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Students' Understanding Of Quadratic Functions: Learning From Students' Voices

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STUDENTS' UNDERSTANDING OF QUADRATIC FUNCTIONS:
LEARNING FROM STUDENTS' VOICES

A Dissertation Presented

by

Jennifer Suzanne Stokes Parent

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Dissertation Examination Committee

Regina Toolin, Ph.D., Advisor
Susan Comerford, Ph.D., Chairperson
Sean M. Hurley, Ph.D.
Alan S. Tinkler, Ph.D.
Cynthia J. Forehand, Ph.D., Dean of the Graduate College

Abstract

The objective of this multiple case study was to examine how three pairs of high school students from a northern Vermont high school approached quadratic functions through traditional and multiple representation tasks. Four research questions were examined: 1) How do students think about the quadratic function as they work on a series of tasks? 2) What mathematical strategies do students employ when they work on a series of tasks related to the quadratic function? 3) How does the type of task, traditional versus multiple representation, impact students' understanding of the quadratic function? 4) What kinds of knowledge (procedural or conceptual) do students utilize when completing a series of tasks about the quadratic function? Qualitative research methods that utilized think-aloud protocols while students were engaged in four tasks pertaining to the quadratic function were employed in this study.

Results suggested that students tend to think about isolated parts of the problem when solving quadratic problems. Early on in their learning about quadratics, students primarily relied on procedural strategies such as think-alouds, gestures, algebraic formulas, converting equation forms, process of elimination, dissecting problems, backtracking, and drawing pictures. In addition, students preferred the standard form to the vertex form when solving quadratics and often confused the y-intercept of the standard form with the y-coordinate of the vertex when the function was in vertex form. Results also indicated that students preferred to algebraically solve a problem versus tabular or graphical strategies. By exploring how students approach the quadratic function through their own voices, this study offers some insight into the conceptions and strategies that students use for solving problems that involve the quadratic function as well as possibilities for how quadratics may be taught in high school.

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Chapter One: Introduction

How students learn functions in mathematics has been a topic of interest for many decades. As a teacher of mathematics for over 10 years, I have been particularly interested in not only *how* my students understand quadratic functions, but also *why* they choose certain strategies and procedures for solving quadratic functions. I was interested in researching the common misconceptions that students have about quadratic functions and the most effective teaching strategies that will help them understand quadratics more fully. What thought sequences are going through their heads when they are doing quadratics? I wanted to inquire into how students make sense and/or develop meaning and understanding about quadratic functions.

I. Vermont's Need for Improvement in Mathematics

Mathematics in Vermont is not holding strong. Since 2005 the New England Common Assessment Program (NECAP) tests have been administered to students in their junior year of high school in New Hampshire, Rhode Island, Maine, and Vermont. The results of the NECAP tests are used for school improvement and accountability under No Child Left Behind (No Child Left Behind (NCLB) Act of 2001, 2002). NECAPs are designed to measure students' level of understanding of the Vermont grade expectations during grade 11, as well as the standards in Maine, New Hampshire and Rhode Island. The test designers, Measured Progress, are contracted to design a test that measures the student achievement of the common grade level expectations (GEs) and standards from the states involved in one test. The test items are also broken down by degree of difficulty or depth, known as the Depth of Knowledge indicators. The Mathematics

NECAP consists of multiple choice and short answer questions. In high school, the

students are tested in the fall of their junior year (grade 11); therefore the test is intended

to measure student achievement on the mathematics standards learned between grades 8

and 10. Essentially students should have completed a rigorous Algebra I and Geometry

course by the beginning of their junior year in high school in order to meet proficiency or

higher on the exam. Proficient with Distinction has a scaled score of 1152-1180, while

Proficient scaled score is 1140-1151, Partially Proficient has a scaled score of 1134-1139,

and lastly, Substantially Below Proficient scaled score is 1100-1133 (NECAP, 2013-

2014, p. 5). Simply one question can determine whether or not a student is deemed

proficient or partially proficient. From Fall of 2009 to Fall of 2012, NECAP mathematics

scores (proficient or proficient with distinction) for high school juniors has ranged from

35-38% proficient (Vermont State Department of Education, 2011-2012, p. 5). These

findings suggest that the state, as a whole, is doing poorly in understanding essential math

concepts amongst high school juniors (See Table 1). Something needs to be done if only

35% of the high school juniors are being deemed proficient in a particular core content

area. "No matter the psychological or socioeconomic reasons, poor mathematical ability

has serious consequences, and as educators we must address the question of why so many

students are failing" (Jones, Hopper, & Franz, 2008, p. 307).

Table 1

Vermont NECAP Grade 11 Mathematics Results

Academic Year	Tested	Level 4 – Proficiency With Distinction		Level 3 – Proficiency		Level 2 – Partially Proficient		Level 1 – Substantially Below Proficient	
	N	N	%	N	%	N	%	N	%
Vermont 2009-2010	6,732	193	3	2,173	32	1,868	28	2,498	37
Study School 2009-2010	245	5	2	94	38	52	21	94	38
Vermont 2010-2011	6,830	190	3	2,399	35	1,754	26	2,487	36
Study School 2010-2011	251	7	3	81	31	70	28	93	37
Vermont 2011-2012	6,408	212	3	2,118	33	1,561	24	2,517	39
Study School 2011-2012	262	4	2	75	29	69	26	114	44
Vermont 2012-2013	6,426	223	3	2,240	35	1,545	24	2,418	38
Study School 2012-2013	260	5	2	87	33	57	22	111	43
Vermont Cumulative total	26,396	818	3	8,930	34	6,728	25	9,920	38
Study School Cumulative total	1,018	21	2	337	33	248	24	412	40

The NECAPs are a method that allows educators to assess the ability of the students at their particular school. One of the topics that are assessed in the Grade 11 NECAPs is the quadratic function. Quadratic functions are a core entity within the Algebra 2 curriculum, which leads to pre-calculus, and eventually calculus.

> Within the high school curriculum, the highest level of mathematics a
> student has studied has the strongest effect on degree upon completion.
> Finishing a course beyond the level of Algebra 2 (for example
> trigonometry or pre-calculus) more than doubles the odds that a student
> who enters college will complete a bachelor's degree. (Singham, 2003, p.
> 507)

Therefore, if knowledge gained from this study allows me, and teachers in general, to eventually "piecemeal" together a more accurate blend between traditional tasks and multiple representations, and have a better sense of where students are not conceptually understanding the quadratic in addition to procedural errors, then this will have been a worthy study.

The study school is in Northern Vermont and has a population of roughly 1150 students in grades 9-12. It is a predominantly Caucasian English speaking school which obtains its students from nine various feeder schools.

NECAP scores for individual students are provided to the school as well as the school average. NECAP test results can help teachers, as well as administrators, formulate plans for school improvement as the school can identify students who did not achieve proficiency and work towards increasing skills and math competency for students in subsequent years. This is particularly true in grades 3 through 8 as teachers can follow

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the cohort and can attempt to close the achievement gap. However, after eighth grade

there is no testing for two years and with students taking different levels of mathematics,

following the cohort is less successful. However, high schools can observe which

students attain math proficiency, examine the courses they followed and work to improve

achievement for subsequent cohorts. Inherent in this model is the necessity of teachers

using informed practice while teaching, presenting similar materials and administering

common local assessments, without which it is difficult to see gaps and strengths in the

methods and courses. If possible, having horizontal and vertical alignment of the

curricula would be helpful as well.

Also problematic is that the public, when observing scores for a fifth grade in

2010, and then seeing the score for a fifth grade in 2011 at the same school, may falsely

compare the two years seeing improvement or a lack thereof. The same cohort is not

being observed in the press. It would be more fruitful to follow the same cohort to see if

school improvement is working.

Over the past eight years, NECAP scores have helped to identify students' ability

to achieve standards in each state. In spring of 2015, students in their junior year of high

school were required to take a new test that is aligned with the Common Core Standards.

These standards have been developed so that all of the states that adopt them will be able

to have a "united effort" in preparing students for their future academic schooling. Since

2012, 45 states, including Vermont, have adopted the Common Core Standards. One

goal is to allow these states to be on the same playing field when it comes to assessing

students. Prior to the Common Core Standards, states could have their own standards and

assessments without being held accountable to the state next door, let alone a state across

the nation. The common scores or levels of achievement will now be a comparison at the same proficiency level. In other words, when a student is proclaimed proficient in one of these 45 states, they should in turn be proclaimed proficient in the other 44 states as well.

It is interesting to note that the Common Core Mathematics Standards has a domain specific to Functions including: Interpreting Functions, Building Functions, Linear, Quadratic, & Exponential Models, and Trigonometric Functions (National Governors Association Center for Best Practices, 2010). The Common Core in mathematics is generally different from the current standards for most states in the following ways: 1) mathematics in the elementary grades has a greater focus on standard operations and much less on statistics and data collection; 2) mathematics in the middle grades tends to use more application; and 3) in high school, modeling is heavily emphasized. Hopefully with students having a stronger base in operations and measurement it will allow for quicker retrieval of procedural knowledge and the ability to apply the operations and concepts of measurement to problems solving and modeling. Based on the various states previously each having their own set of state standards, there is not one manner in which to write standards. Although, with the adaptation of the Common Core Standards, it seems that time is of the essence to truly get to the heart of what is working for students and what is not working when being introduced to quadratic functions.

II. Purpose

In high school, students are introduced to higher levels of mathematics. They meet new classes of functions, additional geometric perspectives, and different ways of analyzing data. Mathematical form and structure begin to connect in ways that did not

occur to students in the lower grades. One connection is that all quadratic functions share

certain properties, as do all functions of other classes – linear, periodic, or exponential.

The National Council of Teachers of Mathematics (2000) states that students need to

"learn to use a wide range of explicitly and recursively defined functions to model the

world around them. Moreover, their understanding of the properties of those functions

will give them insights into the phenomena being modeled" (p. 288).

Learning algebra in high school should help all students to "come to understand

the concept of a class of functions and learn to recognize the characteristics of various

classes" (National Council of Teachers of Mathematics, 2000, p. 297). The overall goal

and purpose of high school mathematics is to give students the capability to explore and

solve common issues in the real world. With this point it is important to learn quadratics

because aspects of the quadratic function are used later on in higher mathematics classes,

especially when dealing with higher polynomial functions (Afamasaga-Fuata'i, 1992;

Curran, 1995), as well as being in students' lives once they leave high school. Although

not restricted to these examples, quadratics relate to the mathematical thinking and

reasoning in the real world due to being involved in describing the paths of projectiles

(Brown, Breunlin, Wiltjer, Degner, Eddins, & Edwards, 2007; Center, 2012), appearing

on suspension bridges, being the cross-section of automobile headlights, satellite dishes,

and radio telescopes (Brown et al., 2007), being used by the military when predicting

where artillery shells will hit the earth (Center, 2012), to describe the orbits along which

the planets move, and the link between quadratic equations and acceleration (Budd &

Sangwin, 2004). In business, quadratics can be used to maximize profit. More than that

though, functions (primarily linear and quadratic) are the first place where students learn

to solve complex higher order thinking problems that involve graphical attributes, such as slope, rather than simply answer problems. By learning the properties and behaviors of quadratic (and in turn eventually polynomial) functions, students learn to look at details and how to formulate the necessary solution(s). This prowess is used in everyday life.

As previously stated, the quadratic function and functions in general are key components to Algebra I and II. For this research study I was interested in investigating what helps and/or hinders students' understanding of quadratic functions when simply given tasks to complete without the influence of a teacher or coach. What approaches and strategies did students utilize when the teacher's approach and manner of speaking is not an influence and the students are left with only the material? Specific aspects of traditional tasks may support learning for the students in this study. On the other hand, specific aspects of multiple representations tasks, such as an explicit representation, may support students as well.

By researching the conceptions and misconceptions that students have about quadratics while attempting to independently solve quadratic problems, I hope to better understand their mathematical thinking and abilities and be better able to address these conceptions in my teaching practice. I aspired to find justification along the process behind traditional and/or multiple representations tasks to emphasize in the future when teaching quadratic functions. The purpose of this study was not to introduce a new method of teaching quadratic functions, but rather to investigate the advantages and disadvantages of how it is already being taught to students. By recording and analyzing student mathematical thinking, the goal is to reveal what the students are thinking as they approach quadratic functions. To be able to expose the nature of their conceptual and

procedural knowledge of quadratic functions could possibly help when planning future curriculum.

III. The Breakdown of the Quadratic Function

The concept of function has various formal and informal definitions. For the purpose of this study, function is going to be described by how it is defined in the Holt Algebra 2 book (Burger, Chard, Hall, & Kennedy, 2007). This book is used in two out of the three Algebra 2 courses taught at the study school. Function is described as, "A relation in which the first coordinate is never repeated. There is only one output for each input, so each element of the domain is mapped to exactly one element in the range" (Burger et al., p. 45). In other words, for every x value there can only be one y value. If there was the coordinate point (2, 1) on a function, there could not also be the point (2, 7) due to the fact that the input x-value 2 has more than one output y-value. Although the concept of function can go much more in depth, for the purpose of this study, function is narrowed down to quadratics.

Quadratic functions are most commonly defined in standard form as $f(x) = ax^2 + bx + c$ when $a \neq 0$. There are different ways to approach the quadratic function. The quadratic function can also be expressed in factored form as $f(x) = a(x-x_1)(x-x_2)$ and vertex form $f(x) = a(x-h)^2 + k$. For this study, quadratic functions will be seen in the standard and vertex forms. This is due to not addressing factoring in this particular study.
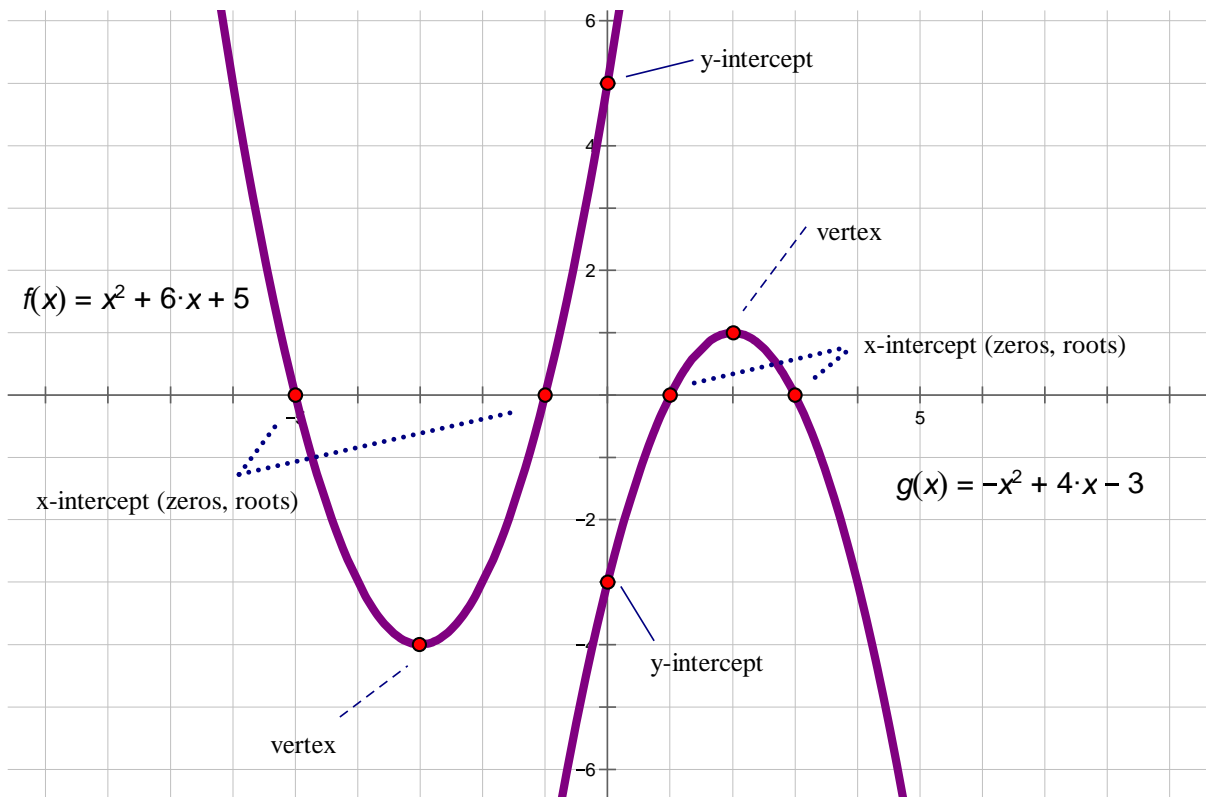
Figure 1: Quadratic Functions Graph

The graph of a quadratic function is called a parabola, which is recognized for its U-shaped formation (see Figure 1: Quadratic Functions Graph). The quadratic function is called the quadratic equation when the standard form is set equal to zero which gives the format of $ax^2 + bx + c = 0$. The "solutions" of the equation is when "x" is solved. This can be done through completing the square, the quadratic formula, factoring, and depending on the equation, graphing. The "solutions" are called the roots, zeros, or x-intercepts, and are essentially when the graph intervenes with the x-axis. The vertex of the parabola is the turning point on the graph. This would be the highest point on the U-shaped graph if the quadratic was negative and opening down, versus when the lowest point on the graph of the U-shaped graph was positive and opening upwards. One of the

most crucial components in understanding the quadratic function is for students to learn how to read the graph. By learning how to read the graph and the different components of the graph, they will heighten their awareness of functions in general. This will be helpful when they move on to pre-calculus and calculus. By understanding the quadratic function and finding the minimums, maximums, and limits, where a function increases versus decreases and when a function is positive versus negative, it should help them when being introduced to higher order polynomials.

IV. Potential Significance of the Study

Teaching involves three basic operations: careful observations of students and diagnosis of individual needs and interests; setting up the physical, social, emotional, and intellectual environment in which the students can learn; and facilitating students' growth by intervening between them and their environment (Schiro, 2008, p. 109). By being able to draw on students' previous knowledge and incorporate the manner in which they approach new problems – with a particular focus on quadratic functions in this study – I hoped to create learning experiences through the various tasks that the students were active in that would deepen the context in which students understand quadratic functions. In this study, this will be demonstrated during the recorded conversations as students work in pairs to solve quadratic problems.

This study extends the research on student learning in the areas of functions and graphs, particularly those related to the quadratic function. This study has the potential to contribute to the mathematical education community's knowledge base on how students develop a conceptual understanding of the graph of the quadratic function. Results from

this research may inform further questions about student understanding of functions and graphs, and hence, will hopefully influence future research in this domain.

Results of this study will serve to influence future classroom instruction in the area of quadratic functions and graphing. Ideally, changes in pedagogy and curriculum units can be linked to knowledge obtained about the development of students' understandings of the quadratic function. Teachers, who are interested in using research, can provide their classrooms, and schools, with informed best practices. If a teacher is willing to look outside their current classroom curricula, they can maximize their use of time within the classroom.

V. Research Statement/Question and Sub Questions

The objective of this research study is to examine how six high school students approach and understand quadratic functions through traditional and/or multiple representation tasks. Qualitative research methodologies that utilize think aloud protocols while students were engaged in either traditional and/or multiple representation tasks were employed. The idea was to develop cases for pairs of students that would ultimately demonstrate the nature of students' conceptual and procedural knowledge of quadratic functions; what they are thinking about as they are approaching problems that involve the quadratic function. The primary research questions that this study addressed are:

1) How do students think about the quadratic function as they work on a series of tasks?

2) What mathematical strategies do students employ when they work on a series of tasks dealing with the quadratic function?

3) How does the type of task, traditional versus multiple representation, impact

    students understanding of the quadratic function?

4) What kinds of knowledge (procedural or conceptual) do students utilize when

    completing a series of tasks about the quadratic function?

Chapter Two: Literature Review

Functions in general are one of the most important topics in all of mathematics (Cooney & Wilson, 1993; Dreyfus & Eisenberg, 1984; Romberg, Carpenter, & Fennema, 1993; Zaslavsky, 1997). In fact, "In the last century much has been said about functions. Magazine articles, convention speeches, and some of the newer text material have devoted considerable effort to present, clearly, this important, unifying mathematical topic" (Hight, 1968, p. 575). Cooney and Wilson noted, "The emphasis on functions as a unifying mathematical concept, as a representation of real-world phenomena, and as an important mathematical structure remains central to contemporary discussions" (p. 144). However, there are many questions about learning quadratics that are still left unanswered. Although research has been conducted on functions in general (Afamasaga-Fuata'i, 1992; Clement, 1989; Dreyfus & Eisenberg, 1984; Eisenberg, 1991; Eisenberg & Dreyfus, 1994; Hartter, 2009; Hatisaru & Erbas, 2010; Hitt, 1998; Leinhardt, Zaslavsky, & Stein, 1990), about linear functions specifically (Chiu, Kessel, Moschkovich, & Munoz-Nunez, 2001; Knuth, 2000; Reiken, 2008), including both linear and quadratic functions in the study (Afamasaga-Fuata'i; Schorr, 2003), as well as functions that have a degree greater than two (Curran, 1995), the participants in the aforementioned studies have primarily been studied post learning the material (see Appendix A: Function Studies Matrix).

An interesting outcome arose when Dreyfus and Eisenberg (1984) were conducting a study with 127 seventh and eighth graders that focused on functions in general. They noticed that high ability students tended to solve problems using a

graphical approach while low ability students were attracted more to pictorial and tabular presentations of problems. In 2000 though, Eric Knuth, while conducting a 284 participant study with high school aged students that focused mainly on linear functions found the opposite. His results showed that the participants relied heavily on algebraic solutions versus graphical, that they seemed to have a ritualistic procedure for solving problems similar to those in the study, and that participants had difficulty when dealing with problems that were in the graph-to-equation direction. When Curran (1995) conducted a study with three upper division high school students in Northern New England, two of her findings were that all three students found describing graphs difficult and that a student's personality, motivation, and attributes play an important role in the degree to which the student will become engaged.

Research on teaching and learning quadratic functions (Didis, Bas, & Erbas; Ellis & Grinstead, 2008; Eraslan, 2008; Metcalf, 2007; Strickland, 2011; Vaiyavutjamai, Ellerton, & Clements; Zaslavsky, 1997) has involved students post learning about the specific function(s) being studied. One of these quadratic function studies, Metcalf (2007), was conducted with three undergraduate pre-calculus students at a New England State University. She found that one of her participants could perform several procedures, but showed limited relational understanding of the concepts. Unfortunately though, none of her participants showed much flexibility in moving between the representations. In addition to this, they all exhibited difficulties with communication dealing with the quadratic function.

Joseph Reiken (2008) investigated 16 high school ninth grade students when learning about slope and the Cartesian Connection while they were engaged in either

traditional or multiple representation tasks. Although his study was focused on specific attributes of the linear function, this research has influenced my study with respect to how students approach tasks when they are initially being introduced to specific mathematic concepts; in this case, the quadratic function. My goal in the study was to investigate what hinders/triggers understanding about the quadratic function from the students' perspective at the point of initial introduction. I did not want to have the students perform the tasks multiple chapters after being initially introduced to quadratic functions, but rather before they received too much follow-up on assessments that would alter their fundamental thoughts.

This literature review is divided into five sections. In the first section I offer a general overview of research on mathematical understanding. The second section follows with a discussion of the research related to the constructivist approaches to teaching and learning mathematics. In the third section, I build on understanding mathematics by relating the theories and research involving multiple representations. The fourth section is a discussion of the research about students' misconceptions and difficulties with functions and their multiple representations, and the fifth section concludes with a synopsis of the think-aloud method.

I. From Learning to Understanding: The Transfer of Knowledge

When a student is taught how to do something, it does not necessarily mean that they know how to do it on their own or apply this knowledge in different contexts. In addition, learning mathematics does not necessarily connote an understanding of mathematics. "Because of the complexity of the functional domain, it is difficult to describe exhaustively the constellation of procedural and conceptual understandings that

underlie competent performance" (Williams, 1993, p. 328). Simply put, procedural understanding is *how* to get something done; conceptual understanding is *why* things are being done. Procedural understandings can be used to solve a mathematical problem quickly and easily, especially as the procedure becomes more automatic. It is slimmer in its applicability though since it is hard to be reflected upon (Briars, 1982). In other words, it is difficult to change a tactic if you do not know why you are doing it in the first place, and do not know where it fits in the bigger schema of the concept. Conceptual understanding, though, allows one to revisit their process and modify it if necessary. There is an interplay that comes from these two understandings that are not necessarily mutually exclusive. Conceptual understanding prompts one to consider whether an answer makes sense. An individual can recall and develop, if necessary, procedural strategies from other existing strategies in their long-term memory (Kotsopoulos, 2007). Although there are obstacles in identifying and studying covert behavior, cognitive psychologists not only believe that it can happen, but that it should happen and that it is worth the effort (Lester, 1982). In other words, if someone continuously finds solutions to the same type of mathematical problem only by using procedural knowledge, without developing the conceptual knowledge for it, it is very easy for this to become habituated since it is getting positive results. Procedural knowledge can allow a student to pass a class, but conceptual knowledge combined with the procedural knowledge will allow the student to be prepared for the next mathematical level, as well as math literacy in the real world.

In the attempt to gain a more complete understanding of the cognitive processes in which students engage in functions and graphs representing functions, careful analysis

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of behavior on mathematical tasks will most certainly need to come into play (Williams, 1993). Once again, "One hears often the distinction between 'doing' and 'understanding.' It is a distinction applied to the case, for example, of a student who presumably understands a mathematical idea but does not know how to use it in computation" (Bruner, 1999, p. 29). For example, students may or may not recognize a function. On the occasion that they do understand, they may not have a complete understanding of all of the elements or be able to transfer the function between different representations of it – ordered pairs, table, equation, graph, etc. If a student only understands a particular form of function, due to that being the only one used in a course, that student will only retain that particular form. In this way, "The student unconsciously accepts the particular form as the definition" (Malik, 1980, p. 491) and is unknowingly blind to the other forms possible of function. "An important ingredient in understanding something is to know where it belongs in a larger scheme and to become familiar with its parts" (Haskell, 2001, p. 29). Knowing what kinds of tools (equations, functions, etc.) are available to a particular problem is important since they lead to one's depth of knowledge, as well as one's ability to successfully complete the problem. Why is there disconnect when it comes to transferring mathematics material from one problem to another? Especially since, as previously stated, functions are such a central entity to the content of Algebra.

With functions being such an important piece of the mathematical puzzle, it is important that students have the background knowledge to do the more mundane, or simplistic mathematical tasks when recognizing and solving functions. Daniel Willingham (2006) relates that, "Students with a rich base of factual knowledge find it

easier to learn more – the rich get richer" (p. 30). This is due to previous knowledge

enhancing thinking. Effective transfer of knowledge requires a sufficient degree of

original knowledge (Bransford & Schwartz, 1999). When a student is solving a problem

in math, space is freed up in their working (short-term) memory and they can focus on

the problem at hand versus the background processes necessary to get to the problem. In

other words, when a student's previous procedural knowledge is sound, it allows the

student to concentrate on the new material, instead of having to recall and sometimes

relearn the prerequisite material. For example, if students know their multiplication

tables, they can learn the concept of factoring polynomials to obtain the roots of a

function easier. Students, unlike computers, cannot simply input information and output

the correct answer when dealing with a new concept. Students must be engaged in

meaningful and contextualized learning experiences in order to retain deep understanding

of the subject matter at hand. In this way, the mathematical procedure is related to other

knowledge, and understood (Mayer, 1982). Rather than focusing on inputs and outputs,

the focus needs to be on the transformation of information and the processes by which

that occurs within the students' thinking (Briars, 1982). The transfer of knowledge

occurs when previous learning and experience is used in order to more quickly and

efficiently learn a new skill, or mathematics content (Haskell, 2001).

One process to examine the transfer of knowledge is to investigate schemas

present within the student's thinking. Simply said, a schema is a collection of memory

that comes to mind when a concept comes to be questioned. An example could be

"visiting the zoo." For a lot of people this pastime would possibly conjure up the vision

of buying tickets, seeing various classes of bear, going into a reptile building, and

perhaps helping to feed a seal. Having a schema come to mind helps "provide a prototypical description of the concept it represents that can be used to interpret a range of specific instances of that concept, and also to infer features of the concept that are not explicitly described" (Briars, 1982, p. 42).

Taking the time to have students conceptually understand, develop, and apply schemata for various functions would be worthwhile for the students to later be able to draw upon when answering various mathematical questions. Having a schema that is adaptable to the current problem at hand could in fact allow for a deeper understanding in the content as a whole and lead to quicker and more accurate analogical transfer (Novick & Holyoak, 1991). Hopefully, by gaining insight on what helps/hinders students from forming their own schema of a quadratic function, this study can enrich the current pool of knowledge surrounding functions in general.

II. Multiple Representations

Engaging in tasks involving multiple representations of a function may be a beneficial way to facilitate the connections between the different representations of the quadratic function. These connections are essential for understanding the various parts of the quadratic that will be explored with the students in this study. The National Council of Teachers of Mathematics (2000) recommends that high school students should be able to "create and use tabular, symbolic, graphical, and verbal representations and to analyze and understand patterns, relations and functions" (p. 297).

Working with different representations of the quadratic is one way to promote what has been called "flexible competence" by Moschkovich, Schoenfeld, and Arcavi (1993), which emphasizes conceptually understanding a domain rather than procedural

mastery. Many other researchers have commented on the importance of students being able to move back and forth between the various representations of each function at hand (Ellis & Grinstead, 2008; Knuth, 2000; Leinhardt et al., 1990). Flexible competence exhibits that the student has a strong conceptual knowledge base of the content and is not simply demonstrating short-term memory superficial procedures.

In a study that Knuth (2000) conducted with 284 high school students ranging from first year algebra through Advanced Placement calculus, he concluded that although "students often appear to understand connections between equations and graphs, particularly given the nature of the tasks that they typically encounter…their actual understanding of the connections is often superficial as best" (Knuth, p. 53). Knuth found that: 1) students relied heavily on algebraic solution methods versus graphical solution methods, even if the graphical would have been quicker; 2) students seemed to have developed a ritualistic procedure for solving problems similar to those in the study; and that 3) students may have difficulties dealing with the graph-to-equation direction of solving problems. These observations indicate that students are dependent on rote procedural understanding versus obtaining and using conceptual understanding. An important question about student conceptual understanding emerges – To what extent are students accessing conceptual knowledge when solving problems? More importantly, what is their conceptual knowledge and understanding?

Students often learn mathematics through textbook problems that all look the same. When students are asked to do other problems within the same domain, but that appear different, the students are lost with what to do (Schoenfeld, 1985b). Students who are asymmetrically stronger in procedural knowledge in a domain over conceptual

knowledge have a harder time transferring the knowledge versus those that are just as strong, if not stronger, in their conceptual understanding (Rittle-Johnson & Alibali, 1999). Conceptual knowledge should be generalizable; it should be flexible enough to flow between different problems within the same domain. By emphasizing conceptual understanding, a person can reconstruct a procedure that they may have forgotten. In other words, they have more to work with, not just a procedure (Schwartz, 2008). Procedural knowledge is at one end of the knowledge spectrum where superficial limitation is automatized and fully compiled, whereas conceptual knowledge is at the other end of the spectrum where the content is understood and easily transferable. On the procedural end, students have in their minds how to superficially solve similar looking problems without much thought going into the process. On the conceptual end, students are able to reassign their thoughts to other problems that may be in different formats, may be asking the question differently, or asking the student to go more in depth by not simply asking for a calculation, but by asking for an interpretation. There is, of course, every possible mixture of the two forms of knowledge as well. Having a greater conceptual knowledge allows the student to apply and adjust the procedure to fit the problem at hand (Alibali, 2005; Rittle-Johnson & Alibali, 1999; Star, 2000).

III. The Constructivist Approach

Constructivism is rooted in the educational perspective that learning occurs through experimental, real life experiences that construct and conditionalize knowledge over time. Although the "tasks" given in this study are not "experimental, real life experiences," a constructivist approach allows for students to develop a deeper conceptual knowledge of the domain. Afamasaga-Fuata'i (1992) indicated how the

constructivism paradigm is present when students take an active and participatory role in their own learning through meaning-making activities. In order to assist students in making meaning during a learning activity, it seems necessary to be deliberate in helping them connect what they are learning to their prior knowledge of the educational content. Prior knowledge of the subject can be dependent on many factors, one of which may be instruction. Instruction in many current classrooms emphasizes "knowing that" instead of "knowing how" to answer a proposed problem (Romberg, 1992). Often, "knowing how" in mathematics is linked with traditional instruction, a focus on memorization of rules and algorithms, and less of a focus on the development of conceptual understanding. Traditional instruction falls short in failing to take into account the social and cultural processes that each student brings to the classroom. It is important to acknowledge these processes and their influences over one's learning.

> Individual students are seen as actively contributing to the development of
> the classroom micro-culture that both allows and constrains their
> individual mathematical activities. This reflexive relation implies that
> neither an individual students' mathematical activity nor the classroom
> micro-culture can be adequately accounted for without considering the
> other. (Paul & Yackel, 1998, p. 161)

Learning does not occur in isolation. It is an experience dependent upon interaction; interaction with the learning activity, prior knowledge, and culture. It seems important, then, to design learning activities that facilitate interaction with others, dialog about mathematics, with the deliberate intention of leading students toward constructing mathematical meaning.

IV. Misconceptions about functions

Previous research has identified various misconceptions or issues that have blocked students' understanding of functions. In mathematics, "Misconceptions are identified as incorrect features of student knowledge that are repeatable and explicit" (Leinhardt et al., 1990, p. 30), but yet the misconception "must have a reasonably well-formulated system of ideas, not simply a justification for an error" (Zaslavsky, 1997, p. 5). In other words, the student has to have purposefully solved an answer while thinking that he or she was right the entire time. When Leinhardt and colleagues did a review of research and theory to teaching and learning domains, functions, graphs, and graphing for the age range of 9-14, they came up with eight subheadings of where misconceptions and difficulties arose. They found that students had misconceptions about: 1) what is and is not a function, 2) correspondence within a function, 3) over generalizing the properties of linear functions, 4) continuous versus discrete graphs, 5) various representations of the same function, 6) relative reading and interpretations, 7) the concept of variable within the equation, and 8) notation within the graph of a function itself.

While Leinhardt et al. (1990) looked at functions and their graphs in general for misconceptions and difficulties, some researchers look at specific functions due to where their focuses are in their investigations. Donna Kotsopoulos (2007) found that secondary students experience many difficulties when factoring quadratics. The difficulties arise due to students being challenged with having to recall basic multiplication facts.

> Given that the factoring of quadratics is the writing of polynomials as a product of polynomials, students need to have both a strong conceptual understanding of multiplication of polynomials as well as the procedural

knowledge to retrieve basic multiplication facts effectively. (Kotsopoulos,

p. 22)

As previously mentioned, there are three different forms of the quadratic function: the

standard form, factored form, and the vertex form. Kotsopoulos points out that students

get confused when quadratics are shown in variations of these forms and not exactly like

the students are used to seeing them. She gives the example of $x^2 + 3x + 1 = x + 4$ being

not in standard form and causing students trouble when asked to perform various tasks

with it.

It has been generalized by Ellis and Grinstead (2008) that when working with

quadratic functions, students' issues mainly appear with 1) connections between

algebraic, tabular, and graphical representations, 2) a view of graphs as whole objects, 3)

struggles to correctly interpret the role of parameters, and 4) a tendency to incorrectly

generalize from linear functions. They found difficulties with connections between

algebraic and graphical representations of quadratic functions. Two-thirds of the students

interviewed described the role of the parameter *a* in $y = ax^2 + bx + c$ as the "slope" of a

quadratic function. In actuality, slope is not a part of the quadratic function; it is a linear

function concept.

Zaslavsky (1997) researched the misconceptions that impeded students'

understanding of quadratics. When dealing with the misconceptions, she coined the

phrase *conceptual obstacles*. Conceptual obstacles are "obstacles that have a cognitive

nature and that can be explained in terms of the mathematical structures and concepts that

underlie students' earlier learning experiences" (p. 20). Her study comprised of over 800

students from eight economically advantaged high schools in Israel. Through a series of

problem sets, Zaslavsky investigated student choice of strategies in problem-solving

possibilities. She identified five conceptual obstacles that impeded the students'

understanding of the quadratic function: 1) interpretation of graphical information

(pictorial entailments), 2) relation between a quadratic function and a quadratic equation,

3) analogy between a quadratic functions and a linear function, 4) seeming change in

form of a quadratic function whose parameter is zero, and 5) over-emphasis on only one

coordinate of special points (ex...vertex).

V. Think-Aloud Method

By using the think-aloud method, this study aims to find out *what,* or the kind of,

conceptions that are occurring for the participants involved. The goal is to develop cases

for pairs of students that will ultimately demonstrate the nature of students' conceptual

and procedural knowledge of quadratic functions.

The think-aloud method asks participants to say whatever they are looking at,

doing, feeling, thinking, understanding, etc. as they are engaged in their task(s). Usually

participants are audio and/or video recorded while they are partaking in the activity. In

this manner their "thinking" is recorded for later analysis. Since this protocol can

provide data about "both sophisticated and less sophisticated cognitive processes that are

difficult to obtain by other means…it is an essential method for areas such as cognitive

psychology, educational science and knowledge acquisition" (Someren, Barnard, &

Sandberg, 1994, p. 7). It should be noted though, that this method is only relevant if the

properties of the solution process are relevant to the study. I believe that the properties of

the solution process are relevant to this study since the focus is to attempt to understand

students' conceptual and procedural knowledge are as they are being introduced to the

quadratic function, the "play by play" of their thoughts are key. "The think aloud method is a means to validate or construct theories of cognitive processes" (Someren et al., p. 9). Although being able to *see* as well as *hear* how the students get a correct answer will be helpful, seeing how students construct an incorrect answer could be just as, if not more, relevant to the study since it could lead to *why* they are constructing an incorrect answer.

Although the prevailing assumption is that single-person protocols allow for the purest cognitions, since the material in the task would be theoretically in-vivo to the students, I believe that having one other person to discuss the tasks with would be beneficial. Vygotsky (1978) stresses the importance of social interactions in the development of complex thinking. The question then arises of why not place students in triads or in groups of even higher numbers. Why stick to dyads? According to Schoenfeld (1985a), if the focus of an investigation is largely cognitive, having groups larger than two should be negative. With larger groups there tend to be an increase in the degree of social interactions, making it "more difficult to tease out the purely cognitive aspects of students' behavior" (Schoenfeld, p. 78). By giving the participants a voice through the think-aloud method, I hope to be able to address the research questions through the analysis of their conversations (César & Santos, 2006).

By having pairs, one of three talk modes can occur: 1) disputation talk, where the participants may disagree with each other, 2) cumulative talk, where they construct a common knowledge between the two of them, and 3) exploratory talk, where the participants critically challenge each other, but are supportive of one another as well (Wegerif & Mercer, 1997). During the same task, a pair could even go between the various types of talks.

Once again, the purpose of this study is to see how the students *approach* the various topics within the quadratic function, not the success rate of each pair, although that will be looked at as well. In other words, what preliminary steps do students take towards completing the various tasks given to them? What are the participants thinking of as they approach the problems, as well as what strategies they use in attempting to solve the problems. Through analyzing the data there is always the possibility that I can learn more from when the students obtain a wrong answer than a correct answer.

Chapter Three: Research Methodolgy

A multiple case study approach for three pairs of students was utilized to investigate questions regarding students' understanding of quadratic functions. The study was bounded by time and location with the results only pertaining to these students and no assumptions have been made on other students (Creswell, 1998). The goal of analysis is to investigate themes within each case and then compare the cases to each other to look for common and contrasting themes.

I. Research Design

This study is an investigation of the effects that traditional and multiple representation tasks have on how students think about the quadratic function, specifically the axis of symmetry, vertex, the location of roots, whether the parabola opens up or down, the maximum/minimum point, the y-intercept and the main translations of the function itself when graphed. The specific methods of factoring for roots are not a part of this study. Due to the boundary of time, this concept would be a study all to itself.

Upon enrollment, students were paired according to the best fit of their schedule. (This will be discussed more in the *Participants* section.) Utilizing a "think-aloud" protocol, each pair participated in the same four tasks. The tasks varied, with one being more traditionally worded, one focused on more multiple representations, and then a combination of the two for two mixed methods tasks.

Students participated in the study tasks over a four-day period (before or after school) for a maximum duration of 45 minutes each day. In order to test and familiarize the students with the technology, (i.e., cameras and audio devices) and study procedures,

a pilot was conducted prior to the study. For two of the pairs, this study coincided with the timing to which the curriculum was taught that relates to quadratic equations. Due to scheduling, the third pair participated just after the chapter assessment, but prior to them getting feedback on the assessment. The majority of quadratic exploration is done during Chapter 5 of the book *Holt, Rinehart and Winston Algebra 2* (2007) that is used at the study school for the various Algebra 2 courses. I, of course, cannot claim that the students had similar knowledge or were exposed to similar instructional practices prior to enrolling in Algebra 2 during study year.

The constructivist methodology was ideally suited to the purpose of investigating the processes by which students might construct mathematical knowledge. It tends, however, to emphasize the cognition of individual students at the expense of social interaction. In other words, it does not fully take into account what the social and cultural interactions can provide and solely concentrates on the students' thinking processes (Cobb, Wood, & Yackel, 1990). Even so, since the students worked in pairs, students' performance on these tasks cannot be analyzed without considering their social and cultural component that will be brought to the interaction. The social constructivist lens seems most appropriate for this type of data analysis. Through the dyads each pair of students would have the possibility of thinking on their own, listening to their partner, and possibly re-evaluating their own thoughts.

It has been documented, "That student errors are seldom random or capricious – they have a rationality and functionality of their own" (Confrey & Smith, 1994, p. 135). This only strengthens the importance that teachers must "pay close attention to how a mathematics problem is conceptualized, worked on, and evaluated by students" (Confrey

& Smith, p. 135). Constructivism in a social setting will allow the understanding, and perhaps the development of knowledge, through the students' activity with mathematical tasks in a mathematical community (Davis, Maher, & Noddings, 1990). By using the think-aloud protocol during, and just after, the chapter in which the participants are being first introduced to the quadratic function, I hope to *catch* these constructivist moments in action.

II. Participants

All six study participants are enrolled in the high school (grades 9-12) where I teach that is located in northern Vermont with a population of approximately 1150 students. There were four males and two females in the study, two of the males and one female were sophomores (10) and the other three were juniors (11) in high school, making an even split.

To minimize outside mathematical influences, the recruitment was to intentionally be drawn from my class. However, due to the small size of my study year's Algebra 2 class, I ended up having to also recruit from other Algebra 2 classes. In order to keep control over the curriculum though, the recruitment was done from the same level of college prep Algebra 2 courses from the study school. Once the students and parents had agreed in writing to the study, the students also gave verbal assent. The first six students who signed and returned the paperwork became participants in the study (see Appendix B: Consent Form).

The students were paired based on convenience and familiarity. The convenience of the pairing came from the order in which the students volunteered to participate in the

study. Familiarity was based on the fact that the pairs already knew each other from the mathematics class and would feel comfortable when discussing the content.

In order to protect confidentiality, the students self-selected pseudonyms. A spreadsheet was maintained with each participant's name, pseudonym, date of consent/assent, and a code for transcription purposes. Codes and/or pseudonyms were utilized to identify people, places and case studies including other schools and personnel that were only peripherally related to the study.

Each day, while the students performed their respective tasks, the pairs were audio taped and videotaped, which was then transcribed for coding and analyzing. The tapes/disks and transcriptions were kept locked throughout the study and the data (both electronic and hardcopy) will be destroyed at the completion of my dissertation.

Although I did not foresee any reason for a student to withdraw from the study, there was a protocol for this purpose. All participants had the right to withdraw from the study without penalty at any point. If one of the students decided to withdraw from the study, any previous data collected up to that point would still be available to be used for research. The partner of the student withdrawing from the study would have had the option to continue with the study by him/herself or withdraw as well. Any data previously collected from the partner would have also been available to be used for research. The reason behind keeping the research was due to the beginning number of six students. If one pair withdrew for any reason that would have resulted in one third of the data disappearing, which could have been detrimental to the study. I requested that I would have been informed of a withdrawal request; luckily none occurred.

The students were compensated with $10 gift cards to the local pizza restaurant.

III. Role of Researcher

The selected participants worked through tasks that used a non-directive approach. In other words, although I was present, I did not assist the students with the mathematics content. I was present only to keep the students on task and talking openly.

My plan was to analyze the individual students' construction of mathematical knowledge as they interacted with their partner. The audio and video recordings were transcribed and then compared side by side with the written work that the students constructed. From these sources I looked for any themes or "aha!" moments between the students that would lead me to insights about how the students approached and understood the quadratic function tasks.

In order to ensure study validity, a standard protocol was in place for when I interacted with the students during the study (see Appendix C: Research Protocol). The protocol was so that even though I was in the room with the participants, there was limited interaction that I would have with them; to simply have me help to keep them on task. To ensure confidentiality and to maintain a positive rapport with the students, as researcher, I was the only person in the room during the study.

IV. Tasks

By the end of the study the three pairs of students had completed the same four tasks (see Tables 2 and 3), though not in the same order. The first two tasks were either traditional (see Appendix D: Traditional Task #1) or multiple representations (see Appendix E: Multiple Representations Task #1) in nature, while the last two were a combination of the two methods (see Appendix F: Mixed Methods Tasks #1 and #2). The tasks were specifically for this study, but based off of the students' curriculum, the

standards provided by the National Council of Teachers of Mathematics, the current

Vermont Mathematics Standards, and the upcoming Common Core Standards in

Mathematics.

Table 2

Daily Agenda

	Day 1	**Day 2**	**Day 3**	**Day 4**
Pair A	Traditional Task #1	Multiple Representation Task #1	Mixed Methods Task #1	Mixed Methods Task #2
Pair B	Multiple Representation Task #1	Traditional Task #1	Mixed Methods Task #1	Mixed Methods Task #2
Pair C	Traditional Task #1	Multiple Representation Task #1	Mixed Methods Task #1	Mixed Methods Task #2

The reasoning behind switching between the *Traditional Task* and the *Multiple*

Representations Task in days one and two is to counterbalance the effect that it may have

on the task in days three and four. The reason for counterbalancing is to control for the

impact the task would have for learning the material. This was to see if there was a

benefit to the order in which the students received the material. If all of the pairs are

given the *Traditional Task* first and then the *Multiple Representations Task*, the

traditional task may in fact "teach" the pairs of students the concept at hand. It would be

hard to determine if the students' thinking on the second task is a result of the task itself,

or the method, or a reflection of their thinking based on the first task.

Table 3

Task Breakdown

Content	Traditional Task #1	Multiple Representation Task #1	Mixed Methods Task #1	Mixed Methods Task #2
Axis of Symmetry	Problem #4	Problems #1-4, 9, 10	Problems #4, 5	Problems #3, 4-7
Vertex	Problems #4, 5, 6-12	Problems #8, 9, 10	Problems #4, 5	Problems #3, 6, 7
Graph Orientation	Problems #1-3, 4	Problems #6, 9, 10		Problem #4
y-intercept	Problem #4, 5	Problems #9, 10	Problems #4, 5	Problems #6, 7
Graph Transformations	Problem #4, 6-11	Problem #6		Problems #1-4
Maximum/minimum point		Problems #7-9		Problems #4
Location of roots			Problems #1, 2, 3	

Traditional task versus multiple representations task – Days 1 and 2. The

Traditional Task was designed prior to the *Multiple Representations Task*. As with

Reiken (2008), a point was made to use a limited number of representations and to

carefully select representations most often found in the students' Algebra 2 text. I

wanted to exhibit only the representations that the participants had recently been exposed

to in their Algebra 2 course. The *Traditional Task* closely followed the structure and

format of the workbook exercises that are presented in the Algebra 2 texts utilized for

instruction. Attention was given to designing open-ended problems to solve. This is

important due to not wanting to lead the participants in one direction or another. I also

did not want them to feel finished after writing one answer down, if more were

applicable. Since the *Multiple Representations Task* problems are merely modified

versions of the *Traditional Task* problems, the *Multiple Representation Task* problems

were designed after the *Traditional Task* problems. In the *Multiple Representations Task*

there was an increase of representations.

Mixed methods – Days 3 and 4. The mixed methods tasks were given to all

three pairs in the same order, *Mixed Methods Task 1* on the third day and *Mixed Methods*

Task 2 on the fourth day. This was to see if switching the order of the previous two tasks

had an affect on the participants' understanding and performance. The *Mixed Methods*

Tasks incorporated elements from both methods (traditional and multiple representations)

to examine whether any of the pairs approached the new problems in different ways.

V. Information Collection

Data for this study came from students' written work for the various tasks, as well

as from the analyses of both the video and audio transcripts.

- Audio and video recordings: Videotaping students made it possible to study

  individual student cognitive growth in a social setting (Davis, Maher, & Martino,

  1992). All sessions were videotaped using a single Kodak Playsport video

  camera per pair of students, secured to a tripod and focused on the students as

  they worked on the various tasks. This was for the comfort of the students. There

  were also audio tape recordings. The data was secured due to the fact that I

  personally owned the video cameras' disks as well as the tape recorders' tapes. I

  took out the disk/tapes of the respective recorders once taping was completed for

  confidentiality.

The audio recordings would back-up the video recordings in what the students said during the tasks. Both of the audio and video recordings were carefully transcribed, coded and analyzed. The details from the video recordings proved to be a rich source of data allowing me to see the mathematical activity of the students, facial expressions and body language that could not have been picked up by simply doing the audio recordings.

- Written work (tasks): The students' individual work while paired up was compared with the transcripts from the audio and video recordings for triangulation.

VI. Procedures

The students participated in four different sessions over a four-day window of time. If one of the pairs had to miss a day of the study due to unforeseen circumstances (illness, death in family, etc.), a make-up day was put in place as soon as possible (which was convenient for all parties involved) to keep the pair of students on track.

Each session was less than 45 minutes long (see Table 4). On the first day the students were read an introduction to the study (see Appendix G: First Day Protocol). Although I was present, it was simply to keep students on task as well as to encourage them to continually "think out loud" while engaging in the tasks. There was a protocol to follow so that the results of the sessions were not compromised. I did not assist the students in the tasks. Intervention only occurred when the students stopped talking. At this point I followed the protocol simply to prompt the students back into "thinking out loud".

Table 4

Participant Session Times

	Traditional Task	Multiple Representations	Mixed Methods #1	Mixed Methods #2
Katy and Zoe	0:30:30	0:22:53	0:22:17	0:26:14
Oliver and George	0:21:20	0:21:35	0:34:35	0:32:27
Jamal and Mohammed	0:10:05	0:19:48	0:12:13	0:12:00

The pairs of students worked in a modified "think-aloud" protocol. This protocol asked participants to say whatever they are looking at, thinking, doing, feeling, etc., as they are engaged in their task(s). Since the students were audio and video recorded I was able to hear firsthand the process of task completion from the students' perspective instead of simply the final product.

VII. Information Analysis

The audio and video recordings were matched up with the students' work artifacts for triangulation. At this point I looked for general interactions between the pairs of students, my own interventions to keep students on task by general prompting, as well as facial and body language as the students were completing the various tasks. I also looked to see if any mathematical gesture-speech mismatch between the students' tasks took place. Studies have shown that although students' gestures may suggest the same information that has been expressed in speech, this is not always the case (Alibali, Flevares, & Goldin-Meadow, 1997; Alibali & Goldin-Meadows, 1993). For example,

through gestures, a participant might reveal that the characteristic that indicates the y-intercept in a function, they are actually interpreting it to be the x-intercept.

Throughout the data analysis I was constantly referring back to previous studies' results, Knuth (2000), Leinhardt et al. (1990), Kotsopoulos (2007), Ellis and Grinstead (2008), and Zaslavsky (1997), to see where misconceptions had been found prior to this study. I wanted to be able to witness and identify *when* misconceptions impacted learning. I wanted to see *what* hindered/triggered understanding for the student involved in the study, and I wanted to see if their understanding was conceptual of procedural.

Although I was looking for correct and incorrect answers throughout the tasks, data was also coded for specific mathematical ideas pertaining to the tasks, and whether the students were specifically answering the traditional task, the multiple representations task, or one of the mixed methods tasks. Although I initially had a few ideas of possible codes when I started analyzing the data, I purposely left my mind open to look for *new* ideas of what the students were thinking. I was also keeping in mind the study's questions so that I could try to answer them. I wanted the codes to emerge from the data. The codes increased from 6 broad codes to 34 specific ones (see Appendix H: Codes).

Also, as the themes/codes were emerging I would have to go back and recode at times due to a particular notion that came up while triangulating the data. Sometimes it would deal with a hand gesture, or the thought of which quadratic form the participants were looking at versus what they thought they were looking at. I would then have to reflow that thought through the data. As Glesne (1999) puts it, "Coding is a progressive process of sorting and defining and defining and sorting those scraps of collected data that are applicable to your research purpose" (p. 135).

VIII. Trustworthiness and Credibility

Many efforts were taken into consideration in order to ensure the trustworthiness of the study. Data was triangulated between the audio and video recordings that were then transcribed and compared with the written artifacts of the students. Also, for control over the consistency of what was said to the students, there was a protocol in place (Appendix C: Research Protocol) for me to follow which said what I could or could not say and when I could say it. For example "I may read the question to the pair (upon request), but I am not to help define any of the words that are in the question." In addition to this, coding was checked through an external reader and compared to the researcher's coding for credibility.

Being that the social constructivist lens was present in this qualitative research, there is a chance that, although the study could be duplicated, the results may not be duplicated. This is due to the fact that different participants can answer the very same tasks differently and that the conversations within new dyads could also have a different outcome. These results can only be said to have occurred for these six participants during this time period. Also, by having the open-ended perspective in constructivism and engaging multiple methods of analysis was important to the trustworthiness of this study.

Chapter Four: Results

I. Overview of Connections between Students and the Quadratic Function

This results section is divided into four individual sections. Each section is devoted to answering one of the research questions. The first section focuses on components of the quadratic function itself, specifically addressing the first research question, "How do students think about the quadratic function as they work through a series of tasks?" The second section addresses the findings to answer, "What mathematical strategies do students employ when they work on a series of tasks dealing with the quadratic function?" The third section looks to see, "How does the type of task, traditional versus multiple representation, impact students understanding of the quadratic function?" Finally, the fourth section focuses on the question, "What kinds of knowledge (procedural or conceptual) do students utilize when completing a series of tasks about the quadratic function?" For analytical purposes, please note that the pair Katy and Zoe, as well as the pair Oliver and George, were the two pairs that participated in the study in-vivo, while Mohammed and Jamal were the pair that participated shortly after taking their chapter assessment, but prior to receiving their assessment results. It is interesting because Mohammed and Jamal took the least amount of time overall on the study. A future study could be conducted which focused on: 1) the amount of time a pair uses to complete the study, 2) the place in the curriculum when the pair has been introduced to the content in the study, and 3) the pairs' task(s) results.

II. Section 1: How Students Think About the Quadratic Function

Brown (2000) relates that students who comprehend a concept "first seek to develop an understanding of problems, and this often involves thinking in terms of core

concepts or big ideas." The knowledge of the students who are newer to the material, or more unsure of themselves, are "much less likely to be organized around big ideas, they are more likely to approach problems by searching for correct formulas and pat answers that fit their everyday intuitions" (p. 49). Therefore, to tackle the research question, "How do students think about the quadratic function as they work on a series of tasks?" I have broken down this section into seven sections, each one focusing on one of the initial core concepts of the quadratic function that students are asked to understand. Part 1 will address the axis of symmetry, part 2 will focus on vertex, part 3 will speak to graph orientation, part 4 will attend to the y-intercept, part 5 will conquer the transformation of graph, part 6 will deal with the maximum/minimum point, and finally part 7 will concentrate on the location of roots. The point in breaking down the quadratic into the core concepts that are being addressed in this study is to focus (as much as possible) on how the participants are thinking about each concept individually, as well as within the big picture or scheme of the quadratic functions.

Axis of symmetry. The axis of symmetry is the line that runs vertically through the x-coordinate of the vertex in a quadratic function. If the function's graph was folded over this line, then the two halves of the function would be the mirror image of each other (see Figure 2: Axis of Symmetry, Vertex, Maximum/Minimum).
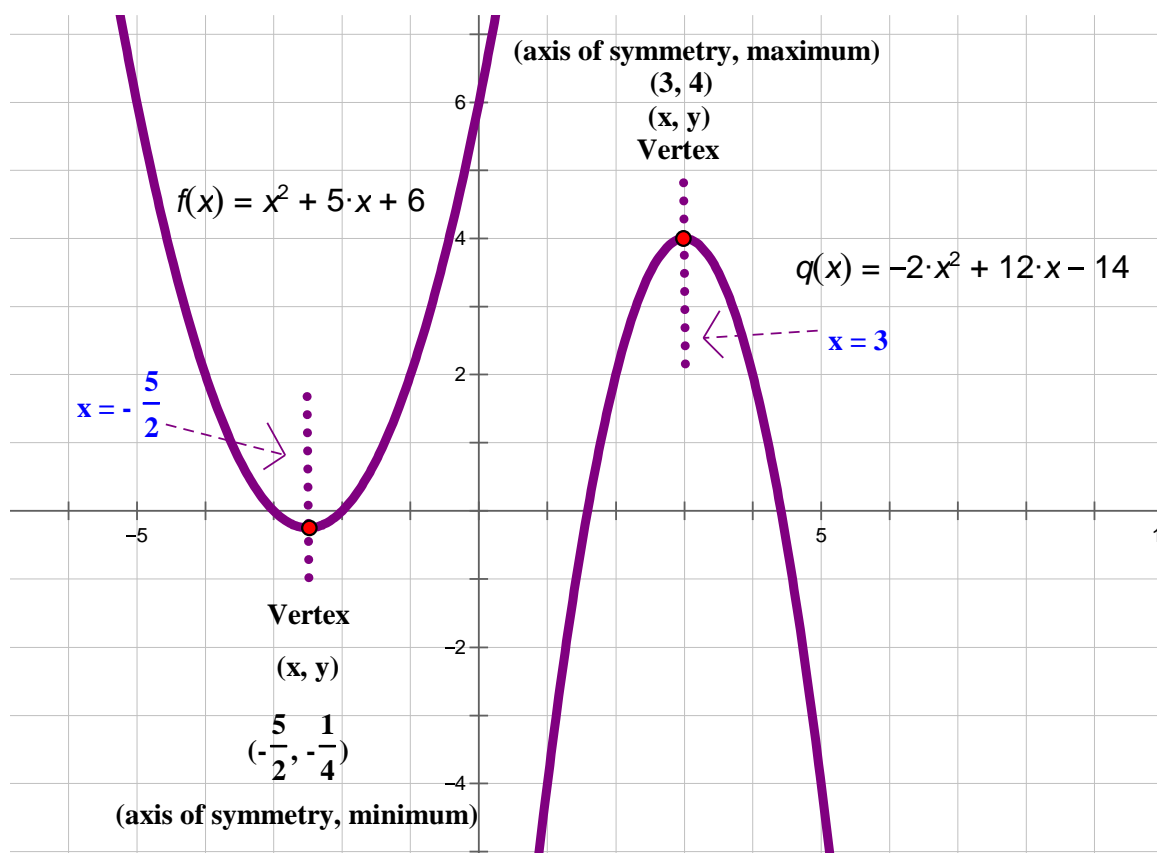
Figure 2: Axis of Symmetry, Vertex, Maximum/Minimum

The analysis of the axis of symmetry began with looking at how the pairs

responded to problem #4 on the *Traditional Task*, problems #1-4, 9, 10 on the *Multiple

Representations Task*, problems #4-5 on the *Mixed Methods Task #1*, and problems #3-7

on the *Mixed Methods Task #2*. I particularly focused on problems #1-4 on the *Multiple

Representations Task* since this was the first place where the participants were asked

questions that specified the axis of symmetry, and that it was not one of many concepts

needed to answer a problem. The participants seemed to understand the axis of

symmetry in 2 ways: 1) as a line that bisects the vertex (and the graph) as a whole, and 2)

as a number derived from the formula. Although one of the pairs initially showed their

procedural understanding of the axis of symmetry being derived from a formula, they

demonstrated limited conceptual knowledge at the time. I will conclude with that

example. Although all three pairs eventually commented on the axis of symmetry in both

ways of thinking, for this purpose I am presenting the pairs under how they initially

presented themselves in thinking about and understanding the axis of symmetry.

Axis of symmetry as a bisecting line.

Example 1: When George and Oliver approached the axis of symmetry, their

conversation implied that they thought of the axis of symmetry as a graphic physical

characteristic.

Oliver: Okay. So what is the axis of symmetry? How do you find it? Is there more than

one way to find it?

George: Well, the axis of symmetry is like something that could be reflected equally on

both sides?

Oliver: uh hum.

George: What is it finding…if you had a mirror, set it down, and if it looks the same, in

the mirror and the piece that it's the axis of symmetry, sure.

Oliver: Also find the vertex could be a way of finding the axis of symmetry, through the

vertex. Find it that way, the mirror way. You could, yea, so do you want to write that

down?

George: You can write it down.

Oliver: So what did we say the axis of symmetry was?

George: The mirror of the axis is the same on both sides. Exactly the same.

Oliver: Okay. So a line, right, it's a line that reflects something else on the other side. Make a line of a parabola. How do you find it?

George: The mirror way, the line, what did you say?

Oliver: Of the vertex.

George: Yes

Oliver: …Vertical line through the vertex. Is there more than one way to find it?

George: We just gave two ways, so yes.

Through George's and Oliver's cumulative talk, they were able to construct together a common way of thinking about the axis of symmetry. Their focus was on that the axis of symmetry was a line that could split the parabola into two congruent parts. Later on, when asked to solve for the axis of symmetry in subsequent problems, they used the formula itself.

Axis of symmetry as a number.

Example 2: When Katy and Zoe were asked to describe the axis of symmetry, they initially went straight to the formula of $x = -\dfrac{b}{2a}$. They also referred to it as the "x portion (of the vertex) of the function."

Katy: Number 1: What is the axis of symmetry? How do you find it? Is there more than one way to find it? So one way is…

Both: -b over 2a

Zoe: I don't think there's another way that at least we've done.

Katy: I don't think there's another way either.

Zoe: So what is the axis of symmetry?

Katy: The axis of symmetry is…

(thinking with time gap)

Proctor: Don't forget to say everything that you are thinking

Katy: I'm trying to think of what I'm going to say.

Zoe: Is the *x*.

Katy: …Is finding the *x* portion of the function.

Zoe: I agree with that.

Katy: For numbers 2-4, please identify the axis of symmetry for the graph of each function.

Zoe: First we have to identify the *a*, *b*, and *c*.

Katy: *a* is 1, *b* is -4, and *c* is 2. So we're going to take…

Zoe: x = -b…

Katy: over 2a.

Katy: So we're going to do –*b* which would end up being a positive 4.

Zoe: A negative times a negative would be a positive.

Zoe: Over 2 x 1 and that would be 4 over 2, which would be 2.

1. What is the axis of symmetry? How do you find it? Is there more than one way to find it?

$\dfrac{-B}{2A}$ FINDING THE X

PORTION OF THE FUNCTION

For numbers 2-4, please identify the axis of symmetry for the graph of each function.

2. $g(x) = x^2 - 4x + 2$

$A = 1$
$B = -4$
$C = 2$

$\dfrac{\pm 4}{2 \cdot 1} = \dfrac{4}{2} = \boxed{2}$
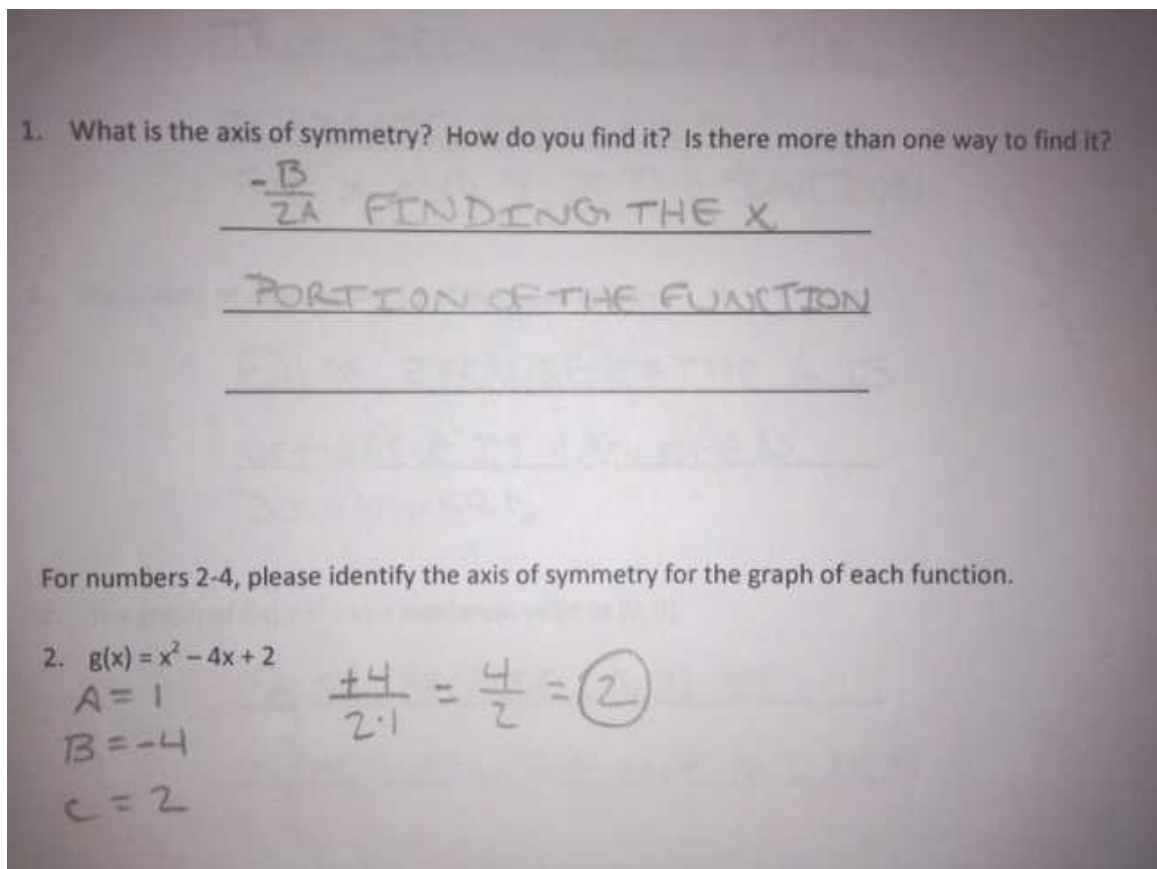
Figure 3: Katy and Zoe Multiple Representation Task problems 1 & 2

Zoe and Katy both felt confident in applying the formula $x = -\dfrac{b}{2a}$ to algebraically solve

for the axis of symmetry. (See Figure 3: Katy and Zoe Multiple Representation Task

problems 1 & 2.) Later they would also show the conceptual knowledge of reflecting the

y-intercept over the axis of symmetry for "another" point.

Example #3: Jamal and Mohammed initially confused the formula for the axis of

symmetry with the quadratic formula, $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, and the equation of a line, $y =$

$mx + b$. The quadratic formula is used to find the roots (x-intercepts, zeros, etc.) of the

quadratic function, and the equation of the line is a part of the linear function, not the

quadratic function. Jamal and Mohammed eventually backtracked after completing other

problems to realize that their initial thought was incorrect. They did not, however, revisit

all of the problems that they had already answered incorrectly while using the quadratic

formula instead of the axis of symmetry formula. See Figure 4: Jamal and Mohammed

Multiple Representation Task problems 1 & 2.

1. What is the axis of symmetry? How do you find it? Is there more than one way to find it?

$$x = \frac{-b}{2a}$$

For numbers 2-4, please identify the axis of symmetry for the graph of each function.

2. $g(x) = x^2 - 4x + 2$

$$x = \frac{4 \pm \sqrt{-4^2 - 4(1)(2)}}{2(1)}$$

$$x = \frac{4 \pm \sqrt{16 - 8}}{2}$$

$$x = \frac{4 \pm \sqrt{8}}{2}$$

$$x = 2 \pm \sqrt{8}$$

$$x = 4 \pm \sqrt{2}$$

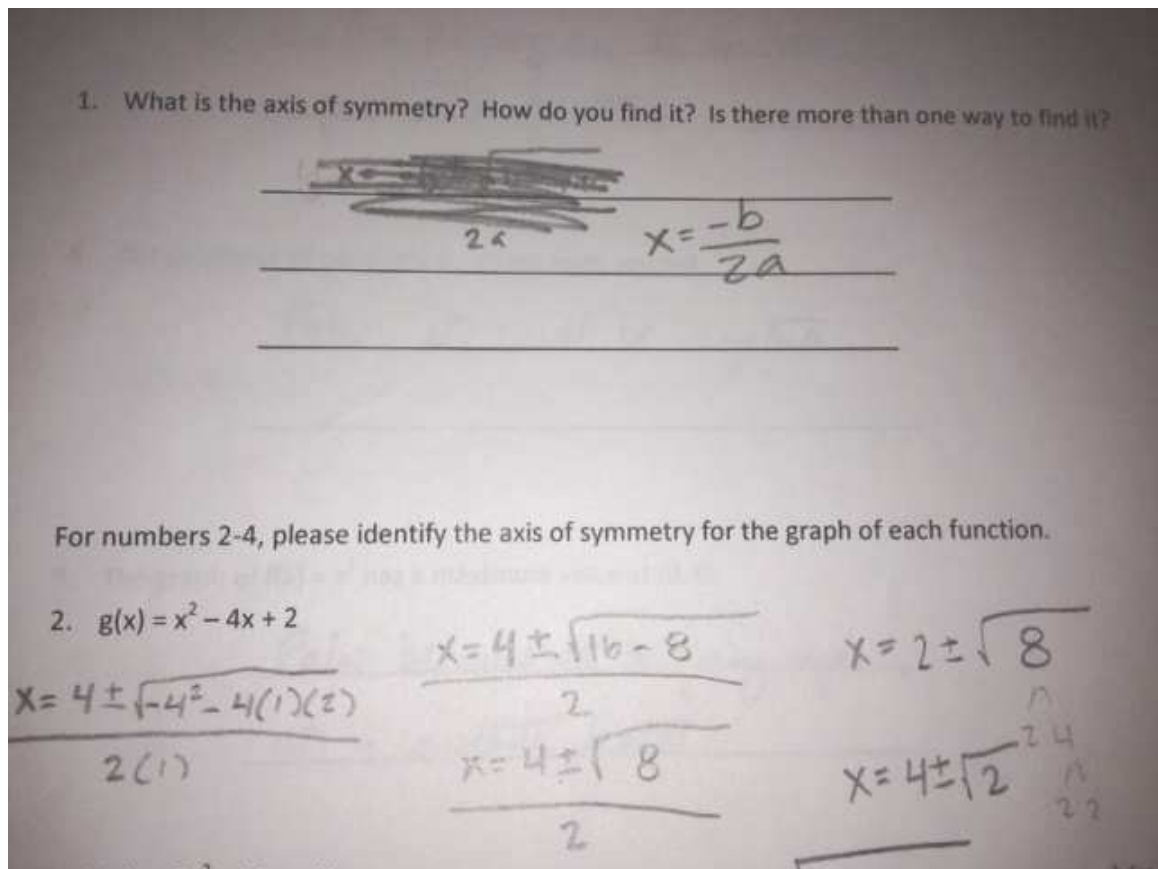*Figure 4*: Jamal and Mohammed Multiple Representation Task problems 1 & 2

Figure 4 shows how Jamal and Mohammed initially wrote the quadratic formula for the

axis of symmetry but eventually backtracked to fix number 1, but not the other problems

that had been incorrectly attempted. It should be noted though, that in subsequent problems when they were asked to solve problems which involved the axis of symmetry and were given the formula, they completed the task, but when they were again asked about the axis of symmetry without the formula, they once again portray themselves as having limited understanding (both procedural and conceptual) of the concept. In that instance they consistently gave the last number of the function. If the function was in the standard form they gave the y-intercept; if the function was in the vertex form they gave the y-coordinate of the vertex.

Although it could be argued that students should know both manners before being thought of as proficient, I believe that both of the pairs, George and Oliver and Zoe and Katy, were headed in the correct direction with their initial take (as well as their subsequent take) on the axis of symmetry.

Vertex. The analysis of the vertex began with looking at how the pairs responded to problems #4-12 on the Traditional Task, problems #8-10 on the Multiple Representation Task, problems #4-5 on the Mixed Methods Task #1, and problems #3, #6, and #7 on the Mixed Methods Task #2. I particularly focused on problems #9 and #10 on the Multiple Representations Task since this was the first place where the participants were asked questions that specified the vertex. All of the participants considered the vertex to be the highest most (or lowest most) coordinate pair (x,y), depending on which direction the parabola was orientated. They also all agreed with the process of algebraically finding the vertex.

Example #4: This example is when Katy and Zoe were approaching problem #9 on the Multiple Representations Task. They had already identified their *a*, *b*, and *c*, as well had found their axis of symmetry.

Katy: So find out vertex, we have to plug our axis of symmetry back into the equation. So it would be f(x) = (2)2 – 4(2) + 3 and that would equal 4 – 8 + 3 so it would be -4 + 3 and that would be -1. So vertex is (2, -1).

Zoe: I would agree.

All three pairs concurred in the thinking that they would have to obtain the axis of symmetry prior to the vertex. Then, by plugging the axis of symmetry back into the original equation for the "x", they would be able to chug out the y, which would be the y-coordinate of the vertex.

  Graph orientation. Graph orientation deals with whether or not the parabola opens up or down. It is indicated by the "a" when the quadratic function is put in standard form (ax^2 + bx + c). If the "a" is positive, then the graph will open up. If the "a" is negative, then the graph will open down.

  The analysis of the graph orientation began with looking at how the pairs responded to problems #1-4 on the *Traditional Task*, problems #6, 9, and 10 on the *Multiple Representations Task*, and problem #4 on the *Mixed Methods Task #2*. If one were to strictly consider right and wrong answers, overall, the three pairs got 91.3% of the questions which referred to the graph's orientation correct. The small percentage that were answered incorrectly were all done by Jamal and Mohammed, and due to there not

being any "talk" about those particular questions, it appears that those questions were

ones that were rushed through in order to get done with the task. Either Jamal or

Mohammed would answer the question while the other one simply watched, or looked off

into the air. For this section I will show one example that exemplifies what all three pairs

wrote for this same question, *Multiple Representations Task* problem #6, and then discuss

a common misconception that appeared during the study.

> ***Common correct understanding.***

Example #5: Katy and Zoe's conversation during the *Multiple Representations Task*

problem #6 about graph orientation was straight to the point.

Katy: 6) The graphs of all quadratic functions open upward. False.

Zoe: False. Because you can have a parabola going downwards.

Katy: Because the *a* can be negative or positive.

Zoe: Making a parabola

Katy: Going up or down.

Zoe: I would agree with that

Katy: False, because if *a* is negative, and that's a downward (parabola).

Zoe: I would agree with that...

By looking at the leading coefficient of the parabola the students were able to determine

the orientation of the graph. Katy and Zoe confirm that this particular characteristic of

the quadratic function is a very straightforward concept.

Common misconception.

Example #6: In this example Oliver and George used the term "slope" to describe "a".

Slope is actually a linear concept and does not have a role in the quadratic function.

Oliver: The graphs of all quadratic functions open upward.

George: No, slope could so it could be negative.

Oliver: Slope could be negative? Is that the answer?

George: Do you agree with that?

Oliver: Yea. It's be the *a* because remember if you're trying to create a graph, you

always use the *a* for up and down.

George: Yea.

In this discussion, Oliver questions George's choice of vocabulary, and in the

video Oliver even looks at George with a questioning face, but cannot pinpoint what is

incorrect since he does agree that the *a*, the leading coefficient, indicates the orientation

of the graph. This example reaffirms what Zaslavsky (1997) and Ellis and Grinstead

(2008) said about students having a tendency to create an analogy between quadratic

functions and linear functions. Two-thirds of the students interviewed by Ellis and

Grinstead described the role of the parameter *a* in $y = ax^2 + bx + c$ as the "slope" of a

quadratic function. When, once again, in actuality, slope is not a part of the quadratic

function; it is a linear function concept.

Y-Intercept. The analysis of the y-intercept began with looking at how the pairs

responded to problems #4-5 on the *Traditional Task*, problem #9 on the *Multiple*

Representations Task, problem #4 on the *Mixed Methods Task #1*, and problem #6 on the

Mixed Methods Task #2. There were two thinking positions that the y-intercept analysis clearly fell in, having success understanding the y-intercept, and not having success understanding the y-intercept.

      Understanding the y-intercept. In theory, finding the y-intercept should be a straightforward part of graphing a quadratic function, especially when it is "c" when the function is in standard form. At some point throughout the tasks, each of the pairs did refer to the y-intercept in this manner. In fact, it was not usual, that while the participants were dissecting the various quadratic functions, they would identify the y-intercept first.

Example #7: Jamal and Mohammed did not waste any time locating the y-intercept while attempting *Traditional Task* problem #4.

Mohammed: We have to graph all of our functions Mr. Jamal.

Jamal: All right, so $x^2 - 3$.

Mohammed: is 3

Jamal: Well, 3 is the y-intercept, so you go down 3 and…

Mohammed: it's right there. (Pointing with the pencil)

Example #8: For this example I will use an excerpt from the conversation between Katy and Zoe while they were solving problem #9 on the *Multiple Representations Task*. At this point they had already solved for the axis of symmetry and the vertex. Katy and Zoe not only demonstrated that they knew how to identify the y-intercept, but that also knew how to use it to help them (theorectically) graph a parabola.

Katy: And our y-intercept is 3.

Zoe: Do we have more than one number for vertex?

Katy: Vertex?

Zoe: Yea, the vertex is the very bottom of the curved line and then it goes up by 3.

Katy: What do you mean?

Zoe: Yes, when you're doing your graph or whatever, the number that you lead off of

and then you find where you're suppose to go, right?

Katy: And *c* is our y-intercept and that's 3.

Zoe: I would agree

Katy: An then we graph it, so the vertex is (2, -1) and it is opening up so that points

would be at zero 1, no, 1 zero and the other point is at 3 zero, And then we can

take, find the arms…

Zoe: Okay. How do we find the arms?

Katy: Do we need to find like two more points or do we just…

Zoe: The y-intercept? So it's something that we have but, we haven't really used.

Katy. Okay, so do we plug in the y-intercept equation to find one? (a point)

Zoe: Ummm, I think that we have to hit the y-intercept. I don't know; I was never good

at this part.

Katy: Oh yea, it would be a y-intercept and then you reflect it (over the axis of

symmetry).

Zoe: Right and you figure out where that went from there.

Through cumulative talk, Zoe and Katy were able to both identify the y-intercept as well as identify another use for it, to reflect over the axis of symmetry line in order to find another point on the parabola.

Example #9: For this example I will use the third pair, Oliver and George, so that there is an example showing understanding for each of the pairs, before showing examples of when some of the participants did not show understanding of the y-intercept. In this example Oliver and George have already solved for the vertex in problem 5a on the *Mixed Methods Task #1*.

George: Right. And then your y-intercept. Isn't that just the last one. Is that what it is?

Oliver: Uh hum.

George: Okay

Oliver: Right?

George: I think so.

Oliver: Your y-intercept is usually *c*. Right?

George: Yea, I think so. I hope so.

George and Oliver were the third and final pair to show understanding (at some point) of the y-intercept in the quadratic function. Knowing this information, it makes the examples of when the pairs did not show understanding of the y-intercept all that more peculiar.

Not understanding the y-intercept. Two specific examples surfaced which

demonstarted a lack of understanding the y-intercept. The first example shows a serious

confusion between the y-intercept when the function is in standard form $(f(x) = ax^2 + bx$

$+ c)$ versus the y-coordinate of the vertex when the function is in the vertex form $(f(x) =$

$a(x - h)^2 + k)$.

Example #10: Zoe and Katy got confused with these two formats when pertaining to this

concept (see Figure 5: Katy and Zoe Mixed Methods #2 problem #7). Although the girls

started out using the process of elimination and correctly eliminating the far right option

due to an incorrect y-intercept, they ultimately got confused between the vertex format

and the standard format.

7. Given the three equations:

$$f(x) = 2(x-3)^2 - 4 \qquad g(x) = 2x^2 + 3x - 4 \qquad h(x) = \frac{1}{2}x^2 - 6x + 8$$

a) Which equation is represented by the graph shown below?

$$F(x) = 2(x-3)^2 - 4$$

b) Explain how you decided on your answer in part (a).
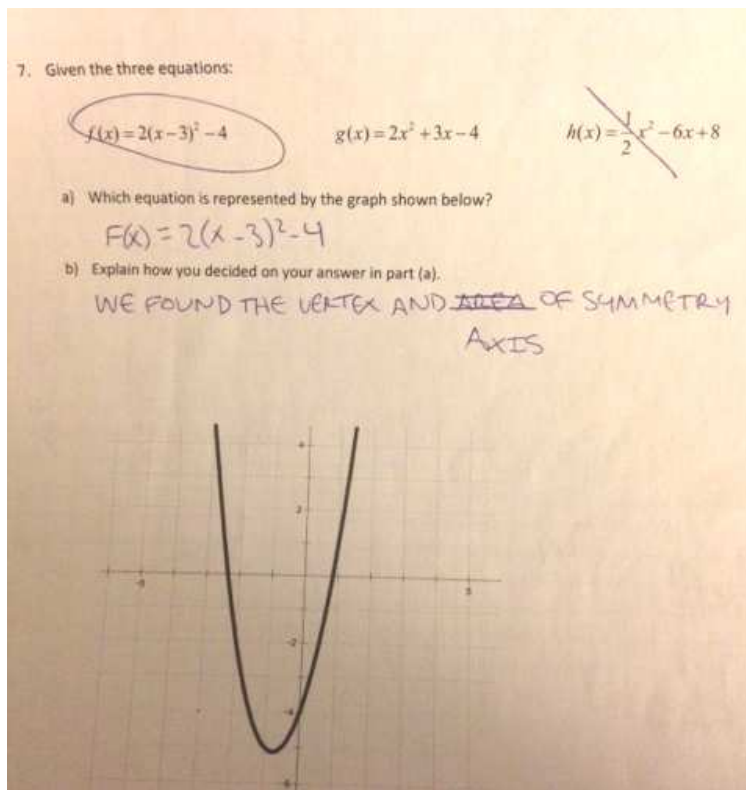
WE FOUND THE VERTEX AND ~~AREA~~ OF SYMMETRY

AXIS

Figure 5: Katy and Zoe Mixed Methods #2 problem #7

Zoe: Well…It's not the last one. Oh wait…that would be indicating the vertex.

Zoe: We have the vertex, right? That would be *c*?

Katy: What do you mean?

Zoe: How did you find the vertex?

Zoe: What is *c*?

Katy: With the area of symmetry.

Zoe: You mean the axis of symmetry??

Katy: Oh yea, right.

Zoe: With axis of symmetry, right, but *c* would be the other half of that right?

Katy: This? (pointing to the y-intercept of the third problem.)

Zoe: No, no

Katy: Oh the y-intercept?

Zoe: Yea

Katy: That's what it has to go through, (Indicating that the y-intercept goes through the
   y-axis on the graph).

Zoe: I thought that we would find the axis of symmetry by plugging it back in through
   (while flipping back through the pages.) Okay, I see what you're saying.
   (Stopping the flipping and putting her hands down again satisfied.)

Katy: So, one of them goes through -4, (indicating that the y-intercept has to be -4 based
   on the graph), so it's between these two now (indicating the first two functions)
   because this one +8 (indicating the last function) and that's not emphasizing that
   (as the y-intercept) on the graph.

Zoe: Yup, so…

Katy: So negative three over two times two…it would be negative three fourths…so it's this one.

Zoe: Okay…I would agree with that. (They actually picked the wrong one; it is the middle one.)

Once again, Katy and Zoe were on the right path when they used the process of elimination by eliminating the "obvious" wrong answer in the option due to the y-intercept not being correct. Unfortunately though, they were confused by the two different formats between the left option (vertex format) and the middle option (standard format). What the girls took as their "b" as if it was standard format, was really an "h" due to being in vertex format (see Table 5: Solving a Function with Two Forms).

Table 5

Solving a Function with Two Forms

Problem	$f(x) = 2(x\text{-}3)^2 - 4$	$g(x) = 2x^2 + 3x - 4$
Format	Vertex $f(x) = a(x - h)^2 + k$	Standard $g(x) = ax^2 + bx + c$
Axis of symmetry	Set $(x - h) = 0$ and solve for x. $x - 3 = 0$ $x = 3$ axis of symmetry $= 3$	$x = -b/2a$ $x = -3/2(2)$ $x = -\frac{3}{4}$ axis of symmetry $= -\frac{3}{4}$
Vertex	(h, k) $(3, -4)$	Plug $-\frac{3}{4}$ back into the original equation for x and solve for y (or $g(x)$). $(-\frac{3}{4}, -5\frac{1}{8})$
y-intercept	Option 1: plug zero in for x and solve for y (or $f(x)$). Option 2: FOIL out $(x\text{-}3)^2$, distribute the 2 and combine like terms in order to convert the vertex form into the standard form to be able to identify c. $C = 14$	C $C = -4$
Correct Answer	No	Yes

Example #11: The other manner in which the y-intercept was confusing was when

Oliver and George found the y-intercept last, instead of starting with it (see Figure 6:

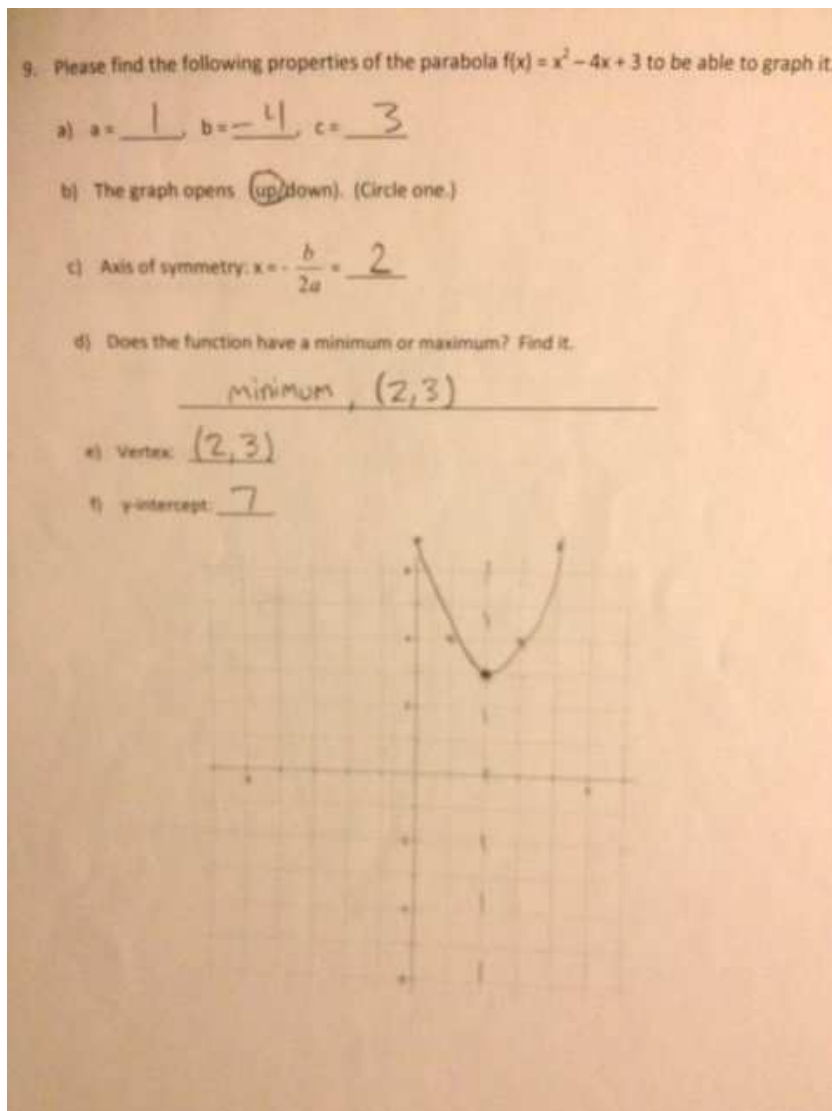Oliver and George Multiple Representations Problem #9).

9. Please find the following properties of the parabola $f(x) = x^2 - 4x + 3$ to be able to graph it.

a) $a =$ __1__, $b =$ __-4__, $c =$ __3__

b) The graph opens (up/down). (Circle one.)

c) Axis of symmetry: $x = -\dfrac{b}{2a} =$ __2__

d) Does the function have a minimum or maximum? Find it.

   __minimum__, $(2,3)$

e) Vertex: $(2,3)$

f) y-intercept: __7__

Figure 6: Oliver and George Multiple Representations Problem #9

George: Please find following properties of the parabola. $F(x) = x^2 - 4x + 3$ to be able to graph it. So *a* is 1, -4, 3 (stating the *a*, *b*, and *c* respectively)…so the graph opens down? No up. Right? Because *a* is positive. Axis of symmetry, -b…4 over 2 so it's 2? As a…

Oliver: Minimum or maximum

George: Going up the…is minimum and its y part would be…

Oliver: 2, because that's the vertex.

George: Yup. But we also have to get the y too, right?

Oliver: Uh hum, which is 3.

George: All right, (2, 3)?

Oliver: Uh hum.

George: The y intercept is…

Oliver: Isn't your y intercept *c*?

George: That's just where it is, like the end point of the…

In this example Oliver and George confused the y-intercept of the standard form with the y-coordinate of the vertex of the vertex form. They did get the correct axis of symmetry of two, (which is also the x-coordinate of the vertex), but to get the y-coordinate of the vertex they needed to plug the two back into the original function to solve for y…which would have given them the vertex (2, -1). They had literally just stated that the y-intercept is the *c*, indicating that perhaps it was a procedural error versus a conceptual error.

Also, instead of simply taking their y-intercept (*c*) of three that they had already found and graphing it on the coordinate plane (which possibly would have indicated to them that their vertex was incorrect due to placement), they decided to go in a round-a-bout manner in obtaining the (wrong) y-intercept. They tried relating the points other then the vertex to the parent graph of $f(x) = x^2$. When a one is the leading coefficient (number in front of the x^2), there is a shortcut that can be performed. From the vertex,

when you move to the right one unit you can move up $(1)^2$…so one unit up. When you move to the right two units, you can move up $(2)^2$…so four units up. When you move to the right three units, you can move up $(3)^2$…so nine units up. These points can then be reflected across the axis of symmetry (see Figure 7: Quadratic Parent Function).
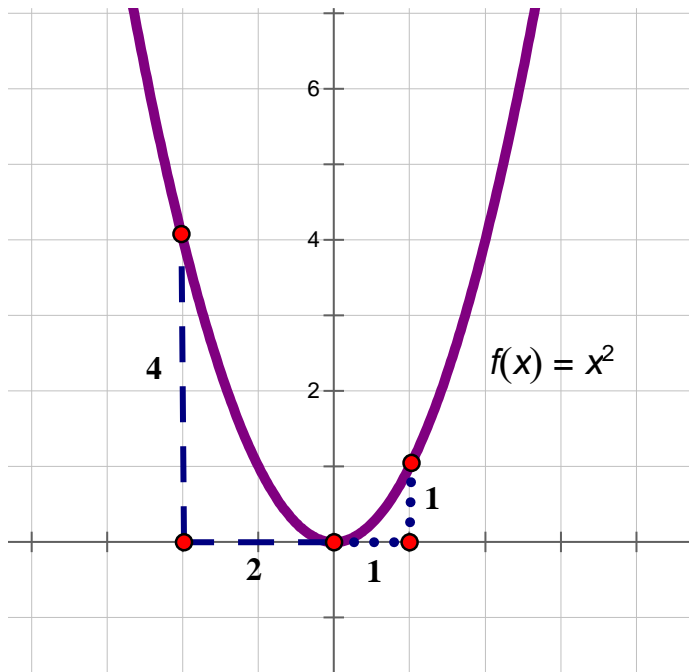

Figure 7: Quadratic Parent Function

George and Oliver continue…

George: Well, this we can at least graph…in creating both vertices, so (2, 3)…and then all goes up by 1 so it's out over 1. Is that right?

Oliver: Okay, you go over 1, 1 squared x 1 is 1, so yeah.

George: And the next one is 4.

Oliver: Go over 2.

George: Over 2 and it's up 4. So it'd be right around there. So the y-intercept would be

7 if this is right. (reflects it) That's good.

From their incorrect vertex of (2, 3) they did apply the correct parent graph symmetry with leading coefficient of one, but since their vertex was incorrect, they got an incorrect y-intercept from this method.

Transformation of graph. The analysis of the transformation of the graph began with looking at how the pairs responded to problem**s** #4**,** 6-11 on the *Traditional Task***,** problem #6 on the *Multiple Representations Task*, and problems #1-4 on the *Mixed Methods Task #2*. I particularly focused on problems #6-11 on the *Traditional Task*, and problems #1-2 on the *Mixed Methods Task* #2 since I believe that these problems gave better insight to what the participants' thinking about transformations were when dealing with the quadratic function. The transformations appeared to give all three pairs issues. If one were to strictly consider correct and incorrect answers, Oliver and George got 70.8% of the transformation questions correct, Jamal and Mohammed got 41.7% correct, and Zoe and Katy got 50% correct. Of the problems that the students got wrong, it was due to conceptual errors; of the ones that they got correct, I believe that it was due to those questions being more simplistic in nature. They were ones that had closer appearance similarities to linear functions, which the participants would be more familiar with. The students knew that they had to move the functions from the parent function of $y = x^2$, but they forgot their basic foundational rationale behind how to move it, and which way to move it.

Example #12: In this example Katy had an insightful moment when she and Zoe were discussing questions #6-11 on the *Traditional Task*. Unfortunately, the insightful "aha!"

moment did not completely pan out into them portraying full understanding on the concept of transformations. Although it is a lengthy excerpt, it demonstrates the girls' cumulative thinking.

Katy: The quadratic parent function if $f(x) = x^2$. Its graph is a parabola with its vertex at the origin $(0, 0)$. Describe each transformation from the parent function. Ummmm…

Zoe: I don't remember learning this…

Teacher: Try looking at the vocabulary with the question to see if that would help.

Katy: I don't remember learning about a parent function (rest her head in her hand).

Zoe: …well, we know that x squared is a…

Katy: But I don't know what a parent function is…so…that is not really…do we have to do like all of these?

Teacher: Yup, that is only one word in the directions…what about the other words in the directions?

Zoe: Well since the parabola is a U shape and the vertex is at $(0, 0)$…so the parent function is one…and the other one is negative one (indicating problem #6) so that deals with the arms…

Katy: But we are suppose to describe the transformation from the parent function.

Zoe: So the transformation is that it…

Katy: So apparently it transforms it…

Zoe: Do you know how to do that?

Katy: What I don't get is that this one is $f(x)$ and all of the rest are $g(x)$…does that mean

that they have been transformed?...oh wait, f…g…they come after one

another…(wiggling her head).

Zoe: (smiles)

Katy: That is the part that I get…oh wait, does that mean when they are rotated, and like

flipped and turned..(happy with her new epiphany) okay

Zoe: What do we have to do?

Katy: You know (getting excited and using hand movements0…when we had the graph

rotated, flipped and turned…like when if it is positive it is like this (indicating an

open up parabola with hands) and when it is negative it does this (indicating an

open down parabola).

Zoe: Oh, okay

Katy: Isn't there a mathematical word that we have to use though…like for each one?

Proctor: Describe your thoughts in your own terms.

Katy: (writes "flipped down" for number 6)…x – 1…so it was slid…because it is at x –

1, it is not at the (0,0) anymore.

Katy is correct in her thinking that transformation means to "flip" and "slide" the parent

graph in various directions so that the vertex is (possibly) in a new coordinate location

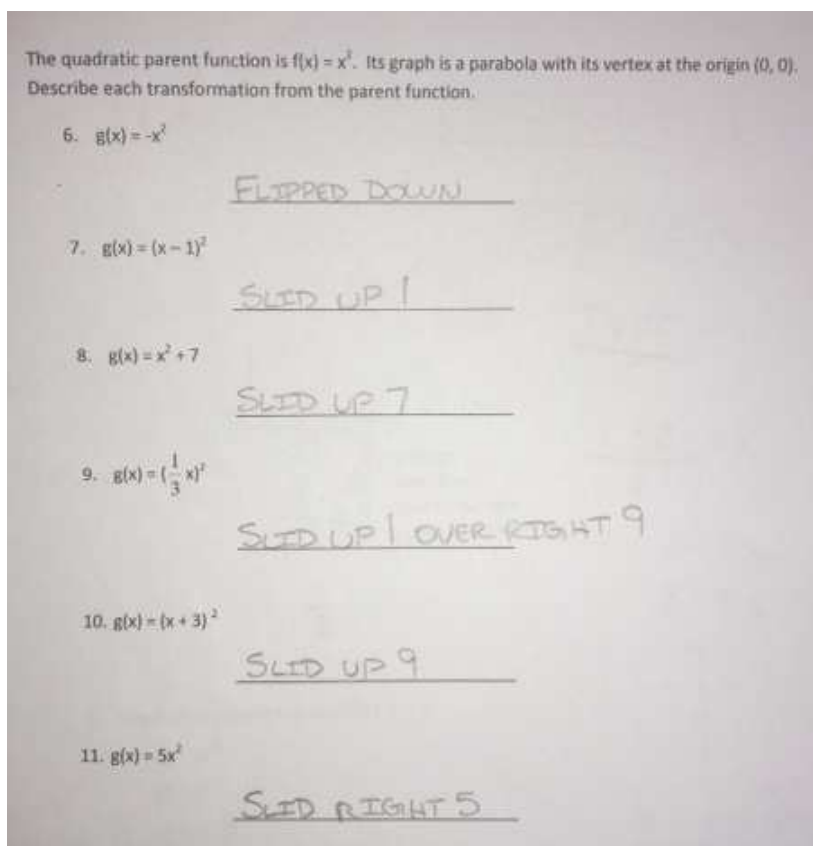(see Figure 8: Katy and Zoe Traditional Task Problems #6-11).

The quadratic parent function is f(x) = x². Its graph is a parabola with its vertex at the origin (0, 0). Describe each transformation from the parent function.

6. g(x) = -x²

    FLIPPED DOWN

7. g(x) = (x − 1)²

    SLID UP 1

8. g(x) = x² + 7

    SLID UP 7

9. g(x) = ($\frac{1}{3}$ x)²

    SLID UP 1 OVER RIGHT 9

10. g(x) = (x + 3)²

    SLID UP 9

11. g(x) = 5x²

    SLID RIGHT 5

Figure 8: Katy and Zoe Traditional Task Problems #6-11

Although Katy and Zoe got to the thinking of transformation meaning to flip or slide,

it also means to stretch or make the arms of the parabola steeper, as well as to compress,

to widen or make the arms more shallow than the original parent graph f(x) = x². When

looking at the correct versus incorrect problems in figure #8, only problems #6 and 8

would be counted as correct. The girls were not consistent with knowing which way a

function slid, if it even was supposed to slide in the first place.

Example #13:   In these two examples demonstrating Oliver's and George's thinking of

the transformations of the quadratic function, they knew in both tasks that a leading

coefficient other than one would change the arms of the function, and not just the location of the vertex.

This is the first of the two examples from Oliver and George. In this one they conclude incorrectly. This excerpt is taken out of the conversation between Oliver and George as they are starting problem #8 on the *Traditional Task*.

Oliver: Okay. Again, is not in standard form. So you want to foil that out?

George: Yea, sure, (laughing) I'll do the fractions. What is it? 1/3x?

Oliver: Uh hum.

George: And that's all they give us? Okay.

Oliver: Remember to have a multiplier.

George: Yea. So it's just, is it 1/6th? Is that it? Or is it…

Oliver: Well, 3 x 3 is 9

George: Oh yea... So it's 1/9th or is it 2/9th? (nervous laugh) That helps us a whole

lot. So + 0x + 0. Wow that is a lot of empty spaces.

Oliver: So umm…

George; We're missing $+ 0 + 0x^2$.

Oliver: Oh wait.

George: Oh yea. All right. (Changes it to $1/9x^2 + 0x + 0$) Okay. Do you know what

happens to the graph?

Oliver: Well, if the parent function goes out like that (indicating with his hands that it is

a wider opening up parabola)…oh my God. I don't know.

George: Yay…hmmm…it goes diagonal (joking)

Oliver: Oh, the graph would just be thinner…

George: yea, you're right.

Oliver: Because remember when you go over, you have to multiply it by this (indicating the 1/3).

George: Okay

Oliver: So the graph would be thinner.

George: Thinner instead of thicker. Yup

Oliver and George understood that the leading coefficient changed the arms of the parabola. Although this pair eventually recorded the answer to #9 being that the parabola became "thinner", at one point, Oliver gestured with his hands that the arms of the parabola would widen the parent graph (see Figure 9: Oliver and George Traditional Task problems #9-11). When the leading coefficient is less than one, the arms of the parabola widen, or become more shallow. When the leading coefficient is greater the one, the arms of the parabola become thinner, allowing the arms to be steeper, or stretch faster.
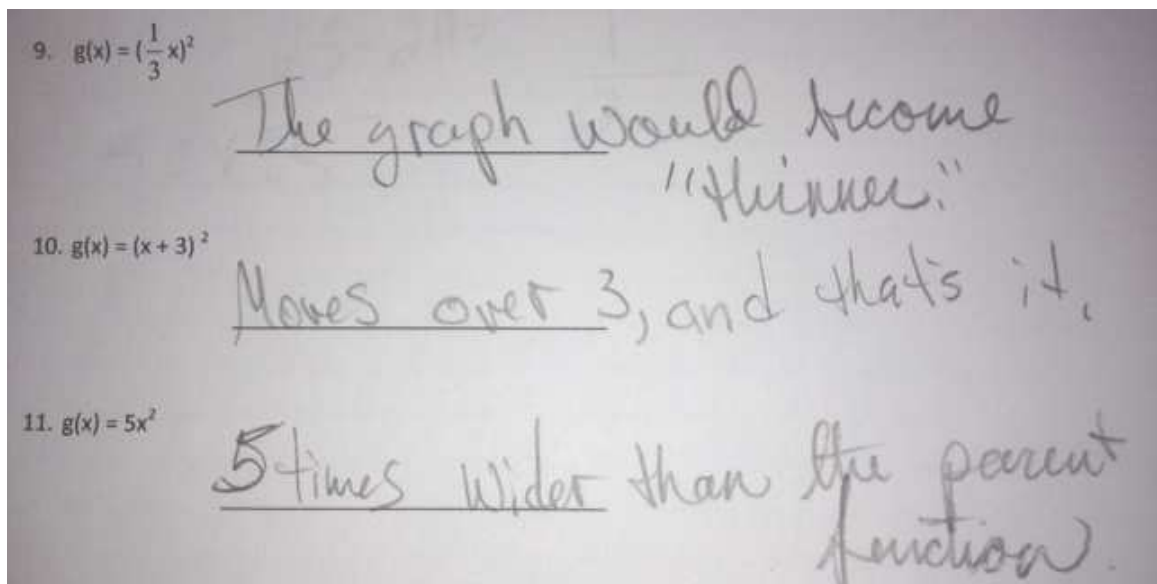
9. $g(x) = (\frac{1}{3}x)^2$

The graph would become "thinner."

10. $g(x) = (x+3)^2$

Moves over 3, and that's it.

11. $g(x) = 5x^2$

5 times wider than the parent function).

Figure 9: Oliver and George Traditional Task problems #9-11

Oliver and George have the opposite thinking when they approached problem #11. This time the leading coefficient was greater than one, and instead of correctly identifying the arms as being thinner, the pair identified them as being "wider."

   In a subsequent problem though, Oliver and George appeared to correctly understand this particular concept of the arms becoming "thinner" or "wider".

Example #14: In this second of the two examples from Oliver and George, they are still looking at the transformations of the quadratic function.

George: All right, #1: Circle the function that produces the widest parabola? How do you know? Please explain your reason.

Oliver: Okay, so this is in standard form. This one wouldn't be…right? Because…

George: I think that it would …because 1/5th, doesn't a fraction make it wider than the

higher up number you have makes it thinner?

Oliver: I think you're right.

George: Because, it does that because like it's how you, you do the number times 5th. I
don't know how to explain it.

Oliver: I know what you mean.

George: Yea.

Oliver: So wouldn't we do the up and over method? It's closer to the vertex.

At this point Oliver is indicating that they are only looking at the "a" portion of the
functions, which is the indicator of whether or not the arms stretch or compress. If by
chance a function did not slide, or leave the coordinate location point of (0, 0), then when
the arms are widened they would in fact become closer to the vertex as well as the x-axis,
versus stretching and becoming closer to the y-axis.

Excerpt continues…

Oliver: The point. (shows with hands that the parabola would be wider…George circles
the middle option.)

Oliver: Explain your reasoning? The *a* is less than 1.

George: Yea.

Oliver: And the closer it is…

George: Yea.

Oliver: So pretty much…So I don't know how you'd explain. When it's a fraction, it…

George: Because, like if, when using the up and over method, it only goes up 1 and out

5, close to coming up. Like if it was like if we were doing this one (indicating the

far right function), it would go up 3 and only out 1,…If that makes sense.

Oliver: That makes sense.

George: So that's why it's wider.

As the researcher I would wonder why, on day two, Oliver and George had an issue with

transformation when the leading coefficient was not the number one, but they so

eloquently explained the concept on day four. I will have to go off of the assumption that

they either recalled the correct conceptual knowledge, or that they simply got mixed up

previously. There is also the possibility that the process of engaging in the tasks initiated

new learning for Oliver and George.

Example #15 In this transformation example, Jamal's and Mohammed's thinking started

off correctly but got sidetracked somehow. This problem was asking to describe the

transformation of the function $g(x) = (x + 3)^2$ from the parent graph of $f(x) = x^2$.

Jamal: $(x + 3)^2$

Mohammed: So making it equal to zero would be…

Jamal: $(x - 3)^2$ which would be 9, and just go um…

Mohammed: That way (indicating moving the graph to the left)

Jamal: No, let's go up

Mohammed: Oh okay

The boys started out conceptually knowing that they should set the portion inside the parenthesis equal to zero and solve for x. This would have given them negative (-) three which would 1) be the x-coordinate of the vertex, and 2) verify that Mohammed was correct that the graph would move to the left three units.

   Maximum/minimum point. The analysis of the maximum/minimum point began with looking at how the pairs responded to problems #7-9 on the *Multiple Representation Task*, and problem #4 on the *Mixed Methods Task #2*. I particularly focused on problems #7-9 on the *Multiple Representations Task* since this was the first place where the participants were asked questions that specified the maximum/minimum point.

   When Mohammed and Jamal were thinking out loud about the maximum and minimum point, they had the consistent thought that it had to deal with an "x" concept in the function. By this they were indicating that one needed to use either the "quadratic formula" or use the "axis of symmetry" to help find the maximum.or minimum point. Upon further analysis they think that both a minimum and maximum value(s) exist. Everything that is less than the axis of symmetry is the minimum and everything that is greater than the axis of symmetry if the maximum values.

   When Katy and Zoe were discussing the maximum and/or minimum point, they came to the final conclusion of, "To find the max or the min you have to use the vertex and arms of the parabola." While this is not incorrect, it does not specify which way the arms must go for the y-coordinate of the vertex to be a maximum or a minimum. In fact, when the video and audio are triangulated with the written paper artifact, it shows that the

girls could not come up with a specific explanation of they were thinking so they wrote a general explanation.

Oliver and George were able to say their thoughts more succintly and with a less generalized tone

Find the vertex, then depending on if the parabola opens up or down, that is you min or max value of the function. If the parabola opens upward due to a positive equation then it's a minimum value. If the parabola opens downward due to a negative equations then it's the maximum value.

Although Oliver and George were more specific in how to find the maximum or minimum of a quadratic function than the previous two pairs, all three pairs left out a key part in their thinking to understand where the actual minimum or maximum of the quadratic is. Yes, you have to have the vertex, but the minimum or maximum of the quadratic is only the y-coordinate of the vertex, not the x-coordinate, and not the entire (x, y) coordinate pair. None of the three pairs were specific with this characteristic.

Location of roots. The analysis of the axis of symmetry began with looking at how the pairs responded to problems #1-3 on the *Mixed Methods Task* #1. Due to the data analyzed, this section has been split into two subsections: 1) What are roots?, and 2) Finding the location of roots. As previously stated, the specific methods of factoring procedurally for roots are not a part of this study. Due to the boundary of time, this concept would be a study all to itself.

What are roots? According to the three pairs of participants, roots are: 1) the most broken down an answer can get (Mohammed and Jamal), and 2) The "x" that comes out of using the quadratic formula (Katy/Zoe and Oliver/George). At the initial asking none of the pairs could think of any other names for "roots" although they referred to the "x's" that would result from using the quadratic formula which could be termed as the 'x-intercepts."

Finding the location of roots. At this point in Algebra 2 they are focused more on the real roots than the imaginary ones. So when they are asked how many roots there are by looking at a diagram, they are not being asked how many complex roots there are (meaning both real and imaginary ones), but rather how many real roots are there. How many times does the function cross or touch the x-axis in general. In the figure below (Figure 10: Examples of number of (real) roots in various quadratic function), function A crosses the x-axis twice, therefore it has two (real) roots. Function B does not touch the x-axis at all, so it has zero (real) roots. Meanwhile, function C simply kisses the x-axis in one spot and returns in the direction that it originally came from, indicating that it has one (real) root.
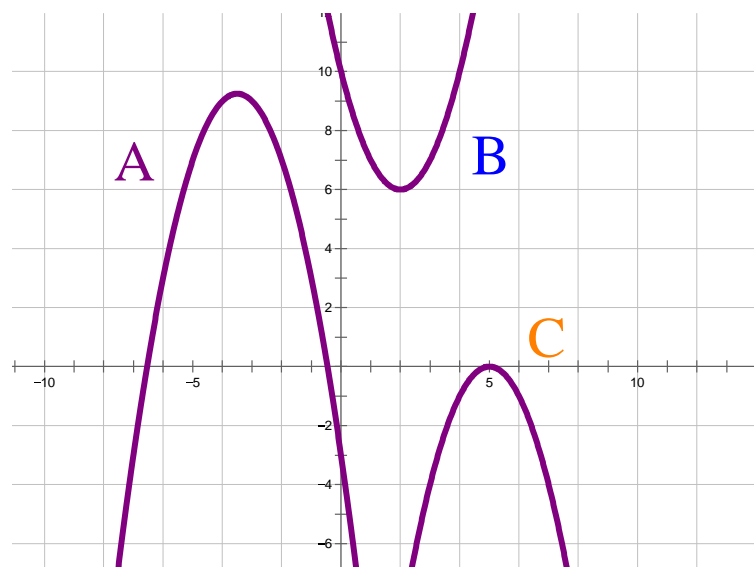
Figure 10: Examples of number of (real) roots in various quadratic functions

Example #16: In Mixed Methods #1 task problem #3, Katy and Zoe conceptually

obtained the number of (real) roots inaccurately (see Figure 11: Zoe and Katy Mixed

Methods #1 Problem #3). Going into this problem, the pair had just finished answering

questions asking for them to explain what the roots of a quadratic function were, how to

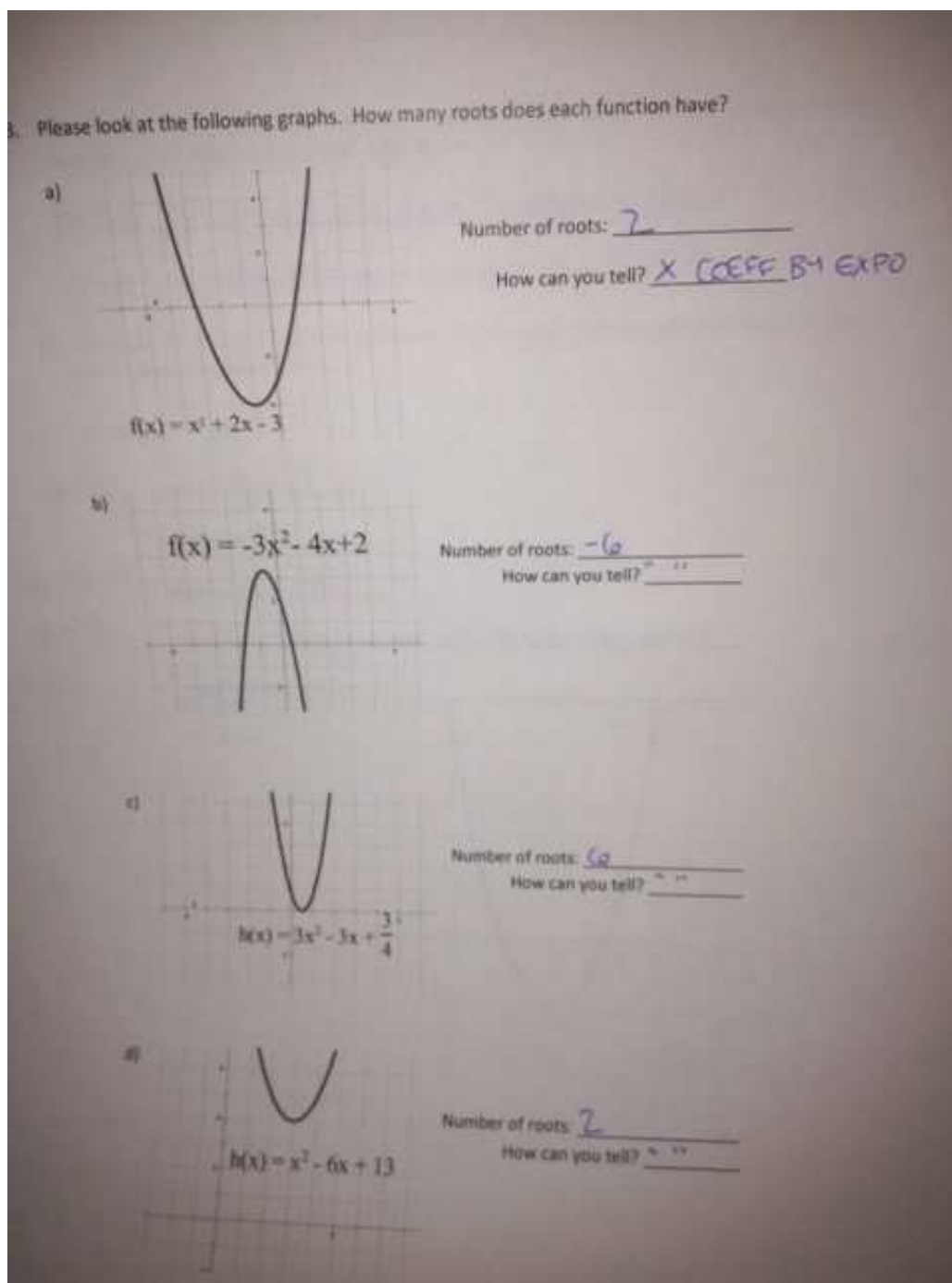find them, different names for them, etc.

3. Please look at the following graphs. How many roots does each function have?

a)

Number of roots: 2

How can you tell? X COEFF BY EXPO

$f(x) = x^2 + 2x - 3$

b)

$f(x) = -3x^2 - 4x + 2$

Number of roots: -6

How can you tell?

c)

Number of roots: 6

How can you tell?

$h(x) = 3x^2 - 3x + \frac{3}{4}$

d)

Number of roots: 2

How can you tell?

$h(x) = x^2 - 6x + 13$

Figure 11: Katy and Zoe Mixed Methods #1 Problem #3

Zoe: So what do you think the root would be?

Katy: I have no clue.

Zoe: Okay, well it has to be some part of the graph so there's the vertex, or the arms, the

      vertex is well…

Katy: Well, apparently some of these like one or some of them have more than one root

      so I mean, also they are like the *x* intercepts to something (YES!!!) I don't know.

(Thinking)

Proctor: Tell me about what you are thinking about.

Katy: I'm thinking about how frustrating this is?

.

.

Zoe: Maybe it's not talking about roots. It's talking about this. Like where it is square

rooted by two. (It is never squared rooted by two, but she is indicating where the term

that is "squared" is located.)

Katy: So what, like multiplying the coefficient by the exponent?

Zoe: Sure

Katy: I don't know but we don't have anything to put down besides that so we might as

well.

Zoe: Okay. Let's put that down.

      Even though both a graph and the function were present, the most direct route

would have been to simply look at the functions (in graphing form) and count how many

times the function crossed (or touched) the x-axis. Instead, Katy and Zoe came up with

the idea of multiplying the coefficient by the degree. The irony is that there is not any

concept with quadratic function in which you would do this process in order to get a

specific answer. While talking through the problem, this pair did actually talk about the possibility that the roots were when it touched the x-axis, but they dismissed it quickly.

In a brainstorming fashion, Katy had said, "Apparently some of these like one or some of them have more than one root so I mean, also they are like the *x* intercepts to something."

On the complete other end of the spectrum, Oliver and George got problem 3a, 3b, and 3c, correct and would have gotten 3d correct except for a slight procedural error at the end. They, too, did not solve the problems in the quick manner of simply looking at the function graphed and seeing how many times it crossed (or touched) the x-axis. Oliver and George did the very long procedural (but conceptually sound alternative) way of plugging each function into the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

and algebraically solving for the number of (real) roots that each function had. In other words, it did not matter to them that there was a graph in front of them; they went back to their comfort zone of finding the answer algebraically. This reiterates when Leinhardt, Zaslavsky, and Stein (1990) previously stated that one of the students' misconceptions dealing with functions in general was moving between representations of functions and how it poses different psychological processes. As previously stated in Chapter Two, students prefer one form of a function over the rest, and perhaps do not see the bigger picture about how they all belong to a larger scheme in the world of mathematics. This also emphasizes Knuth's (2000) concept that students rely heavily on algebraic solutions in general versus graphical methods, even if the graphical solution may in fact be quicker. This was definitely the case with this problem.

Example #17: The last pair of participants, Mohammed and Jamal, did not answer what

the question was asking. Again, the question asked for the participants to state the

number of roots for each function in a graph. Even though they had implied knowledge

on the concept of roots previously in stating that the roots were the most broken down the

quadratic function could get and that one "would use the quadratic formula" to find the

roots, they did try to go the easier direction of looking at the graph versus performing the

task algebraically.

Jamal: Please look at the following graphs. How many roots does each function have? I

don't even know how to find that.

Mohammed: Well, don't we have to use the thing…find *a*, *b*, and *c* and then plug them

into the quadratic formula?...But how do you tell…I feel like it is not…there has

to be another way to tell…(pointing to "number of roots" with pencil)

Jamal: You mean like from the graph (pointing to the graph with pencil)?

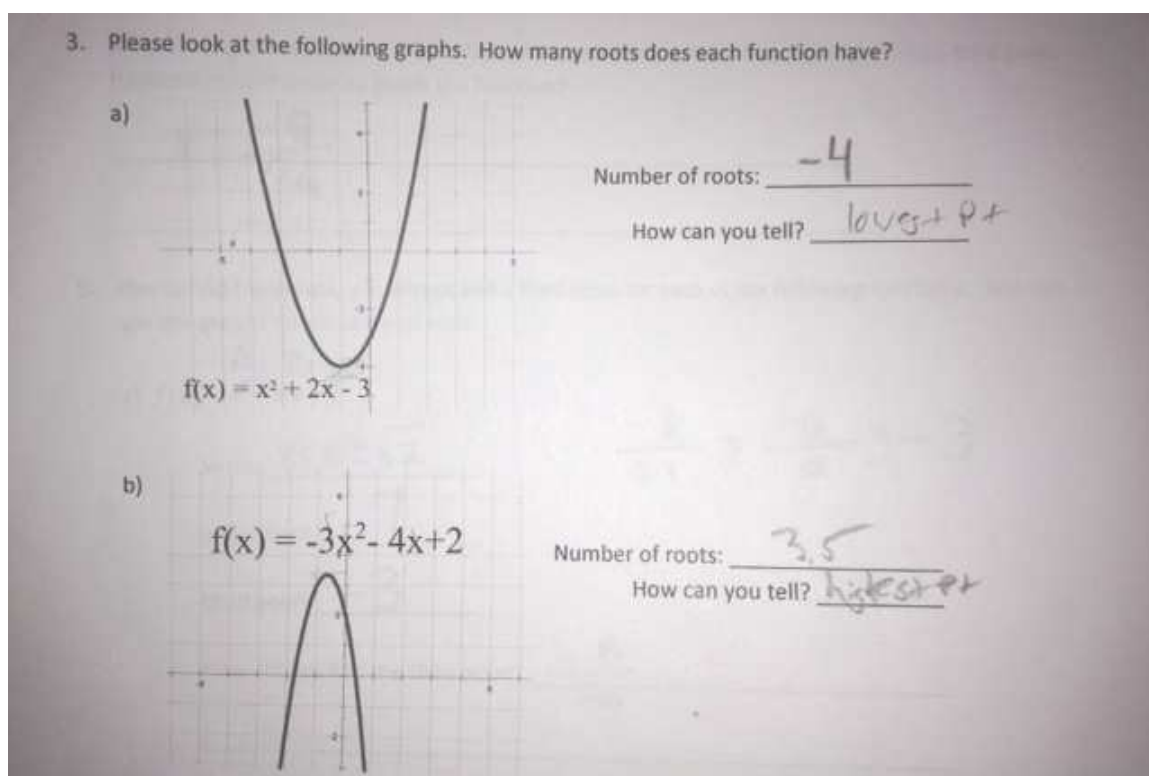**Mohammed:** Yea. (starts filling in the numbers for the roots)

3. Please look at the following graphs. How many roots does each function have?

a)

$f(x) = x^2 + 2x - 3$

Number of roots: _____ -4

How can you tell? _____ lovest Pt

b)

$f(x) = -3x^2 - 4x + 2$

Number of roots: _____ 3,5

How can you tell? _____ highest Pt

Figure 12: Jamal and Mohammed Mixed Methods #1 problem #3

It was difficult to discern why Mohammed chose these numbers until Jamal reminded

him to go back and explain (see Figure 12: Jamal and Mohammed Mixed Methods #1

problem #3). Mohammed reasoned that the "number of roots" pertained to the highest or

lowest point on the y with the vertex. In other words, Mohammed identified the

maximum/minimum point for each graph instead of the "number of roots." With their

confusion in thinking about the roots versus maximum/minimum point in these problems,

it might explain a little more why they were confused previously when trying to

understand the maximum/minimum point and thinking that it related to "x".

With the students relying so heavily on the algebraic solutions, it makes me

question the students' difficulty with directionality between the representations of the

quadratic function. With the above set of questions related to roots, only one pair

(Mohammed and Jamal, albeit incorrectly) thought to look at the graph to solve the

problem instead of trying to algebraically solve the problem by manipulating the equation

of the function.

III. Section 2: Students' Use of Strategies in Understanding the Quadratic Function

This section focuses on the second research question, "What mathematical

strategies do students employ when they work on a series of tasks dealing with the

quadratic function?" The participants engaged in discussion about the problems

concentrated on the vocabulary in the task, or at times, would attempt to explain or prove

a point by gesturing in the air, instead of on the paper. Other strategies that students

employed included 1) jumping straight into using a formula, 2) converting the current

form to a more familiar form (which was usually the standard form), or simply having

issues with the representation present in general 3) engaging in a process of elimination,

4) dissecting the problem, 5) backtracking to the same problem more than once or

backtracking to a similar problem that they had already seen, and 6) drawing or sketching

a picture. It is hard to say, though, that only one strategy was used per problem.

Therefore, except for talking and dissection, which are strategies that are incorporated

with problem solving, the examples will show the initial strategy used by the participants.

Talking. For the most part the conversations would fit into the category of

cumulative talk, where the participants constructed their answers together and were

courteous to each other in listening to one another. Occasionally a dyad would expand to

exploratory talk, where they would critically challenge each other. On the other end,

though, sometimes one participant would talk over the other participant and unfortunately

would lose out on a pertinent viewpoint. An example of this occurred with Katy and Zoe

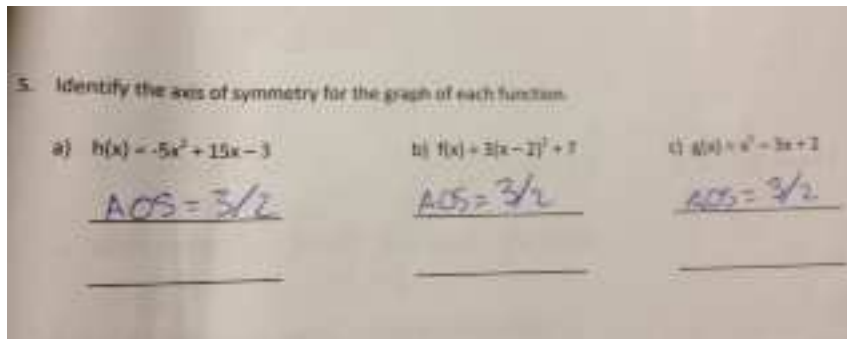(see Figure 13: Katy and Zoe Mixed Methods Task #2 problem #5).

Figure 13: Katy and Zoe Mixed Methods Task #2 problem #5

Example #18. In this example, Zoe and Katy have already found the axis of symmetry

for problem (a) and (c); backtracked to (b) and looked more closely at it.

Katy: I think that that would be 3 over 2 too, just to show us…like all the differences.

　　(giving a swooping indication of the three problems)

Zoe: I don't know. I don't see how that would be three halves (back to 5b)…because I

　　can see where the x would equal 2. (She was at the right answer already, but Katy

　　kept going.)

Katy: What about this (indicating the 7)

Zoe: Well that's a c so that wouldn't be in her form. (-b over 2a for the axis of

　　symmetry)…x = 2 and that is a 2 (indicating the number within the parenthesis),

　　but I'm not sure, it might be a 3 over it.

Zoe: Sure, that kind of looks…

Zoe: Okay…I can see where you get it. I'm not sure how to do it. But I see where you

would get that.

Katy: Do you agree?

Zoe: I would agree.

Although Zoe is making a point (albeit incorrect), Katy continued with her own thought

process and talked over her and missed Zoe's point.

     Gestures. Gestures were utilized by the participants to communicate a

mathematical concept during their conversation. Surprisingly there was not a

mathematical gesture/speech mismatch throughout the study. The recorded gestures

included: drawing parabolas in the air or on the table, pointing in a particular direction,

indicating movement of the graph, and pointing to something in reference to their speech.

Example #19: During problem #12 on the *Traditional Task*, Oliver and George

demonstrated using hand gestures to indicate parabola movement.

Oliver: The vertex form of a quadratic function is….The parent function f(x) = x squared

is translated two moves left and three moves up. Find the quadratic function in

vertex form…translated two moves left, so that means our x is…

George: This way? (Indicating with his pencil moving to the left)

Oliver: On 2, -2, and our y is 3, so yea, we have to create a parent function, write the

quadratic function.

In this example George used his pencil controlled by a hand movement to indicate that

the graph's vertex was going to slide from the parent graph's vertex of (0, 0) to the left so

that the x-coordinate of the new graph was going to be at -2.

Example #20: At other times, the participants would use their fingers to indicate the

shape of the parabola while trying to explain a concept to their partner. In this example,

Katy and Zoe describe to each other the orientation of the parabola while pointing out

key features that have led them to that belief. This is the beginning of problem #4 from

Mixed Methods Task #2.

Katy: The height that a baseball reaches when it is thrown can be modeled by the

function $h(t) = -16(t-1.5)^2 + 10$. (Katy looks at Zoe with an aspirated look.)

What is the shape of the ball's path?

Zoe: So you have to get t = …

Katy: This is the ummm vertex form, isn't it? (looks at Zoe with questioning eyes.)

Zoe: Yea. This is in the form that we just saw so it just has to be t = … ahhh…+ 1.5 and

… 10…so…and it

Katy: The shape of the ball's path…(almost whispering)

Zoe: Well no…it would be the then upside down U. (Draws an upside down U on the

desk with her finger.)

Katy: It wouldn't be straight because it would be…ah… well it would be curved.

Zoe: But it would also be upside down at the a…(Indicating the negative sign in front of

the 16 in the function)

Katy: So it would be going like this. (Showing a downwards U with her finger0

Zoe: It would be going down.

Katy: What do you mean?

Zoe: Like the U down

Katy: That's what I was saying, go like this. (Indicating with her hands again).

Through gestures and repeating the conversation the girls were able to agree on the

general shape of the flight of the ball.

      Formula. Another strategy that was used, was to undeviatingly select a formula.

By jumping straight into the formula, it is indicating that one (if not both) of the

participants did not think that a conversation was necessary and that they felt confident

enough to continue on with the problem.

Example #21: Katy and Zoe made this evident when asked to identify the axis of

symmetry for various functions on the *Multiple Representations Task* (see Figure 14: Zoe

and Katy Multiple Representations #2).

Katy: For numbers 2-4, please identify the axis of symmetry for the graph of each function.

Zoe: First we have to identify *a*, *b*, and *c*.

Katy: *a* is 1, *b* is -4, and *c* is 2. So we're going to take…

Zoe: x = -b…

Katy: Over 2a

Katy: So we're going to do *–b* which would end up being a positive 4.

Zoe: A negative times a negative would be a positive.

Zoe: Over 2 x 1 and that would be 4 over 2, which would be 2.

2. $g(x) = x^2 - 4x + 2$
$A = 1$
$B = -4$
$c = 2$

$\frac{\pm 4}{2 \cdot 1} = \frac{4}{2} = \boxed{2}$

Figure 14: Katy and Zoe Multiple Representations problem #2

In this case there was no discussion about what was being asked of them in the problem. The students did not need to discuss the problem before proceeding with the problem itself. Katy and Zoe felt confident enough to simply jump to the formula and solve the question. With this particular problem, though, problem #2 on the *Multiple Representations Task*, Katy and Zoe did not have any other options to solve the problem.

Since the function was given in the standard form, and not the vertex form, and without a

graph of the function already drawn, their only choice was to solve for a, b, and c and

plug the correct coefficients into the axis of symmetry formula. In this next example,

though, Oliver and George had options other than going straight to a formula, yet still

chose to go the formula route.

Example #22: While completing *Traditional Task* problem #4, Oliver and George

decided to algebraically solve for various attributes of the function, before graphing it

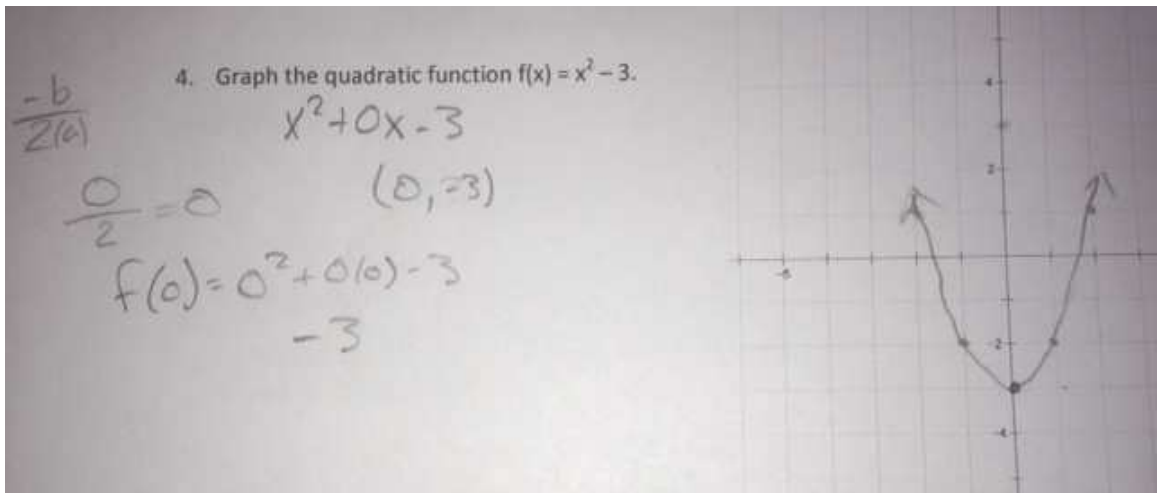(see Figure 15: Oliver and George Traditional Task Problem #4.

Figure 15: Oliver and George Traditional Task problem #4

In this problem, instead of recognizing that $f(x) = x^2 - 3$ was just a transformation of the

parent quadratic graph of $f(x) = x^2$, they went into a long (albeit correct), sequence of

finding the axis of symmetry, plugging the axis of symmetry back into the function to

finish finding the vertex, and locking down the arms with specific coordinate points that lied on the graph.

Converting forms. The representation of the function presented an issue to the participants 10.9% of the time. One of the main concerns that kept occurring is that when a function was given in vertex form. Instead of addressing the problem in the form given, the participants would convert the problem to the standard form for (assumedly) comfort purposes. This was particularly prevalent with Oliver and George as well as Katy and Zoe. This confirms Kotsopoulos' (2007) findings that students get confused when the quadratic is in various forms.

Vertex form versus standard form. If the students' work compared the standard form of the quadratic function ($f(x) = ax^2 + bx + c$) with the vertex form of the quadratic function ($f(x)-a(x-h)^2 + k$), the standard form was better understood. Out of 24 problems that had some aspect of the problem presented in the vertex form, the participants converted 11 of the problems to the standard form while attempting to solve the problem. (45.8% occurence) The following problems demonstrate how the students preferred the standard form over the vertex form.

Example #23: Finding the axis of symmetry from the vertex form was not as easily obtainable for the students as when they were finding it from the standard form. In fact, they would rather do as both Zoe and Katy, as well as Oliver and George, did in Problem #4 on the *Multiple Representations Task* and change the vertex form to the standard form before applying the axis of symmetry formula. They did this instead of simply finding

the axis of symmetry while the function was in vertex form (see Figure 16: Multiple

Representations #1 Problem #4 KEY).

Figure 16: Multiple Representations #1 Problem #4 KEY

    Due to errors in procedure, the students answered this problem incorrectly.

Recognizing that the lead coefficient was negative (-) four instead of a positive (+) four,

they may have answered the problem correctly (see Figure 17: Oliver and George

Multiple Representations #1 problem #4).

Figure 17: Oliver and George Multiple Representations problem #4

This emphasizes what Knuth (2000) stated, in which students seem to have a ritualistic procedure for solving problems when they are similar. They may go in a complete circle before they get to the right answer even though there is a more direct root that would have been quicker. The data from this study provides some support for the idea that students are more confident in understanding the standard form than the vertex form. This is supported by the fact that in Algebra 2 level students see most functions in standard form more than any other form. It does not appear that the participants have flexible competence when it comes to the concept of the axis of symmetry. In other words the participants did show proof of being able to find the axis of symmetry in various forms.

Example #24: In this next example, Katy and Zoe were working on problem #3 on the

Mixed Methods Task to find the vertex of each function, when giving in vertex form, and

then describe the transformation of each function.

Katy: And then we have to find the area (axis) of symmetry for that one. (Indicating 3b.)

So it would be $x^2 + 9 + 2$. Do you agree?

Zoe: Uh hum. Where did you get that from?

Katy: I'm just distributing it. (Mis-distributing the square into the binomial – common

procedural error.)

Zoe: It would be *x*, this is one of those that might be x = 3 because you make these into

zero so that x = 0 or x – 3 = 0, then you add three, so *x* would just be positive 3.

Although Katy and Zoe continued to successfully finish finding the y-coordinate of the

vertex, Katy's first impulse was to convert the vertex form of the function into a standard

form of the function in order to obtain the vertex. If Zoe did not speak out about the

fastest route of obtaining the vertex from its present state, they might have in fact gotten

the wrong answer since Katy did not convert the form correctly. Unfortunately though,

the vertex form of the quadratic function was not the only representation issue that

appeared in this study.

   Representation issues. Other representation concerns occurred when the pairs

were asked to "Fill in the table" with ordered pairs for a particular function (see Figure

18: Mixed Methods #2 Problem #6) or to find three different solutions (ordered pairs) for

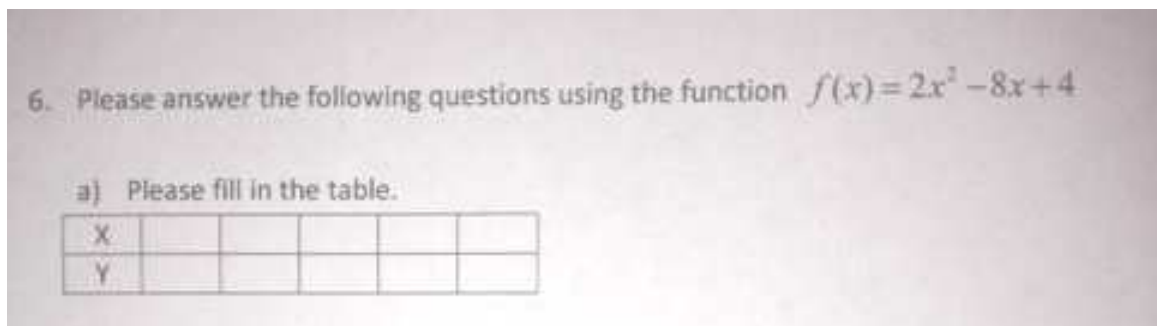a particular function (see Figure 20: Traditional Task Problem #5).

6. Please answer the following questions using the function $f(x) = 2x^2 - 8x + 4$

a) Please fill in the table.

X					
Y					

Figure 18: Mixed Methods #2 Problem #6

Example #25: During the *Mixed Methods Task* #2 on problem #6, the participants were asked to fill in a table for a particular quadratic function. None of the pairs were fully successful on this problem. Oliver and George were conceptually correct when they started out the problem by stating:

George: Please answer the following questions using the functions $f(x) = 2x^2 - 8x + 4$.

Oliver: Okay. This is easy. We've been doing this all along. Okay. We have to start with 0 because 0…

(Thinking)

Oliver: So we'll start with 0 for our x and so you take the 0 and you just plug into the formula and that's our y.

George: Okay.

As George and Oliver went through the procedural process of obtaining the pairs, they made a slight miscalculation at the end which resulted in one of the pairs being incorrect.

Example #26: In the same problem, Katy and Zoe, on the other hand, did not grasp the concept that *any* number could be plugged in for *x* and one would be able to procedurally calculate the *y* counterpart for it. Ironically, they did, however, understand that specific ordered pairs that laid on the function could be entered into the table. Though they came up with ordered pairs for the y-intercept, x-intercepts, and vertex, they only wrote down the vertex, but with a missing negative sign on the y-coordinate of the vertex.

Katy's and Zoe's actions exemplify Zaslavsky's (1997) thought that students put an overarching emphasis on only one coordinate of special points (ex.,…vertex). This also fits together with Ellis and Grinstead's (2008) work that found that students had misconceptions about connections between algebraic, tabular, and graphical representations.

Example #27: Lastly, for problem #6 on the *Mixed Methods Task* #2, Jamal and Mohammed, who demonstrated limitations in their conceptual and procedural knowledge in their attempt to solve this problem.

Mohammed: Please answer the following questions using the function $2x^2 - 8x + 4$. So…

Jamal: Please fill in the table, I don't know what we are supposed to do with the table…

Mohammed: So if the *x* was 1, then the y intercept would be…

Jamal: 4

Mohammed indicated that when you input a one for the x-coordinate of an ordered pair, you would be able to output the y-coordinate of the same ordered pair. Unfortunately, though, when Jamal heard *y intercept*, he interpreted that Mohammed was asking where the function going to cross the y-axis itself, and answered four. Jamal is correct that the y-intercept of the function $f(x) = 2x^2 - 8x + 4$ would be four, specifically (0, 4) as a coordinate pair. Due to this miscommunication, the conversation continued as follows:

Mohammed: …and then if it was 2, it would be 8. (Fills in the table but with the wrong rule for a thought process)

Instead of plugging each x into the function to output the y, he simply multiplied each x by four, which would actually result in a linear graph if plotted on an xy coordinate plane, instead of a quadratic graph. This consequently was the resulting table (see Figure 19: Jamal and Mohammed Mixed Methods Task #2 Problem #6). This reiterates what both Ellis and Grinstead (2008) and Zaslavsky (1997) have found in their studies that students make mistakes by incorrectly generalizing, or making analogies with quadratic function characteristics from linear function characteristics.
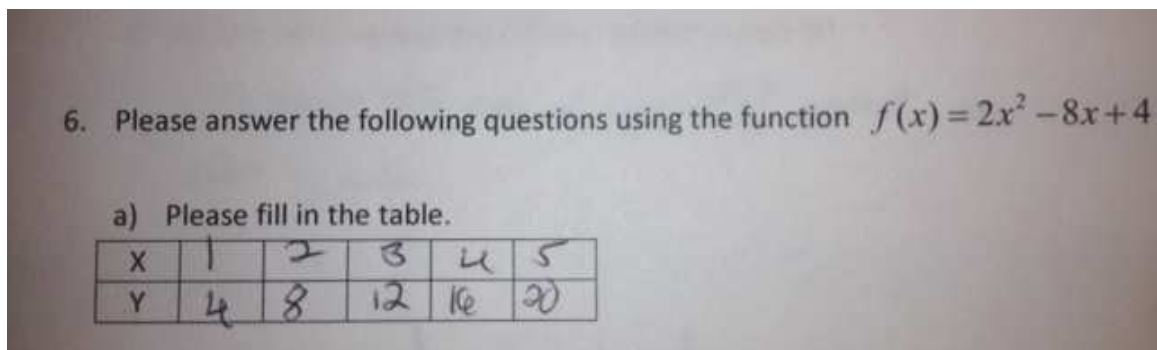
6. Please answer the following questions using the function $f(x) = 2x^2 - 8x + 4$

a) Please fill in the table.

X	1	2	3	4	5
Y	4	8	12	16	20

Figure 19: Jamal and Mohammed Mixed Methods Task #2 problem #6

As previously noted, the participants also had misconceptions when asked to find three solutions (ordered pairs), that laid on a specific function, when not given any other representation to look at. This required them to understand that any (x, y) pair that made the equation true would be acceptable answers as solutions. As with the above table format, the participants could have plugged in any number for the *x*, procedurally solved for a *y*, placed into the coordinate pair format of (x, y), instead of the table format, and they would have had a solution (see Figure #20: Traditional Task Problem #5).

5. Write down three solutions to $y = x^2 + 5x + 6$.
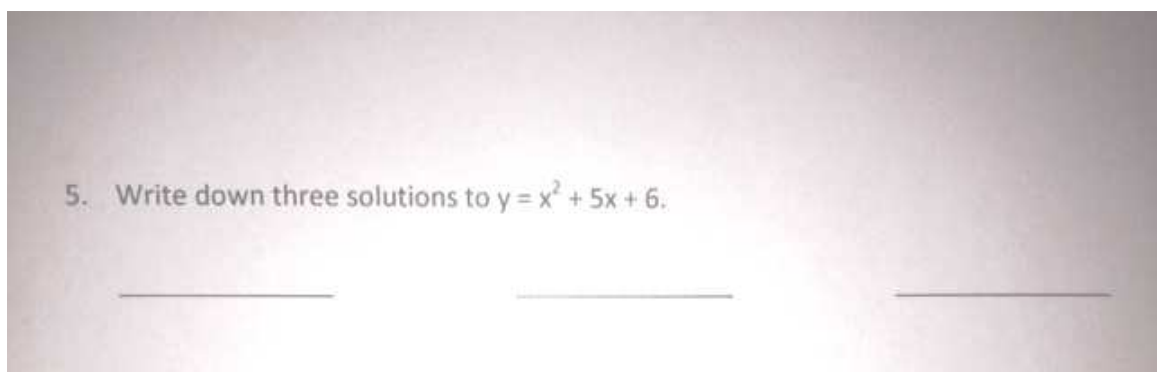
_____   _____   _____

Figure 20: Traditional Task Problem #5

Example #28: When Katy and Zoe attempted problem #5 on the *Traditional Task*, they initially skipped it, only to backtrack and return to it towards the end of their session (see Figure 21: Katy and Zoe Traditional Task #5). In the end, they did remember that they could factor and solve for the x-intercepts, but they did not put them in the (x, y) coordinate form of (-2, 0) and (-3, 0), which would have resulted in two of their three solutions. The x = ½ came from the pair arbitrarily combining the quadratic formula with the axis of symmetry in an attempt to find a third solution.

5. Write down three solutions to $y = x^2 + 5x + 6$.

$(x+3)(x+2)$ $x=-2, x=-3$ $x = 1/2$
QUADRATIC

Figure 21: Katy and Zoe Traditional Task #5

Example #29: Oliver and George were the only pair that did not backtrack to *Traditional Task* #5 in order to finish it (see Figure 22: Oliver and George Traditional Task #5). Although they were the only pair to understand that a (x, y) coordinate pair was needed in order to have a solution for the function, they did not utilize the correct procedures when solving for the y-coordinate of the vertex. They made errors in squaring -2.5 as well as confusing positive and negative signs while solving.

Figure 22: Oliver and George Traditional Task #5

The pairs had difficulty with directionality between various representations of the
quadratic function, as well as between different formats of the quadratic function. In
other words, coming up with (x, y) coordinate solutions from a function is one direction,
whereas being given a graph and asked to extract (x, y) coordinate solutions is another.
The students seemed to be more capable of solving problems successfully if they were
given the formulas and asked in a step-by-step procedure rather than to simply solve for a
specific piece of the function without any aid. This may be due to the newness of the
quadratic function in the participants' mathematics career, and perhaps one of the only
methods that was modeled and reinforced throughout their previous mathematics classes.

Process of elimination. Issues arose when participants were asked to go between
various function representations. Problem #7 on the *Mixed Methods* #2 task proved to be
one that two of the pairs (Jamal/Mohammed and Katy/Zoe) struggled with, and
eventually got wrong (see Figure 23: Jamal and Mohammed Mixed Methods Task #2
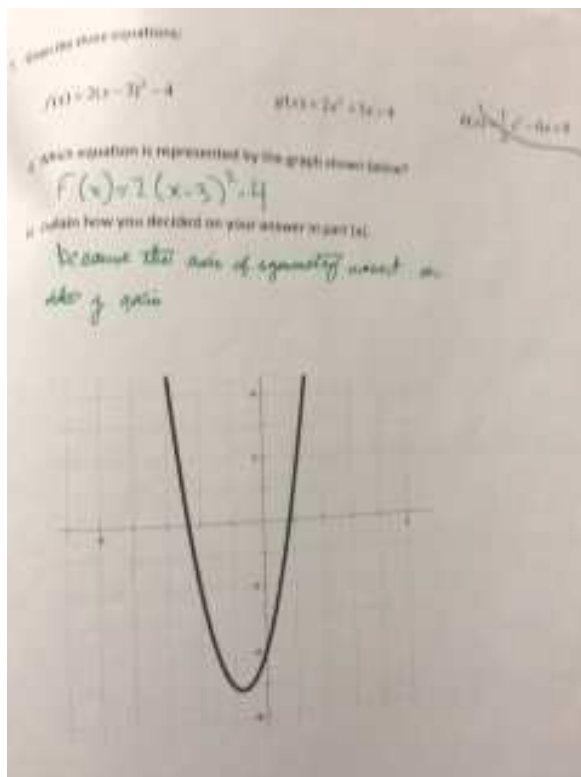Problem #7).

Figure 23: Jamal and Mohammed Mixed Methods Task #2 Problem #7

Example #29: As with all of the groups, Jamal and Mohammed decided that they wanted

to use process of elimination to decide which function best fit the given graph. They first

eliminated the most obvious wrong answer (farthest to the right) upon realizing that a

positive (+) eight would have to be involved in the graph as the y-intercept.

Jamal: Given the three equations…which equation is represented by the graph shown

below?

Mohammed: So this one is not it (slashes through the third equation)…because this is

an 8 (indicating the y-intercept.) These are very similar except…

Jamal: One is quantity squared and the other's…

Mohammed: So the one that's quantity squared…

Jamal: I think that it is this one (indicating the equation farthest left) because when it is written like that I think that it is actually on the y-axis. (They did pick the wrong one, but they did get rid of the most obviously wrong one first.)

The boys confused the vertex form (farthest left equation) with the standard form (middle equation) as well as the y-coordinate of the vertex form with the y-intercept of the standard form. The irony is that what I thought was conceptual understanding may have in fact have been procedural knowledge since they got confused when narrowing down between the first two equations. Though, this was not the only place where these two had an issue with being consistent with their thoughts.

Dissecting problems. In every problem presented to the pairs, there was one strategy that was used by all of them – dissection of the problem. The dissecting of the problems fell into two categories: 1) by dissecting the vocabulary in the problem itself, or 2) by dissecting the equation in order to answer the problem.

Vocabulary. When the participants dissected a problem by looking at the vocabulary, they would break apart the instructions until they were able to make connections with other concepts.

Example #31: In this example, *Traditional Task* problem #1, Katy and Zoe had a cumulative discussion about the vocabulary within the problem itself in order to be able to answer the problem.

Katy: Problems 1-3: For each quadratic, would the parabola open up, down, to the left

or to the right? Please explain your reasoning. 1)…$y = 3x^2 + 6x + 8$.

Zoe: Well I don't remember what a parabola is, but since it has to open up, down, left, or

right, it is that U shaped thing (indicating a U with her finger).

Katy: So of the 3x squared the 3 is our *a* so the 6 would be our *b* and the 8 is our *c*. The

3x squared is positive so our *a* is positive so it would open up.

Zoe: I would agree.

Even though initially Zoe claims to not "remember what a parabola is," after revisiting

the vocabulary, they were able to successfully work through the problem.

 Equation. Along with dissecting the vocabulary within a problem, the

participants would dissect the equation itself in order to answer the question.

Example #32: In this example, mixed methods task #2 problem 2, Oliver and George

have been asked to describe the differences and similarities between the two function f(x)

$= x^2 – 1$ and f(x) $= (x – 1)^2$. After converting the second function from vertex form to

standard form, they set about comparing the two functions.

Oliver: Alright, so…so looking at it, this is in standard form, right?

George: Uh hum.

Oliver: And this is not. That's the difference between the two functions. They both

have…

George: x^2

Oliver: Yup

George: They're different because this one is a quadratic function and, well this is a

quadratic function too, it's just missing the middle *b*. So it's different because

one is missing the *b* and the other one has it, once you do it out.

Oliver: And the vertexes of them (are different). This one is (-1, 0) (pointing to the first

function). That one is –b so zero, and actually this is (0, -1) (correcting himself)

so they have different vertexes.

George: Same numbers just flipped?

Oliver: Yup. What else? They both have different y-intercepts. This one has a -1 y-

intercept where this one has a zero y-intercept (it actually has a zero y-coordinate

to its vertex and a y-intercept of 1).

The boys continued on in the dissection of the problem, breaking apart the various pieces

of the two functions while finding their similarities and differences in order to feel

comfortable writing their answer.

   Backtracking. The students did not always feel comfortable answering problems

on the initial read. In fact, 14.5% of the time, the participants went ahead to other

problems on the task and then backtracked to the previous problems in attempts to solve it again.

Example #33: When Jamal and Mohammed were reading *Traditional Task* #1, they did not have the conceptual knowledge to be able to complete the problem.

Mohammed: So what does this say…Write down three solutions for $x^2 + 5x + 6$. What do you think a solution would be?

(Thinking)

Mohammed: Maybe we should go about solving this one.

Jamal: I don't know. What's your opinion?

Mohammed: My opinion is (noises and giggling, looking awkward about being stuck…)(Jamal fidgeting with hat)

Mohammed: Ahhh…we should just describe the transformation. (Indicationg to move on to the next set of problems #6-11.) Let's do that.

(Switches problem and returns later)

Jamal: Should we (go back to) write down the solutions? (Referring to backtracking to problem #5 that they skipped over.)

Mohammed: Yes, we have to write down the three solutions to this stuff.

Jamal: Oh yeah…ummm…

Mohammed: Three solutions (muttering it)

Jamal: I have no idea.

Mohammed: Oh jeezum

Proctor: Don't forget to speak up, otherwise it will be hard to hear what you have

previously said when I'm looking at the tapes.

Mohammed: All right. (louder) So we have to write down three solutions to $x^2 + 5x + 6$,

so what do you think we're solving for Jamal? Like…

Jamal: Probably y… (They could have used the y-intercept in an ordered pair form)

Mohammed: So we would have to …. (mouth noises), how should we start this

problem? This is pretty tricky.

Jamal: Yea. I don't know how to do this one.

Mohammed: Honestly, I kind of spaced out on this.

Mohammed and Jamal would again leave this problem and return to it a third

time, only to eventually allow it to be the only problem that they left blank in any of the

tasks. In this case I believe that the overarching concept that any ordered pair (x, y) that

lies on the quadratic qualifies to be a *solution*, was not yet a point that was conceptually

sound in their minds.

       Drawing pictures. Upon analyzing the data, I was expecting the students to draw

or sketch pictures/graphs more often than they did. There were only five instances where

a pair drew some sort of picture to help with the problem at hand. When looking back though, after coding and analyzing the data, I am not surprised that there are so few pictures initiated by the participants since the participants seemed to be solving problems primarily algebraically.

Example #34: During the *Multiple Representations Task* problem #8, Zoe and Katy were asked to "describe how to find the minimum or maximum of a function." Their general response of having "to use the vertex and arms of the parabola" has already been commented upon previously in Section II Part Maximum/minimum Point earlier in this chapter. What was not discussed, though, was that Katy suggested that whether a parabola had a minimum or maximum partially depended on which quadrant it was in.

Katy: So we could say, if the vertex and the arms are more on the negative quadrant of the graph, then the positive quadrant of the graph, then it would be a minimum, not a maximum.

Zoe: I don't think so, I think that it depends on which way it is flipped…

Katy: Because, I mean if the vertex, if it's going down, if it's flipped down, and it's like (0, 0), then it's going to be more negative. (using hand motions)

Zoe: Right, but even it it were like here, (starting to draw sketch of parabolas) and even if that's negative if it's flipped the other way, it still could be the minimum. It depends on which way it is flipped, not what quadrant it is in.

Katy: Right, Well I was saying on this side the quadrant so that it could be more specific

(not wanting to give up on her point).

**Zoe:** But does where it's placed matter?

Katy: As long as it's up or down it is saying whether it is a, has a minimum or

maximum.

While attempting to solve this problem, Katy was trying to explain to Zoe which

quadrant the parabola was in made it more or less of a minimum or maximum. Through

drawing two different parabolas (see Figure 24: Multiple Representations Problem #8 -

Zoe) that both encompassed minimum points, Zoe was able to persuade Katy to

gracefully switch her answer so that they were in agreement with one another.
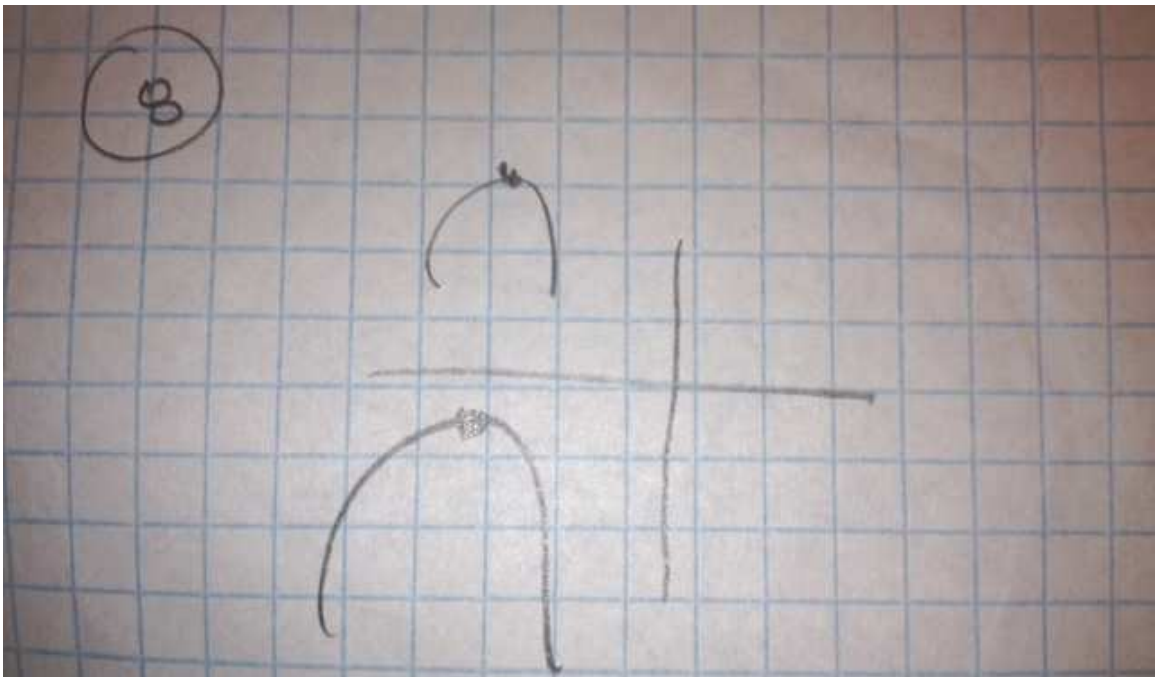
Figure 24: Multiple Representations problem #8 – Zoe

Example #34: In this next example, Oliver used a drawing to help process the problem #1

on the multiple representations task which asked "What is the axis of symmetry? How

do you find it? Is there more than one way to find it?" By referring to the drawing, he

and George were able to bounce ideas off of each other and cumulatively come to a

consensus about their answer before writing it down (see Figure 25: Multiple

Representations Task Problem #1 – Oliver)
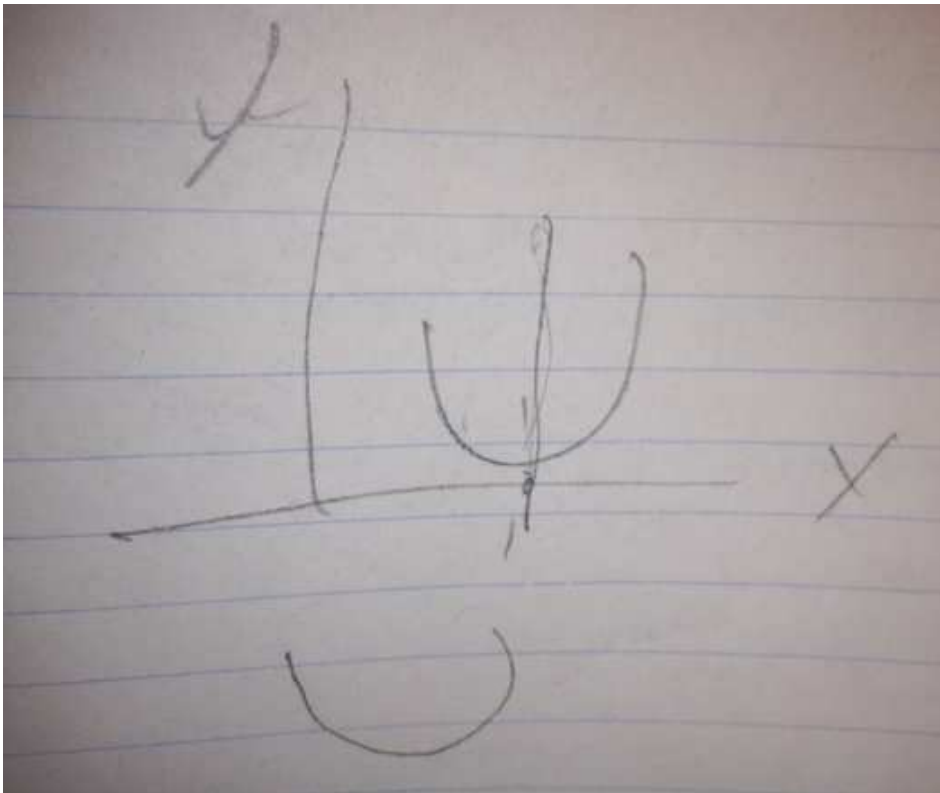
Figure 25: Multiple Representations Task problem #1 - Oliver

In this figure, Oliver is both verbally and pictorially processing the multiple manners in

which an axis of symmetry can be described, including: vertical line, mirror, line of

reflection, etc.

IV. Section 3: Differences in Understanding the Quadratic Function due to

Instructional Strategies.

The third research question was, "How does the type of task, traditional versus

multiple representation, impact students understanding of the quadratic function?" As

previously stated, by the end of this study all participants were given the same four tasks

but not necessarily in the same order (see Table 6: Daily Agenda with Participants).

Table 6

Daily Agenda with Participants

	Day 1	Day 2	Day 3	Day 4
Katy and Zoe	Traditional Task #1 53%	Multiple Representation Task #1 88%	Mixed Methods Task #1 69.5%	Mixed Methods Task #2 63.75%
Oliver and George	Multiple Representation Task #1 81%	Traditional Task #1 70%	Mixed Methods Task #1 87.4%	Mixed Methods Task #2 97.5%
Jamal and Mohammed	Traditional Task #1 40%	Multiple Representation Task #1 52%	Mixed Methods Task #1 20%	Mixed Methods Task #2 36.25%

The reason behind switching between the *Traditional Task* and the *Multiple*

Representations Task in days one and two was to counterbalance the effect that it may

have on the task in days three and four. Although in terms of performances, there were

differences in understanding between the three pairs, I believe that had to do more with

who the individuals were within the three pairs versus the order in which they were given

the tasks. Except for when the pairs participated in the *Multiple Representations Task*,

Oliver and George consistently performed the highest, Katy and Zoe were a close second, and Jamal and Mohammed were a more distant third when it came to strictly performance data. In both day three and day four of the pairs' individual tasks, their scores stayed par to the course, indicating that "counterbalancing" day one and day two was not necessary.

V. Section 4: Student Understanding the Quadratic Function Using Procedural Knowledge Versus Conceptual Knowledge

The fourth research question was, "What kinds of knowledge (procedural or conceptual) do students utilize when completing a series of tasks about the quadratic function?" Currently, there has not been a standardized method developed for assessing procedural and conceptual knowledge. With this being stated, it has become the general agreement that students use both forms of knowledge, and that they are interactive and bi-directional (Rittle-Johnson, 2012). For this study I am defining procedural knowledge as something (like steps) that can be broken down and followed. For conceptual knowledge I am defining it as the big picture, or as Hiebert and Lefevre (1986) referred to it as "a connected web of knowledge" (p. 3). When analyzing the data for procedural versus conceptual knowledge, I initially looked for the correct or incorrect answers, which does not define either procedural or conceptual knowledge. It is important to note though that one can have correctness without understanding. If the pair got the answer wrong, I then went back to the problem by looking at the transcriptions, the video, and the task artifact itself to analyze why they got the problem wrong. I was interested to know if they 1) started out (or eventually came to) the conceptual big picture for the problem and then procedurally did a miscalculation, or 2) if they did not have the conceptual big picture, or there were flaws in the big picture. In other words, they had

pieces of the web of knowledge, but due to the study being conducted towards the beginning of coming to terms with the content, the students, were still connecting the web pieces. It is even possible that the students will have only a partial understanding of the conceptual knowledge once the unit had been completed.

Mohammed and Jamal answered 16.7% of the problems correctly that pertained to the axis of symmetry, with 93.3% due to conceptual errors. Initially, when asked about the axis of symmetry in the Multiple Representations Task, they utilized the quadratic formula, which finds the root(s) of the function. They did this for the first four problems of that particular task before being presented with the actual axis of symmetry formula, $x = -\dfrac{b}{2a}$; after which they got the next two axis of symmetry problems correct, with one minor procedural negative sign error. They also went back and corrected the first of the initial four problems, but conceptually they did not start off with the correct knowledge. For these boys, they understood how to "plug and chug" out the answer when prompted (by the task) but demonstrated little conceptual knowledge of the problems.

Table 7 illustrates the breakdown of incorrect answers due to conceptual and procedural knowledge. When the student pairs obtained incorrect answers, they were primarily due to conceptual errors from the beginning.

Table 7

Incorrect Answers Due to Conceptual and Procedural Errors

Content	Overall % Incorrect	% of Incorrect Due to Conceptual Error	% of Incorrect Due to Procedural Error
Axis of Symmetry	35.2%	94.7%	5.3%
Vertex	38%	87.5%	12.5%
Graph Orientation	12.5%	66.7%	33.3%
y-Intercept	37.5%	100%	0%
Transformations	45.8%	100%	0%
Maximum/Minimum	25%	100%	0%
Location of Roots	55.6%	100%	0%
Overall	36.4%	94.8%	5.2%

One issue that arose concerning conceptual knowledge was the inability to understand concepts in various situations. The pairs demonstrated difficulties in solving the various forms of the quadratic function. Also, when the students answered the problem incorrectly, it demonstrated that there were gaps in the linking relationships between quadratic functions concepts not being as prominent as the discrete bits of information that would be used to produce a procedurally correct answer (Ben-Hur, 2006).

Rittle-Johnson and Alibali (1999) have found that when students learn procedural knowledge only, they have a harder time transferring the information; yet when students learn conceptual knowledge that is then reinforced by procedural practice, the students find it easier to grasp the broader mathematical picture. Since the participants were in the middle to the end of initially learning the material, I did not find it too surprising that they had conceptual knowledge errors. It would be interesting to check the findings if this

study had been repeated with the same participants, but a chapter or two later in their

Algebra 2 course. Perhaps they would have had a more conceptual hold on the various

concepts.

Chapter Five: Discussion

My goal in conducting this research was to gain insight into how students approach the quadratic function. Due to functions being such a broad topic, I chose to focus on the following research questions:

1) How do students think about the quadratic function as they work on a series of tasks?

2) What mathematical strategies do students employ when they work on a series of tasks dealing with the quadratic function?

3) How does the type of task, traditional versus multiple representations, impact students understanding of the quadratic function?

4) What kinds of knowledge (procedural or conceptual) do students utilize when completing a series of tasks about the quadratic function?

The six students that participated in the study each shared their thought processes as they approached the various problems within each task dealing with the quadratic function. By analyzing the data collected from the six students that participated in the study, I feel that I have been able to add to the existing literature on students and their thoughts as they approach the quadratic function.

In this chapter I will synthesize data from the study to provide information in response to the research questions. The titles of the section headings indicate their relation to the research questions listed in Chapter 1. Before my concluding thoughts, recommendations suggested by the current study are cited for future teaching as well as future research.

I. How Do Students Think about the Quadratic Function?

   As the students approached the problems within the four tasks, I attempted to breakdown and isolate the seven specific core contents that were being addressed in this study. These were: 1) axis of symmetry, 2) vertex, 3) graph orientation, 4) y-intercept, 5) graph transformations, 6) maximum/minimum point, and 7) location of roots. Ironically, I found that the students thought about the quadratic function in parts, rather than as a whole unit.

   When dealing with the axis of symmetry, the participants viewed it in two ways: 1) as a line that bisects the vertex (and the graph as a whole), and 2) as a number derived from the formula. As for the vertex, the participants considered the vertex to be the highest most (or lowest most) coordinate pair (x, y), depending on which direction the parabola was orientated. They also all agreed with the process of algebraically finding the vertex by plugging the axis of symmetry into the original function for the *x*, and then solving for the *y*.

   As the pairs addressed the graph orientation, they generally understood that if the function was positive, then the graph's parabola opened up, and if the function was negative, then the graph's parabola opened down. The term "slope" did come up though in the conversations. Slope is actually a linear concept and does not have a role in the quadratic function. This reaffirms what Zaslavsky (1997) and Ellis and Grinstead (2008) found in their studies about students having a tendency to create an analogy between quadratic functions and linear functions.

   Transforming the graphs appeared to give all of the pairs issues during the various tasks. After conversing as pairs, each student pair remembered that to transform a graph

means to slide, flip, compress, stretch, etc. It appeared, though, that the students forgot, or did not yet conceptually understand, the basic foundational rationale behind how to move the functions, and which way to move them.

When discussing the maximum/minimum point, Katy and Zoe came to the final general conclusion that "you have to use the vertex and arms of the parabola" to find the point. Oliver and George were more specific when they included the orientation of the graph indicating if the point was going to be a maximum or minimum point. None of the three pairs were specific that the y-coordinate of the vertex was the maximum/minimum point, and not the entire (x, y) vertex.

When the pairs were solving problems about the location of roots, they approached this in two manners: 1) what the roots were, and 2) find the roots themselves. All three pairs generally stated that the roots were a broken down x component of the quadratic function. When looking at various graphs, though, and indicating the location, and how many (real) roots there were on the graphs, all three of the pairs appeared to not understand the task and what was being asked of them. Although both Zoe and Katy and Mohammed and Jamal had fleeting moments of discussion that would have led them to the correct answer, only George and Oliver executed a thought process, albeit more complex and time involved, that resulted in some correct answers. Instead of simply looking at the quadratic graphs and observing how many times the functions touched or crossed the x-axis, they proceeded to use the quadratic formula to algebraically solve for the roots.

II. What Mathematical Strategies Do Students Employ?

As the participants attempted to solve problems on the four tasks, they employed various strategies along the way. These included: 1) conversation, 2) gestures, 3) undeviatingly using a formula, 4) converting the current quadratic form to a more familiar form or having an issue with the representation in general, 5) process of elimination, 6) dissecting the problem, 7) backtracking, and 8) sketching a picture. Talking through the problems of course was key, due to using the Think-Aloud method. The students used a combination of strategies for the problems.

Two key observations came from the students using strategy #4, converting the current quadratic form to a more familiar form or having an issue with the representation in general. The first one is that not once did the students convert the standard form of the quadratic to the vertex form, but rather they seemed to always convert the vertex form to the standard form. The second observation being that the pairs would prefer to algebraically solve a problem rather than observe a pre-made graph, or change the representation of the function from an equation to an x, y table.

When the pairs used the fifth strategy, process of elimination, another interesting strategy came about. When the students were looking at both the vertex form and the standard form, they confused the y-coordinate of the vertex form with the y-intercept of the standard form. This confusion could eventually lead to mis-graphing the quadratic function at a later point.

III. What Effect Do Various Instructional Strategies Have?

By the end of the study, all three pairs of participants were given the same tasks but not necessarily in the same order. (See Table 6: Daily Agenda with Participants.) The

four tasks varied between being traditional, multiple representations, and two that contained mixed methods. The general outcome of the tasks was that Oliver and George *performed* the highest, and also put the most time into each task overall, and Jamal and Mohammed performed the lowest and spent the least amount of time on the tasks overall. I believe that the instructional strategies did not have a wavering effect on the outcome. A future study could consist of three separate groups, with the first group only being given traditional tasks, the second group only being given multiple representations tasks, and the third group mixed methods. Perhaps if this was done over a four day (or longer) period one would see a difference between the instructional strategy outcomes.

IV. What Procedural Knowledge Versus Conceptual Knowledge Do Students Use?

The National Council of Teachers of Mathematics (2000) recommends that high school students should be able to "create and use tabular, symbolic, graphical, and verbal representations and to analyze and understand patterns, relations and functions" (p. 297). The data from this study reveals that the participants were limited in both their conceptual and procedural understanding of the quadratic function. The participants illustrated a variety of misconceptions when presented with standard problems related to the quadratic function. But, when given *hints* through graphs, a function, a formula, etc., they were more successful in solving the problem. In addition, the participants had higher confidence in their answers if the problems were presented in the quadratic standard form where they could algebraically solve for the answer.

One does have to remember that the students were involved in this study during or just after the time period that they were initially introduced to the quadratic function. The fact that they were not all sure of themselves in every situation is to be expected, and the

reason why the study occurred when it did in their curriculum. I did not want rehearsed and finely tuned answers; this study was an attempt to capture their initial thoughts about the quadratic function.

With quadratic functions being such an important piece of the mathematical puzzle, it is important that students have the background knowledge to do the more mundane mathematical tasks when recognizing and solving these functions, which will then bridge to other functions. On the occasion that students did understand the function presented to them, they may not have a complete understanding of all of the elements or be able to transfer the function between different representations of it – ordered pairs, table, equation, graph, etc. If a student only understands a particular form of function, due to that being the only one used in a course, that student will only retain that particular form. Procedural knowledge can allow a student to pass a class, but conceptual knowledge combined with the procedural knowledge will allow the student to be prepared for the next mathematical level.

As noted previously, the pairs preferred to convert the vertex form of the quadratic to the standard form in order to solve the problem. This is primarily due to the fact that students see the standard form of most functions more than any other form while taking Algebra 2. During the academic year of Algebra 2, each function presented to the students is usually presented as a single entity. Normally the connections between the various functions are not made until the pre-calculus curriculum. This is in line with Knuth's (2000) belief that students have a ritualistic procedure for solving problems when they are similar. Students will even go as far as engaging in extra procedural steps, although a more direct route would have been quicker, due to not understanding the

overarching schema of the concept. According to Ghazali (2011), students are more likely to use mathematical procedures rather than knowing how the mathematical procedures are achieved. In other words, they would rather focus on the calculation procedures then finding out how the conceptual pieces are intertwined.

V. Implication for Teaching

This study has the potential to offer many teaching suggestions. At the beginning it calls attention to the quadratic function itself along with key aspects of it. Students often overlook the connections between these concepts as they only see one method of getting the answer to specific questions. Perhaps students are too focused on "the tree" and are not seeing that the tree is a part of an entire forest. As a result, I believe that the students are missing key conceptual bridges not only when they are initially learning the quadratic function, but also later as they discover other functions with similar attributes.

The lack of conceptual knowledge demonstrated from these three pairs of students could be a rationale for why a constructivist approach would benefit students in learning the quadratic function. I believe that a hands-on cooperative learning style, as well as the use of multiple representations during instruction is beneficial for the students' mathematical learning and understanding over time. Currently, standard curriculum may ask students whether five quadratic functions open up or down, and then it may ask them to transfer five quadratic functions into graphs, and then to identify from the graphs of another five quadratic functions their y-intercept. It does not challenge them to dissect the function, while providing evidence to demonstrate how the parts interplay with one another, and *then* bear out this information with the rest of their classmates. I **will** be utilizing this approach the next time I introduce the quadratic function to students.

I believe that mathematics educators need to build connections not only between the various concepts within the quadratic function, but between the classes of polynomial functions as well. An interesting outcome from this study is the finding that the participants confused the vertex form and the standard form of the quadratic, specifically when addressing the y-coordinate of the vertex and the y-intercept. As a mathematics teacher, one solution in which to tackle this issue is to be mindful of specifying which form the problem is in, as well as being explicit when referring to the y-*coordinate of the vertex* versus the y-*intercept*, not just calling both a universal *y*. Another recommendation is to allow the students to compare the two forms with guided direction from the teacher so that they are active in constructing their own understanding of the differences between the two forms more concretely.

Many researchers and educators have advocated the use of multiple representations as a way to enhance conceptual understanding of many mathematical concepts. By using research-based teaching methods, one does not have to constantly engage in trial and error practices in their own classroom. Methods have already been tested. It may not work for all classrooms, but it gives a sound place to start, which would hopefully advance (and even change) the practice. This study, however, did not find a difference in understanding or achievement (based on the number of correct on the tasks) between the multiple representation tasks and the traditional tasks. One reason may be that the students alternated between the tasks and groups that were not presented solely with one method or the other. Another reason may be that there were a limited number of students (three pairs) that performed the tasks in only four days. Upon

reflection, perhaps the addition of pre- and post-tests would have yielded additional information about students understanding of quadratics.

VI. Implications for Future Research

The current study has extended the research that investigates how students approach the quadratic function with specific attention to the axis of symmetry, vertex, orientation of graph, y-intercept, transformations when the function is graphed, the maximum/minimum point, and the location of roots. This study has the potential to add to the mathematical community's knowledge of how students develop a conceptual understanding of the entire spectrum of the quadratic function. This research may be replicated and additional tasks could be designed that would continue to move towards enhancing connections among the aspects of the quadratic function.

One item that was not included in this study was the use of technology. When creating the tasks for this study, I purposefully used "easy" numbers since I wanted to know about the students' conceptual understanding about the quadratic function, not about how well they can manipulate a (graphing) calculator. In addition, if a student had misconceptions about a problem, I wanted it to be based on their mathematical conceptions, and not on a possible procedural issue related to technology. Future research could include the same tasks where groups are evenly numbered with one set of pairs being allowed to use calculators and one set of pairs not being allowed to use calculators. This would also open up the issue being able to document the students' technological procedural processes to be analyzed in addition to the other data.

Due to the structure of the tasks and their design, I was not able to systematically investigate students' understanding of the quadratic function from multiple directions.

This is directly related to the use of worksheet tasks as the method of delivery instead of conducting an interview or a mini teaching in order to obtain a collection of data. In other words, should students be presented with the algebraic, graphical, or table versions of the same function first? How would students best learn the entire spectrum of the quadratic overall? Perhaps another area of study for future research could explore the direction of how traditional and multiple representations should be presented to students initially. For this, one would need a much larger study population, along with additional teachers to administer task protocols with different curricula for the various groups of students.

Lastly, a common standardized mathematical tool could be developed to assess procedural knowledge versus conceptual knowledge as a student is solving problems. There are various opinions on how to assess the differences between these two forms of knowledge, but it is difficult to accurately compare the strategies since they are not lock in step with one another. I do not know what this tool would look like per se, but it should be a tool that clearly and efficiently assesses the various forms of knowledge.

VII. Concluding Thoughts

As I was proctoring the tasks and then analyzing the data, I was surprised by the fact that students did not know the material. Somehow their understanding was not demonstrated in the tasks, or was it? Was their knowledge so heavily procedural due to the limited time that was allowed through the curriculum timeline that when they were assessed without the aid of an instructor, that they were unable to answer the questions? I was not surprised that the results suggested that the students tended to think about isolated parts of the problem when solving the quadratic problems, but I was surprised

with the lack of conceptual knowledge that was shown. In the end, I was disappointed by the lack of conceptual knowledge demonstrated by the students in the study.

While analyzing the data, the following strategic and misconception observations were thought to be key:

- Participants preferred the standard form over the vertex form.

- Participants confused the y-intercept of the standard form versus the y-coordinate of the vertex when the function was in vertex form.

- Participants preferred algebraically solving a problem versus tabular or graphical.

- The linear function term of "slope" came up when students were discussing the transformations of the quadratic graph.

- The students interpreted the maximum/minimum point of the quadratic function to be the entire (x, y) point of the vertex instead of solely the y-coordinate of the vertex.

It was surprising how difficult it was to recruit six student participants for the study. I accredit this to the fact that the students were not accustomed to teachers asking for volunteers for this type of study. Teachers at the study school primarily work on earning their Master's degree. Most conduct action research studies, without needing to require the students' parent's permission, nor go through the Institutional Review Board (IRB) process. They only have to remove any student identifiable information from their study. There is one other person at the study school who went through this very same doctoral program, but due to the nature of his study, his participants were adults.

Therefore, to my knowledge, other than when I did other coursework, asking students to be a part of a study that involved the IRB process was very new to the study school.

Another aspect of the study that I had not considered was the difficulty entailed in playing the full role of researcher vs. teacher. With knowing the students, I knew that a word *here*, a focused finger point on the page *there*, or a quick example on the board would set them straight when I could see them getting off course or frustrated during a task. As a teacher, it was especially difficult when the students would look up with a, "Can you help us?" look on their faces, or even ask if they could ask me a question. I was consciously trying to document when these pauses in discussion were occurring so that I could bring it back to my "own" instruction when possible in the future and not leave it in the study.

I fully agree with Curran (1995) that a student's motivation, personality, and attitude play an important role in the extent to which the student will become engaged in subject matter and eventually whether a student will be successful in that subject matter or not. This speaks to larger issues surrounding curriculum and instruction though, and could also be an avenue for future research. With that being said, each student's understanding in this study was uniquely affected by the prior knowledge and experiences that they brought to the study.

Through this study, I am more aware of the importance of focusing on both my students' conceptual understanding and their procedural understanding. Perhaps I can help students to better understand mathematics when they too are able to tell the difference between the two forms of knowledge. A perfect example of this is when students are solving problems with positive and negative signs with numbers. The wrong

sign a third of the way through the problem can offset the entire remaining portion of the problem, yet conceptually they could have been solving it correctly.

The research on functions, constructivism, conceptual and procedural knowledge and other key studies have truly informed my curriculum and teaching. I am constantly looking to see if a student's mistake is due to a procedural issue (from the past) or a newer conceptual issue that would take longer to clarify. I have found that students actually appreciate knowing the misconception that led to the mistake (especially if it is procedural) and go forth with the math content with higher self-confidence. I am alert to the research findings that students do not show much flexibility in moving between the representations of functions, as well as the difficulties exhibited when describing the graphs of functions. In the future, I will be vigilant to the concept that Dreyfus and Eisenberg (1984) discovered that high ability students tend toward a graphical approach to functions, while low ability students are more attracted to pictorial and tabular representations. I have yet to make any connections between this and my various students, but am keeping it as a reference.

As a result of my research findings and the literature on student understanding of quadratics, I have taken the following steps to modify my daily teaching practice.

1. Become more conscious of the knowledge used when my students make mistakes,

2. Overemphasize the difference between the y-intercept versus the y-coordinate of a vertex, when a function is in any form,

3. Place more emphasize in the similarities of polynomial functions of the various classes: linear, quadratic, cubic, etc,

4. Attempt to have my students describe in words, whether written or verbally, their process (both conceptual and procedural) as they are approaching problems,

5. Encourage students to solve a function from many different directions,

6. Provide more hands-on discovery time with (and for) my students.

I want my students to see that some problems have various pathways to get to the answer; that as long as they can conceptually explain their reasoning so that I understand their chosen pathway, that they will get the answer correct. I want students to be able to work backwards and forward when solving problems, and to be able to explain their reasoning.

As a result of this study, I have become more aware of the benefits of utilizing multiple representations for the same problem when solving quadratics. In the future, I hope to address what Schoenfeld (1985b) has found in his various studies, that students rely heavily on textbook problems and when given a problem that deals with the same material but does not quite fit the *mold* that they are used to, they are lost in trying to solve it. My intent is to address these issues with my own students, colleagues, and future curriculum. By integrating the key study findings into my curriculum and teaching, my goal is for students to be able to obtain and use a higher level of conceptual knowledge earlier on in their learning, have higher confidence when approaching quadratic function problems, and ultimately, have a higher success in their overall mathematical understanding.

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Appendix A: Function Studies Matrix

Author(s) and date	Quant or qual	Location, sample N	Age or grade, gender	Specific function	Study prior to content, during or post?	Purpose of study	Results
Dreyfus & Eisenberg, 1984	quant	127	7th / 8th grade, coed	Functions in general	Post	Assessment of students' intuitions on the mathematical notion of functions.	High ability students tended toward a graphical approach to functions, while low ability students were attracted to pictorial and tabular presentations.
Afamasaga-Fuata'i, 1992	Qual	4 students	High school	Contextualized problems dealing in particular with linear and quadratics functions	post	To investigate the nature of students' conceptualizations of functional relationships as they emerged from solving contextual problems, and how they dealt with problems that they encountered.	Students' conceptualizations of mathematics concepts in realistic situations represented legitimate, and viable alternatives to formal views traditionally taught in school mathematics.

| Curran, 1995 | Qual | Observed 25 students, interviewed 1 teacher and 3 students, northern New England | 11th/12th, coed | Functions in general, emphasizing polynomial functions with degree greater than two | post | Determining how the students' ability to interpret the graphs of polynomial functions of degree greater than two depend and builds on their understanding of the graphs of linear and quadratic functions. | • Students made connections between the classes of polynomial functions that are inherent to the graphs of all functions
• Found contributing/inhibiting factors when making the transition to polynomial functions of higher degrees.
• All three students found describing graphs difficult.
• Students enjoyed using graphing calculators.
• A student's personality, motivation, and attitudes play an important role in the degree to which the student will become engaged. |
| Zaslavsky, 1997 | Qual | >800, in 25 classrooms, eight high schools in economically well-established areas in | 10th / 11th grade, coed, | quadratics | post | Reveal students' misconceptions surrounding quadratic functions and to possibly identify possible roots in students' earlier | Five obstacles that were identified:
1. The interpretation of graphical information (pictorial entailments)
2. The relation between a quadratic function and a quadratic equation
3. The analogy between a quadratic function and a linear equation |

						learning experiences.	4. The seeming change in the algebraic form of a quadratic function whose parameter is zero 5. The over-emphasis on only one coordinate of special points.
Knuth, 2000	Qual	284, large suburban high school	College-prep students ranging from first-year algebra to AP calculus, coed	Mainly linear	Post	Foster insight into students' understanding of connections between equations and graphs	Students relied heavily on algebraic solution methods versus graphical. Students seemed to have developed a ritualistic procedure for solving problems similar to those in the study. Students may have difficulties dealing with the graph-to-equation direction.
Schorr, 2003	Qual	Inner-city middle school in New Jersey, between 8 and 11 students met for each session	7th and 8th grade	Linear and quadratic	post	Problem-solving sessions as students interpreted graphical representations involving constant and linearly changing velocities.	Meaningful mathematical experiences in the mathematics of motion are possible for students at the middle school level.

| Metcalf, 2007 | Qual | 3 students, at a NE state university | Undergraduate Pre-calculus | Quadratic | post | What is the nature of students' understanding of algebraic and graphical representations of quadratics? | One student could perform several procedures, but showed limited relational understanding of the concepts. None of the participants showed much flexibility in moving between the representations, they also exhibited difficulties with communication. |
| Ellis & Grinstead, 2008 | Qual | Secondary students, classroom observations and 8 student interviews | Algebra II/Trigonometry class | Quadratic | post | Focus on connections between algebraic and graphical representations of quadratic functions, specifically on the roles of the parameters a, b, and c in the general form of $y = ax^2 + bx + c$. | Surprisingly two-thirds of the students in the study identified a as the "slope" of the parabola. |

Reiken, 2008	Qual	16 students, Southern California	9th grade	Linear	Pre and during	An investigation of the effects that various tasks have on how student think about slope and the Cartesian connection	Students understand the Cartesian connection from two perspectives, while they understand slope as a number in five different ways.
Strickland, 2011	Mixed	5 students idendified as having a learning disability of having difficulties in mathematics	High school	Quadratic	During	To determine the effect of blending instructional practices from both the special education and the mathematics education literature.	All of the students improved their algebraic accuracy on tasks involving quadratic expressions embedded within an area content.
Hatisaru & Erbas, 2012	Mixed	11 – 10 item diagnostic test 3 interviewed	10th grade	Concept of function	Post	Investigate perceptions on the concept of functions in vocational high schools on industry	Students have weak perception of the concept of function.

Appendix B: Consent Form

Student Participant Informed Consent

**Title of Research Project: Students' Understanding of Quadratic Functions:
Learning From Students' Voices**

Principal Investigator: Jennifer Parent

Faculty Sponsor: Regina Toolin

When reading this form, please note that the words 'you' and "your" refer to the person in the study rather than to a parent or guardian or legally authorized representative who might sign this form on behalf of the person in the study.

You are being invited to take part in this research study because as an Algebra II high school student you have been introduced to mathematical functions, but have not gone in depth specifically with quadratic functions. By participating in this study hopefully you will be able to provide insight on how to improve the instructional experience for students in the future when dealing with the quadratic function. The research study is being done as part of the requirements for the completion of a doctoral degree in Educational Leadership and Policy Studies at the University of Vermont.

You are encouraged to ask questions and take the opportunity to discuss the study with anybody you think can help you make this decision.

<u>**Why is This Research Study Being Conducted?**</u>
The question of how students learn mathematics has been a topic of interest for many decades. As a teacher of mathematics for over ten years, I have been particularly interested in not only *how* my students understand quadratic functions, but *why* they choose certain strategies and procedures for solving quadratic functions. I want to research any common misconceptions that the students may have about quadratics. In addition, I am interested in various formats and strategies for teaching quadratics that may help students learn concepts more fully. I want to inquire into how students' make sense of or develop meaning and understanding about quadratic functions. I am interested in the thought patterns and sequences that occur for students when they are engaged in learning quadratic functions and equations. In summary, this investigation is interested in the effects that traditional and multiple representation tasks have on how students think about the quadratic function.

How Many People Will Take Part In The Study?
Six high school Algebra II students will be taking part in the study.

What Is Involved In The Study?
- Six students will be asked to participate in the study. The study will occur after school over four days with a max of 45 minutes per day after school.
- The four days will be consecutive in one week.
- The week prior to the study you will be asked to come in for roughly 15 minutes to test the methods and camera.
- The students will be placed in pairs and complete the same four tasks through a "think aloud" protocol. This protocol asks participants to say whatever they are looking at, thinking, doing, feeling, etc, as they are engaged in their task(s). I will investigate first hand the process of task completion through the students' eyes (and cognitive thinking) instead of simply the final product. The tasks will be audio and video recorded for the purposes of transcription.

What Are The Risks and Discomforts Of The Study?
We will do our best to protect the information we collect from you. Information that identifies you will be kept secure and restricted. However, there is a potential risk for an accidental breach of confidentiality. Your participation in this research study will have no affect on your Mathematics grade in any manner.

What Are The Benefits of Participating In The Study?
There may be no direct benefit to you however; your participation may help your peers in the future. As a result of participation in this research, it is possible though that you may obtain a different and possibly better understanding of the quadratic function!

Are There Any Costs?
The only cost associated with participating in this study is your time.

What Is the Compensation?
Your will receive compensation in the form of a $10 gift card to Mimmo's Italian Restaurant.

Can You Withdraw or Be Withdrawn From This Study?
Your participation in this study is completely voluntary. You have the right to say no, and you may also change your mind at any time and for any reason and withdraw by contacting the researcher. If you decide to withdraw, previous data collected up to that point will still be used for the research study. Your partner will then be given the option to continue singly or stop as well. If your partner decides to withdraw, then the data collected previously from your partner will also have the possibility of be used for the study.

What About Confidentiality?

To protect your confidentiality you will be asked to self-select a pseudonym. A master list that links your identity to the pseudonym will be kept in a locked filing cabinet and the only person who will have access to it is the Principal Investigator.

All research material (audio and video recordings, tasks data, consent form) collected during the "think aloud" protocol while working on the math tasks will be stored on a password-protected computer or in a locked filing cabinet. This data will not include your name or the actual names of the other participants. The four tasks will be audio and video recorded to be transcribed, coded and analyzed. Those recordings, along with the transcriptions will be destroyed at the end of the study. Final results will be published without identifying information, only pseudonyms.

Upon request the Institutional Review Board will be granted direct access to your research record for verification of data collection methods and/or data.

Contact Information

You may contact Jennifer Parent, the Principal Investigator in charge of this study, at 802-527-6545 for more information about this study. If you have any questions about your rights as a participant in a research project or for more information on how to proceed should you believe that you have been injured as a result of your participation in this study you should contact Nancy Stalnaker, the Director of the Research Protections Office at the University of Vermont at 802-656-5040.

Statement of Consent

You have been given and have read or have had read to you a summary of this research study. Should you have any further questions about the research, you may contact the person conducting the study at the address and telephone number given below. Your participation is voluntary and you may refuse to participate or withdraw at any time without penalty or prejudice.

You agree to participate in this study and you understand that you will receive a signed copy of this form.

This form is valid only if the Committees on Human Research's current stamp of approval is shown below.

_____
Name of Participant Printed

_____
Signature of Legal Guardian or Legally Authorized Representative   Date
(applicable for children and subjects unable to provide consent)

_____
Name of Legal Guardian or Legally Authorized Representative Printed

_____
Signature of Principal Investigator or Designee   Date

_____
Name of Principal Investigator or Designee Printed

Name of Principal Investigator: Jennifer Parent, M.A.
Address: 71 South Main Street A-314, Saint Albans, Vermont, 05478
Telephone Number: 802-527-6545

Name of Faculty Sponsor: Regina Toolin, Ph.D
Address: Waterman Building 409A, University of Vermont
Telephone Number: 802-656-1024

Research Protocol

General Instructions (not to be read aloud)

- The pairs must work on the task for the entire 45 minutes. If after this time they have not complete the entire task I may tell them that they may stop.
- Each pair will be allowed to spend as much time as necessary on each problem, if they are actively "working through talking" and that the 45 minutes has not come to an end. If the pair gets stuck on a problem and it appears that they have reached a standoff, allow the pair to struggle for 5 minutes. At this point, have each student write down their individual thoughts on the lined paper provided and move on to the next problem. In addition, if the paper appears to become visibly distressed from working on a problem (after at least 10 minutes have passed) have them follow the same procedures mentioned above and move on to the next problem.
- Be sure that prompts **BOLDED** on the master tasks are asked of the pair where and when appropriate.
- I may read the question to the pair (upon request), but I am not to help define any of the words that are in the question.
- Each day the students will be provided with the task for the day (one per pair), a piece of lined paper for writing on (one per student), and pencil (one per student). At no point will they be given a calculator to use.
- While they are working on the tasks, it may be necessary to encourage the discussion or probe their thinking further. Only the following questions may be used to facilitate the discussion, elicit deeper thinking from the pair, or have them further explain their thinking. At no point am I to assist the pair in any way towards the solution to the problem.

Allowable Questions and Phrases

- "Please continue talking." I will use this to remind them to continue their discussion.
- "Tell me what you are thinking about?" (directed towards one member of the pair). This prompt should be used if it appears that one member of the pair seems confused or is not talking as much as the other member of the pair.
- "Tell me what you think about what 'your partner' said?" (directed towards one member of the pair). This prompt should be used if it appears that one member of the pair disagrees with the other, either by observing their facial expressions or through their discussion.
- "If you are stuck, try to think about the vocabulary within the question." This prompt should be used if both partners seem stumped and do not know where to start.

BOLDED Prompts on specific questions:

Task	Number	Prompt
Traditional #1	4	Think about what the graph may look like.
Traditional #1	5	How could you get a solution to any function?
Multiple Representations #1	1	If you are stuck try dissecting the vocabulary within the question.
Multiple Representation #1	8	Try to think if the maximum or minimum connect to anything else on the function that may trigger something helpful.
Mixed Methods #1	1	If you are stuck, try dissecting the vocabulary within the question.
Mixed Methods #2	2	What is alike and what is different between the functions?

Appendix D: Traditional Task #1

Traditional Task 1 (Problems 1-12)

Problems 1-3: for each quadratic, would the parabola open up, down, to the left or to the right? Please explain your reasoning.

1. $y = 3x^2 + 6x + 8$

    a) Open up
    b) Open down
    c) Open to the right
    d) Open to the left

2. $y = -x^2 - x - 6$

    a) Open up
    b) Open down
    c) Open to the right
    d) Open to the left

3. $y = -2x^2 + 7x - 9$

    a) Open up
    b) Open down
    c) Open to the right
    d) Open to the left

4. Graph the quadratic function $f(x) = x^2 - 3$.
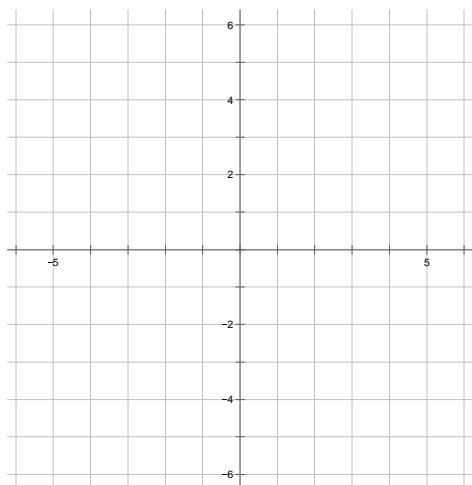
5. Write down three solutions to $y = x^2 + 5x + 6$.

_____   _____   _____

The quadratic parent function is $f(x) = x^2$. Its graph is a parabola with its vertex at the origin (0, 0). Describe each transformation from the parent function.

6. $g(x) = -x^2$

_____

7. $g(x) = (x - 1)^2$

_____

8. $g(x) = x^2 + 7$

_____

9. $g(x) = (\frac{1}{3}x)^2$

_____

10. $g(x) = (x + 3)^2$

_____

11. $g(x) = 5x^2$

_____

12. The vertex form of a quadratic function is $f(x) = a(x - h)^2 + k$.

The parent function $f(x) = x^2$ is translated 2 units left and 3 units up. Write the quadratic function in vertex form.

_____

Appendix E: Multiple Representations Task #1

Multiple Representations

Task 1 (Problems 1-10)

1. What is the axis of symmetry? How do you find it? Is there more than one way to find it?

_____

_____

_____

For numbers 2-4, please identify the axis of symmetry for the graph of each function.

2. $g(x) = x^2 - 4x + 2$

3. $h(x) = -8x^2 + 12x - 11$

4. $k(x) = -4(x + 3)^2 + 9$

For numbers 5 – 7, please tell whether each statement is true or false. Then explain why.

5. The graph of a quadratic function is **always** a parabola.

_____

_____

6. The graphs of **all** quadratic functions open upward.

_____

_____

7. The graph of f(x) = x^2 has a **maximum** value at (0, 0).

_____

_____

Please answer the following:

8. Describe how to find the minimum or maximum of a function.

_____

_____

9. Please find the following properties of the parabola $f(x) = x^2 - 4x + 3$ to be able to graph it.

a) a = _____, b = _____, c = _____

b) The graph opens (up/down). (Circle one.)

c) Axis of symmetry: x = - $\dfrac{b}{2a}$ = _____

d) Does the function have a minimum or maximum? Find it.

_____

e) Vertex: _____
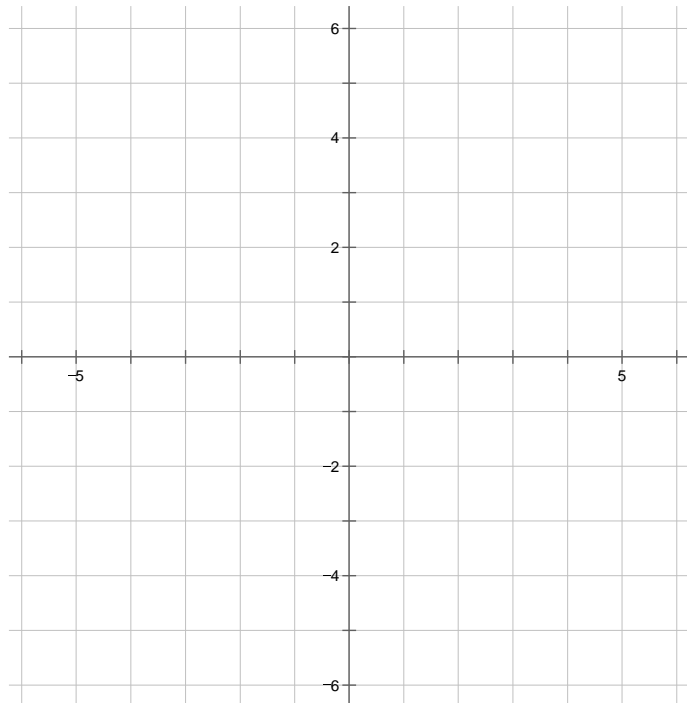
f) y-intercept: _____

10. Use the properties of a parabola to answer the following questions and then to graph it.

Please use the function $g(x) = 2x^2 + 4x - 2$

a. The graph opens (up/down). (Circle one.)

b. Axis of symmetry   _____

c. Vertex   _____

d. y-intercept   _____
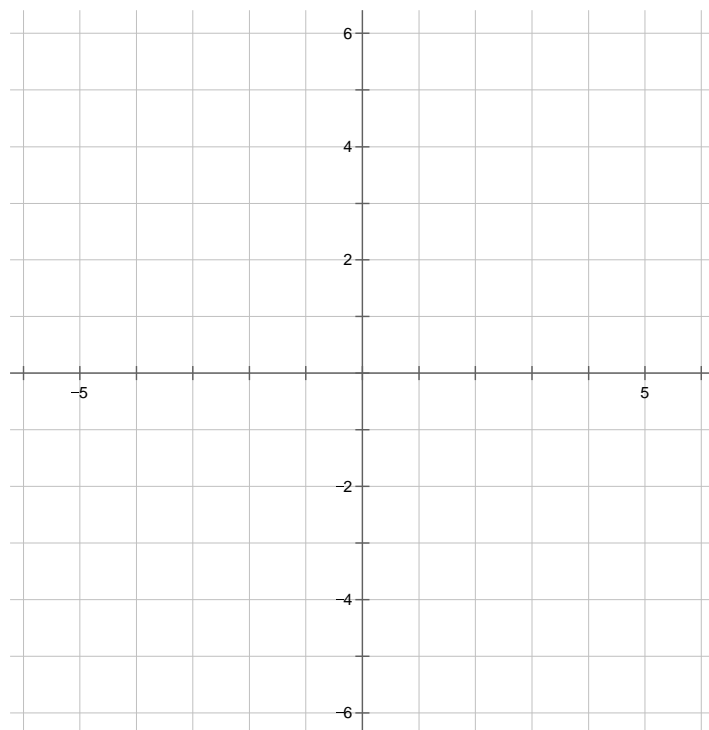
Mixed Methods Task #1 (1-5)

1. What are the *roots* of a quadratic function? What are other names for the *roots*?

    _____

    _____

    _____

2. How can you find the *roots* of a quadratic function? Is there more then one way? Can you state/describe other ways to find the roots of a quadratic function?

    _____

    _____

    _____
    You may draw pictures in addition to your written explanation if you wish.

3. Please look at the following graphs. How many roots does each function have?

a)

Number of roots: _____
How can you tell? _____

$f(x) = x^2 + 2x - 3$

b)

$f(x) = -3x^2 - 4x + 2$
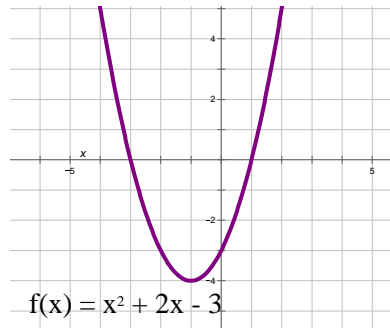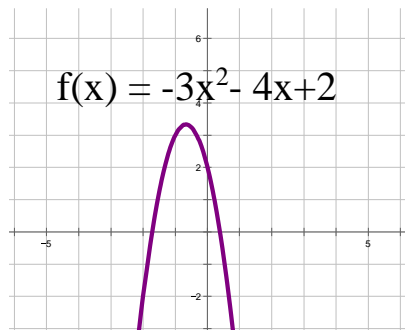
Number of roots: _____
How can you tell? _____

c)

Number of roots: _____
How can you tell? _____

$h(x) = 3x^2 - 3x + \dfrac{3}{4}$

d)

$h(x) = x^2 - 6x + 13$
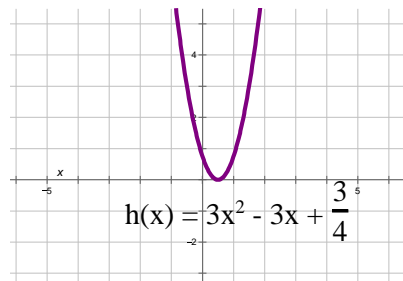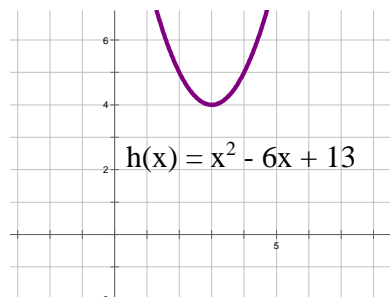
Number of roots: _____
How can you tell? _____

4. After finding the vertex and y-intercept of a quadratic function, how could you find a third point (ordered pair) in order to graph the function?

    _____

    _____

5. Please find the vertex, y-intercept and a third point for each of the following functions. You can use the graphs attached if you wish.

    a) $f(x) = x^2 - 6x + 7$

        vertex _____

        y-intercept _____

        third point _____

        How did you find the third point?

        _____

        _____

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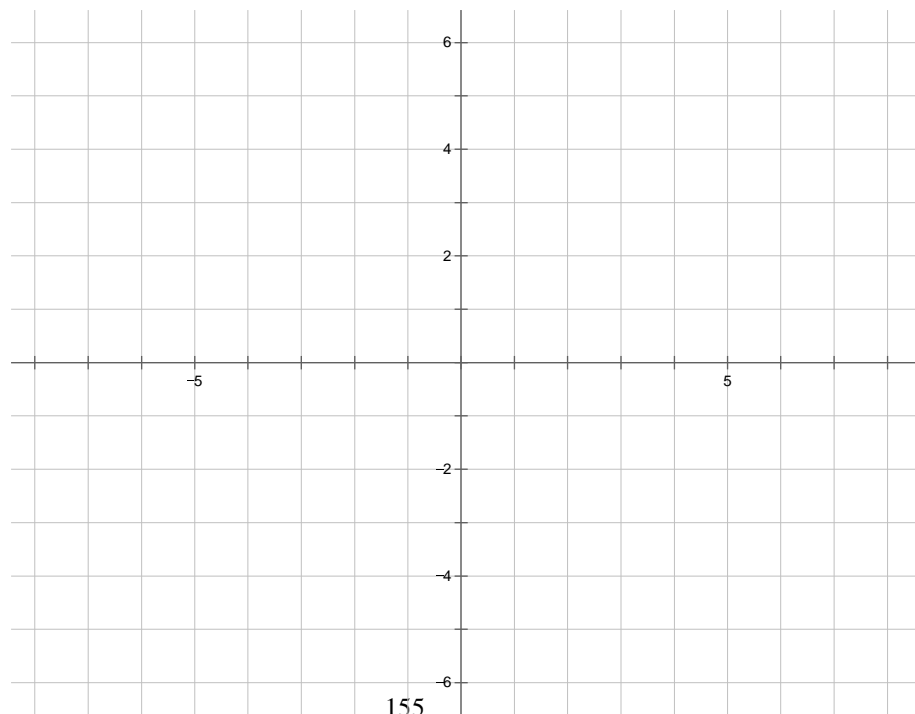
b) $g(x) = \dfrac{1}{2}x^2 - 2x - 1$

vertex _____

y-intercept _____

third point _____

How did you find the third point?

_____

_____
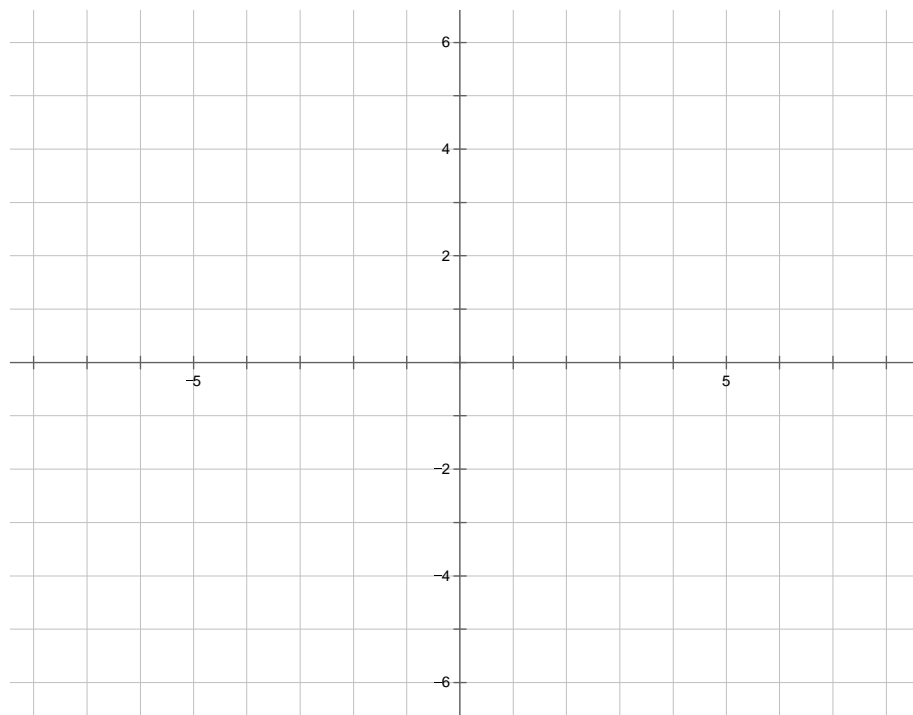
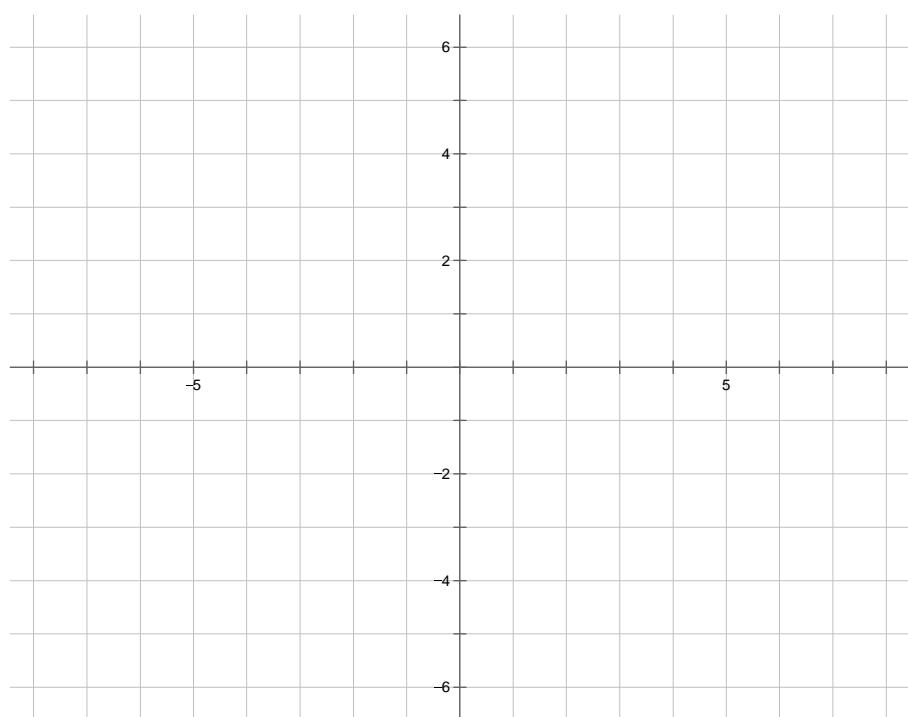c) $h(x) = -\dfrac{1}{4}x^2 + x + 2$

     vertex _____

     y-intercept _____

     third point _____

     How did you find the third point?

     _____

     _____

Mixed Methods Task #2 (1-7)

Please show all work!!

Answer the following questions about functions and transformations.

1. Circle the function that produces the widest parabola. How do you know? Please **explain** your reasoning.

   $f(x) = 2x^2 - 4$ $\qquad\qquad$ $g(x) = -\dfrac{1}{5}x^2 + 2$ $\qquad$ $h(x) = -3(x-1)^2$

2. **Describe** the difference(s) and similarity(ies) between these two functions: $f(x) = x^2 - 1$ and $f(x) = (x-1)^2$.
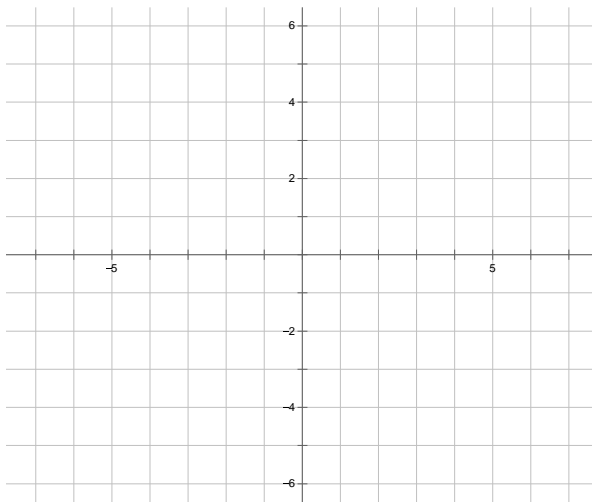
3. Use the graph of $f(x) = x^2$ as a guide. *Find* the vertex of each translation. *Graph* each function and then *describe* the transformation.
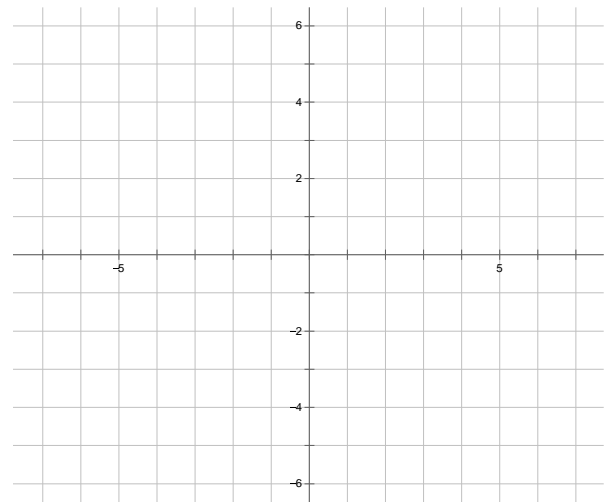
   a) $g(x) = (x + 1)^2 - 3$ $\qquad\qquad\qquad$ b) $h(x) = (x-3)^2 + 2$

   Vertex: (-1, _____) $\qquad\qquad\qquad$ Vertex: (_____ , _____)

   Transformation Description $\qquad\qquad$ Transformation Description

   _____ _____

4. The height that a baseball reaches when it is thrown can be modeled by the function
   $h(t) = -16(t - 1.5)^2 + 10$.

   a) What is the shape of the ball's path?

   _____

   b) What happens to the ball between t = 0 and t = 1.5 seconds?
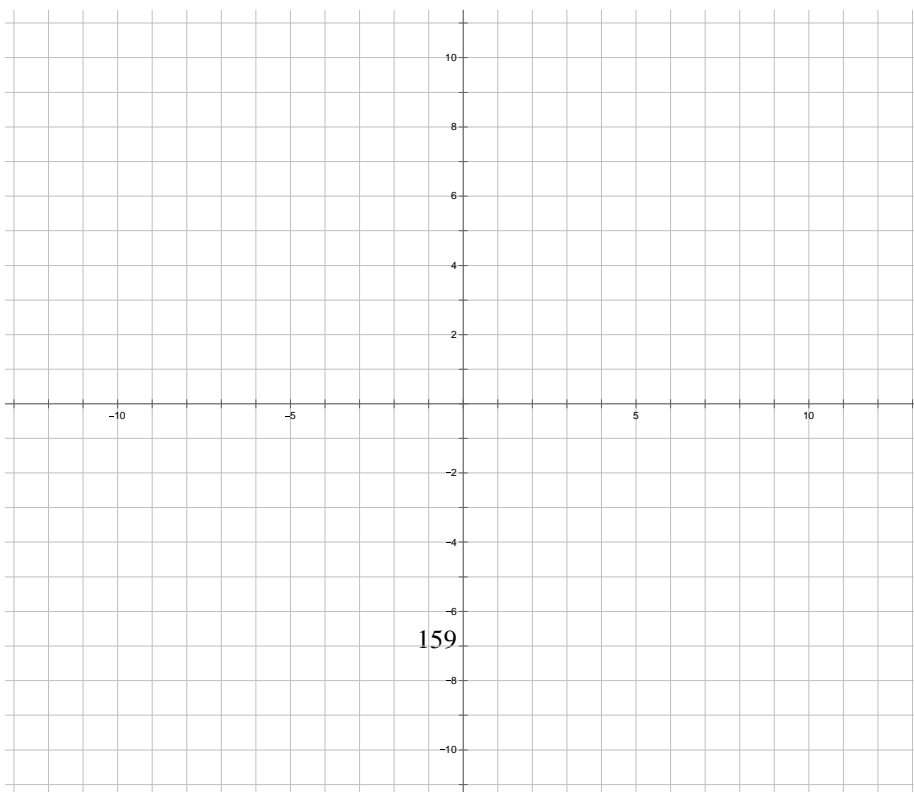
   _____

   _____

   _____

   c) Describe the transformation of h from the parent function $f(t) = t^2$.

   _____

   _____

   _____

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5. Identify the axis of symmetry for the graph of each function.

   a) $h(x) = -5x^2 + 15x - 3$ b) $f(x) = 3(x - 2)^2 + 7$ c) $g(x) = x^2 - 3x + 2$

   _____   _____   _____

   _____   _____   _____

6. Please answer the following questions using the function $f(x) = 2x^2 - 8x + 4$
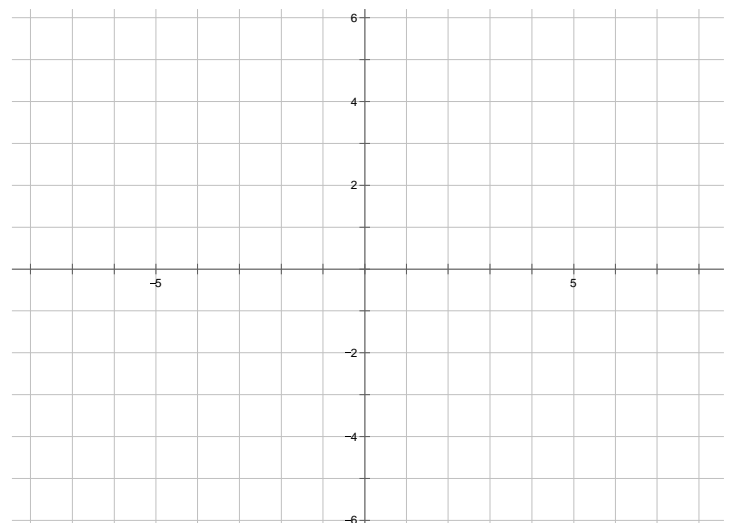
   a) Please fill in the table.

X					
Y					

   b) Please find the vertex.

   _____

   c) Please find the y-intercept

   _____

   d) Please graph the function

7. Given the three equations:
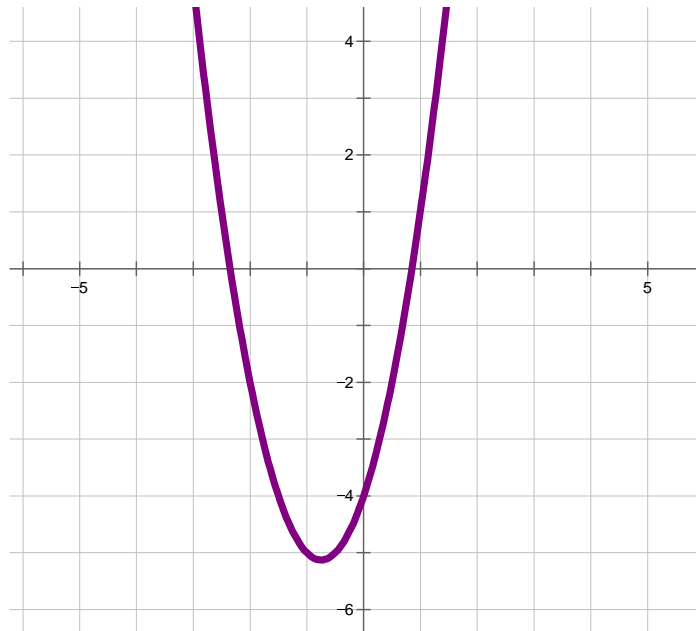
$$f(x) = 2(x-3)^2 - 4 \qquad g(x) = 2x^2 + 3x - 4 \qquad h(x) = \frac{1}{2}x^2 - 6x + 8$$

a) Which equation is represented by the graph shown below?

b) Explain how you decided on your answer in part (a).

First Day Protocol

To be read to all participants on the first day:

Thank you for participating in this study. I hope that you find what you learn here helpful in your Algebra II studies. This study will last a total of four days for about 45 minutes a day. I want to remind you that if at any point you feel uncomfortable for any reason or simply wish to stop participating in the study, you may do so at any time with no penalty. This study is not evaluative and will not impact your course grade in any way.

I'm going to present you with a series of tasks that have to do with the quadratic function. I'd like you to work on the tasks together with your partner and do so out-loud so that I can hear the discussion that you are having for recording purposes. If you wish for me to read a question to you I may upon request, but I will not be able to define any of the words in the question to you.

Each day you will be provided with the task for that day for you and your partner to share, one piece of scrap lined paper for each of you to use as you wish, and a pencil each. You can write whatever you would like on the paper. At no point may you use a calculator while completing the various tasks.

During you and your partner's discussion, I would like you to try to agree on an answer to one question before moving on to the next. In addition, I may ask you questions that are designed to help me understand what you are thinking. I will be videotaping and audio taping both of you as you are working on the tasks to help me remember what you both do at different points in the problems. Only I, Mrs. Parent, and some University of Vermont's professors will have access to the tapes. They will not be shared with anyone else. Are you ready to start? Go ahead and read the directions out loud and begin.

<h1 style="text-align:center">Appendix H: Codes</h1>

How students think…

Core Concepts

- Axis of Symmetry…(AOS)

- Vertex…(V)

- Whether the parabola opened up or down (orientation)…(O)

- Y-intercept…(y-int)

- Transformations when function is graphed…(Trans)

- Maximum/minimum point…(MP)

- Location of roots…(R)

Accuracy of what students thought

- Whether the students knew how to get to the correct answer with (CAWD) or without discussing it in depth. (In depth ≥ 30 seconds post reading initial question.)

- Whether the students knew how to get to the correct answer without (CAWOD) discussing it in depth. (In depth ≥ 30 seconds post reading initial question.)

- Indicated that they knew the right answer but wrote the wrong answer…(IK)

- Did not understand what to do at first, but came back and got minimum parts correct…(DNUGPRL)

- Did not understand what to do at first, but came back and got maximum parts correct…(DNUGPRM)

- Did not understand at first read and never got it…(DUNG)

- Got the right answer but explanation was not accurate…(AWA)

- Thought they understood at first read and did completely…(UQ)

- Thought they understood at first read, but did not at all…(UDNU)

- Thought they understood at first read but did not. They got minimal parts correct (<50%)…(UDNUPL)

- Thought they understood at first read but did not. They got most parts correct, but not all (≥50%)…(UDNUPM)

- Correct versus incorrect answer…graded according to 1, 2, 3, 4, or 5 points for each part of a question.

Strategies that students used…

- Put the problem into a more familiar form (F)

- Jump to formula (JF)

- Process of elimination (E)

- Dissected the problem (D)

- Draw a picture (P)

- Mathematical Body Gesture…(G)

- Gesture-Speech mismatch…(GSM)

- Representations of functions caused an issue for the participants…(RCI)

Procedural versus conceptual knowledge…

- Procedural understanding…(P)…systematic, step by step, as if following a recipe

- Conceptual understanding…(C)…the bigger picture, the web of knowledge, the ins and outs of a concept

Dyads

- Disputational talk (DT)

- Cumulative talk (CT)

- Exporatory talk (ET)

- Overtalk (OT)

- Not talk (OT)