12-1-2012

Maximal Specht Varieties of Monoids

Edmond W. H. Lee
Nova Southeastern University, edmond.lee@nova.edu

Moscow Mathematical Journal **12** (2012) 787–802

MAXIMAL SPECHT VARIETIES OF MONOIDS

EDMOND W. H. LEE

Abstract. A variety of algebras is a *Specht variety* if all its subvarieties are finitely based. This article presents the first example of a maximal Specht variety of monoids. The existence of such an example is counterintuitive since it is long known that maximal Specht varieties of semigroups do not exist. This example permits a characterization of Specht varieties in the following four classes based on identities that they must satisfy and varieties that they cannot contain: (1) overcommutative varieties, (2) varieties containing a certain monoid of order seven, (3) varieties of aperiodic monoids with central idempotents, and (4) subvarieties of the variety generated by the Brandt monoid of order six. Other results, including the uniqueness or nonexistence of limit varieties within the aforementioned four classes, are also deduced. Specifically, overcommutative limit varieties of monoids do not exist. In contrast, the limit variety of semigroups, discovered by M. V. Volkov in the 1980s, is an overcommutative variety.

2010 Math. Subj. Class. 20M07.

Key words and phrases. Monoids, varieties, Specht varieties, limit varieties, finitely based, hereditarily finitely based.

1. Introduction

In 1950, while investigating polynomial identities of rings, Specht questioned whether or not every variety of associative rings over a field of characteristic zero is finitely based [31]. This question was affirmatively answered by Kemer [17] in the 1980s. Nowadays, the finite basis problem for all the main varieties of universal algebras is commonly known as the *Specht problem*. Unlike the aforementioned positive result of Kemer [17], not every variety of algebras is finitely based in general; there exist non-finitely based varieties of associative rings over a field of any positive characteristic [1], [8], [29] some of which are even nilpotent [5], [10]. For results and comments regarding the Specht problem for varieties of groups and varieties of semigroups, refer to the surveys of Gupta and Krasilnikov [9] and Volkov [34] respectively. The present article is solely concerned with varieties of monoids, with references made to varieties of semigroups only for comparison. Unless otherwise specified, all varieties are varieties of monoids.

A variety that contains only finitely based subvarieties is called a *Specht variety*,[1] while a monoid that generates a Specht variety is called a *Specht monoid*. A variety is *periodic* if it satisfies a nontrivial identity of the form $x^{n+k} \approx x^n$; if $k = 1$ in this identity, then the variety does not contain any nontrivial groups and is said to be *aperiodic*. It is well known and easily shown that a non-periodic variety must contain the variety **Com** of commutative monoids and so is also said to be *overcommutative*. The variety **Com** is itself Spechtian [11] and vacuously overcommutative. An example of an overcommutative Specht variety that properly contains **Com**, due to Pollák, is the variety defined by the identity $xyx \approx x^2y$ [25]. Apparently, no other larger overcommutative Specht variety containing Pollák's variety is known. The main objective of this article is to present an example that is not only new but also maximal.

Theorem 1.1. (i) *The variety* $\mathbf{O}$ *of monoids defined by the identities*

$$xhxyty \approx xhyxty, \qquad xhytxy \approx xhytyx \tag{0}$$

is a Specht variety.

(ii) *The variety* $\mathbf{O}$ *and its dual variety* $\mathbf{O}^\delta$ *are maximal Specht varieties.*

(iii) *The varieties* $\mathbf{O}$ *and* $\mathbf{O}^\delta$ *are the only maximal overcommutative Specht varieties.*

Theorem 1.1(ii) is quite counterintuitive since it is long known that maximal Specht varieties of semigroups do not exist [30, Proposition 15.2]. Presently, the varieties $\mathbf{O}$ and $\mathbf{O}^\delta$ are the only known examples of maximal Specht varieties of monoids.

Corollary 1.2. *The varieties in the intervals* $[\mathbf{Com}, \mathbf{O}]$ *and* $[\mathbf{Com}, \mathbf{O}^\delta]$ *are precisely all overcommutative Specht varieties of monoids.*

Let $\mathcal{X}^*$ denote the free monoid over a countably infinite alphabet $\mathcal{X}$. Elements of $\mathcal{X}^*$ are called *words*. For any set $\mathcal{W}$ of words, let $\mathrm{S}(\mathcal{W})$ denote the Rees quotient of $\mathcal{X}^*$ over the ideal of all words that are not factors of any word in $\mathcal{W}$. Equivalently, $\mathrm{S}(\mathcal{W})$ can be treated as the monoid that consists of every factor of every word in $\mathcal{W}$, together with a zero element 0, with binary operation $\cdot$ given by

$$\mathbf{u} \cdot \mathbf{v} = \begin{cases} \mathbf{uv}, & \text{if } \mathbf{uv} \text{ is a factor of some word in } \mathcal{W}; \\ 0, & \text{otherwise.} \end{cases}$$

The empty factor, more conveniently written as 1, is the identity element of the monoid $\mathrm{S}(\mathcal{W})$. If $\mathcal{W} = \{\mathbf{w}_1, \ldots, \mathbf{w}_r\}$, then write $\mathrm{S}(\mathcal{W}) = \mathrm{S}(\mathbf{w}_1, \ldots, \mathbf{w}_r)$.

Rees quotients of free monoids constitute a significant source of examples in the study of the finite basis problem for semigroups and monoids. In 1969, Perkins published the first two examples of non-finitely based finite semigroups: the well-known Brandt monoid

$$B_2^1 = \langle a, b, 1 \mid a^2 = b^2 = 0,\ aba = a,\ bab = b \rangle$$

[1] In the study of varieties of groups and varieties of semigroups, Specht varieties are also commonly called *hereditarily finitely based varieties*. See, for instance, Bryant and Newman [3], Pollák [25], and Shevrin and Volkov [30].

of order six and the monoid

$$P_{25} = \mathrm{S}(xyzyx,\ xzyxy,\ xyxy,\ x^2z)$$

of order 25 [24]. More recent work of Jackson [14], Sapir [28], and their collaboration [16] shed more light on the finite basis problem for Rees quotients of free monoids and demonstrated how non-finitely based monoids can be located. In the early 2000s, Jackson proved that the varieties $\mathbf{J_1}$ and $\mathbf{J_2}$, generated by the monoids

$$J_1 = \mathrm{S}(xhxyty) \quad \text{and} \quad J_2 = \mathrm{S}(xhytxy,\ xyhxty)$$

respectively, are minimal non-finitely based varieties [15], or *limit varieties*.

Remark 1.3. Presently, the varieties $\mathbf{J_1}$ and $\mathbf{J_2}$ remain the only explicitly known examples of non-group limit varieties of monoids. In contrast, Kozhevnikov demonstrated that continuum many limit varieties of monoids consisting of groups exist [18], but none has yet been explicitly described; see Gupta and Krasilnikov [9]. As for non-group limit varieties of semigroups, countably infinitely many belonging to a few classes have been explicitly described [23], [26], [27], [32].

By Zorn's lemma, any non-finitely based variety contains some limit subvariety. It follows that a variety is Spechtian if and only if it does not contain any limit subvarieties. Therefore the more complete the classification of limit varieties, the better the understanding of Specht varieties. It is evident from Lemmas 2.1 and 2.2 that the identity system (0) or its dual system is satisfied by any variety that contains the monoid $\mathrm{S}(xyx)$ but excludes the varieties $\mathbf{J_1}$ and $\mathbf{J_2}$; it is precisely this result that inspired the formulation of Theorem 1.1(i). More information on the varieties $\mathbf{J_1}$ and $\mathbf{J_2}$ and other related preliminary results are given in Section 2.

In Section 3, Theorem 1.1(i) is applied to show that all Specht varieties from the following classes are characterized precisely by the exclusion of the limit varieties $\mathbf{J_1}$ and $\mathbf{J_2}$:

($\mathcal{O}_{\mathrm{com}}$) overcommutative varieties;
($\mathcal{C}_{xyx}$) varieties that contain the monoid $\mathrm{S}(xyx)$;
($\mathcal{A}_{\mathrm{cent}}$) varieties of aperiodic monoids with central idempotents;
($\mathcal{S}_{\mathbf{B_2^1}}$) subvarieties of the variety $\mathbf{B_2^1}$ generated by the Brandt monoid B_2^1.

Parts (ii) and (iii) of Theorem 1.1 then follow from this characterization. Other results are also deduced; these include the existence and uniqueness of limit varieties within the classes $\mathcal{C}_{xyx}$, $\mathcal{A}_{\mathrm{cent}}$, and $\mathcal{S}_{\mathbf{B_2^1}}$, and their nonexistence from the class $\mathcal{O}_{\mathrm{com}}$:

Theorem 1.4. *Overcommutative limit varieties of monoids do not exist.*

This result is again unexpected since the very first published example of a limit variety of semigroups, due to Volkov [32], is overcommutative.

Sections 4 and 5 are devoted to proving Theorem 1.1(i). Specifically, identities that can be used to define noncommutative subvarieties of $\mathbf{O}$ are described in Section 4; based on this description and results of Head [11], Higman [12], and Volkov [33], the finite basis property of every subvariety of $\mathbf{O}$ is established in Section 5.

Acknowledgments. The author is grateful to the following colleagues for valuable comments and suggestions: the anonymous reviewers, I. Dolinka, M. G. Jackson, and M. V. Volkov.

2. Preliminaries

Let x be any letter and $\mathbf{w}$ be any word. The number of times x occurs in $\mathbf{w}$ is denoted by $\mathrm{occ}(x, \mathbf{w})$. If $\mathrm{occ}(x, \mathbf{w}) = 1$, then x is *simple* in $\mathbf{w}$; if $\mathrm{occ}(x, \mathbf{w}) \geqslant 2$, then x is *non-simple* in $\mathbf{w}$. Denote by $\mathrm{sim}(\mathbf{w})$ the set of simple letters of $\mathbf{w}$ and by $\mathrm{non}(\mathbf{w})$ the set of non-simple letters of $\mathbf{w}$. The *content* of a word $\mathbf{w}$ is the set of all letters occurring in $\mathbf{w}$ and is denoted by $\mathrm{con}(\mathbf{w})$. Define the relation $\stackrel{\circ}{=}$ on $\mathcal{X}^*$ by $\mathbf{u} \stackrel{\circ}{=} \mathbf{v}$ if $\mathrm{occ}(x, \mathbf{u}) = \mathrm{occ}(x, \mathbf{v})$ for all $x \in \mathcal{X}$.

An identity is written as $\mathbf{u} \approx \mathbf{v}$, where $\mathbf{u}$ and $\mathbf{v}$ are nonempty words. A monoid M *satisfies* an identity $\mathbf{u} \approx \mathbf{v}$ if, for any substitution φ from $\mathcal{X}$ into M, the elements $\mathbf{u}\varphi$ and $\mathbf{v}\varphi$ of M coincide. A variety $\mathbf{V}$ *satisfies* an identity $\mathbf{u} \approx \mathbf{v}$ if every monoid in $\mathbf{V}$ satisfies $\mathbf{u} \approx \mathbf{v}$. The variety *defined* by a set Σ of identities is the class of all monoids that satisfy all identities in Σ; in this case, Σ is a *basis* for the variety. A variety is *finitely based* if it possesses a finite basis. For any variety $\mathbf{V}$, the subvariety of $\mathbf{V}$ defined by a set Σ of identities is denoted by $\mathbf{V}\{\Sigma\}$. Refer to the monograph of Burris and Sankappanavar [4] for more information on varieties and universal algebra.

A word $\mathbf{w}$ is an *isoterm* for a variety if the variety does not satisfy any nontrivial identity of the form $\mathbf{w} \approx \mathbf{v}$. The set of all isoterms for a variety $\mathbf{V}$ is denoted by $\mathrm{iso}(\mathbf{V})$. The notion of an isoterm is a convenient method to determine when a monoid of the form $\mathrm{S}(\mathcal{W})$ belongs to a variety.

Lemma 2.1 (Jackson [15, Lemma 3.3])**.** *Let* $\mathbf{V}$ *be any variety and* $\mathcal{W}$ *be any set of words. Then* $\mathrm{S}(\mathcal{W}) \in \mathbf{V}$ *if and only if* $\mathcal{W} \subseteq \mathrm{iso}(\mathbf{V})$.

Lemma 2.2. *Let* $\mathbf{V}$ *be any variety such that* $xyx \in \mathrm{iso}(\mathbf{V})$.

(i) *If* $xhxyty \notin \mathrm{iso}(\mathbf{V})$, *then* $\mathbf{V}$ *satisfies the first identity in* (0).
(ii) *If* $xhytxy \notin \mathrm{iso}(\mathbf{V})$, *then* $\mathbf{V}$ *satisfies the second identity in* (0).

Proof. It suffices to verify part (i) since part (ii) follows similarly. Suppose that $xhxyty \notin \mathrm{iso}(\mathbf{V})$. Then the variety $\mathbf{V}$ satisfies a nontrivial identity of the form $xhxyty \approx \mathbf{w}$. The assumption $xyx \in \mathrm{iso}(\mathbf{V})$ implies that

(a) $\mathrm{occ}(x, \mathbf{w}) = \mathrm{occ}(y, \mathbf{w}) = 2$ and $\mathrm{occ}(h, \mathbf{w}) = \mathrm{occ}(t, \mathbf{w}) = 1$;
(b) the h in $\mathbf{w}$ is sandwiched between the two occurrences of x in $\mathbf{w}$;
(c) the t in $\mathbf{w}$ is sandwiched between the two occurrences of y in $\mathbf{w}$.

If t occurs before h in $\mathbf{w}$, then the variety $\mathbf{V}$ satisfies the identity $ht \approx th$ and so also the system (0). If h occurs before t in $\mathbf{w}$, then it follows from (a)–(c) that $\mathbf{w} = xhyxty$. ☐

Lemma 2.3. (i) *The monoid* $\mathrm{S}(xyx)$ *belongs to the varieties* $\mathbf{J_1}$ *and* $\mathbf{J_2}$.

(ii) *The monoids* J_1 *and* J_2 *belong to the variety* $\mathbf{B_2^1}$.

(iii) *For any* $\mathcal{W} \subseteq \mathcal{X}^*$, *the monoid* $\mathrm{S}(\mathcal{W})$ *is aperiodic and contains only central idempotents.*

Proof. (i) This follows from Jackson [15, Section 5]; see also Figure 1.

(ii) It is easily verified that $xyx \in \operatorname{iso}(\mathbf{B_2^1})$, and that the variety $\mathbf{B_2^1}$ does not satisfy any identity from the system (0) or its dual system. Hence $\mathrm{S}(xyx) \in \mathbf{B_2^1}$ by Lemma 2.1, and $xhxyty$, $xhytxy$, $xyhxty \in \operatorname{iso}(\mathbf{B_2^1})$ by Lemma 2.2. Consequently, J_1, $J_2 \in \mathbf{B_2^1}$ by Lemma 2.1.

(iii) This follows from the easy observation that the elements 0 and 1 of any monoid $\mathrm{S}(\mathcal{W})$ are the only idempotents. $\square$

The subvarieties of $\mathbf{J_1}$ and $\mathbf{J_2}$ constitute the lattices in Figure 1 [15, Section 5]. In this figure, each ◉ represents a non-finitely based variety, each • represents a finitely based variety, $\mathbf{S}(\mathcal{W})$ denotes the variety generated by the monoid $\mathrm{S}(\mathcal{W})$, and $\mathbf{0}$ denotes the variety of trivial monoids.

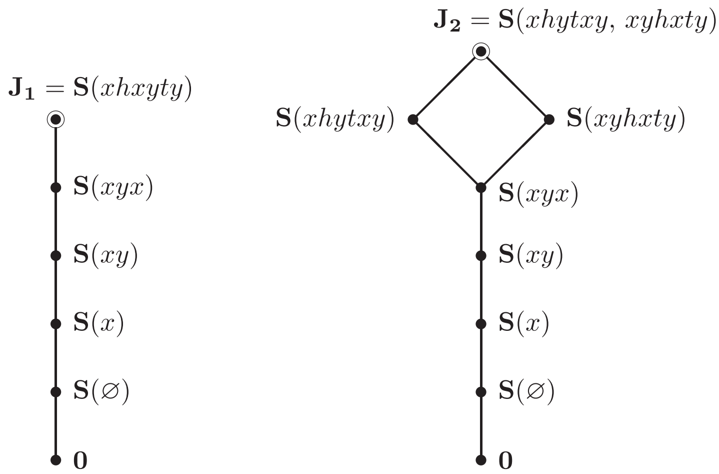

FIGURE 1. Subvarieties of $\mathbf{J_1}$ and $\mathbf{J_2}$

Remark 2.4. The monoid $\mathrm{S}(xyx)$, when treated as a semigroup, generates a variety of semigroups with continuum many subvarieties [13]. It follows that with respect to varieties of semigroups, neither the identity system (0) defines a Specht variety nor the monoids $\mathrm{S}(xhxyty)$ and $\mathrm{S}(xhytxy,\, xyhxty)$ generate limit varieties. Consequently and unfortunately, none of the results from the present article applies to varieties of semigroups.

3. Other Results

3.1. Specht varieties in $\mathcal{O}_{\mathbf{com}}$, $\mathcal{C}_{xyx}$, $\mathcal{A}_{\mathbf{cent}}$, and $\mathcal{S}_{\mathrm{B}_2^1}$. For a characterization of Specht varieties from the four classes stated in Section 1, it is convenient to define the *distinguished* class $\mathfrak{D}$ to consist of all varieties $\mathbf{V}$ such that $\mathrm{S}(xyx) \in \mathbf{V}$ or $\mathbf{V} \subseteq \mathbf{O}$ or $\mathbf{V} \subseteq \mathbf{O}^\delta$.

Lemma 3.1. *The following statements on any variety* $\mathbf{V} \in \mathfrak{D}$ *are equivalent*:

(I) $\mathbf{V}$ *is Spechtian*;
(II) $\mathbf{J_1}, \mathbf{J_2} \nsubseteq \mathbf{V}$;
(III) $\mathbf{V} \subseteq \mathbf{O}$ *or* $\mathbf{V} \subseteq \mathbf{O}^\delta$.

Proof. The implication (III) $\Rightarrow$ (I) follows from Theorem 1.1(i) while the implication (I) $\Rightarrow$ (II) needs no comment. Suppose that (II) holds. If $\mathrm{S}(xyx) \notin \mathbf{V}$, then (III) holds by the definition of $\mathfrak{D}$. Hence assume that $\mathrm{S}(xyx) \in \mathbf{V}$. By (II) and Lemma 2.1, either $xhxyty$, $xhytxy \notin \mathrm{iso}(\mathbf{V})$ or $xhxyty$, $xyhxty \notin \mathrm{iso}(\mathbf{V})$. By Lemma 2.2, the variety $\mathbf{V}$ satisfies the identity system (0) or its dual system. □

Theorem 3.2. *The statements* (I)–(III) *are equivalent for any variety* $\mathbf{V}$ *from the classes* $\mathcal{O}_{\mathrm{com}}$, $\mathcal{C}_{xyx}$, $\mathcal{A}_{\mathrm{cent}}$, *and* $\mathcal{S}_{\mathbf{B_2^1}}$.

Proof. By Lemma 3.1, it suffices to show that the four classes are contained in $\mathfrak{D}$. This is achieved by showing that any variety $\mathbf{V}$ from $\mathcal{O}_{\mathrm{com}} \cup \mathcal{C}_{xyx} \cup \mathcal{A}_{\mathrm{cent}} \cup \mathcal{S}_{\mathbf{B_2^1}}$ belongs to $\mathfrak{D}$. If $\mathbf{V} \in \mathcal{C}_{xyx}$, then $\mathbf{V} \in \mathfrak{D}$ because the inclusion $\mathcal{C}_{xyx} \subseteq \mathfrak{D}$ holds by definition. Therefore further assume that $\mathbf{V} \notin \mathcal{C}_{xyx}$, whence $\mathrm{S}(xyx) \notin \mathbf{V}$. By Lemma 2.1, the variety $\mathbf{V}$ satisfies a nontrivial identity of the form $xyx \approx \mathbf{w}$.

Case 1. $\mathbf{V} \in \mathcal{O}_{\mathrm{com}}$. Then $\mathrm{occ}(x, \mathbf{w}) = 2$ and $\mathrm{occ}(y, \mathbf{w}) = 1$ so that $\mathbf{w} \in \{x^2y, yx^2\}$. It follows that either $\mathbf{V} \subseteq \mathbf{O}$ or $\mathbf{V} \subseteq \mathbf{O}^\delta$, whence $\mathbf{V} \in \mathfrak{D}$.

Case 2. $\mathbf{V} \in \mathcal{A}_{\mathrm{cent}}$. Then arguments of Jackson [15, Proof of Lemma 4.1] can be repeated to show that $\mathbf{w} \in \{x^2y, yx^2\}$. It follows that either $\mathbf{V} \subseteq \mathbf{O}$ or $\mathbf{V} \subseteq \mathbf{O}^\delta$, whence $\mathbf{V} \in \mathfrak{D}$.

Case 3. $\mathbf{V} \in \mathcal{S}_{\mathbf{B_2^1}}$. It is easily checked that the monoid B_2^1 satisfies the identities

$$x^3 \approx x^2, \quad x^2yx \approx xyx^2, \quad x^2y^2 \approx y^2x^2 \tag{3.1}$$

so that the variety $\mathbf{V}$ also satisfies (3.1). Since the variety $\mathbf{V}$ satisfies the identity $xyx \approx \mathbf{w}$, it is routinely shown that $\mathbf{V}$ satisfies the identities

$$xyx \approx x^2yx \approx xyx^2. \tag{3.2}$$

Since

$$\begin{aligned} xhxyty &\overset{(3.2)}{\approx} xhx^2y^2ty \overset{(3.1)}{\approx} xhy^2x^2ty \overset{(3.2)}{\approx} xhyxty, \\ xhytxy &\overset{(3.2)}{\approx} xhytx^2y^2 \overset{(3.1)}{\approx} xhyty^2x^2 \overset{(3.2)}{\approx} xhytyx, \end{aligned}$$

the variety $\mathbf{V}$ satisfies the identity system (0) so that $\mathbf{V} \subseteq \mathbf{O}$. Hence $\mathbf{V} \in \mathfrak{D}$. □

3.2. Maximal Specht varieties: proof of Theorem 1.1 parts (ii) and (iii). Theorem 1.1(iii) follows from the equivalence of (I) and (III) of Lemma 3.1. As for Theorem 1.1(ii), any variety properly containing either $\mathbf{O}$ or $\mathbf{O}^\delta$ is overcommutative and so cannot be Spechtian by the equivalence of (I) and (III) of Lemma 3.1.

3.3. Uniqueness of limit varieties. The varieties $\mathbf{J_1}$ and $\mathbf{J_2}$ are known to be the only finitely generated, limit varieties in the class $\mathcal{A}_{\mathrm{cent}}$ [21]. This result can now be generalized to other classes that also include non-finitely generated varieties.

Theorem 3.3. *The varieties* $\mathbf{J_1}$ *and* $\mathbf{J_2}$ *are the only limit varieties in the classes* $\mathcal{C}_{xyx}$, $\mathcal{A}_{\mathrm{cent}}$, *and* $\mathcal{S}_{\mathbf{B_2^1}}$.

Proof. By Lemma 2.3, the varieties $\mathbf{J_1}$ and $\mathbf{J_2}$ belong to the classes $\mathcal{C}_{xyx}$, $\mathcal{A}_{\mathrm{cent}}$, and $\mathcal{S}_{\mathbf{B}_2^1}$. The uniqueness of these varieties then follows from the equivalence of (I) and (II) of Lemma 3.1. □

Let $\mathbf{P_{25}}$ denote the variety generated by the monoid P_{25} introduced in Section 1.

Corollary 3.4. *The varieties* $\mathbf{J_1}$ *and* $\mathbf{J_2}$ *are the only limit subvarieties of* $\mathbf{P_{25}}$.

Proof. By Lemma 2.3(iii), the variety $\mathbf{P_{25}}$ belongs to the class $\mathcal{A}_{\mathrm{cent}}$. The arguments in the proof of Lemma 2.3(ii) can be repeated to show that $J_1, J_2 \in \mathbf{P_{25}}$. The result now follows from Theorem 3.3. □

3.4. Nonexistence of overcommutative limit varieties: proof of Theorem 1.4. By Theorem 3.2, any overcommutative variety that excludes the limit varieties $\mathbf{J_1}$ and $\mathbf{J_2}$ must be Spechtian and so cannot be a limit variety.

3.5. Rees quotients of $\boldsymbol{\mathcal{X}}^*$. Sapir [28] has shown that for any word $\mathbf{w} \in \{x, y\}^*$, the monoid $\mathrm{S}(\mathbf{w})$ is finitely based if and only if $\mathbf{w}$ is one of the following words:

$$x^m y^n, \quad y^m x^n, \quad x^m y x^n, \quad y^m x y^n, \quad m, n \geqslant 0. \tag{3.3}$$

It is easily checked whether or not the monoid $\mathrm{S}(\mathbf{w})$ is Spechtian for any word $\mathbf{w}$ from (3.3). If $m, n \geqslant 2$, then the monoids $\mathrm{S}(x^m y^n)$ and $\mathrm{S}(y^m x^n)$ do not satisfy the first identity in (0) and so are not Spechtian by Theorem 3.2. For any word $\mathbf{w}$ from (3.3) that contains neither x^2y^2 nor y^2x^2 as a factor, the monoid $\mathrm{S}(\mathbf{w})$ satisfies the identity system (0) and so is Spechtian by Theorem 1.1(i).

In general, it is unknown if there exists an algorithm that decides, given a finite set $\mathcal{W}$ of words, whether or not the monoid $\mathrm{S}(\mathcal{W})$ is finitely based; see Shevrin and Volkov [30, Question 7.1]. Now by Lemma 2.3(iii), the monoid $\mathrm{S}(\mathcal{W})$ generates a variety in the class $\mathcal{A}_{\mathrm{cent}}$. Therefore by Lemma 3.1, the monoid $\mathrm{S}(\mathcal{W})$ is Spechtian if and only if it satisfies the identity system (0) or its dual system; checking this latter condition is a problem of complexity $O\big(|\mathrm{S}(\mathcal{W})|^4\big)$.

3.6. Direct products of monoids. It is easily seen from Figure 1 that the monoids $\mathrm{S}(xhytxy)$ and $\mathrm{S}(xyhxty)$ are Spechtian, while their direct product generates the limit variety $\mathbf{J_2}$ and so is not Spechtian. The following result provides a more general method for obtaining similar examples.

Proposition 3.5. (i) *Let M be any monoid such that $M \notin \mathbf{O} \cup \mathbf{O}^\delta$. Then the direct product $M \times \mathrm{S}(xyx)$ is not Spechtian.*

(ii) *For any cancellative monoid M, the direct product $M \times \mathrm{S}(xyx)$ is Spechtian if and only if M is commutative.*

Proof. (i) The variety $\mathbf{V}$ generated by the monoid $M \times \mathrm{S}(xyx)$ clearly contains the monoid $\mathrm{S}(xyx)$ so that $\mathbf{V} \in \mathcal{C}_{xyx}$. Since $\mathbf{V} \nsubseteq \mathbf{O}$ and $\mathbf{V} \nsubseteq \mathbf{O}^\delta$ by assumption, the variety $\mathbf{V}$ is not Spechtian by Lemma 3.1.

(ii) It is routinely verified that $\mathrm{S}(xyx) \in \mathbf{O}$. If the monoid M is commutative, then the monoid $M \times \mathrm{S}(xyx)$ satisfies the identity system (0) and so is Spechtian by Theorem 1.1(i). Conversely, suppose that the monoid M is noncommutative. Since the monoid M is cancellative, it cannot satisfy any identity from the identity system (0) or its dual system. By Lemma 3.1, the monoid $M \times \mathrm{S}(xyx)$ is not Spechtian. $\square$

Example 3.6. The monoid

$$A_0^1 = \langle a,\, b,\, 1 \mid a^2 = a,\ b^2 = b,\ ba = 0 \rangle$$

of order five is Spechtian [20]. Since the monoid A_0^1 does not satisfy the identity $xhxyty \approx xhyxty$, the direct product $A_0^1 \times \mathrm{S}(xyx)$ is not Spechtian by Proposition 3.5(i). The word $xhxyty$ is an isoterm for the variety $\mathbf{A_0^1} \vee \mathbf{S}(xyx)$ generated by $A_0^1 \times \mathrm{S}(xyx)$ so that by Lemma 2.1, the limit variety $\mathbf{J_1}$ is contained in $\mathbf{A_0^1} \vee \mathbf{S}(xyx)$.

Example 3.7. The regular band

$$ReB = \langle a,\, b,\, 1 \mid a^2 = a,\ b^2 = b,\ aba = a,\ bab = b \rangle$$

of order five is well known to be Spechtian [2], [6], [7]. Since the monoid ReB does not satisfy the identities $xhytxy \approx xhyxyx$ and $xyhxty \approx yxhxty$, the direct product $ReB \times \mathrm{S}(xyx)$ is not Spechtian by Proposition 3.5(i). The words $xhytxy$ and $xyhxty$ are isoterms for the variety $\mathbf{ReB} \vee \mathbf{S}(xyx)$ generated by $ReB \times \mathrm{S}(xyx)$ so that by Lemma 2.1, the limit variety $\mathbf{J_2}$ is contained in $\mathbf{ReB} \vee \mathbf{S}(xyx)$.

Remark 3.8. Even though the monoid $A_0^1 \times ReB$ does not satisfy any identity from the system (0) or its dual system, it is Spechtian not only as a monoid but also as a semigroup [22]. The variety of semigroups generated by $A_0^1 \times ReB$ contains countably infinitely many subvarieties, while the variety of monoids generated by $A_0^1 \times ReB$ contains only 28 subvarieties [19].

4. Identities of Noncommutative Subvarieties of O

The present section establishes restrictions on the type of identities that can be used to define noncommutative subvarieties of **O**. This result will then be used in Section 5 to prove that the variety **O** is Spechtian.

Proposition 4.1. *Each noncommutative subvariety of* **O** *can be defined by the identities* (0) *together with some of the following identities*:

$$x^{e_0} \prod_{i=1}^{r} (h_i x^{e_i}) \approx x^{f_0} \prod_{i=1}^{r} (h_i x^{f_i}), \tag{4.1A}$$

where $e_0,\, f_0,\, \ldots,\, e_r,\, f_r \geqslant 0$ *and* $r \geqslant 0$;

$$x^{e_0} y^{f_0} \prod_{i=1}^{r} (h_i x^{e_i} y^{f_i}) \approx y^{f_0} x^{e_0} \prod_{i=1}^{r} (h_i x^{e_i} y^{f_i}), \tag{4.1B}$$

where $e_0,\, f_0 \geqslant 1,\, e_1,\, f_1,\, \ldots,\, e_r,\, f_r \geqslant 0,\ \sum_{i=0}^{r} e_i \geqslant 2,\ \sum_{i=0}^{r} f_i \geqslant 2,$ *and* $r \geqslant 0$.

A list of results is developed in Sections 4.1 and 4.2 to put identities satisfied by subvarieties of **O** into specific forms. The proof of Proposition 4.1 is then given in Section 4.3.

4.1. Canonical form. Suppose that x is any letter and that $\mathbf{w}$ is any word written in the form $\mathbf{w}_0 \prod_{i=1}^{r}(x^{e_i}\mathbf{w}_i)$, where $e_1, \ldots, e_r \geqslant 1$ and $\mathbf{w}_0, \ldots, \mathbf{w}_r$ are words that do not involve the letter x, with $\mathbf{w}_0$ or $\mathbf{w}_r$ possibly being empty. Then the factors $x^{e_1}, \ldots, x^{e_r}$ are called *x-stacks*, or simply *stacks*, of $\mathbf{w}$. Specifically, x^{e_1} is the *primary x-stack* of $\mathbf{w}$, while $x^{e_2}, \ldots, x^{e_r}$ are *secondary x-stacks* of $\mathbf{w}$.

For any set $\mathcal{Y}$ of letters from $\mathcal{X}$, define

$$\mathcal{Y}^{Ⓢ} = \big\{x_1^{e_1} \cdots x_r^{e_r} \mid x_1, \ldots, x_r \in \mathcal{Y} \text{ are distinct, } e_1, \ldots, e_r \geqslant 1, \text{ and } r \geqslant 0\big\}.$$

Note that every stack of every word in $\mathcal{Y}^{Ⓢ}$ is primary.

Example 4.2. (i) $\{x\}^{Ⓢ} = \{\varnothing, x, x^2, \ldots\}$.

(ii) $\{x, y\}^{Ⓢ} = \{\varnothing, x^e, y^f, x^e y^f, y^e x^f \mid e, f \geqslant 1\}$.

A word $\mathbf{w}$ is said to be in *canonical form* if

$$\mathbf{w} = \mathbf{w}_0 \prod_{i=1}^{r}(h_i \mathbf{w}_i) \tag{4.2}$$

for some $r \geqslant 0$ such that $\mathrm{sim}(\mathbf{w}) = \{h_1, \ldots, h_r\}$ and $\mathbf{w}_0, \ldots, \mathbf{w}_r \in \mathrm{non}(\mathbf{w})^{Ⓢ}$.

Lemma 4.3. *Any word can be converted by the identities* (0) *into a word in canonical form.*

Proof. It is clear that any word $\mathbf{w}$ can be written in the form (4.2) for some $r \geqslant 0$ with $\mathrm{sim}(\mathbf{w}) = \{h_1, \ldots, h_r\}$ and $\mathbf{w}_0, \ldots, \mathbf{w}_r \in \mathrm{non}(\mathbf{w})^*$. If a letter $x \in \mathrm{non}(\mathbf{w})$ occurs more than once in some $\mathbf{w}_i$, then the identities (0) can be used to gather any non-first occurrence of x in $\mathbf{w}_i$ with the first occurrence of x in $\mathbf{w}_i$. Repeat this gathering on every letter of every $\mathbf{w}_i$ results in $\mathbf{w}_0, \ldots, \mathbf{w}_r \in \mathrm{non}(\mathbf{w})^{Ⓢ}$. □

4.2. Regular identities and well-balanced identities. An identity $\mathbf{u} \approx \mathbf{v}$ is said to be *regular* if

(R1) $\mathbf{u}$ and $\mathbf{v}$ are in canonical form,
(R2) $\mathrm{sim}(\mathbf{u}) = \mathrm{sim}(\mathbf{v})$ and $\mathrm{non}(\mathbf{u}) = \mathrm{non}(\mathbf{v})$,
(R3) the order of appearance of the simple letters of $\mathbf{u}$ coincides with the order of appearance of the simple letters of $\mathbf{v}$.

The words $\mathbf{u}$ and $\mathbf{v}$ that form a regular identity can therefore be written as

$$\mathbf{u} = \mathbf{u}_0 \prod_{i=1}^{r}(h_i \mathbf{u}_i) \quad \text{and} \quad \mathbf{v} = \mathbf{v}_0 \prod_{i=1}^{r}(h_i \mathbf{v}_i), \tag{4.3}$$

where $\mathrm{sim}(\mathbf{u}) = \mathrm{sim}(\mathbf{v}) = \{h_1, \ldots, h_r\}$ and $\mathbf{u}_i, \mathbf{v}_i \in \mathrm{non}(\mathbf{u})^{Ⓢ} = \mathrm{non}(\mathbf{v})^{Ⓢ}$ for all i. Such a regular identity is said to be *well-balanced* if $\mathbf{u}_i \stackrel{\circ}{=} \mathbf{v}_i$ for all i. The identity $\mathbf{u} \approx \mathbf{v}$ is *not well-balanced at x* if $\mathrm{occ}(x, \mathbf{u}_i) \neq \mathrm{occ}(x, \mathbf{v}_i)$ for some i.

Lemma 4.4. *Let $\sigma\colon \mathbf{u} \approx \mathbf{v}$ be any regular identity that is not well-balanced. Then*

$$\mathbf{O}\{\sigma\} = \mathbf{O}\{\mathsf{A}^\sigma, \sigma'\}$$

for some finite set A^σ of identities from (4.1A) *and some well-balanced identity σ'.*

Proof. By assumption, the words $\mathbf{u}$ and $\mathbf{v}$ can be assumed to be those from (4.3). Suppose that the identity σ is not well-balanced at precisely the letters $x_1, \ldots, x_m$. For each $i \in \{0, \ldots, r\}$, let $e_i = \operatorname{occ}(x_1, \mathbf{u}_i)$ and $f_i = \operatorname{occ}(x_1, \mathbf{v}_i)$ so that $\mathbf{u}_i = \mathbf{a}_i x_1^{e_i} \mathbf{b}_i$ and $\mathbf{v}_i = \mathbf{c}_i x_1^{f_i} \mathbf{d}_i$ for some $\mathbf{a}_i, \mathbf{b}_i, \mathbf{c}_i, \mathbf{d}_i \in \mathcal{X}^*$. Then

$$\mathbf{u} = \mathbf{a}_0 x_1^{e_0} \mathbf{b}_0 \prod_{i=1}^{r} (h_i \mathbf{a}_i x_1^{e_i} \mathbf{b}_i) \quad \text{and} \quad \mathbf{v} = \mathbf{c}_0 x_1^{f_0} \mathbf{d}_0 \prod_{i=1}^{r} (h_i \mathbf{c}_i x_1^{f_i} \mathbf{d}_i).$$

The identity $\alpha_1 \colon x^{e_0} \prod_{i=1}^{r} (h_i x^{e_i}) \approx x^{f_0} \prod_{i=1}^{r} (h_i x^{f_i})$ from (4.1A) is a consequence of σ, whence $\mathbf{O}\{\sigma\} = \mathbf{O}\{\alpha_1, \sigma\}$. Let $\mathbf{v}^{(1)} = \mathbf{c}_0 x_1^{e_0} \mathbf{d}_0 \prod_{i=1}^{r} (h_i \mathbf{c}_i x_1^{e_i} \mathbf{d}_i)$. The identity $\mathbf{v} \approx \mathbf{v}^{(1)}$ is clearly a consequence of α_1 so that

$$\mathbf{O}\{\sigma\} = \mathbf{O}\{\alpha_1, \sigma^{(1)}\},$$

where $\sigma^{(1)} : \mathbf{u} \approx \mathbf{v}^{(1)}$ is a regular identity that is not well-balanced at precisely the letters $x_2, \ldots, x_m$.

Now for any j with $1 \leqslant j < m$, suppose that the identity $\sigma^{(j)}$ is not well-balanced at precisely the letters $x_{j+1}, \ldots, x_m$. Then the argument in the previous paragraph can be repeated on the letter x_{j+1} to obtain

$$\mathbf{O}\{\sigma^{(j)}\} = \mathbf{O}\{\alpha_{j+1}, \sigma^{(j+1)}\},$$

where α_{j+1} is some identity from (4.1A) and $\sigma^{(j+1)}$ is a regular identity that is not well-balanced at precisely the letters $x_{j+2}, \ldots, x_m$. Hence

$$\mathbf{O}\{\sigma\} = \mathbf{O}\{\alpha_1, \sigma^{(1)}\} = \mathbf{O}\{\alpha_1, \alpha_2, \sigma^{(2)}\} = \cdots = \mathbf{O}\{\alpha_1, \ldots, \alpha_m, \sigma^{(m)}\}.$$

Since $\sigma^{(m)}$ is a well-balanced identity, the lemma holds with $\mathsf{A}^\sigma = \{\alpha_1, \ldots, \alpha_m\}$. □

Lemma 4.5. *Let σ be any nontrivial, well-balanced identity. Then*

$$\mathbf{O}\{\sigma\} = \mathbf{O}\{\mathsf{B}^\sigma\}$$

for some finite set B^σ of identities from (4.1B).

This lemma can be established in the same manner as Lee [21, Lemma 4.11]. Full details are provided here for the sake of completeness.

For the rest of this subsection, let $\sigma \colon \mathbf{u} \approx \mathbf{v}$ be the nontrivial, well-balanced identity in Lemma 4.5. If either $\mathbf{u}$ or $\mathbf{v}$ is a simple word, then by (R2) and (R3), the identity σ is contradictorily trivial. Therefore both words $\mathbf{u}$ and $\mathbf{v}$ are non-simple, and they can be assume to be those from (4.3). The prefixes $\mathbf{u}_0$ and $\mathbf{v}_0$ are clearly products of primary stacks. For each $i \in \{1, \ldots, r\}$, any secondary stack of $\mathbf{u}_i$ can be moved by the identities (0) to any position within $\mathbf{u}_i$. Specifically, all secondary stacks of $\mathbf{u}_i$ can be gathered to the left and arranged in alphabetical order. Hence the word $\mathbf{u}$ can be rewritten as

$$\mathbf{u} \overset{(0)}{=} \mathbf{p}_0 \prod_{i=1}^{r} (h_i \mathbf{s}_i \mathbf{p}_i),$$

where the word $\mathbf{p}_i$ is a possibly empty product of some primary stacks of $\mathbf{u}$ that occur in $\mathbf{u}_i$, the word $\mathbf{s}_i$ is a possibly empty product of some secondary stacks of

$\mathbf{u}$ that occur in $\mathbf{u}_i$, and the stacks in $\mathbf{s}_i$ are arranged in alphabetical order. Since $\mathbf{u}_i \stackrel{\circ}{=} \mathbf{v}_i$ for all i, gathering the stacks of $\mathbf{v}$ in a similar manner results in

$$\mathbf{v} \stackrel{(0)}{=} \mathbf{q}_0 \prod_{i=1}^{r} (h_i \mathbf{s}_i \mathbf{q}_i), \tag{4.4}$$

where the word $\mathbf{q}_i$ is a possibly empty product of some primary stacks of $\mathbf{v}$ that occur in $\mathbf{v}_i$.

Remark 4.6. (i) $\mathbf{p}_i \stackrel{\circ}{=} \mathbf{q}_i$ for all i since $\mathbf{u} \approx \mathbf{v}$ is well-balanced.
(ii) $\mathbf{p}_r = \mathbf{q}_r = \varnothing$ since $\mathbf{u}_r$ and $\mathbf{v}_r$ cannot contain any primary stacks.

It is convenient to call $\mathbf{p}_i$ the *i-th primary stack product* of $\mathbf{u}$, and $\mathbf{q}_i$ the *i-th primary stack product* of $\mathbf{v}$.

Lemma 4.7. *Let ℓ be the least integer such that the ℓ-th primary stack products of* $\mathbf{u}$ *and* $\mathbf{v}$ *are different, that is,* $\mathbf{p}_0 = \mathbf{q}_0, \ldots, \mathbf{p}_{\ell-1} = \mathbf{q}_{\ell-1},$ *and* $\mathbf{p}_\ell \neq \mathbf{q}_\ell$. *Then*

$$\mathbf{O}\{\sigma\} = \mathbf{O}\{\mathsf{B},\, \mathbf{u} \approx \mathbf{v}^\dagger\}$$

for some set B *of identities from* (4.1B) *and some word* $\mathbf{v}^\dagger$ *of the form* (4.4) *such that for each* $i \in \{0, \ldots, \ell\}$, *the i-th primary stack products of* $\mathbf{u}$ *and* $\mathbf{v}^\dagger$ *are identical.*

Proof. Let $\mathbf{z} \in \mathcal{X}^*$ be the longest suffix that is common to both $\mathbf{p}_\ell$ and $\mathbf{q}_\ell$. Then $\mathbf{p}_\ell = \mathbf{a} y^f \mathbf{z}$ and $\mathbf{q}_\ell = \mathbf{b} y^f x_1^{e_1} \cdots x_s^{e_s} \mathbf{z}$ for some $\mathbf{a}, \mathbf{b} \in \mathcal{X}^*$ and distinct stacks $x_1^{e_1}, \ldots, x_s^{e_s}, y^f$ with $s \geqslant 1$. It suffices to show how $\mathbf{v}$ can be rewritten, by invoking some identities of $\mathbf{O}\{\sigma\}$, into a word $\mathbf{v}'$ such that the ℓ-th primary stack products of $\mathbf{u}$ and $\mathbf{v}'$ share the longer prefix $y^f \mathbf{z}$. This procedure can then be repeated to obtain the required word $\mathbf{v}^\dagger$.

Recall from Remark 4.6(i) that $\mathbf{p}_\ell \stackrel{\circ}{=} \mathbf{q}_\ell$. Hence

(a) the stacks $x_1^{e_1}, \ldots, x_r^{e_r}$ of $\mathbf{q}_\ell$ must appear in the factor $\mathbf{a}$ of $\mathbf{p}_\ell$.

Further, the secondary y-stacks and secondary x_1-stacks of both $\mathbf{u}$ and $\mathbf{v}$ must occur in some of $\mathbf{s}_{\ell+1}, \ldots, \mathbf{s}_m$. Therefore, for any $i > \ell$, if either an x_1-stack (say x_1^p) or a y-stack (say y^q) or both occur in $\mathbf{s}_i$, then the identities in (0) can be applied to gather these stacks to the left of $\mathbf{s}_i$ resulting in $\mathbf{w}_i \mathbf{s}_i'$, where $\mathbf{w}_i \in \{x_1^p, y^q, x_1^p y^q\}$ and $\mathbf{s}_i'$ is obtained from $\mathbf{s}_i$ by eliminating all occurrences of x_1 and y. Therefore

$$\begin{aligned}
\mathbf{u} &\stackrel{(0)}{=} \mathbf{p}_0 \left(\prod_{i=1}^{\ell-1} (h_i \mathbf{s}_i \mathbf{p}_i) \right) h_\ell \mathbf{s}_\ell \mathbf{a} y^f \mathbf{z} \prod_{i=\ell+1}^{r} (h_i \mathbf{w}_i \mathbf{s}_i' \mathbf{p}_i),\\
\mathbf{v} &\stackrel{(0)}{=} \mathbf{q}_0 \left(\prod_{i=1}^{\ell-1} (h_i \mathbf{s}_i \mathbf{q}_i) \right) h_\ell \mathbf{s}_\ell \mathbf{b} y^f x_1^{e_1} x_2^{e_2} \cdots x_s^{e_s} \mathbf{z} \prod_{i=\ell+1}^{r} (h_i \mathbf{w}_i \mathbf{s}_i' \mathbf{q}_i).
\end{aligned} \tag{4.5}$$

Recall from (a) that the stack $x_1^{e_1}$ appears in the factor $\mathbf{a}$. Therefore retaining only the letters $x_1, y, h_{\ell+1}, \ldots, h_r$ in the identity $\mathbf{u} \approx \mathbf{v}$ results in the identity

$$\beta_1 \colon x_1^{e_1} y^f \prod_{i=\ell+1}^{r} (h_i \mathbf{w}_i) \approx y^f x_1^{e_1} \prod_{i=\ell+1}^{r} (h_i \mathbf{w}_i)$$

from (4.1B). Hence $\mathbf{O}\{\sigma\} = \mathbf{O}\{\beta_1, \sigma\}$. Now it is easily seen that the identity β_1 of $\mathbf{O}\{\sigma\}$ can be used to interchange the primary y-stack and primary x_1-stack of $\mathbf{v}$ in (4.5) to obtain

$$\mathbf{v}^{(1)} = \mathbf{q}_0\left(\prod_{i=1}^{\ell-1}(h_i\mathbf{s}_i\mathbf{q}_i)\right)h_\ell\mathbf{s}_\ell\mathbf{b}x_1^{e_1}y^f x_2^{e_2}\cdots x_s^{e_s}\mathbf{z}\prod_{i=\ell+1}^{r}(h_i\mathbf{w}_i\mathbf{s}_i'\mathbf{q}_i).$$

Therefore $\mathbf{O}\{\sigma\} = \mathbf{O}\{\beta_1, \mathbf{u} \approx \mathbf{v}^{(1)}\}$. If the factor $x_2^{e_2}\cdots x_s^{e_s}$ of $\mathbf{v}^{(1)}$ is nonempty, then the procedure in this paragraph can be repeated to interchange the primary stacks y^f and $x_2^{e_2}$ in $\mathbf{v}^{(1)}$. Specifically, $\mathbf{O}\{\mathbf{u} \approx \mathbf{v}^{(1)}\} = \mathbf{O}\{\beta_2, \mathbf{u} \approx \mathbf{v}^{(2)}\}$ for some identity β_2 from (4.1B) and

$$\mathbf{v}^{(2)} = \mathbf{q}_0\left(\prod_{i=1}^{\ell-1}(h_i\mathbf{s}_i\mathbf{q}_i)\right)h_\ell\mathbf{s}_\ell\mathbf{b}x_1^{e_1}x_2^{e_2}y^f x_3^{e_3}\cdots x_s^{e_s}\mathbf{z}\prod_{i=\ell+1}^{r}(h_i\mathbf{w}_i\mathbf{s}_i'\mathbf{q}_i).$$

Continuing in this manner, the stack y^f can be moved to the right until it immediately precedes the factor $\mathbf{z}$, that is, $\mathbf{O}\{\mathbf{u} \approx \mathbf{v}^{(s-1)}\} = \mathbf{O}\{\beta_s, \mathbf{u} \approx \mathbf{v}^{(s)}\}$ for some identity β_s from (4.1B) and

$$\mathbf{v}^{(s)} = \mathbf{q}_0\left(\prod_{i=1}^{\ell-1}(h_i\mathbf{s}_i\mathbf{q}_i)\right)h_\ell\mathbf{s}_\ell\mathbf{b}x_1^{e_1}\cdots x_s^{e_s}y^f\mathbf{z}\prod_{i=\ell+1}^{r}(h_i\mathbf{w}_i\mathbf{s}_i'\mathbf{q}_i).$$

Hence $\mathbf{O}\{\sigma\} = \mathbf{O}\{\mathsf{B}, \mathbf{u} \approx \mathbf{v}'\}$, where $\mathsf{B} = \{\beta_1, \ldots, \beta_s\}$ and $\mathbf{v}' = \mathbf{v}^{(s)}$. □

Proof of Lemma 4.5. Since ℓ in Lemma 4.7 is arbitrary, the result can be repeated so that $\mathbf{O}\{\sigma\} = \mathbf{O}\{\mathsf{B}^\sigma, \mathbf{u} \approx \mathbf{v}^\dagger\}$ for some set B^σ of identities from (4.1B) and some word $\mathbf{v}^\dagger$ of the form (4.4) such that for any ℓ, the ℓ-th primary stack products of $\mathbf{u}$ and $\mathbf{v}^\dagger$ are identical. The identity $\mathbf{u} \approx \mathbf{v}^\dagger$ is then trivial so that $\mathbf{O}\{\sigma\} = \mathbf{O}\{\mathsf{B}^\sigma\}$. □

4.3. Proof of Proposition 4.1. Let $\mathbf{V}$ be any noncommutative subvariety of $\mathbf{O}$. Then $\mathbf{V} = \mathbf{O}\{\Sigma\}$ for some set Σ of identities. To show that the identities in Σ can be chosen from (4.1), it suffices to show that if $\sigma : \mathbf{u} \approx \mathbf{v}$ is any identity in Σ, then $\mathbf{O}\{\sigma\} = \mathbf{O}\{\mathsf{A}^\sigma, \mathsf{B}^\sigma\}$ for some set A^σ of identities from (4.1A) and some set B^σ of identities from (4.1B). Since any group that satisfies the identity system (0) is commutative, the variety $\mathbf{V}$ is not generated by groups, whence $\mathrm{con}(\mathbf{u}) = \mathrm{con}(\mathbf{v})$.

Case 1. $\mathrm{sim}(\mathbf{u}) = \mathrm{sim}(\mathbf{v})$. Then the identity σ satisfies both (R2) and (R3). By Lemma 4.3, the words $\mathbf{u}$ and $\mathbf{v}$ can be chosen to be in canonical form so that (R1) holds and the identity σ is regular. It then follows from Lemmas 4.4 and 4.5 that $\mathbf{O}\{\sigma\} = \mathbf{O}\{\mathsf{A}^\sigma, \mathsf{B}^\sigma\}$ for some finite set A^σ of identities from (4.1A) and some finite set B^σ of identities from (4.1B).

Case 2. $\mathrm{sim}(\mathbf{u}) \neq \mathrm{sim}(\mathbf{v})$. Then the identity σ implies the identity $x^k \approx x$ for some $k \geqslant 2$. Let $\mathbf{u}'$ be the word obtained by replacing every simple letter x in $\mathbf{u}$ with x^k, and let $\mathbf{v}'$ be the word similarly obtained from $\mathbf{v}$. Then $\mathbf{O}\{\sigma\} = \mathbf{O}\{x^k \approx x, \sigma'\}$, where the identity $\sigma' \colon \mathbf{u}' \approx \mathbf{v}'$ satisfies $\mathrm{sim}(\mathbf{u}') = \mathrm{sim}(\mathbf{v}')$ and the identity $x^k \approx x$ belongs to (4.1A). It follows from Case 1 that $\mathbf{O}\{\sigma'\} = \mathbf{O}\{\mathsf{A}^{\sigma'}, \mathsf{B}^{\sigma'}\}$ for some finite set $\mathsf{A}^{\sigma'}$ of identities from (4.1A) and some finite set $\mathsf{B}^{\sigma'}$ of identities from (4.1B). Hence $\mathbf{O}\{\sigma\} = \mathbf{O}\{\mathsf{A}^\sigma, \mathsf{B}^\sigma\}$, where $\mathsf{A}^\sigma = \{x^k \approx x\} \cup \mathsf{A}^{\sigma'}$ and $\mathsf{B}^\sigma = \mathsf{B}^{\sigma'}$.

5. Finite Basis Property for Subvarieties of $\mathbf{O}$

Lemma 5.1 (Head [11]). *Every variety of commutative monoids is finitely based.*

Lemma 5.2 (Volkov [33, Corollary 2]). *Any set of identities from* (4.1A) *defines a finitely based variety.*

The main aim of the present section is to establish the finite basis property of every subvariety of $\mathbf{O}$. Let $\mathbf{V}$ be any proper subvariety of $\mathbf{O}$. By Lemma 5.1, the variety $\mathbf{V}$ can be assumed noncommutative. By Proposition 4.1, there exists some set Σ of identities from (4.1) such that $\mathbf{V} = \mathbf{O}\{\Sigma\}$. By Lemma 5.2, it suffices to show that any set of identities from (4.1B) defines a finitely based subvariety of $\mathbf{O}$. This result is established in Proposition 5.7.

A *quasi-order* on a set X is a binary relation $\leqslant$ on X that is reflexive and transitive; in this case, $(X, \leqslant)$ is said to be a *quasi-ordered set*. The *direct product* of two quasi-ordered sets $(X', \leqslant')$ and $(X'', \leqslant'')$ is $(X' \times X'', \leqslant)$, where $\leqslant\, = \,\leqslant' \times \leqslant''$ is given by $(a', a'') \leqslant (b', b'')$ if $a' \leqslant' b'$ and $a'' \leqslant'' b''$.

A quasi-order on a set X is a *well-quasi-order* if any nonempty subset Y of X contains finitely positively many elements minimal in Y.

Lemma 5.3 (Higman [12]). *The direct product of two well-quasi-ordered sets is well-quasi-ordered.*

In this section, let $\leqslant$ denote the usual ordering on the set $\mathbb{N}_0$ of nonnegative integers. In what follows, since finite sequences of symbols from the direct product $\mathbb{N}_0^2 = \mathbb{N}_0 \times \mathbb{N}_0$ will be involved, notation will become less cumbersome if each element (e, f) of $\mathbb{N}_0^2$ is abbreviated by ef. For instance the order $\leqslant_2\, = \,\leqslant \times \leqslant$ on $\mathbb{N}_0^2$ is given by $ef \leqslant_2 pq$ if $e \leqslant p$ and $f \leqslant q$. It follows from Lemma 5.3 that $(\mathbb{N}_0^2, \leqslant_2)$ is a well-quasi-ordered set.

Lemma 5.4. *Let*

$$\varepsilon\colon\ x^{e_0} y^{f_0} \prod_{i=1}^{r} (h_i x^{e_i} y^{f_i}) \approx y^{f_0} x^{e_0} \prod_{i=1}^{r} (h_i x^{e_i} y^{f_i}),$$

$$\pi\colon\ x^{p_0} y^{q_0} \prod_{i=1}^{s} (h_i x^{p_i} y^{q_i}) \approx y^{q_0} x^{p_0} \prod_{i=1}^{s} (h_i x^{p_i} y^{q_i})$$

be any identities from (4.1B). *Suppose that*

(a) $e_0 f_0 \leqslant_2 p_0 q_0$,

(b) $r \leqslant s$ *and there exists a subsequence* $j_1, \ldots, j_r$ *of* $1, \ldots, s$ *such that*

$$e_1 f_1 \leqslant_2 p_{j_1} q_{j_1}, \ \ldots, \ e_r f_r \leqslant_2 p_{j_r} q_{j_r}.$$

Then the inclusion $\mathbf{O}\{\varepsilon\} \subseteq \mathbf{O}\{\pi\}$ *holds.*

Proof. It follows from (a) that $p_0 - e_0, q_0 - f_0 \in \mathbb{N}_0$. First suppose that $r = 0$ so that the identity ε is simply $x^{e_0} y^{f_0} \approx y^{f_0} x^{e_0}$. Since

$$x^{p_0} y^{q_0} \overset{(0)}{\approx} x^{e_0} y^{f_0} x^{p_0 - e_0} y^{q_0 - f_0} \overset{\varepsilon}{\approx} y^{f_0} x^{e_0} x^{p_0 - e_0} y^{q_0 - f_0} \overset{(0)}{\approx} y^{q_0} x^{p_0},$$

the identities (0) and ε imply the identity $x^{p_0} y^{q_0} \approx y^{q_0} x^{p_0}$, which in turn implies the identity π. Hence the inclusion $\mathbf{O}\{\varepsilon\} \subseteq \mathbf{O}\{\pi\}$ holds.

Now suppose that $r \geqslant 1$. Let φ be the substitution $h_1 \mapsto x^{p_0-e_0}y^{q_0-f_0}h_1$. Then

$$\begin{aligned}
x^{p_0}y^{q_0}\prod_{i=1}^{r}(h_i x^{e_i}y^{f_i}) &\overset{(0)}{\approx} x^{e_0}y^{f_0}x^{p_0-e_0}y^{q_0-f_0}\prod_{i=1}^{r}(h_i x^{e_i}y^{f_i}) \\
&= \left(x^{e_0}y^{f_0}\prod_{i=1}^{r}(h_i x^{e_i}y^{f_i})\right)\varphi \\
&\overset{\varepsilon}{\approx} \left(y^{f_0}x^{e_0}\prod_{i=1}^{r}(h_i x^{e_i}y^{f_i})\right)\varphi \\
&= y^{f_0}x^{e_0}x^{p_0-e_0}y^{q_0-f_0}\prod_{i=1}^{r}(h_i x^{e_i}y^{f_i}) \\
&\overset{(0)}{\approx} y^{q_0}x^{p_0}\prod_{i=1}^{r}(h_i x^{e_i}y^{f_i}).
\end{aligned}$$

Hence the identities (0) and ε imply the identity

$$\varepsilon' \colon x^{p_0}y^{q_0}\prod_{i=1}^{r}(h_i x^{e_i}y^{f_i}) \approx y^{q_0}x^{p_0}\prod_{i=1}^{r}(h_i x^{e_i}y^{f_i}).$$

By (b), it is easily seen that there is a substitution ψ that fixes both x and y such that $\big(\prod_{i=1}^{r}(h_i x^{e_i}y^{f_i})\big)\psi$ is a prefix of $\prod_{i=1}^{s}(h_i x^{p_i}y^{q_i})$, that is,

$$\left[\left(\prod_{i=1}^{r}(h_i x^{e_i}y^{f_i})\right)\psi\right]\mathbf{t} = \prod_{i=1}^{s}(h_i x^{p_i}y^{q_i})$$

for some $\mathbf{t} \in \mathcal{X}^*$. Then

$$\begin{aligned}
x^{p_0}y^{q_0}\prod_{i=1}^{s}(h_i x^{p_i}y^{q_i}) &= \left[\left(x^{p_0}y^{q_0}\prod_{i=1}^{r}(h_i x^{e_i}y^{f_i})\right)\psi\right]\mathbf{t} \\
&\overset{\varepsilon'}{\approx} \left[\left(y^{q_0}x^{p_0}\prod_{i=1}^{r}(h_i x^{e_i}y^{f_i})\right)\psi\right]\mathbf{t} \\
&= y^{q_0}x^{p_0}\prod_{i=1}^{s}(h_i x^{p_i}y^{q_i}).
\end{aligned}$$

Therefore the identity ε' implies the identity π, whence the inclusion $\mathbf{O}\{\varepsilon\} \subseteq \mathbf{O}\{\pi\}$ holds. $\square$

Let $(\mathbb{N}_0^2)^*$ denote the set of all finite sequences of symbols from $\mathbb{N}_0^2$. For any two sequences $\mathbf{e} = (e_1f_1, \ldots, e_rf_r)$ and $\mathbf{p} = (p_1q_1, \ldots, p_sq_s)$ in $(\mathbb{N}_0^2)^*$, define $\mathbf{e} \leqslant_2^* \mathbf{p}$ if there exists a subsequence $j_1, \ldots, j_r$ of $1, \ldots, s$ such that

$$e_1f_1 \leqslant_2 p_{j_1}q_{j_1}\ , \ \ldots, \ e_rf_r \leqslant_2 p_{j_r}q_{j_r}.$$

Since $(\mathbb{N}_0^2, \leqslant_2)$ is a well-quasi-ordered set, it follows from the well-known Higman's lemma [12, Theorem 4.3] that $((\mathbb{N}_0^2)^*, \leqslant_2^*)$ is also a well-quasi-ordered set.

Lemma 5.5. *The direct product* $(\mathbb{N}_0^2 \times (\mathbb{N}_0^2)^*, \preccurlyeq)$, *where* $\preccurlyeq\ = \ \leqslant_2 \times \leqslant_2^*$, *is a well-quasi-ordered set.*

Proof. This follows from Lemma 5.3 since $(\mathbb{N}_0^2, \leqslant_2)$ and $((\mathbb{N}_0^2)^*, \leqslant_2^*)$ are well-quasi-ordered sets. $\square$

Now associate each identity

$$\varepsilon\colon x^{e_0}y^{f_0}\prod_{i=1}^{r}(h_i x^{e_i}y^{f_i}) \approx y^{f_0}x^{e_0}\prod_{i=1}^{r}(h_i x^{e_i}y^{f_i})$$

from (4.1B) with the element $\vec{\varepsilon} = (e_0 f_0, (e_1 f_1, \ldots, e_r f_r))$ in $\mathbb{N}_0^2 \times (\mathbb{N}_0^2)^*$. Then the following result is a consequence of Lemma 5.4

Lemma 5.6. *Let ε and π be any identities from* (4.1B). *If $\vec{\varepsilon} \preccurlyeq \vec{\pi}$, then the inclusion $\mathbf{O}\{\varepsilon\} \subseteq \mathbf{O}\{\pi\}$ holds.*

Proposition 5.7. *Let B be any set of identities from* (4.1B). *Then the variety $\mathbf{O}\{\mathsf{B}\}$ is finitely based.*

Proof. Since $\left(\mathbb{N}_0^2 \times (\mathbb{N}_0^2)^*, \preccurlyeq\right)$ is a well-quasi-ordered set by Lemma 5.5, the subset $\vec{\mathsf{B}} = \{\vec{\varepsilon} \mid \varepsilon \in \mathsf{B}\}$ of $\mathbb{N}_0^2 \times (\mathbb{N}_0^2)^*$ contains finitely positively many elements that are minimal in $\vec{\mathsf{B}}$. Let $\varepsilon_1, \ldots, \varepsilon_m \in \mathsf{B}$ be such that $\vec{\varepsilon}_1, \ldots, \vec{\varepsilon}_m$ are all the elements minimal in $\vec{\mathsf{B}}$. The inclusion $\mathbf{O}\{\mathsf{B}\} \subseteq \mathbf{O}\{\varepsilon_1, \ldots, \varepsilon_m\}$ holds vacuously. If $\pi \in \mathsf{B}$, then $\vec{\varepsilon}_i \preccurlyeq \vec{\pi}$ for some i so that by Lemma 5.6, the identity π is satisfied by the variety $\mathbf{O}\{\varepsilon_i\}$. The inclusion $\mathbf{O}\{\varepsilon_1, \ldots, \varepsilon_m\} \subseteq \mathbf{O}\{\mathsf{B}\}$ thus follows. $\square$

References

[1] A. Y. Belov, *On non-Specht varieties*, Fundam. Prikl. Mat. **5** (1999), no. 1, 47–66 (Russian). MR 1799544

[2] A. P. Birjukov, *Varieties of idempotent semigroups*, Algebra i Logika **9** (1970), 255–273 (Russian). MR 0297897. English translation in Algebra and Logic **9** (1970), 153–164.

[3] R. M. Bryant and M. F. Newman, *Some finitely based varieties of groups*, Proc. London Math. Soc. (3) **28** (1974), 237–252. MR 0340422

[4] S. Burris and H. P. Sankappanavar, *A course in universal algebra*, Graduate Texts in Mathematics, vol. 78, Springer-Verlag, New York, 1981. MR 648287

[5] S. Crvenković, I. Dolinka, and V. Tasić, *A locally finite variety of rings with an undecidable equational theory*, Q. J. Math. **57** (2006), no. 3, 297–307. MR 2253588

[6] C. Fennemore, *All varieties of bands*, Semigroup Forum **1** (1970), no. 2, 172–179. MR 0271257

[7] J. A. Gerhard, *The lattice of equational classes of idempotent semigroups*, J. Algebra **15** (1970), 195–224. MR 0263953

[8] A. V. Grishin, *Examples of T-spaces and T-ideals of characteristic 2 without the finite basis property*, Fundam. Prikl. Mat. **5** (1999), no. 1, 101–118 (Russian). MR 1799541

[9] C. K. Gupta and A. Krasilnikov, *The finite basis question for varieties of groups—some recent results*, Illinois J. Math. **47** (2003), no. 1-2, 273–283. MR 2031321.

[10] C. K. Gupta and A. N. Krasilnikov, *A non-finitely based system of polynomial identities which contains the identity $x^6 = 0$*, Q. J. Math. **53** (2002), no. 2, 173–183. MR 1909509

[11] T. J. Head, *The varieties of commutative monoids*, Nieuw Arch. Wisk. (3) **16** (1968), 203–206. MR 0237696

[12] G. Higman, *Ordering by divisibility in abstract algebras*, Proc. London Math. Soc. (3) **2** (1952), 326–336. MR 0049867

[13] M. Jackson, *Finite semigroups whose varieties have uncountably many subvarieties*, J. Algebra **228** (2000), no. 2, 512–535. MR 1764577

[14] M. Jackson, *On the finite basis problem for finite Rees quotients of free monoids*, Acta Sci. Math. (Szeged) **67** (2001), no. 1-2, 121–159. MR 1830137

[15] M. Jackson, *Finiteness properties of varieties and the restriction to finite algebras*, Semigroup Forum **70** (2005), no. 2, 159–187. MR 2129773

[16] M. Jackson and O. Sapir, *Finitely based, finite sets of words*, Internat. J. Algebra Comput. **10** (2000), no. 6, 683–708. MR 1809378

[17] A. R. Kemer, *Solution of the problem as to whether associative algebras have a finite basis of identities*, Dokl. Akad. Nauk SSSR **298** (1988), no. 2, 273–277 (Russian). MR 937115. English translation in Soviet Math. Dokl. **37** (1988), no. 1, 60–64.

[18] P. A. Kozhevnikov, *Varieties of groups of prime exponent and identities with high powers*, Ph.D. thesis, Moscow State University, 2000.

[19] E. W. H. Lee, *Varieties generated by* 2*-testable monoids*, to appear in Studia Sci. Math. Hungar.

[20] E. W. H. Lee, *On the variety generated by some monoid of order five*, Acta Sci. Math. (Szeged) **74** (2008), no. 3-4, 509–537. MR 2487929

[21] E. W. H. Lee, *Finitely generated limit varieties of aperiodic monoids with central idempotents*, J. Algebra Appl. **8** (2009), no. 6, 779–796. MR 2597280

[22] E. W. H. Lee, *Finite basis problem for 2-testable monoids*, Cent. Eur. J. Math. **9** (2011), no. 1, 1–22. MR 2753878

[23] E. W. H. Lee and M. V. Volkov, *Limit varieties generated by completely 0-simple semigroups*, Internat. J. Algebra Comput. **21** (2011), no. 1-2, 257–294. MR 2787461

[24] P. Perkins, *Bases for equational theories of semigroups*, J. Algebra **11** (1969), 298–314. MR 0233911

[25] G. Pollák, *On two classes of hereditarily finitely based semigroup identities*, Semigroup Forum **25** (1982), no. 1–2, 9–33. MR 663167

[26] G. Pollák, *A new example of limit variety*, Semigroup Forum **38** (1989), no. 3, 283–303. MR 982010

[27] M. V. Sapir, *On Cross semigroup varieties and related questions*, Semigroup Forum **42** (1991), no. 3, 345–364. MR 1092964

[28] O. Sapir, *Finitely based words*, Internat. J. Algebra Comput. **10** (2000), no. 4, 457–480. MR 1776051

[29] V. V. Shchigolev, *Examples of infinitely based T-ideals*, Fundam. Prikl. Mat. **5** (1999), no. 1, 307–312. MR 1799533

[30] L. N. Shevrin and M. V. Volkov, *Identities of semigroups*, Izv. Vyssh. Uchebn. Zaved. Mat. (1985), no. 11, 3–47, 85 (Russian). MR 829099. English translation in Soviet Math. (Iz. VUZ) **29** (1985), no. 11, 1–64.

[31] W. Specht, *Gesetze in Ringen. I*, Math. Z. **52** (1950), 557–589. MR 0035274

[32] M. V. Volkov, *An example of a limit variety of semigroups*, Semigroup Forum **24** (1982), no. 4, 319–326. MR 657909

[33] M. V. Volkov, *A general finite basis condition for systems of semigroup identities*, Semigroup Forum **41** (1990), no. 2, 181–191. MR 1057589

[34] M. V. Volkov, *The finite basis problem for finite semigroups*, Sci. Math. Jpn. **53** (2001), no. 1, 171–199. MR 1821612

Division of Math, Science, and Technology, Nova Southeastern University, Fort Lauderdale, Florida 33314, USA

Email address: `edmond.lee@nova.edu`