J. Plasma Physics (1978), *vol.* 20, *part* 3, *pp.* 329–344

Printed in Great Britain

Clebsch representations in the theory of minimum energy equilibrium solutions in magnetohydrodynamics

By HANNO RUND

Department of Mathematics, University of Arizona, Tucson, Arizona 85721

DANIEL R. WELLS

AND LAWRENCE CARL HAWKINS

Department of Physics, University of Miami, Coral Gables, Florida 33124

(Received 15 February 1977, and in revised form 16 November 1977)

The introduction of Clebsch representations allows one to formulate the problem of finding minimum energy solutions for a magneto-fluid as a well-posed problem in the calculus of variations of multiple integrals. When the latter is subjected to integral constraints, the Euler–Lagrange equations of the resulting isoperimetric problem imply that the fluid velocities are collinear with the magnetic field. If, in particular, one constraint is abolished, Alfvén velocities are obtained. In view of the idealized nature of the model treated here, further investigations of more sophisticated structures by means of Clebsch representations are anticipated. Preliminary results of a similar calculation utilizing a modified two fluid model are discussed.

1. Introduction

It is important to find solutions of the variational problem posed by a determination of minimum free energy bounded plasma structures. These solutions correspond to force-free or quasi force-free configurations. The concept of force-free fields has played a major role in both astrophysics and the theory of stable laboratory plasmas. It has proved to be very useful in investigating possibly naturally occurring stable states in magnetically confined high temperature plasmas (Wells & Norwood 1968; Wells 1970, 1976; Taylor 1974). Experimentally it is found that force-free and quasi force-free configurations are very stable structures, even if the force-freeness is not exact (quasi force-free) or if they are surrounded by or surround plasma structures that are not force-free.

Since the energy associated with region G is generally expressed as an integral over G, the problem of the derivation of minimum energy states may be formulated as a variational principle. However, in many cases the nature of the integrand, as presented *ab initio* by purely physical considerations, is such that these are not well-posed problems from the point of view of the calculus of variations: this prohibits the direct application of the simple and yet rigorous techniques of that discipline. Very often the *ad hoc* evaluation of the first variation is unconvincing in that it depends on boundary conditions which cannot be motivated physically or mathematically. Moreover, the conclusions resulting from the

Downloaded from https:/www.cambridge.org/core. Open University Libraryy, on 10 Jan 2017 at 00:18:38, subject to the Cambridge Core terms of use, available at https:/www.cambridge.org/core/terms. http://dx.doi.org/10.1017/S0022377800023898

requirement that the first variation vanish cannot be used directly to distinguish between possible maxima and minima: this in turn entails an appeal to the second variation, the derivation of which is not only subject to similar objections, but is often also extremely complicated. These difficulties are severely compounded when the system is subjected to constraints as prescribed by physical considerations, which gives rise to the so-called problems of Lagrange and/or isoperimetric problems, the known theory of which can be safely applied only to properly defined problems in the calculus of variations.

It is fortunate that many of these difficulties can be overcome almost effortlessly, and yet with perfect rigour, by the introduction of suitably chosen Clebsch representations. The simplest example of such a representation is the decomposition of an arbitrary vector field on a three-dimensional Euclidean space into the sum of a gradient and a scalar multiple of another gradient. Although such representations have been used sporadically in the past in connexion with the equations of motion of fluid dynamics (Lamb 1932; Bateman 1929, 1930; Eckart 1960; Seliger & Whitham 1968), and also in electromagnetic theory (Rund 1976, 1977), it would appear that their applicability to the aforementioned complex of ideas has not yet been exploited. It is hoped that the present paper represents a step in this direction.

In order to substantiate our opening remarks, a brief review is given of some basic concepts of the calculus of variations of multiple integrals, including a relatively simple criterion for the distinction between maxima and minima which does not depend on a direct appeal to the second variation. The power of the method of Clebsch representations is illustrated by means of a very simple example (Rund 1976), which, however, may not be entirely devoid of physical interest. In §2 Clebsch potentials are used to construct an acceptable Lagrangian for the energy integral corresponding to a single-species magnetofluid; the explicit form of the Euler–Lagrange equations is derived, and it is shown that the aforementioned criterion indicates that minimum energy solutions are then obtained. Since physical considerations appear to prescribe the introduction of integral constraints, the same problem is treated once more in §3 as an isoperimetric problem in the calculus of variations. The resulting Euler–Lagrange equations imply that the velocity field is collinear with the magnetic field; moreover, if one of the constraints is removed, Alfvén velocities are thus obtained. In view of the simplicity of the model presented here and its rather close qualitative agreement with experimental observations (Wells 1976), it is evident that more sophisticated models, particularly structures based on many-species magnetofluids, should be analysed in this manner.

The following observations should serve to clarify the general point of view adopted in this paper. Let θ^A denote† a set of m class C^2 functions of n independent variables x^j. In the $(n+m)$-dimensional configuration space X_{n+m} of the variables (x^j, θ^A) a system of m equations of the form

$$\theta^A = \theta^A(x^j), \tag{1.1}$$

† For the present, lower case indices j, h, ... range from 1 to n, while capital indices A, B, ... range from 1 to m. For the physical applications discussed in subsequent sections $n = 3$. A comma followed by a subscript denotes partial differentiation; e.g. $\theta^A_{,j} = \partial\theta^A/\partial x^j$.

Downloaded from https:/www.cambridge.org/core. Open University Libraryy, on 10 Jan 2017 at 00:18:38, subject to the Cambridge Core terms of use, available at https:/www.cambridge.org/core/terms. http://dx.doi.org/10.1017/S0022377800023898

defines an n-dimensional subspace C_n of X_{n+m}, whose tangent spaces are spanned by the n vectors $(\delta^h_j, \theta^A_{,j})$. Let us suppose also that we are given a class C^2 Lagrangian $L = L(x^j, \theta^A, \theta^A_{,j})$, together with a finite, simply-connected region G in the domain X_n of the independent variables x^j. In general, the value of the n-fold integral

$$I(C_n) = \int_G L(x^j, \theta^A, \theta^A_{,j})\, d(x), \tag{1.2}$$

where
$$d(x) = dx^1 \wedge \ldots \wedge dx^n, \tag{1.3}$$

depends on the choice of the functions θ^A as arguments of the integrand, that is, on the choice of the subspace C_n (as anticipated by our notation). More precisely, let $\bar{C}_n : \bar{\theta}^A = \bar{\theta}^A(x^j)$ represent any class C^2 subspace of X_{n+m} which coincides with C_n on the $(n-1)$-dimensional boundary ∂G of G, i.e.

$$\bar{\theta}^A(x^j) = \theta^A(x^j) \quad \text{for all} \quad x^j \in \partial G. \tag{1.4}$$

Unless the Lagrangian is a divergence (a case specifically excluded here), one has $I(\bar{C}_n) \neq I(C_n)$, and the problem in the calculus of variations as defined by the integral (1.2) is concerned with the formulation of the conditions which the subspace C_n must satisfy in order that $I(C_n)$ assumes an extreme value relative to all other subspaces $\bar{C}_n$ subject to (1.4). As is well known, a first necessary condition for an extremum of (1.2) is represented by the m Euler–Lagrange equations

$$E_{\theta^A}(L) = 0, \tag{1.5}$$

where
$$E_{\theta^A}(L) = \frac{d}{dx^j}\left(\frac{\partial L}{\partial \theta^A_{,j}}\right) - \frac{\partial L}{\partial \theta^A}, \tag{1.6}$$

in which the differential operator d/dx^j is defined by

$$\frac{d}{dx^j} = \frac{\partial}{\partial x^j} + \frac{\partial \theta^B}{\partial x^j}\frac{\partial}{\partial \theta^B} + \frac{\partial^2 \theta^B}{\partial x^j \partial x^h}\frac{\partial}{\partial \theta^B_{,h}}. \tag{1.7}$$

Thus, when written out in full, the Euler–Lagrange equations are

$$\frac{\partial^2 L}{\partial x^j \partial \theta^A_{,j}} + \frac{\partial^2 L}{\partial \theta^B \partial \theta^A_{,j}}\frac{\partial \theta^B}{\partial x^j} + \frac{\partial^2 L}{\partial \theta^B_{,h} \partial \theta^A_{,j}}\frac{\partial^2 \theta^B}{\partial x^j \partial x^h} - \frac{\partial L}{\partial \theta^A} = 0, \tag{1.8}$$

which is a system of m second-order partial differential equations for the functions θ^A which define the subspace C_n (the latter being called an extremal whenever (1.8) is satisfied).

Even if an extremum is attained (which is by no means guaranteed by (1.8)), one can distinguish between maxima and minima only by invoking further criteria, of which the simplest is the generalized Weierstrass excess function of Weyl (Weyl 1935; Rund 1973). At any given point (x^j, θ^A) of X_{n+m} two n-dimensional planes are spanned by a given pair of vectors (δ^h_j, ϕ^A_j), $(\delta^h_j, \bar{\phi}^A_j)$: the excess function relative to these planes is defined to be

$$E(x^j, \theta^A, \phi^A_j, \bar{\phi}^A_j) = L(x^j, \theta^A, \bar{\phi}^A_j) - L(x^j, \theta^A, \phi^A_j) - \frac{\partial L(x^j, \theta^A, \phi^A_j)}{\partial \theta^B_h}(\bar{\phi}^B_h - \phi^B_h). \tag{1.9}$$

If C_n affords an extreme value to the integral (1.2) one generally has

$$E(x^j, \theta^A, \theta^A_{,j}, \bar{\theta}^A_{,j}) \gtrless 0 \tag{1.10}$$

Downloaded from https:/www.cambridge.org/core. Open University Libraryy, on 10 Jan 2017 at 00:18:38, subject to the Cambridge Core terms of use, available at https:/www.cambridge.org/core/terms. http://dx.doi.org/10.1017/S0022377800023898

according as this extremum is a minimum or a maximum, where $\theta^A_{,j}$ refers to C_n, while $\bar{\theta}^A_{,j}$ refers to any other subspace $\bar{C}_n$ satisfying the boundary condition (1.4).† There are, of course, other criteria such as various forms of the Legendre condition; these are generally expressed in terms of the non-vanishing or otherwise of $nm \times nm$ determinants and therefore involve tedious calculations.

The point that we wish to stress here, however, is that the entire analysis is meaningless unless the Lagrangian depends explicitly on at least some of the derivatives $\theta^A_{,j}$. For, if $\partial L/\partial\theta^A_{,j} = 0$ for *all* θ^A, the Euler–Lagrange equations (1.8) reduce to $\partial L/\partial\theta^A = 0$, $A = 1, \ldots, m$, which obviously do not represent a system of differential equations for the functions $\theta^A(x^j)$ which define the subspace C_n. Similarly, the condition (1.10) is rendered meaningless under these circumstances, while the individual entries of the aforementioned $nm \times nm$ determinants associated with the various Legendre conditions vanish identically. (It should be remarked, however, that these objections are not necessarily valid if $\partial L/\partial\theta^A_{,j} = 0$ for *some*, but not all θ^A.)

Fortunately, in many cases these difficulties can be overcome with great ease and elegance by the use of Clebsch representations, namely if some or all of the dependent functions θ^A can be expressed in terms of Clebsch potentials. For, under these conditions, the substitution of θ^A into the Lagrangian endows the latter with the required derivatives, namely the derivatives of the Clebsch potentials, the potentials themselves now playing the role of the dependent functions. This device then allows one to bring the entire theory of the calculus of variations to bear on the problem at hand.

The following simple example serves to illustrate this technique. Let x^j ($j = 1, 2, 3$) represent the Cartesian co-ordinates of a three-dimensional Euclidean space E_3, on which there is defined a differentiable vector field with components X_h, which are required to assume prescribed values on the boundary ∂G of some region G in E_3. It is now required to find the conditions which the field X_h must satisfy in order that the integral

$$I = \frac{1}{2}\int_G X_h X_h \, d(x) \tag{1.11}$$

assume an extreme value subject to the given boundary conditions. When the problem is thus stated, the integral (1.11) does not give rise to a well-posed problem in the calculus of variations in the sense of the above remarks. However, it is now recalled that any differentiable vector field in E_3 admits a Clebsch representation of the form

$$X_j = \frac{\partial\psi}{\partial x^j} + Q\frac{\partial P}{\partial x^j}, \tag{1.12}$$

where ψ, Q, P are suitably chosen class C^2 Clebsch potentials. Accordingly the integrand in (1.11) can be expressed in the form

$$L(Q, \psi_{,j}, P_{,j}) = \frac{1}{2}\left(\frac{\partial\psi}{\partial x^h} + Q\frac{\partial P}{\partial x^h}\right)\left(\frac{\partial\psi}{\partial x^h} + Q\frac{\partial P}{\partial x^h}\right), \tag{1.13}$$

† There are exceptions to this rule (Rund 1973, p. 283). This is because, strictly speaking, one should only construct excess functions relative to appropriate fields of extremals. Since there are n ways of constructing such fields (Rund 1973, Supplementary Appendix), there are n distinct excess functions, of which (1.9) is the simplest. However, this function is adequate for our present purposes.

Downloaded from https:/www.cambridge.org/core. Open University Libraryy, on 10 Jan 2017 at 00:18:38, subject to the Cambridge Core terms of use, available at https:/www.cambridge.org/core/terms. http://dx.doi.org/10.1017/S0022377800023898

which is of the type examined above, with $n = 3$, $m = 3$, and $\theta^1 = \psi$, $\theta^2 = P$, $\theta^3 = Q$. Also,

$$\frac{\partial L}{\partial \psi_{,j}} = X_h \frac{\partial X_h}{\partial \psi_{,j}} = X_h \delta_{hj} = X_j, \tag{1.14}$$

$$\frac{\partial L}{\partial P_{,j}} = X_h \frac{\partial X_h}{\partial P_{,j}} = QX_h \delta_{hj} = QX_j, \tag{1.15}$$

and accordingly the Euler–Lagrange equations (1.8) now assume the form

$$\frac{\partial X_j}{\partial x^j} = 0, \quad \frac{\partial}{\partial x^j}(QX_j) = 0, \quad -X_j \frac{\partial P}{\partial x^j} = 0,$$

which can be combined to yield

$$\frac{\partial X_j}{\partial x^j} = 0, \quad X_j \frac{\partial Q}{\partial x^j} = 0, \quad X_j \frac{\partial P}{\partial x^j} = 0. \tag{1.16}$$

Of these, the last pair indicates that X_j is parallel to the intersection of the level surfaces $P = \text{const.}$, $Q = \text{const.}$ in E_3. Thus the most general solution of this pair of equations is given by

$$X_j = \sigma \epsilon_{jhk} \frac{\partial Q}{\partial x^h} \frac{\partial P}{\partial x^k}, \tag{1.17}$$

where $\sigma(x^h)$ is some scalar function of position, while ϵ_{jhk} denotes the three-dimensional permutation symbol. Because of the skew-symmetry properties of the latter, we can express (1.17) in the form

$$X_j = \sigma \epsilon_{jhk} \frac{\partial}{\partial x^h}\left(Q \frac{\partial P}{\partial x^k}\right) = \sigma \epsilon_{jhk} \frac{\partial}{\partial x^h}\left(\frac{\partial \psi}{\partial x^k} + Q \frac{\partial P}{\partial x^k}\right),$$

or, if we use (1.12) once more,

$$X_j = \sigma \epsilon_{jhk} \frac{\partial X_k}{\partial x^h},$$

which is equivalent to

$$\mathbf{X} = \sigma \operatorname{curl} \mathbf{X}. \tag{1.18}$$

Moreover, the first member of (1.16) is simply

$$\operatorname{div} \mathbf{X} = 0, \tag{1.19}$$

which guarantees the existence of a vector field $\mathbf{Y}$ such that

$$\mathbf{X} = \operatorname{curl} \mathbf{Y}. \tag{1.20}$$

The equations (1.18), (1.19) constitute necessary conditions which the field X_h must satisfy in order that the integral (1.11) assume an extreme value.†

Let us now turn to the excess function (1.9), which must be evaluated at each

† If $\mathbf{X}$ is identified with a magnetic field $\mathbf{B}$, the integral (1.11) represents the energy of the field in the region G; thus our problem is a minimum energy problem. The latter had been considered in an entirely different manner and under far more restrictive hypotheses by S. Chandrasekhar & L. Woltjer (1958), and L. Woltjer (1958), who also arrived at the conclusion that $\mathbf{B}$ is parallel to curl $\mathbf{B}$ (see (1.18)). However, the treatment of these authors presupposes the representation $\mathbf{B} = \operatorname{curl} \mathbf{A}$, which in our case, follows directly from the Euler–Lagrange equations (see (1.20)).

Downloaded from https:/www.cambridge.org/core. Open University Libraryy, on 10 Jan 2017 at 00:18:38, subject to the Cambridge Core terms of use, available at https:/www.cambridge.org/core/terms. http://dx.doi.org/10.1017/S0022377800023898

point $(x^1, x^2, x^3, \psi, Q, P)$ of the configuration space X_6. In terms of our present notation (1.9) becomes

$$E(x^h, \psi, Q, P, \psi_{,j}, P_{,j}, \overline{\psi}_{,j}, \overline{P}_{,j}) = L(x^h, \psi, Q, P, \overline{\psi}_{,j}, \overline{P}_{,j}) - L(x^h, \psi, Q, \psi_{,j}, P_{,j}) - \frac{\partial L}{\partial \psi_{,h}}(\overline{\psi}_{,h} - \psi_h) - \frac{\partial L}{\partial P_{,h}}(\overline{P}_{,h} - P_{\,h}),$$

or, if we use (1.12)–(1.14), with $\overline{X}_h = \overline{\psi}_{,h} + Q\overline{P}_{,h}$,

$$E = \tfrac{1}{2}(\overline{X}_h \overline{X}_h - X_h X_h) - X_h(\overline{\psi}_{,h} - \psi_{,h}) - QX_h(\overline{P}_{,h} - P_{,h}) = \tfrac{1}{2}(\overline{X}_h \overline{X}_h + X_h X_h - 2X_h \overline{X}_h) = \tfrac{1}{2}|\overline{\mathbf{X}} - \mathbf{X}|^2,$$

so that $E \geqslant 0$ for all $\overline{\psi}_{,j}$, $\overline{P}_{,j}$ with equality if and only if $\overline{\psi}_{,j} = \psi_j, \overline{P}_{,j} = P_{,j}$. Thus, any extremum of the integral (1.11), attained by means of (1.18), (1.19), will be a minimum.

The method outlined above will be applied to a more complicated problem in the next section.

2. An unconstrained minimum energy problem

We shall consider a single species plasma with mass and charge density ρ and ρ_e respectively, whose velocity field is denoted by $\mathbf{v}$. We consider this fluid to carry all currents and mass motions. There is, in addition, a fluid of opposite electrical charge which is stationary in the laboratory frame and whose mass is negligible compared to moving single species fluid. Then, if ν is the particle density, we may write

$$\rho = m\nu, \quad \rho_e = e\nu, \tag{2.1}$$

where m and e are constants. Also, if ϵ_0 and μ_0 denote the (constant) permittivity† and permeability, respectively, while $\mathbf{E}$ and $\mathbf{B}$ respectively represent the electric field strength and the magnetic induction, the (4, 4) component of the energy-momentum tensor density prescribes the following expression (Rund 1978) for the energy density:

$$W = \tfrac{1}{2}(\epsilon_0 E^2 + B^2/\mu_0) + \tfrac{1}{2}\rho v^2 + \rho_e \phi + \rho V + \nu U(\nu), \tag{2.2}$$

where ϕ, V denote the electrostatic and gravitational potentials respectively, while U represents a pressure term which is derivable from the equation of state $p = f(\nu)$ as a solution of the equation

$$\nu^2 U'(\nu) = f(\nu). \tag{2.3}$$

In accordance with standard practice in magnetohydrodynamics we shall neglect (Kippenhahn & Möllenhoff 1975) the terms E^2, $\rho_e \phi$ in (2.2). Moreover, if the equation of state is of the form $p = M\nu^\gamma = f(\nu)$, with constant M and γ, it follows from (2.3) that

$$\nu U(\nu) = p/(\gamma - 1), \tag{2.4}$$

and, under these circumstances, the expression (2.2) reduces precisely to the expression for the energy density given by Bernstein *et al.* (1958). However, in

† The rationalized MKSQ system is used for all units.

Downloaded from https:/www.cambridge.org/core. Open University Libraryy, on 10 Jan 2017 at 00:18:38, subject to the Cambridge Core terms of use, available at https:/www.cambridge.org/core/terms. http://dx.doi.org/10.1017/S0022377800023898

the present context, the relation (2.4) will *not* be assumed, but the gravitational potential V will be neglected. Thus we shall accept

$$W = \tfrac{1}{2}B^2/\mu_0 + \tfrac{1}{2}\rho v^2 + \nu U(\nu) \tag{2.5}$$

as the appropriate expression for the energy density. Also, since it is consistent with the approximations made above to neglect the displacement current $\epsilon_0\,\partial\mathbf{E}/\partial t$, Maxwell's equations in the form

$$\operatorname{div}\mathbf{B} = 0, \quad \operatorname{curl}\mathbf{B} = \mu_0\mathbf{j} \tag{2.6}$$

will be used below, where $\mathbf{j}$ denotes the current density

$$\mathbf{j} = \rho_e\mathbf{v}. \tag{2.7}$$

The total energy contained in a region G is given by

$$I = \int_G W d(x), \tag{2.8}$$

and we shall now endeavour to obtain the conditions which our field variables must satisfy in order that this integral assume a minimum value. From the remarks of the previous section it is evident that the expression (2.5) cannot serve as a suitable Lagrangian for this problem unless we introduce Clebsch potentials. Following a general procedure described elsewhere (Rund, 1978), we therefore use a Clebsch representation of the generalized momenta

$$p_j = mv_j + eA_j, \tag{2.9}$$

where $\mathbf{A}$ is the vector potential of the electromagnetic field; that is, we put

$$mv_j + eA_j = \partial\psi/\partial x^j + Q\,\partial P/\partial x^j, \tag{2.10}$$

to which we adjoin the usual representation

$$\mathbf{B} = \operatorname{curl}\mathbf{A}. \tag{2.11}$$

By means of these relations the expressions B^2 and v^2 can be eliminated from (2.5), which give rise to an acceptable Lagrangian of the type

$$L(\nu, Q, A_h, \psi_{,j}, P_{,j}, A_{h,j}) = \tfrac{1}{2}\mu_0^{-1}(\epsilon_{hkl}A_{l,k})(\epsilon_{hkl}A_{l,k}) + \tfrac{1}{2}\rho m^{-2}(\psi_{,j} + QP_{,j} - eA_j)(\psi_{,j} + QP_{,j} - eA_j) + \nu U(\nu). \tag{2.12}$$

Thus, in the notation of the previous section, we have $n = 3$, $m = 7$, and we shall now derive the explicit forms of the Euler–Lagrange equations (1.8).

To this end we note that, by virtue of (2.11) and (2.12),

$$\frac{\partial L}{\partial A_{h,j}} = \mu_0^{-1}B_l\frac{\partial B_l}{\partial A_{h,j}} = \mu_0^{-1}B_l\frac{\partial}{\partial A_{h,j}}(\epsilon_{lrs}A_{s,r}) = \mu_0^{-1}\epsilon_{ljh}B_l, \tag{2.13}$$

so that
$$\frac{\partial}{\partial x^j}\left(\frac{\partial L}{\partial A_{h,j}}\right) = \mu_0^{-1}\epsilon_{ljh}\frac{\partial B_l}{\partial x^j} = -\mu_0^{-1}(\operatorname{curl}\mathbf{B})_h, \tag{2.14}$$

while
$$\frac{\partial L}{\partial A_h} = \frac{1}{m}\rho v_j(-e\delta_{jh}) = -\frac{e}{m}\rho v_h,$$

or, if we use (2.1) and (2.7),

$$\partial L/\partial A_h = -\rho_e v_h = -j_h. \tag{2.15}$$

Downloaded from https:/www.cambridge.org/core. Open University Libraryy, on 10 Jan 2017 at 00:18:38, subject to the Cambridge Core terms of use, available at https:/www.cambridge.org/core/terms. http://dx.doi.org/10.1017/S0022377800023898

In the notation (1.6) we therefore have

$$E_{A_h}(L) = -(\mu_0^{-1}\operatorname{curl}\mathbf{B}-\mathbf{j})_h. \tag{2.16}$$

Accordingly the three Euler–Lagrange equations corresponding to the dependent functions A_h yield nothing new: they are merely the second set of Maxwell's equations as displayed in (2.6). On the other hand, since

$$\frac{\partial L}{\partial \psi_{,h}} = \rho v_j \frac{\partial v_j}{\partial \psi_{,h}} = \frac{\rho}{m} v_j \delta_{jh} = \nu v_h \tag{2.17}$$

and

$$\frac{\partial L}{\partial P_{,h}} = \rho v_j \frac{\partial v_j}{\partial P_{,h}} = \frac{\rho}{m} Q\delta_{jh} = \nu Q v_h, \tag{2.18}$$

we obtain

$$E_\psi(L) = \frac{\partial}{\partial x^h}(\nu v_h), \tag{2.19}$$

together with

$$E_P(L) = \frac{\partial}{\partial x^h}(\nu Q v_h), \tag{2.20}$$

$$E_Q(L) = -\nu v_h \frac{\partial P}{\partial x^h} \tag{2.21}$$

and

$$E_\nu(L) = -[\tfrac{1}{2}mv^2 + U(\nu) + \nu U'(\nu)]. \tag{2.22}$$

The Euler–Lagrange equations

$$E_\psi(L) = 0, \quad E_P(L) = 0, \quad E_Q(L) = 0, \tag{2.23}$$

may be combined to yield

$$\frac{\partial}{\partial x^h}(\nu v_h) = 0, \quad v_j \frac{\partial Q}{\partial x^j} = 0, \quad v_j \frac{\partial P}{\partial x^j} = 0. \tag{2.24}$$

Following the procedure of the previous section, we deduce from the latter pair of these equations that

$$v_j = \sigma\epsilon_{jhk}\frac{\partial Q}{\partial x^h}\frac{\partial P}{\partial x^k} = \sigma\epsilon_{jhk}\frac{\partial}{\partial x^h}\left(\frac{\partial \psi}{\partial x^k} + Q\frac{\partial P}{\partial x^k}\right),$$

or, if we use (2.10) and (2.11),

$$\mathbf{v} = \sigma \operatorname{curl}(m\mathbf{v} + e\mathbf{A}) = m\sigma \operatorname{curl}\mathbf{v} + e\sigma\mathbf{B}. \tag{2.25}$$

The first member of (2.24) is simply

$$\operatorname{div}(\nu\mathbf{v}) = 0, \tag{2.26}$$

while the Euler–Lagrange equation corresponding to (2.22) obviously reflects some kind of Bernoullian theorem by virtue of (2.3).

Let us now turn to an examination of the excess function (1.9) for this particular problem. From (2.12) and (2.10) it follows that, at any point

$$(x^1, x^2, x^3, \nu, \psi, Q, P, A_1, A_2, A_3)$$

of the configuration space X_{10},

$$L(\nu, Q, A_h, \bar{\psi}_{,j}, \bar{P}_{,j}, \bar{A}_{h,j}) - L(\nu, Q, A_h, \psi_{,j}, P_{,j}, A_{h,j}) = \tfrac{1}{2}\mu_0^{-1}(\bar{B}^2 - B^2) + \tfrac{1}{2}\rho(\bar{v}^2 - v^2). \tag{2.27}$$

Downloaded from https:/www.cambridge.org/core. Open University Libraryy, on 10 Jan 2017 at 00:18:38, subject to the Cambridge Core terms of use, available at https:/www.cambridge.org/core/terms. http://dx.doi.org/10.1017/S0022377800023898

Also, using (2.13) and (2.11), we find that

$$\frac{\partial L}{\partial A_{h,j}}(\bar{A}_{h,j}-A_{h,j}) = \mu_0^{-1}\epsilon_{ljh}B_l(\bar{A}_{h,j}-A_{h,j})$$
$$= \mu_0^{-1}\mathbf{B}.(\operatorname{curl}\mathbf{A}-\operatorname{curl}\bar{\mathbf{A}}) = \mu_0^{-1}\mathbf{B}.\bar{\mathbf{B}}-\mu_0^{-1}B^2, \qquad (2.28)$$

while (2.17) and (2.18) yield

$$\frac{\partial L}{\partial \psi_{,h}}(\bar{\psi}_{,h}-\psi_{,h})+\frac{\partial L}{\partial \bar{P}_{,h}}(\bar{P}_{,h}-\bar{P}_{,h}) = \nu v_h(\bar{\psi}_{,h}+Q\bar{P}_{,h})-\nu v_h(\psi_{,h}+QP_{,h})$$
$$= \nu v_h(m\bar{v}_h+eA_h)-\nu v_h(mv_h+eA_h)$$
$$= \rho\bar{\mathbf{v}}.\mathbf{v}-\rho v^2. \qquad (2.29)$$

When (2.27)–(2.29) are substituted in the definition (1.9) of the excess function, it is found that the latter may be reduced to the form

$$E = \tfrac{1}{2}\mu_0^{-1}|\bar{\mathbf{B}}-\mathbf{B}|^2+\tfrac{1}{2}\rho|\bar{\mathbf{v}}-\mathbf{v}|^2, \qquad (2.30)$$

from which it follows that $E \geqslant 0$, with equality if and only if $\bar{B}_h = B_h$ and $\bar{v}_h = v_h$. Thus, if an extremum of the energy integral (2.8) is obtained as a consequence of the equations (2.25) and (2.26) which govern the motion, this extremum must be a minimum.

3. The constrained minimum energy problem as an isoperimetric problem

It is likely that the problem treated in the previous section reflects an over-simplification from a physical point of view since no constraints whatsoever were imposed on the system. As has been observed elsewhere (Woltjer 1960; Taylor 1974; Wells 1976), a more realistic model may be obtained when two integral constraints are introduced, namely

$$I_1 = \text{const.}, \quad I_2 = \text{const.}, \qquad (3.1)$$

where

$$I_1 = \int_G \mathbf{A}.\mathbf{B}\,d(x), \qquad (3.2)$$

$$I_2 = \int_G \mathbf{B}.\mathbf{v}\,d(x). \qquad (3.3)$$

(The weakest condition under which the conditions (3.1) are applicable will be discussed in the Appendix.)

The more general problem concerning the extreme values of the energy integral (2.8) subject to the integral constraints (3.1) is nothing other than an isoperimetric problem in the calculus of variations, provided that the latter is formulated as a well-posed problem by means of Clebsch potentials. This has already been accomplished as far as the energy integral (2.8) is concerned; therefore it merely remains to do the same in a consistent manner for the constraint integrals (3.2) and (3.3). Under these circumstances, then, the general theory of multiple

Downloaded from https:/www.cambridge.org/core. Open University Libraryy, on 10 Jan 2017 at 00:18:38, subject to the Cambridge Core terms of use, available at https:/www.cambridge.org/core/terms. http://dx.doi.org/10.1017/S0022377800023898

integral isoperimetric problems (Rund 1972) may be invoked. Thus, if in accordance with (3.2) and (3.3) we put

$$L_1 = \mathbf{A}.\mathbf{B} \tag{3.4}$$

and

$$L_2 = \mathbf{B}.\mathbf{v}, \tag{3.5}$$

this theory entails that a necessary condition for an extreme value of the integral (2.8) subject to the constraints (3.1) is given by the system of equations

$$E_z(L) + \lambda_1 E_z(L_1) + \lambda_2 E_z(L_2) = 0, \tag{3.6}$$

where z denotes in turn each of the dependent functions $A_1, A_2, A_3, \psi, Q, P$, while λ_1, λ_2 represent Lagrange multipliers which are constant on each extremal (that is, on each solution of the entire system (3.6)).

The explicit form of (3.6) will now be derived. When (2.11) is substituted in (3.4) it is seen that

$$L_1 = \mathbf{A}.\operatorname{curl}\mathbf{A} = \epsilon_{klp} A_k A_{p,l},$$

so that

$$\frac{\partial L_1}{\partial A_{h,j}} = \epsilon_{kjh} A_k, \quad \frac{\partial L_1}{\partial A_h} = \epsilon_{hkj} A_{j,k}, \tag{3.7}$$

which gives

$$E_{A_h}(L_1) = 2\epsilon_{kjh} A_{k,j} = -2\,(\operatorname{curl}\mathbf{A})_h = -2B_h, \tag{3.8}$$

this being the only contribution to (3.6) from L_1. With respect to L_2, we must substitute not only from (2.11) but also from the original Clebsch representation (2.10), thus obtaining

$$L_2 = \epsilon_{lkp} A_{p,k} v_l = m^{-1}\epsilon_{lkp} A_{p,k}(\psi_{,l} + QP_{,l} - eA_l). \tag{3.9}$$

Hence

$$\frac{\partial L_2}{\partial A_{h,j}} = \epsilon_{ljh} v_l, \quad \frac{\partial L_2}{\partial A_h} = -\frac{e}{m} B_l \delta_{lh} = -\frac{e}{m} B_h, \tag{3.10}$$

which yields

$$E_{A_h}(L_2) = -\left[\operatorname{curl}\mathbf{v} - \frac{e}{m}\mathbf{B}\right]_h. \tag{3.11}$$

Also,

$$\frac{\partial L_2}{\partial \psi_{,h}} = B_l \frac{\partial v_l}{\partial \psi_{,h}} = \frac{1}{m} B_l \delta_{lh} = \frac{1}{m} B_h, \tag{3.12}$$

so that, by virtue of the first member of (2.6),

$$E_\psi(L_2) = m^{-1}\operatorname{div}\mathbf{B} = 0 \tag{3.13}$$

identically. Similarly,

$$\frac{\partial L_2}{\partial P_{,h}} = B_l \frac{\partial v_l}{\partial P_{,h}} = \frac{1}{m} B_l Q \delta_{lh} = \frac{1}{m} Q B_h \tag{3.14}$$

and

$$\frac{\partial L_2}{\partial Q} = B_h \frac{\partial v_h}{\partial Q} = \frac{1}{m} B_h P_{,h}. \tag{3.15}$$

Hence, if the first member of (2.6) is used once more,

$$E_P(L_2) = \frac{1}{m}\frac{\partial}{\partial x^h}(QB_h) = \frac{1}{m}\mathbf{B}.\nabla Q, \tag{3.16}$$

together with

$$E_Q(L_2) = -\frac{1}{m} B_h P_{,h} = -\frac{1}{m}\mathbf{B}.\nabla P. \tag{3.17}$$

Downloaded from https:/www.cambridge.org/core. Open University Libraryy, on 10 Jan 2017 at 00:18:38, subject to the Cambridge Core terms of use, available at https:/www.cambridge.org/core/terms. http://dx.doi.org/10.1017/S0022377800023898

We are now in a position to write (3.6) explicitly. Recalling that $E_{A_h}(L) = 0$ identically because of the second member of (2.6),† it is seen with the aid of (3.8) and (3.11) that (3.6), with $z = A_h$, is simply

$$-2\lambda_1 \mathbf{B} - \lambda_2 (\operatorname{curl} \mathbf{v} - (e/m)\mathbf{B}) = 0,$$

or

$$(2\lambda_1 - (e/m)\lambda_2)\mathbf{B} + \lambda_2 \operatorname{curl} \mathbf{v} = 0. \tag{3.18}$$

According to (3.13), the case when $z = \psi$ in (3.6), does not affect the latter, so that we are left with (2.19) unchanged:

$$\operatorname{div}(\rho \mathbf{v}) = 0. \tag{3.19}$$

From (2.21) and (3.17) it follows that, with $z = Q$, (3.6) becomes

$$(\rho \mathbf{v} + \lambda_2 \mathbf{B}).\nabla P = 0. \tag{3.20}$$

Finally, with the aid of (2.20) and (3.16), it is found for the case when $z = P$, that (3.6) reduces to

$$\operatorname{div}(\nu Q \mathbf{v}) + (\lambda_2/m)\mathbf{B}.\nabla Q = 0. \tag{3.21}$$

Thus the Euler–Lagrange equations (3.6) of our isoperimetric problem imply the system (3.18)–(3.21), on which the entire subsequent analysis is to be based.

First, we observe that an application of (3.19) to (3.22) reduces the latter to the form

$$(\rho \mathbf{v} + \lambda_2 \mathbf{B}).\nabla Q = 0, \tag{3.22}$$

and this, together with (3.20), implies the existence of a scalar function $\mu(x^h)$ such that

$$\rho \mathbf{v} + \lambda_2 \mathbf{B} = \mu(\nabla Q \times \nabla P). \tag{3.23}$$

Second, it follows directly from the Clebsch representation (2.10) that

$$m \operatorname{curl} \mathbf{v} + e \operatorname{curl} \mathbf{A} = \operatorname{curl}(\nabla \psi + Q \nabla P) = (\nabla Q \times \nabla P),$$

so that

$$\nabla Q \times \nabla P = m \operatorname{curl} \mathbf{v} + e\mathbf{B}. \tag{3.24}$$

A comparison of this with (3.23) yields

$$\mu m \operatorname{curl} \mathbf{v} = \rho \mathbf{v} + (\lambda_2 - \mu e)\mathbf{B}. \tag{3.25}$$

Without loss of generality we may assume that $\lambda_2 \neq 0$ (for, if we were to put $\lambda_2 = 0$, this would entail the removal of the second of the constraints (3.1)). We may then write (3.18) as

$$\operatorname{curl} \mathbf{v} = (e/m - 2\lambda_1/\lambda_2)\mathbf{B}, \tag{3.26}$$

which may be combined with (3.25) to give

$$\rho \mathbf{v} = \alpha \mathbf{B}, \tag{3.27}$$

where

$$\alpha = 2\mu(e - m\lambda_1/\lambda_2) - \lambda_2. \tag{3.28}$$

Moreover, it follows from (3.26) that the flow is irrotational if the constant

$$\beta = e/m - 2\lambda_1/\lambda_2 \tag{3.29}$$

† It must be noted that this result depends explicitly on (2.7), which is applicable only to a single-species fluid. The result for a multi-species fluid will, in general, be more complex than (3.18).

Downloaded from https:/www.cambridge.org/core. Open University Libraryy, on 10 Jan 2017 at 00:18:38, subject to the Cambridge Core terms of use, available at https:/www.cambridge.org/core/terms. http://dx.doi.org/10.1017/S0022377800023898

vanishes; since we wish to exclude this case, we shall assume that $\beta \neq 0$. Also, the second member of (2.6) can be expressed in the form

$$\rho \mathbf{v} = m\mu_0^{-1}e^{-1}\,\mathrm{curl}\,\mathbf{B}, \tag{3.30}$$

and accordingly the relations (3.26), (3.27) and (3.30) can be combined into the form

$$\rho \mathbf{v} = \alpha \mathbf{B} = \alpha\beta^{-1}\,\mathrm{curl}\,\mathbf{v} = m\mu_0^{-1}e^{-1}\,\mathrm{curl}\,\mathbf{B}. \tag{3.31}$$

Thus, unless the flow is irrotational, the Euler–Lagrange equations of our isoperimetric problem imply that the four vector fields $\mathbf{v}$, $\mathbf{B}$, curl $\mathbf{v}$, curl $\mathbf{B}$ *are collinear.* In this connexion it should be noted that in general α is not a constant; this is due to the appearance of the scalar μ in (3.28). In fact, it we take the divergence of (3.27), at the same time noting (3.19) and the first member of (2.6), we obtain

$$\nabla\alpha . \mathbf{B} = 0, \tag{3.32}$$

which shows that the gradient field of α is normal to the general direction of the flow. Similarly, by taking the curl of (3.27), we find that

$$\alpha\,\mathrm{curl}\,\mathbf{B} = \rho\,\mathrm{curl}\,\mathbf{v} + \nabla\rho \times \mathbf{v} - \nabla\alpha \times \mathbf{B},$$

or, if we use (3.31),

$$\alpha^2\beta^{-1}\,\mathrm{curl}\,\mathbf{B} = m\mu_0^{-1}e^{-1}\rho\,\mathrm{curl}\,\mathbf{B} + \alpha\beta^{-1}(\nabla\rho \times \mathbf{v} - \nabla\alpha \times \mathbf{B}),$$

from which it follows, again because of (3.31),

$$(\alpha^2\beta^{-1} - m\mu_0^{-1}e^{-1}\rho)\,(\mathrm{curl}\,\mathbf{B}.\mathbf{v}) = 0.$$

Since $\mathbf{v}.\mathrm{curl}\,\mathbf{B} \neq 0$, it is evident that the scalar α is given by

$$\alpha = \pm k\rho^{\frac{1}{2}}, \tag{3.33}$$

where

$$k = [m\mu_0^{-1}\beta/e]^{\frac{1}{2}} \tag{3.34}$$

is a constant. Accordingly the relation (3.27) can now be expressed in the form

$$\mathbf{v} = \pm k\rho^{-\frac{1}{2}}\mathbf{B}, \tag{3.35}$$

while (3.32), combined with (3.33), shows that

$$\nabla\rho . \mathbf{B} = 0 \tag{3.36}$$

and

$$\nabla\rho . \mathbf{v} = 0, \tag{3.37}$$

indicating that the mass density gradient is normal to the direction of the flow. Both α and ρ are constant on a flux or stream tube.

Remark 1. As noted above, the analysis following (3.25) depends on the assumption that $\lambda_2 \neq 0$. On the other hand, if the first of the constraints (3.1) is removed, which is tantamount to setting $\lambda_1 = 0$, no significant consequences arise. Indeed, the only change is reflected in (3.29), which then reduces to $\beta = e/m$, in which case the constant (3.34) becomes

$$k = \mu_0^{-\frac{1}{2}} \tag{3.38}$$

and in this case (3.35) *corresponds to Alfvén velocities*

$$\mathbf{v} = \pm(\mu_0\rho)^{-\frac{1}{2}}\mathbf{B}. \tag{3.39}$$

Downloaded from https:/www.cambridge.org/core. Open University Libraryy, on 10 Jan 2017 at 00:18:38, subject to the Cambridge Core terms of use, available at https:/www.cambridge.org/core/terms. http://dx.doi.org/10.1017/S0022377800023898

Remark 2. The simplest solution of the equations (3.20) and (3.22) is represented by

$$\rho \mathbf{v} + \lambda_2 \mathbf{B} = 0. \tag{3.40}$$

This is the solution considered by Wells (1976), and, according to the general relation (3.23), this would correspond to the case $\mu = 0$. However, under these circumstances it follows from (3.28) that $\alpha = -\lambda_2$, which is a constant for any configuration that is consistent with the Euler–Lagrange equations, and accordingly (3.33) implies that the mass density ρ is constant. It is therefore concluded that the special solution (3.40) would correspond to an incompressible flow, while the result (3.31) corresponds to 'isochoric flow' defined by (3.37).

If one considers a simple two-fluid model instead of the single-species fluid analysed above, one has

$$\mathbf{J} = ne(\mathbf{v}_i - \mathbf{v}_e);$$

$$\rho_0 \mathbf{u} = \rho_i \mathbf{v}_i + \rho_2 \mathbf{v}_e,$$

$$\mathbf{u} = \frac{\rho_i}{\rho_0}\mathbf{v}_i + \frac{\rho_2}{\rho_0}\mathbf{v}_e \quad \text{but} \quad \rho_2 \ll \rho_0.$$

Hence

$$u \approx v_i.$$

If one now assumes that all of the kinetic energy is carried by the heavy positive ions, then the Lagrange density is unchanged and (2.12) is still valid. If one then proceeds to determine the resulting Euler–Lagrange equations one finds

$$\mathbf{B} = \beta^{-1} \operatorname{curl} \mathbf{u},$$

$$\rho \mathbf{u} = \alpha \mathbf{B} + (\mu\rho/\lambda_2)\, \mathbf{v}_e,$$

$$\alpha/\beta \operatorname{curl} \mathbf{u} = m\mu_0^{-1} e^{-1} \operatorname{curl} \mathbf{B} + \rho \mathbf{v}_e \gamma,$$

$$\alpha \mathbf{B} = m\mu_0^{-1} e^{-1} \operatorname{curl} \mathbf{B} + \rho\gamma \mathbf{v}_e \quad \text{where} \quad \gamma = (\mu_0 e^+/m^+ - \mu/\lambda^2).$$

One sees that force-freeness and collinearity are still maintained if the assumption is made that $\mathbf{v}_i$ and $\mathbf{v}_e$ are everywhere parallel. In that case (3.31) is somewhat modified but the essential collinearity of all four vector fields $\mathbf{u}$, $\mathbf{B}$, curl $\mathbf{u}$ and curl $\mathbf{B}$ is maintained.

The more general case in which the kinetic energy terms of both the heavy positive ions and electrons are maintained in the Lagrange density will be the subject of a later paper.

This research was sponsored in part by NSF GP 43070 and in part by NSF PHY74 08237 A02.

Appendix. The constraints

In this Appendix the constraint equations (3.1) will be derived from first principles, with special emphasis on the weakest conditions which must be satisfied by the electromagnetic field in order that these equations be valid. No appeal whatsoever is made in this analysis to the approximations introduced at the beginning of § 2.

Downloaded from https:/www.cambridge.org/core. Open University Libraryy, on 10 Jan 2017 at 00:18:38, subject to the Cambridge Core terms of use, available at https:/www.cambridge.org/core/terms. http://dx.doi.org/10.1017/S0022377800023898

We shall begin with the integral I_1 as given by (3.2). In terms of the usual 4-potential representation

$$\mathbf{B} = \operatorname{curl}\mathbf{A}, \quad \mathbf{E} = -\nabla\phi - \partial\mathbf{A}/\partial t, \tag{A 1}$$

we have

$$\begin{aligned}\operatorname{div}[(\mathbf{E}\times\mathbf{A})+\phi\mathbf{B}] &= \mathbf{A}.\operatorname{curl}\mathbf{E} - \mathbf{E}.\operatorname{curl}\mathbf{A} + \nabla\phi.\mathbf{B} \\ &= -\mathbf{A}.\operatorname{curl}[\partial\mathbf{A}/\partial t] - 2\mathbf{E}.\mathbf{B} - \mathbf{B}.\partial\mathbf{A}/\partial t,\end{aligned}$$

so that

$$\frac{\partial}{\partial t}(\mathbf{A}.\mathbf{B}) = -2\mathbf{E}.\mathbf{B} - \operatorname{div}[(\mathbf{E}\times\mathbf{A})+\phi\mathbf{B}], \tag{A 2}$$

which is merely an identity resulting from the representation (A 1). When (A 2) is integrated over a fixed region G, the unit normal to the boundary ∂G of G being denoted by $\mathbf{n}$, it follows from the divergence theorem that the integral (3.2) is constant in time, provided that

$$\mathbf{E}.\mathbf{B} = 0 \quad \text{throughout} \quad G \tag{A 3}$$

and

$$\mathbf{n}.[(\mathbf{E}\times\mathbf{A})+\phi\mathbf{B}] = 0 \quad \text{on} \quad \partial G. \tag{A 4}$$

Since the direction of the vector $\mathbf{A}$ is essentially arbitrary by virtue of the possibility of introducing gauge transformations, the latter condition can be satisfied in general if and only if

$$\mathbf{n}.\mathbf{B} = 0 \quad \text{on} \quad \partial G, \tag{A 5}$$

and $\mathbf{n}.(\mathbf{E}\times\mathbf{A}) = 0$ for *any* vector field $\mathbf{A}$, which implies that $\mathbf{E}$ must be parallel to $\mathbf{n}$, that is,

$$\mathbf{E} = \tau\mathbf{n} \quad \text{on} \quad \partial G, \tag{A 6}$$

where τ denotes some scalar. *Thus the conditions* (A 3), (A 5) *and* (A 6) *ensure that the integral* (3.2) *is constant in time.*† In passing we observe also that (A 5) is sufficient to establish the gauge-invariance of that integral. For, under the gauge transformation $\mathbf{A}\rightarrow\bar{\mathbf{A}} = \mathbf{A}+\nabla\chi$, we have

$$\begin{aligned}\int_G \bar{\mathbf{A}}.\mathbf{B}\,d(x) &= \int_G (\mathbf{A}.\mathbf{B}+\mathbf{B}.\nabla\chi)\,d(x) \\ &= \int_G \mathbf{A}.\mathbf{B}\,d(x) + \int_{\partial G} \operatorname{div}(\chi\mathbf{B})\,d(x),\end{aligned} \tag{A 7}$$

in which the last integral vanishes by virtue of (A 5). Also the conditions (A 5) and (A 6) imply that $\mathbf{E}.\mathbf{B} = 0$ on ∂G, which is consistent with (A 3). Equation (A 5) implies a perfect fluid where $\mathbf{E} = -\mathbf{v}\times\mathbf{B}$.

Let us now turn to the integral I_2 as given by (3.3). For the discussion of the latter we shall require not only the representation (A 1), but also the Eulerian equations of motion

$$\rho d\mathbf{v}/dt = \rho_e(\mathbf{E}+\mathbf{v}\times\mathbf{B}) - \nabla p, \tag{A 8}$$

in which d/dt denotes the convected derivative, together with the Maxwell equation

$$\operatorname{curl}\mathbf{E} = -\partial\mathbf{B}/\partial t. \tag{A 9}$$

† It should be remarked that these conditions are considerably weaker than those imposed by Woltjer (1960) in the course of his derivation of the constraint equations (3.1). This is related to the fact that Woltjer uses a special gauge.

Downloaded from https:/www.cambridge.org/core. Open University Libraryy, on 10 Jan 2017 at 00:18:38, subject to the Cambridge Core terms of use, available at https:/www.cambridge.org/core/terms. http://dx.doi.org/10.1017/S0022377800023898

Because of the identity

$$d\mathbf{v}/dt = \partial\mathbf{v}/\partial t + \tfrac{1}{2}\nabla v^2 - \mathbf{v}\times\operatorname{curl}\mathbf{v}, \tag{A 10}$$

we can write (A 8) in the form

$$\partial\mathbf{v}/\partial t = \mathbf{v}\times\operatorname{curl}\mathbf{v} - \tfrac{1}{2}\nabla v^2 + \bar{k}(\mathbf{E}+\mathbf{v}\times\mathbf{B}) - \rho^{-1}\nabla p, \tag{A 11}$$

where $\bar{k} = e/m$. Hence

$$\mathbf{B}.\partial\mathbf{v}/\partial t = \mathbf{B}.(\mathbf{v}\times\operatorname{curl}\mathbf{v}) - \tfrac{1}{2}\mathbf{B}.\nabla v^2 + \bar{k}\mathbf{E}.\mathbf{B} - \rho^{-1}\mathbf{B}.\nabla p. \tag{A 12}$$

Now, with the aid of the equation of state: $p = f(\nu)$, taken in conjunction with (2.3), it is easily verified that

$$\rho^{-1}\nabla p = \nabla[F(\nu)], \tag{A 13}$$

where the function $F(\nu)$ is given by

$$mF(\nu) = U(\nu) + \nu U'(\nu). \tag{A 14}$$

Thus
$$\rho^{-1}\mathbf{B}.\nabla p = \mathbf{B}.\nabla[F(\nu)] = \operatorname{div}[F(\nu)\,\mathbf{B}], \tag{A 15}$$

while
$$\mathbf{B}.\nabla v^2 = \operatorname{div}(v^2\mathbf{B}). \tag{A 16}$$

When (A 15) and (A 16) are substituted in (A 12), the latter becomes

$$\mathbf{B}.\partial\mathbf{v}/\partial t = \mathbf{B}.(\mathbf{v}\times\operatorname{curl}\mathbf{v}) + \bar{k}\mathbf{E}.\mathbf{B} - \operatorname{div}[\tfrac{1}{2}v^2\mathbf{B} + F(\nu)\,\mathbf{B}]. \tag{A 17}$$

Also, from (A 9) it follows that

$$\begin{aligned}\mathbf{v}.\partial\mathbf{B}/\partial t &= -\mathbf{v}.\operatorname{curl}\mathbf{E}\\ &= \operatorname{div}(\mathbf{v}\times\mathbf{E}) - \mathbf{E}.\operatorname{curl}\mathbf{v},\end{aligned} \tag{A 18}$$

which, when added to (A 17), yields

$$\begin{aligned}(\partial/\partial t)(\mathbf{B}.\mathbf{v}) = \mathbf{B}.(\mathbf{v}\times\operatorname{curl}\mathbf{v}) + \bar{k}\mathbf{E}.\mathbf{B} - \mathbf{E}.\operatorname{curl}\mathbf{v}&\\ - \operatorname{div}[\tfrac{1}{2}v^2\mathbf{B} + F(\nu)\,\mathbf{B} - (\mathbf{v}\times\mathbf{E})]&,\end{aligned}$$

or
$$\begin{aligned}(\partial/\partial t)(\mathbf{B}.\mathbf{v}) = [\mathbf{E} + (\mathbf{v}\times\mathbf{B})].(\bar{k}\mathbf{B} - \operatorname{curl}\mathbf{v})&\\ - \operatorname{div}[\tfrac{1}{2}v^2\mathbf{B} + F(\nu)\,\mathbf{B} - (\mathbf{v}\times\mathbf{E})]&.\end{aligned} \tag{A 19}$$

This is the expression which we have been seeking. The constancy in time of the integral (3.3) is ensured whenever

$$[\mathbf{E} + (\mathbf{v}\times\mathbf{B})].(\bar{k}\mathbf{B} - \operatorname{curl}\mathbf{v}) = 0 \quad \text{throughout} \quad G, \tag{A 20}$$

and
$$[\tfrac{1}{2}v^2 + F(\nu)](\mathbf{B}.\mathbf{n}) - (\mathbf{v}\times\mathbf{E}).\mathbf{n} = 0 \quad \text{on} \quad \partial G. \tag{A 21}$$

However, the latter condition is satisfied automatically if (A 5) and (A 6) hold, while (A 20) is certainly valid if Ohm's law in the form

$$\mathbf{E} = -(\mathbf{v}\times\mathbf{B}) \tag{A 22}$$

is applicable. But this, in turn, implies the condition (A 3). It is therefore concluded that *the conditions* (A 5), (A 6) and (A 22) *are sufficient to guarantee the validity of the constraint equations* (3.1). Moreover, when (A 22) and (A 6) are combined, one obtains

$$\tau\mathbf{n} = -(\mathbf{v}\times\mathbf{B}), \tag{A 23}$$

and hence
$$\mathbf{v}.\mathbf{n} = 0, \tag{A 24}$$

Downloaded from https:/www.cambridge.org/core. Open University Libraryy, on 10 Jan 2017 at 00:18:38, subject to the Cambridge Core terms of use, available at https:/www.cambridge.org/core/terms. http://dx.doi.org/10.1017/S0022377800023898

which is precisely what one would expect from a purely physical point of view. Note that (A 22) implies a fluid with zero viscosity. The constraint I_1 depends critically on the assumption of zero Ohmic resistance while I_2 depends on zero Ohmic resistance *and* zero viscosity.

REFERENCES

Bateman, H. 1929 *Proc. R. Soc.* A **125**, 598.

Bateman, H. 1930 *Proc. Nat. Acad. Sci. U.S.A.* **16**, 816.

Bernstein, I. B., Frieman, E. A., Kruskal, M. D. & Kulsrud, R. M. 1958 *Proc. Roy. Soc.* A **244**, 17.

Chandrasekhar, S. & Woltjer, L. 1958 *Proc. Nat. Acad. Sci. U.S.A.* **44**, 285.

Eckart, C. 1960 *Phys. Fluids*, **3**, 421.

Kippenhahn, R. & Möllenhoff, C. 1975 *Plasmaphysik*. Bibliographisches Institut, Zürich.

Lamb, H. 1932 *Hydrodynamics*, 6th ed. Cambridge University Press.

Rund, H. 1972 *Utilitas Math.* **1**, 157.

Rund, H. 1973 *The Hamilton–Jacobi Theory in the Calculus of Variations*, augmented and revised reprint. Krieger Publications.

Rund, H. 1976 *Topics in Differential Geometry* (ed. H. Rund and W. F. Forbes). Academic.

Rund, H. 1977 *J. Math. Phys.* **18**, 84.

Rund, H. 1978 *Arch. Rat. Mech. Anal.*, **67**, 305.

Seliger, R. L. & Whitham, G. B. 1968 *Proc. R. Soc.* A **305**, 1.

Taylor, J. B. 1974 *Phys. Rev. Lett.* **33**, 1139.

Wells, D. R. 1970 *J. Plasma Phys.* **4**, 645.

Wells, D. R. 1976 *Pulsed High Beta Plasmas* (ed. D. E. Evans). Pergamon.

Wells, D. R. & Norwood, J., Jr. 1968 *J. Plasma Phys.* **11**, 1582.

Weyl, H. 1935 *Ann. Math.* (2) **36**, 607.

Woltjer, L. 1958 *Proc. Nat. Acad. Sci. U.S.A.* **44**, 489.

Woltjer, L. 1960 *Rev. Mod. Phys.* **32**, 914.

Downloaded from https:/www.cambridge.org/core. Open University Libraryy, on 10 Jan 2017 at 00:18:38, subject to the Cambridge Core terms of use, available at https:/www.cambridge.org/core/terms. http://dx.doi.org/10.1017/S0022377800023898