Dynamics of Continuous, Discrete and Impulsive Systems
Series **A:** *Mathematical Analysis* **22 (2015)** 47-60

http://www.watam.org

A DIFFERENCE EQUATION WITH ANTI-PERIODIC BOUNDARY CONDITIONS

Jeffrey W. Lyons[1] and Jeffrey T. Neugebauer[2]

[1]Division of Math, Science and Technology
Nova Southeastern University, Fort Lauderdale, FL 33314 USA

[2]Department of Mathematics and Statistics
Eastern Kentucky University, Richmond, Kentucky 40475 USA

Abstract. In this paper, we apply an extension of the Leggett-Williams fixed point theorem to the second order difference equation $\Delta^2 u(k)+f(u(k+1)) = 0$, $k \in \{0, 1, \dots, N\}$, satisfying the anti-periodic boundary conditions $u(0) + u(N + 2) = 0$, $\Delta u(0) + \Delta u(N + 1) = 0$. Two important results of this paper involve providing the Green's function for $-\Delta^2 u(k) = 0$ satisfying $u(0) + u(N + 2) = 0$, $\Delta u(0) + \Delta u(N + 1) = 0$ and showing this Green's function satisfies a concavity like property. An example is also given.

Keywords. Fixed Point Theorem, Difference Equation, Antiperiodic, Antisymmetric, Functional.

AMS (MOS) subject classification: 39A10.

1. Introduction

In this paper, we will apply an Avery, Anderson, and Henderson [2] fixed point theorem that is an extension of the Leggett-Williams fixed point theorem [10]. Avery and others have been providing fixed point theorems that relax some of the assumptions of the original Leggett-Williams fixed point theorem; for examples, see [1, 3, 4, 5, 6].

Here, we consider the second order difference equation

$$\Delta^2 u(k) + f(u(k + 1)) = 0,\ k \in \{0, 1, \dots, N\}, \tag{1.1}$$

satisfying anti-periodic boundary conditions

$$u(0) + u(N + 2) = 0,\ \Delta u(0) + \Delta u(N + 1) = 0, \tag{1.2}$$

where $f : \mathbb{R} \to \mathbb{R}$ is continuous and $\Delta u(k) = u(k + 1) - u(k)$ is the forward difference operator. We will show that if f satisfies certain conditions, then (1.1), (1.2) has an antisymmetric solution in the sense that $u(N + 2 - k) = -u(k)$.

While there has been much work done on ordinary differential equations satisfying antiperiodic boundary values (for some examples, see [7, 8, 9]),