle type, $l_i \cdot D_{ijkl} = D_{ij} + D_{ik} - D_{jk} = 0$ and $l_l \cdot D_{ijkl} = D_{li} + D_{lj} - D_{ij} = 2$
For star type ,$l_i \cdot D_{ijkl} = D_{ij} + D_{ik} - D_{jk} = -1$ and $l_j \cdot D_{ijkl} = D_{ji} + D_{jk} - D_{ik} = 0$.
We need to check $D \cdot D_{IJKL} = \sum_{i=1}^{n} a_i (l_i \cdot D_{IJKL}) > 0,\ \forall D_{IJKL} \neq 0$. So we will have two cases,
Case I Triangle case: In this case we set

$$m_L + m_J > \frac{m}{2},\ m_L + m_K > \frac{m}{2},\ m_J + m_K > \frac{m}{2}$$

and

$$l_i \cdot D_{IJKL} = \begin{cases} 2 & \text{for } i \in I \\ 0 & \text{for } i \notin I \end{cases}$$

Case II Star case: In this case

$$m_I + m_L > \frac{m}{2},\ m_I + m_J > \frac{m}{2},\ m_I + m_K > \frac{m}{2}.$$

So

$$l_i \cdot D_{IJKL} = \begin{cases} -1 & \text{for } i \in I \\ 1 & \text{for } i \notin I \end{cases}$$

In order to obtain $D \cdot D_{IJKL} > 0$, in the first case we must have $a_I > 0$ and in the second case $a_I < a_J + a_K + a_L$.□
Corollary : The anticanonical class $c_1(\mathcal{M}_n) = \sum l_i$ is ample if and only if for all star type degenerations of any quadruple D_{IJKL} we have

$$|I| < |J| + |K| + |L|.$$

Definition : $\mathcal{M}$ is Fano if the first anti-canonical class(first Chern class) is ample.

In other words, $\mathcal{M}$ is Fano if and only if $c_1(\mathcal{M}) \cdot D_{IJKL} > 0$ for all quadrangles D_{IJKL}.
Definition : A maximal degeneration in $\mathcal{C}_n(m)$ is a cycle consisting of configurations in which $p_i = p_j$, for all $i, j \in I$ and I is the maximal set. We will denote a maximal degeneration by $\mathcal{M}_I$.

Note that $\mathcal{M}_I \subset \mathcal{C}_n(m)$ and actually $\mathcal{M}_I \simeq \mathbb{P}^k$ with $k = n - |I| - 2$.
The maximal sets I are characterized by inequalities

$$\begin{aligned} m_I + m_s &> \frac{m}{2} \\ m_I &< \frac{m}{2} \end{aligned}$$

Theorem :$\mathcal{C}_n(m)$ is Fano if and only if any maximal degeneration $\mathcal{M}_I$ is either a point or has dimension greater than $\frac{n-4}{2}$
Proof: We know that $\mathcal{C}_n(m)$ is Fano $\Leftrightarrow$ the polygon has no quadrangle degenerations of "star" type i.e, we won't have degenerations given by

$$m_I + m_J > \frac{m}{2},\ m_I + m_K > \frac{m}{2},\ m_I + m_L > \frac{m}{2}$$

with $|I| \geq \frac{n}{2}$.
We can successively move a side $p_s,\ s \notin I$ to the set of edges whose indices are contained in I as long as it is possible i.e, we can move it as long as $m_I < \frac{m}{2}$. As a result, we arrive to the maximal degeneration $m_I < \frac{m}{2}$ and $m_I + m_s > \frac{m}{2},\ \forall s \notin I$. There are two possibilities;
i) The maximal degeneration is a point i.e, $|I| = n-2$,
ii) The maximal degeneration is of positive dimension.

$$dim\mathcal{M}_I = n - |I| - 2 < \frac{n}{2} - 2$$

for the ones which are not Fano(i.e, quadrangles of star type with $|I| \geq \frac{n}{2}$)
Therefore, for Fano we have

$$dim\mathcal{M}_I > \frac{n-4}{2}. \quad \square$$

Corollary : For $n \in \{4, 5\}$ we have Fano polygon spaces

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Vehbi Emrah Paksoy, Claremont McKenna College, emrah.paksoy@cmc.edu