CURVATURE

A geometric villain that ruins our instinctive perception of nature

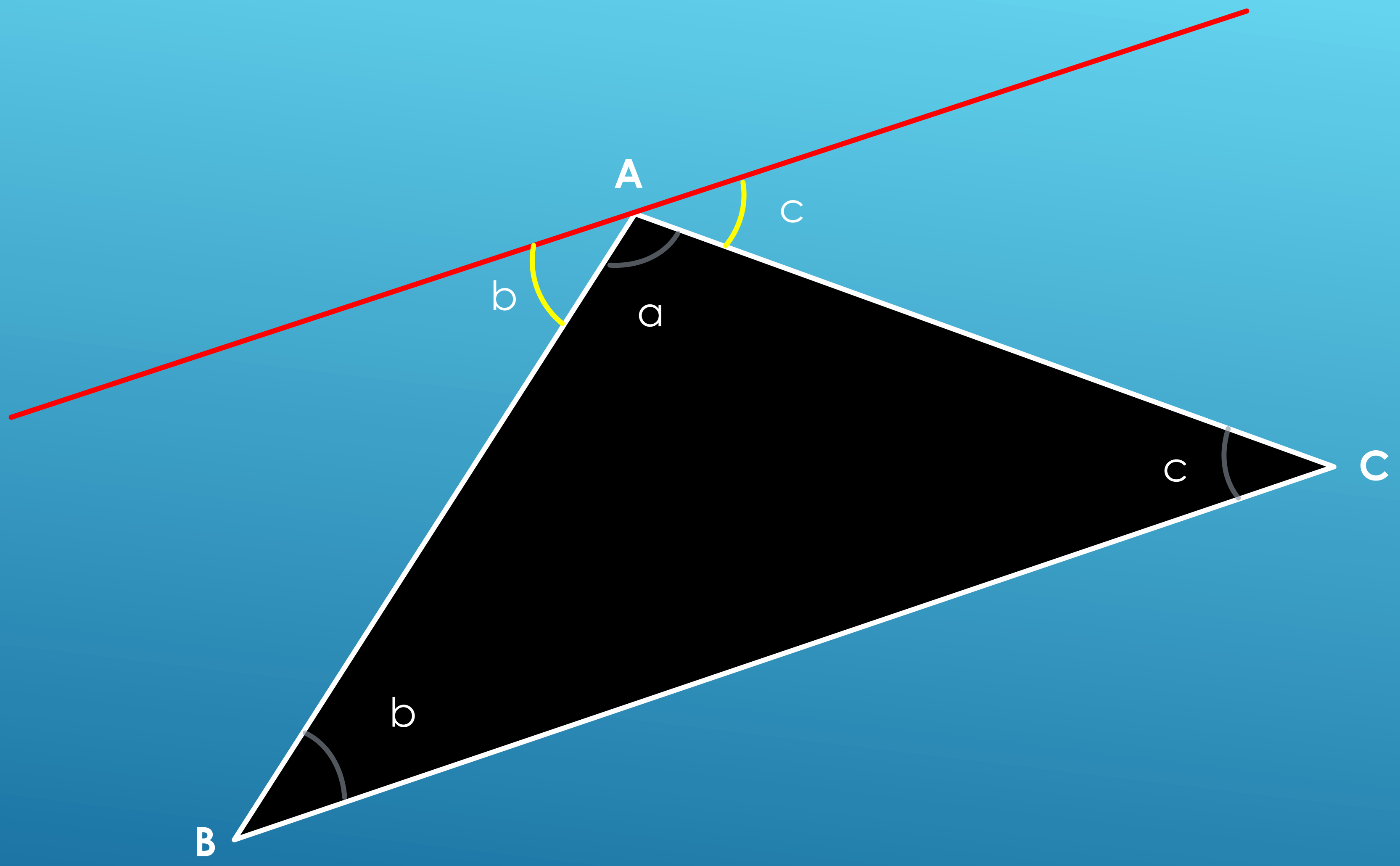
Vehbi Paksoy, HCNSO

THE PLAN

1. Is obvious really obvious?

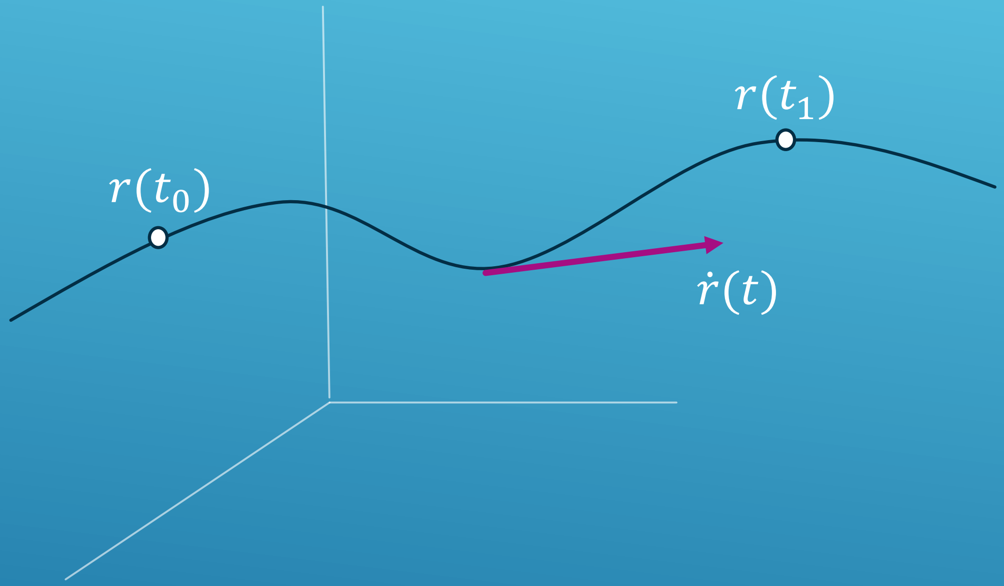
2. Curvature of curves in 3D

3. Curvature of Surfaces in 3D


A
B
C
a
b
c
b
c

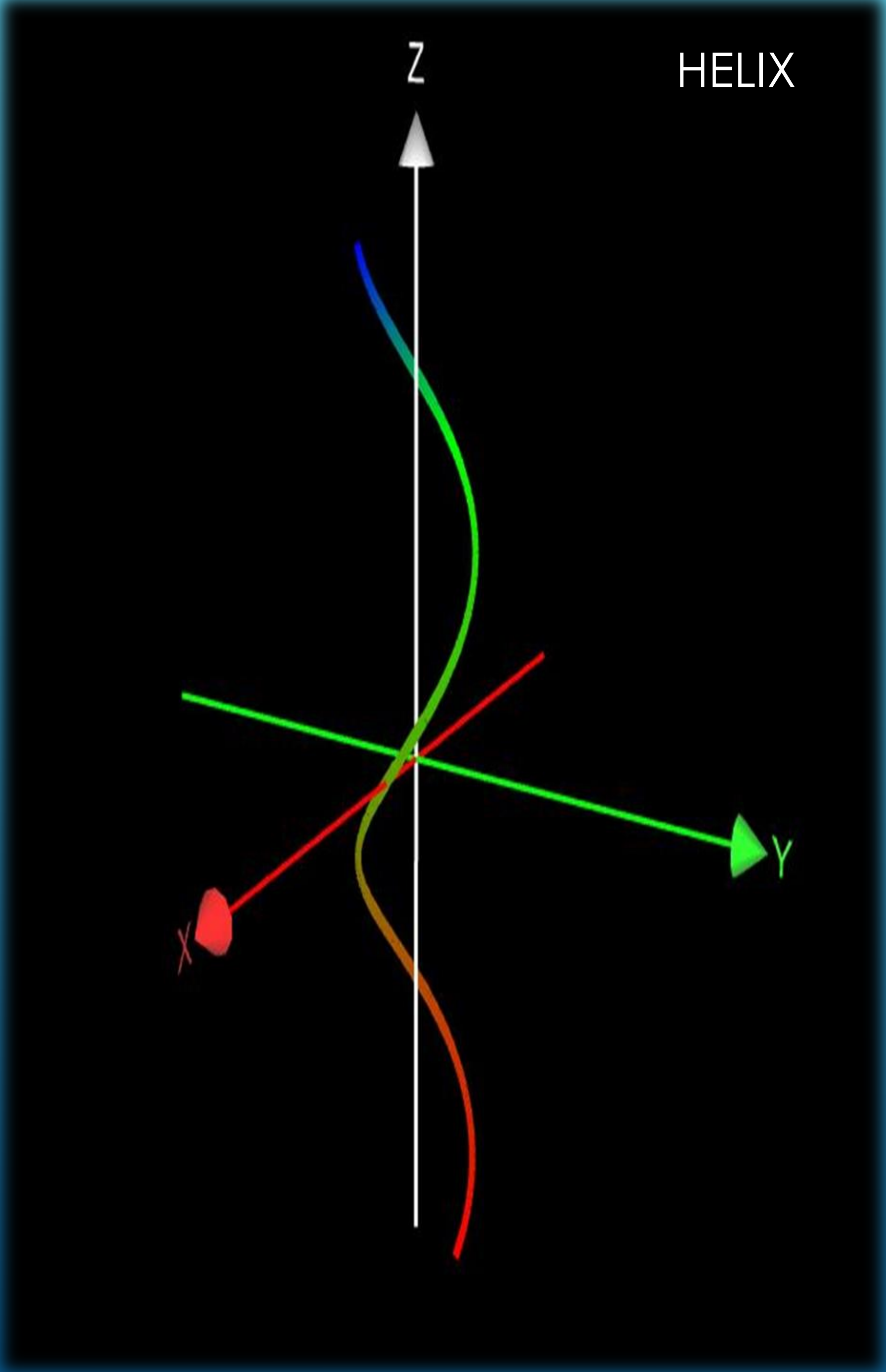
CURVES IN 3D

- Given parametrically : $r(t) = (x(t), y(t), z(t))$



- Parametrization by arc-length : $r(s) = (x(s), y(s), z(s))$

$$s = \int_{t_0}^{t_1} \|\dot{r}(t)\| dt = \int_{t_0}^{t_1} \sqrt{(\dot{x})^2 + (\dot{y})^2 + (\dot{z})^2} dt$$

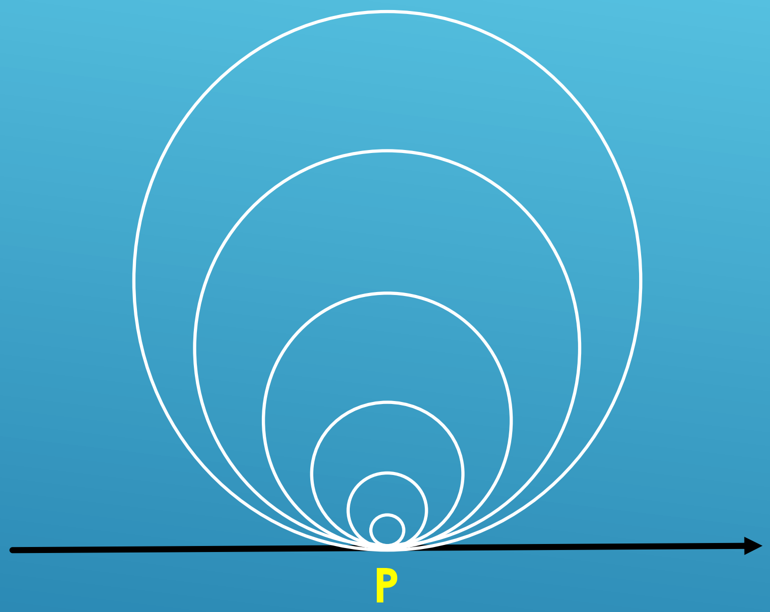


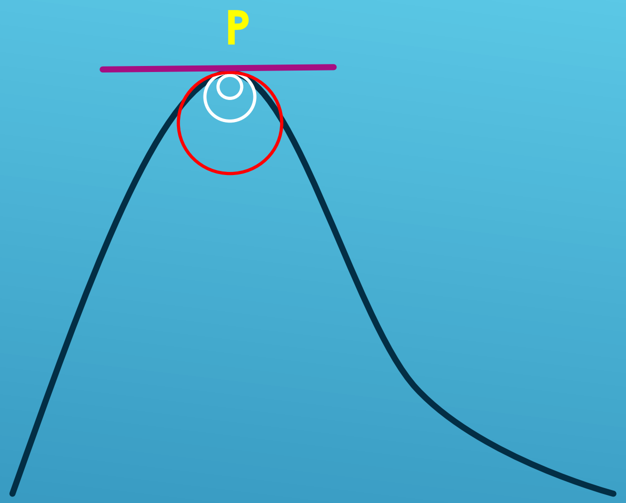
- Standard parametrization : $r(t) = (\cos(t)\,, \sin(t)\,, 2t)$

- Arc-length parametrization : $r(s) = (\cos\left(\frac{s}{\sqrt{5}}\right), \sin\left(\frac{s}{\sqrt{5}}\right), \frac{2s}{\sqrt{5}})$

- $\|\dot{r}(s)\| = 1$

WHAT IS THE DIFFERENCE

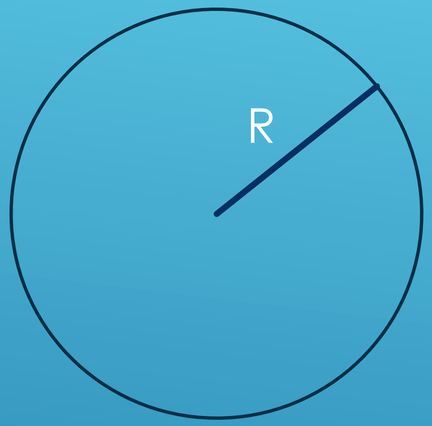




- Curvature $\kappa = \dfrac{1}{R}$

- *R is the radius of the circle which* gives the best approximation of the curve near the point.

❖ Circle of radius R

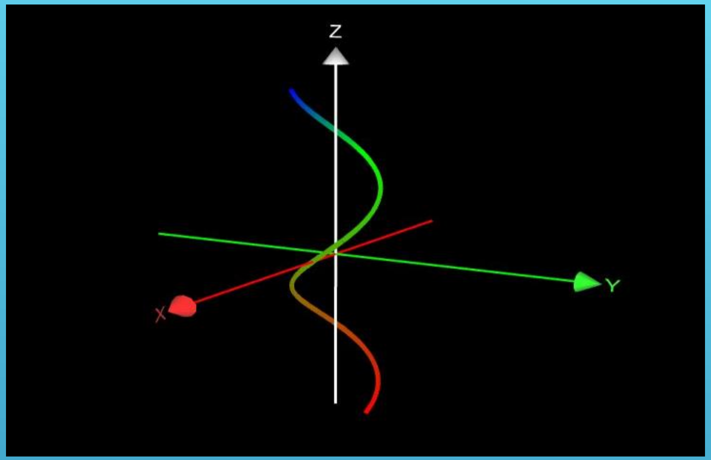


κ =1/R

❖ Line

κ = 0

❖ Helix



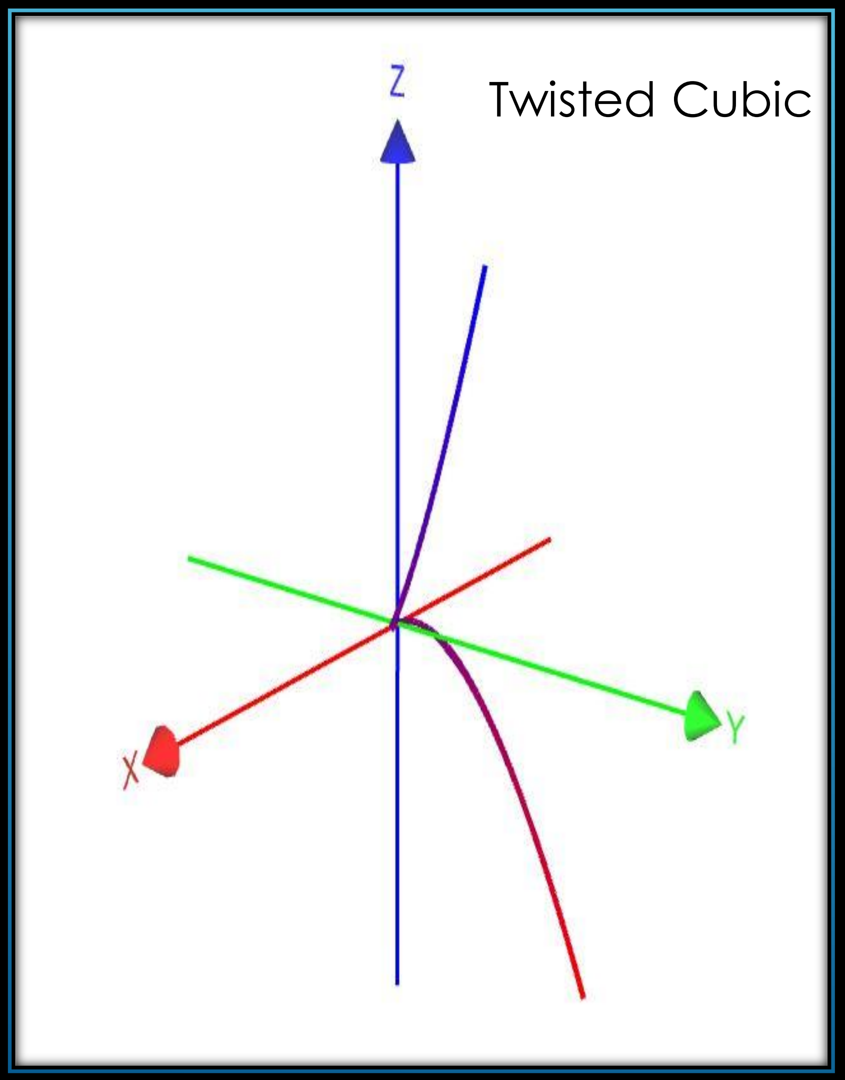
κ = ?

❖ κ is the measure of the rate of change of tangent vector at a point as we travel along the curve.

$$\boxed{\kappa(s) = \|\ddot{r}(s)\|}$$

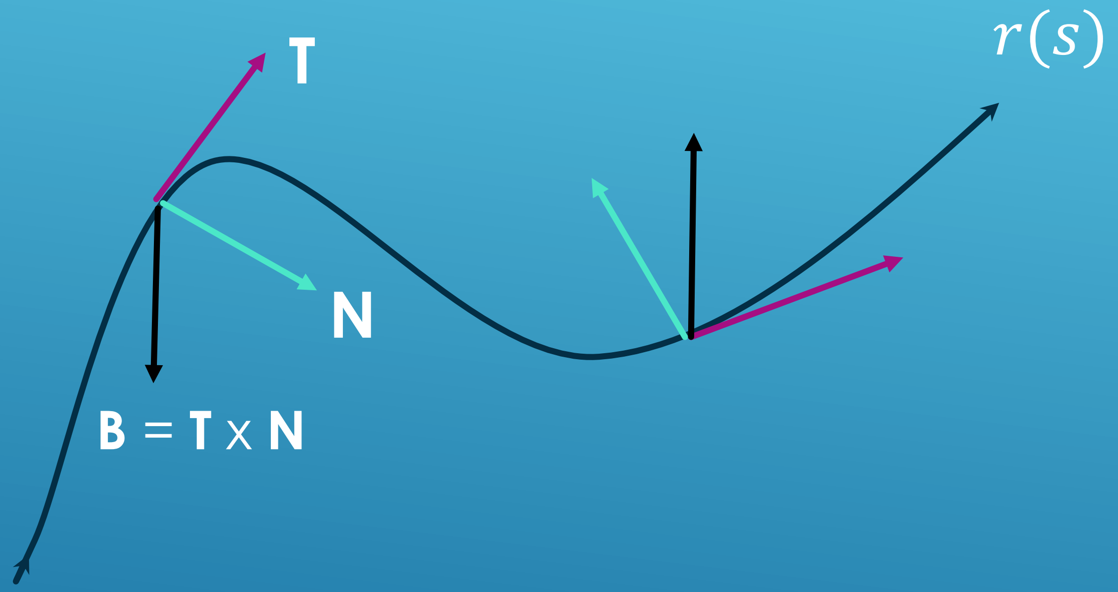
COMPUTING THE CURVATURE

- Arc-length parametrization can be tedious



- $r(t) = (t, t^2, t^3)$
- $s = \int_0^t \sqrt{1 + 4u^2 + 9u^4}\, du = ??$
- $\kappa = \dfrac{\|\dot{r}(t) \times \ddot{r}(t)\|}{\|\dot{r}(t)\|^3}$

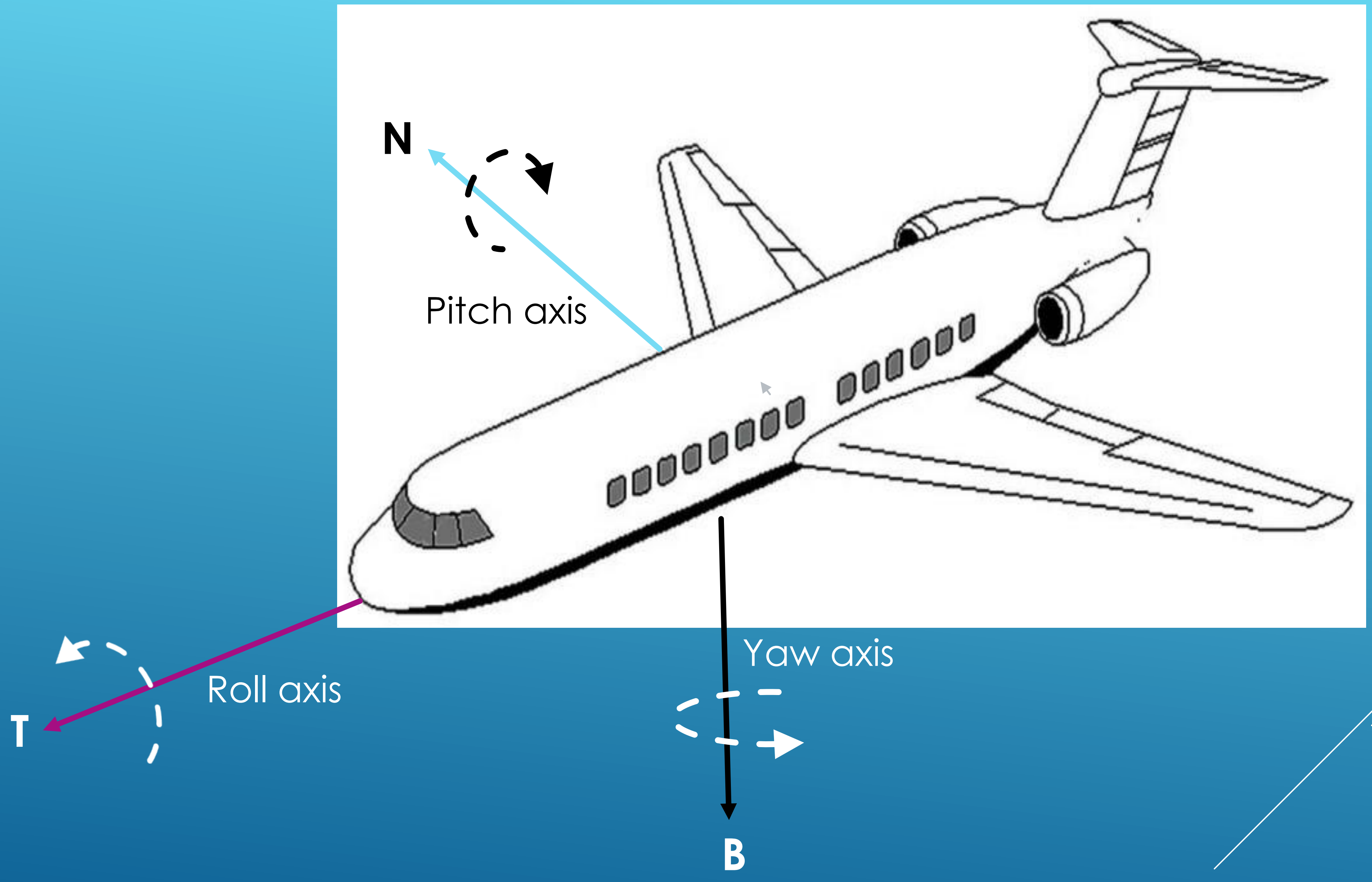
SERRE-FRENET FRAME



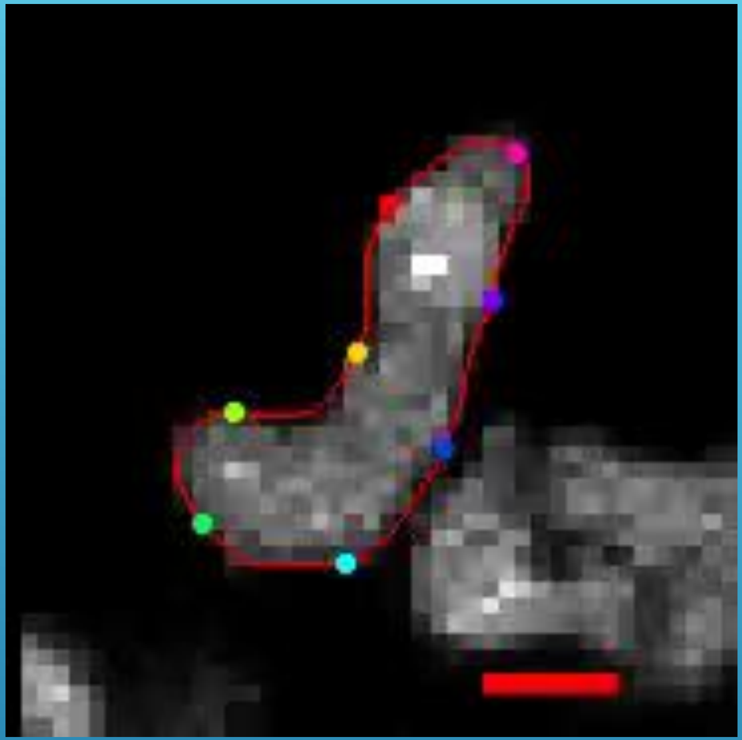
$$\dot{T} = \kappa N$$

$$\dot{N} = -\kappa T + \tau B$$

$$\dot{B} = -\tau N$$


N
Pitch axis
Roll axis
T
Yaw axis
B

Cell Shape Dynamics: From Waves to Migration ***



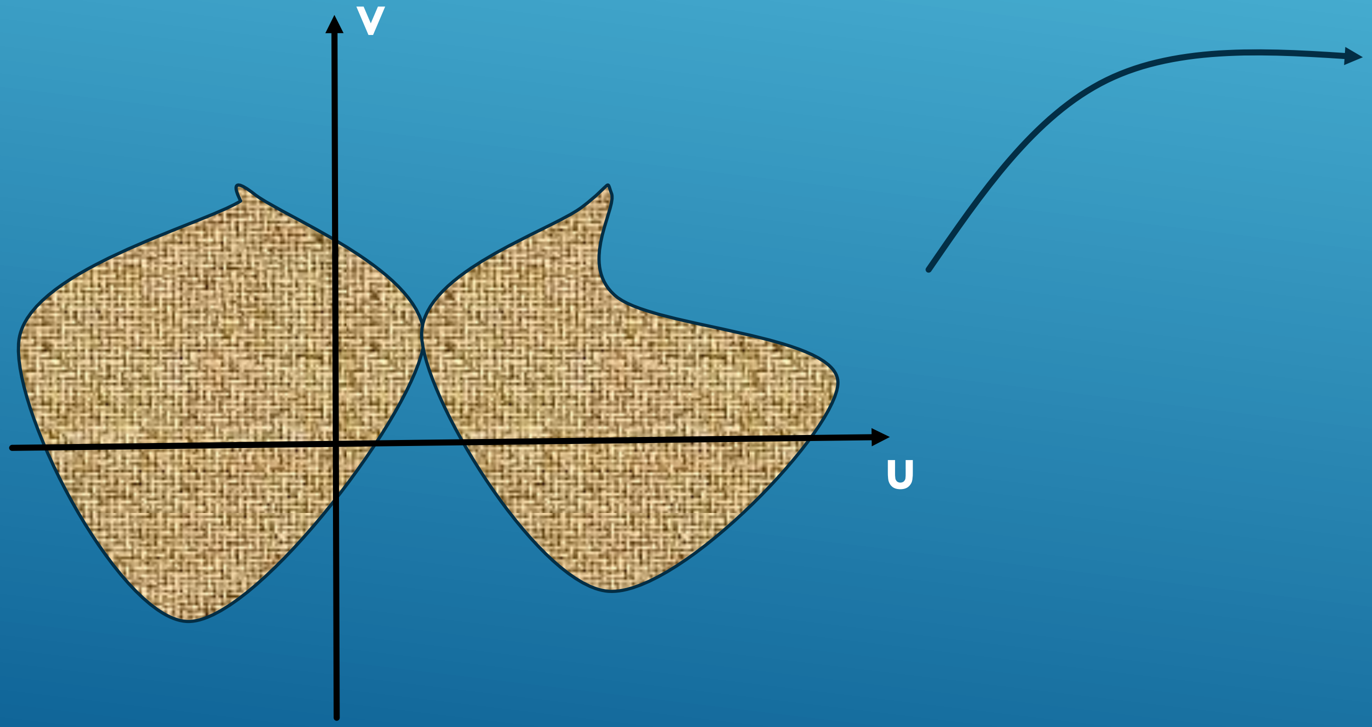
❖ Curvature in the study of wave-like characteristics of amoeba migration.

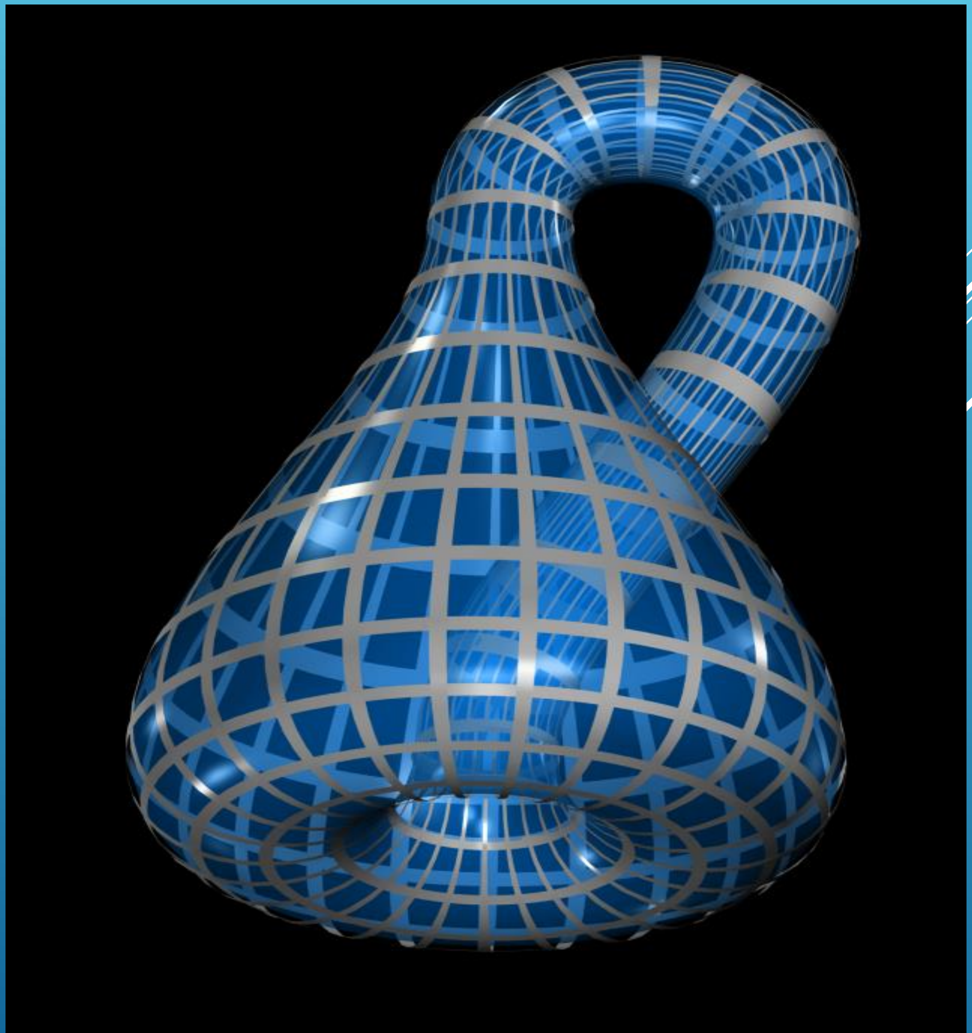
Meghan K. Driscoll,[1] Colin McCann,[1,2] Rael Kopace,[1] Tess Homan,[1] John T. Fourkas,[3,4] Carole Parent,[2] and Wolfgang Losert, PLOS, 2012

PARAMETRIC SURFACES

❖ A surface M in space is a 2 dimensional object, usually given parametrically.

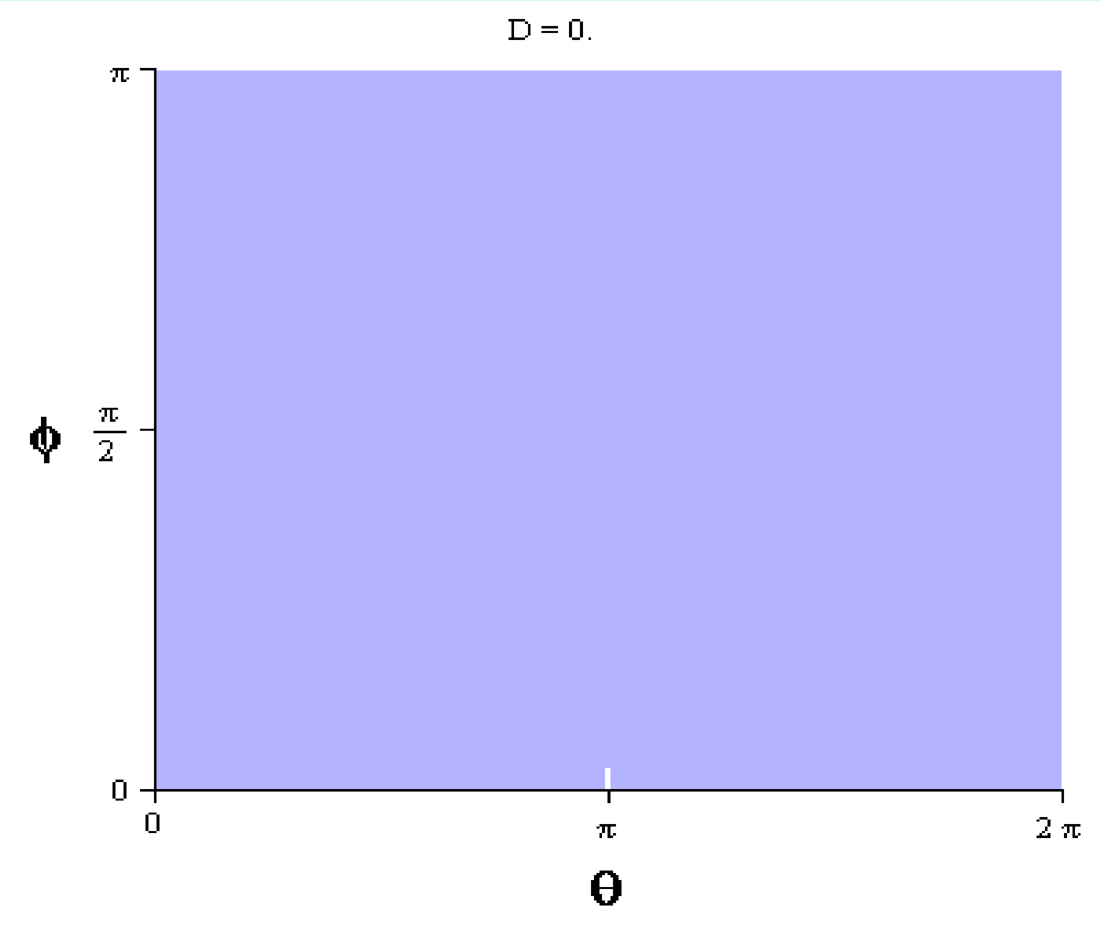
❖ $r = (x(u,v), y(u,v), z(u,v))$

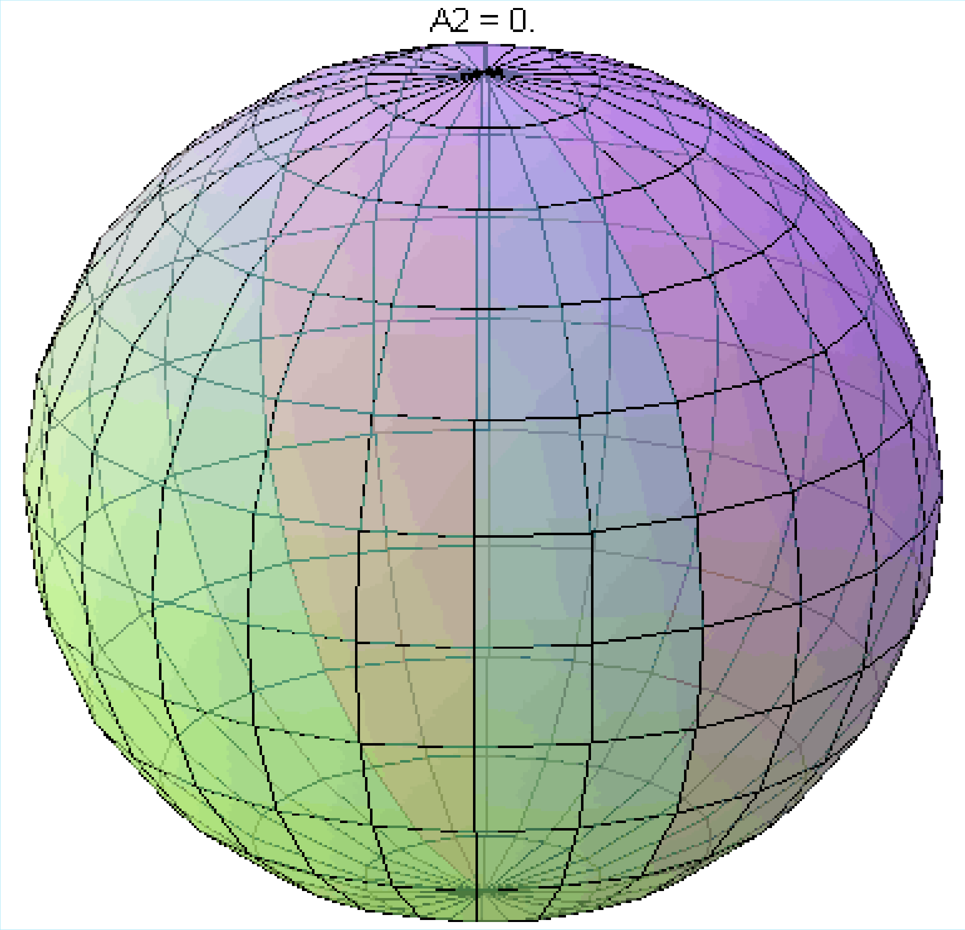




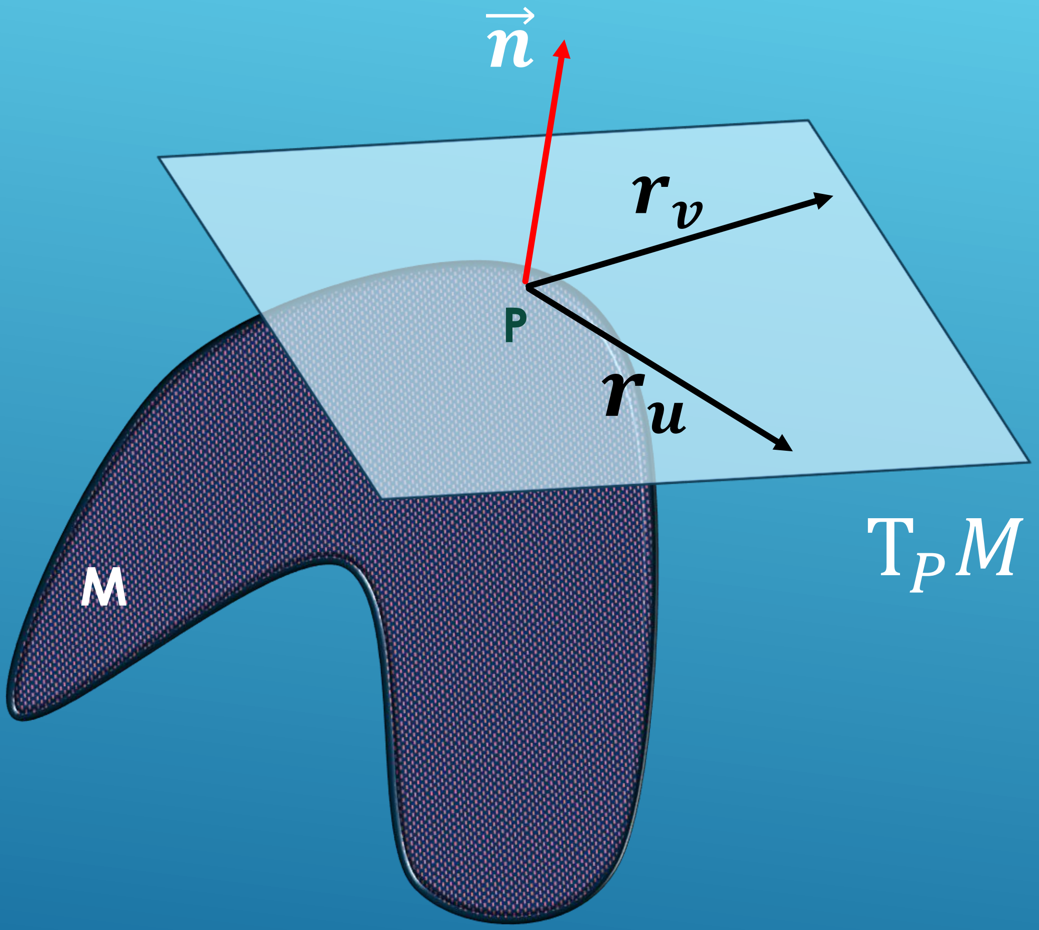
SPHERE

$$r = (cos(\theta)sin(\varphi), sin(\theta)sin(\varphi), cos(\varphi))$$



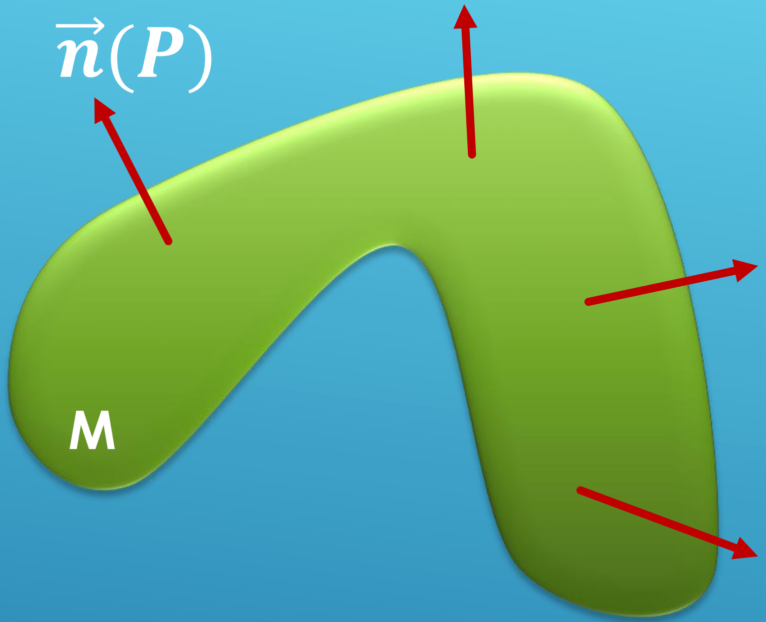


FROM NON-LINEAR TO LINEAR



$$r = (x(u,v), y(u,v), z(u,v))$$

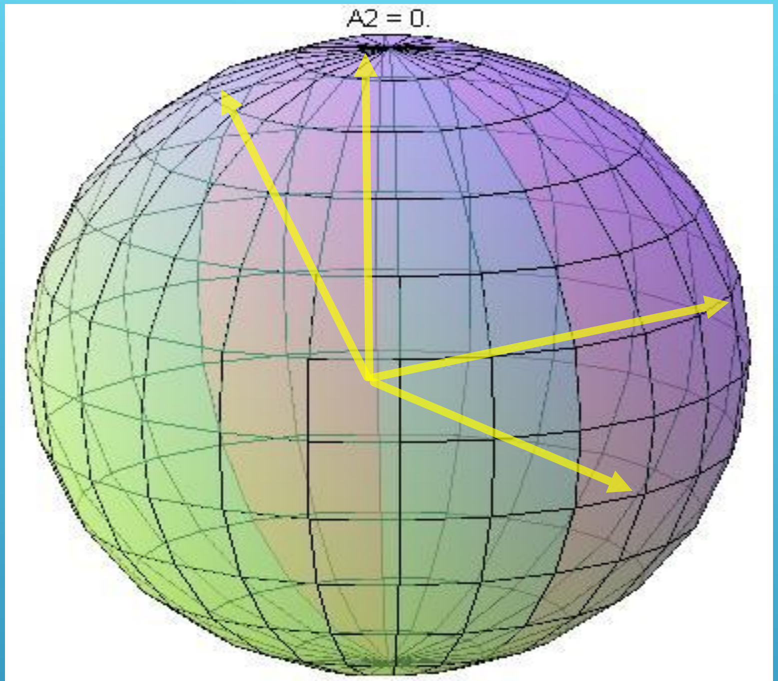
- $r_u = (x_u, y_u, z_u)$
- $r_v = (x_v, y_v, z_v)$
- $\vec{n} = \dfrac{r_u \times r_v}{\|r_u \times r_v\|}$



Gauss Map

$$G: M \to S^2$$

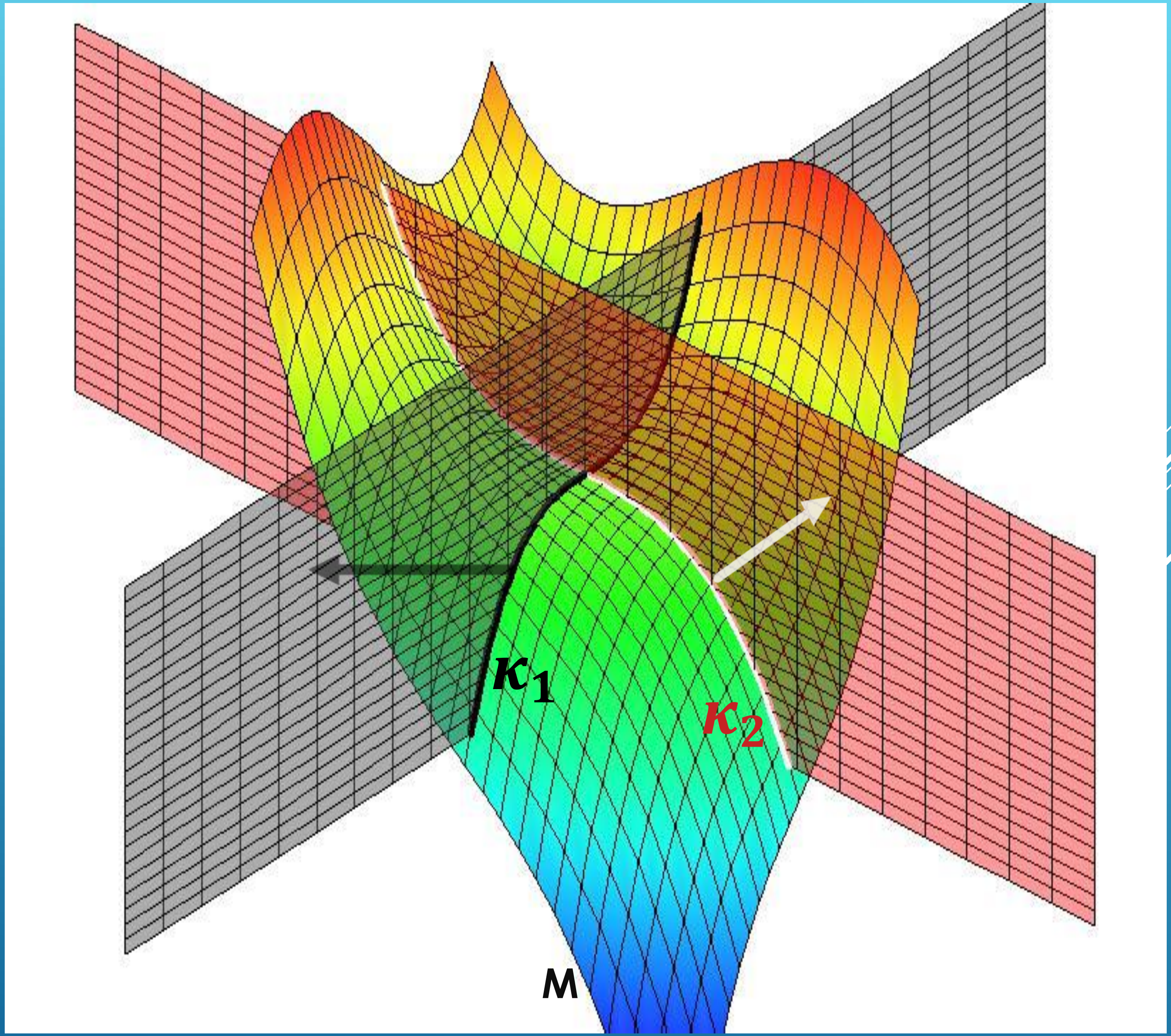
$$P \to \vec{n}(P)$$

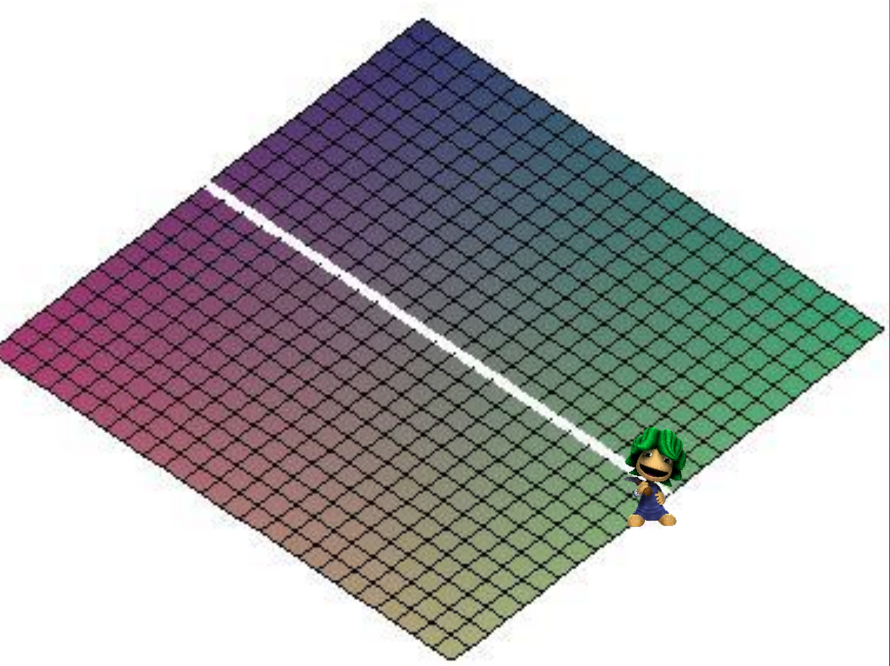


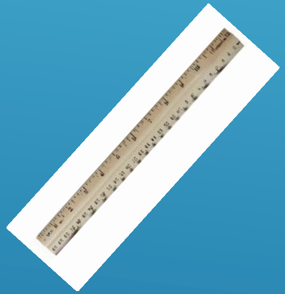
❖ **Shape operator** is the negative of the derivative of the Gauss map.

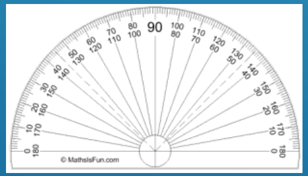
CURVATURE FOR SURFACES

- $\kappa_1 = minimum$
- $\kappa_2 = maximum$
- **Gauss Curvature** : $\mathrm{K} = \kappa_1 \cdot \kappa_2$
- **Mean Curvature** : $\mathrm{H} = \frac{\kappa_1 + \kappa_2}{2}$
- Gauss and Mean curvatures *are determinant and half of the trace of* the shape operator.

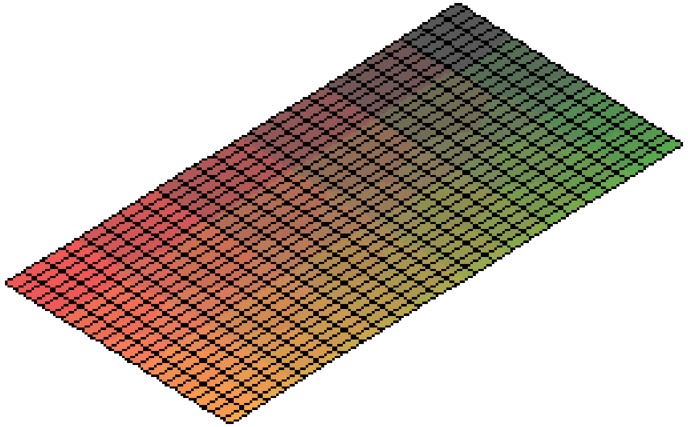








q = 0.



GAUSS'S THEOREM EGREGIUM

Two dimensional creatures cannot compute $\boldsymbol{\kappa_1}$ and $\boldsymbol{\kappa_2}$ using infinitesimal ruler and protractor BUT they can determine $\mathrm{K} = \boldsymbol{\kappa_1} \cdot \boldsymbol{\kappa_2}$. This means, 2D creatures can determine the shape of their world without stepping out to 3rd dimension!

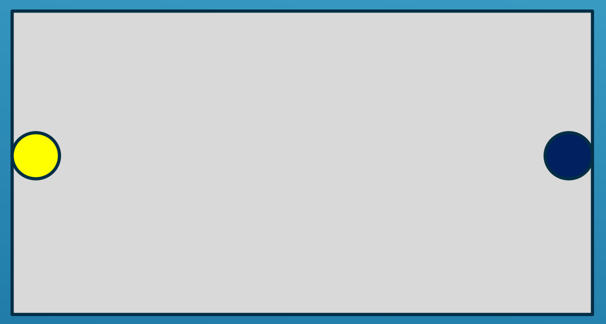
$$K = -\frac{1}{2\sqrt{EG}}\left(\frac{\partial}{\partial v}\left(\frac{E_v}{\sqrt{EG}}\right) + \frac{\partial}{\partial u}\left(\frac{G_u}{\sqrt{EG}}\right)\right)$$

$$\boldsymbol{E} = \boldsymbol{r_u} \cdot \boldsymbol{r_u}$$

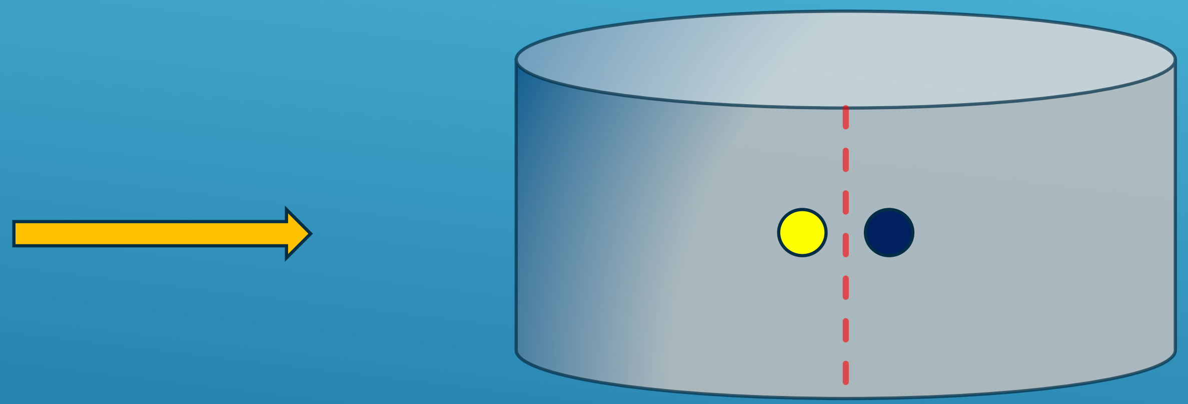
$$\mathbf{G} = \boldsymbol{r_v} \cdot \boldsymbol{r_v}$$

- **Rotating, moving or bending the surface does not change the Gauss curvature but, stretching or breaking does.**

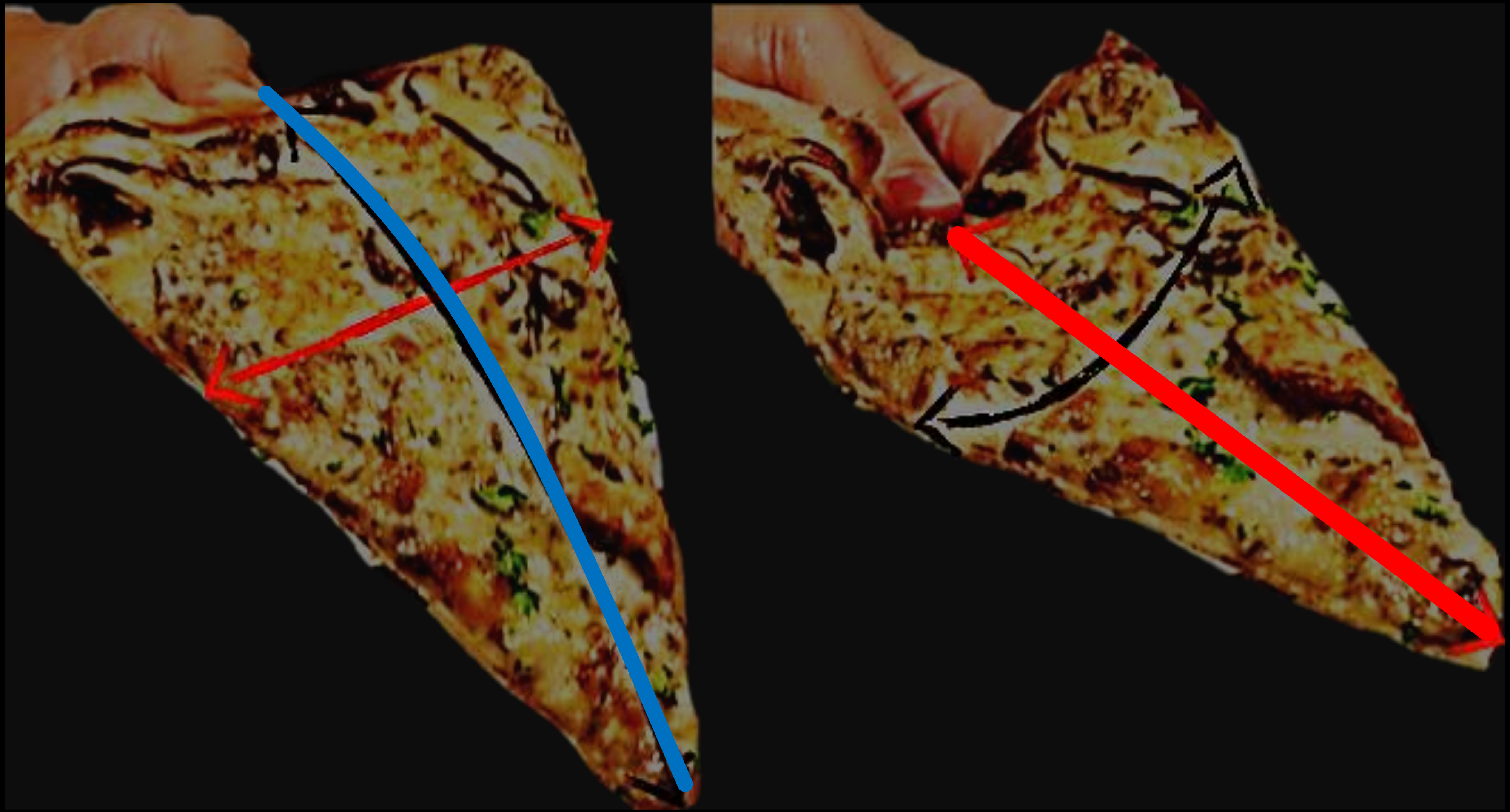
- Two surfaces with the same Gauss curvature are "locally" the same, but not globally!!!



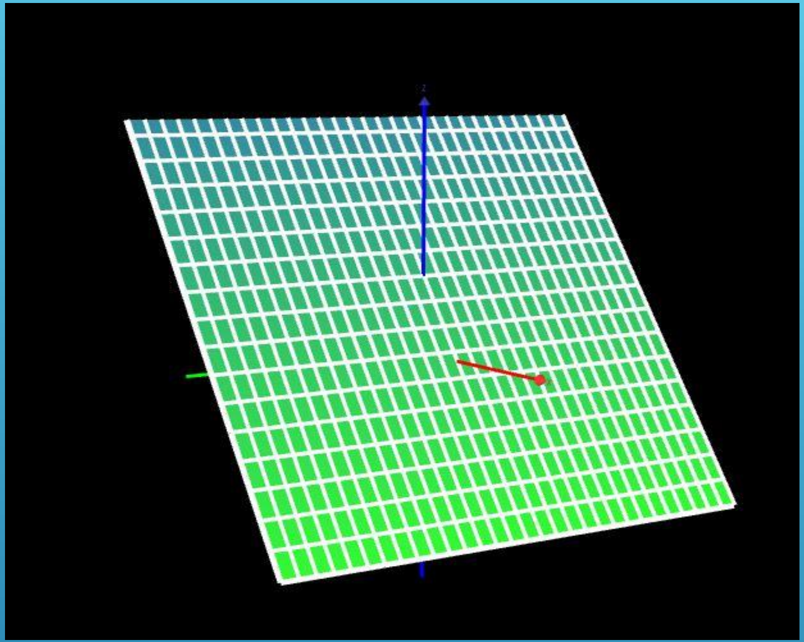
K=0

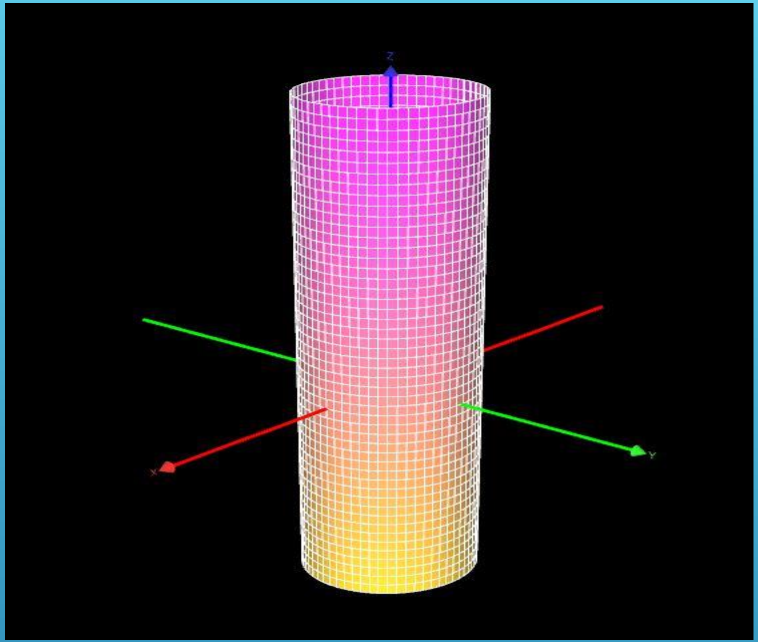


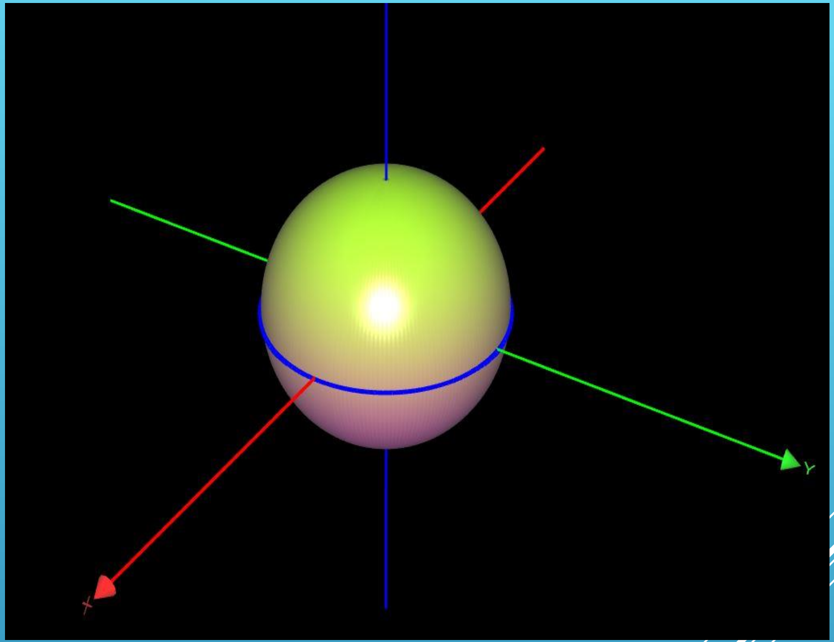
K=0

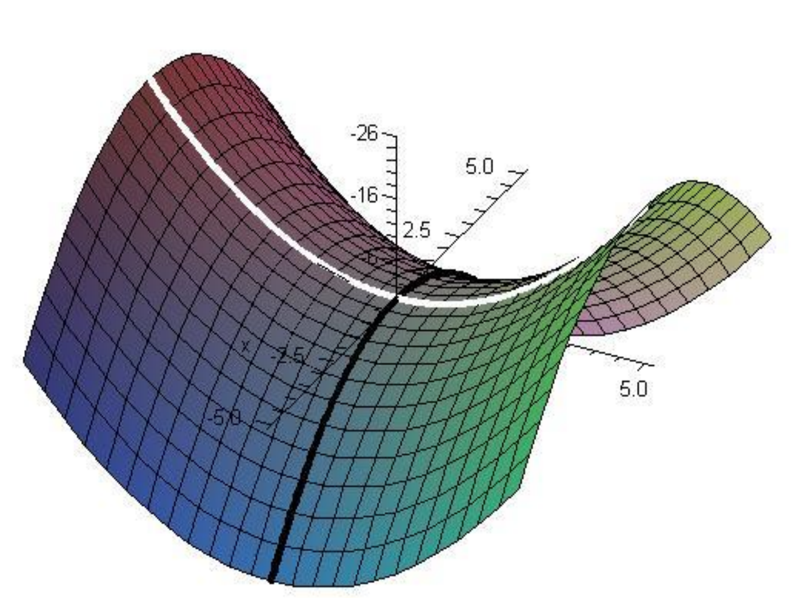


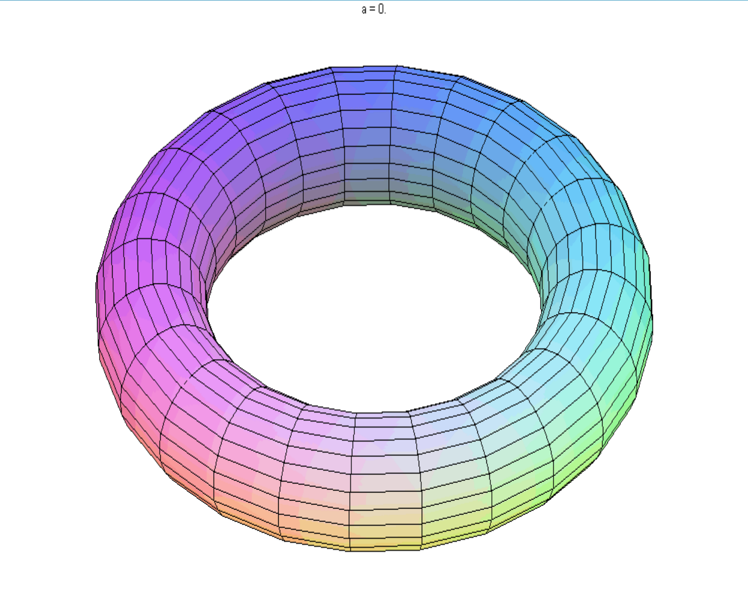
VARIOUS SURFACES

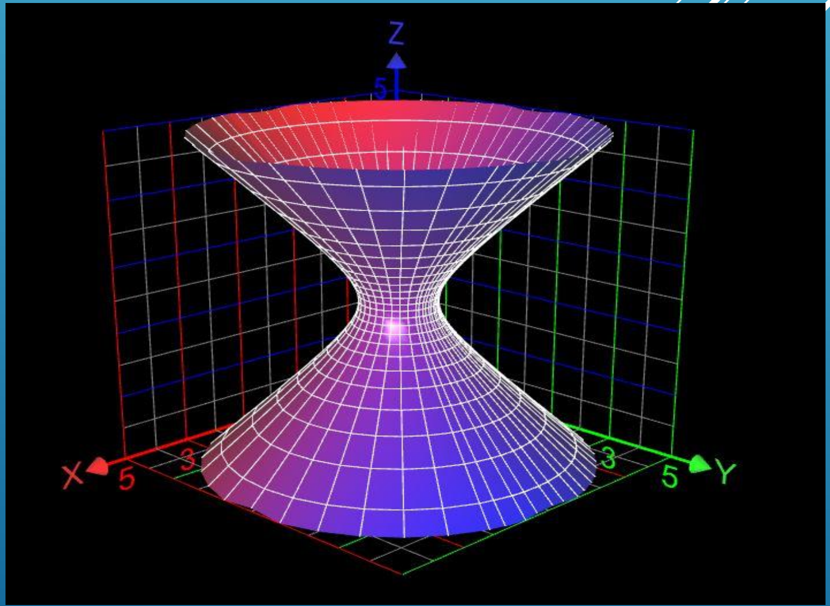




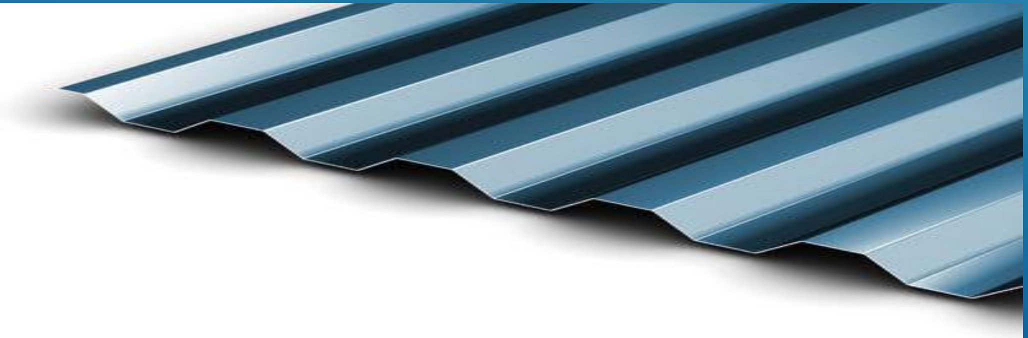






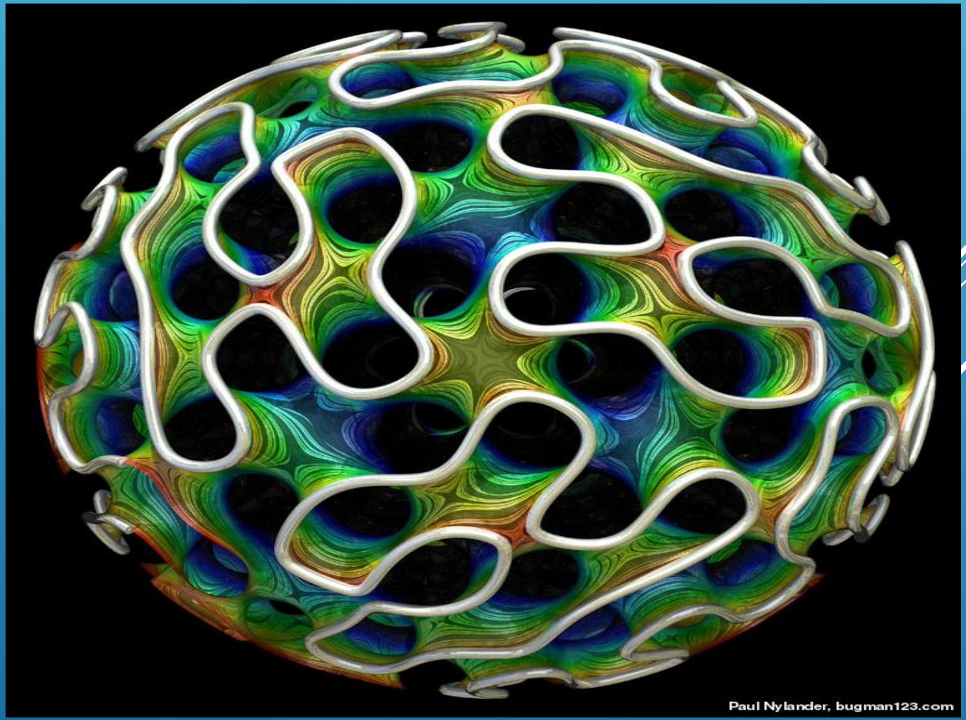


CURVATURE IN ARCHITECTURE AND DESIGN

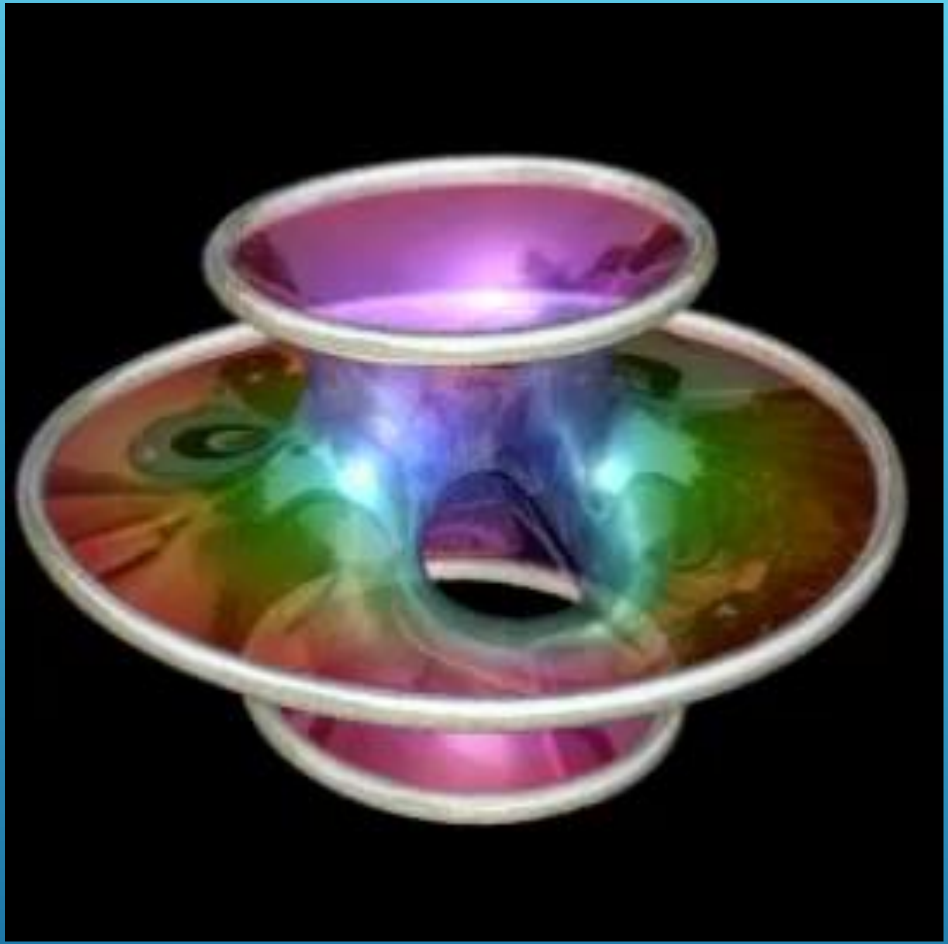


MINIMAL SURFACES

- A surface M is minimal if $\mathbf{H} = \frac{\boldsymbol{\kappa_1}+\boldsymbol{\kappa_2}}{2} = 0$
- Any planar surface is minimal (NOT INTERESTING)
- A Gyroid (VERY INTERESTING)
- Gyroid structures are found in certain surfactant or lipid mesophases and block copolymers.
- **THEOREM: Every soap film is a physical model of a minimal surface.**



COSTA-HOFFMAN-MEEK SURFACE

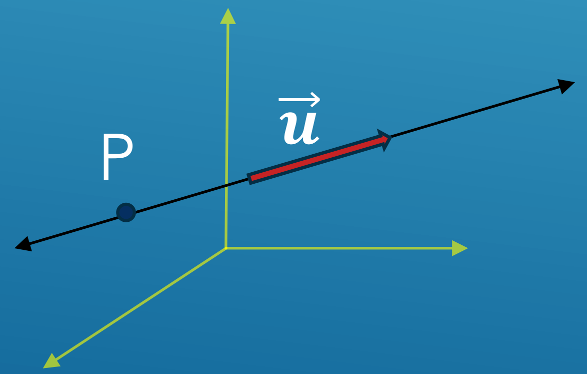


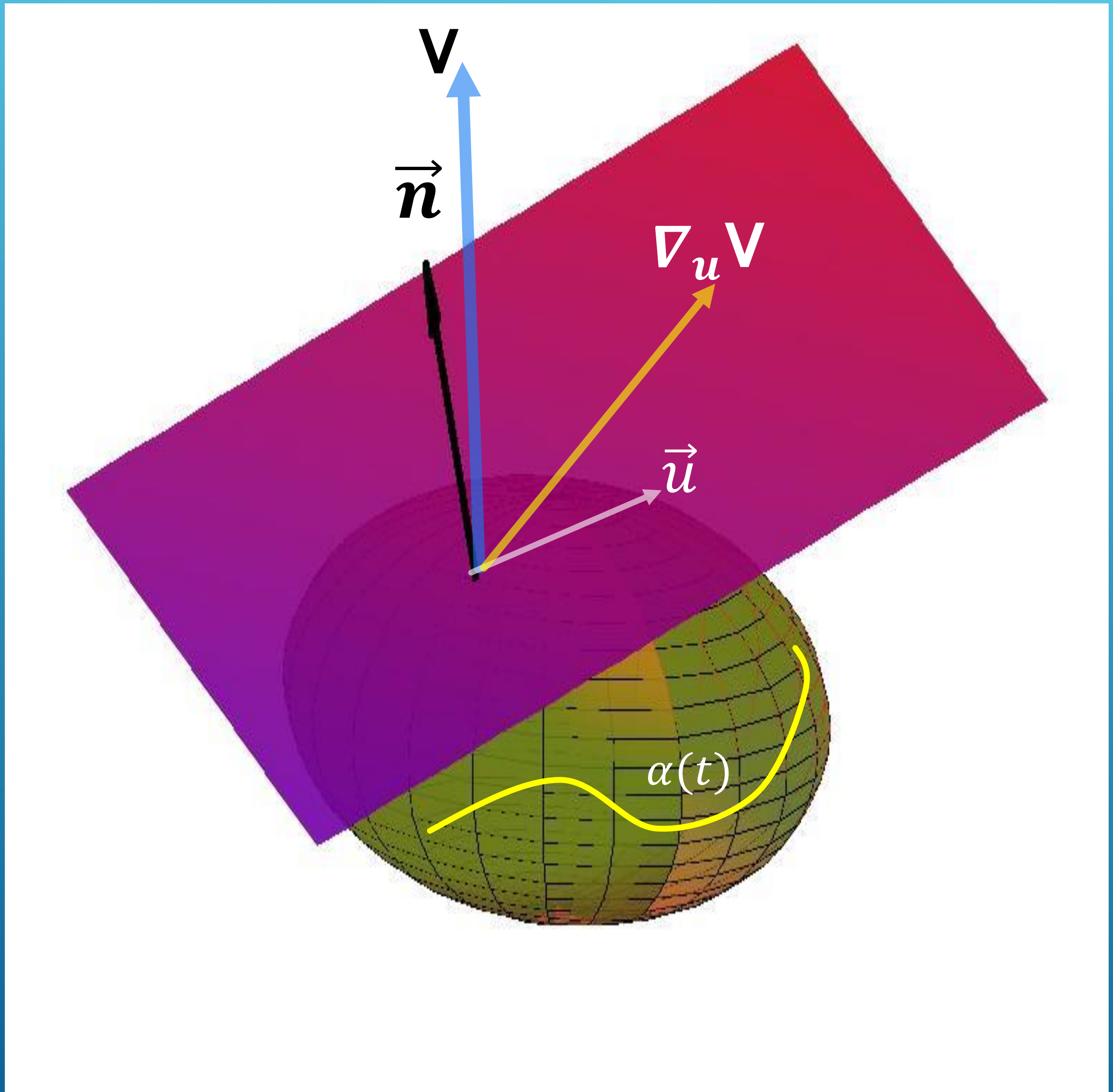
The interpretation of the Costa-Hoffman-Meeks minimal surface as insertion of multiple directional holes connecting the top to the water and the water at the bottom to the sky provided a single gesture combining all aspects. -Tobias Walliser

CONCEPT OF A LINE

- $\underbrace{y=mx+b}_{\text{slope-intercept}}$ $\longrightarrow$ $\underbrace{\boldsymbol{r(t) = (t, mt + b)}}_{\text{parametric}}$

- In general, a line in space is given by $r(t) = P + t\vec{u}$. So, $\ddot{r} = \mathbf{0}$





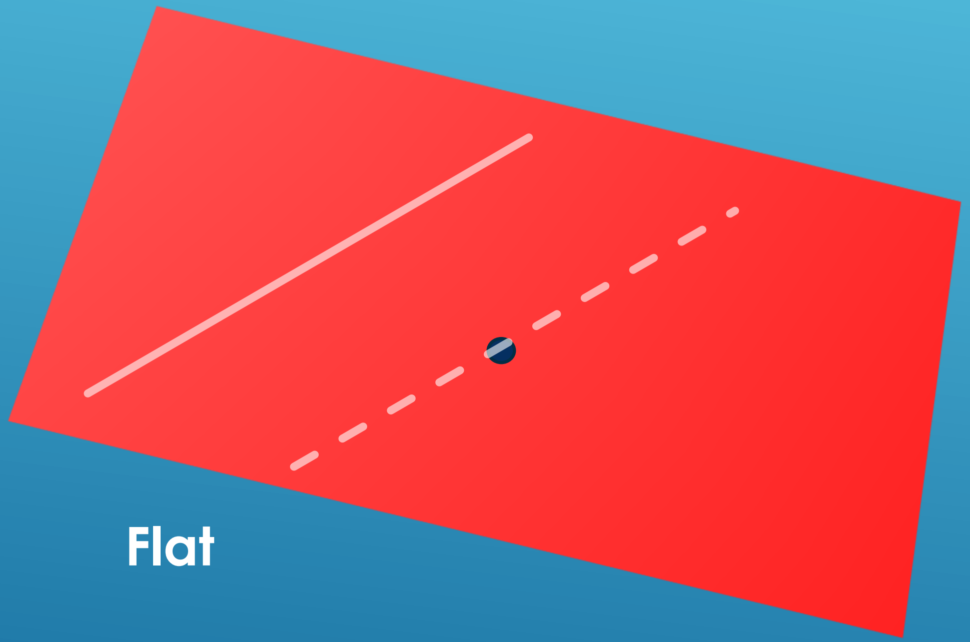
- The covariant derivative $\nabla_u \mathbf{V}$ of vector field V is the projection of the change of vector field in $\vec{u}$ direction onto the tangent plane.

- A curve $\alpha(t)$ on the surface is called a "Geodesic" if $\nabla_{\dot{\alpha}}\, \dot{\alpha} = 0$

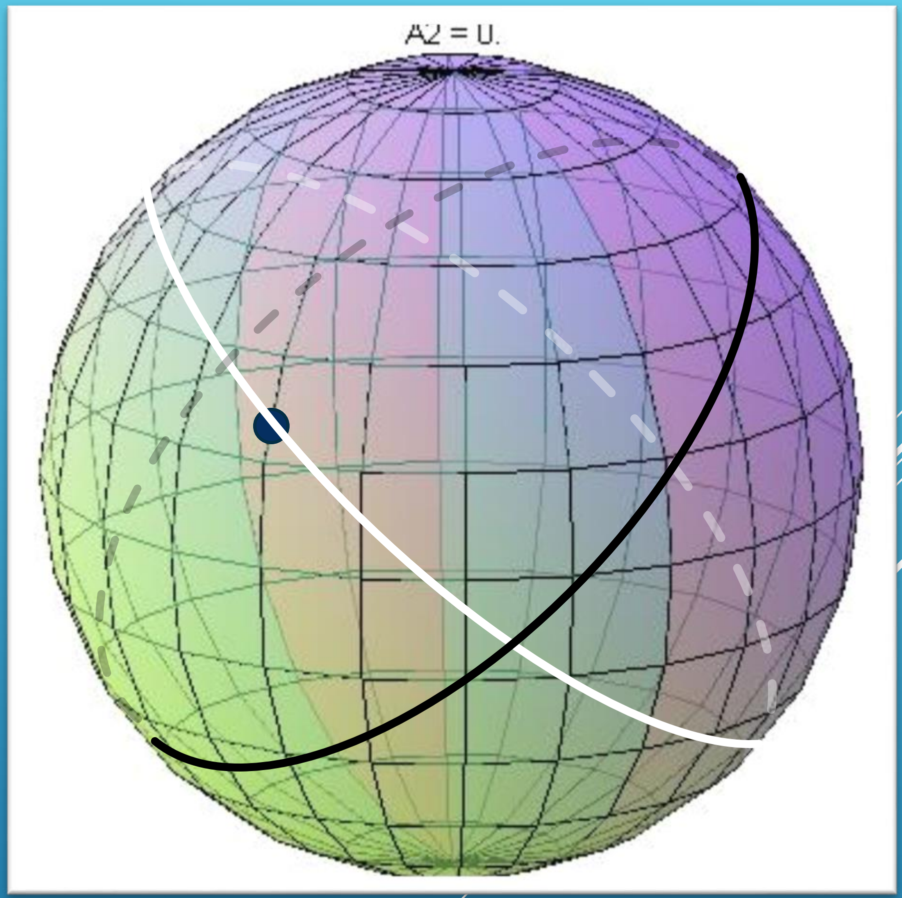
- **Geodesics are the "lines" of curved spaces**

- V is parallel along a curve $\alpha(t)$ if

$$\nabla_{\dot{\alpha}} \mathbf{V} = \mathbf{0}$$



Flat



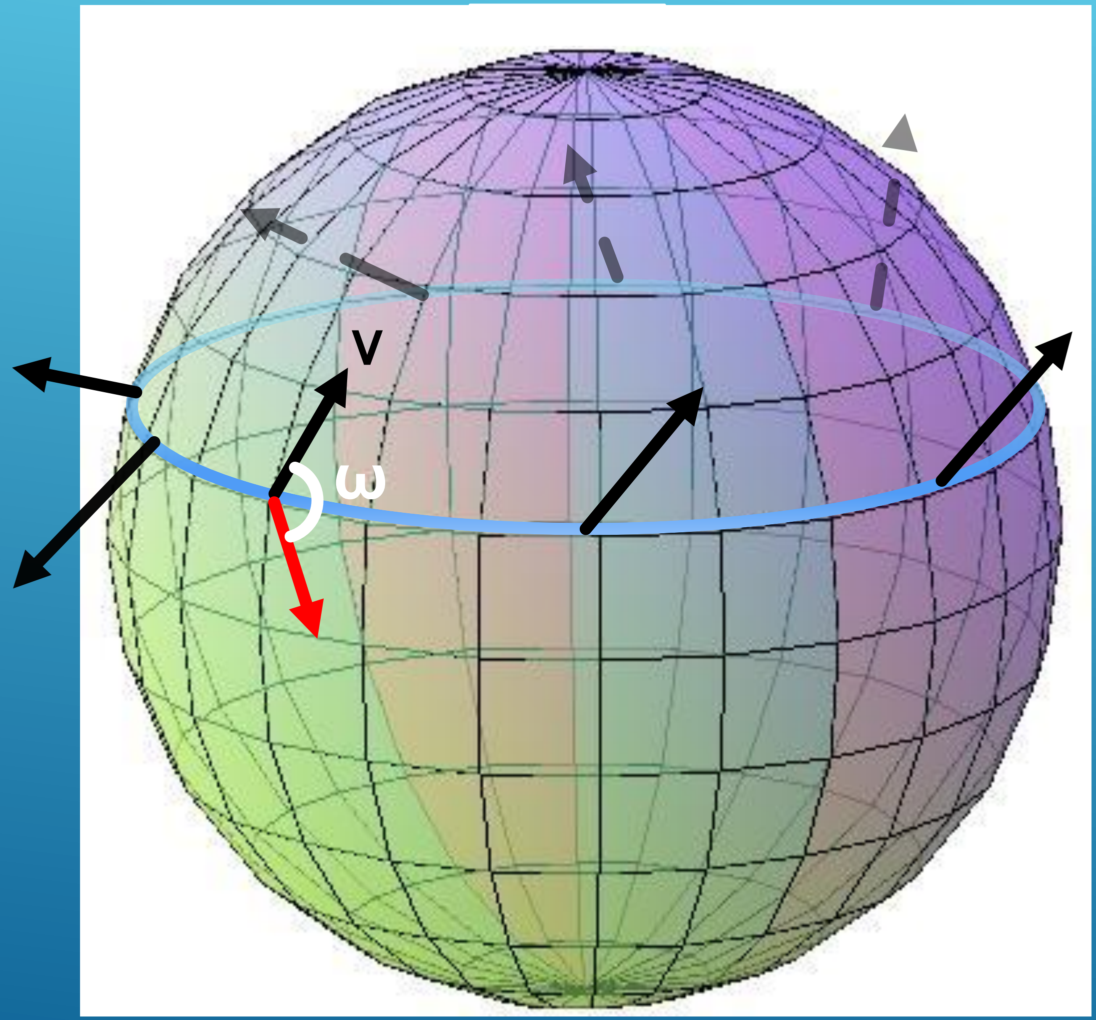
Curved

MERCATOR PROJECTION

$K \neq 0$

$K = 0$

PARALLEL TRANSPORT AND HOLONOMY



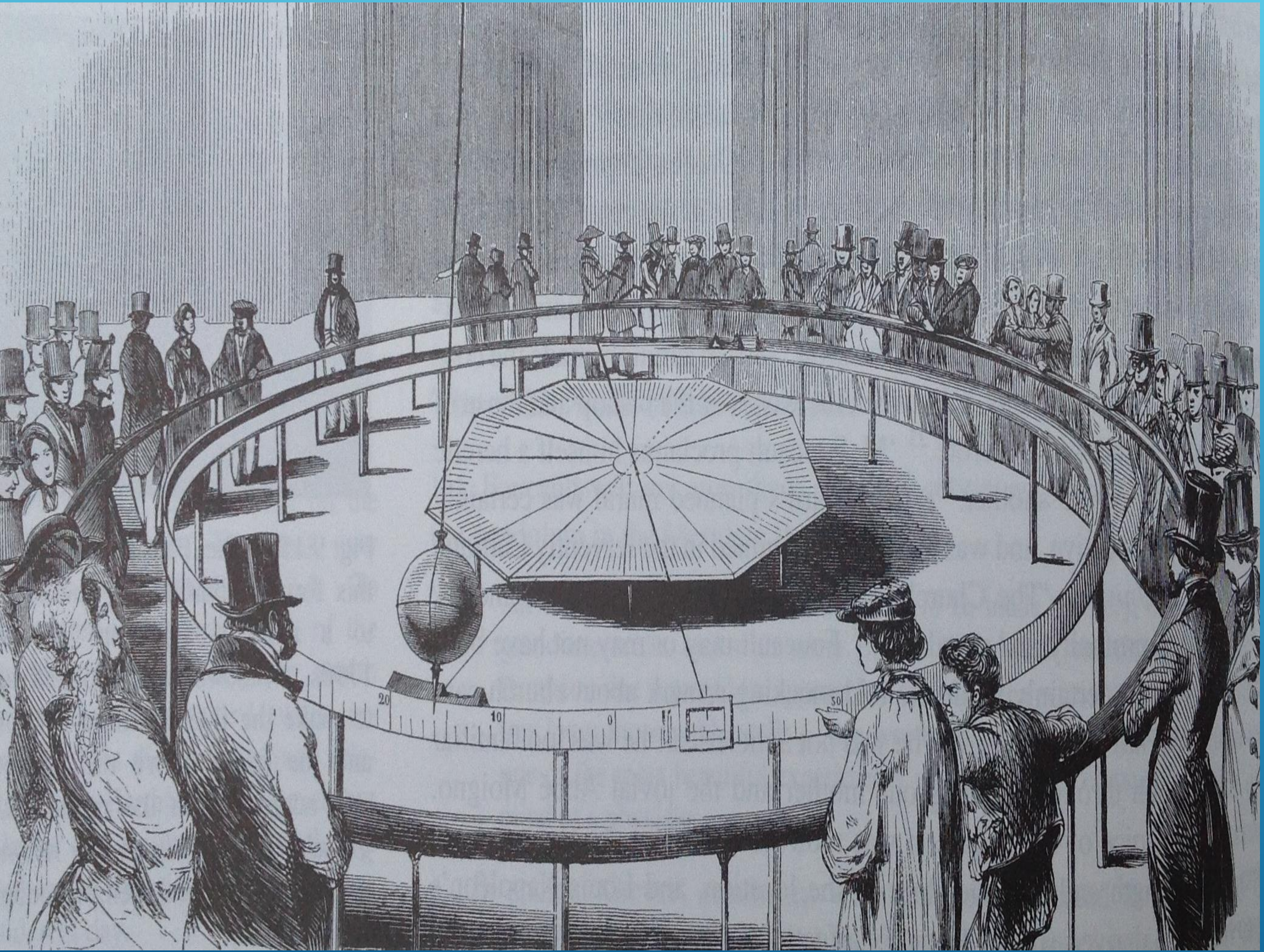
❖ The parallel vector V is rotated by **ω**

as it moved along the latitude v_0 .

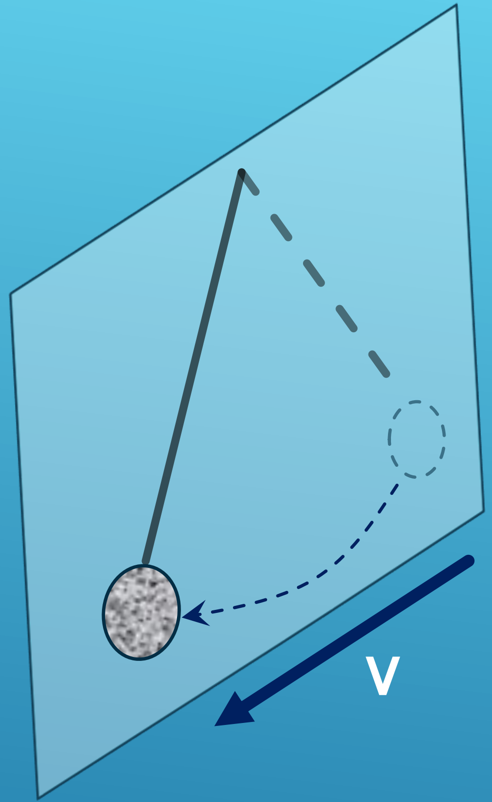
❖ But, 2D inhabitants of the sphere could not observe the rotation since V is parallel. ***For them, vector field moves "parallel" along the latitude***.

❖ **ω = - 2 π sin (v_0)**

"YOU ARE INVITED TO SEE THE EARTH IS SPINNING"

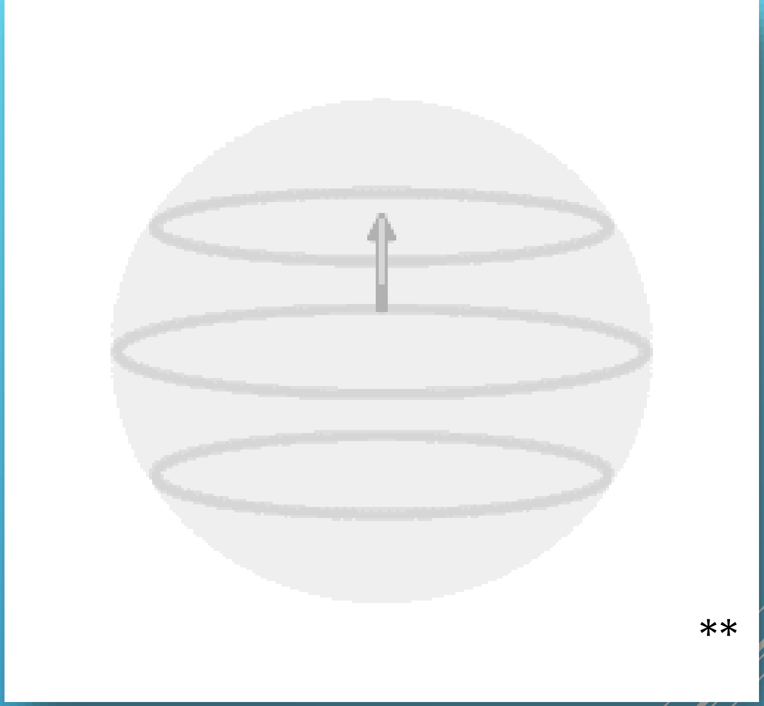


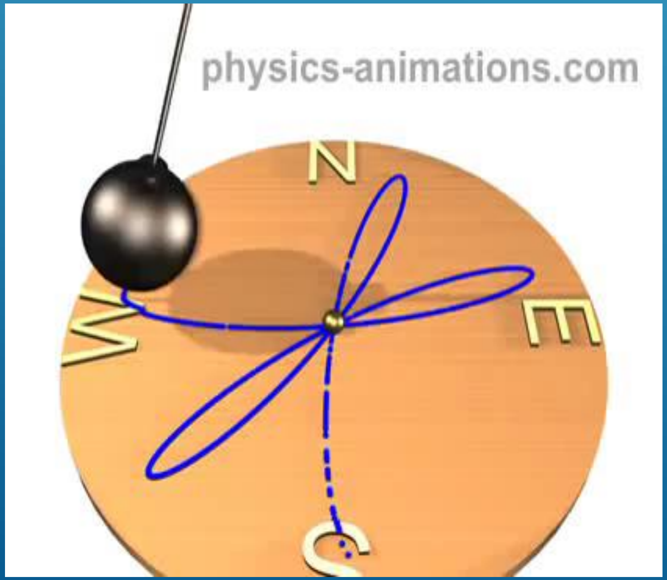
- An iron ball of 28 kg is suspended by a 67 meter (about 220 ft) wire.
- Using this experiment, in 1851, Foucault proved that the earth is spinning.



❖ Rod is long. So, swings can be seen as tangential to the sphere.

❖ Pendulum moves slowly around latitude so, we ignore centripetal force on it. Only gravitation acts on the pendulum .





❖ V is parallel along the latitude. It has holonomy

$$\omega = -2\pi \sin(v_0)$$

THEOREM: Earth rotates along its latitude circles.

**By Cleon Teunissen

GAUSS CURVATURE

- INTRINSIC
- INVARIANT UNDER CERTAIN DEFORMATIONS
- STRENGHT, RESISTANCE
- MOST FUNDAMENTAL GEOMETRIC PROPERTY

MEAN CURVATURE

- DEPENDS ON HOW SURFACE IS PLACED
- NOT INVARIANT
- SURFACE TENSION, AREA MINIMIZING
- GREAT TOOL FOR NOISE REDUCTION IN DIGITAL IMAGING

CURVATURE IN HIGHER DIMENSIONS

$$\underbrace{R_{\mu v} - \frac{1}{2} R g_{\mu v} + \Lambda g_{\mu v}}_{\text{Curvature}} = \underbrace{\frac{8\pi G}{c^4} T_{\mu v}}_{\text{Matter \& energy}}$$