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# Ordering and Reordering: Using Heffter Arrays to Biembed Complete Graphs

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# Ordering and Reordering: Using Heffter Arrays to Biembed Complete Graphs

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# Abstract

In this paper we extend the study of Heffter arrays and the biembedding of graphs on orientable surfaces first discussed by Archdeacon in 2014. We begin with the definitions of Heffter systems, Heffter arrays, and their relationship to orientable biembeddings through current graphs. We then focus on two specific cases. We first prove the existence of embeddings for every $K_{6n+1}$ with every edge on a face of size 3 and a face of size $n$. We next present partial results for biembedding $K_{10n+1}$ with every edge on a face of size 5 and a face of size $n$. Finally, we address the more general question of ordering subsets of $\mathbb{Z}_n \setminus \{0\}$. We conclude with some open conjectures and further explorations.

# Acknowledgements

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# Chapter 1

# Introduction and definitions

Recently there has been much interest in biembeddings of complete graphs on surfaces. In this thesis we continue this study; specifically we look at orientable embeddings of the complete graph on $2mn + 1$ vertices with each edge on both an $m$-cycle and an $n$-cycle. In particular, we will concentrate on the cases where $m = 3$ and $m = 5$. Such an embedding is called a biembedding. Note that such an embedding is necessarily 2-colorable with the faces that are $m$-cycles receiving one color while those faces that are $n$-cycles receive the other color. So each pair of vertices occur together in exactly one $m$-cycle and one $n$-cycle. Hence this is a simultaneous embedding of an $m$-cycle system and an $n$-cycle system on $2mn + 1$ vertices.

There has been extensive work done in the area of biembedding 3-cycle systems, or so called Steiner triple systems. In 2004, both Bennet, Grannell, and Griggs [5] and Grannell and Korzhik [10] published papers on *nonorientable* biembeddings of pairs of Steiner triple systems. In [9] the eighty Steiner triple systems of order 15 were also proven to have *orientable* biembeddings. In addition, Granell and Koorchik [11] gave methods to construct orientable biembeddings of two cyclic Steiner triple systems from current assignments on Möbius ladder graphs. Brown [7] constructed a class of biembeddings where one face is a triangle and one face is a quadrilateral. A useful survey on biembeddings of Steiner triple systems can be found in [15]. Most recently, Forbes, Griggs, Psomas, and Širáň [8] proved the existence of biembeddings of pairs of Steiner triple systems in orientable pseudosurfaces with one pinch point, and McCourt [13] gave nonorientable biembeddings for the complete graph on $n$ vertices with a Steiner triple system of order $n$ and a Hamiltonian cycle for all $n \equiv 3 \pmod{36}$ with $n \geq 39$. Example 1.1 shows a biembedding of the complete graph on 7 vertices using a pair of Steiner triple systems.

**Example 1.1.** *Each edge of $K_7$ in Figure 1.1 is on one black face and one white face, each a triangle. This is an example of biembedding $K_7$ on the torus using a pair of Steiner triple systems.*

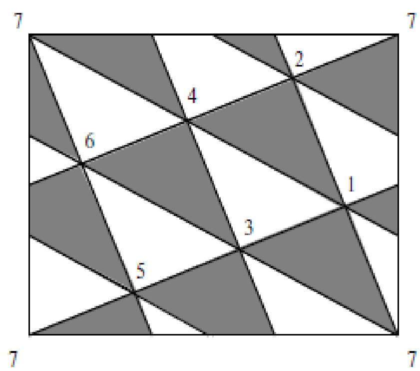


*Figure 1.1: Biembedding $K_7$ on the torus with two Steiner triple systems.*

Despite numerous results using Steiner triple systems to biembed complete graphs, this thesis presents the first biembeddings of the complete graph on $6n + 1$ vertices using a Steiner triple system and an $n$-cycle system. In other words, for $n \geq 3$ we use a special array, called a Heffter array, to prove the existence of an orientable biembedding of $K_{6n+1}$ such that every edge is on both a 3-cycle and an $n$-cycle. We also use Heffter arrays to explore biembedding the complete graph on $10n + 1$ vertices with each edge on both a 5-cycle and an $n$-cycle. Here we begin with the definition of Heffter systems and Heffter arrays summarized from a recent paper by Archdeacon [1].

Let $\mathbb{Z}_r$ be the cyclic group of odd order $r$ whose elements are denoted 0 and $\pm i$ where $i = 1, 2, ..., \frac{r-1}{2}$. A *half-set* $L \subseteq \mathbb{Z}_r$ has $\frac{r-1}{2}$ nonzero elements and contains exactly one of $\{x, -x\}$ for each such pair. A *Heffter system* $D(r, k)$ is a partition of $L$ into parts of size $k$ such that the sum of the elements in each part equals 0 modulo $r$. Two Heffter systems, $D_1 = D(2mn + 1, n)$ and $D_2 = D(2mn + 1, m)$, on the same half-set, $L$, are *orthogonal* if each part (of size $n$) in $D_1$ intersects each part (of size $m$) in $D_2$ in a single element. A *Heffter array* $H(m, n)$ is an $m \times n$ array whose rows form a $D(2mn + 1, n)$, call it $D_1$, and whose columns form a $D(2mn + 1, m)$, call it $D_2$. Furthermore, since each cell $a_{i,j}$ contains the shared element in the $i^{th}$ part of $D_1$ and the $j^{th}$ part of $D_2$, these row and column Heffter systems are orthogonal. So an $H(m, n)$ is equivalent to a pair of orthogonal Heffter systems. In Example 1.2 we give orthogonal Heffter systems $D_1 = D(31, 5)$ and $D_2 = D(31, 3)$ along with the resulting Heffter array $H(3, 5)$.

**Example 1.2.** *A Heffter system $D_1 = D(31, 5)$ and a Heffter system $D_2 = D(31, 3)$:*
$D_1 = \{\{6, 7, -10, -4, 1\}, \{-9, 5, 2, -11, 13\}, \{3, -12, 8, 15, -14\}\}$,
$D_2 = \{\{6, -9, 3\}, \{7, 5, -12\}, \{-10, 2, 8\}, \{-4, -11, 15\}, \{1, 13, -14\}\}$.

*The resulting Heffter array $H(3, 5)$:*

$$\begin{bmatrix} 6 & 7 & -10 & -4 & 1 \\ -9 & 5 & 2 & -11 & 13 \\ 3 & -12 & 8 & 15 & -14 \end{bmatrix}.$$

Let $A$ be a subset of $\mathbb{Z}_r \setminus \{0\}$ with $\sum_{a \in A} a \equiv 0 \pmod{m}$ such that no pair $\{x, -x\}$ appears in $A$. Let $(a_1, ..., a_k)$ be a cyclic ordering of the elements in $A$ and let $s_i = \sum_{j=1}^{i} a_j \pmod{m}$ be the $i^{th}$ partial sum. The ordering is *simple* if $s_i \neq s_j$ for $i \neq j$. A Heffter system $D(r, k)$ is *simple* if and only if each part has a simple ordering. Further, a Heffter array $H(m, n)$ is *simple* if and only if its row and column Heffter systems are simple. A *$k$-cycle system* on $r$ points is a collection of $k$-cycles with the property that any pair of points appears in a unique $k$-cycle. The following proposition [1] describes the connection between Heffter systems and $k$-cycle systems.

**Proposition 1.3.** *[1] The existence of a simple Heffter system $D(r, k)$ implies the existence of a simple $k$-cycle system decomposition of the edges $E(K_r)$. Furthermore, the resulting $k$-cycle system is cyclic.*

*Proof.* Let $\{a_1, ..., a_k\}$ be a part of the $D(r, k)$ and assume the ordering $(a_1, ..., a_k)$ is simple. Form a walk $(0, s_1, s_2, ..., s_k)$ in the complete graph $K_r$ with vertex set $\mathbb{Z}_r$. Develop this walk modulo $r$ and repeat the process for each part of $D(r, k)$. Since each ordering is simple, the $k$-walks contain no vertex twice and hence form simple $k$-cycles. Moreover, because each pair $\{x, -x\}$ has exactly one element in $D(r, k)$, each difference appears only once, and hence the simple $k$-cycles partition $E(K_r)$. Clearly this construction yields a cyclic $k$-cycle system. $\square$

**Example 1.4.** *Let $r = 19$, $k = 3$, and $D(19, 3) = \{\{8, 2, 9\}, \{7, -3, -4\}, \{1, 5, -6\}\}$. Note that the orderings presented are indeed simple. Then let $K_{19}$ have vertex set $\mathbb{Z}_{19}$. We partition the edges of $K_{19}$ into 3-cycles by following the procedure presented in the above proof.*

| | | |
|---|---|---|
| *(0, 8, 10)* | *(0,7,4)* | *(0,1,6)* |
| *(1,9,11)* | *(1,8,5)* | *(1,2,7)* |
| *(2, 10, 12)* | *(2, 9, 6)* | *(2,3,10)* |
| ... | ... | ... |
| *(18,7,9)* | *(18,6,3)* | *(18,0,5)* |

Let $D_1 = D(r, k_1)$ and $D_2 = D(r, k_2)$ be two orthogonal Heffter systems with orderings $\omega_1$ and $\omega_2$ respectively. The orderings are *compatible* if their composition $\omega_1 \circ \omega_2$ is a cyclic permutation on the half-set. The importance of compatible orderings will be discussed in the next chapter.

In Chapter 2 we relate Heffter arrays to biembeddings of complete graphs using current graphs, establishing the motivation for the remainder of the paper. In Chapter 3 we use Heffter arrays to prove the existence of orientable biembeddings of the complete graph on $6n+1$ vertices with each edge on a 3-cycle and an $n$-cycle. Chapter 4 discusses partial results on the existence of orientable biembeddings of the complete graph on $10n+1$ vertices. In Chapter 5 we discuss a more general conjecture concerning the sequencing of subsets of $\mathbb{Z}_n \setminus \{0\}$. Finally, we conclude with further study and open conjectures.

# Chapter 2

# Relating Heffter arrays and biembeddings

In this chapter we describe the relationship between Heffter arrays and biembeddings of graphs on orientable surfaces using current graphs, summarized from Archdeacon [1]. We assume some basic knowledge of the reader pertaining to graphs, current graphs, and derived embeddings. For more detailed information and explanations see [12].

## 2.1 Orientable embeddings and current graphs

Consider a graph $G$ and for every edge let $e^+$ and $e^-$ denote its two possible directions. Let $D(G)$ be the set of all directed edges, and define $\tau$ as the function swapping $e^+$ and $e^-$ for every edge. Let $D_v$ denote the set of edges directed out of $v$. A *local rotation* $\rho_v$ is a cyclic permutation of $D_v$. Selecting a local rotation for each vertex collectively gives a *rotation*, $\rho$, of $D(G)$. Given a rotation on $G$ we can use $\rho \circ \tau$ to calculate the face boundaries of a cellular embedding of $G$ on an orientable surface. This process is called the *face-tracing algorithm*. A rotation $\rho$ such that $\rho \circ \tau$ gives a single cycle is called a *monofacial rotation*; such an embedding (with a single face) is called a *monofacial embedding*.

A *current assignment* on $G$ with currents from $\mathbb{Z}_r$ is a function $\kappa : D(G) \to \mathbb{Z}_r$ such that $\kappa(e^+) = -\kappa(e^-)$. A current assignment on a monofacial embedding of a graph is frequently used to construct a rotation on a complete graph. We often require that (1) $\kappa$ is a bijection between $D(G)$ and $\mathbb{Z}_r \setminus \{0\}$, and (2) $\kappa$ satisfies *Kirchoff's Current Law (KCL)*, which states that for every vertex $v, \sum_{e \in D_v} \kappa(e) \equiv 0 \pmod{r}$.

## 2.2 Heffter arrays and biembeddings

An $(s,t) -$ *biregular* graph with *biorder* $(m,n)$ is a bipartite graph with one part having $m$ vertices of degree $s$ and the other part having $n$ vertices of degree $t$. A *biembedding* of a graph is one that is face 2-colorable. The following theorems from [1] lead us to the relation between Heffter arrays and biembeddings of graphs on orientable surfaces.

**Theorem 2.1.** *[1] Let $G$ be an $(s,t)$-biregular graph of biorder $(m,n)$. Suppose that $G$ has a rotation $\rho$ giving a monofacial embedding and a bijective current assignment $\kappa : D(G) \to \mathbb{Z}_{2ms+1}$ satisfying KCL. Furthermore, assume each local rotation on $G$ is simple with respect to $\kappa$. Then there exists an embedding of $K_{2ms+1}$ on an orientable surface such that each edge lies on a simple $s$-cycle face and a simple $t$-cycle face.*

*Proof.* We use the standard construction of a derived embedding from a current graph. The vertex set of $K_{2ms+1}$ consists of the elements of $\mathbb{Z}_{2ms+1}$. Let $e_1, e_2, ..., e_{2ms}$ denote the directed edges traversed in the single face of the embedding of $G$. Define the local rotation at vertex $i \in \mathbb{Z}_{2ms+1}$ as $(\kappa(e_1)+i, \kappa(e_2)+i, ..., \kappa(e_{2ms})+i)$. We use the face-tracing algorithm to show that a vertex of degree $d$ in $G$ satisfying $KCL$ corresponds to $2ms+1$ faces of size $d$ in the embedding of $K_{2ms+1}$. Since the graph is $(s,t)$-biregular, each edge of $K_{2ms+1}$ lies on a face of size $s$ and a face of size $t$. Moreover, since each local rotation is simple, then the corresponding faces are simple cycles. $\square$

For the remainder of the paper we will assume that $t = m$ and $s = n$, as we have $m \times n$ Heffter arrays with $m$ elements in each column and $n$ elements in each row. It is interesting to notice what orientable surface we are biembedding on. Euler's formula,

$V - E + F = 2 - 2g$, can be used to determine the genus of the surface. It is easy to compute that for $K_{2mn+1}$, the number of vertices is $V = 2mn + 1$, the number of edges is $E = \binom{2mn+1}{2}$, and the number of faces is $F = \binom{2mn+1}{2}(1/m + 1/n)$. Substituting these values into Euler's formula we get the following proposition.

**Proposition 2.2.** *For $m, n \geq 3$ and using the construction from Theorem 2.1, $K_{2mn+1}$ can be biembedded on the orientable surface with genus*

$$g = 1 - 1/2\Big[2mn + 1 + \binom{2mn+1}{2}(1/m + 1/n - 1)\Big].$$

**Example 2.3.** *Using Proposition 2.2 we can compute the genus of the surface on which we biembed $K_{31}$, where $m = 3$ and $n = 5$:*

$$g = 1 - 1/2\Big[31 + \binom{31}{2}(1/3 + 1/5 - 1)\Big] = 1 - 1/2(31 + 465(-7/15)) = 94.$$

*So $K_{31}$ can be embedded on an orientable surface with genus 94 such that every edge is on both a 3-cycle and a 5-cycle.*

The following proposition relates Heffter arrays to current assignments.

**Proposition 2.4.** *[1] A Heffter array $H = H(m, n)$ is equivalent to a bjective current assignment $\kappa$ on an $(n, m)$-biregular graph $G$ of biorder $(m, n)$. Two compatible simple orderings $\omega_r$ and $\omega_c$ are equivalent to a monofacial rotation $\rho$ on $G$, where $\rho$ is simple with respect to $\kappa$.*

*Proof.* Let $H(m, n)$ be such a Heffter array with compatible simple orderings $\omega_r$ and $\omega_c$. Form a bipartite graph $G$ whose vertex set consists of the rows of $H$ in one part and and the columns of $H$ in the other. For each cell $a_{i,j}$ in $H$ add an edge in $G$ labeled with current $a_{i,j}$ directed from the vertex corresponding to the $i$th row of $H$ to the vertex corresponding to the $j$th column of $H$. Since $H$ has $n$ entries per row and $m$ per column, $G$ is an $(n, m)$-biregular graph of biorder $(m, n)$. Furthermore, each row and column of $H$ sums to 0 by definition and thus $G$ satisfies $KCL$. Finally, the entries of $H$ form a half-set, $L$, and so $G$ has a bijective current assignment $\kappa$.

Now define $\tau : \mathbb{Z}_{2mn+1} \setminus \{0\} \to \mathbb{Z}_{2mn+1} \setminus \{0\}$ such that $\tau(a) = -a$. We use $\tau$ along with the compatible orderings $\omega_r$ and $\omega_c$ to define $\gamma : \mathbb{Z}_{2mn+1} \setminus \{0\} \to Z_{2mn+1} \setminus \{0\}$ by:

$$\gamma(a) = \begin{cases} \omega_r(a), & a \in L, \\ \tau \circ \omega_c \circ \tau(a), & a \notin L. \end{cases}$$

Note that if $a \in L$, $(\gamma \circ \tau)^2(a) = \omega_r \circ \omega_c(a)$. Since the orderings are compatible, $(\gamma \circ \tau)^2(a)$ acts cyclically on $L$. Also the odd powers of $\gamma \circ \tau$ act cyclically on $-L$ and thus $\rho = \gamma \circ \tau$ acts cyclically on $\mathbb{Z}_{2mn+1} \setminus \{0\}$ and the embedding is monofacial. The reverse of the construction above gives the equivalence. $\square$

**Example 2.5.** *Here we show the bipartite graph created using the process above for the $H(3,5)$ from Example 1.2. Note that the graph satisfies KCL and that all of the current assignments are distinct.*

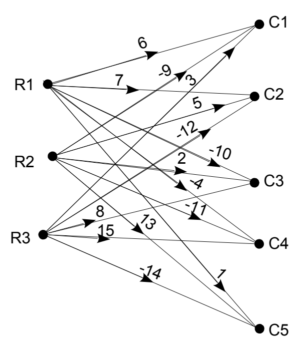


*Figure 2.1: Bipartite current graph for $H(3,5)$.*

**Corollary 2.6.** *[1] Given a Heffter array $H(m, n)$ with simple compatible orderings $\omega_r$ on $D(2mn + 1, n)$ and $\omega_c$ on $D(2mn + 1, m)$, there exists an embedding of $K_{2mn+1}$ on an orientable surface such that every edge is on a simple cycle face of size $m$ and a simple cycle face of size $n$.*

*Proof.* Let $H$ be such a Heffter array. Proposition 2.4 gives us a bijective current assignment $\kappa$ on an $(n, m)$-biregular graph $G$ of biorder $(m, n)$ and a monofacial rotation $\rho$ on $G$, where $\rho$ is simple with respect to $\kappa$. We then apply Theorem 2.1 to embed $K_{2mn+1}$ on an orientable surface with each edge lying on a simple cycle face of size $m$ and a simple cycle face of size $n$. □

It is our goal to find Heffter arrays which fulfill these conditions: namely, simple Heffter arrays with compatible orderings. Chapters 3 and 4 discuss this project for $3 \times n$ Heffter arrays $5 \times n$ Heffter arrays, respectively.

# Chapter 3

# $3 \times n$ Heffter arrays

In this chapter we give constructions for $3 \times n$ Heffter arrays; we divide them into cases modulo eight. We then give row reorderings of each construction which yield simple Heffter arrays. Finally, using these reorderings, we prove that there exists a biembedding of $K_{6n+1}$ using a Steiner triple system and an $n$-cycle system.

## 3.1 Constructing $3 \times n$ Heffter arrays

The following theorem gives a construction of $3 \times n$ Heffter arrays for all $n \geq 3$ with cases for $n$ modulo 8. Here we give only the constructions; details of the proof can be found in [2].

**Theorem 3.1.** *[2] There exists a $3 \times n$ Heffter array for all $n \geq 3$.*

*Proof.* We start with specific constructions for $3 \times 3$ and $3 \times 4$ Heffter arrays followed by general constructions for $n \equiv 0, 1, \ldots, 7 \pmod 8$.

**n = 3:** The following is a $3 \times 3$ Heffter array:

$$\begin{bmatrix} -8 & -2 & -9 \\ 7 & -3 & -4 \\ 1 & 5 & -6 \end{bmatrix}.$$

**n = 4:** The following is a $3 \times 4$ Heffter array:

$$\begin{bmatrix} 1 & 2 & 3 & -6 \\ 8 & -12 & -7 & 11 \\ -9 & 10 & 4 & -5 \end{bmatrix}.$$

$\boldsymbol{n \equiv 0 \pmod 8, n \geq 8}$**:** In this case define $m = \frac{n-8}{8}$. The first four columns are:

$$A = \begin{bmatrix} -12m-13 & -10m-11 & 4m+6 & 4m+3 \\ 4m+4 & -8m-7 & 18m+17 & 18m+19 \\ 8m+9 & 18m+18 & -22m-23 & -22m-22 \end{bmatrix}.$$

For each $0 \leq r \leq 2m$ define

$$A_r = (-1)^r \begin{bmatrix} (8m+r+10) & (-8m+2r-8) & (14m-r+14) & (-4m+2r-1) \\ (8m-2r+5) & (-16m-r-16) & (-4m+2r-2) & (-18m-r-20) \\ (-16m+r-15) & (24m-r+24) & (-10m-r-12) & (22m-r+21) \end{bmatrix}.$$

Add on the remaining $n-4$ columns by concatenating the $A_r$ arrays for each value of $r$ between 0 and $2m$. So the final array will be:

$$\begin{bmatrix} A & A_0 & A_1 & \cdots & A_{2m} \end{bmatrix}.$$

$\boldsymbol{n \equiv 1 \pmod 8, n \geq 9}$**:** Here $m = \frac{n-9}{8}$. The first five columns are:

$$A = \begin{bmatrix} 8m+7, & 10m+12 & 16m+18 & 4m+6 & 4m+3 \\ 8m+10 & 8m+9 & -12m-14 & -22m-26 & 18m+22 \\ -16m-17 & -18m-21 & -4m-4 & 18m+20 & -22m-25 \end{bmatrix}.$$

For each $0 \leq r \leq 2m$ define

$$A_r = (-1)^r \begin{bmatrix} (-8m+2r-5), & (-10m-r-13) & (-24m+r-27) & (-4m+2r-1) \\ (16m-r+16) & (-4m+2r-2) & (8m-2r+8) & (-18m-r-23) \\ (-8m-r-11) & (14m-r+15) & (16m+r+19) & (22m-r+24) \end{bmatrix}.$$

Add on the remaining $n-5$ columns by concatenating the $A_r$ arrays for each value of $r$ between 0 and $2m$.

$\boldsymbol{n \equiv 2 \pmod 8, n \geq 10}$**:** In this case $m = \frac{n-10}{8}$. The first six columns are:

$$A = \begin{bmatrix} 24m+30 & 16m+21 & 10m+13 & 8m+8 & 4m+5 & 8m+9 \\ 24m+29 & -8m-11 & -10m-14 & 12m+16 & 16m+20 & 12m+17 \\ 2 & -8m-10 & 1 & -20m-24 & -20m-25 & -20m-26 \end{bmatrix}.$$

For each $0 \leq r \leq 2m$ define

$$A_r = (-1)^r \begin{bmatrix} (-8m+2r-7) & (10m+r+15) & (-22m+r-27) & (-8m+2r-6) \\ (16m-r+19) & (4m-2r+3) & (4m-2r+4) & (-16m-r-22) \\ (-8m-r-12) & (-14m+r-18) & (18m+r+23) & (24m-r+28) \end{bmatrix}.$$

Add on the remaining $n-6$ columns by concatenating the $A_r$ arrays for each value of $r$ between 0 and $2m$.

**$n \equiv 3 \pmod 8, n \geq 11$:** Define $m = \frac{n-11}{8}$. The first seven columns are:

$$A = \begin{bmatrix} 24m+33 & 8m+11 & 8m+13 & 4m+6 & 1 & -12m-17 & 8m+10 \\ 24m+32 & -16m-23 & -12m-18 & 10m+15 & 20m+27 & -8m-9 & 14m+20 \\ 2 & 8m+12 & 4m+5 & -14m-21 & -20m-28 & 20m+26 & -22m-30 \end{bmatrix}.$$

For each $0 \leq r \leq 2m$ define

$$A_r = (-1)^r \begin{bmatrix} (-16m+r-22) & (24m-r+31) & (4m-2r+4) & (-4m+2r-3) \\ (8m-2r+8) & (-8m+2r-7) & (-22m+r-29) & (-10m-r-16) \\ (8m+r+14) & (-16m-r-24) & (18m+r+25) & (14m-r+19) \end{bmatrix}.$$

Add on the remaining $n-7$ columns by concatenating the $A_r$ arrays for each value of $r$ between 0 and $2m$.

**$n \equiv 4 \pmod 8, n \geq 12$:** Let $m = \frac{n-12}{8}$. The first eight columns are:

$$A = \begin{bmatrix} 8m+13 & 10m+16 & 22m+34 & -4m-5 & 4m+7 & -22m-35 & -12m-18 & -1 \\ 4m+6 & 8m+11 & -4m-8 & 22m+33 & -14m-22 & 4m+10 & -2 & -20m-30 \\ -12m-19 & -18m-27 & -18m-26 & -18m-28 & 10m+15 & 18m+25 & 12m+20 & 20m+31 \end{bmatrix}.$$

For $0 \leq r \leq 2m$ define

$$A_r = (-1)^r \begin{bmatrix} (-16m+r-23) & (-8m+2r-12) & (14m-r+21) & (4m-2r+3) \\ (8m+r+14) & (-16m-r-24) & (-10m-r-17) & (18m+r+29) \\ (8m-2r+9) & (24m-r+36) & (-4m+2r-4) & (-22m+r-32) \end{bmatrix}.$$

Add on the remaining $n-8$ columns by concatenating the $A_r$ arrays for each value of $r$ between 0 and $2m$.

$\boldsymbol{n \equiv 5 \pmod 8, n \geq 5}$ : Here $m = \frac{n-5}{8}$. The first five columns are:

$$A = \begin{bmatrix} 8m+6 & 10m+7 & -16m-10 & -4m-4 & 4m+1 \\ -16m-9 & 8m+5 & 4m+2 & -18m-11 & 18m+13 \\ 8m+3 & -18m-12 & 12m+8 & 22m+15 & -22m-14 \end{bmatrix}.$$

For each $0 \leq r \leq 2m-1$ define

$$A_r = (-1)^r \begin{bmatrix} (-8m+2r-1) & (-14m+r-8) & (16m+r+11) & (4m-2r-1) \\ (16m-r+8) & (4m-2r) & (8m-2r+4) & (18m+r+14) \\ (-8m-r-7) & (10m+r+8) & (-24m+r-15) & (-22m+r-13) \end{bmatrix}.$$

Add on the remaining $n-5$ columns by concatenating the $A_r$ arrays for each value of $r$ between 0 and $2m-1$.

$\boldsymbol{n \equiv 6 \pmod 8, n \geq 6}$**:** In this case, $m = \frac{n-6}{8}$. The first six columns are:

$$A = \begin{bmatrix} 24m+18 & -16m-13 & -1 & 8m+4 & -4m-3 & -8m-5 \\ 2 & 8m+6 & -10m-8 & -20m-14 & -16m-12 & -12m-11 \\ 24m+17 & 8m+7 & 10m+9 & 12m+10 & 20m+15 & 20m+16 \end{bmatrix}.$$

For each $0 \leq r \leq 2m-1$ define

$$A_r = (-1)^r \begin{bmatrix} (-8m+2r-3) & (-4m+2r-1) & (-4m+2r-2) & (8m-2r+2) \\ (16m-r+11) & (-10m-r-10) & (22m-r+16) & (16m+r+14) \\ (-8m-r-8) & (14m-r+11) & (-18m-r-14) & (-24m+r-16) \end{bmatrix}.$$

Add on the remaining $n-6$ columns by concatenating the $A_r$ arrays for each value of $r$ between 0 and $2m-1$.

$\boldsymbol{n \equiv 7 \pmod 8, n \geq 7}$**:** Now $m = \frac{n-7}{8}$. The first seven columns are:

$$A = \begin{bmatrix} 24m+21 & 16m+15 & 4m+3 & -4m-4 & -20m-18 & -12m-11 & -8m-6 \\ 2 & -8m-8 & -12m-12 & 14m+14 & 1 & 20m+16 & -14m-13 \\ 24m+20 & -8m-7 & 8m+9 & -10m-10 & 20m+17 & -8m-5 & 22m+19 \end{bmatrix}.$$

For each $0 \leq r \leq 2m-1$ define

$$A_r = (-1)^r \begin{bmatrix} (-16m+r-14) & (-8m+2r-3) & (-18m-r-16) & (4m-2r+1) \\ (8m+r+10) & (-16m-r-16) & (22m-r+18) & (10m+r+11) \\ (8m-2r+4) & (24m-r+19) & (-4m+2r-2) & (-14m+r-12) \end{bmatrix}.$$

Add on the remaining $n-7$ columns by concatenating the $A_r$ arrays for each value of $r$ between 0 and $2m-1$.

This concludes the constructions of the $3 \times n$ arrays. To prove these are Heffter arrays, simply check that each element of the half set occurs, and calculate each row and column sum, verifying they are equivalent to $0 \pmod{6n+1}$. Details of this step can be found in [2]. $\square$

## 3.2 Reordering the Heffter arrays

Suppose $H = (h_{ij})$ is any Heffter array given by the constructions in Theorem 3.1. We first note that each column in $H$ is simple just using the standard ordering, which is simply a cycle ordering on each column, $\omega_c = (h_{11}, h_{21}, h_{31})(h_{12}, h_{22}, h_{32}) \dots (h_{1n}, h_{2n}, h_{3n})$. Thus we must only reorder the three rows so they have distinct partial sums, thereby making $H$ simple. In each of the following lemmas we present a single reordering that makes the standard ordering, which is again simply a cycle ordering on each row, $\omega_r = (h_{11}, h_{12}, \dots, h_{1n})(h_{21}, h_{22}, \dots, h_{2n})(h_{31}, h_{32}, \dots, h_{3n})$ simple. In finding a *single* reordering which works for all three rows in $H$, we are actually rearranging the order of the columns without changing the elements which appear in the rows and columns. For notation, when we say use the ordering $\{a_1, a_2, \ldots, a_n\}$ this means that in the resulting array $H'$, column $a_i$ of $H$ will appear in column $i$ of $H'$. In Example 3.2 we give $H(3,8)$ (the original form), the reordering for the rows, and $H'(3,8)$ (the reordered form).

**Example 3.2.** *The original* $3 \times 8$ *Heffter array:*

$$\begin{bmatrix} -13 & -11 & 6 & 3 & 10 & -8 & 14 & -1 \\ 4 & -7 & 17 & 19 & 5 & -16 & -2 & -20 \\ 9 & 18 & -23 & -22 & -15 & 24 & -12 & 21 \end{bmatrix}.$$

*Note that in row 1,* $s_1 = s_6 = -13 \equiv 36 \pmod{49}$*, and so* $\omega_r$ *is not simple. So we use the reordering* $R = \{1, 2, 6, 8, 5, 3, 4, 7\}$*. The reordered* $3 \times 8$ *Heffter array:*

$$\begin{bmatrix} -13 & -11 & -8 & -1 & 10 & 6 & 3 & 14 \\ 4 & -7 & -16 & -20 & 5 & 17 & 19 & -2 \\ 9 & 18 & 24 & 21 & -15 & -23 & -22 & -12 \end{bmatrix}.$$

*We list the partial sums for each row as their smallest positive equivalence modulo 49:*
*Row 1:* $\{36, 25, 17, 16, 26, 32, 35, 0\}$,
*Row 2:* $\{4, 46, 30, 10, 15, 32, 2, 0\}$,
*and Row 3:* $\{9, 27, 2, 23, 8, 34, 12, 0\}$.
*Now all the partial sums are distinct, and so* $\omega_r$ *is simple.*

As the reader can see, the simultaneous reordering of the three rows results in a reordering of the columns with the elements in each row and column remaining the same. Again, the cases are broken up modulo 8 and we will consider each individually. For every case we will write the partial sums as their lowest positive equivalence modulo $6n + 1$.

**Lemma 3.3.** *There exist simple $H(3,3)$ and $H(3,4)$.*

*Proof.* First recall that the column Heffter system of the $H(3,3)$ from Theorem 3.1 is simple. Furthermore, the partial row sums $\pmod{19}$ are as follows: row 1: $\{11, 9, 0\}$, row 2: $\{7, 4, 0\}$, and row 3: $\{1, 6, 0\}$. Clearly each row has distinct partial sums, and therefore the Heffter array is simple.

Similarly, we see that that $H(3,4)$ from Theorem 3.1 is also simple. Here the partial sums $\pmod{25}$ are as follows: row 1: $\{1, 3, 6, 0\}$, row 2: $\{8, 21, 14, 0\}$, and row 3: $\{16, 1, 5, 0\}$. Clearly each row has distinct partial sums and thus the Heffter array is simple. □

Now we consider each of the constructed equivalence classes modulo 8. Let $H$ be the $H(3,n)$ constructed in Theorem 3.1 and let $H'$ be the $3 \times n$ Heffter array where the columns of $H$ have been reordered as given in each lemma. For the following lemmas we introduce the notation $[a,b] = \{a, a+1, a+2, \ldots, b\}$ and $[a,b]_2 = \{a, a+2, a+4, \ldots, b\}$.

**Lemma 3.4.** *Suppose $n \equiv 0 \pmod 8$ and $n \geq 8$. For $m = 0$ the ordering $\{1, 2, 6, 8, 5, 3, 4, 7\}$ yields a simple $3 \times 8$ Heffter array. For $m \geq 1$, the ordering*
$R = \{9, 13, ..., n-3, 1, 11, 15, .., n-1, 2, 10, 14, .., n-2, 6, 8, 12, 16, ..., n, 5, 3, 7, 4\}$
*yields a simple $3 \times n$ Heffter array.*

*Proof.* First we will directly prove the case for $m = 0$. The original and reordered constructions of the $3 \times 8$ matrix can be seen in Figures 3.1 and 3.2

$$\begin{bmatrix} -13 & -11 & 6 & 3 & 10 & -8 & 14 & -1 \\ 4 & -7 & 17 & 19 & 5 & -16 & -2 & -20 \\ 9 & 18 & -23 & -22 & -15 & 24 & -12 & 21 \end{bmatrix}$$

*Figure 3.1: The original $3 \times 8$ Heffter array.*

After reordering the original rows with the ordering $\{1, 2, 6, 8, 5, 3, 4, 7\}$, the partial row sums are as follows: row 1: $\{36, 25, 17, 16, 26, 32, 35, 0\}$, row 2: $\{4, 46, 30, 10, 15, 32, 2, 0\}$,

$$\begin{bmatrix} -13 & -11 & -8 & -1 & 10 & 6 & 3 & 14 \\ 4 & -7 & -16 & -20 & 5 & 17 & 19 & -2 \\ 9 & 18 & 24 & 21 & -15 & -23 & -22 & -12 \end{bmatrix}$$

*Figure 3.2: The reordered* $3 \times 8$ *Heffter array.*

and row 3: $\{9, 27, 2, 23, 8, 34, 12, 0\}$. Clearly the partial sums for each row are distinct, and thus $H'$ is simple.

Now suppose $m \geq 1$. So $n = 8m + 8$. For each $i = 1, 2, 3$ define $P_i$ as the set of partial sums of row $i$. Now divide each $P_i$ into four subsets based on the columns of $H$: $P_{i,1}$ is the set of partial sums of row $i$ and columns $\{9, 13, ..., n-3\}$ from $H$, $P_{i,2}$ is the set of partial sums of row $i$ and columns $\{1, 11, 15, .., n-1\}$ from $H$, $P_{i,3}$ is the set of partial sums of row $i$ and columns $\{2, 10, 14, .., n-2\}$ from $H$, and $P_{i,4}$ is the set of partial sums of row $i$ and columns $\{6, 8, 12, 16, ..., n, 5, 3, 7, 4\}$ from $H$. Then for $i = 1$ we have:

$$\begin{aligned} P_{1,1} =& [39m + 39, 40m + 38] \cup [1, m], \\ P_{1,2} =& [36m + 36, 37m + 36] \cup [23m + 23, 24m + 22], \\ P_{1,3} =& [26m + 25, 28m + 25]_2 \cup [32m + 33, 34m + 31]_2, \text{ and} \\ P_{1,4} =& [16m + 16, 18m + 16]_2 \cup [18m + 17, 20m + 17]_2 \cup \{26m + 26, 30m + 32, 0\}. \end{aligned}$$

First note that the elements of each $P_{1,j}$ lie in the range 1 to $48m + 49 = 6n + 1$, so we need only worry about equality in $\mathbb{Z}$ (as opposed to $\mathbb{Z}_{6n+1}$). Also notice that each set of partial sums covers two disjoint ranges of numbers ($P_{1,4}$ contains four additional numbers). For example, $P_{1,1}$ contains the range $39m + 39$ to $40m + 38$ and the range 1 to $m$. Within these ranges the sets of partial sums either contain every number in the range, or every other number in the range. Thus for each $j$ the partial sums in $P_{1,j}$ are distinct. Next note that the only overlap of the ranges occurs with $[26m+25, 28m+25]$ in $P_{1,3}$ and $26m+26$ in $P_{1,4}$. But $P_{1,3}$ only contains only the odd numbers within the range, and $26m + 26$ is even;

thus they are distinct. Therefore the partial sums in row 1 are distinct. Now we consider row 2:

$$
\begin{aligned}
P_{2,1} =& [40m+46, 42m+44]_2 \cup [46m+49, 48m+47]_2 \\
P_{2,2} =& [2m+4, 4m+6]_2 \cup [4m+8, 6m+4]_2 \\
P_{2,3} =& [12m+14, 13m+13] \cup [43m+46, 44m+45] \\
P_{2,4} =& [27m+30, 28m+30] \cup [8m+10, 9m+10] \cup \{16m+15, 34m+32, 30m+30, 0\}.
\end{aligned}
$$

Again, each set of partial sums $P_{2,j}$ lies in the range 1 to $48m+49 \in \mathbb{Z}$ and covers two disjoint ranges of numbers ($P_{2,4}$ contains four extra numbers). Within these ranges the sets of partial sums either contain every number in the range, or every other number in the range. Next note that these ranges do not overlap and thus the partial sums in row 2 are distinct. Finally, consider when $i = 3$:

$$
\begin{aligned}
P_{3,1} =& [1, m] \cup [15m+15, 16m+14] \\
P_{3,2} =& [8m+9, 9m+9] \cup [19m+22, 20m+21] \\
P_{3,3} =& [2m+4, 3m+3] \cup [25m+27, 26m+27] \\
P_{3,4} =& [m+2, 2m+2] \cup [22m+22, 23m+23] \cup \{6m+8, 32m+34, 0\}.
\end{aligned}
$$

Note that each set of partial sums lies in the range 1 to $48m+49 \in \mathbb{Z}$ and covers two disjoint ranges of numbers ($P_{3,4}$ contains three numbers in addition to this). This time, within these ranges the sets of partial sums contain every number in the range. Also, these ranges do not overlap and thus the partial sums in row 3 are distinct. Therefore, if we reorder each row in $H$ by $R$ to get $H'$, then $\omega_r$ is simple for each row, concluding the proof. $\square$

**Lemma 3.5.** *For $n \equiv 1 \pmod 8$ and $n \geq 9$, the row ordering*
$R = \{8, 12, 16, ..., n-1, 3, 7, 11, 15, ..., n-2, 5, 6, 10, 14, ..., n-3, 1, 9, 13, 17, ..., n, 2, 4\}$
*yields a simple $3 \times n$ Heffter array.*

*Proof.* In this case $n = 8m+9$ and so we are working modulo $48m+55 = 6n+1$. For each $i = 1, 2, 3$ define $P_i$ as the set of partial sums of row $i$. Now divide each $P_i$ into four subsets based on the columns of $H$: $P_{i,1}$ is the set of partial sums of row $i$ and columns $\{8, 12, 16, ..., n-1\}$ from $H$, $P_{i,2}$ is the set of partial sums of row $i$ and columns $\{3, 7, 11, 15, ..., n-2\}$ from $H$, $P_{i,3}$ is the set of partial sums of row $i$ and columns $\{5, 6, 10, 14, .., n-3\}$ from $H$, and $P_{i,4}$ is the set of partial sums of row $i$ and columns $\{1, 9, 13, 17, ..., n, 2, 4\}$ from $H$. Then for $i = 1$ we see:

$$
\begin{aligned}
P_{1,1} =& [24m+28, 25m+28] \cup [47m+55, 48m+54] \\
P_{1,2} =& [41m+46, 42m+46] \cup [30m+33, 31m+33] \\
P_{1,3} =& [32m+36, 34m+36]_2 \cup [26m+31, 28m+31]_2 \\
P_{1,4} =& [34m+38, 36m+38]_2 \cup [32m+37, 34m+37]_2 \cup \{44m+49, 0\}.
\end{aligned}
$$

Note that each $P_{1,j}$ lies in the range 1 to $48m+55 \in \mathbb{Z}$ and contains two ranges of numbers ($P_{1,4}$ covers three more numbers). Within these ranges the sets of partial sums either contain every number in the range, or every other number in the range. Next note that the only ranges which overlap are in $P_{1,3}$ and $P_{1,4}$ from $32m+36$ to $34m+37$. But $P_{1,3}$ contains only the even numbers in this range while $P_{1,4}$ contains the odd numbers.

Therefore the partial sums in row 1 are distinct. Now consider row 2:

$$\begin{aligned}
P_{2,1} =& [2, 2m] \cup [6m + 8, 8m + 8] \\
P_{2,2} =& [38m + 47, 40m + 47]_2 \cup [40m + 49, 42m + 49]_2 \\
P_{2,3} =& [10m + 14, 11m + 14] \cup [25m + 30, 26m + 30] \\
P_{2,4} =& [14m + 17, 15m + 17] \cup [33m + 40, 34m + 40] \cup \{22m + 26, 0\}.
\end{aligned}$$

Again, each set of partial sums lies in the range 1 to $48m + 55 \in \mathbb{Z}$ and covers two ranges of numbers ($P_{2,4}$ covers two additional numbers). Within these ranges the sets of partial sums either contain every number in the range, or every other number in the range. Also note that these ranges do not overlap and thus the partial sums in row 2 are distinct. Finally, consider when $i = 3$:

$$\begin{aligned}
P_{3,1} =& [47m + 54, 48m + 54] \cup [16m + 19, 17m + 19] \\
P_{3,2} =& [13m + 15, 14m + 15] \cup [26m + 30, 27m + 30] \\
P_{3,3} =& [43m + 49, 44m + 49] \cup [4m + 5, 5m + 5] \\
P_{3,4} =& [27m + 32, 28m + 32] \cup [1, m + 1] \cup \{30m + 35, 0\}.
\end{aligned}$$

First note that every $P_{3,j}$ lies in the range 1 to $48m + 55 \in \mathbb{Z}$ and covers two ranges of numbers ($P_{3,4}$ contains two numbers in addition to this). Moreover, within these ranges the sets of partial sums contain every number in the range. Finally, these ranges do not overlap and thus the partial sums in row 3 are distinct. Therefore, if we reorder each row in $H$ by $R$, then $\omega_r$ is simple for each row, concluding the proof. $\square$

**Lemma 3.6.** *For $n \equiv 2 \pmod 8$ and $n \geq 10$, the ordering*

$R = \{10, 14, ..., n, n-3, n-7, ..., 7, 4, 6, 8, 12, ..., n-2, 5, 9, 13, ..., n-1, 2, 3, 1\}$

*yields a simple $3 \times n$ Heffter array.*

*Proof.* Note that here $n = 8m+10$ and we are working modulo $48m+61$. For each $i = 1, 2, 3$ define $P_i$ as the set of partial sums of row $i$. Now divide each $P_i$ into four subsets based on the columns of $H$: $P_{i,1}$ is the set of partial sums of row $i$ and columns $\{10, 14, ..., n\}$, $P_{i,2}$ is the set of partial sums of row $i$ and columns $\{n-3, n-7, ..., 7, 4\}$, $P_{i,3}$ is the set of partial sums of row $i$ and columns $\{6, 8, 12, ..., n-2\}$, and $P_{i,4}$ is the set of partial sums of row $i$ and columns $\{5, 9, 13, ..., n-1, 2, 3, 1\}$. For $i = 1$ we have:

$$P_{1,1} = [40m+55, 42m+55]_2 \cup [46m+61, 48m+59]_2$$

$$P_{1,2} = [36m+48, 38m+48]_2 \cup [42m+57, 44m+55]_2 \cup \{44m+56\}$$

$$P_{1,3} = [3m+4, 4m+4] \cup [14m+19, 15m+19]$$

$$P_{1,4} = [18m+24, 19m+24] \cup [45m+58, 46m+58] \cup \{14m+18, 24m+31, 0\}.$$

First note that each set of partial sums lies in the range 1 to $48m+61 \in \mathbb{Z}$ and contains two disjoint ranges of numbers ($P_{1,4}$ contains three additional numbers). Within these ranges the sets of partial sums either contain every number in the range, or every other number in the range. Next note that these ranges do not overlap and therefore the partial sums in row 1 are distinct. Now we consider row 2:

$$P_{2,1} = [1, m] \cup [31m+39, 32m+39]$$

$$P_{2,2} = [45m+58, 46m+58] \cup [30m+39, 31m+38] \cup \{10m+13\}$$

$$P_{2,3} = [22m+30, 24m+30]_2 \cup [24m+33, 26m+33]_2$$

$$P_{2,4} = [40m+53, 42m+53]_2 \cup [42m+57, 44m+57]_2 \cup \{34m+46, 24m+32, 0\}.$$

Note that each $P_{2,j}$ lies in the range 1 to $48m + 61$ and covers two ranges of numbers ($P_{2,2}$ covers one additional number and $P_{2,4}$ covers three other numbers). Within these ranges the sets of partial sums either contains every number in the range, or every other number in the range. Also note that these ranges do not overlap and thus the partial sums in row 2 are distinct. Finally, consider $i = 3$:

$$\begin{aligned}
P_{3,1} =& [1, m] \cup [23m + 28, 24m + 28] \\
P_{3,2} =& [13m + 16, 14m + 16] \cup [22m + 28, 23m + 27] \cup \{42m + 53\} \\
P_{3,3} =& [8m + 9, 9m + 9] \cup [21m + 27, 22m + 27] \\
P_{3,4} =& [7m + 7, 8m + 7] \cup [36m + 45, 37m + 45] \cup \{48m + 58, 48m + 59, 0\}.
\end{aligned}$$

Here each set of partial sums, $P_{3,j}$, lies in the range 1 to $48m + 61 \in \mathbb{Z}$ and covers two ranges of numbers ($P_{3,2}$ contains one number and $P_{3,4}$ contains three numbers in addition to this). This time, within these ranges the partial sums cover every number in the range. Note that these ranges do not overlap and thus the partial sums in row 3 are distinct. Therefore, if we reorder each row of $H$ by $R$, then $H'$ is simple and this concludes the proof. $\square$

**Lemma 3.7.** *For $n \equiv 3 \pmod 8$ and $n \geq 11$, the ordering*
$R = \{9, 13, ..., n-2, 8, 12, ..., n-3, 1, 11, 15, .., n, 7, 6, 10, 14, ..., n-1, 5, 2, 3, 4\}$
*yields a simple $3 \times n$ Heffter array.*

*Proof.* Note in this case $n = 8m + 11$ and thus we are working modulo $48m + 67$. Now for each $i = 1, 2, 3$ define $P_i$ as the set of partial sums of row $i$. Now divide each $P_i$ into four subsets based on the columns of $H$:$P_{i,1}$ is the set of partial sums of row $i$ and columns $\{9, 13, ..., n-2\}$, $P_{i,2}$ is the set of partial sums of row $i$ and columns $\{8, 12, ..., n-3\}$, $P_{i,3}$ as the set of partial sums of row $i$ and columns $\{1, 11, 15, .., n\}$, and $P_{i,4}$ as the set of partial sums of row $i$ and columns $\{7, 6, 10, 14, ..., n-1, 5, 2, 3, 4\}$. Then for $i = 1$ we have:

$$\begin{aligned}
P_{1,1} =& [1, m] \cup [23m + 31, 24m + 31] \\
P_{1,2} =& [7m + 9, 8m + 9] \cup [22m + 31, 23m + 30] \\
P_{1,3} =& [28m + 39, 30m + 39]_2 \cup [30m + 42, 32m + 42]_2 \\
P_{1,4} =& \{38m + 49\} \cup [26m + 32, 28m + 32]_2 \cup [28m + 38, 30m + 36]_2 \\
& \cup \{28m + 36, 28m + 37, 36m + 48, 44m + 61, 0\}.
\end{aligned}$$

First note that each $P_{1,j}$ lies in the range 1 to $48m + 67 \in \mathbb{Z}$ and covers two ranges of numbers ($P_{1,4}$ covers four additional numbers). Within these ranges the sets of partial sums contains either every number in the range, or every other number in the range. Also, these ranges do not overlap and therefore the partial sums in row 1 are distinct. Now we consider when $i = 2$:

$$\begin{aligned}
P_{2,1} =& [40m + 60, 42m + 60]_2 \cup [46m + 67, 48m + 65]_2 \\
P_{2,2} =& [42m + 62, 44m + 60]_2 \cup [1, 2m + 1]_2 \\
P_{2,3} =& [13m + 17, 14m + 17] \cup [24m + 33, 25m + 33] \\
P_{2,4} =& \{27m + 37\} \cup [45m + 66, 46m + 66] \cup [18m + 28, 19m + 28] \\
& \cup \{18m + 26, 2m + 3, 38m + 52, 0\}.
\end{aligned}$$

Here each set of partial sums lies in the range 1 to $48m+67 \in \mathbb{Z}$ and contains two ranges of numbers ($P_{2,4}$ covers five more numbers). Within these ranges the sets of partial sums either contain every number in the range, or every other number in the range. Furthermore, these ranges do not overlap and thus the partial sums in row 2 are distinct. Finally, consider

$i = 3$:

$$\begin{aligned}
P_{3,1} =&[1, m] \cup [31m + 43, 32m + 43]\\
P_{3,2} =&[30m + 43, 31m + 42] \cup [39m + 57, 40m + 57]\\
P_{3,3} =&[40m + 59, 41m + 59] \cup [5m + 11, 6m + 11]\\
P_{3,4} =&\{31m + 48\} \cup [2m + 7, 3m + 7] \cup [21m + 32, 22m + 32]\\
&\cup \{2m + 4, 10m + 16, 14m + 21, 0\}.
\end{aligned}$$

Each $P_{3,j}$ lies in the range 1 to $48m + 67 \in \mathbb{Z}$ and covers two ranges of numbers ($P_{3,4}$ contains five additional numbers). Within these ranges the sets of partial sums contain every number in the range. Now note that these ranges do not overlap and thus the partial sums in row 3 are distinct. If we reorder every row in $H$ by $R$, then $\omega_r$ is simple for each row and $H'$ is simple, concluding the proof. □

**Lemma 3.8.** *For $n \equiv 4 \pmod 8$ and $n \geq 12$, the ordering*
$R = \{9, 13, ..., n-3, 11, 15, ..., n-1, 4, 10, 14, ..., n-2, 12, 16, ..., n, 1, 2, 6, 5, 7, 8, 3\}$
*yields a simple $3 \times n$ Heffter array.*

*Proof.* In this case $n = 8m + 12$ and we are working modulo $48m + 73$. Again for each $i = 1, 2, 3$ define $P_i$ as the set of partial sums of row $i$ and divide each $P_i$ into four subsets based on the columns of $H$. Define $P_{i,1}$ as the set of partial sums of row $i$ and columns $\{9, 13, ..., n-3\}$ from $H$, $P_{i,2}$ as the set of partial sums of row $i$ and columns $\{11, 15, ..., n-1\}$ from $H$, $P_{i,3}$ as the set of partial sums of row $i$ and columns $\{4, 10, 14, ..., n-2\}$ from $H$, and $P_{i,4}$ as the set of partial sums of row $i$ and columns $\{12, 16, ..., n, 1, 2, 6, 5, 7, 8, 3\}$ from

$H$. Then for $i = 1$:

$$
\begin{aligned}
P_{1,1} =& [32m + 50, 33m + 50] \cup [47m + 73, 48m + 72] \\
P_{1,2} =& [46m + 71, 47m + 71] \cup [33m + 51, 34m + 50] \\
P_{1,3} =& [34m + 54, 36m + 54]_2 \cup [40m + 66, 42m + 66]_2 \\
P_{1,4} =& [38m + 59, 40m + 57]_2 \cup [36m + 56, 38m + 54]_2 \\
& \cup \{38m + 57, 38m + 58, 46m + 70, 8m + 13, 34m + 51, 26m + 40, 26m + 39, 0\}
\end{aligned}
$$

First note that each set of partial sums lies in the range 1 to $48m + 73 \in \mathbb{Z}$ and covers two disjoint ranges of numbers ($P_{1,4}$ contain six more numbers in addition to this). Within these ranges the sets partial sums either contain every number in the range, or every other number in the range. Also note that these ranges do not overlap and therefore the partial sums in row 1 are distinct. Now we consider row 2:

$$
\begin{aligned}
P_{2,1} =& [8m + 14, 9m + 14] \cup [47m + 73, 48m + 72] \\
P_{2,2} =& [9m + 15, 10m + 14] \cup [46m + 70, 47m + 70] \\
P_{2,3} =& [3m + 6, 4m + 6] \cup [20m + 30, 21m + 30] \\
P_{2,4} =& [2m + 6, 3m + 5] \cup [21m + 35, 22m + 35] \\
& \cup \{26m + 41, 34m + 52, 38m + 62, 24m + 40, 24m + 38, 4m + 8, 0\}.
\end{aligned}
$$

Here each $P_{2,j}$ lies in the range 1 to $48m + 73 \in \mathbb{Z}$ and contains two ranges of numbers ($P_{2,4}$ contains seven additional numbers). Within these ranges the sets of partial sums either cover every number in the range, or every other number in the range. These ranges do not overlap and thus the partial sums in row 2 are distinct. Finally, consider when $i = 3$:

$$\begin{aligned}
P_{3,1} =& [2, 2m]_2 \cup [6m + 9, 8m + 7]_2 \\
P_{3,2} =& [2m + 5, 4m + 5]_2 \cup [4m + 9, 6m + 7]_2 \\
P_{3,3} =& [9m + 13, 10m + 13] \cup [34m + 50, 35m + 50] \\
P_{3,4} =& [8m + 13, 9m + 12] \cup [35m + 54, 36m + 54] \\
& \cup \{24m + 35, 6m + 8, 24m + 33, 34m + 48, 46m + 38, 18m + 26, 0\}.
\end{aligned}$$

First note that each set of partial sums lies in the range 1 to $48m + 73 \in \mathbb{Z}$ and covers two ranges of numbers ($P_{3,4}$ contains seven extra numbers). This time, within these ranges the partial sums cover either every number or every other number in the range. Next note that these ranges do not overlap and thus the partial sums in row 3 are distinct. So if we reorder each row in $H$ by $R$, then $\omega_r$ is simple for each row and this concludes the proof. $\square$

**Lemma 3.9.** *For* $n \equiv 5 \pmod 8$*, the row ordering*
$R = \{9, 13, ..., n, 5, 6, 10, ..., n - 3, 3, 7, 11, ..., n - 2, 1, 8, 12, ..., n - 1, 4, 2\}$
*yields a simple* $3 \times n$ *Heffter array.*

*Proof.* Note in this case $n = 8m + 5$ and so we are working modulo $48m + 31$. For each $i = 1, 2, 3$ define $P_i$ as the set of partial sums of row $i$. Now divide each $P_i$ into four subsets based on the columns of $H$: $P_{i,1}$ is the set of partial sums of row $i$ and columns $\{9, 13, ..., n\}$, $P_{i,2}$ is the set of partial sums of row $i$ and columns $\{5, 6, 10, ..., n - 3\}$, $P_{i,3}$ is the set of partial sums of row $i$ and columns $\{3, 7, 11, ..., n - 2\}$, and $P_{i,4}$ is the set of partial sums of row $i$ and columns $\{1, 8, 12, ..., n - 1, 4, 2\}$. Then for $i = 1$ we have:

$$P_{1,1} = [2, 2m]_2 \cup [2m+1, 4m-1]_2$$

$$P_{1,2} = [46m+31, 48m+29]_2 \cup [4m+1, 6m+1]_2$$

$$P_{1,3} = [35m+22, 36m+22] \cup [22m+14, 23m+13]$$

$$P_{1,4} = [42m+28, 43m+28] \cup [11m+8, 12m+7] \cup \{38m+24, 0\}$$

Each set of partial sums, $P_{1,j}$, lies in the range 1 to $48m+31$, so we need only be concerned with equality in $\mathbb{Z}$. Furthermore, each set of partial sums covers two ranges of numbers ($P_{1,4}$ contains two numbers in addition to this). Within these ranges the sets of partial sums either contain every number in the range, or every other number in the range. Next note that these ranges do not overlap and therefore the partial sums in row 1 are distinct. Now we consider row 2:

$$P_{2,1} = [47m+31, 48m+30] \cup [18m+14, 19m+13]$$

$$P_{2,2} = [17m+13, 18m+13] \cup [32m+22, 33m+21]$$

$$P_{2,3} = [22m+15, 24m+15]_2 \cup [24m+17, 26m+15]_2$$

$$P_{2,4} = [8m+6, 10m+6]_2 \cup [14m+12, 16m+10]_2 \cup \{40m+26, 0\}.$$

Note for all $j$, $P_{2,j}$ lies in the range 1 to $48m+31 \in \mathbb{Z}$ and contains two disjoint ranges of numbers ($P_{2,4}$ contains two additional numbers). Within these ranges the partial sums either cover every number in the range, or every other number in the range. Also note that these ranges do not overlap and thus the partial sums in row 2 are distinct. Finally, consider $i = 3$:

$$P_{3,1} =[47m + 31, 48m + 30] \cup [26m + 18, 27m + 17]$$

$$P_{3,2} =[16m + 11, 17m + 10] \cup [25m + 17, 26m + 17]$$

$$P_{3,3} =[2, m + 1] \cup [37m + 25, 38m + 25]$$

$$P_{3,4} =[21m + 13, 22m + 12] \cup [44m + 28, 45m + 28] \cup \{18m + 12, 0\}.$$

Here we have that each set of partial sums lies in the range 1 to $48m + 31 \in \mathbb{Z}$ and covers two ranges of numbers ($P_{3,4}$ also contains two other numbers). Within these ranges the sets of partial sums contain every number in the range. Next note that these ranges do not overlap and thus the partial sums in row 3 are distinct. Therefore if we reorder the rows of $H$ by the permutation $R$, then $H'$ is simple, concluding the proof □

**Lemma 3.10.** *For $n \equiv 6 \pmod 8$, the row ordering*
$R = \{10, 14, ..., n, 2, 9, 13, ..., n - 1, 4, 7, 11, ..., n - 3, 1, 8, 12, ..., n - 2, 5, 3, 6\}$
*yields a simple $3 \times n$ Heffter array.*

*Proof.* Here $n = 8m + 6$ and we are working modulo $48m + 37$. For each $i = 1, 2, 3$ define $P_i$ as the set of partial sums of row $i$. Now divide each $P_i$ into four subsets based on the columns of $H$: $P_{i,1}$ is the set of partial sums of row $i$ and columns $\{10, 14, ..., n\}$, $P_{i,2}$ is the set of partial sums of row $i$ and columns $\{2, 9, 13, ..., n - 1\}$, $P_{i,3}$ is the set of partial sums of row $i$ and columns $\{4, 7, 11, ..., n - 3\}$, and $P_{i,4}$ is the set of partial sums of row $i$ and columns $\{1, 8, 12, ..., n - 2, 5, 3, 6\}$. For $i = 1$ we see:

$$\begin{aligned}
P_{1,1} =& [2, 2m]_2 \cup [6m + 4, 8m + 2]_2 \\
P_{1,2} =& [30m + 22, 32m + 20]_2 \cup [32m + 24, 34m + 24]_2 \\
P_{1,3} =& [32m + 25, 34m + 23]_2 \cup [38m + 28, 40m + 28]_2 \\
P_{1,4} =& [10m + 8, 12m + 6]_2 \cup [12m + 9, 14m + 9]_2 \cup \{8m + 6, 8m + 5, 0\}.
\end{aligned}$$

First note that each $P_{1,j}$ lies in the range 1 to $48m + 37 \in \mathbb{Z}$. Furthermore, each set of partial sums contains two ranges of numbers ($P_{1,4}$ contains three additional numbers). Within these ranges the sets of partial sums contain every other number in the range. Next note that the only ranges which overlap are $P_{1,2}$ and $P_{1,3}$ from $32m + 24$ to $34m + 24$. But $P_{1,2}$ covers only the even numbers in this range while in $P_{1,3}$ the partial sums cover the odd numbers. Therefore the partial sums in row 1 are distinct. Now consider when $i = 2$:

$$\begin{aligned}
P_{2,1} =& [16m + 14, 7m + 13] \cup [47m + 37, 48m + 36] \\
P_{2,2} =& [7m + 6, 8m + 6] \cup [28m + 23, 29m + 22] \\
P_{2,3} =& [36m + 29, 37m + 29] \cup [3m + 4, 4m + 3] \\
P_{2,4} =& [26m + 22, 27m + 21] \cup [37m + 31, 38m + 31] \cup \{22m + 19, 12m + 11, 0\}.
\end{aligned}$$

First note that each set of partial sums lies in the range 1 to $48m + 37 \in \mathbb{Z}$ and covers two disjoint ranges of numbers ($P_{2,4}$ covers three numbers in addition to this). Within these ranges the partial sums cover every number in the range. Next note that these ranges do not overlap and thus the partial sums in row 2 are distinct. Finally, we consider when $i = 3$:

$$P_{3,1} =[24m+21, 25m+20] \cup [47m+37, 48m+36]$$
$$P_{3,2} =[36m+31, 37m+30] \cup [7m+7, 8m+7]$$
$$P_{3,3} =[20m+17, 21m+17] \cup [11m+10, 12m+9]$$
$$P_{3,4} =[10m+9, 11m+8] \cup [45m+34, 46m+34] \cup \{18m+12, 28m+21, 0\}.$$

For $j = 1, 2, 3$ and $4$, $P_{2,j}$ lies in the range 1 to $48m + 37 \in \mathbb{Z}$ and covers two ranges of numbers ($P_{3,4}$ contains three numbers in addition to this). Within these ranges the sets of partial sums contain every number in the range. Also note that these ranges do not overlap and therefore the partial sums in row 3 are distinct. Thus by reordering each row of $H$ by $R$ we have made $\omega_r$ simple for each row, concluding the proof □

**Lemma 3.11.** *For $n \equiv 7 \pmod 8$, the row ordering*
$R = \{10, 14, ..., n-1, 2, 8, 12, ..., n-3, 6, 11, 15, ..., n, 3, 9, 13, ..., n-2, 1, 4, 5, 7\}$
*yields a simple $3 \times n$ Heffter array.*

*Proof.* Note in this case $n = 8m + 7$ and so we are working modulo $48m + 43$. For each $i = 1, 2, 3$ define $P_i$ as the set of partial sums of row $i$. Now divide each $P_i$ into four subsets based on the columns of $H$. Define $P_{i,1}$ as the set of partial sums of row $i$ and columns $\{10, 14, ..., n-1\}$ of $H$, $P_{i,2}$ as the set of partial sums of row $i$ and columns $\{2, 8, 12, ..., n-3\}$ of $H$, $P_{i,3}$ as the set of partial sums of row $i$ and columns $\{6, 11, 15, ..., n\}$ of $H$, and $P_{i,4}$ as the set of partial sums of row $i$ and columns $\{3, 9, 13, ..., n-2, 1, 4, 5, 7\}$ of $H$. For $i = 1$ we see:

$$P_{1,1} = [1, m] \cup [29m + 28, 30m + 27]$$
$$P_{1,2} = [m + 1, 2m] \cup [16m + 15, 17m + 15]$$
$$P_{1,3} = [6m + 7, 8m+]_2 \cup [4m + 4, 6m + 4]_2$$
$$P_{1,4} = [2m + 4, 4m + 2]_2 \cup [8m + 7, 10m + 7]_2 \cup \{32m + 28, 28m + 24, 8m + 6, 0\}.$$

Each set of partial sums lies in the range 1 to $48m + 43$, and each set of partial sums covers two ranges of numbers ($P_{1,4}$ covers four numbers in addition to this). Within these ranges the sets of partial sums either contain every number in the range, or every other number in the range. Next note that these ranges do not overlap and therefore the partial sums in row 1 are distinct. Now we consider row 2:

$$P_{2,1} = [1, m] \cup [21m + 19, 22m + 18]$$
$$P_{2,2} = [m + 2, 2m + 1] \cup [40m + 35, 41m + 35]$$
$$P_{2,3} = [11m + 8, 12m + 8] \cup [22m + 19, 23m + 18]$$
$$P_{2,4} = [47m + 39, 48m + 39] \cup [30m + 24, 31m + 23] \cup \{48m + 41, 14m + 12, 14m + 13, 0\}.$$

First note $P_{2,j}$ lies in the range 1 to $48m + 43 \in \mathbb{Z}$ and covers two ranges of numbers ($P_{2,4}$ contains four additional numbers). Within these ranges the partial sums cover every number in the range. Also, these ranges do not overlap and thus the partial sums in row 2 are distinct. Finally, consider $i = 3$:

$$
\begin{aligned}
P_{3,1} =& [46m+43, 48m+41]_2 \cup [44m+41, 46m+39]_2 \\
P_{3,2} =& [44m+42, 46m+40]_2 \cup [38m+36, 40m+36]_2 \\
P_{3,3} =& [18m+19, 19m+18] \cup [31m+31, 32m+31] \\
P_{3,4} =& [14m+17, 15m+16] \cup [39m+40, 40m+40] \cup \{16m+17, 6m+7, 26m+24, 0\}.
\end{aligned}
$$

Here each set of partial sums lies in the range 1 to $48m + 43 \in \mathbb{Z}$ and contains two ranges of numbers ($P_{3,4}$ also contains four other numbers). Within these ranges the sets of partial sums contain either every or every other number in the range. Next note that the only ranges which overlap are in $P_{3,1}$ and $P_{3,2}$ from $44m+41$ to $46m+40$. But in $P_{3,1}$ the partial sums are only the odd numbers in this range while in $P_{3,2}$ the partial sums are only the even numbers. Thus the partial sums in row 3 are distinct. If we reorder the rows of $H$ by $R$, then $H'$ is simple, concluding the proof $\square$

We summarize the results of this section in the following theorem.

**Theorem 3.12.** *There exists a simple Heffter array $H(3, n)$ for all $n \geq 3$.*

## 3.3 Biembedding $K_{6n+1}$

Now that we have established simple row and column orderings for each original construction of a $3 \times n$ Heffter array we are prepared to prove that each reordered $3 \times n$ Heffter array gives a biembedding of $K_{6n+1}$ such that every edge is on a face of size 3 and a face of size $n$. Below is the specific application of Theorem 2.6 for $3 \times n$ Heffter arrays.

**Corollary 3.13.** *Given a Heffter array $H = H(n,3)$ with compatible simple orderings $\omega_r$ on the rows of $H$ $(D(6n+1,n))$ and $\omega_c$ on the columns of $H$ $(D(6n+1,3))$, there exists a biembedding of $K_{6n+1}$ such that every edge is on a simple cycle face of size $n$ and a simple cycle face of size 3.*

**Theorem 3.14.** *There exists a biembedding of $K_{6n+1}$ such that every edge is on an $n$-cycle and a 3-cycle for $n \geq 3$.*

*Proof.* By Theorem 3.12, given any $n \in \mathbb{Z}^+$, $n \geq 3$, there exists a $3 \times n$ simple Heffter array, call it $H = (h_{ij})$. Let $L$ be the half-set of elements in $\mathbb{Z}_{6n+1}$ contained in $H$. Recall that the rows of $H$ form a $D(2mn+1, n)$ and the columns of $H$ form a $D(2mn+1, m)$. Then using Corollary 3.13 it suffices to show that the orderings $\omega_r = (h_{11}, h_{12}, \ldots, h_{1n})(h_{21}, h_{22}, \ldots, h_{2n})$ $(h_{31}, h_{32}, \ldots, h_{3n})$ and $\omega_c = (h_{11}, h_{21}, h_{31})(h_{12}, h_{22}, h_{32}) \ldots (h_{1n}, h_{2n}, h_{3n})$ are compatible on the row and column Heffter systems. We must consider two cases: $n \equiv 1, 2 \pmod 3$ and $n \equiv 0 \pmod 3$.

First assume $n \equiv 1, 2 \pmod 3$. In this case we do not change $\omega_r$ or $\omega_c$. Given an element $h_{ij}$, $\omega_c \circ \omega_r(h_{ij}) = h_{i+1,j+1}$ where the row subscript is modulo 3 and the column subscript is modulo $n$. Since 1 and 2 are relatively prime with 3, the permutation created by continuously applying $\omega_c \circ \omega_r$ is of length $3n$. Therefore, this is a cyclic permutation of $L$ and the orderings are compatible.

Now assume $n \equiv 0 \pmod 3$. In this case we leave $\omega_r$ as presented but change the ordering $\omega_c$ to be $(h_{11}, h_{21}, h_{31})(h_{12}, h_{22}, h_{32}) \ldots (h_{3c}, h_{2c}, h_{1c})$. Note that in changing only the final cycle, the row and column Heffter systems associated with $H$ remain simple. Then given an element $h_{ij}$, we have that $\omega_c \circ \omega_r(h_{ij}) = h_{i+1,j+1}$ for $j < n$ and $\omega_c \circ \omega_r(h_{i,j}) = h_{i-1,j+1} = h_{i-1,1}$ for $j = n$ where the row position is modulo 3 and the column position is modulo $n$. Starting with $h_{11}$ and continuously applying $\omega_c \circ \omega_r$ we obtain the permutation $(h_{11}, h_{22}, \ldots, h_{3n}, h_{21}, h_{32}, \ldots, h_{2n}, h_{31}, h_{12}, \ldots, h_{1n})$. This is a cyclic permutation of $L$, and therefore the orderings are compatible. $\square$

# Chapter 4

# $5 \times n$ Heffter arrays

In this section we discuss partial results for the reordering of $5 \times n$ Heffter arrays. As before, we first establish the starting constructions for $5 \times n$ Heffter arrays broken up into cases modulo 8. We then show general reorderings of each construction which yield distinct partial sums for the first three rows. Next we discuss why we were unable to find general orderings for the last two rows, and finally we present specific reorderings for $n \leq 100$ (see Appendix A).

## 4.1 Constructing $5 \times n$ Heffter arrays

The construction for $5 \times n$ Heffter arrays follows a similar pattern to the construction of $3 \times n$ Heffter arrays. The following theorem gives a construction of a $5 \times n$ Heffter array for all $n \geq 3$ with cases for $n = 0, 1, \ldots, 7 \pmod 8$. Details of the proof can again be found in [2].

**Theorem 4.1.** *[2] There exists a $5 \times n$ Heffter array for $n \geq 3$.*

*Proof.* We start with specific constructions for the $5 \times 3$, $5 \times 4$, $5 \times 5$, and $5 \times 6$ Heffter arrays, followed by general constructions for $n \equiv 0, 1, \ldots 7 \pmod 8$. We consider each case individually.

$\boldsymbol{n = 3}$**:** The following is a $5 \times 3$ Heffter array:

$$\begin{bmatrix} 6 & -15 & 9 \\ 14 & -1 & -13 \\ -10 & 12 & -2 \\ -3 & 8 & -5 \\ -7 & -4 & 11 \end{bmatrix}.$$

$\boldsymbol{n = 4}$**:** The following is a $5 \times 4$ Heffter array:

$$\begin{bmatrix} 7 & -16 & -10 & 19 \\ -12 & 15 & 17 & -20 \\ -2 & 9 & -18 & 11 \\ 6 & 5 & 3 & -14 \\ 1 & -13 & 8 & 4 \end{bmatrix}.$$

$\boldsymbol{n = 5}$**:** The following is a $5 \times 5$ Heffter array:

$$\begin{bmatrix} 1 & 5 & 6 & 7 & -19 \\ 2 & 8 & 12 & 15 & 14 \\ 3 & 9 & -21 & 22 & -13 \\ 4 & 11 & -25 & -24 & -17 \\ -10 & 18 & -23 & -20 & -16 \end{bmatrix}.$$

$\boldsymbol{n = 6}$**:** The following is a $5 \times 6$ Heffter array:

$$\begin{bmatrix} 1 & -8 & -7 & 15 & 26 & -27 \\ -2 & 20 & -11 & 24 & -25 & -6 \\ 29 & -19 & 17 & -4 & -10 & -13 \\ 30 & -9 & -21 & -23 & -5 & 28 \\ 3 & 16 & 22 & -12 & 14 & 18 \end{bmatrix}.$$

$\boldsymbol{n \equiv 0 \pmod 8, n \geq 8}$**:** Define $m = \frac{n-8}{8}$ and create eight set columns:

$$A = \begin{bmatrix} 8m+10 & -4m-5 & -12m-15 & -8m-8 & -8m-13 & -24m-35 & 24m+26 & 24m+40 \\ 8m+11 & -8m-12 & 24m+30 & -24m-31 & -24m-33 & 24m+36 & 24m+38 & -24m-39 \\ 1 & 24m+28 & -8m-9 & 22m+25 & -8m-7 & -10m-14 & -16m-21 & -4m-3 \\ -2 & 12m+16 & -24m-29 & -14m-18 & 16m+19 & -4m-4 & -8m-6 & 22m+24 \\ -16m-20 & -24m-27 & 20m+23 & 24m+32 & 24m+34 & 14m+17 & -24m-37 & -18m-22 \end{bmatrix}.$$

For each $0 \leq r \leq m-1$ define

$$A_r = \begin{bmatrix} -4r+8m+5 & 4r-4m-2 & -4r+8m+4 & 4r-4m-1 & 4r-8m-3 & -4r+4m & 4r-8m-2 & -4r+4m-1 \\ 2r+8m+14 & -2r-10m-15 & 2r+16m+22 & -2r-18m-23 & -2r-8m-15 & 2r+10m+16 & -2r-16m-23 & 2r+18m+24 \\ 2r-16m-18 & -2r+14m+16 & 2r-24m-25 & -2r+22m+23 & -2r+16m+17 & 2r-14m-15 & -2r+24m+24 & 2r-22m-22 \\ 2r+24m+41 & -2r-26m-41 & 2r+28m+41 & -2r-30m-41 & -2r-32m-41 & 2r+34m+41 & -2r-36m-41 & 2r+38m+41 \\ -2r-24m-42 & 2r+26m+42 & -2r-28m-42 & 2r+30m+42 & 2r+32m+42 & -2r-34m-42 & 2r+36m+42 & -2r-38m-42 \end{bmatrix}.$$

Add on the remaining $n-8$ columns by concatenating the $A_r$ arrays for each value of $r$ between 0 and $m-1$. So the resulting array is:

$$\begin{bmatrix} A & A_0 & A_1 & \cdots & A_{m-1} \end{bmatrix}.$$

$\boldsymbol{n \equiv 1 \pmod 8, n \geq 9}$**:** Here $m = \frac{n-9}{8}$. The first nine columns are:

$$A = \begin{bmatrix} 1 & -8m-11 & -8m-10 & 24m+33 & -24m-34 & -4m-5 & 20m+23 & -24m-40 & 24m+43 \\ -2 & -12m-14 & -10m-13 & -18m-20 & 24m+27 & 24m+37 & 24m+39 & -8m-12 & -24m-42 \\ 3 & -24m-28 & 18m+22 & -24m-32 & -8m-9 & -18m-21 & -12m-16 & -8m-6 & -4m-4 \\ 40m+45 & 24m+29 & -24m-30 & -4m-7 & -16m-19 & -24m-36 & -8m-8 & 24m+41 & -12m-15 \\ 40m+44 & 20m+24 & 24m+31 & 22m+26 & 24m+35 & 22m+25 & -24m-38 & 16m+17 & 16m+18 \end{bmatrix}.$$

For each $0 \leq r \leq m-1$ define

$$A_r = \begin{bmatrix} -4r+8m+7 & 4r-4m-2 & -4r+4m+3 & 4r-8m-4 & 4r-8m-5 & -4r+4m & 4r-4m-1 & -4r+8m+2 \\ 2r+16m+20 & -2r-18m-23 & 2r+10m+14 & -2r-8m-13 & -2r-16m-21 & 2r+18m+24 & -2r-10m-15 & 2r+8m+14 \\ 2r-24m-26 & -2r+22m+24 & 2r-14m-16 & -2r+16m+16 & -2r+24m+25 & 2r-22m-23 & -2r+14m+15 & 2r-16m-15 \\ 2r+24m+44 & -2r-26m-44 & 2r+28m+44 & -2r-30m-44 & -2r-32m-44 & 2r+34m+44 & -2r-36m-44 & 2r+38m+44 \\ -2r-24m-45 & 2r+26m+45 & -2r-28m-45 & 2r+30m+45 & 2r+32m+45 & -2r-34m-45 & 2r+36m+45 & -2r-38m-45 \end{bmatrix}.$$

Add on the remaining $n-9$ columns by concatenating the $A_r$ arrays for each value of $r$ between 0 and $m-1$.

$\boldsymbol{n \equiv 2 \pmod 8, n \geq 10}$**:** In this case $m = \frac{n-10}{8}$. The first ten columns are:

$$A = \begin{bmatrix} 1 & -8m-12 & -8m-11 & -12m-16 & 24m+38 & 24m+40 & 16m+20 & -8m-9 & -4m-4 & -24m-47 \\ -2 & -12m-15 & -14m-19 & 20m+25 & 20m+27 & -14m-18 & 24m+42 & -24m-43 & -24m-45 & 24m+48 \\ 3 & 24m+32 & 22m+29 & -24m-35 & -4m-7 & -4m-6 & -24m-41 & 24m+44 & -10m-14 & -4m-5 \\ 40m+49 & -24m-31 & -24m-33 & -8m-10 & -24m-37 & 18m+23 & -8m-13 & -16m-22 & 24m+46 & 22m+28 \\ 40m+50 & 20m+26 & 24m+34 & 24m+36 & -16m-21 & -24m-39 & -8m-8 & 24m+30 & 14m+17 & -18m-24 \end{bmatrix}.$$

For each $0 \leq r \leq m-1$ define

$$A_r = \begin{bmatrix} -4r+8m+6 & 4r-8m-7 & -4r+4m+2 & 4r-4m-3 & 4r-8m-4 & -4r+8m+5 & 4r-4m & -4r+4m+1 \\ 2r+8m+14 & -2r-16m-23 & 2r+10m+15 & -2r-18m-25 & -2r-8m-15 & 2r+16m+24 & -2r-10m-16 & 2r+18m+26 \\ 2r-16m-19 & -2r+24m+29 & 2r-14m-16 & -2r+22m+27 & -2r+16m+18 & 2r-24m-28 & -2r+14m+15 & 2r-22m-26 \\ 2r+24m+49 & -2r-26m-49 & 2r+28m+49 & -2r-30m-49 & -2r-32m-49 & 2r+34m+49 & -2r-36m-49 & 2r+38m+49 \\ -2r-24m-50 & 2r+26m+50 & -2r-28m-50 & 2r+30m+50 & 2r+32m+50 & -2r-34m-50 & 2r+36m+50 & -2r-38m-50 \end{bmatrix}.$$

Add on the remaining $n-10$ columns by concatenating the $A_r$ arrays for each value of $r$ between 0 and $m-1$.

$\boldsymbol{n \equiv 3 \pmod 8, n \geq 11}$**:** Now $m = \frac{n-11}{8}$ and create eleven set columns:

$$A = \begin{bmatrix} 8m+13 & -4m-7 & -8m-12 & -4m-5 & -16m-26 & -4m-6 & 16m+24 & 22m+32 & -24m-50 & -10m-18 & 24m+55 \\ 10m+16 & -12m-19 & -12m-20 & 22m+33 & -8m-10 & 24m+45 & -24m-46 & -24m-48 & -16m-27 & 24m+53 & 16m+23 \\ -18m-28 & 16m+25 & -24m-38 & -24m-40 & 24m+35 & -10m-17 & 24m+47 & -18m-30 & 24m+34 & 14m+21 & -8m-9 \\ -2 & -24m-36 & 20m+31 & 24m+41 & 24m+43 & -24m-44 & -8m-14 & -4m-3 & 24m+51 & -24m-52 & -8m-15 \\ 1 & 24m+37 & 24m+39 & -18m-29 & -24m-42 & 14m+22 & -8m-11 & 24m+49 & -8m-8 & -4m-4 & -24m-54 \end{bmatrix}.$$

For each $0 \leq r \leq m-1$ define

$$A_r = \begin{bmatrix} -4r+4m+1 & 4r-8m-6 & -4r+4m+2 & 4r-8m-7 & 4r-4m+1 & -4r+8m+4 & 4r-4m & -4r+8m+5 \\ 2r+18m+31 & -2r-16m-28 & 2r+10m+19 & -2r-8m-16 & -2r-18m-32 & 2r+16m+29 & -2r-10m-20 & 2r+8m+17 \\ 2r-22m-31 & -2r+24m+33 & 2r-14m-20 & -2r+16m+22 & -2r+22m+30 & 2r-24m-32 & -2r+14m+19 & 2r-16m-21 \\ 2r+24m+56 & -2r-26m-56 & 2r+28m+56 & -2r-30m-56 & -2r-32m-56 & 2r+34m+56 & -2r-36m-56 & 2r+38m+56 \\ -2r-24m-57 & 2r+26m+57 & -2r-28m-57 & 2r+30m+57 & 2r+32m+57 & -2r-34m-57 & 2r+36m+57 & -2r-38m-57 \end{bmatrix}.$$

Add on the remaining $n-11$ columns by concatenating the $A_r$ arrays for each value of $r$ between 0 and $m-1$.

$\boldsymbol{n \equiv 4 \pmod 8, n \geq 12}$**:** Define $m = \frac{n-12}{8}$ and create twelve set columns:

$$A = \begin{bmatrix} 8m+14 & -4m-7 & -8m-13 & -8m-12 & -8m-17 & 14m+24 & -24m-49 & -24m-51 & -8m-9 & 14m+23 & 24m+37 & 24m+60 \\ 8m+15 & -8m-16 & -12m-21 & -14m-25 & 16m+27 & -10m-19 & 24m+50 & -4m-5 & 24m+54 & 24m+56 & -24m-57 & -24m-59 \\ -16m-28 & 12m+22 & -24m-41 & 22m+36 & -24m-45 & 24m+48 & 24m+38 & 24m+52 & -24m-53 & -24m-55 & 24m+58 & -18m-32 \\ 1 & -24m-39 & 24m+42 & 24m+44 & -8m-11 & -24m-47 & -16m-29 & 22m+35 & -8m-18 & -4m-4 & -8m-8 & 22m+34 \\ -2 & 24m+40 & 20m+33 & -24m-43 & 24m+46 & -4m-6 & -8m-10 & -18m-31 & 16m+26 & -10m-20 & -16m-30 & -4m-3 \end{bmatrix}.$$

For each $0 \leq r \leq m-1$ define

$$A_r = \begin{bmatrix} -4r+8m+7 & 4r-4m-2 & -4r+8m+6 & 4r-4m-1 & 4r-8m-5 & -4r+4m & 4r-8m-4 & -4r+4m-1 \\ 2r+8m+19 & -2r-10m-21 & 2r+16m+31 & -2r-18m-33 & -2r-8m-20 & 2r+10m+22 & -2r-16m-32 & 2r+18m+34 \\ 2r-16m-25 & -2r+14m+22 & 2r-24m-36 & -2r+22m+33 & -2r+16m+24 & 2r-14m-21 & -2r+24m+35 & 2r-22m-32 \\ 2r+24m+61 & -2r-26m-61 & 2r+28m+61 & -2r-30m-61 & -2r-32m-61 & 2r+34m+61 & -2r-36m-61 & 2r+38m+61 \\ -2r-24m-62 & 2r+26m+62 & -2r-28m-62 & 2r+30m+62 & 2r+32m+62 & -2r-34m-62 & 2r+36m+62 & -2r-38m-62 \end{bmatrix}.$$

Add on the remaining $n-12$ columns by concatenating the $A_r$ arrays for each value of $r$ between 0 and $m-1$.

**$n \equiv 5 \pmod 8, n \geq 13$:** In this case $m = \frac{n-13}{8}$. The first thirteem columns are:

$$A = \begin{bmatrix} 1 & -8m-15 & -8m-14 & -4m-9 & -8m-13 & -4m-7 & 20m+33 & 16m+25 & 24m+55 & 24m+57 & -18m-32 & -10m-19 & -24m-62 \\ -2 & -12m-20 & -10m-18 & -18m-29 & -16m-27 & -18m-30 & -24m-50 & -24m-52 & -4m-6 & -24m-56 & 22m+35 & 24m+61 & 24m+63 \\ 3 & 20m+34 & 18m+31 & -24m-44 & 24m+47 & 24m+49 & -8m-12 & -8m-16 & -12m-21 & -16m-28 & -24m-58 & 14m+23 & -8m-8 \\ 40m+65 & -24m-40 & -24m-42 & 24m+45 & 24m+39 & -24m-48 & -12m-22 & -8m-10 & -24m-54 & -8m-11 & 24m+59 & -4m-5 & 16m+24 \\ 40m+64 & 24m+41 & 24m+43 & 22m+37 & -24m-46 & 22m+36 & 24m+51 & 24m+53 & 16m+26 & 24m+38 & -4m-4 & -24m-60 & -8m-17 \end{bmatrix}.$$

For each $0 \leq r \leq m-1$ define

$$A_r = \begin{bmatrix} -4r+8m+9 & 4r-4m-2 & -4r+4m+3 & 4r-8m-6 & 4r-8m-7 & -4r+4m & 4r-4m-1 & -4r+8m+4 \\ 2r+16m+29 & -2r-18m-33 & 2r+10m+20 & -2r-8m-18 & -2r-16m-30 & 2r+18m+34 & -2r-10m-21 & 2r+8m+19 \\ 2r-24m-37 & -2r+22m+34 & 2r-14m-22 & -2r+16m+23 & -2r+24m+36 & 2r-22m-33 & -2r+14m+21 & 2r-16m-22 \\ 2r+24m+64 & -2r-26m-64 & 2r+28m+64 & -2r-30m-64 & -2r-32m-64 & 2r+34m+64 & -2r-36m-64 & 2r+38m+64 \\ -2r-24m-65 & 2r+26m+65 & -2r-28m-65 & 2r+30m+65 & 2r+32m+65 & -2r-34m-65 & 2r+36m+65 & -2r-38m-65 \end{bmatrix}.$$

Add on the remaining $n-13$ columns by concatenating the $A_r$ arrays for each value of $r$ between 0 and $m-1$.

**$n \equiv 6 \pmod 8, n \geq 14$:** Here $m = \frac{n-14}{8}$ and create fourteen set columns:

$$A = \begin{bmatrix} 1 & -8m-16 & -8m-15 & -8m-14 & -4m-9 & -4m-8 & -8m-12 & -8m-13 & -4m-6 & -4m-7 & -8m-10 & -16m-31 & 24m+66 & -24m-67 \\ -2 & -12m-21 & -14m-26 & -12m-22 & -16m-29 & -14m-25 & -8m-17 & -16m-30 & -24m-57 & 24m+60 & 16m+27 & -24m-63 & -4m-4 & 24m+68 \\ 3 & 20m+36 & 22m+40 & 20m+35 & 20m+37 & 18m+32 & 16m+28 & 24m+56 & -10m-19 & -24m-59 & 24m+62 & -8m-11 & -24m-65 & -18m-34 \\ 40m+70 & 24m+44 & -24m-45 & -24m-47 & -24m-49 & -24m-51 & -24m-53 & 24m+42 & 24m+58 & -18m-33 & -8m-18 & 24m+64 & 14m+23 & -4m-5 \\ 40m+69 & -24m-43 & 24m+46 & 24m+48 & 24m+50 & 24m+52 & 24m+54 & -24m-55 & 14m+24 & 22m+39 & -24m-61 & 24m+41 & -10m-20 & 22m+38 \end{bmatrix}.$$

For each $0 \leq r \leq m-1$ define

$$A_r = \begin{bmatrix} -4r+8m+8 & 4r-8m-9 & -4r+4m+2 & 4r-4m-3 & 4r-8m-6 & -4r+8m+7 & 4r-4m & -4r+4m+1 \\ 2r+8m+19 & -2r-16m-32 & 2r+10m+21 & -2r-18m-35 & -2r-8m-20 & 2r+16m+33 & -2r-10m-22 & 2r+18m+36 \\ 2r-16m-26 & -2r+24m+40 & 2r-14m-22 & -2r+22m+37 & -2r+16m+25 & 2r-24m-39 & -2r+14m+21 & 2r-22m-36 \\ 2r+24m+69 & -2r-26m-69 & 2r+28m+69 & -2r-30m-69 & -2r-32m-69 & 2r+34m+69 & -2r-36m-69 & 2r+38m+69 \\ -2r-24m-70 & 2r+26m+70 & -2r-28m-70 & 2r+30m+70 & 2r+32m+70 & -2r-34m-70 & 2r+36m+70 & -2r-38m-70 \end{bmatrix}.$$

Add on the remaining $n-14$ columns by concatenating the $A_r$ arrays for each value of $r$ between 0 and $m-1$.

$\boldsymbol{n \equiv 7 \pmod 8, n \geq 7}$**:** Define $m = \frac{n-7}{8}$. The first seven columns are:

$$A = \begin{bmatrix} 8m+9 & -4m-5 & 24m+27 & -4m-3 & -24m-31 & -24m-32 & 24m+35 \\ 10m+11 & 16m+17 & -24m-26 & 22m+22 & -24m-23 & 24m+33 & -24m-34 \\ -18m-19 & -12m-13 & -8m-8 & 24m+29 & 8m+6 & 14m+15 & -8m-10 \\ 1 & -24m-24 & 20m+21 & -18m-20 & 16m+18 & -10m-12 & 16m+16 \\ -2 & 24m+25 & -12m-14 & -24m-28 & 24m+30 & -4m-4 & -8m-7 \end{bmatrix}.$$

For each $0 \leq r \leq m-1$ define

$$A_r = \begin{bmatrix} -4r+4m+1 & 4r-8m-4 & -4r+4m+2 & 4r-8m-5 & 4r-4m+1 & -4r+8m+2 & 4r-4m & -4r+8m+3 \\ 2r+18m+21 & -2r-16m-19 & 2r+10m+13 & -2r-8m-11 & -2r-18m-22 & 2r+16m+20 & -2r-10m-14 & 2r+8m+12 \\ 2r-22m-21 & -2r+24m+22 & 2r-14m-14 & -2r+16m+15 & -2r+22m+20 & 2r-24m-21 & -2r+14m+13 & 2r-16m-14 \\ 2r+24m+36 & -2r-26m-36 & 2r+28m+36 & -2r-30m-36 & -2r-32m-36 & 2r+34m+36 & -2r-36m-36 & 2r+38m+36 \\ -2r-24m-37 & 2r+26m+37 & -2r-28m-37 & 2r+30m+37 & 2r+32m+37 & -2r-34m-37 & 2r+36m+37 & -2r-38m-37 \end{bmatrix}$$

Add on the remaining $n-7$ columns by concatenating the $A_r$ arrays for each value of $r$ between 0 and $m-1$.

Clearly these are all $5 \times n$ arrays. To prove they are Heffter arrays simply sum each row and column, verifying each row and column sum is $0 \pmod{10n+1}$. Details of this step can be found in [2] $\square$

## 4.2 Reordering the Heffter arrays

Suppose $H = (h_{ij})$ is any Heffter array given by the constructions in Theorem 4.1. We first note that each column in $H$ is a simple ordering just using the standard ordering $\omega_c = (h_{11}, h_{21}, h_{31}, h_{41}, h_{51})(h_{12}, h_{22}, h_{32}, h_{42}, h_{52}) \dots (h_{1n}, h_{2n}, h_{3n}, h_{4n}, h_{5n})$. Thus we must only reorder the rows so they have distinct partial sums. To find general reorderings for any $m \times n$ Heffter array we need to find a pattern. The obvious, and seemingly only, place to look is in the columns developed by $A_r$, which expand in a patterned way (the first set of columns doesn't expand at all). In the case of the $5 \times n$ Heffter arrays, this means finding a way to interleave the columns of $A_r$. Unfortunately, the first three rows follow the same pattern, while the last two rows follow a different pattern. For this reason, we were unable to find a single general reordering for all five rows. However, we were able to find general reorderings for the first 3 rows in the same way we found reorderings for the $3 \times n$ Heffter arrays. For now we will only discuss the eight general cases as the four specific cases for $n = 3, 4, 5, 6$ will be discussed at the end of this chapter. We will not provide proofs for these lemmas, as they follow the same outline as in the $3 \times n$ cases. For the following lemmas, let $H$ be a $5 \times n$ Heffter array with original construction from Theorem 4.1.

**Lemma 4.2.** *For* $n \equiv 0 \pmod 8, n \geq 8$*, the row ordering*
$R = \{9, 13, ..., n-3, 11, 15, ..., n-1, 7, 10, 14, ..., n-2, 6, 12, 16, ..., n, 4, 8, 2, 5, 1, 3\}$
*yields a* $5 \times n$ *Heffter array with distinct partial row sums in the first three rows and whose columns form a simple* $D(10n + 1, 5)$*.*

**Lemma 4.3.** *For* $n \equiv 1 \pmod 8, n \geq 9$*, the row ordering*
$R = \{10, 14, ..., n-3, 12, 16, ..., n-1, 7, 11, 15, ..., n-2, 5, 13, 17, ..., n, 2, 1, 3, 8, 4, 6, 9\}$
*yields a* $5 \times n$ *Heffter array with distinct partial row sums in the first three rows and whose columns form a simple* $D(10n + 1, 5)$*.*

**Lemma 4.4.** *For* $n \equiv 2 \pmod 8, n \geq 10$*, the row ordering*
$R = \{11, 15, ..., n-3, 13, 17, ..., n-1, 5, 12, 16, ..., n-2, 14, 18, ..., n, 2, 1, 3, 6, 7, 8, 9, 4, 10\}$
*yields a* $5 \times n$ *Heffter array with distinct partial row sums in the first three rows and whose columns form a simple* $D(10n + 1, 5)$*.*

**Lemma 4.5.** *For* $n \equiv 3 \pmod 8, n \geq 11$*, the row ordering*
$R = \{12, 16, ..., n-3, 14, 18, ..., n-1, 5, 13, 17, ..., n-2, 15, 19, ..., n, 2, 1, 8, 6, 3, 9, 11, 7, 10, 4\}$
*yields a* $5 \times n$ *Heffter array with distinct partial row sums in the first three rows and whose columns form a simple* $D(10n + 1, 5)$*.*

**Lemma 4.6.** *For* $n \equiv 4 \pmod 8, n \geq 12$*, the row ordering*

$R = \{13, 17, ..., n-3, 15, 19, ..., n-1, 2, 14, 18, ..., n-2, 5, 16, 20, ..., n, 12, 7, 3, 6, 10, 1, 4, 11, 9, 8\}$

*yields a* $5 \times n$ *Heffter array with distinct partial row sums in the first three rows and whose columns form a simple* $D(10n+1, 5)$.

**Lemma 4.7.** *For* $n \equiv 5 \pmod 8, n \geq 13$*, the row ordering*

$R = \{14, 18, ..., n-3, 16, 20, ..., n-1, 2, 15, 19, ..., n-2, 4, 17, 21, ..., n, 3, 1, 7, 6, 9, 5, 8, 10, 13, 12, 11\}$

*yields a* $5 \times n$ *Heffter array with distinct partial row sums in the first three rows and whose columns form a simple* $D(10n+1, 5)$.

**Lemma 4.8.** *For* $n \equiv 6 \pmod 8, n \geq 14$*, the row ordering*

$R = \{15, 19, ..., n-3, 17, 21, ..., n-1, 2, 16, 20, ..., n-2, 18, 22, ..., n, 4, 3, 1, 5, 10, 7, 6, 8, 12, 11, 9, 13, 14\}$

*yields a* $5 \times n$ *Heffter array with distinct partial row sums in the first three rows and whose columns form a simple* $D(10n+1, 5)$.

**Lemma 4.9.** *For* $n \equiv 7 \pmod 8, n \geq 7$*, the row ordering*

$R = \{8, 12, ..., n-3, 10, 14, ..., n-1, 7, 9, 13, ..., n-2, 11, 15, ..., n, 3, 4, 2, 6, 1, 5\}$

*yields a* $5 \times n$ *Heffter array with distinct partial row sums in the first three rows and whose columns form a simple* $D(10n+1, 5)$.

## 4.3 The problem with the last two rows

It is valid to find individual, or separate, orderings for different columns, or rows, as long as $\omega_r$ and $\omega_c$ are compatible. As such, we attempted to find a separate general reordering for the last two rows in the $5 \times n$ Heffter arrays. Unfortunately, we did not succeed because of the way the last two rows are constructed. To illustrate the problem we had with generalizing a reordering for the last two rows we will consider a specific case modulo eight. Consider the case when $n \equiv 7 \pmod 8$. As stated before, we need only consider the concatenated columns (i.e. the last $n-7$ columns) to establish a pattern. In the case $n \equiv 7$

(mod 8) we concatenate the following columns for values of $r$ between $r$ and $m-1$ (note here we only display rows 1, 4, and 5):

$$\begin{bmatrix} -4r+4m+1 & 4r-8m-4 & -4r+4m+2 & 4r-8m-5 & 4r-4m+1 & -4r+8m+2 & 4r-4m & -4r+8m+3 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 2r+24m+36 & -2r-26m-36 & 2r+28m+36 & -2r-30m-36 & -2r-32m-36 & 2r+34m+36 & -2r-36m-36 & 2r+38m+36 \\ -2r-24m-37 & 2r+26m+37 & -2r-28m-37 & 2r+30m+37 & 2r+32m+37 & -2r-34m-37 & 2r+36m+37 & -2r-38m-37 \end{bmatrix}.$$

In all five rows as $m$ grows larger, each concatenated column adds additional numbers which are equally spaced from the previous number. For example, column 8 of row 1 expands as follows for $0 \le r \le m-1$: $m=0:\{\emptyset\}, m=1:\{5\}, m=2:\{9,5\}, m=3:\{13,9,5\}$, etc. Column 8 of row 5 expands as follows for $0 \le r \le m-1$: $m=0:\{\emptyset\}, m=1:\{-61\}, m=2:\{-85,-87\}, m=3:\{-109,-111,-113\}$, etc. We cannot however place these columns next to each other in the reordering since the partial sums would then move outside of the modulus range, allowing for possibility of overlap (which indeed happens in certain cases). To eliminate this problem we attempted to pair each column with one of similar absolute value, but opposite sign. In the first three rows, this method worked perfectly as the partial sums remain the same distance apart regardless of how large $m$ is. However, in the last two rows the partial sums grow further apart as $m$ grows larger, creating eventual overlap of the partial sums. We still maintain that it is possible to reorder every $5 \times n$ matrix, but are unsure as to whether a general solution is possible to find.

## 4.4 Specific solutions

Although unable to find a general reordering for the $5 \times n$ Heffter arrays, we found specific reorderings for $H(5,n)$ with $3 \le n \le 100$. The reorderings were found using Mathematica and are presented as cyclic permutations. The following example demonstrates how to use the reorderings presented in Appendix A.

**Example 4.10.** *For $n = 5$ the original construction of $H(5,5)$ is:*

$$\begin{bmatrix} 1 & 5 & 6 & 7 & -19 \\ 2 & 8 & 12 & 15 & 14 \\ 3 & 9 & -21 & 22 & -13 \\ 4 & 11 & -25 & -24 & -17 \\ -10 & 18 & -23 & -20 & -16 \end{bmatrix}$$

*Figure 4.1: The original* $5 \times 5$ *Heffter array*

*The reordering for $n = 5$ in Appendix A is (3 4 5 2 1). So we reorder each row by the permutation $R = \{3, 4, 5, 2, 1\}$ to get the array:*

$$\begin{bmatrix} 6 & 7 & -19 & 5 & 1 \\ 12 & 15 & 14 & 8 & 2 \\ -21 & 22 & -13 & 9 & 3 \\ -25 & -24 & -17 & 11 & 4 \\ -23 & -20 & -16 & 18 & -10 \end{bmatrix}$$

*Figure 4.2: The reordered* $5 \times 5$ *Heffter array*

*One can easily check that the partial sums for each row are distinct.*

The following theorem states the existence of these orderings and biembeddings of $K_{10n+1}$.

**Theorem 4.11.** *(a) There exists a simple $5 \times n$ Heffter array for $3 \leq n \leq 100$.*
*(b) There exists a biembedding of $K_{10n+1}$ with every edge on a 5-cycle and an $n$-cycle.*

*Proof.* Let $H = (h_{ij})$ be the $H(5, n)$ presented in 4.1 for $3 \leq n \leq 100$.

(a) In Appendix A we present a simultaneous reordering for every row so that the rows (and columns) of $H$ form simple Heffter systems. Therefore there exists a simple $5 \times n$ Heffter array for $3 \leq n \leq 100$.

(b) Given the simple Heffter array $H$ from above, let $L$ be the half-set of elements in $\mathbb{Z}_{10n+1}$ contained in $H$. Recall that the rows of $H$ form a $D(2mn+1, n)$ and the columns of $H$ form a $D(2mn+1, m)$. Then using Corollary 2.6 it suffices to show that the orderings $\omega_r = (h_{11}, h_{12}, \ldots, h_{1n})(h_{21}, h_{22}, \ldots, h_{2n}) \ldots (h_{51}, h_{52}, \ldots, h_{5n})$ and $\omega_c = (h_{11}, h_{21}, h_{31}, h_{41}, h_{51})(h_{12}, h_{22}, h_{32}, h_{42},$ are compatible on the row and column Heffter systems. We must consider two cases: $n \equiv 1, 2, 3, 4 \pmod 5$ and $n \equiv 0 \pmod 5$.

First assume $n \equiv 1, 2, 3, 4 \pmod 5$. In this case we do not change $\omega_r$ or $\omega_c$. Given an element $h_{ij}$, $\omega_c \circ \omega_r(h_{ij}) = h_{i+1,j+1}$ where the row subscript is modulo 5 and the column subscript is modulo $n$. Since 1,2,3, and 4 are relatively prime with 5, the permutation created by continuously applying $\omega_c \circ \omega_r$ is of length $5n$. Therefore, this is a cyclic permutation of $L$ and the orderings are compatible.

Now assume $n \equiv 0 \pmod 5$. In this case we leave $\omega_r$ as presented but change $\omega_c$ so the last cycle $(h_{5n}, h_{4n}, h_{3n}, h_{2n}, h_{1n})$. Note that in changing the $\omega_c$, the row and column Heffter systems associated with $H$ remain simple. Then given an element $h_{ij}$, $\omega_c \circ \omega_r(h_{ij}) = h_{i+1,j+1}$ for $j < n$ and $\omega_c \circ \omega_r(h_{i,j}) = h_{i-1,j+1} = h_{i-1,1}$ for $j = n$ where the row position is modulo 5 and the column position is modulo $n$. Similar to the $3 \times n$ case, this is a cyclic permutation of $L$, and therefore the orderings are compatible. $\square$

# Chapter 5

# Partial sums in cyclic groups

In this chapter we discuss a conjecture about the ordering of subsets of $\mathbb{Z}_n \setminus \{0\}$.

**Conjecture 5.1.** *Let $\mathscr{A} \subseteq \mathbb{Z}_n$. There exists an ordering of the elements of $\mathscr{A}$ such that the partial sums are all distinct, i.e., for all $1 \leq j \leq k$, $s_i \neq s_j$.*

Conjecture 5.1 was first discussed by Archdeacon, Dinitz, Mattern, and Stinson in [3]. Alspach was interested in a similar decomposition problem, but with paths of length $k$ instead of $k$-cycles. The following slightly different conjecture was made several years ago by Alspach, see [6]:

**Conjecture 5.2.** *(Alspach) Suppose $A = \{a_1, \ldots, a_k\} \subseteq \mathbb{Z}_n \setminus \{0\}$ has the property that $\sum_{a \in A} a \neq 0$. Then there exists an ordering of the elements of $A$ such that the partial sums are all distinct and nonzero.*

In the following proposition, we show that Conjecture 5.2 implies Conjecture 5.1

**Proposition 5.3.** *[3] Conjecture 5.2 implies Conjecture 5.1.*

*Proof.* Assume that Conjecture 5.2 is true. Let $A = \{a_1, \ldots, a_k\} \subseteq \mathbb{Z}_n \setminus \{0\}$. If $\sum_{a \in A} a \neq 0$, then by Conjecture 2 there is an ordering of the elements of $A$ such that the partial sums are all distinct, proving Conjecture 1 in this case.

So assume that $\sum_{a \in A} a = 0$. It follows that $\sum_{i=1}^{k-1} a_i \neq 0$. So by Conjecture 2 there is an ordering $(a'_1, a'_2, \ldots, a'_{k-1})$ of $\{a_1, \ldots, a_{k-1}\}$ where all of the partial sums are distinct and nonzero. Now reinsert $a_k$ at the end of the ordering to get $(a'_1, a'_2, \ldots, a'_{k-1}, a_k)$. The only new partial sum is $s_k = 0 = \sum_{a \in A} a$ and since all of the earlier partial sums are nonzero (and distinct), we have that all the partial sums are now distinct. This proves Conjecture 5.1. $\square$

These two conjectures are natural generalizations of sequenceable groups. A *sequenceable* group is one which has an ordering of *all* the group elements such that all the partial sums are distinct. It is well known that $(\mathbb{Z}_n, +)$ is sequenceable if and only if it had a unique element of order 2. More generally, the following list gives a summary of known sequenceable groups. For references to the proofs of these results, see the survey by Ollis [14].

1. Abelian groups with a unique element of order 2.
2. Dihedral groups of order at least 10.
3. Non-abelian groups of order $n$ where $10 \leq n \leq 32$.
4. Some groups and direct product of groups of order $pq$ where $p$ and $q$ are odd primes.
5. $A_5$ and $S_5$.

We note that if Conjecture 5.1 was proven for all $n$, there would be no need to find reorderings for any Heffter arrays, as given an $H(m, n)$ there would then exist a simple $D(2mn + 1, n)$ and a simple $D(2mn + 1, m)$. For the biembedding problem, one would still need to prove the compatibility of the orderings on these simple Heffter systems. The proof of Conjecture 5.1 for $k \leq 6$ is given in [3], in the following proof we give the details only for $k = 6$.

**Theorem 5.4.** *Conjecture 5.1 is true when $k \leq 6$.*

*Proof.* See [3] for the cases where $k \leq 5$. For the remainder of this proof we let $\mathscr{A} = \{a_1, a_2, a_3, a_4, a_5, a_6\}$, and let $s_i$ be the partial sum of the first $i$ numbers in an arrangement of $\mathscr{A}$. Let $p$ be the number of pairs $\{x, -x\}$ in $\mathscr{A}$; so $p = 0, 1, 2,$ or $3$. First note that $s_i \neq s_{i+1}$ for any $1 \leq i \leq 5$ since $0 \notin \mathscr{A}$. Also note that if $\mathscr{A}$ is arranged such that for all $i$, $a_i \neq -a_{i+1}$, then $s_i \neq s_{i+2}$ for any $1 \leq i \leq 4$. Assuming this, we must only check the cases $s_1 = s_4, s_1 = s_5, s_1 = s_6, s_2 = s_5, s_2 = s_6$, and $s_3 = s_6$.

Assume $p = 0$, and let $\mathscr{A} = \{u, v, w, x, y, z\}$. Arrange $\mathscr{A}$ as $A = (u, v, w, x, y, z)$, renaming if necessary, so that $s_1, s_2, s_3$, and $s_4$ are distinct. In this case since there are no occurrences of a pair $\{x, -x\}$, the only conditions that can fail are the following six possibilities: (1) $s_1 = s_5$ and $s_3 = s_6$, (2) $s_1 = s_5$ and $s_3 \neq s_6$, (3) $s_1 \neq s_5$ and $s_3 = s_6$, (4) $s_1 = s_6$, (5) $s_2 = s_5$, or (6) $s_2 = s_6$. It is straightforward to show that in each of these cases the other possibilities are mutually exclusive. We will look at each case individually. For all cases, let $s'_i$ and $s''_i$ denote the $i^{th}$ partial sum after one ($'$) or two ($''$) changes of ordering, denoted $A'$ and $A''$ respectively.

1. ($s_1 = s_5$ and $s_3 = s_6$): In this case we have $v+w+x+y = 0 = x+z+y$. Now arrange $\mathscr{A}$ as follows: $A' = (u, v, x, w, z, y)$. Here both $s_3$ and $s_5$ have changed. Clearly, $s'_1 \neq s'_5$ as $s'_1 = s_1 = s_5 \neq s'_5$. Also, $s'_2 \neq s'_5$ since $s'_2 = s'_5$ would imply $x + w + z = 0$; however, since $x + z + y = 0$ this means $w = y$, a contradiction. Finally, $s'_3 \neq s'_6$ since $s'_3 \neq s_3 = s_6 = s'_6$.

2. ($s_1 = s_5$ and $s_3 \neq s_6$): In this case we have that $v + w + x + y = 0$. Now arrange $\mathscr{A}$ as follows: $A' = (u, v, w, x, z, y)$. First note that only $s_5$ has changed, and so we only need to check conditions containing $s'_5$. Clearly, $s'_1 \neq s'_5$ since $s'_1 = s_1 = s_5 \neq s'_5$. We could however have $s'_2 = s'_5$. If this is the case, then $v + w + x + y = 0 = w + x + z$. Then arrange $\mathscr{A}$ as follows: $A'' = (u, w, v, x, z, y)$. Here only $s'_2$ has changed from the previous arrangement, so we need only check conditions containing $s''_2$. We see

that $s''_2 \neq s''_5$ as $s''_2 \neq s'_2 = s'_5 = s''_5$. We also see that $s''_2 \neq s''_6$ since if not, then we get $v + x + z + y = 0$; however, since $v + w + x + y = 0$ we would have $w = z$, a contradiction.

3. ($s_1 \neq s_5$ and $s_3 = s_6$): In this case $x + y + z = 0$. Now arrange $\mathscr{A}$ as follows: $A' = (u, v, x, w, y, z)$. Here only $s_3$ has changed, but $s'_3 \neq s'_6$ as $s'_3 \neq s_3 = s_6 = s'_6$.

4. ($s_1 = s_6$): Here we have that $v + w + x + y + z = 0$. Arrange $\mathscr{A}$ as follows: $A' = (v, u, w, x, y, z)$. Note that only $s_1$ has changed, so we only need to check the conditions containing $s'_1$, including $s'_1 = s'_4$. Clearly, $s'_1 \neq s'_6$ since $s'_1 \neq s_1 = s_6 = s'_6$. However, it is possible for $s'_1 = s'_4$ or $s'_1 = s'_5$, but note that these cases are mutually exclusive.

   (a) ($s_1 = s_6$ and $s'_1 = s'_4$): In this case we get $v+w+x+y+z = 0 = u+w+x$. Then arrange $\mathscr{A}$ as follows: $A'' = (v, u, w, y, x, z)$. Note that only $s'_4$ has changed from $A'$. Thus we only check $s''_1 = s''_4$. But this is impossible since $s''_1 = s'_1 = s'_4 \neq s''_4$.

   (b) ($s_1 = s_6$ and $s'_1 = s'_5$): In this case we get $v + w + x + y + z = 0 = u + w + x + y$. Arrange $\mathscr{A}$ as follows: $A'' = (v, u, w, y, z, x)$. Here only $s'_4$ and $s'_5$ have changed from the previous arrangement. We see that $s''_1 \neq s''_4$ since equality implies that $u + w + y = 0$ and since $u + w + x + y = 0$ we would have $x = 0$, a contradiction. Also, $s''_1 \neq s''_5$ since if not, then we have that $u + w + y + z = 0$; however, since $u + w + x + y = 0$, this implies $z = x$, which is impossible. Finally, $s''_2 \neq s''_5$ as equality would imply that $w + y + z = 0$, but since $v + w + x + y + z = 0$ we would have $v = -x$, which is a contradiction.

5. ($s_2 = s_5$): In this case we have $w + x + y = 0$. Now arrange $\mathscr{A}$ as follows: $A' = (u, v, w, x, z, y)$. Note that only $s_5$ has changed so we only need to check those cases involving $s'_5$. Clearly $s'_2 \neq s'_5$ as $s'_2 = s_2 = s_5 \neq s'_5$. However, it is possible for $s'_1 = s'_5$. In this case we get $w + x + y = 0 = v + w + x + z$. Arrange $\mathscr{A}$ as follows: $A'' = (u, v, w, z, y, x)$. Here only $s'_4$ and $s'_5$ have changed from $A'$. We see $s''_1 \neq s''_4$ as

equality would imply that $v+w+z=0$ and since $v+w+x+z=0$ this would imply $x=0$, which is impossible. Also, $s_1'' \neq s_5''$ as $s_1''=s_1'=s_5' \neq s_5''$. Finally, $s_2'' \neq s_5''$ as $s_2''=s_2=s_5 \neq s_5''$.

6. ($s_2=s_6$): In this case $w+x+y+z=0$. Arrange $\mathscr{A}$ as follows: $A'=(u,w,v,x,y,z)$. Here only $s_2$ has changed and thus we need only check the cases containing $s_2'$. We see that $s_2' \neq s_6'$ as $s_2' \neq s_2 = s_6 = s_6'$. It is possible for $s_2'=s_5'$. In this case we have $w+x+y+z=0=v+x+y$. Now arrange $\mathscr{A}$ as follows: $A''=(u,w,v,x,z,y)$. Here only $s_5'$ has changed from $A'$. We see $s_1'' \neq s_5''$ as equality would imply $w+v+x+z=0$; however, since $w+x+y+z=0$ this would mean $v=y$, a contradiction. Clearly, $s_2'' \neq s_5''$ as $s_2''=s_2'=s_5' \neq s_5'$. This completes the case for $p=0$.

Next assume that $p=1$. Let $\mathscr{A}=\{x,-x,v,w,y,z\}$ and arrange $\mathscr{A}$ as follows: $A=(x,v,-x,w,y,z)$. Since $x$ is not adjacent to $-x$, the only conditions that can fail are the following nine possibilities: (1) $s_1=s_4$ and $s_2=s_6$ (2) $s_1=s_4$ and $s_3=s_6$, (3) $s_1=s_4$, $s_2 \neq s_6$, and $s_3 \neq s_6$, (4) $s_1 \neq s_4$ and $s_2=s_6$, (5) $s_1 \neq s_4$, $s_1 \neq s_5$, and $s_3=s_6$, (6) $s_1=s_5$ and $s_3=s_6$, (7) $s_1=s_5$ and $s_3 \neq s_6$, (8) $s_1=s_6$, or (9) $s_2=s_5$. It is straightforward to show that all other combinations are not possible. We consider each case individually and define $s_i'$ and $s_i''$ as before.

1. ($s_1=s_4$ and $s_2=s_6$): In this case we have $x=v+w=w+y+z$. Arrange $\mathscr{A}$ as follows: $A'=(x,w,y,v,-x,z)$. Here $s_2,s_3$, and $s_4$ have changed. Clearly $s_1' \neq s_4'$ as $s_1'=s_1=s_4 \neq s_4'$ and $s_2' \neq s_6'$ as $s_2' \neq s_2=s_6=s_6'$. Also, $s_2' \neq s_5'$ since equality would imply that $x=y+v$ and since $x=v+w$ this implies that $w=y$, which is impossible. Finally, $s_3' \neq s_6'$ since if $s_3'=s_6'$, then $x=v+z$, and since $x=v+w$ this would mean $w=z$, a contradiction.

2. ($s_1=s_4$ and $s_3=s_6$): In this case $x=v+w$ and $w+y+z=0$. Then arrange $\mathscr{A}$ as follows: $A$‘ $=(x,v,w,y,-x.z)$. Here only $s_3$ and $s_4$ have changed. Clearly $s_1' \neq s_4'$

since $s'_1 = s_1 = s_4 \neq s'_4$ and similarly $s'_3 \neq s'_6$ as $s'_3 \neq s_3 = s_6 = s'_6$.

3. ($s_1 = s_4$ and $s_2 \neq s_6$ and $s_3 \neq s_6$): In this case we get $x = v + w$. We now arrange $\mathscr{A}$ as follows: $A' = (x, v, w, y, -x, z)$. Here only $s_3$ and $s_4$ have changed, so we need only check cases containing $s'_3$ and $s'_4$. Clearly, $s'_1 \neq s'_4$ as $s'_1 = s_1 = s_4 \neq s'_4$. However, it is possible for $s'_3 = s'_6$. In this case we have $x = v + w = y + z$. Now arrange $\mathscr{A}$ as follows: $A'' = (x, w, y, v, -x, z)$. Here $s'_2$ and $s'_3$ have changed from the previous arrangement. We see $s''_2 \neq s''_5$ as equality would imply $x = y + v$; however, since $x = v + w$ this would mean $w = y$, which is impossible. Also, if $s''_2 = s''_6$, then we would have $x = y + v + z$ and since $x = y + z$ this would imply $v = 0$, a contradiction. Hence $s''_2 \neq s''_6$ . Finally, $s''_3 \neq s''_6$ since $s''_3 \neq s'_3 = s'_6 = s''_6$.

4. ($s_1 \neq s_4$ and $s_2 = s_6$): In this case we have that $x = w + y + z$. Now arrange $\mathscr{A}$ as $A' = (x, w, v, -x, y, z)$. Here only $s_2$ and $s_3$ have changed. Clearly, $s'_2 \neq s'_6$ since $s'_2 \neq s_2 = s_6 = s'_6$. Also, $s'_3 \neq s'_6$ as equality would imply $x = z + y$; however, since $x = w + y + z$, this would mean $w = 0$, which is impossible. It is possible however for $s'_2 = s'_5$. In this case we get $x = w + y + z = v + y$. Arrange $\mathscr{A}$ as follows: $A' = (x, w, v, -x, z, y)$. Here only $s'_5$ has changed. We see $s''_1 \neq s''_5$ since if $s''_1 = s''_5$, then $x = w + y + z$ and since $x = w + y + z$ we get that $y = v$, a contradiction. Finally, $s''_2 \neq s''_5$ since $s''_2 = s'_2 = s'_5 \neq s''_5$.

5. ($s_1 \neq s_4$, $s_1 \neq s_5$, and $s_3 = s_6$): In this case we get $w + y + z = 0$. Arrange $\mathscr{A}$ as follows: $A' = (x, v, w, -x, y, z)$. Here only $s_3$ has changed. Clearly, $s'_3 \neq s'_6$ since $s'_3 \neq s_3 = s_6 = s'_6$.

6. ($s_1 = s_5$ and $s_3 = s_6$): In this case we get $x = v + w + y$ and $w + y + z = 0$. We arrange $\mathscr{A}$ as $A' = (x, v, w, -x, z, y)$. Here only $s_3$ and $s_5$ have changed. We see $s'_1 \neq s'_5$ since $s'_1 = s_1 = s_5 \neq s'_5$. Similarly, $s'_3 \neq s'_6$ as $s'_3 \neq s_3 = s_6 = s'_6$. Also, $s'_2 \neq s'_5$ as equality would mean that $x = w + z$. But since $w + y + z = 0$, we have that $w + z = -y$.

Together these imply that $x = -y$, a contradiction.

7. ($s_1 = s_5$ and $s_3 \neq s_6$): In this case $x = v + w + y$. Arrange $\mathscr{A}$ as follows: $A' = (x, v, -x, w, z, y)$. Here only $s_5$ has changed. Clearly, $s'_1 \neq s'_5$ since $s'_1 = s_1 = s_5 \neq s'_5$. It is possible however for $s'_2 = s'_5$. In this case we get $x = v + w + y = w + z$. Now arrange $\mathscr{A}$ as follows: $A'' = (x, v, z, -x, y, w)$. Here $s'_3, s'_4$, and $s'_5$ have changed from $A'$. We see $s''_3 \neq s''_6$ as equality would imply $x = y + w$, but since $x = w + z$ this would mean $z = y$, which is a contradiction. Also, $s''_1 \neq s''_4$ since if $s''_1 = s''_4$, then $x = v + z$; however, since $x = w + z$ this would mean $w = v$, which is impossible. Furthermore, $s''_1 \neq s''_5$ as equality would imply $x = v + z + y$ and since $x = v + w + y$ this would imply that $w = z$, a contradiction. Finally, $s''_2 \neq s''_5$ since $s''_2 = s'_2 = s'_5 \neq s''_5$.

8. ($s_1 = s_6$): In this case we get $x = v + w + y + z$. Then arrange $\mathscr{A}$ as follows: $A' = (v, x, w, -x, y, z)$. Here only $s_1$ and $s_3$ have changed. We see $s'_1 \neq s'_4$ since this would imply $w = 0$, which is impossible. Also, $s'_1 \neq s'_5$ as this means $w = -y$, a contradiction. Clearly, $s'_1 \neq s'_6$ since $s'_1 \neq s_1 = s_6 = s'_6$. Finally, $s'_3 \neq s'_6$ as equality would imply $x = y + z$ and since $x = v + w + y + z$ this would mean $v = -w$, a contradiction.

9. ($s_2 = s_5$): In this case $x = w + y$. Arrange $\mathscr{A}$ as $A' = (x, v, -x, w, z, y)$. Here only $s_5$ has changed. Clearly, $s'_2 \neq s'_5$ as $s'_2 = s_2 = s_5 \neq s'_5$. However, it is possible for $s'_1 = s'_5$. In this case we get $x = w + y = v + w + z$. Now arrange $\mathscr{A}$ as $A'' = (v, x, w, -x, z, y)$. Here only $s'_1$ and $s'_3$ have changed. We see $s''_1 \neq s''_4$ as equality would imply $w = 0$, a contradiction. Clearly, $s''_1 \neq s''_5$ since $s''_1 \neq s'_1 = s'_5 = s''_5$. Also, if $s''_1 = s''_6$, then $w + z + y = 0$, but since $x = w + y$ this would mean $-x = z$, which is impossible. Hence $s''_1 \neq s''_6$. Finally, $s''_3 \neq s''_6$ as equality would mean that $x = z + y$; however, since $x = w + y$ this would imply $w = z$, a contradiction. This completes the case $p = 1$.

Now assume $p = 2$. Let $\mathscr{A} = \{x, -x, y, -y, w, z\}$ and arrange $\mathscr{A}$ as follows: $A = (x, y, -x, -y, w, z)$. Since neither $x, -x$ nor $y, -y$ are adjacent in $\mathscr{A}$, we need only check those partial sums at least three apart. Clearly, $s_1 \neq s_4$ since that implies $x = 0$, and $s_1 \neq s_5$ since that yields $x = w$. The only conditions which could fail are the following four possibilities: (1) $s_1 = s_6$, (2) $s_2 = s_5$, (3) $s_2 = s_6$, and (4) $s_3 = s_6$. It is straightforward to show that if any one of these conditions hold, then the other three do not hold. We look at each individual case.

1. ($s_1 = s_6$): In this case we have that $x = w+z$. We arrange $\mathscr{A}$ as $A' = (w, x, y, -x, z, -y)$. Here every partial sum except $s_6$ has changed. Clearly, $s'_1 \neq s'_4$ since this would mean $y = 0$, $s'_1 \neq s'_5$ since this would mean $y = -z$, $s'_1 \neq s'_6$ since this would mean $z = 0$, and $s'_2 \neq s'_6$ since this would mean $x = z$. We also see $s'_2 \neq s'_5$ as equality would imply $x = y + z$ and since $x = w + z$, this would mean $y = w$. Finally, we see $s'_3 \neq s'_6$ since if $s'_3 = s'_6$, then $z = x + y$ and since $x = w + z$ this would imply $y = -w$, a contradiction.

2. ($s_2 = s_5$): In this case $w = x+y$. Then arrange $\mathscr{A}$ as follows: $A' = (x, y, -x, -y, z, w)$. Here only $s_5$ has changed. We see $s'_1 \neq s'_5$ as equality would mean that $x = z$, a contradiction. Also, $s'_2 \neq s'_5$ since $s'_2 = s_2 = s_5 \neq s'_5$.

3. ($s_2 = s_6$): In this case $x = w + z - y$. Arrange $\mathscr{A}$ as $A' = (x, z, -y, w, y, -x)$. Here $s_2, s_3, s_4$, and $s_5$ have all changed. We see $s'_2 \neq s'_5$ as this would imply $w = 0$ and $s'_2 \neq s'_6$ as this would imply $x = w$. Also, $s'_1 \neq s'_4$ as equality would imply $z - y + w = 0$; however, since $x = w + z - y$ this would mean $x = 0$. Furthermore, $s'_1 \neq s'_5$ as equality would imply $z = -w$, a contradiction. It is however possible for $s'_3 = s'_6$. In this case we get $x = w + z - y$ and $x = w + y$, which implies $z = 2y$. Now arrange $\mathscr{A}$ as follows: $A'' = (x, w, y, z, -x, -y)$. Again, $s'_2, s'_3, s'_4$, and $s'_5$ have all changed from the previous arrangement. We see $s''_2 \neq s''_5$ as equality would imply $x = y + z$ and since $x = w + z - y$ this means $w = 2y$. But since $z = 2y$ this implies $w = z$. Also,

$s_2'' \neq s_6''$ since here equality would imply $x = z$ and $s_3'' \neq s_6''$ since $s_3'' \neq s_3' = s_6' = s_6''$. Furthermore, $s_1'' \neq s_4''$ since if $s_1'' = s_4''$, then this would imply that $w + y + z = 0$. But since $x = w + z - y$ we get that $x = -2y$; however, since $z = 2y$ this would mean $x = -z$. Finally, $s_1'' \neq s_5''$ since equality would imply $x = w + y + z$; however, since $x = w + z - y$ this means $y = -y$, a contradiction.

4. ($s_3 = s_6$): In this case we have $y = w + z$. We now arrange $\mathscr{A}$ as follows: $A' = (w, x, y, -x, z, -y)$. Here everything but $s_6$ has changed. Clearly, $s_1' \neq s_4'$ as this would imply $y = 0$, $s_1' \neq s_5'$ since this would mean $y = -z$, and $s_1' \neq s_6'$ as this implies $z = 0$. Also, $s_2' \neq s_6'$ as equality would imply $x = z$. Furthermore, $s_3' \neq s_6'$ as equality means $y = z - x$ and since $y = w + z$ this would imply $w = -x$. It is possible however for $s_2' = s_5'$. In this case we get $y = w + z$ and $x = z + y$. Now arrange $\mathscr{A}$ as follows: $A'' = (x, y, w, -x, z, -y)$. Here $s_1'$ and $s_2'$ have changed from the arrangement $A'$. We see $s_1'' \neq s_4''$ as equality would imply $x = y + w$ and since $x = z + y$ this means $z = w$. Also, $s_1'' \neq s_5''$ since equality implies that $x = y + w + z$; however, since $x = z + y$ this implies $w = 0$, a contradiction. Furthermore, $s_1'' \neq s_6''$ since equality would imply $x = w + z$ and since $y = w + z$ we have that $x = y$, which is impossible. Clearly, $s_2'' \neq s_5''$ since $s_2'' \neq s_2' = s_5' = s_5''$. Finally, $s_2'' \neq s_6''$ since equality implies $x + y = w + z$; however, since $y = w + z$, this would mean $x = 0$, which is a contradiction. This completes the case for $p = 2$.

Finally, assume that $p = 3$. Let $\mathscr{A} = \{x, -x, y, -y, z, -z\}$ and arrange $\mathscr{A}$ as follows: $A = (x, y, z, -x, -y, -z)$. Since no pair of additive inverses appears in adjacent positions, we only need to check the partial sums that are least three apart. Clearly, $s_1 \neq s_5$ since this would imply $x = z$, $s_1 \neq s_6$ as this would imply $x = 0$, and $s_2 \neq s_6$ since this would imply $x = -y$. The only conditions that can fail are the following three possibilities: (1) $s_1 = s_4$, (2) $s_2 = s_5$, or (3) $s_3 = s_6$. It is straightforward to show these possibilities are mutually exclusive. We consider each case individually.

1. ($s_1 = s_4$): In this case we get $x = y+z$. Then arrange $\mathscr{A}$ as $A' = (x, y, z, -y, -x, -z)$. Here only $s_4$ has changed and clearly $s_1' \neq s_4'$ since $s_1' = s_1 = s_4 \neq s_4'$.

2. ($s_2 = s_5$): In this case we have that $z = x + y$. Now arrange $\mathscr{A}$ as follows: $A' = (x, -y, z, y, -x, -z)$. Here $s_2, s_3$, and $s_4$ have changed. We see $s_1' \neq s_4'$ since this would imply $z = 0$. Also, $s_2' \neq s_5'$ as $s_2' \neq s_2 = s_5 = s_5'$ Furthermore, $s_2' \neq s_6'$ as this would imply $x = y$. Finally, $s_3' \neq s_6'$ as equality implies that $x - y + z = 0$. But since $z = x + y$, then $x = z - y$, which implies that $z = y$, a contradiction.

3. ($s_3 = s_6$): In this case $x + y + z = 0$. Then arrange $\mathscr{A}$ as $A' = (x, y, -z, -x, z, -y)$. Here $s_3, s_4$, and $s_5$ have changed. Clearly, $s_3' \neq s_6'$ as $s_3' \neq s_3 = s_6 = s_6'$. Also, $s_1' \neq s_4'$ as equality would imply $x = y - z$; however, since $x + y + z = 0$ this means $y = -y$, a contradiction. Furthermore, $s_1' \neq s_5'$ as equality would imply that $x = y$ and $s_2' \neq s_5'$ as equality here would imply $x = 0$. This completes the proof.

$\square$

# Chapter 6

# Conclusion

Prior to this thesis, some results have been proven regarding the biembedding of Steiner triple systems on both orientable and non-orientable surfaces. Despite this, the question of biembedding complete graphs on $6n+1$ vertices with every face on both a 3-cycle and an $n$-cycle had never before been considered.

To address this question we first studied Heffter arrays and their relationship to current graphs and biembeddings of complete graphs on orientable surfaces. We then used this relationship to prove that for every $n \geq 3$ there exists a biembedding of $K_{6n+1}$ using a Steiner triple system and an $n$-cycle system.

We also extended the question to biembedding complete graphs on $10n+1$ vertices with every edge on a face of size 5 and a face of size $n$. Although unable to completely solve this question, we gave general reorderings for the first three rows of every $5 \times n$ Heffter array and discussed the reasons why we were unable to find reorderings for the fourth and fifth rows of these arrays. In Appendix A we list reorderings for $5 \times n$ Heffter arrays which lead to a biembedding of $K_{10n+1}$ usuing a cyclic 5-cycle system and a cyclic $n$-cycle system for all $3 \leq n \leq 100$. Finally, in Chapter 5 we discussed a related conjecture generalizing sequenceable groups.

To continue our study of Heffter arrays and the biembedding of complete graphs, we hope for a new idea to find a general reordering of the last two rows in the $5 \times n$ Heffter arrays. We also hope to find a general reordering for *all* of the rows in the $5 \times n$ Heffter arrays in order to construct a simple $H(5, n)$ for all $n$. We can also expand this project to the next case of Heffter arrays, $7 \times n$, or to Heffter arrays with empty cells. To read more about Heffter arrays with empty cells see [4]. In terms of sequencing subsets of $\mathbb{Z}_n \setminus \{0\}$, it does not seem fruitful to extend the proof used in $k \leq 6$ to the case $k = 7$, although we certainly believe we could. Despite this we still believe Conjecture 5.1 to be true.

# Appendix A

# Specific solutions for reordering $5 \times n$ Heffter arrays

We provide the following cyclic permutations as reorderings for each of the $5 \times n$ Heffter arrays given in Theorem 4.1 for al $3 \leq n \leq 100$. The resulting arrays are simple Heffter arrays.

$\boldsymbol{n = 3}$**:** (3 2 1)

$\boldsymbol{n = 4}$**:** (2 1 4 3)

$\boldsymbol{n = 5}$**:** (3 4 5 2 1)

$\boldsymbol{n = 6}$**:** (3 6 2 4 1 5)

$\boldsymbol{n = 7}$**:** (1 6 7 5 2 4 3)

$\boldsymbol{n = 8}$**:** (8 3 6 7 1 2 4 5)

$\boldsymbol{n = 9}$**:** (4 2 9 5 3 6 7 8 1)

$\boldsymbol{n = 10}$**:** (2 9 7 5 3 4 1 10 8 6)

$\boldsymbol{n = 11}$**:** (2 3 5 4 9 6 8 1 7 10 11)

$\boldsymbol{n = 12}$**:** (2 5 8 10 4 1 3 12 6 11 9 7)

$\boldsymbol{n = 13}$**:** (8 6 9 1 4 3 13 11 5 2 7 12 10)

$\boldsymbol{n = 14}$**:** (7 13 4 2 11 10 5 9 12 14 1 8 3 6)

$\boldsymbol{n = 15}$**:** (4 3 12 2 14 15 9 13 8 11 7 1 6 5 10)

$\boldsymbol{n = 16}$**:** (16 10 5 13 15 8 3 11 14 6 2 9 1 7 12 4)

$\boldsymbol{n = 17}$**:** (15 2 14 16 12 4 10 5 11 3 6 8 13 17 9 7 1)

$\boldsymbol{n = 18}$**:** (10 7 14 6 8 9 12 15 4 3 5 18 1 16 13 17 2 11)

$\boldsymbol{n = 19}$**:** (2 4 5 9 16 14 1 7 3 19 12 13 8 15 6 11 18 10 17)

$\boldsymbol{n = 20}$**:** (4 8 11 9 14 3 6 20 7 18 13 19 10 12 2 15 5 16 1 17)

$\boldsymbol{n = 21}$**:** (19 12 2 4 8 10 16 18 3 20 17 5 15 1 21 13 14 11 9 7 6)

$\boldsymbol{n = 22}$**:** (20 17 3 16 8 10 6 22 9 14 15 13 12 18 2 1 19 21 4 7 5 11)

$\boldsymbol{n = 23}$**:** (15 16 6 23 22 11 8 13 3 10 19 7 17 4 20 12 21 1 5 18 9 14 2)

$\boldsymbol{n = 24}$**:** (2 1 16 18 6 23 13 8 4 10 15 21 5 20 17 19 12 24 3 11 7 9 22 14)

$\boldsymbol{n = 25}$**:** (21 12 6 20 23 24 18 5 19 10 15 2 17 9 7 25 16 11 1 22 14 4 3 8 13)

$\boldsymbol{n = 26}$**:** (23 20 14 3 2 10 5 9 12 18 15 16 26 19 7 21 13 11 1 17 25 8 6 4 22 24)

$\boldsymbol{n = 27}$**:** (13 8 18 23 2 25 15 19 9 17 26 16 3 4 12 20 10 6 1 11 24 14 22 5 21 7 27)

$\boldsymbol{n = 28}$**:** (11 26 14 10 17 6 22 27 28 23 9 21 12 15 2 8 1 18 4 5 7 24 13 20 3 25 16 19)

$\boldsymbol{n = 29}$**:** (4 26 18 7 15 13 19 21 1 14 25 3 29 28 24 2 20 12 17 23 16 8 5 9 27 10 6 22 11)

$\boldsymbol{n = 30}$**:** (29 28 26 16 7 27 20 12 21 23 2 22 8 13 17 19 18 24 5 14 1 3 25 4 6 15 11 10 30 9)

$\boldsymbol{n} = \mathbf{31}$**:** (10 21 16 14 28 19 17 2 13 26 25 6 9 29 4 5 18 27 31 1 8 23 24 3 30 15 7 11 12 20 22)

$\boldsymbol{n} = \mathbf{32}$**:** (9 11 1 31 4 3 23 14 24 7 16 8 27 13 25 15 21 2 32 18 6 19 22 5 17 20 12 10 28 29 30 26)

$\boldsymbol{n} = \mathbf{33}$**:** (7 11 33 28 6 31 9 25 29 4 21 12 16 26 13 27 19 14 15 17 3 18 30 24 23 22 1 5 20 8 2 32 10)

$\boldsymbol{n} = \mathbf{34}$**:** (17 15 7 22 2 32 34 6 33 18 29 5 14 16 13 19 12 9 25 24 10 4 8 28 27 21 26 31 30 23 20 3 1 11)

$\boldsymbol{n} = \mathbf{35}$**:** (10 29 2 4 8 15 24 32 18 31 35 3 16 20 21 19 33 23 27 17 11 6 5 28 22 12 13 14 30 7 1 34 26 25 9)

$\boldsymbol{n} = \mathbf{36}$**:** (13 9 29 23 18 31 24 36 28 7 12 10 1 25 2 20 35 11 5 32 26 27 19 33 3 14 34 17 6 15 16 30 8 4 21 22)

$\boldsymbol{n} = \mathbf{37}$**:** (10 7 17 36 33 4 26 27 19 13 14 3 11 34 23 1 37 29 32 24 15 28 31 30 9 16 8 21 25 20 2 18 22 6 5 35 12)

$\boldsymbol{n} = \mathbf{38}$**:** (28 15 29 3 24 2 35 21 30 13 25 23 16 6 5 32 27 14 38 8 37 12 4 11 20 34 10 22 26 19 1 18 36 7 33 17 9 31)

$\boldsymbol{n} = \mathbf{39}$**:** (15 21 29 31 37 2 27 10 6 19 4 23 3 34 8 26 9 17 5 33 30 16 25 28 12 18 14 36 11 38 7 32 1 22 35 39 24 20 13)

$\boldsymbol{n} = \mathbf{40}$**:** (2 37 19 7 39 17 30 40 5 16 27 8 35 22 28 12 13 1 34 31 4 20 23 15 10 36 24 3 33 25 21 14 29 18 38 11 32 26 9 6)

$\boldsymbol{n} = \mathbf{41}$**:** (18 15 20 4 34 16 17 32 26 24 41 7 28 10 36 31 22 30 21 37 25 13 35 5 38 33 11 23 29 12 19 39 27 9 8 40 14 3 6 1 2)

$\boldsymbol{n = 42}$**:** (21 27 40 5 11 32 37 41 36 1 18 24 25 6 39 8 10 14 12 26 3 20 22 4 28 38 19 35 31 2 13 16 17 15 30 7 42 23 29 33 34 9)

$\boldsymbol{n = 43}$**:** (9 10 2 16 29 22 21 36 40 18 32 24 12 13 20 34 37 14 5 33 41 17 23 39 26 4 11 25 27 19 30 35 1 28 42 6 43 3 8 7 31 15 38)

$\boldsymbol{n = 44}$**:** (20 30 31 36 29 21 12 28 17 24 14 23 26 11 22 37 34 25 9 4 40 44 7 33 10 35 2 13 27 5 39 3 1 41 38 6 15 32 19 42 16 8 43 18)

$\boldsymbol{n = 45}$**:** (29 22 45 8 44 42 17 35 2 3 1 32 9 37 28 33 26 7 43 5 39 4 12 24 10 27 18 6 20 41 23 14 34 40 13 21 31 38 36 16 15 19 25 11 30)

$\boldsymbol{n = 46}$**:** (42 31 25 36 40 38 13 34 33 15 19 12 28 39 10 32 29 8 45 37 6 3 2 1 24 11 17 35 4 9 21 41 44 18 14 27 43 46 16 26 22 7 20 23 5 30)

$\boldsymbol{n = 47}$**:** (44 42 10 28 36 33 7 17 34 19 16 23 46 9 15 6 30 20 24 40 11 38 4 31 5 45 14 41 39 22 27 25 26 37 1 35 21 8 32 13 43 47 29 18 3 2 12)

$\boldsymbol{n = 48}$**:** (43 46 25 24 8 27 7 12 29 40 36 6 10 2 15 37 33 26 9 23 35 3 31 20 42 21 34 47 45 38 32 14 18 39 30 28 11 1 13 44 16 48 19 17 5 4 41 22)

$\boldsymbol{n = 49}$**:** (16 29 11 34 24 9 2 15 33 37 30 13 46 21 45 40 22 27 25 44 23 32 47 5 1 36 35 7 14 4 8 48 12 10 18 6 31 26 17 3 20 49 38 41 42 43 19 39 28)

$\boldsymbol{n = 50}$**:** (38 47 18 48 31 9 14 45 29 41 15 2 42 17 35 36 50 34 13 25 44 1 5 4 10 28 33 32 3 12 43 24 26 49 37 19 21 7 11 6 8 27 46 22 30 20 16 40 23 39)

$\boldsymbol{n = 51}$**:** (6 14 46 27 28 48 33 11 44 19 13 4 37 29 41 2 7 40 20 10 34 24 31 30 50 15 47 17 12 5 43 9 38 16 3 18 39 25 49 26 45 42 35 8 22 51 23 1 32 36 21)

$\boldsymbol{n = 52}$**:** (19 3 9 39 45 11 10 43 40 34 44 46 12 26 47 13 37 17 15 36 21 23 29 5 7 28 18 35 14 2 25 38 27 41 51 31 32 30 22 8 20 1 6 52 49 24 48 4 16 50 33 42)

$\boldsymbol{n=53}$**:** (2 35 4 1 40 24 16 41 36 28 22 52 53 15 12 47 6 38 19 32 39 37 20 26 25 3 14 21 13 48 18 8 7 5 27 9 42 17 45 44 43 46 30 31 49 34 10 29 50 11 23 51 33)

$\boldsymbol{n=54}$**:** (4 37 48 32 39 6 19 31 23 10 1 25 36 30 41 42 49 33 15 28 9 35 2 52 44 54 12 11 24 27 53 20 18 50 26 38 43 21 14 13 16 34 45 17 40 7 46 8 5 47 29 51 3 22)

$\boldsymbol{n=55}$**:** (15 30 25 27 8 31 46 6 3 33 36 50 54 51 5 18 19 22 47 23 44 40 14 12 38 37 29 4 39 53 10 20 42 49 21 2 32 24 7 34 1 11 45 55 43 13 16 9 48 52 35 26 41 17 28)

$\boldsymbol{n=56}$**:** (55 12 24 5 18 40 27 51 37 56 46 39 21 42 28 35 54 11 53 47 2 45 22 19 38 43 15 41 49 32 17 33 44 1 14 48 36 31 16 4 25 50 30 8 13 3 29 6 7 9 26 34 52 23 10 20)

$\boldsymbol{n=57}$**:** (14 49 48 37 3 39 45 43 54 1 19 32 11 18 28 15 23 31 40 52 34 35 7 44 22 12 42 25 55 10 33 2 8 4 53 21 20 47 9 50 30 36 24 27 5 16 26 51 56 46 13 17 29 38 41 57 6)

$\boldsymbol{n=58}$**:** (54 18 3 9 24 10 53 52 11 27 51 26 43 42 57 37 28 38 4 5 19 1 56 50 16 29 31 40 21 45 7 2 44 30 47 58 48 8 32 13 55 6 15 35 25 23 14 33 34 46 39 49 22 41 12 36 20 17)

$\boldsymbol{n=59}$**:** (26 34 13 14 30 12 38 40 47 32 53 58 2 16 15 49 25 43 24 33 9 6 37 5 3 46 19 54 44 51 23 35 22 27 17 55 20 42 28 4 50 41 36 7 57 31 10 59 8 29 39 45 18 56 1 11 21 48 52)v

$\boldsymbol{n=60}$**:** (8 59 31 9 20 56 21 26 25 3 51 45 22 17 42 32 4 34 43 12 35 28 57 48 40 19 10 33 36 24 53 11 15 41 7 37 1 5 55 44 46 30 50 27 14 39 38 29 54 16 2 47 58 60 52 18 23 6 13 49)

$\boldsymbol{n=61}$**:** (5 7 46 34 41 21 44 23 28 51 32 38 39 31 36 48 29 18 43 11 26 45 3 4 42 24 14 25 12 57 16 60 27 13 20 8 58 10 15 22 1 52 30 19 53 33 61 9 59 6 55 2 56 54 40 35 49 47 37 17 50)

$\boldsymbol{n=62}$**:** (4 28 17 27 8 9 32 54 44 57 12 62 53 24 45 42 60 23 11 18 26 16 37 39 22 34 20 7 30 36 48 1 59 47 2 38 13 3 58 40 49 43 10 21 41 61 5 15 14 6 31 29 52 33 55 50 46 51 56 19 25 35)

$\boldsymbol{n} = \mathbf{63}$**:** (11 31 20 18 43 25 49 45 52 24 5 62 48 29 41 6 39 16 15 36 54 59 34 33 17 61 4 13 37 21 51 23 14 28 50 35 60 53 26 12 55 57 56 3 38 8 7 22 30 63 32 27 44 58 2 47 10 19 9 42 1 46 40)

$\boldsymbol{n} = \mathbf{64}$**:** (4 32 33 39 45 36 57 26 55 52 19 16 24 59 17 1 38 12 58 50 5 11 18 46 37 30 25 28 61 47 56 8 2 21 40 42 31 64 43 14 63 41 49 48 54 27 51 6 15 44 62 34 35 7 53 23 13 20 3 9 10 22 29 60)

$\boldsymbol{n} = \mathbf{65}$**:** (14 23 59 64 18 24 2 65 37 60 27 32 43 29 10 11 38 34 8 56 42 13 55 40 7 9 50 6 44 4 48 30 52 46 62 36 5 28 33 45 17 31 58 47 15 3 25 57 53 41 35 51 16 21 19 22 20 63 39 1 49 54 61 12 26)

$\boldsymbol{n} = \mathbf{66}$**:** (3 14 6 55 65 63 36 4 19 2 62 1 47 59 10 50 9 37 15 45 46 53 16 18 22 41 26 5 34 54 44 39 13 7 57 32 66 21 64 24 58 48 43 12 52 60 61 38 31 56 23 8 40 33 11 51 35 17 42 20 49 30 29 27 28 25)

$\boldsymbol{n} = \mathbf{67}$**:** (16 66 60 19 20 18 41 43 6 46 36 26 64 47 7 27 24 48 52 13 3 11 44 59 22 56 40 1 4 61 51 39 37 42 14 33 38 65 17 54 29 9 34 55 67 28 63 30 62 5 8 15 58 21 35 23 12 10 50 32 49 2 53 31 45 57 25)

$\boldsymbol{n} = \mathbf{68}$**:** (5 49 3 55 66 16 54 23 36 19 22 42 34 35 48 24 39 12 17 20 9 15 51 38 18 57 10 52 21 43 31 47 63 61 1 50 40 45 27 28 56 59 32 65 60 53 64 2 68 58 41 37 25 46 44 14 4 7 33 13 26 29 30 6 8 67 11 62)

$\boldsymbol{n} = \mathbf{69}$**:** (5 49 65 3 27 63 13 37 35 69 55 42 46 34 32 67 52 14 60 28 54 17 19 15 2 38 41 4 31 61 68 50 6 23 53 48 36 16 30 9 39 1 62 25 7 56 45 24 43 10 57 40 20 12 33 51 22 11 47 66 59 58 18 44 26 29 8 21 64)

$\boldsymbol{n} = \mathbf{70}$**:** (24 5 54 3 36 20 6 43 33 52 60 40 55 41 45 29 23 68 13 38 37 42 14 51 21 62 69 58 30 15 59 50 34 25 1 53 66 28 35 27 18 46 22 16 32 67 47 64 17 10 39 9 49 7 61 31 26 8 19 70 12 48 57 4 44 56 2 65 11 63)

$\boldsymbol{n = 71}$**:** (8 59 67 68 45 69 33 10 16 42 22 40 70 61 15 1 25 43 28 63 35 6 60 18 29 56 9 14 26 3 58 34 55 54 46 20 44 36 12 27 37 53 5 2 30 38 64 66 24 31 19 11 52 39 71 32 7 62 41 23 57 50 21 17 48 4 51 65 13 47 49)

$\boldsymbol{n = 72}$**:** (13 66 10 34 24 5 12 57 51 7 39 61 25 60 16 69 68 47 2 19 71 67 22 9 14 62 58 53 70 41 50 45 6 29 28 31 26 21 56 32 4 38 36 46 59 65 44 63 48 15 27 43 42 30 11 49 23 1 35 54 18 55 64 52 40 72 3 20 8 17 33 37)

$\boldsymbol{n = 73}$**:** (1 28 23 51 39 11 17 50 20 10 5 47 48 52 49 59 35 65 9 21 37 61 8 6 70 22 36 33 16 31 30 68 44 45 41 73 29 67 13 34 3 62 40 26 42 63 2 18 71 24 25 55 4 12 27 69 56 15 53 14 66 58 19 32 64 60 38 54 7 46 72 43 57)

$\boldsymbol{n = 74}$**:** (1 45 33 70 42 37 25 58 16 5 66 28 47 41 22 72 20 35 31 71 46 11 23 73 61 55 51 15 3 27 19 34 67 64 10 65 60 14 53 52 43 36 24 30 40 69 56 26 9 68 39 18 48 12 38 17 6 13 4 57 74 2 8 59 29 7 62 63 44 21 32 54 49 50)

$\boldsymbol{n = 75}$**:** (13 73 66 34 14 43 69 38 18 33 8 25 63 75 30 29 71 2 42 40 39 62 64 19 37 46 12 4 54 28 23 11 21 58 74 57 26 72 1 52 22 48 67 10 49 45 59 61 41 3 65 5 44 16 47 35 36 20 15 7 70 51 55 60 32 17 9 24 31 6 56 53 50 27 68)

$\boldsymbol{n = 76}$**:** (23 74 10 66 58 67 55 69 14 32 31 49 2 46 27 36 53 47 73 29 33 4 65 48 34 18 42 26 7 13 22 56 28 5 50 15 21 52 16 25 57 20 61 71 76 30 17 24 19 72 68 43 6 54 37 70 11 75 44 8 39 51 64 63 59 1 60 41 9 35 62 12 38 40 3 45)

$\boldsymbol{n = 77}$**:** (13 3 47 57 1 42 67 64 62 45 31 12 76 5 2 63 7 46 60 59 65 6 23 32 50 11 4 53 70 14 15 33 27 68 9 71 55 73 34 56 38 22 44 61 69 74 48 54 26 16 43 20 41 77 49 40 18 58 30 66 17 29 36 37 39 35 24 8 51 28 19 10 25 72 21 52 75)

$\boldsymbol{n = 78}$**:** (24 61 8 11 47 37 2 29 73 70 27 26 42 6 76 38 17 7 13 68 12 32 60 33 28 57 50 54 41 64 67 9 21 51 75 43 55 69 18 16 53 58 66 65 22 19 1 20 49 45 77 10 52 31 5 40 15 56 71 48 46 4 72 39 35 44 63 36 3 14 62 30 74 25 34 59 78 23)

$n = 79$**:** (36 17 43 7 78 72 47 31 76 13 66 48 32 56 14 63 52 23 25 79 24 77 40 30 55 71 27 44 8 4 15 60 73 65 2 69 6 18 59 16 9 75 61 70 22 54 19 49 29 67 3 45 10 20 46 34 51 37 5 1 68 50 21 12 28 26 53 62 39 41 35 11 38 33 74 57 58 42 64)

$n = 80$**:** (12 1 54 36 50 65 22 21 71 32 23 67 57 56 35 73 79 64 28 31 15 58 77 3 20 43 53 61 42 19 10 74 48 55 38 49 18 45 72 5 51 13 7 52 39 46 17 33 78 59 25 9 14 6 76 47 69 62 30 34 44 80 4 11 66 2 16 41 24 27 26 60 37 68 70 63 8 29 40 75)

$n = 81$**:** (12 45 43 60 69 50 37 27 23 11 13 75 20 2 33 52 73 42 18 24 17 63 6 1 14 72 62 28 54 47 38 30 78 65 58 68 44 36 51 16 19 59 55 56 64 10 39 3 34 70 4 77 40 61 49 15 5 46 67 7 29 53 31 21 57 22 8 25 26 71 81 80 35 76 74 66 32 41 9 48 79)

$n = 82$**:** (55 61 80 5 21 9 67 32 26 23 41 52 20 27 16 15 11 28 60 10 2 79 8 13 77 31 34 56 22 40 24 43 51 66 70 48 30 17 50 75 76 42 18 69 6 29 14 82 4 45 59 57 81 58 3 62 35 53 47 1 36 74 64 71 12 25 33 68 73 38 54 7 78 49 63 44 39 65 19 46 72 37)

$n = 83$**:** (33 45 58 24 10 79 66 21 26 52 57 80 32 72 29 62 40 74 18 59 13 83 23 12 6 67 60 20 34 1 8 46 22 43 68 31 5 2 54 37 73 82 30 70 9 28 15 25 75 7 64 48 49 53 61 71 50 42 14 56 69 44 55 36 11 19 63 17 3 38 65 47 77 76 78 35 51 16 27 4 39 41 81)

$n = 84$**:** (35 44 78 62 47 7 53 2 36 6 84 5 38 74 79 41 13 54 3 9 28 40 69 82 55 83 4 17 48 51 39 22 1 76 10 81 8 72 70 16 42 25 37 65 49 45 27 31 15 21 52 43 23 64 34 24 32 60 12 33 50 26 66 14 61 68 11 58 73 29 59 19 75 80 67 71 30 46 18 63 56 57 20 77)

$n = 85$**:** (4 22 47 19 54 43 79 78 5 28 11 66 58 56 76 10 53 8 50 37 31 20 12 35 13 23 45 3 33 83 44 16 48 17 24 46 71 85 7 57 9 65 68 18 30 39 51 42 69 29 38 82 26 52 74 73 81 70 34 40 77 2 75 49 61 21 63 32 41 6 55 60 27 72 67 59 36 64 62 25 84 15 14 1 80)

$n = 86$**:** (47 30 37 2 39 52 73 81 80 55 76 71 83 43 69 58 48 10 51 11 15 19 7 86 8 67 54 32 27 5 75 82 9 85 84 65 6 14 68 28 22 57 33 25 34 18 38 40 62 50 60 70 29 1 16 3 64 4 31 77 74 13 42 44 23 63 72 79 61 46 20 66 45 21 24 49 56 53 59 35 17 26 36 12 41 78)

$\boldsymbol{n} = \mathbf{87}$**:** (55 30 45 61 67 87 82 39 62 48 77 56 73 76 5 64 69 79 53 40 44 7 54 4 6 1 46 15 11 20 14 70 21 38 12 10 86 85 35 33 78 25 8 50 68 34 26 18 43 27 19 22 63 75 32 71 17 24 47 42 80 13 74 37 83 65 9 60 84 36 28 16 49 3 57 52 41 31 2 58 66 29 23 59 81 51 72)

$\boldsymbol{n} = \mathbf{88}$**:** (7 61 21 37 75 59 39 87 80 26 3 73 16 48 28 5 2 29 6 66 38 17 65 33 78 9 27 84 13 70 46 42 35 57 8 32 22 53 64 1 10 58 31 52 43 77 23 69 14 45 56 49 41 55 47 20 76 62 74 4 40 67 12 88 63 72 86 82 15 83 85 50 44 79 25 54 34 36 24 11 60 51 68 30 18 19 71 81)

$\boldsymbol{n} = \mathbf{89}$**:** (24 54 82 50 11 44 10 39 63 1 64 68 40 4 27 42 73 84 57 35 43 56 55 71 52 21 38 22 28 72 48 87 30 49 31 34 45 74 80 2 33 66 77 29 41 51 12 85 5 89 9 23 37 17 7 8 47 76 46 16 69 6 75 81 79 3 78 59 88 19 25 18 60 70 53 36 86 58 83 20 26 65 32 67 61 62 13 14 15)

$\boldsymbol{n} = \mathbf{90}$**:** (23 41 7 59 67 82 35 53 32 26 71 60 6 15 57 11 85 28 2 16 42 88 49 69 48 5 12 21 19 17 20 63 30 75 65 50 36 37 90 72 10 47 46 3 38 87 44 22 25 77 34 56 76 83 81 33 27 4 64 80 52 1 66 62 18 31 13 84 54 43 29 40 39 55 73 51 79 58 8 74 68 70 78 45 89 61 9 24 14 86)

$\boldsymbol{n} = \mathbf{91}$**:** (21 5 82 84 28 77 80 61 45 22 50 34 7 65 72 17 29 71 90 89 47 64 62 48 66 42 19 16 1 13 51 67 31 20 23 52 79 60 44 55 12 74 6 70 91 78 36 2 15 27 30 33 32 73 49 54 39 24 63 58 14 56 40 81 59 76 11 35 9 26 43 18 38 10 87 25 68 57 8 88 41 69 85 4 37 3 46 53 75 83 86)

$\boldsymbol{n} = \mathbf{92}$**:** (9 3 80 30 32 63 56 37 73 46 42 53 65 5 82 6 70 87 39 69 59 41 4 27 28 92 35 45 29 13 58 34 18 77 51 90 68 38 75 55 91 84 33 57 67 79 89 12 88 15 66 74 71 60 2 22 23 36 14 40 81 7 20 1 31 85 25 21 24 76 78 72 62 19 26 86 8 11 61 47 44 10 48 50 64 54 52 83 16 17 43 49)

$\boldsymbol{n} = \mathbf{93}$**:** (17 62 71 37 90 19 10 84 82 4 61 29 78 70 59 1 73 26 75 81 72 38 13 85 20 23 43 45 2 89 15 77 69 42 92 12 53 6 3 68 33 50 48 21 30 36 27 80 25 83 49 93 40 22 31 52

47 91 66 60 56 24 18 7 55 28 88 67 39 74 58 32 51 65 86 8 16 41 87 9 11 64 5 54 34 46 14 35 44 79 63 57 76)

$\boldsymbol{n}$ **= 94:** (1 5 37 34 4 35 30 69 66 20 2 12 45 84 43 74 36 49 88 63 89 3 44 41 78 33 93 14 51 81 64 60 40 8 28 71 29 50 82 6 11 53 38 32 65 75 31 25 46 13 10 73 26 15 9 61 55 56 39 85 23 58 19 80 94 18 86 16 91 72 22 76 62 24 92 83 42 21 27 48 52 87 57 17 70 59 67 77 68 7 47 54 79 90)

$\boldsymbol{n}$ **= 95:** (11 8 18 15 45 81 69 93 76 6 52 58 23 62 9 68 12 55 41 14 21 44 5 32 75 25 38 53 91 7 19 30 95 33 26 64 65 63 78 82 42 84 40 61 88 73 22 80 87 57 39 24 46 72 92 83 27 90 51 20 29 50 56 70 59 94 89 79 17 10 86 85 60 16 54 66 74 28 47 77 71 34 2 43 37 1 31 36 35 4 49 3 13 48 67)

$\boldsymbol{n}$ **= 96:** (7 67 91 26 44 11 8 42 52 83 51 3 81 64 25 56 72 14 30 33 35 43 40 32 86 5 50 85 23 73 61 76 75 63 58 92 55 68 41 89 65 71 9 34 2 38 13 79 87 19 96 46 59 10 74 57 78 28 66 49 53 45 54 12 16 15 21 80 6 84 93 95 27 37 31 4 60 69 77 47 20 62 90 94 36 17 48 24 1 39 18 82 29 70 22 88)

$\boldsymbol{n}$ **= 97:** (18 23 48 11 67 32 5 44 3 28 69 94 12 21 77 96 29 85 1 81 89 47 34 60 7 58 14 20 4 50 36 70 65 66 56 41 10 30 82 71 15 31 17 43 37 26 49 86 95 38 9 75 92 19 79 62 72 51 42 55 87 63 91 40 8 27 73 35 84 54 24 45 61 93 68 22 13 76 64 57 2 33 46 80 16 53 97 74 39 52 78 6 90 83 25 59 88)

$\boldsymbol{n}$ **= 98:** (14 12 53 50 83 87 93 18 30 46 81 9 82 65 4 42 24 7 5 44 36 47 56 72 94 68 15 31 16 89 21 29 11 71 23 26 97 91 35 57 98 74 69 77 78 63 88 2 32 86 1 59 25 51 52 79 58 33 92 49 90 28 85 61 3 8 34 84 66 17 67 95 39 38 73 10 22 60 55 64 96 54 45 43 41 27 62 70 19 13 48 76 37 20 6 40 75 80)

$\boldsymbol{n}$ **= 99:** (95 57 18 21 56 16 76 5 29 34 50 90 54 89 3 32 22 77 10 66 17 84 12 53 27 62 68 55 40 35 71 6 67 42 61 63 37 74 91 36 78 15 80 43 85 19 28 94 51 30 79 70 48 87 64

88 39 38 97 2 65 93 92 81 59 73 41 52 98 75 46 60 20 25 23 82 99 1 72 26 33 9 4 47 44 11 86 24 13 58 7 69 8 31 14 96 49 45 83)

$\boldsymbol{n = 100}$**:** (30 6 7 85 3 75 68 80 78 20 92 52 55 73 91 16 24 43 71 48 32 38 11 19 63 96 72 56 15 49 97 46 28 66 34 14 83 87 25 1 77 13 18 40 54 57 76 29 33 31 99 93 41 50 53 10 4 2 94 22 26 39 70 45 35 17 62 21 51 60 84 47 81 88 95 79 12 36 86 61 44 67 37 98 100 58 74 23 90 9 82 65 42 89 8 64 27 59 5 69)

# Bibliography


[1] D.S. Archdeacon, Heffter arrays and biembedding graphs on surfaces, *Electron. J. Combin.* **22** (2015), #P1.74.

[2] D. Archdeacon, T. Boothby, and J.H. Dinitz, Tight Heffter arrays exist for all possible values, In preparation.

[3] D. Archdeacon, J.H. Dinitz, A. Mattern, and D.R. Stinson, On partial sums in syclic groups, Preprint, 2015, (arXiv:1501.06872 [math.CO] 27 Jan 2015 ).

[4] D. Archdeacon, J. Dinitz, D. Donovan, and S. Yazici, Square integer Heffter arrays with empty cells, *Des. Codes Crypt.* (to appear), (arXiv:1412.8409 [math.CO] 29 Dec 2014).

[5] G.K. Bennet, M.J. Grannell, and T.S. Griggs, Non-orientable biembeddings of Steiner triple systems of order 15, *Acta Math. Univ. Comenian. (N.S.)* **73**, (2004), 101–106.

[6] J.-P. Bode and H. Harborth, Directed paths of diagonals within polytopes, *Discrete Math.* **299** (2005), 3–10.

[7] M. Brown, Surface embeddings of families of combinatorial designs, Ph.D. Thesis, The University of Vermont, 2010.

[8] A.D. Forbes, T.S. Griggs, and C. Psomas, J. Širáň, Biembeddings of Steiner triple systems in orientable pseudosurfaces with one pinch point, *Glasg. Math J.* **56**, (2014), 252–260.

[9] M.J. Grannell, T.S. Griggs, and M. Knor, Orientable biembeddings of Steiner triple systems of order 15, *J. Combin. Math. Combin. Comput.* **68**, (2009), 129–136.

[10] M. J. Grannell and V.P. Korzhik, Non-orientable biembeddings of Steiner triple systems, *Discrete Math* **285**, (2004), 121–126.

[11] M.J. Grannell and V.P. Koorchik, Orientable biembeddings of cyclic Steiner triple systems from current assignments on Möbius ladder graphs, *Discrete Math* **309**, (2009), 2847–2860.

[12] J. L. Gross and T. W. Tucker, *Topological Graph Theory*, General Publishing Company, Ltd., Toronto, Ontario, 2001.

[13] Thomas A. McCourt, Biembedding a Steiner triple system with a Hamilton cycle decomposition of a complete graph, *J. Graph Theory* **77**, (2014), 68–87.

[14] M. A. Ollis, Sequenceable groups and related topics, *Electron. J. Combin.* **20** (2013), #DS10v2.

[15] M.J. Grannell and T.S. Griggs, Designs and topology. In Surveys in Combinatorics 1997, A. Hilton and J. Talbot, eds. London Mathematical Society Lecture Note Series 346. Cambridge, UK: Cambridge University Press, (2007), 121-174.