4553

GEOMETRY AND SINGULARITIES

OF SPATIAL AND SPHERICAL CURVES

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ABSTRACT

In the first part of this dissertation the spherical evolute, the spherical involute, the spherical orthotomic and the spherical antiorthotomic are investigated and their local diffeomorphic types are determined. The concept of the spherical conic is introduced. It is proven that the incident angle and reflection angle are equal for the spherical conic. The necessary and sufficient conditions for the spherical conic to be a circle are given.

In the second part of this dissertation the ruled surfaces of normals and binormals of a regular space curve are locally classified under the left-right action according to the types of the curve. For this purpose some results are obtained on the relationship of the powers of terms in the Taylor series of an invertible function and its inverse.

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Part I

Spherical Curves and Singularities

CHAPTER 1

Characterization and Properties of a Spherical Curve

1.1 Geodesic Distance on the Unit Sphere

Let $S^2 \subset \mathbb{R}^3$ be the unit sphere centered at the origin and $\gamma(s)$ be a smooth unit speed curve lying on S^2, where s is the arclength of γ. We may measure the contact between γ and a (geodesic) circle on S^2. To this end we consider the geodesic distance between two points on S^2. Given two points P and Q on S^2, let θ be the angle subtended at the center of the unit sphere by PQ and $d(P,Q)$ denote the *geodesic distance* between P and Q. Then by the law of cosines we have

$$|P-Q|^2 = 1^2 + 1^2 - 2 \times 1 \times 1 \times \cos\theta,$$

where $|P-Q|$ is the length of the chord PQ. So

$$d(P,Q) = \arccos\left(1 - \frac{1}{2}|P-Q|^2\right). \tag{1.1}$$

1.2 Characterization of a Spherical Curve

Let $\gamma : I \to \mathbb{R}^3$ be a unit speed curve and $\mathbf{t}, \mathbf{n}, \mathbf{b}$ be the Frenet-Serret trihedron. It's well known that supposing the torsion $\tau \neq 0$, γ is a spherical curve if and only if $R\tau + (TR')' = 0$ (see [16, Page 32]), where $R = \dfrac{1}{\kappa}$, κ is the curvature of γ and $T = \dfrac{1}{\tau}$ and $'$ denotes the derivative relative to the arclength s in this paper. Here we want to say a few words about the case $\tau(s) = 0$ for some $s \in I$. Suppose γ is a unit speed curve on the sphere centered at the origin of radius a and γ is not planar, i.e., $\tau(s)$ is not identically equal to 0. Then

$\gamma(s) \cdot \gamma(s) = a^2$. Differentiation gives

$$\mathbf{t} \cdot \gamma = 0 \;\Rightarrow\; \kappa(\mathbf{n} \cdot \gamma) + \mathbf{t} \cdot \mathbf{t} = 0 \;\Rightarrow\; \kappa(\mathbf{n} \cdot \gamma) = -1, \tag{1.2}$$

which implies $|\kappa| \geq \dfrac{1}{|\mathbf{n}| \cdot |\gamma|} = \dfrac{1}{a} > 0$ for all $s \in I$. Throughout this paper we take an orientation such that $\kappa > 0$. Further differentiation yields

$$\begin{aligned} \kappa' \mathbf{n} \cdot \gamma + \kappa(-\kappa \mathbf{t} + \tau \mathbf{b}) \cdot \gamma = 0 \quad &\Rightarrow \kappa' \mathbf{n} \cdot \gamma + \kappa \tau \mathbf{b} \cdot \gamma = 0 \\ &\Rightarrow \tau \mathbf{b} \cdot \gamma = \frac{\kappa'}{\kappa^2}. \end{aligned}$$

The set $\{s \mid \tau(s) \neq 0\}$ is open in I. For each non-internal point $s^* \in \{s \mid \tau(s) = 0\}$ (i.e., the point s^* such that there is no interval $I_1 \subset \{s \mid \tau(s) = 0\}$ with $s^* \in I_1$.) we have a sequence $\{s_i\}_{i=1}^{\infty}$ such that $\tau(s_i) \neq 0$ and $\lim s_i = s^*$.

$$\text{If } \tau(s) = 0 \text{ then } \tau \mathbf{b} \cdot \gamma = \frac{\kappa'}{\kappa^2} \Rightarrow \kappa'(s) = 0. \tag{1.3}$$

Suppose s' is a point in $\{s \mid \tau(s) = 0\}$ such that we have an interval $I' \subset \{s \mid \tau(s) = 0\}$ with $s' \in I'$ and I' is maximal with such property. On the interval I', $\kappa'(s) = \tau(s) = 0$. So the restriction $\gamma|I'$ of γ to I' is a segment of a circle. Let $\widetilde{s}$ be one of the boundary points of I' or the isolated point in $\{s \mid \tau(s) = 0\}$, then we have a sequence $\{s_i\}_{i=1}^{\infty}$ such that $\tau(s_i) \neq 0$ and $\lim s_i = \widetilde{s}$. So for $\tau(s) \neq 0$,

$$\mathbf{b} \cdot \gamma = \frac{\kappa'}{\tau \kappa^2} = -TR' \;\Rightarrow\; \lim_{i \to \infty} \frac{\kappa'}{\tau \kappa^2}(s_i) = \lim_{i \to \infty} (\mathbf{b} \cdot \gamma)(s_i) = \mathbf{b}(\widetilde{s}) \cdot \gamma(\widetilde{s}).$$

Here we compute $\mathbf{b} \cdot \gamma$ in terms of κ, τ. Suppose $\tau(\widetilde{s}) = \tau'(\widetilde{s}) = \cdots = \tau^{(n)}(\widetilde{s}) = 0$, but $\tau^{(n+1)}(\widetilde{s}) \neq 0$. By (1.3) $\tau(\widetilde{s}) = 0 \Rightarrow \kappa'(\widetilde{s}) = 0$. By L'Hospital's Rule,

$$-\kappa^2(\mathbf{b} \cdot \gamma)(\widetilde{s}) = \lim_{s \to \widetilde{s}} \frac{\kappa'}{\tau} = \lim_{s \to \widetilde{s}} \frac{\kappa''}{\tau'}. \tag{1.4}$$

So

$$\tau'(\widetilde{s}) = 0 \;\Rightarrow\; \kappa''(\widetilde{s}) = 0. \tag{1.5}$$

Similarly we have

$$-\kappa^2(\mathbf{b}\cdot\boldsymbol{\gamma})(\widetilde{s}) = \lim_{s\to\widetilde{s}} \frac{\kappa'(s)}{\tau(s)} = \cdots = \lim_{s\to\widetilde{s}} \frac{\kappa^{(n+2)}(s)}{\tau^{(n+1)}(s)} = \frac{\kappa^{(n+2)}(\widetilde{s})}{\tau^{(n+1)}(\widetilde{s})}.$$

So

$$(\mathbf{b}\cdot\boldsymbol{\gamma})(\widetilde{s}) = -\frac{\kappa^{(n+2)}(\widetilde{s})}{\kappa^2(\widetilde{s})\tau^{(n+1)}(\widetilde{s})}. \tag{1.6}$$

Since $\boldsymbol{\gamma}$ is a spherical curve the osculating sphere of $\boldsymbol{\gamma}$ coincides with the sphere wherein $\boldsymbol{\gamma}$ lies (see [16, page 32]) and for $\tau(s) \neq 0$ we have

$$R^2 + (\mathbf{b}\cdot\boldsymbol{\gamma})^2 = R^2 + (TR')^2 = a^2, \tag{1.7}$$

noting $R = \dfrac{1}{\kappa}$ and $T = \dfrac{1}{\tau}$ (see [16, Page 32]). $|\mathbf{b}\cdot\boldsymbol{\gamma}| = |TR'|$ *is the distance from the center of the sphere (the origin) to the osculating plane of* $\boldsymbol{\gamma}$. When $\tau(\widetilde{s}) = 0$ we just treat TR' as a single thing and define $TR'(\widetilde{s}) = (\mathbf{b}\cdot\boldsymbol{\gamma})(\widetilde{s})$, the distance from the center of the sphere (the origin) to the osculating plane of $\boldsymbol{\gamma}$ at $\widetilde{s}$.

The curve $\boldsymbol{\gamma}$ is *great* at s if its circle of curvature is a great circle on the sphere, (see [13, page 100]) i.e., the osculating plane of $\boldsymbol{\gamma}$ at s passes through the center of the sphere. $\mathbf{b}\cdot\boldsymbol{\gamma} = 0 \Leftrightarrow \boldsymbol{\gamma} = -R\mathbf{n} \Leftrightarrow \boldsymbol{\gamma} = -a\mathbf{n}$ since $\boldsymbol{\gamma}\cdot\boldsymbol{\gamma} = a^2$ and $R = \dfrac{1}{\kappa} > 0$ (also see (2.4) in the following). So we have

Proposition 1 *The spherical curve* $\boldsymbol{\gamma}$ *is great at* $s \Leftrightarrow \mathbf{b}\cdot\boldsymbol{\gamma} = -TR' = 0$ *at* $s \Leftrightarrow \boldsymbol{\gamma} = -a\mathbf{n}$ *at* s. ■

Over the interval I', $\kappa'(s) = 0 \Rightarrow R(s) = R(\widetilde{s})$ for $s \in I'$. So for $s \in I'$, $(\mathbf{b}(s)\cdot\boldsymbol{\gamma}(s))^2 = [(TR')(s)]^2 = a^2 - R(\widetilde{s})^2$. i.e.,

$$\mathbf{b}\cdot\boldsymbol{\gamma} = TR' = \text{constant over the interval } I'. \tag{1.8}$$

$$\begin{aligned} &\text{For } \tau(s) \neq 0, \mathbf{b}\cdot\gamma = -TR' \\ \Rightarrow\quad & -\tau\mathbf{n}\cdot\gamma + \mathbf{b}\cdot\mathbf{t} = (-TR')' \\ \Rightarrow\quad & \frac{\tau}{\kappa} + (TR')' = 0 \end{aligned}$$

$\Rightarrow$

$$R\tau + (TR')' = 0. \tag{1.9}$$

For each non-internal point $s^* \in \{s \mid \tau(s) = 0\}$ (i.e., the point s^* such that there is no interval $I_1 \subset \{s \mid \tau(s) = 0\}$ with $s^* \in I_1$.) we have a sequence $\{s_i\}_{i=1}^{\infty}$ such that $\tau(s_i) \neq 0$ and $\lim s_i = s^*$. So by (1.9)

$$\lim_{i\to\infty} (TR')'(s_i) = \lim_{i\to\infty} (-R\tau(s_i)) = -R\tau(s^*) = 0. \tag{1.10}$$

On the other hand, from $TR' = \textit{constant}$ over the interval I' we know that $(TR')' = 0$ over the interval I' and we can define $(TR')'(s^*) = 0$, so $(TR')'$ is continuous over I and (1.9) holds for $\tau(s) = 0$ too.

Contrarily, suppose a non-planar curve γ satisfies (1.9). Then for $\tau(s) \neq 0$ by $(8-11)$ on [16, Page 32] one of the corresponding parts of γ lies on a sphere of radius, say a and $R^2 + (TR')^2 = a^2$. For $\tau(s) = 0$, $(TR')' = 0 \Rightarrow TR'$ is constant on $\{s \mid \tau(s) = 0\}$ by (1.10). Let $I' \subset \{s \mid \tau(s) = 0\}$ be as above and $\widetilde{s}$ be one of the boundary points of I'. Then using the same argument as above (there exists a sequence $\{s_i\}_{i=1}^{\infty}$ such that $\tau(s_i) \neq 0$ and $\lim s_i = \widetilde{s}$) we have $R(s)^2 = R(\widetilde{s})^2 = a^2 - (TR')^2(\widetilde{s})$ for $s \in I'$. This implies that the part of γ corresponding to I' is a part of a circle of radius $\sqrt{a^2 - (TR')^2(\widetilde{s})}$ and it is on the sphere of radius a centered at the origin. The different non-circular parts of γ are connected (separated) by the circular parts and each adjacent part of γ share the same sphere. So the entirety of γ is on the sphere of radius a centered at the origin.

1.3 Geodesic Curvature

For a regular curve γ on the unit sphere, its *geodesic curvature* $\kappa_g(s)$ is defined to be the tangential component of the curvature of γ at s (see [13, page 88-89]). Let θ be the half angle subtended at the center of the unit sphere by any diameter of the circle cut out by the osculating plane of γ (the circle of curvature). Then $\kappa_g = \kappa \cos\theta = -\kappa T R' = \dfrac{\kappa'}{\tau\kappa}$ (c.f. [13, page 89]). So $\kappa_g = 0 \Leftrightarrow TR' = 0$. Especially,

if $\tau \neq 0$, then $\kappa_g = 0 \Leftrightarrow \kappa' = 0$;

if $\tau(\tilde{s}) = \tau'(\tilde{s}) = \cdots = \tau^{(n)}(\tilde{s}) = 0$ but $\tau^{(n+1)}(\tilde{s}) \neq 0$, then by (1.6) we have $\kappa_g(\tilde{s}) = \frac{\kappa^{(n+2)}(\tilde{s})}{\kappa(\tilde{s})\tau^{(n+1)}(\tilde{s})}$.

So in this case $\kappa_g = 0 \Leftrightarrow \kappa^{(n+2)} = 0$. And by Proposition 1,

$$\gamma \text{ is great} \Leftrightarrow TR' = 0$$

$$\Leftrightarrow \kappa' = 0 \text{ if } \tau \neq 0 \text{ or } \kappa^{(n+2)} = 0 \text{ if } \tau(\tilde{s}) = \tau'(\tilde{s}) = \cdots = \tau^{(n)}(\tilde{s}) = 0 \text{ but } \tau^{(n+1)}(\tilde{s}) \neq 0.$$

CHAPTER 2

Contact with the Circle on S^2 and the Spherical Evolute

2.1 Contact with the Circle on S^2

From now on we always assume $\boldsymbol{\gamma}$ is a unit speed curve on the unit sphere.

For a unit speed curve $\boldsymbol{\gamma} : I \to S^2$ and a point $\mathbf{u} \in S^2$, we define a *geodesic distance* function between $\boldsymbol{\gamma}(s)$ and $\mathbf{u}$ by $d_g : I \times S^2 \to \mathbb{R}$,

$$
\begin{aligned}
d_g(s, \mathbf{u}) &= d(\boldsymbol{\gamma}(s), \mathbf{u}) \\
&= \arccos\left[1 - \frac{1}{2}(\boldsymbol{\gamma}(s) - \mathbf{u}) \cdot (\boldsymbol{\gamma}(s) - \mathbf{u})\right] \\
&= \arccos(\boldsymbol{\gamma}(s) \cdot \mathbf{u}),
\end{aligned}
$$

by $\boldsymbol{\gamma}(s) \cdot \boldsymbol{\gamma}(s) = \mathbf{u} \cdot \mathbf{u} = 1$.

Consider all the points $\mathbf{x}$ on S^2 having the same geodesic distance C_1 from $\mathbf{u}$, i.e., $d(\mathbf{x}, \mathbf{u}) = C_1$. Let $C = \cos C_1$, then $\mathbf{x} \cdot \mathbf{u} = C$. We call the set of all such $\mathbf{x}$ *the geodesic circle* with *the geodesic center* $\mathbf{u}$ and *cosine of the geodesic radius* C.

We state some definitions.

Definition 2 *The spherical normal to $\boldsymbol{\gamma}$ at $\boldsymbol{\gamma}(s)$ is the great circle passing through $\boldsymbol{\gamma}(s)$ and normal to $\boldsymbol{\gamma}$ at $\boldsymbol{\gamma}(s)$ and is given by:*

$$
\begin{cases}
\mathbf{x} \cdot \mathbf{x} = 1, \\
\mathbf{x} \cdot \mathbf{t}(s) = 0.
\end{cases}
\tag{2.1}
$$

Definition 3 *The spherical tangent to γ at $\gamma(s)$ is the great circle which is tangent to γ at $\gamma(s)$ and is given by*

$$\begin{cases} \mathbf{x}\cdot\mathbf{x}=1, \\ \mathbf{x}\cdot(\gamma(s)\times\mathbf{t}(s))=0. \end{cases} \tag{2.2}$$

In the following proposition, for simplicity we omit the dependence of $\gamma, \mathbf{t}, \mathbf{n}$ and $\mathbf{b}$ on s.

Proposition 4 *Let γ be a unit speed curve on S^2 and $\mathbf{u}\in S^2$, then*

(i) $d_g'(s,\mathbf{u})=0 \Leftrightarrow \mathbf{u}=\lambda\mathbf{n}+\mu\mathbf{b}$ for some $\lambda,\mu\in\mathbb{R}$, i.e., $\mathbf{u}$ lies on the spherical normal to γ at $\gamma(s)$.

(ii) $d_g'(s,\mathbf{u})=d_g''(s,\mathbf{u})=0 \Leftrightarrow \mathbf{u}=\pm\mathbf{b}$.

(iii) $d_g'(s,\mathbf{u})=d_g''(s,\mathbf{u})=d_g'''(s,\mathbf{u})=0 \Leftrightarrow \mathbf{u}=\pm\mathbf{b}$ and $\tau(s)=0$.

(iv) $d_g'(s,\mathbf{u})=d_g''(s,\mathbf{u})=d_g'''(s,\mathbf{u})=d_g^{(IV)}(s,\mathbf{u})=0 \Leftrightarrow \mathbf{u}=\pm\mathbf{b}, \tau(s)=0$ and $\tau'(s)=0$.

Proof.

(i)

$$d_g'(s,\mathbf{u})=\frac{-\mathbf{u}\cdot\gamma'}{[1-(\gamma\cdot\mathbf{u})^2]^{\frac{1}{2}}}=0 \Leftrightarrow \mathbf{u}\cdot\gamma'=0 \Leftrightarrow \mathbf{t}\perp\mathbf{u} \Leftrightarrow \mathbf{u}=\lambda\mathbf{n}+\mu\mathbf{b}$$

for some $\lambda,\mu\in\mathbb{R} \Leftrightarrow \mathbf{u}$ lies on the great circle cut out by the normal plane of $\gamma(s)$, i.e., the spherical normal to γ at $\gamma(s)$ since $\mathbf{u}\in S^2$.

(ii)

$$d_g''(s,\mathbf{u})=\frac{(-\mathbf{u}\cdot\gamma')'\,[1-(\gamma\cdot\mathbf{u})^2]^{\frac{1}{2}}-(-\mathbf{u}\cdot\gamma')\left([1-(\gamma\cdot\mathbf{u})^2]^{\frac{1}{2}}\right)'}{1-(\gamma\cdot\mathbf{u})^2}.$$

So

$$\begin{aligned} & d_g'(s,\mathbf{u}) = d_g''(s,\mathbf{u}) = 0 \\ \Leftrightarrow \quad & \mathbf{u}\cdot\boldsymbol{\gamma}' = (-\mathbf{u}\cdot\boldsymbol{\gamma}')' = 0 \\ \Leftrightarrow \quad & \mathbf{u} = \lambda\mathbf{n} + \mu\mathbf{b} \text{ and } (\lambda\mathbf{n}+\mu\mathbf{b})\cdot\kappa\mathbf{n} = 0 \text{ for some } \lambda,\mu\in\mathbb{R}. \\ \Leftrightarrow \quad & \mathbf{u} = \mu\mathbf{b} \ \text{ for some } \mu\in\mathbb{R} \ \text{ since } \kappa\neq 0. \end{aligned}$$

And $\mathbf{u}\in S^2 \Rightarrow (\mu\mathbf{b})^2 = 1 \Rightarrow \mu = \pm 1$. So

$$d_g'(s,\mathbf{u}) = d_g''(s,\mathbf{u}) = 0 \quad \Leftrightarrow \quad \mathbf{u} = \pm\mathbf{b}.$$

(iii)

$$\begin{aligned} & d_g'''(s,\mathbf{u}) \\ = \ & \frac{(-\mathbf{u}\cdot\boldsymbol{\gamma}')''\left[1-(\boldsymbol{\gamma}\cdot\mathbf{u})^2\right]^{\frac{1}{2}} - (-\mathbf{u}\cdot\boldsymbol{\gamma}')\left(\left[1-(\boldsymbol{\gamma}\cdot\mathbf{u})^2\right]^{\frac{1}{2}}\right)''}{1-(\boldsymbol{\gamma}\cdot\mathbf{u})^2} \\ & + \left\{(-\mathbf{u}\cdot\boldsymbol{\gamma}')'\left[1-(\boldsymbol{\gamma}\cdot\mathbf{u})^2\right]^{\frac{1}{2}} - (-\mathbf{u}\cdot\boldsymbol{\gamma}')\left(\left[1-(\boldsymbol{\gamma}\cdot\mathbf{u})^2\right]^{\frac{1}{2}}\right)'\right\}\left(\frac{1}{1-(\boldsymbol{\gamma}\cdot\mathbf{u})^2}\right)' \end{aligned}$$

So

$$\begin{aligned} & d_g'(s,\mathbf{u}) = d_g''(s,\mathbf{u}) = d_g'''(s,\mathbf{u}) = 0 \\ \Leftrightarrow \quad & -\mathbf{u}\cdot\boldsymbol{\gamma}' = -\mathbf{u}\cdot\boldsymbol{\gamma}'' = -\mathbf{u}\cdot\boldsymbol{\gamma}''' = 0 \\ \Leftrightarrow \quad & \mathbf{u} = \pm\mathbf{b} \text{ and } -\mathbf{u}\cdot(\kappa\mathbf{n})' = 0 \\ \Leftrightarrow \quad & \mathbf{u} = \pm\mathbf{b} \text{ and } -\mathbf{u}\cdot[\kappa'\mathbf{n} + \kappa(-\kappa\mathbf{t} + \tau\mathbf{b})] = 0 \\ \Leftrightarrow \quad & \mathbf{u} = \pm\mathbf{b} \text{ and } \mp\kappa\tau = 0 \\ \Leftrightarrow \quad & \mathbf{u} = \pm\mathbf{b} \text{ and} \tau = 0 \text{ since} \kappa\neq 0. \end{aligned}$$

(iv)

$$\begin{aligned} & d_g^{(IV)}(s,\mathbf{u}) \\ = \ & \frac{(-\mathbf{u}\cdot\boldsymbol{\gamma}')'''\left[1-(\boldsymbol{\gamma}\cdot\mathbf{u})^2\right]^{\frac{1}{2}} + (-\mathbf{u}\cdot\boldsymbol{\gamma}')''\left(\left[1-(\boldsymbol{\gamma}\cdot\mathbf{u})^2\right]^{\frac{1}{2}}\right)'}{1-(\boldsymbol{\gamma}\cdot\mathbf{u})^2} \\ & -\frac{(-\mathbf{u}\cdot\boldsymbol{\gamma}')'\left(\left[1-(\boldsymbol{\gamma}\cdot\mathbf{u})^2\right]^{\frac{1}{2}}\right)'' + (-\mathbf{u}\cdot\boldsymbol{\gamma}')\left(\left[1-(\boldsymbol{\gamma}\cdot\mathbf{u})^2\right]^{\frac{1}{2}}\right)'''}{1-(\boldsymbol{\gamma}\cdot\mathbf{u})^2} \end{aligned}$$

$$+\left\{(-\mathbf{u}\cdot\gamma')''\left[1-(\gamma\cdot\mathbf{u})^2\right]^{\frac{1}{2}}-(-\mathbf{u}\cdot\gamma')\left(\left[1-(\gamma\cdot\mathbf{u})^2\right]^{\frac{1}{2}}\right)''\right\}\left(\frac{1}{1-(\gamma\cdot\mathbf{u})^2}\right)'$$
$$+\left\{(-\mathbf{u}\cdot\gamma')'\left[1-(\gamma\cdot\mathbf{u})^2\right]^{\frac{1}{2}}-(-\mathbf{u}\cdot\gamma')\left(\left[1-(\gamma\cdot\mathbf{u})^2\right]^{\frac{1}{2}}\right)'\right\}\left(\frac{1}{1-(\gamma\cdot\mathbf{u})^2}\right)''$$

So

$$d_g'(s,\mathbf{u})=d_g''(s,\mathbf{u})=d_g'''(s,\mathbf{u})=d_g^{(IV)}(s,\mathbf{u})=0$$
$$\Leftrightarrow\quad -\mathbf{u}\cdot\gamma'=-\mathbf{u}\cdot\gamma''=-\mathbf{u}\cdot\gamma'''=(-\mathbf{u}\cdot\gamma)^{(IV)}=0$$
$$\Leftrightarrow\quad \mathbf{u}=\pm\mathbf{b},\tau=0 \text{ and } -\mathbf{u}\cdot[\kappa'\mathbf{n}+\kappa(-\kappa\mathbf{t}+\tau\mathbf{b})]'=0$$
$$\Leftrightarrow\quad \mathbf{u}=\pm\mathbf{b},\tau=0 \text{ and } -\mathbf{u}\cdot\left[\kappa''\mathbf{n}+2\kappa'(-\kappa\mathbf{t}+\tau\mathbf{b})+\kappa(-\kappa'\mathbf{t}-\kappa^2\mathbf{n}+\tau'\mathbf{b}-\tau^2\mathbf{n})\right]=0$$
$$\Leftrightarrow\quad \mathbf{u}=\pm\mathbf{b},\tau=0 \text{ and } \kappa\tau'=0$$
$$\Leftrightarrow\quad \mathbf{u}=\pm\mathbf{b},\tau=0 \text{ and } \tau'=0 \text{ since } \kappa\neq 0.$$

■.

Remark 1 $\mathbf{u}=\gamma+\frac{1}{\kappa}\mathbf{n}-\mu\mathbf{b},\mu\in\mathbb{R}$ *is the focal line (or polar axis) of* γ*. If* $\mathbf{u}\in S^2$ *then*

$$(\gamma+\frac{1}{\kappa}\mathbf{n}-\mu\mathbf{b})\cdot(\gamma+\frac{1}{\kappa}\mathbf{n}-\mu\mathbf{b})=1$$
$$\Rightarrow\quad \gamma\cdot\gamma+\frac{1}{\kappa^2}\mathbf{n}\cdot\mathbf{n}+\mu^2\mathbf{b}\cdot\mathbf{b}+2\gamma\cdot(\frac{1}{\kappa}\mathbf{n}-\mu\mathbf{b})-2\frac{1}{\kappa}\mu\mathbf{n}\cdot\mathbf{b}=1$$

therefore,

$$\mu^2-2(\gamma\cdot\mathbf{b})\mu+\frac{2}{\kappa}\gamma\cdot\mathbf{n}+\frac{1}{\kappa^2}=0. \tag{2.3}$$

By (7-6) in [16, Page 25] the center of S^2 *(the origin) is expressed as*

$$\gamma+R\mathbf{n}+TR'\mathbf{b}=0. \tag{2.4}$$

So $\gamma\cdot\mathbf{b}=-TR'$ *and* $\gamma\cdot\mathbf{n}=-R$*. Then from (2.3) we have*

$$\mu^2+2TR'\mu-R^2=0 \tag{2.5}$$

The solutions to (2.5) are

$$\mu=-TR'\pm\sqrt{(TR')^2+R^2}=-TR'\pm 1$$

since $(TR')^2 + R^2 = 1$ *by (1.7). So* $\mathbf{u} = \gamma + \frac{1}{\kappa}\mathbf{n} - \mu\mathbf{b} = \gamma + R\mathbf{n} + TR'\mathbf{b} \pm \mathbf{b} = \pm\mathbf{b}$ *by (2.4).*

So $\{\mathbf{u} \mid d_g'(s,\mathbf{u}) = d_g''(s,\mathbf{u}) = 0\} = \{\pm\mathbf{b}\}$ is the intersection of the focal line with the unit sphere. It's called the *focal center* of γ at s, denoted by $\mathbf{e} = \pm\mathbf{b}$ (c.f. [13, page 89]). It's also the center of the osculating geodesic circle of γ at s. Porteous calls the curve $\mathbf{e}$ of the focal centers the *evolute* of γ. So we proved that for a spherical curve γ *the evolute of* γ *is the binormal indicatrix of* γ. We'd like to add the word "*spherical*" before the evolute and call $\mathbf{e}$ the *spherical evolute* of γ. As a result, γ is called the *spherical involute* of $\mathbf{e}$.

Remark 2 *By (i) of Proposition 4, the geodesic circle on* S^2 *has* 2*-point contact with* γ *at any point* s *iff the geodesic center of the geodesic circle passing through* $\gamma(s)$ *is on the spherical normal to* γ *at* $\gamma(s)$*. And the geodesic circle on* S^2 *has* 3*-point contact with* γ *at any point* s *iff the geodesic center of the geodesic circle passing through* $\gamma(s)$ *is one of the focal centers of* γ *at* s*.*

Remark 3 *In this proposition, if we replace* $d_g : I \times S^2 \to \mathbb{R}$ *by the Euclidean distance-squared function* $d_e : I \times S^2 \to \mathbb{R}$ *defined by* $d_e(s,\mathbf{u}) = (\gamma(s) - \mathbf{u})^2 = 2(1 - \gamma(s)\cdot\mathbf{u})$*, we still get the same result, i.e.,*

$$d_g'(s,\mathbf{u}) = 0 \;\Leftrightarrow\; d_e'(s,\mathbf{u}) = 0;$$

$$d_g'(s,\mathbf{u}) = d_g''(s,\mathbf{u}) = 0 \;\Leftrightarrow\; d_e'(s,\mathbf{u}) = d_e''(s,\mathbf{u}) = 0;$$

$$d_g'(s,\mathbf{u}) = d_g''(s,\mathbf{u}) = d_g'''(s,\mathbf{u}) = 0 \;\Leftrightarrow\; d_e'(s,\mathbf{u}) = d_e''(s,\mathbf{u}) = d_e'''(s,\mathbf{u}) = 0;$$

$$\begin{aligned} & d_g'(s,\mathbf{u}) = d_g''(s,\mathbf{u}) = d_g'''(s,\mathbf{u}) = d_g^{(IV)}(s,\mathbf{u}) = 0 \\ \Leftrightarrow\; & d_e'(s,\mathbf{u}) = d_e''(s,\mathbf{u}) = d_e'''(s,\mathbf{u}) = d_e^{(IV)}(s,\mathbf{u}) = 0. \end{aligned}$$

Remark 4 *For a general regular spherical curve (not necessarily unit speed) $\boldsymbol{\gamma}(t)$, then $\widetilde{\gamma}(t) = \gamma(s^{-1}(t))$ is a unit-speed reparametrization, and $\gamma(t) = \widetilde{\gamma}(s(t))$, $s'(t) = \|\gamma'(t)\|$. Let $\widetilde{\kappa}(t), \widetilde{\tau}(t)$, $\widetilde{\mathbf{t}}(t)$, $\widetilde{\mathbf{n}}(t)$ and $\widetilde{\mathbf{b}}(t)$ be the curvature, torsion, unit tangent, unit normal and unit binormal of $\widetilde{\gamma}(t)$, then the curvature, torsion, tangent, normal and binormal of $\gamma(t)$ are defined to be $\kappa(t) = \widetilde{\kappa}(s(t))$, $\tau(t) = \widetilde{\tau}(s(t))$, $\mathbf{t}(t) = \widetilde{\mathbf{t}}(s(t))$, $\mathbf{n}(t) = \widetilde{\mathbf{n}}(s(t))$ and $\mathbf{b}(t) = \mathbf{t}(t) \times \mathbf{n}(t)$.*

Consider

$$d_g(t, \mathbf{u}) = \arccos\left[1 - \frac{1}{2}(\boldsymbol{\gamma}(t) - \mathbf{u}) \cdot (\boldsymbol{\gamma}(t) - \mathbf{u})\right] = \arccos(\boldsymbol{\gamma}(t) \cdot \mathbf{u}).$$

Then

$$\begin{aligned}
d_g'(t, \mathbf{u}) &= -\frac{\mathbf{u} \cdot \boldsymbol{\gamma}'(t)}{[1 - (\boldsymbol{\gamma}(t) \cdot \mathbf{u})^2]^{\frac{1}{2}}} = -\frac{s'(t)\left[\mathbf{u} \cdot \widetilde{\mathbf{t}}(s(t))\right]}{[1 - (\boldsymbol{\gamma}(t) \cdot \mathbf{u})^2]^{\frac{1}{2}}}, \\
d_g''(t, \mathbf{u}) &= -\frac{s''(t)\left[\mathbf{u} \cdot \widetilde{\mathbf{t}}(s(t))\right] + [s'(t)]^2 \widetilde{\kappa}(s(t))\, [\mathbf{u} \cdot \widetilde{\mathbf{n}}(s(t))]}{[1 - (\boldsymbol{\gamma}(t) \cdot \mathbf{u})^2]^{\frac{1}{2}}} \\
&\quad - s'(t)\left[\mathbf{u} \cdot \widetilde{\mathbf{t}}(s(t))\right]\left(\frac{1}{[1 - (\boldsymbol{\gamma}(t) \cdot \mathbf{u})^2]^{\frac{1}{2}}}\right)',
\end{aligned}$$

and

$$\begin{aligned}
d_g'''(t, \mathbf{u}) = &-\frac{s'''(t)\left[\mathbf{u} \cdot \widetilde{\mathbf{t}}(s(t))\right] + [s'(t)]^2 \widetilde{\kappa}(s(t)))'\left[\mathbf{u} \cdot \widetilde{\mathbf{t}}(s(t))\right]}{[1 - (\boldsymbol{\gamma}(t) \cdot \mathbf{u})^2]^{\frac{1}{2}}} \\
&+\frac{[s'(t)]^2 \widetilde{\kappa}(s(t))\left[\mathbf{u} \cdot [-\widetilde{\kappa}(s(t))]\, \widetilde{\mathbf{t}}(s(t))\right] + \widetilde{\tau}(s(t))\widetilde{\mathbf{b}}(s(t)))}{[1 - (\boldsymbol{\gamma}(t) \cdot \mathbf{u})^2]^{\frac{1}{2}}} \\
&-\left\{s''(t)\left[\mathbf{u} \cdot \widetilde{\mathbf{t}}(s(t))\right] + [s'(t)]^2 \widetilde{\kappa}(s(t))\, [\mathbf{u} \cdot \widetilde{\mathbf{n}}(s(t))]\right\}\left(\frac{1}{[1 - (\boldsymbol{\gamma}(t) \cdot \mathbf{u})^2]^{\frac{1}{2}}}\right)' \\
&-s'(t)\left[\mathbf{u} \cdot \widetilde{\mathbf{t}}(s(t))\right]\left(\frac{1}{[1 - (\boldsymbol{\gamma}(t) \cdot \mathbf{u})^2]^{\frac{1}{2}}}\right)''.
\end{aligned}$$

So

$$\begin{aligned}
d_g'(t, \mathbf{u}) = 0 \quad &\Leftrightarrow \quad \mathbf{u} = \lambda\widetilde{\mathbf{n}}(s(t)) + \mu\widetilde{\mathbf{b}}(s(t)) \text{ for some } \lambda, \mu \in \mathbb{R}; \\
d_g'(t, \mathbf{u}) = d_g''(t, \mathbf{u}) = 0 \quad &\Leftrightarrow \quad \mathbf{u} = \pm\mathbf{b}(t) = \pm\widetilde{\mathbf{b}}(s(t));
\end{aligned}$$

$$d_g'(t,\mathbf{u}) = d_g''(t,\mathbf{u}) = d_g'''(t,\mathbf{u}) = 0$$

$$\Leftrightarrow \quad \mathbf{u} = \pm\mathbf{b}(t) = \pm\widetilde{\mathbf{b}}(s(t)) \text{ and } \tau(t) = \widetilde{\tau}(s(t)) = 0.$$

2.2 Some Properties of the Spherical Evolute

As in the case of the plane evolute we can show:

Proposition 5 *The spherical evolute* $\mathbf{e}$ *of* γ *is the envelope of the family of the spherical normals to* γ *and the spherical evolute has a cusp at* s *iff* $\tau(s) = 0$.

Proof. The family of the spherical normals to γ is:

$$\begin{cases} \mathbf{x}\cdot\mathbf{x} = 1, \\ \mathbf{x}\cdot\mathbf{t}(s) = 0. \end{cases}$$

Then the envelope is

$$\begin{cases} \mathbf{x}\cdot\mathbf{x} = 1, \\ \mathbf{x}\cdot\mathbf{t}(s) = 0, \\ \mathbf{x}\cdot\kappa(s)\mathbf{n}(s) = 0. \end{cases}$$

Since $\kappa(s) \neq 0$, we have $\mathbf{x}\cdot\mathbf{n}(s) = 0$. Considering $\mathbf{x}\cdot\mathbf{t}(s) = 0$, $\mathbf{x} = \mu(s)\mathbf{b}(s)$ for some μ. But $\mathbf{x}\cdot\mathbf{x} = 1 \Rightarrow (\mu(s))^2 = 1 \Rightarrow \mathbf{x} = \pm\,\mathbf{b}(s)$, which is the spherical evolute $\mathbf{e}$ of γ. $\mathbf{x}' = \pm\tau(s)\mathbf{n}(\mathbf{s})$, so the spherical evolute has a cusp at s iff $\tau(s) = 0$. ■

Remark 5 *Let* $\theta(s)$ *be the angle made by* $\pm\mathbf{b}(s)$ *and* $\gamma(s)$. *(See Figure 1 for the case of* $\mathbf{b}(s)$ *and* $\gamma(s)$*).*

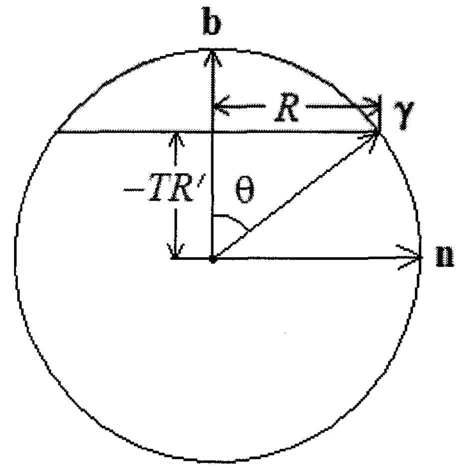


Figure 1: Angle between $\pm\mathbf{b}(s)$ and $\gamma(s)$

Then $\sin\theta = R\ (=\frac{1}{\kappa})$. The arclength $l(s)$ of the portion of the great circle connecting $\mathbf{b}(s)$ $(-\mathbf{b}(s))$ and $\gamma(s)$ is $\arcsin R(s)$ $(\pi - \arcsin R(s)\ )$, which is the geodesic radius of the osculating geodesic circle of γ at s. We call $l(s)$ the *geodesic radius of curvature* of $\gamma(s)$ at s. So

$$l'(s) = \pm\frac{R'(s)}{\sqrt{1-R^2(s)}} = \pm\frac{R'(s)}{\sqrt{(TR')^2(s)}} = \pm\tau(s)$$

and

$$l'(s) = 0 \ \Leftrightarrow\ \tau(s) = 0 \ \Leftrightarrow\ \mathbf{b}'(s) = \mp\tau(s)\mathbf{n}(s) = 0.$$

The spherical evolute of γ, therefore, has a cusp at s iff the geodesic radius of curvature of γ has a critical point at s. The evolute of a plane curve γ with $\kappa(s)$ never zero is given by $\varepsilon(s) = \gamma(s) + [1/\kappa(s)]\mathbf{n}(s)$. $\varepsilon'(s) = [1/\kappa(s)]'\mathbf{n}(s)$. So $\mathbf{e}'(s) = \mp\tau(s)\mathbf{n}(s)$ is the analogue for $\mathbf{e}$ of the equation $\varepsilon'(s) = [1/\kappa(s)]'\mathbf{n}(s)$ (c.f. Proposition 5.2 (vi) in [13]). $[1/\kappa(s)]'$ is the derivative of the radius of curvature of γ at s and $\tau(s)$ is the derivative of the geodesic radius of curvature of γ up to sign.

Remark 6 *If $\gamma(t)$ is not a unit speed curve, then the envelope of the spherical normals to γ is:*

$$\begin{cases} \mathbf{x}\cdot\mathbf{x}=1, \\ \mathbf{x}\cdot\gamma'(t)=0, \\ \mathbf{x}\cdot\gamma''(t)=0. \end{cases} \Longrightarrow \begin{cases} \mathbf{x}\cdot\mathbf{x}=1, \\ s'\mathbf{x}\cdot\mathbf{t}(s)=0, \\ \mathbf{x}\cdot(s'^2\kappa(s)\mathbf{n}(s)+s''\mathbf{t}(s))=0, \end{cases}$$

where s is the arclength and $s'(t)=\|\gamma'(t)\|\neq 0$. Therefore, we have

$$\mathbf{x}\cdot\mathbf{x}=1,\ \ \mathbf{x}\cdot\mathbf{t}(s)=0 \ \ \textit{and} \ \ \mathbf{x}\cdot\mathbf{n}(s)=0,$$

because $\kappa(s)\neq 0$. So as above, we have $\mathbf{x}=\pm\,\mathbf{b}(s(t))$. The spherical evolute of γ is its binornmal indicatrix no matter if it is unit speed. $\mathbf{x}'=\pm\,\tau(s(t))\mathbf{b}(s(t))$, and the spherical evolute has a cusp at s iff $\tau(s(t))=0$, i.e., the torsion of the spherical evolute is zero.

In [13, page 89] Porteous defines the concept of an A_k *center.* A point $\mathbf{c}$ of S^2 will be said to be an A_k *center* of the curve γ at s if, for all i such that $1\leq i\leq k$, $\mathbf{c}\cdot\gamma_i(s)=0$, but $\mathbf{c}\cdot\gamma_{k+1}(s)\neq 0$. Here γ_i denotes the i-th derivative of γ. For the evolute $\mathbf{e}$ of γ, $\mathbf{e}\cdot\gamma_1=\pm\mathbf{b}\cdot\mathbf{t}=0$ and $\mathbf{e}\cdot\gamma_2=\pm\mathbf{b}\cdot\kappa\mathbf{n}=0$, so the evolute $\mathbf{e}$ is at least an A_2 center of γ.

Furthermore we have:

Proposition 6 *The evolute $\mathbf{e}$ is an A_{k-1} ($k\geq 3$) center of the spherical curve γ at s iff $\tau(s)=\tau'(s)=\cdots=\tau^{(k-4)}(s)=0$, but $\tau^{(k-3)}(s)\neq 0$.*

Proof. $\gamma_1=\mathbf{t}$, $\gamma_2=\kappa\mathbf{n}$, $\gamma_3=-\kappa^2\mathbf{t}+\kappa'\mathbf{n}+\kappa\tau\mathbf{b}$, $\gamma_4=-3\kappa\kappa'\mathbf{t}+(\kappa''-\kappa^3-\kappa\tau)\mathbf{n}+(2\kappa'\tau+\kappa\tau')\mathbf{b}$.

Claim: $\gamma_k=$ (a polynomial of κ, the derivatives of κ, τ and the derivatives of τ up to the order $k-5$)$\mathbf{t}+$ (the polynomial of κ, the derivatives of κ, τ and the derivatives of τ up to the order $k-4$)$\mathbf{n}+$ ((the polynomial of τ and the derivatives of τ up to the order

$k-4$ with the coefficients the polynomial of κ and the derivatives of κ, which contains no constant term) $+\kappa\tau^{(k-3)})\mathbf{b}$. This is true for $k=3,4.$ Suppose it is true for $k=m$. Then for $k=m+1$,

$\gamma_{m+1}=$ (the polynomial of κ, the derivatives of κ, τ and the derivatives of τ up to the order $m-4)\mathbf{t}+$ (the polynomial of κ, the derivatives of κ, τ and the derivatives of τ up to the order $m-5)(\kappa\mathbf{n})+$ (the polynomial of κ, the derivatives of κ, τ and the derivatives of τ up to the order $m-3)\mathbf{n}+$ (the polynomial of κ, the derivatives of κ, τ and the derivatives of τ up to the order $m-4)(-\kappa\mathbf{t}+\tau\mathbf{b})+$ (the polynomial of τ and the derivatives of τ up to the order $m-3$ with the coefficients the polynomial of κ and the derivatives of κ, which contains no constant term) $+\kappa\tau^{(m-2)})\mathbf{b}+$ (the polynomial of τ and the derivatives of τ up to the order $m-4$ with the coefficients the polynomial of κ and the derivatives of κ, which contains no constant term) $+\kappa\tau^{(m-3)})(-\tau\mathbf{n})$

$=$ (the polynomial of κ, the derivatives of κ, τ and the derivatives of τ up to the order $m-4)\mathbf{t}+$ (the polynomial of κ, the derivatives of κ, τ and the derivatives of τ up to the order $m-3)\mathbf{n}+$ (the polynomial of τ and the derivatives of τ up to the order $m-3$ with the coefficients the polynomial of κ and the derivatives of κ, which contains no constant term) $+\kappa\tau^{(m-2)})\mathbf{b}$. This completes the induction.

So $\mathbf{e}\cdot\gamma_k=\pm\mathbf{b}\cdot$ (the polynomial of τ and the derivatives of τ up to the order $k-4$ with the coefficients the polynomial of κ and the derivatives of κ, which contains no constant term) $+\kappa\tau^{(k-3)})\mathbf{b}$

$=$ the polynomial of τ and the derivatives of τ up to the order $k-4$ with the coefficients the polynomial of κ and the derivatives of κ, which contains no constant term) $+\kappa\tau^{(k-3)}$.

Proof of sufficiency: Since $\kappa \neq 0$ by (1.2), $\tau(s) = \tau'(s) = \cdots = \tau^{(k-4)}(s) = 0$ but $\tau^{(k-3)}(s) \neq 0$ implies that $\mathbf{e} \cdot \boldsymbol{\gamma}_i = 0$ for $1 \leq i \leq k-1$ but $\mathbf{e} \cdot \boldsymbol{\gamma}_k \neq 0$. So $\mathbf{e}$ is an A_{k-1} center of the spherical curve $\boldsymbol{\gamma}$ at s.

Proof of necessity:

$$\mathbf{e} \cdot \boldsymbol{\gamma}_3 = \kappa\tau = 0 \;\;\Rightarrow\;\; \tau = 0 . \mathbf{e} \cdot \boldsymbol{\gamma}_3 = \mathbf{e} \cdot \boldsymbol{\gamma}_4 = 0 \;\;\Rightarrow\;\; \tau = \tau' = 0.$$

So by the expression for $\mathbf{e} \cdot \boldsymbol{\gamma}_k$ above and the induction, that $\mathbf{e}$ is an A_{k-1} $(k \geq 3)$ center of $\boldsymbol{\gamma}$ at s implies $\tau(s) = \tau'(s) = \cdots = \tau^{(k-4)}(s) = 0$, but $\tau^{(k-3)}(s) \neq 0$. ■.

Proposition 7 *Let $\boldsymbol{\gamma} : I \to S^2$ be a unit speed curve. Then the spherical evolute* $\mathbf{e}$ *of $\boldsymbol{\gamma}$ is nowhere great.*

Proof. We know

$$\begin{cases} \mathbf{e}' = \pm\mathbf{b}' = \mp\tau\mathbf{n}, \mathbf{e}'' = \mp\left[\tau'\mathbf{n}+\tau(-\kappa\mathbf{t}+\tau\mathbf{b})\right] \\ \mathbf{e}' \times \mathbf{e}'' = \tau\mathbf{n} \times \tau(-\kappa\mathbf{t}+\tau\mathbf{b}) = \kappa\tau^2\mathbf{b} + \tau^3\mathbf{t} \end{cases},$$

and then

$$\begin{cases} \text{the unit tangent of } \mathbf{e} \text{ is: } \mathbf{t_e} = \dfrac{\mathbf{e}'}{|\mathbf{e}'|} = \pm\mathbf{n}, \\ \text{the unit binormal of } \mathbf{e} \text{ is: } \mathbf{b_e} = \dfrac{\mathbf{e}' \times \mathbf{e}''}{|\mathbf{e}' \times \mathbf{e}''|} = \dfrac{\kappa\mathbf{b} + \tau\mathbf{t}}{\sqrt{\kappa^2 + \tau^2}} \end{cases}.$$

Here if $\tau = 0$, we can still define $\mathbf{t_e}$ and $\mathbf{b_e}$ to be the limit directions of $\dfrac{\mathbf{e}'}{|\mathbf{e}'|}$ and $\dfrac{\mathbf{e}' \times \mathbf{e}''}{|\mathbf{e}' \times \mathbf{e}''|}$ since the curvature κ of $\boldsymbol{\gamma}$ is nonzero. By Proposition 1 and

$$\mathbf{e} \cdot \mathbf{b_e} = \frac{\mathbf{e} \cdot (\mathbf{e}' \times \mathbf{e}'')}{|\mathbf{e}' \times \mathbf{e}''|} = \frac{[\mathbf{e}, \mathbf{e}', \mathbf{e}'']}{|\mathbf{e}' \times \mathbf{e}''|} = \pm\mathbf{b} \cdot \frac{\kappa\mathbf{b} + \tau\mathbf{t}}{\sqrt{\kappa^2 + \tau^2}} = \pm\frac{\kappa}{\sqrt{\kappa^2 + \tau^2}} \neq 0$$

the result follows. ■

CHAPTER 3

Relationship between the Spherical Evolute and the Spherical Involute

3.1 Construct the Spherical Evolute from the Spherical Involute

Definition 8 *Given 2 spherical curves γ and* $\mathbf{e}$, *if* $\mathbf{e}$ *is the spherical evolute of γ then γ is called the spherical involute of* $\mathbf{e}$.

Given a spherical unit speed curve $\gamma(s)$, suppose another spherical curve (not necessarily unit speed) $\gamma_1(s)$ at each point P_1 is tangent to a spherical normal to γ at a corresponding point P. In this section we use the suffix 1 to indicate the elements of γ_1 at P_1 corresponding to those of γ at P. By the above assumption and the fact that γ_1 is in the plane where the spherical tangent to γ_1 at P_1 lies we know $\gamma_1 \perp \mathbf{t} \Rightarrow \gamma_1 = \lambda\mathbf{n}+\mu\mathbf{b}$. Then (in this subsection $' = \frac{d}{ds}$, s is the arclength of γ)

$$\begin{aligned} \gamma_1' &= \lambda'\mathbf{n}+\lambda(-\kappa\mathbf{t}+\tau\mathbf{b}) + \mu'\mathbf{b} + \mu(-\tau\mathbf{n}) \\ &= -\lambda\kappa\mathbf{t} + (\lambda' - \mu\tau)\mathbf{n} + (\lambda\tau + \mu')\mathbf{b}. \end{aligned}$$

That the spherical normal to γ at P is tangent to γ_1 at P_1 implies $\gamma_1' \perp \mathbf{t}$. So $\lambda\kappa = 0$. But $\kappa \neq 0$. Then $\lambda = 0$ and $\gamma_1 = \mu\mathbf{b}$. Since γ_1 is a spherical curve,

$$\gamma_1 \cdot \gamma_1 = (\mu\mathbf{b})\cdot(\mu\mathbf{b}) = 1 \quad \Rightarrow \quad \mu^2 = 1.$$

Hence $\gamma_1 = \pm\mathbf{b}$. And for $\gamma_1 = \pm\mathbf{b}$, $\gamma_1' = \mp\tau\mathbf{n}$, which is the normal to γ. From this construction and Proposition 5 we know

Proposition 9 *The spherical evolute of γ is a spherical curve γ_1 which is tangent to the spherical normal to γ wherever γ_1 meets the spherical normal to γ.* ■

3.2 Construct the Spherical Involute from the Spherical Evolute

Now consider a pair of spherical curves $\boldsymbol{\gamma}$ and $\boldsymbol{\gamma}_1$, we want to seek $\boldsymbol{\gamma}$ as the spherical involute of $\boldsymbol{\gamma}_1$ (in this subsection $' = \frac{d}{ds_1}$, s_1 is the arclength of $\boldsymbol{\gamma}_1$). Suppose at each of its points the spherical tangent of $\boldsymbol{\gamma}$ is normal to a spherical tangent of $\boldsymbol{\gamma}_1$ at a corresponding point and $\boldsymbol{\gamma}(s_1)$ is on the spherical tangent of $\boldsymbol{\gamma}_1$ at $\boldsymbol{\gamma}_1(s_1)$, i.e., $\boldsymbol{\gamma}(s_1)$ is on the plane spanned by $\boldsymbol{\gamma}_1(s_1)$ and $\mathbf{t}_1(s_1)$, so $\boldsymbol{\gamma} = \lambda\boldsymbol{\gamma}_1 + \mu\mathbf{t}_1 \Rightarrow$

$$\begin{aligned}
\boldsymbol{\gamma}' &= \lambda'\boldsymbol{\gamma}_1 + \lambda\mathbf{t}_1 + \mu'\mathbf{t}_1 + \mu\kappa_1\mathbf{n}_1 \\
&= \lambda'(-R_1\mathbf{n}_1 - T_1R_1'\mathbf{b}_1) + (\lambda+\mu')\mathbf{t}_1 + \mu\kappa_1\mathbf{n}_1 \\
&= (\lambda+\mu')\mathbf{t}_1 + (\mu\kappa_1 - \lambda'R_1)\mathbf{n}_1 - \lambda'T_1R_1'\mathbf{b}_1.
\end{aligned}$$

By the assumption $\boldsymbol{\gamma}' \perp \mathbf{t}_1$ we get

$$\lambda + \mu' = 0. \tag{3.1}$$

$$\begin{aligned}
\boldsymbol{\gamma}\cdot\boldsymbol{\gamma} = 1 \quad &\Rightarrow (\lambda\boldsymbol{\gamma}_1 + \mu\mathbf{t}_1)^2 = 1 \\
&\Rightarrow \lambda^2\boldsymbol{\gamma}_1\cdot\boldsymbol{\gamma}_1 + \mu^2\mathbf{t}_1\cdot\mathbf{t}_1 + 2\lambda\mu\boldsymbol{\gamma}_1\cdot\mathbf{t}_1 = 1 \\
&\Rightarrow \lambda^2 + \mu^2 = 1 \\
&\Rightarrow \lambda\lambda' + \mu\mu' = 0
\end{aligned}$$

By (3.1) we have $\lambda(\lambda' - \mu) = 0$, and then

$$\begin{cases} \lambda = 0, \\ \mu = \text{constant}; \end{cases} \tag{3.2}$$

or

$$\begin{cases} \mu = \lambda', \\ \mu' = -\lambda. \end{cases} \tag{3.3}$$

In the case of (3.2), $\boldsymbol{\gamma} = \mu\mathbf{t}_1 . \boldsymbol{\gamma}\cdot\boldsymbol{\gamma} = 1 \;\Rightarrow\; \mu = \pm 1 \;\Rightarrow\; \boldsymbol{\gamma} = \pm\mathbf{t}_1$.

For a general curve its binormal is given by $\mathbf{b} = \frac{1}{\kappa s'^3}\gamma' \times \gamma''$, where $s' = \|\gamma'(t)\|$. If $\gamma = \pm\mathbf{t}_1$ is the spherical involute of γ_1 then $\gamma' \times \gamma''$ would be parallel to γ_1. But for $\gamma = \pm\mathbf{t}_1$,

$$\begin{aligned}
\gamma_1 \times (\gamma' \times \gamma'') &= \gamma_1 \times (\mathbf{t}_1' \times \mathbf{t}_1'') = (\gamma_1 \cdot \mathbf{t}_1'')\mathbf{t}_1' - (\gamma_1 \cdot \mathbf{t}_1')\mathbf{t}_1'' \\
&= (\gamma_1 \cdot (\kappa_1' \mathbf{n}_1 + \kappa_1 [-\kappa_1 \mathbf{t}_1 + \tau_1 \mathbf{b}_1])\mathbf{t}_1' - (\gamma_1 \cdot \kappa_1 \mathbf{n}_1)\mathbf{t}_1'' \\
&= [\kappa_1'(-R_1) + \kappa_1\tau_1(-T_1R_1')]\,\mathbf{t}_1' - \kappa_1(-R_1)\mathbf{t}_1'' \\
&= \left(-\frac{\kappa_1'}{\kappa_1} + \kappa_1 \frac{\kappa_1'}{\kappa_1^2}\right)\mathbf{t}_1' + \mathbf{t}_1'' \\
&= \mathbf{t}_1'' = \kappa_1'\mathbf{n}_1 - \kappa_1^2\mathbf{t}_1 + \kappa_1\tau_1\mathbf{b}_1 \\
&\neq \mathbf{0}
\end{aligned}$$

So $\pm\mathbf{t}_1' \times \pm\,\mathbf{t}_1''$ is not parallel to γ_1 and $\pm\mathbf{t}_1$ is not the spherical involute of γ_1.

In the case of (3.3), $\lambda'' + \lambda = 0$. Thus $\lambda = c_1 \cos s_1 + c_2 \sin s_1$ and $\mu = -c_1 \sin s_1 + c_2 \cos s_1$. $\lambda^2 + \mu^2 = 1 \Rightarrow c_1^2 + c_2^2 = 1$. So we can find an angle θ_0 such that $c_1 = \cos\theta_0$ and $c_2 = \sin\theta_0$. Then

$$\left.\begin{aligned}
\lambda &= \cos\theta_0 \cos s_1 + \sin\theta_0 \sin s_1 \\
&= \cos(\theta_0 - s_1) \\
\mu &= -\cos\theta_0 \sin s_1 + \sin\theta_0 \cos s_1 \\
&= \sin(\theta_0 - s_1)
\end{aligned}\right\} \Rightarrow \gamma(s_1) = \cos(\theta_0 - s_1)\gamma_1(s_1) + \sin(\theta_0 - s_1)\mathbf{t}_1(s_1).$$

θ_0 is the angle made by $\gamma(s_1)$ and $\gamma_1(s_1)$ when $s_1 = 0$. So

$$\begin{aligned}
\gamma'(s_1) &= \sin(\theta_0 - s_1)\,[\gamma_1(s_1) + \kappa_1(s_1)\mathbf{n}_1(s_1)] \\
&= \sin(\theta_0 - s_1)\left\{\frac{1}{\kappa_1(s_1)}\left[\kappa_1^2(s_1) - 1\right]\mathbf{n}_1(s_1) - (T_1R_1')(s_1)\mathbf{b}_1(s_1)\right\} \cdot \\
&= \sin(\theta_0 - s_1)\left\{\kappa_1(s_1)\,[(T_1R_1')(s_1)]^2\,\mathbf{n}_1(s_1) - (T_1R_1')(s_1)\mathbf{b}_1(s_1)\right\}
\end{aligned}$$

So the spherical involute of γ_1 has a cusp when $s_1 = \theta_0$ or $(T_1R_1')(s_1) = 0$ i.e., γ_1 is great at s_1. Here the spherical evolute γ_1 could be great at s_1 because the spherical involute γ

might have a cusp at the corresponding point. This does not contradicts Proposition 7, where the spherical involute γ is a regular curve.

Here we verify that the spherical evolute of $\gamma(s_1) = \cos(\theta_0 - s_1)\gamma_1(s_1) + \sin(\theta_0 - s_1)\mathbf{t}_1(s_1)$ is indeed γ_1 as follows:

$$\begin{aligned}\gamma_1(s_1)\cdot\gamma'(s_1) &= \gamma_1(s_1)\cdot[\sin(\theta_0 - s_1)\gamma_1(s_1) + \kappa_1(s_1)\sin(\theta_0 - s_1)\mathbf{n}_1(s_1)] \\ &= \sin(\theta_0 - s_1) + \kappa_1(s_1)\sin(\theta_0 - s_1)\,[-R_1(s_1)] \\ &= 0\end{aligned}$$

and

$$\begin{aligned}&\gamma_1(s_1)\cdot\gamma''(s_1) \\ = \;& \gamma_1(s_1)\cdot\{-\cos(\theta_0 - s_1)\gamma_1(s_1) - [\kappa_1(s_1)\cos(\theta_0 - s_1) - \kappa_1'(s_1)\sin(\theta_0 - s_1)]\,\mathbf{n}_1(s_1) + \\ & +\sin(\theta_0 - s_1)\left[1 - \kappa_1^2(s_1)\right]\mathbf{t}_1(s_1) + \kappa_1(s_1)\tau_1(s_1)\sin(\theta_0 - s_1)\mathbf{b}_1(s_1)\} \\ = \;& -\cos(\theta_0 - s_1) + R_1(s_1)\,[\kappa_1(s_1)\cos(\theta_0 - s_1) - \kappa_1'(s_1)\sin(\theta_0 - s_1)] \\ & -T_1(s_1)R_1'(s_1)\kappa_1(s_1)\tau_1(s_1)\sin(\theta_0 - s_1) \\ = \;& -\frac{\kappa_1'(s_1)}{\kappa_1(s_1)}\sin(\theta_0 - s_1) - \kappa_1(s_1)\left[-\frac{\kappa_1'(s_1)}{\kappa_1^2(s_1)}\right]\sin(\theta_0 - s_1) \\ = \;& 0\end{aligned}$$

$$\Longrightarrow \qquad \gamma_1 \times (\gamma' \times \gamma'') = (\gamma_1\cdot\gamma'')\gamma' - (\gamma_1\cdot\gamma')\gamma'' = \mathbf{0}.$$

So γ_1 is the spherical evolute of γ and γ is the spherical involute of γ_1.

We summarize this result in the following proposition:

Proposition 10 *Suppose at each of its points the spherical tangent to γ is normal to a spherical tangent to γ_1 at a corresponding point and $\gamma(s_1)$ is on the spherical tangent to γ_1 at $\gamma_1(s_1)$. Then $\gamma(s_1) = \cos(\theta_0 - s_1)\gamma_1(s_1) + \sin(\theta_0 - s_1)\mathbf{t}_1(s_1)$ or $\gamma(s_1) = \pm\mathbf{t}_1(s_1)$, but only $\gamma(s_1) = \cos(\theta_0 - s_1)\gamma_1(s_1) + \sin(\theta_0 - s_1)\mathbf{t}_1(s_1)$ is the spherical involute of γ_1 and γ_1 is* *the spherical evolute of* γ, *where* θ_0 *is the angle made by* $\gamma(s_1)$ *and* $\gamma_1(s_1)$ *when* $s_1 = 0$. *The spherical involute* $\gamma(s_1)$ *has a cusp when* $s_1 = \theta_0$ *or* $(T_1 R_1')(s_1) = 0$ *i.e.,* γ_1 *is great at* s_1. ■

Remark 7 *At each of its points* $\pm\mathbf{t}_1$ *is normal to a spherical tangent of* γ_1 *but* $\pm\mathbf{t}_1$ *is not the spherical involute of* γ_1.

3.3 Local Diffeomorphic Image of the Spherical Evolute

In this section we'll use the same technique as used in [1] on the application of the unfolding in [3] to get the local diffeomorphic image of the spherical evolute of γ. We first state some definitions about the unfolding, the versal unfoldings and the (p)versal unfolding, and A_k-singularity (**Definition 3.6** in [3]) then a theorem on the (p)versal unfolding in [3] for later use.

Definition 11 *Let* $F : \mathbb{R} \times \mathbb{R}^r, (s_0, \mathbf{x}_0) \to \mathbb{R}$ *be a smooth function and* $f = F_{\mathbf{x}_0}, F_{\mathbf{x}_0}(s) = F(s, \mathbf{x}_0)$. *Then* F *is called an* r*-parameter unfolding of* f.

Definition 12 *Let* $G : \mathbb{R} \times \mathbb{R}^p, (s_0, \mathbf{y}_0) \to \mathbb{R}$ *be a* p*-parameter unfolding of the function* $g = G_{\mathbf{y}_0}$. *Let*

$$a : \mathbb{R} \times \mathbb{R}^r, (s_0, \mathbf{x}_0) \to \mathbb{R} \text{ where } a(s, \mathbf{x}_0) = s \text{ close to } s_0$$

$$b : \mathbb{R}^r, \mathbf{x}_0 \to \mathbb{R}^p, \mathbf{y}_0$$

$$c : \mathbb{R}^r, \mathbf{x}_0 \to \mathbb{R}$$

be smooth. Then the unfolding $F : \mathbb{R} \times \mathbb{R}^r, (s_0, \mathbf{x}_0) \to \mathbb{R}$ *defined by*

$$F(s, \mathbf{x}) = G(a(s, \mathbf{x}), b(\mathbf{x})) + c(\mathbf{x})$$

is said to be (p)induced from G. *If* G *is such that every unfolding of* g *is (p)induced from* G *then* G *is called a (p)versal unfolding of* g *at* s_0.

Here (p) mean *potential.* Note that F is an unfolding of f, where $f(s) = g(s) + c(\mathbf{x}_0)$, which is g up to an additive constant.

Definition 13 *Induced and versal unfoldings: this is as in the last definition with* $c \equiv 0$.

Definition 14 *Given a differentiable function* $f : \mathbb{R}, s_0 \to \mathbb{R}$. *For* $k \geq 0$, f *is said to have type* A_k *at* s_0, *or an* A_k*-singularity at* s_0, *if* $f^{(p)}(s_0) = 0$ *for all* p *with* $1 \leq p \leq k$, *and* $f^{(k+1)}(s_0) \neq 0$. *We also say that* f *has type* $A_{\geq k}$ *at* s_0 *when* $f^{(p)}(s_0) = 0$ *for all* p *with* $1 \leq p \leq k$.

Let $F : \mathbb{R}\times\mathbb{R}^r, (s_0, \mathbf{x}_0) \to \mathbb{R}$ be an r-parameter unfolding of the function $f = F_{\mathbf{x}_0}, F_{\mathbf{x}_0}(s) = F(s, \mathbf{x}_0)$, $\mathbf{x} = (x_1, \ldots, x_r)$.

Theorem 15 *(**6.10p** in [3]) Let the* $(k-1)$*-jet of* $\partial F/\partial x_i$ *at* x_0 *be* $j^{k-1}(\partial F/\partial x_i(s, \mathbf{x}_0))(s_0) = \alpha_{1i}s + \alpha_{2i}s^2 + \cdots + \alpha_{k-1,i}s^{k-1}$ *(without the constant term) for* $i = 1, \ldots, r$. *Then* $F(s, \mathbf{x})$ *is (p)versal iff the* $(k-1) \times r$ *matrix of coefficients* (α_{ji}) *has rank* $k-1$. *(This certainly requires* $k - 1 \leq r$, *so the smallest possible value of* r *is* $k-1$.)

In our case, for the curve γ, let $F : I \times S^2 \to \mathbb{R}$ be $F(s, \mathbf{x}) = d_e(s, \mathbf{x}) = (\gamma(s) - \mathbf{x}) \cdot (\gamma(s) - \mathbf{x})$ (noting 3). Then for any fixed $\mathbf{x}_0 \in S^2$, $F(s, \mathbf{x})$ is a 2-parameter unfolding of the function $f(s) = F(s, \mathbf{x}_0)$. And the following theorem is essentially due to Pei, Donghe and Sano, Takashi. (c.f. Theorem 4.2 (2) in [1]).

Theorem 16 *For the unit speed curve* $\gamma = (r_1(s), r_2(s), r_3(s))$ *on the unit sphere* S^2 *and* $k = 2, 3$, *if* $f(s)$ *has the* A_k*-singularity at* $s_0 \in I$, *then* F *is the (p)versal unfolding of* f.

Proof. Since $\mathbf{x} = (x_1, x_2, x_3) \in S^2$, $x_1^2 + x_2^2 + x_3^2 = 1$. x_1, x_2 and x_3 can't be all zero. Without loss of generality, suppose $x_3 \neq 0$. Then by $x_3 = \pm\sqrt{1-(x_1^2+x_2^2)}$, we have

$$\begin{aligned} F(s,\mathbf{x}) &= (\gamma(s)-\mathbf{x})\cdot(\gamma(s)-\mathbf{x}) \\ &= (r_1(s)-x_1)^2 + (r_2(s)-x_2)^2 + \left[r_3(s) \mp \sqrt{1-(x_1^2+x_2^2)}\right]^2. \end{aligned}$$

So

$$\begin{aligned} \frac{\partial F}{\partial x_1} &= 2(x_1 - r_1(s)) + 2\left[r_3(s) \mp \sqrt{1-(x_1^2+x_2^2)}\right]\left[\frac{\pm x_1}{\sqrt{1-(x_1^2+x_2^2)}}\right] \\ &= -2r_1(s) + \frac{2x_1}{x_3} r_3(s), \end{aligned}$$

and

$$\begin{aligned} \frac{\partial F}{\partial x_2} &= 2(x_2 - r_2(s)) + 2\left[r_3(s) \mp \sqrt{1-(x_1^2+x_2^2)}\right]\left[\frac{\pm x_2}{\sqrt{1-(x_1^2+x_2^2)}}\right] \\ &= -2r_2(s) + \frac{2x_2}{x_3} r_3(s). \end{aligned}$$

Let $\mathbf{x}_0 = (x_{01}, x_{02}, x_{03}) \in S^2$ and assume $x_{03} \neq 0$, then $j^1(\partial F/\partial x_i(s, \mathbf{x}_0))(s_0) = -2r_i'(s_0)s + \frac{2x_{0i}}{x_{03}} r_3'(s_0)s$, and

$$j^2\left(\partial F/\partial x_i(s,\mathbf{x}_0)\right)(s_0) = -2r_i'(s_0)s - r_i''(s_0)s^2 + \frac{2x_{0i}}{x_{03}} r_3'(s_0)s + \frac{x_{0i}}{x_{03}} r_3''(s_0)s^2$$

$(k-$jets without the constant term$)$

So the $(2-1)\times 2$ matrix $M_1 = (\alpha_{11}, \alpha_{12})$ of coefficients α as in Theorem 15 is

$$M_1 = \left[-2r_1'(s_0) + \frac{2x_{01}}{x_{03}} r_3'(s_0), -2r_2'(s_0) + \frac{2x_{02}}{x_{03}} r_3'(s_0)\right];$$

the $(3-1)\times 2$ matrix M_2 of coefficients α as in Theorem 15 is

$$M_2 = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix} = \begin{bmatrix} -2r_1'(s_0) + \frac{2x_{01}}{x_{03}} r_3'(s_0) & -2r_2'(s_0) + \frac{2x_{02}}{x_{03}} r_3'(s_0) \\ -r_1''(s_0) + \frac{x_{01}}{x_{03}} r_3''(s_0) & -r_2''(s_0) + \frac{x_{02}}{x_{03}} r_3''(s_0) \end{bmatrix}.$$

When f has the A_k-singularity ($k=2,3$) at $s_0 \in I$, then $F'_{\mathbf{x}_0}(s_0) = F''_{\mathbf{x}_0}(s_0) = 0$ and by Proposition 4

$$\mathbf{x}_0 = (x_{01}, x_{02}, x_{03}) = \pm\mathbf{b}(s_0) = \pm\mathbf{t}(s_0) \times \mathbf{n}(s_0) = \pm\frac{1}{\kappa(s_0)}\gamma'(s_0) \times \gamma''(s_0)$$

$\Rightarrow$

$$\begin{cases} x_{01} &= \pm\frac{1}{\kappa(s_0)}\left[r_2'(s_0)r_3''(s_0) - r_3'(s_0)r_2''(s_0)\right] \\ x_{02} &= \pm\frac{1}{\kappa(s_0)}\left[r_3'(s_0)r_1''(s_0) - r_1'(s_0)r_3''(s_0)\right] \\ x_{03} &= \pm\frac{1}{\kappa(s_0)}\left[r_1'(s_0)r_2''(s_0) - r_2'(s_0)r_1''(s_0)\right]. \end{cases}$$

The determinant of M_2

$$\begin{aligned} &\det(M_2) \\ = \; & 2\left\{r_1'(s_0)r_2''(s_0) - r_2'(s_0)r_1''(s_0) + \frac{x_{02}}{x_{03}}\left[r_3'(s_0)r_1''(s_0) - r_1'(s_0)r_3''(s_0)\right]\right. \\ & \left. + \frac{x_{01}}{x_{03}}\left[r_2'(s_0)r_3''(s_0) - r_3'(s_0)r_2''(s_0)\right]\right\} \\ = \; & \pm 2\kappa(s_0)(x_{03} + \frac{x_{02}}{x_{03}}x_{02} + \frac{x_{01}}{x_{03}}x_1) \\ = \; & \pm\frac{2\kappa(s_0)}{x_{03}} \neq 0 \end{aligned}$$

So by Theorem 15, F is the (p)versal unfolding of f. ■

Remark 8 *If $\gamma = (r_1(s), r_2(s), r_3(s))$ is a general curve on the unit sphere S^2, Theorem 16 still holds: when f has the A_k-singularity ($k=2,3$) at $s_0 \in I$, then $F'_{\mathbf{x}_0}(s_0) = F''_{\mathbf{x}_0}(s_0) = 0$ and by Remark 4 after Proposition 4*

$$\begin{aligned} & \mathbf{x}_0 = (x_{01}, x_{02}, x_{03}) = \pm\mathbf{b}(t_0) = \pm\frac{1}{\|\gamma'(t_0) \times \gamma''(t_0)\|}\gamma'(t_0) \times \gamma''(t_0) \\ \Rightarrow \quad & \det(M_2) = \pm\frac{2\|\gamma'(t_0) \times \gamma''(t_0)\|}{x_{03}} \neq 0 \textit{ because } \kappa(t) = \frac{\|\gamma'(t_0) \times \gamma''(t_0)\|}{\|\gamma'(t)\|^3} \neq 0. \end{aligned}$$

Here we state a theorem in [3] (see 6.16p, 6.17p, 6.18p in [3]):

Theorem 17 *Let* $F : (\mathbb{R}\times\mathbb{R}^r, (s_0, \mathbf{x}_0)) \to \mathbb{R}$ *be an r-parameter unfolding of* $f(s)$ *which has the* A_k*-singularity at* s_0*. Let the bifurcation set of* F *be denoted by*

$$\mathfrak{B}_F = \{x \in \mathbb{R}^r | \text{ there exists } s \text{ with } \frac{\partial F}{\partial s} = \frac{\partial^2 F}{\partial s^2} = 0 \text{ at } (s, x)\}$$

Suppose that F *is an (p)versal unfolding. Then*

(a) *if* $k = 2$, *then* $\mathfrak{B}_F$ *is locally diffeomorphic to* $\{0\}\times\mathbb{R}^{r-1}$;

(b) *if* $k = 3$, *then* $\mathfrak{B}_F$ *is locally diffeomorphic to* $C\times\mathbb{R}^{r-2}$;

(c) *if* $k = 4$, *then* $\mathfrak{B}_F$ *is locally diffeomorphic to* $SW\times\mathbb{R}^{r-3}$,

where $C = \{(x_1, x_2)|x_1^2 = x_2^3\}$ *and* $SW = \{(x_1, x_2, x_3)|x_1 = 3u^4 + u^2v, x_2 = 4u^3 + 2uv, x_3 = v\}$.

So by above Theorem 15, 16, 17, Proposition 4 and Remark 3 after Proposition 4 we have the following theorem (c.f. **4.2 (4)** in [3]):

Theorem 18 *Let* γ *be a unit speed curve on* S^2 *and* $\mathbf{u} \in S^2$, *then the spherical evolute of* γ *is*

(a) *diffeomorphic to a line if* $\tau(s_0) \neq 0$;

(b) *diffeomorphic to the ordinary cusp* C *if* $\tau(s_0) = 0$ *and* $\tau'(s_0) \neq 0$. ■

Remark 9 *This theorem holds for a general curve* γ *by Remark 4 after Proposition 4.*

CHAPTER 4

Spherical Orthotomic and Antiorthotomic

4.1 Spherical Orthotomic

Given a spherical unit speed curve γ and a point $\mathbf{u} \in S^2$. The spherical tangent to γ at s is given by

$$\begin{cases} \mathbf{x} \cdot \mathbf{x} = 1, \\ \mathbf{x} \cdot (\gamma(s) \times \mathbf{t}(s)) = 0. \end{cases}$$

i.e.,

$$\begin{cases} \mathbf{x} \cdot \mathbf{x} = 1, \\ \mathbf{x} \cdot (R\mathbf{b} - TR'\mathbf{n})(s) = 0. \end{cases} \tag{4.1}$$

Then the *spherical orthotomic* of γ relative to $\mathbf{u}$ is defined to be the set of reflections of $\mathbf{u}$ about the planes where the great circles (4.1) lie for all $s \in I$. We denote the spherical orthotomic of γ relative to $\mathbf{u}$ by $\widetilde{\mathbf{u}}$. Then

$$\widetilde{\mathbf{u}} = 2((\gamma - \mathbf{u}) \cdot \mathbf{v})\mathbf{v} + \mathbf{u},$$

where $\mathbf{v} = \frac{\mathbf{b} - (\mathbf{b} \cdot \gamma)\gamma}{\|\mathbf{b} - (\mathbf{b} \cdot \gamma)\gamma\|}$ is the unit tangential projection of $\mathbf{b}$ in the tangent plane to the sphere at γ (see Figure 2)

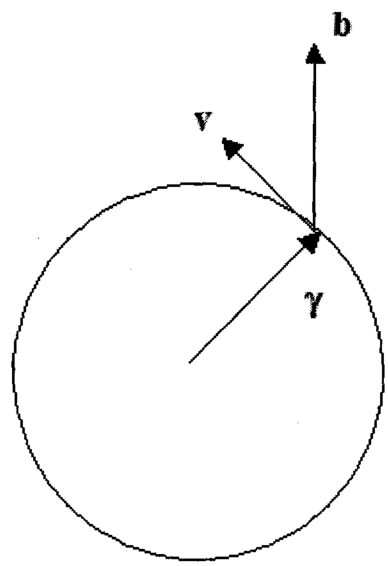


Figure 2: Unit tangential projection of $\mathbf{b}$ in the tangent plane to the sphere at $\boldsymbol{\gamma}$

and

$$\begin{aligned} \mathbf{b}-(\mathbf{b}\cdot\boldsymbol{\gamma})\boldsymbol{\gamma} &= \mathbf{b}+TR'(-R\mathbf{n}-TR'\mathbf{b}) \\ &= (1-(TR')^2)\mathbf{b}-TRR'\mathbf{n} \\ &= R^2\mathbf{b}-TRR'\mathbf{n}, \end{aligned}$$

noting $\boldsymbol{\gamma} = -R\mathbf{n}-TR'\mathbf{b}$ and $R^2+(TR')^2=1$. And $\|\mathbf{b}-(\mathbf{b}\cdot\boldsymbol{\gamma})\boldsymbol{\gamma}\| = \sqrt{R^4+(TRR')^2} = R\sqrt{(R^2+(TR')^2)} = R$. So $\mathbf{v} = R\mathbf{b}-TR'\mathbf{n}$. By

$$\boldsymbol{\gamma}\cdot\mathbf{v} = R\boldsymbol{\gamma}\cdot\mathbf{b}-TR'\boldsymbol{\gamma}\cdot\mathbf{n} = R(-TR')-TR'(-R)=0, \tag{4.2}$$

we have

$$\begin{aligned} \widetilde{\mathbf{u}} &= -2(\mathbf{u}\cdot\mathbf{v})\mathbf{v}+\mathbf{u} \qquad (4.3) \\ &= -2\left[\mathbf{u}\cdot(R\mathbf{b}-TR'\mathbf{n})\right](R\mathbf{b}-TR'\mathbf{n})+\mathbf{u} \\ &= \left[-2R(\mathbf{u}\cdot\mathbf{b})+2TR'(\mathbf{u}\cdot\mathbf{n})\right](R\mathbf{b}-TR'\mathbf{n})+\mathbf{u} \\ &= \mathbf{u}-\left\{2TR'\left[TR'(\mathbf{u}\cdot\mathbf{n})-R(\mathbf{u}\cdot\mathbf{b})\right]\mathbf{n}-2R\left[TR'(\mathbf{u}\cdot\mathbf{n})-R(\mathbf{u}\cdot\mathbf{b})\right]\mathbf{b}\right\}. \end{aligned}$$

Now we compute the derivatives of $\mathbf{v}$ and $\mathbf{u}$.

$$\begin{aligned} \mathbf{v}' &= R'\mathbf{b} - \tau R\mathbf{n} - (TR')'\mathbf{n} - TR'(-\kappa\mathbf{t}+\tau\mathbf{b}) \qquad (4.4) \\ &= -\left[\tau R + (TR')'\right]\mathbf{n} + TR'\kappa\mathbf{t} \\ &= TR'\kappa\mathbf{t}, \end{aligned}$$

noting $\tau R + (TR')' = 0$ (see $(8-12)$ in [16, Page 32].)

$$\begin{aligned} \widetilde{\mathbf{u}}' &= (-2)\left[(\mathbf{u}\cdot\mathbf{v}')\mathbf{v} + (\mathbf{u}\cdot\mathbf{v})\mathbf{v}'\right] \qquad (4.5) \\ &= (-2)\left\{TR'\kappa(\mathbf{u}\cdot\mathbf{t})(R\mathbf{b}-TR'\mathbf{n}) + \left[R(\mathbf{u}\cdot\mathbf{b}) - TR'(\mathbf{u}\cdot\mathbf{n})\right]TR'\kappa\mathbf{t}\right\} \\ &= (-2)\left\{TR'\kappa\left[R(\mathbf{u}\cdot\mathbf{b}) - TR'(\mathbf{u}\cdot\mathbf{n})\right]\mathbf{t} - (TR')^2\kappa(\mathbf{u}\cdot\mathbf{t})\mathbf{n} + TR'(\mathbf{u}\cdot\mathbf{t})\mathbf{b}\right\}. \end{aligned}$$

So $\widetilde{\mathbf{u}}' = 0 \Leftrightarrow$

$$\begin{cases} TR'\kappa(R(\mathbf{u}\cdot\mathbf{b}) - TR'(\mathbf{u}\cdot\mathbf{n})) = 0 \\ (TR')^2\kappa(\mathbf{u}\cdot\mathbf{t}) = 0 \\ TR'(\mathbf{u}\cdot\mathbf{t}) = 0. \end{cases} \qquad (4.6)$$

If $TR'(s) = (\mathbf{b}\cdot\boldsymbol{\gamma})(s) = 0$ then $\widetilde{\mathbf{u}}'(s) = 0$. i.e., if $\boldsymbol{\gamma}$ is great at s then $\widetilde{\mathbf{u}}'(s) = 0$ by Proposition 1. If $TR' \neq 0$ then from (4.6) , we have

$$\begin{cases} R(\mathbf{u}\cdot\mathbf{b}) - TR'(\mathbf{u}\cdot\mathbf{n}) = 0 \\ \mathbf{u}\cdot\mathbf{t} = 0 \end{cases} \qquad (4.7)$$

Since $\mathbf{u}\cdot\mathbf{t} = 0$, we can write $\mathbf{u} = u_1\mathbf{n} + u_2\mathbf{b}$. Then

$$R(\mathbf{u}\cdot\mathbf{b}) - TR'(\mathbf{u}\cdot\mathbf{n}) = 0 \;\Rightarrow\; Ru_2 - TR'u_1 = 0 \;\Rightarrow\; u_2 = \frac{TR'}{R}u_1.$$

Furthermore

$$\begin{aligned} & \mathbf{u} \in S^2 \\ \Rightarrow \quad & u_1^2 + u_2^2 = 1 \\ \Rightarrow \quad & \left[(\frac{TR'}{R})^2 + 1 \right] u_1^2 = 1 \\ \Rightarrow \quad & \left[(TR')^2 + R^2 \right] u_1^2 = R^2 \\ \Rightarrow \quad & u_1^2 = R^2 \text{ since } (TR')^2 + R^2 = 1. \end{aligned}$$

Hence

$$\begin{cases} u_1 = \pm R \\ u_2 = \pm TR'. \end{cases}$$

So

$$\mathbf{u} = u_1 \mathbf{n} + u_2 \mathbf{b} = \pm \left(R\mathbf{n} + TR'\mathbf{b} \right) = \mp \boldsymbol{\gamma} \text{ since } \boldsymbol{\gamma} = -(R\mathbf{n} + TR'\mathbf{b}).$$

And obviously $\mathbf{u} = \pm(R\mathbf{n}+TR'\mathbf{b})$ satisfies (4.7). We have proved the following proposition:

Proposition 19 *Let $\boldsymbol{\gamma}$ be a unit speed curve on S^2, and $\mathbf{u} \in S^2$ is not on $\boldsymbol{\gamma}$ or its antipodal image ($\mathbf{u} \notin \pm\boldsymbol{\gamma}$). Let $\widetilde{\mathbf{u}}$ be the spherical orthotomic of $\boldsymbol{\gamma}$ relative to $\mathbf{u}$. Then $\widetilde{\mathbf{u}}'(s) = 0 \Leftrightarrow TR'(s) = (\mathbf{b} \cdot \boldsymbol{\gamma})(s) = 0$, i.e., $\boldsymbol{\gamma}$ is great at s.* ■

4.2 Spherical Orthotomic as the Envelope of a Family of Geodesic Circles Centered at $\gamma(s)$ and Passing through $\mathbf{u}$

First we state a definition about the discriminant set.

Definition 20 *Let $F : (\mathbb{R}\times\mathbb{R}^r, (s_0, \mathbf{x}_0)) \to \mathbb{R}$ be a r-parameter unfolding of $f(s)$. The discriminant set of F is defined by*

$$\mathfrak{D}_F = \left\{ x \in \mathbb{R}^r \mid \textit{there exists } s \textit{ with } F = \frac{\partial F}{\partial s} = 0 \textit{ at } (s, x) \right\}.$$

Let $\gamma = \gamma(s)$ be a unit speed curve on S^2, $\mathbf{u} \in S^2$ and $F(s, \mathbf{x}) = (\gamma - \mathbf{x}) \cdot (\gamma - \mathbf{x}) - (\gamma - \mathbf{u}) \cdot (\gamma - \mathbf{u})$ for $\mathbf{x} \in S^2$. So $F(s, \mathbf{x}) = 0$ is a family of geodesic circles on S^2 centered at $\gamma(s)$ and passing through the point $\mathbf{u}$. Actually they are the intersection of a family of spheres centered at $\gamma(s)$ and passing through the point $\mathbf{u}$ with S^2.

$$\begin{aligned} F(s, \mathbf{x}) &= \gamma \cdot \gamma - 2\mathbf{x} \cdot \gamma + \mathbf{x} \cdot \mathbf{x} - \gamma \cdot \gamma + 2\mathbf{u} \cdot \gamma - \mathbf{u} \cdot \mathbf{u} \\ &= 2(\mathbf{u} - \mathbf{x}) \cdot \gamma \qquad \textit{since } \mathbf{u} \cdot \mathbf{u} = \mathbf{x} \cdot \mathbf{x} = 1. \end{aligned}$$

Then the envelope of the family $F(s, \mathbf{x}) = 0$, i.e., the discriminant set of F is $\mathfrak{D}_F = \{\mathbf{x} \mid F(s, \mathbf{x}) = \frac{\partial F(s,\mathbf{x})}{\partial s} = 0 \text{ for some } s \in I\}$.

$$F(s, \mathbf{x}) = \frac{\partial F(s, \mathbf{x})}{\partial s} = 0 \;\Leftrightarrow\; (\mathbf{u} - \mathbf{x}) \cdot \gamma = (\mathbf{u} - \mathbf{x}) \cdot \mathbf{t} = 0.$$

By $(\mathbf{u} - \mathbf{x}) \cdot \mathbf{t} = 0$ we can write $\mathbf{u} - \mathbf{x} = v_1 \mathbf{n} + v_2 \mathbf{b}$. So

$$(\mathbf{u} - \mathbf{x}) \cdot \gamma = 0 \;\Rightarrow\; v_1 \mathbf{n} \cdot \gamma + v_2 \mathbf{b} \cdot \gamma = 0 \;\Rightarrow\; -Rv_1 - TR'v_2 = 0.$$

On the other hand

$$\begin{aligned} \mathbf{u} - \mathbf{x} = v_1 \mathbf{n} + v_2 \mathbf{b} \;&\Rightarrow\; \mathbf{x} \cdot \mathbf{x} = \mathbf{u} \cdot \mathbf{u} - 2\left[v_1(\mathbf{u} \cdot \mathbf{n}) + v_2(\mathbf{u} \cdot \mathbf{b})\right] + v_1^2 + v_2^2 \\ &\Rightarrow\; v_1^2 + v_2^2 - 2\left[v_1(\mathbf{u} \cdot \mathbf{n}) + v_2(\mathbf{u} \cdot \mathbf{b})\right] = 0 \text{ since } \mathbf{x} \cdot \mathbf{x} = \mathbf{u} \cdot \mathbf{u} = 1. \end{aligned}$$

By $-Rv_1 - TR'v_2 = 0$ we have

$$\begin{aligned} v_2 = \tfrac{-R}{TR'} v_1 \;&\Rightarrow\; v_1^2\left[1 + (\tfrac{-R}{TR'})^2\right] - 2\left[\mathbf{n} \cdot \mathbf{u} - \tfrac{R}{TR'}(\mathbf{u} \cdot \mathbf{b})\right] v_1 = 0 \\ &\Rightarrow\; v_1^2 - 2TR'\left[TR'(\mathbf{u} \cdot \mathbf{n}) - R(\mathbf{u} \cdot \mathbf{b})\right] v_1 = 0 \text{ since } (TR')^2 + R^2 = 1. \end{aligned}$$

Then

$$\begin{cases} v_1 = 0 \\ v_2 = 0 \end{cases} \quad \text{or} \quad \begin{cases} v_1 = 2TR'\left[TR'(\mathbf{u} \cdot \mathbf{n}) - R(\mathbf{u} \cdot \mathbf{b})\right] \\ v_2 = -2R\left[TR'(\mathbf{u} \cdot \mathbf{n}) - R(\mathbf{u} \cdot \mathbf{b})\right]. \end{cases}$$

Therefore,

$$\mathbf{x} = \mathbf{u} \quad \text{or } \mathbf{x} = \mathbf{u} - \left\{2TR'\left[TR'(\mathbf{u}\cdot\mathbf{n}) - R(\mathbf{u}\cdot\mathbf{b})\right]\mathbf{n} - 2R\left[TR'(\mathbf{u}\cdot\mathbf{n}) - R(\mathbf{u}\cdot\mathbf{b})\right]\mathbf{b}\right\}.$$

If

$$\mathbf{u} = \mathbf{u} - \left(2TR'\left[TR'(\mathbf{u}\cdot\mathbf{n}) - R(\mathbf{u}\cdot\mathbf{b})\right]\mathbf{n} - 2R\left[TR'(\mathbf{u}\cdot\mathbf{n}) - R(\mathbf{u}\cdot\mathbf{b})\right]\mathbf{b}\right)(s)$$

for some $s \in \mathbb{R}$,

(here $\left(2TR'\left[TR'(\mathbf{u}\cdot\mathbf{n}) - R(\mathbf{u}\cdot\mathbf{b})\right]\mathbf{n} - 2R\left[TR'(\mathbf{u}\cdot\mathbf{n}) - R(\mathbf{u}\cdot\mathbf{b})\right]\mathbf{b}\right)(s)$ mean the evaluation of $\left(2TR'\left[TR'(\mathbf{u}\cdot\mathbf{n}) - R(\mathbf{u}\cdot\mathbf{b})\right]\mathbf{n} - 2R\left[TR'(\mathbf{u}\cdot\mathbf{n}) - R(\mathbf{u}\cdot\mathbf{b})\right]\mathbf{b}\right)$ at s and we'll use this notation throughout this paper)

then

$$\left[TR'(\mathbf{u}\cdot\mathbf{n}) - R(\mathbf{u}\cdot\mathbf{b})\right](s) = 0,$$

this implies that $\mathbf{u}$ is on the great circle (4.1). So the discriminant set

$$\begin{aligned}&\mathfrak{D}_F \\ =\ &\{\mathbf{u}\} \cup \left\{\mathbf{u} - \left(2TR'\left[TR'(\mathbf{u}\cdot\mathbf{n}) - R(\mathbf{u}\cdot\mathbf{b})\right]\mathbf{n} - 2R\left[TR'(\mathbf{u}\cdot\mathbf{n}) - R(\mathbf{u}\cdot\mathbf{b})\right]\mathbf{b}\right)(s),\ s \in I\right\}.\end{aligned}$$

By (4.3)

$$\widetilde{\mathbf{u}} = \mathbf{u} - \left\{2TR'\left[TR'(\mathbf{u}\cdot\mathbf{n}) - R(\mathbf{u}\cdot\mathbf{b})\right]\mathbf{n} - 2R\left[TR'(\mathbf{u}\cdot\mathbf{n}) - R(\mathbf{u}\cdot\mathbf{b})\right]\mathbf{b}\right\}.$$

So the discriminant set of F consists of the point $\mathbf{u}$ and the spherical orthotomic of γ relative to $\mathbf{u}$. i.e.,

$$F(s_0, \mathbf{x}) = \frac{\partial F(s_0, \mathbf{x})}{\partial s} = 0 \;\Leftrightarrow\; \mathbf{x} = \mathbf{u} \text{ or } \mathbf{x} = (-2(\mathbf{u}\cdot\mathbf{v})\mathbf{v} + \mathbf{u})(s_0) \tag{4.8}$$

$$
\begin{aligned}
& F(s_0,\mathbf{x}) = \frac{\partial F(s_0,\mathbf{x})}{\partial s} = \frac{\partial^2 F(s_0,\mathbf{x})}{\partial s^2} = 0 \\
\Leftrightarrow\ & (\mathbf{u}-\mathbf{x})\cdot\boldsymbol{\gamma}(s_0) = (\mathbf{u}-\mathbf{x})\cdot\mathbf{t}(s_0) = \kappa(\mathbf{u}-\mathbf{x})\cdot\mathbf{n}(s_0) = 0 \\
\Leftrightarrow\ & \mathbf{x} = \mathbf{u} \text{ or} \\
& \mathbf{x} = \mathbf{u} - \left(2TR'\left[TR'(\mathbf{u}\cdot\mathbf{n}) - R(\mathbf{u}\cdot\mathbf{b})\right]\mathbf{n} - 2R\left[TR'(\mathbf{u}\cdot\mathbf{n}) - R(\mathbf{u}\cdot\mathbf{b})\right]\mathbf{b}\right)(s_0) \\
& \quad \text{and } 2TR'\left(TR'(\mathbf{u}\cdot\mathbf{n}) - R(\mathbf{u}\cdot\mathbf{b})\right)(s_0) = 0 \\
\Leftrightarrow\ & \mathbf{x} = \mathbf{u} \text{ or } \mathbf{x} = \mathbf{u} - 2\left(R^2(\mathbf{u}\cdot\mathbf{b})\mathbf{b}\right)(s_0) \text{ and } TR'(s_0) = 0
\end{aligned}
$$

i.e.,

$$
\begin{aligned}
& F(s_0,\mathbf{x}) = \frac{\partial F(s_0,\mathbf{x})}{\partial s} = \frac{\partial^2 F(s_0,\mathbf{x})}{\partial s^2} = 0 \\
\Leftrightarrow\ & \mathbf{x} = \mathbf{u} \text{ or } \mathbf{x} = \left(-2(\mathbf{u}\cdot\mathbf{v})\mathbf{v} + \mathbf{u}\right)(s_0) = \mathbf{u} - 2\left(R^2(\mathbf{u}\cdot\mathbf{b})\mathbf{b}\right)(s_0) \text{ and } TR'(s_0) = 0 \\
\Leftrightarrow\ & \mathbf{x} = \mathbf{u} \text{ or } \mathbf{x} = \left(-2(\mathbf{u}\cdot\mathbf{v})\mathbf{v} + \mathbf{u}\right)(s_0) = \mathbf{u} - 2\left(R^2(\mathbf{u}\cdot\mathbf{b})\mathbf{b}\right)(s_0) \text{ and } \boldsymbol{\gamma} \text{ is great at } s_0.
\end{aligned}
$$

and

$$
\begin{aligned}
& F(s_0,\mathbf{x}) = \frac{\partial F(s_0,\mathbf{x})}{\partial s} = \frac{\partial^2 F(s_0,\mathbf{x})}{\partial s^2} = \frac{\partial^3 F(s_0,\mathbf{x})}{\partial s^3} = 0 \\
\Leftrightarrow\ & (\mathbf{u}-\mathbf{x})\cdot\boldsymbol{\gamma}(s_0) = (\mathbf{u}-\mathbf{x})\cdot\mathbf{t}(s_0) = \left(\kappa(\mathbf{u}-\mathbf{x})\cdot\mathbf{n}\right)(s_0) \\
& = \left(\kappa'(\mathbf{u}-\mathbf{x})\cdot\mathbf{n} + \kappa(\mathbf{u}-\mathbf{x})\cdot(-\kappa\mathbf{t} + \tau\mathbf{b})\right)(s_0) = 0 \\
\Leftrightarrow\ & (\mathbf{u}-\mathbf{x})\cdot\boldsymbol{\gamma}(s_0) = (\mathbf{u}-\mathbf{x})\cdot\mathbf{t}(s_0) = (\mathbf{u}-\mathbf{x})\cdot\mathbf{n}(s_0) \\
& = (\mathbf{u}-\mathbf{x})\cdot(-\kappa\mathbf{t} + \tau\mathbf{b})(s_0) = 0 \\
\Leftrightarrow\ & \mathbf{x} = \mathbf{u}, \text{ or } \mathbf{x} = \left(-2(\mathbf{u}\cdot\mathbf{v})\mathbf{v} + \mathbf{u}\right)(s_0) = \mathbf{u} - 2\left(R^2(\mathbf{u}\cdot\mathbf{b})\mathbf{b}\right)(s_0), \\
& TR'(s_0) = 0 \text{ and } -2\left(R^2(\mathbf{u}\cdot\mathbf{b})\tau\right)(s_0) = 0 \\
\Leftrightarrow\ & \mathbf{x} = \mathbf{u}, \text{or } \mathbf{x} = \mathbf{u} - 2\left(R^2(\mathbf{u}\cdot\mathbf{b})\mathbf{b}\right)(s_0),\ \boldsymbol{\gamma} \text{ is great at } s_0 \text{ and } \tau(s_0) = 0,
\end{aligned}
$$

i.e.,

$$
\begin{aligned}
& F(s_0,\mathbf{x}) = \frac{\partial F(s_0,\mathbf{x})}{\partial s} = \frac{\partial^2 F(s_0,\mathbf{x})}{\partial s^2} = \frac{\partial^3 F(s_0,\mathbf{x})}{\partial s^3} = 0 \\
\Leftrightarrow\ & \mathbf{x} = \mathbf{u} \text{ or } \mathbf{x} = \mathbf{u} - 2\left(R^2(\mathbf{u}\cdot\mathbf{b})\mathbf{b}\right)(s_0),\ \boldsymbol{\gamma} \text{ is great at } s_0 \text{ and } \tau(s_0) = 0.
\end{aligned}
$$

Obviously, if $\mathbf{x} = \mathbf{u}$ then $F(s, \mathbf{x}) = 0$ and there is no $k \geq 1$ such that $\frac{\partial^k F(s_0, \mathbf{u})}{\partial s^k} \neq 0$ for any $s_0 \in \mathbb{R}$. So from the above we have the following proposition:

Proposition 21 *Given a unit speed curve γ on S^2 and a point $\mathbf{u} \in S^2$. Let*

$$F(s, \mathbf{x}) = (\gamma(s) - \mathbf{x}) \cdot (\gamma(s) - \mathbf{x}) - (\gamma(s) - \mathbf{u}) \cdot (\gamma(s) - \mathbf{u}), \ \mathbf{x} \in S^2.$$

For $\mathbf{x}_0 \in S^2$ define $f(s) = F_{\mathbf{x}_0}(s) = F(s, \mathbf{x}_0)$. Then

f has the A_1-singularity at s_0

$\Leftrightarrow$ $\mathbf{x}_0 = \mathbf{u} - (2TR'[TR'(\mathbf{u} \cdot \mathbf{n}) - R(\mathbf{u} \cdot \mathbf{b})]\,\mathbf{n} - 2R\,[TR'(\mathbf{u} \cdot \mathbf{n}) - R(\mathbf{u} \cdot \mathbf{b})]\,\mathbf{b})\,(s_0)$ *but* $\mathbf{x}_0 \neq \mathbf{u}$ *and γ is not great at s_0*

$\Leftrightarrow$ $\mathbf{x}_0 = \mathbf{u} - (2TR'[TR'(\mathbf{u} \cdot \mathbf{n}) - R(\mathbf{u} \cdot \mathbf{b})]\,\mathbf{n} - 2R\,[TR'(\mathbf{u} \cdot \mathbf{n}) - R(\mathbf{u} \cdot \mathbf{b})]\,\mathbf{b})\,(s_0)$ *but* $(TR'(\mathbf{u} \cdot \mathbf{n}) - R(\mathbf{u} \cdot \mathbf{b}))\,(s_0) \neq 0$ *and γ is not great at s_0;*

and

f has the A_2-singularity at s_0

$\Leftrightarrow$ $\mathbf{x}_0 = \mathbf{u} - \left(2R^2(\mathbf{u} \cdot \mathbf{b})\mathbf{b}\right)(s_0)$ *but* $(R(\mathbf{u} \cdot \mathbf{b}))\,(s_0) \neq 0$, *$\gamma$ is great at s_0 and $\tau(s_0) \neq 0$.*

■

4.3 Local Diffeomorphic Image of the Spherical Orthotomic

In order to investigate the local diffeomorphic image of the spherical orthotomic, we need Theorem 6.10 in [3].

Theorem 22 *(6.10 in [3]) Let $F : (\mathbb{R} \times \mathbb{R}^r, (s_0, \mathbf{x}_0)) \to \mathbb{R}$ be an r-parameter unfolding of $f(s) = F_{\mathbf{x}_0}(s) = F(s, \mathbf{x}_0)$ and assume f has the A_k-singularity at s_0 $(k \geq 1)$. Let $\mathbf{x} = (x_1, \ldots, x_r)$ and the $(k-1)$-jet with constant of $\partial F / \partial x_i$ at $\mathbf{x}_0$ be $j^{k-1}(\partial F / \partial x_i(s, \mathbf{x}_0))(s_0) =$*

$\alpha_{0i} + \alpha_{1i}s + \alpha_{2i}s^2 + \cdots + \alpha_{k-1,i}s^{k-1}$ for $i = 1, \ldots, r$. Then $F(s, \mathbf{x})$ is versal iff the $k \times r$ matrix of coefficients (α_{ji}) for $j = 0, \ldots, k-1$; $i = 1, \ldots, r$ has rank k. (This certainly requires $k \leq r$, so the smallest possible value of r is k.)

Now we will investigate if $F(s, \mathbf{x}) = 2(\mathbf{u} - \mathbf{x}) \cdot \boldsymbol{\gamma}$ is a versal unfolding of $f(s) = F_{\mathbf{x}_0}(s) = 2(\mathbf{u} - \mathbf{x}_0) \cdot \boldsymbol{\gamma}$ for $\mathbf{u} \notin \pm\boldsymbol{\gamma}$.

Theorem 23 *For the unit speed curve $\boldsymbol{\gamma} = (r_1(s), r_2(s), r_3(s))$ on the unit sphere S^2 and $k = 1, 2$, if $f(s)$ as above has the A_k-singularity at $s_0 \in I$ and $\mathbf{x}_0 \neq \mathbf{u}$, then F is the versal unfolding of f.*

Proof. Since $\mathbf{x} = (x_1, x_2, x_3) \in S^2$, $x_1^2 + x_2^2 + x_3^2 = 1$. x_1, x_2 and x_3 can't be all zero. Without loss of generality, we suppose $x_3 \neq 0$. Then by $x_3 = \pm\sqrt{1 - (x_1^2 + x_2^2)}$, we have

$$\begin{aligned} & F(s, \mathbf{x}) = 2(\mathbf{u} - \mathbf{x}) \cdot \boldsymbol{\gamma}(s) \\ \Rightarrow \quad & \frac{\partial F}{\partial x_1} = -2r_1(s) + \frac{2x_1}{x_3} r_3(s), \frac{\partial F}{\partial x_2} = -2r_2(s) + \frac{2x_2}{x_3} r_3(s). \end{aligned}$$

Let $\mathbf{x}_0 = (x_{01}, x_{02}, x_{03}) \in S^2$ and assume $x_{03} \neq 0$, then

$$\begin{aligned} & j^1 \left(\partial F / \partial x_1 (s, \mathbf{x}_0) \right) (s_0) \\ = \quad & -2r_1(s_0) + \frac{2x_{01}}{x_{03}} r_3(s_0) - 2r_1'(s_0)s + \frac{2x_{01}}{x_{03}} r_3'(s_0)s, \end{aligned}$$

and

$$\begin{aligned} & j^1 \left(\partial F / \partial x_2 (s, \mathbf{x}_0) \right) (s_0) \\ = \quad & -2r_2(s_0) + \frac{2x_{02}}{x_{03}} r_3(s_0) - 2r_2'(s_0)s + \frac{2x_{02}}{x_{03}} r_3'(s_0)s. \end{aligned}$$

So the $(2-1) \times 2$ matrix $M_1 = (\alpha_{11}, \alpha_{12})$ of coefficients α as in Theorem 22 (6.10 in [3]) is

$$M_1 = \left[-2r_1(s_0) + \frac{2x_{01}}{x_{03}} r_3(s_0), -2r_2(s_0) + \frac{2x_{02}}{x_{03}} r_3(s_0) \right];$$

the $(3-1)\times 2$ matrix M_2 of coefficients α as in Theorem 22 (6.10 in [3]) is

$$M_2 = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix} = \begin{bmatrix} -2r_1(s_0) + \frac{2x_{01}}{x_{03}} r_3(s_0) & -2r_2(s_0) + \frac{2x_{02}}{x_{03}} r_3(s_0) \\ -2r_1'(s_0) + \frac{2x_{01}}{x_{03}} r_3'(s_0) & -2r_2'(s_0) + \frac{2x_{02}}{x_{03}} r_3'(s_0) \end{bmatrix}.$$

When f has the A_k-singularity ($k = 1, 2$) at $s_0 \in I$, then $F_{\mathbf{x}_0}(s) = F'_{\mathbf{x}_0}(s) = 0$ and by (4.8) $\mathbf{x}_0 = \mathbf{u}$ or

$$\mathbf{x}_0 = \mathbf{u} - \left(2TR'\left[TR'(\mathbf{u}\cdot\mathbf{n}) - R(\mathbf{u}\cdot\mathbf{b})\right]\mathbf{n} - 2R\left[TR'(\mathbf{u}\cdot\mathbf{n}) - R(\mathbf{u}\cdot\mathbf{b})\right]\mathbf{b}\right)(s_0).$$

Now we need to find the condition for M_2 to be nonsingular.

$$\begin{aligned} \det M_2 &= 4\left[r_1(s_0)r_2'(s_0) - r_1'(s_0)r_2(s_0\right] \\ &\quad -\frac{x_{02}}{x_{03}}\left[r_1(s_0)r_3'(s_0) - r_1'(s_0)r_3(s_0)\right] + \frac{x_{01}}{x_{03}}\left[r_2(s_0)r_3'(s_0) - r_2'(s_0)r_3(s_0)\right] \\ &= \frac{4}{x_{03}}\det\left(\mathbf{x}_0, \gamma(s_0), \gamma'(s_0)\right) \\ &= \frac{4}{x_{03}}\det\left(\mathbf{x}_0, \gamma(s_0), \mathbf{t}(s_0)\right) \end{aligned}$$

Therefore,

$$M_2 \text{ is singular } \Leftrightarrow \mathbf{x}_0 = \lambda\gamma(s_0) + \mu\mathbf{t}(s_0) \text{ for some } \lambda, \mu \in \mathbb{R}.$$

Now suppose $\mathbf{x}_0 \neq \mathbf{u}$, then

$$\mathbf{x}_0 = \mathbf{u} - \left(2TR'\left[TR'(\mathbf{u}\cdot\mathbf{n}) - R(\mathbf{u}\cdot\mathbf{b})\right]\mathbf{n} - 2R\left[TR'(\mathbf{u}\cdot\mathbf{n}) - R(\mathbf{u}\cdot\mathbf{b})\right]\mathbf{b}\right)(s_0) \tag{4.9}$$

because $F_{\mathbf{x}_0}(s) = F'_{\mathbf{x}_0}(s) = 0$.

So if M_2 is singular then we have

$$\begin{aligned} &\lambda\gamma(s_0) + \mu\mathbf{t}(s_0) \\ = \;& \mathbf{u} - \left(2TR'\left[TR'(\mathbf{u}\cdot\mathbf{n}) - R(\mathbf{u}\cdot\mathbf{b})\right]\mathbf{n} - 2R\left[TR'(\mathbf{u}\cdot\mathbf{n}) - R(\mathbf{u}\cdot\mathbf{b})\right]\mathbf{b}\right)(s_0) \end{aligned}$$

$\Longrightarrow$

$$
\begin{aligned}
& \left(\lambda(-R\mathbf{n}+TR'\mathbf{b})+\mu\mathbf{t}\right)(s_0) \\
= \; & ((\mathbf{u}\cdot\mathbf{t})\mathbf{t}+(\mathbf{u}\cdot\mathbf{n})\mathbf{n}+(\mathbf{u}\cdot\mathbf{b})\mathbf{b} \\
& -\left[2TR'\left[TR'(\mathbf{u}\cdot\mathbf{n})-R(\mathbf{u}\cdot\mathbf{b})\right]\mathbf{n}-2R\left[TR'(\mathbf{u}\cdot\mathbf{n})-R(\mathbf{u}\cdot\mathbf{b})\right]\mathbf{b}\right])(s_0)
\end{aligned}
$$

$\Longrightarrow$

$$
\begin{aligned}
& (-\lambda R\mathbf{n}+\lambda TR'\mathbf{b}+\mu\mathbf{t})(s_0) \\
= \; & ((\mathbf{u}\cdot\mathbf{t})\mathbf{t}+\left\{\mathbf{u}\cdot\mathbf{n}-2TR'\left[TR'(\mathbf{u}\cdot\mathbf{n})-R(\mathbf{u}\cdot\mathbf{b})\right]\right\}\mathbf{n} \\
& +\left\{\mathbf{u}\cdot\mathbf{b}+2R\left[TR'(\mathbf{u}\cdot\mathbf{n})-R(\mathbf{u}\cdot\mathbf{b})\right]\right\}\mathbf{b})(s_0)
\end{aligned}
$$

$\Longrightarrow$

$$
\begin{cases}
\mathbf{t}(s_0)\cdot\mathbf{u}=\mu \\
\{\mathbf{u}\cdot\mathbf{n}-2TR'[TR'(\mathbf{u}\cdot\mathbf{n})-R(\mathbf{u}\cdot\mathbf{b})]\}(s_0)=-\lambda R(s_0) \\
\{\mathbf{u}\cdot\mathbf{b}+2R[TR'(\mathbf{u}\cdot\mathbf{n})-R(\mathbf{u}\cdot\mathbf{b})]\}(s_0)=\lambda TR'(s_0)
\end{cases}
$$

$\Longrightarrow$

$$
\mathbf{u}\cdot\mathbf{t}(s_0)=\mu, \mathbf{u}\cdot\mathbf{n}(s_0)=-\lambda R(s_0), \mathbf{u}\cdot\mathbf{b}(s_0)=-\lambda TR'(s_0)
$$

$\Longrightarrow$

$$
\begin{aligned}
\mathbf{u} &= ((\mathbf{u}\cdot\mathbf{t})\mathbf{t}+(\mathbf{u}\cdot\mathbf{n})\mathbf{n}+(\mathbf{u}\cdot\mathbf{b})\mathbf{b})(s_0) \\
&= \mu\mathbf{t}(s_0)-\lambda R(s_0)\mathbf{n}-\lambda TR'(s_0)\mathbf{b} \\
&= \mu\mathbf{t}(s_0)+\lambda\gamma(s_0)=\mathbf{x}_0,
\end{aligned}
$$

i.e., $\mathbf{u}=\mathbf{x}_0$. This contradicts the assumption $\mathbf{x}_0\neq\mathbf{u}$. So when $\mathbf{x}_0\neq\mathbf{u}$, M_2 is nonsingular. And M_1 has rank 1. So by Theorem 22 (6.10 in [3]) F is the versal unfolding of f. ■

Here we need a theorem in [3] (see 6.16, 6.17, 6.18 in [3]):

Theorem 24 *Let $F : (\mathbb{R}\times\mathbb{R}^r, (s_0, \mathbf{x}_0)) \to \mathbb{R}$ be an r-parameter unfolding of $f(s)$ which has the A_k-singularity at s_0. Suppose that F is a versal unfolding.*

(a) if $k = 1$, then $\mathfrak{D}_F$ is locally diffeomorphic to $\{0\}\times\mathbb{R}^{r-1}$;

(b) if $k = 2$, then $\mathfrak{D}_F$ is locally diffeomorphic to $C\times\mathbb{R}^{r-2}$;

(c) if $k = 3$, then $\mathfrak{D}_F$ is locally diffeomorphic to $SW\times\mathbb{R}^{r-3}$,

where $C = \{(x_1, x_2) | \, x_1^2 = x_2^3\}$ and $SW = \{(x_1, x_2, x_3) | \, x_1 = 3u^4 + u^2v, x_2 = 4u^3 + 2uv, x_3 = v\}$.

Then by (4.8), Theorem 22, 23, 24 and Proposition 21 we have:

Theorem 25 *Let γ be a unit speed curve on S^2 and $\mathbf{u} \in S^2$, then the spherical orthotomic of γ relative to $\mathbf{u}$ is, around the point corresponding to $s = s_0$,*

(a) locally diffeomorphic to a line if γ is not great at s_0 and

$(TR'(\mathbf{u}\cdot\mathbf{n}) - R(\mathbf{u}\cdot\mathbf{b}))\,(s_0) \neq 0$;

(b) locally diffeomorphic to the ordinary cusp C if γ is great at s_0, but

$(R(\mathbf{u}\cdot\mathbf{b}))(s_0) \neq 0$ *and* $\tau(s_0) \neq 0$.

4.4 Spherical Antiorthotomic

Definition 26 *Given a spherical curve γ and a point $\mathbf{u} \in S^2$, a spherical curve $\boldsymbol{\delta}$ is called the spherical antiorthotomic of γ relative to $\mathbf{u}$ if γ is the spherical orthotomic of $\boldsymbol{\delta}$ relative to $\mathbf{u}$.*

By the definitions of the spherical orthotomic and the spherical antiorthotomic, we know the spherical antiorthotomic of γ relative to $\mathbf{u}$ is just the envelope of the family of the great circles the planes of which perpendicularly bisect the chords connecting the points of γ and $\mathbf{u}$. These great circles are given by the following:

$$\begin{cases} \mathbf{x}\cdot\mathbf{x} = 1, \\ \mathbf{x}\cdot(\gamma - \mathbf{u}) = 0. \end{cases} \tag{4.10}$$

Let $F(s, \mathbf{x}) = \mathbf{x}\cdot(\gamma - \mathbf{u})$ for $\mathbf{x} \in S^2$. Then for $\mathbf{x} \in S^2$, $F(s, \mathbf{x}) = 0$ defines the family of the great circles in (4.10). Then the envelope of the family of great circles $F(s, \mathbf{x}) = 0$, i.e., the discriminant set $\mathfrak{D}_F$ of F is given by

$$\begin{cases} \mathbf{x}\cdot\mathbf{x} = 1, \\ \mathbf{x}\cdot(\gamma - \mathbf{u}) = 0, \\ \mathbf{x}\cdot\mathbf{t} = 0. \end{cases}$$

So we can write $\mathbf{x} = x_1\mathbf{n} + x_2\mathbf{b}$. Thus

$$\begin{aligned} & \mathbf{x}\cdot(\gamma - \mathbf{u}) = 0 \\ \Rightarrow\quad & (x_1\mathbf{n} + x_2\mathbf{b})\cdot(\gamma - \mathbf{u}) = 0 \\ \Rightarrow\quad & x_1\mathbf{n}\cdot\gamma - x_1(\mathbf{u}\cdot\mathbf{n}) + x_2\mathbf{b}\cdot\gamma - x_2(\mathbf{u}\cdot\mathbf{b}) = 0 \\ \Rightarrow\quad & -x_1R - x_1(\mathbf{u}\cdot\mathbf{n}) - x_2TR' - x_2(\mathbf{u}\cdot\mathbf{b}) = 0 \\ \Rightarrow\quad & x_1(R + \mathbf{u}\cdot\mathbf{n}) = -(TR' + \mathbf{u}\cdot\mathbf{b})x_2. \end{aligned}$$

Then

$$\begin{aligned} & x_1^2 + x_2^2 = 1 \\ \Rightarrow\quad & \left[(TR' + \mathbf{u}\cdot\mathbf{b})^2 + (R + \mathbf{u}\cdot\mathbf{n})^2\right] x_2^2 = (R + \mathbf{u}\cdot\mathbf{n})^2 \\ \Rightarrow\quad & \left[(TR')^2 + (\mathbf{u}\cdot\mathbf{b})^2 + 2TR'(\mathbf{u}\cdot\mathbf{b}) + R^2 + (\mathbf{u}\cdot\mathbf{n})^2 + 2R(\mathbf{u}\cdot\mathbf{n})\right] x_2^2 = (R + \mathbf{u}\cdot\mathbf{n})^2 \\ \Rightarrow\quad & \left[1 + 1 - (\mathbf{u}\cdot\mathbf{t})^2 - 2(\mathbf{u}\cdot\gamma)\right] x_2^2 = (R + \mathbf{u}\cdot\mathbf{n})^2 \end{aligned}$$

since $(TR')^2 + R^2 = 1, (\mathbf{u}\cdot\mathbf{t})^2 + (\mathbf{u}\cdot\mathbf{n})^2 + (\mathbf{u}\cdot\mathbf{b})^2 = 1$

and $\mathbf{u}\cdot\gamma = [(\mathbf{u}\cdot\mathbf{t})\mathbf{t} + (\mathbf{u}\cdot\mathbf{n})\mathbf{n} + (\mathbf{u}\cdot\mathbf{b})\mathbf{b}]\cdot(-R\mathbf{n} - TR'\mathbf{b}) = -TR'(\mathbf{u}\cdot\mathbf{b}) - R(\mathbf{u}\cdot\mathbf{n})$.

And

$$(TR' + \mathbf{u}\cdot\mathbf{b})^2 + (R + \mathbf{u}\cdot\mathbf{n})^2 = 0 \quad \Leftrightarrow \quad \mathbf{n}\cdot(\boldsymbol{\gamma} - \mathbf{u}) = \mathbf{b}\cdot(\boldsymbol{\gamma} - \mathbf{u}) = 0$$

$$\Leftrightarrow \quad \boldsymbol{\gamma} - \mathbf{u} \parallel \mathbf{t}$$

$$\Leftrightarrow \quad \boldsymbol{\gamma} - \mathbf{u} = \lambda\mathbf{t} \text{ for some } \lambda\in\mathbb{R}.$$

$$\Leftrightarrow \quad \mathbf{u}\cdot\mathbf{t} = -\lambda,\ \mathbf{u}\cdot\mathbf{n} = -R \text{ and } \mathbf{u}\cdot\mathbf{b} = -TR'.$$

So $\mathbf{u}\cdot\mathbf{t} = 0$ by $(\mathbf{u}\cdot\mathbf{t})^2 + (\mathbf{u}\cdot\mathbf{n})^2 + (\mathbf{u}\cdot\mathbf{b})^2 = 1$ and $(TR')^2 + R^2 = 1$. Hence $\mathbf{u} = -R\mathbf{n} - TR'\mathbf{b} = \boldsymbol{\gamma}$.

If $\mathbf{u}$ is not on $\boldsymbol{\gamma}$, i.e., $2 - (\mathbf{u}\cdot\mathbf{t})^2 - 2(\mathbf{u}\cdot\boldsymbol{\gamma}) \neq 0$, we have

$$\begin{cases} x_1 = \mp\dfrac{TR' + \mathbf{u}\cdot\mathbf{b}}{\sqrt{2 - (\mathbf{u}\cdot\mathbf{t})^2 - 2(\mathbf{u}\cdot\boldsymbol{\gamma})}}, \\ x_2 = \pm\dfrac{R + \mathbf{u}\cdot\mathbf{n}}{\sqrt{2 - (\mathbf{u}\cdot\mathbf{t})^2 - 2(\mathbf{u}\cdot\boldsymbol{\gamma})}}. \end{cases}$$

Thus the spherical antiorthotomic of $\boldsymbol{\gamma}$ relative to $\mathbf{u}$ is:

$$\boldsymbol{\delta} = \pm\left[\frac{TR' + \mathbf{u}\cdot\mathbf{b}}{\sqrt{2 - (\mathbf{u}\cdot\mathbf{t})^2 - 2(\mathbf{u}\cdot\boldsymbol{\gamma})}}\mathbf{n} - \frac{R + \mathbf{u}\cdot\mathbf{n}}{\sqrt{2 - (\mathbf{u}\cdot\mathbf{t})^2 - 2(\mathbf{u}\cdot\boldsymbol{\gamma})}}\mathbf{b}\right]. \tag{4.11}$$

We have

$$\begin{aligned} & F(s_0, \mathbf{x}) = \frac{\partial F(s_0, \mathbf{x})}{\partial s} = 0 \\ \Leftrightarrow \quad & \mathbf{x} = \pm\left(\frac{TR' + \mathbf{u}\cdot\mathbf{b}}{\sqrt{2 - (\mathbf{u}\cdot\mathbf{t})^2 - 2(\mathbf{u}\cdot\boldsymbol{\gamma})}}\mathbf{n} - \frac{R + \mathbf{u}\cdot\mathbf{n}}{\sqrt{2 - (\mathbf{u}\cdot\mathbf{t})^2 - 2(\mathbf{u}\cdot\boldsymbol{\gamma})}}\mathbf{b}\right)(s_0), \end{aligned}$$

$$F(s_0, \mathbf{x}) = \frac{\partial F(s_0, \mathbf{x})}{\partial s} = \frac{\partial^2 F(s_0, \mathbf{x})}{\partial s^2} = 0 \Leftrightarrow \mathbf{x}\cdot(\boldsymbol{\gamma} - \mathbf{u}) = \mathbf{x}\cdot\mathbf{t} = \mathbf{x}\cdot\kappa\mathbf{n} = 0$$

$$\Leftrightarrow \mathbf{x} = \pm\mathbf{b}(s_0) \text{ and } (TR' + \mathbf{u}\cdot\mathbf{b})(s_0) = 0,$$

and

$$F(s_0, \mathbf{x}) = \frac{\partial F(s_0, \mathbf{x})}{\partial s} = \frac{\partial^2 F(s_0, \mathbf{x})}{\partial s^2} = \frac{\partial^3 F(s_0, \mathbf{x})}{\partial s^3} = 0$$

$$\Leftrightarrow \quad \mathbf{x}\cdot(\boldsymbol{\gamma} - \mathbf{u}) = \mathbf{x}\cdot\mathbf{t} = \mathbf{x}\cdot\kappa\mathbf{n} = \mathbf{x}\cdot(\kappa'\mathbf{n} - \kappa^2\mathbf{t} + \kappa\tau\mathbf{b}) = 0$$

$$\Leftrightarrow \quad \mathbf{x} = \pm\mathbf{b}(s_0) \text{ and } (TR' + \mathbf{u}\cdot\mathbf{b})(s_0) = \tau(s_0) = 0.$$

Let $f(s) = F_{\mathbf{x}_0}(s) = F(s, \mathbf{x}_0)$, then from the above we have

Proposition 27 *Given a unit speed curve γ on S^2 and a point $\mathbf{u} \in S^2$ which is not on γ and let $F(s, \mathbf{x}) = \mathbf{x} \cdot (\gamma - \mathbf{u})$, $\mathbf{x} \in S^2$. For $\mathbf{x}_0 \in S^2$ define $f(s) = F_{\mathbf{x}_0}(s) = F(s, \mathbf{x}_0)$. Then*

$$\begin{aligned} & f \text{ has the } A_1\text{-singularity at } s_0 \\ \Leftrightarrow \quad & \mathbf{x}_0 = \pm \left(\frac{TR' + \mathbf{u} \cdot \mathbf{b}}{\sqrt{2 - (\mathbf{u} \cdot \mathbf{t})^2 - 2(\mathbf{u} \cdot \gamma)}} \mathbf{n} - \frac{R + \mathbf{u} \cdot \mathbf{n}}{\sqrt{2 - (\mathbf{u} \cdot \mathbf{t})^2 - 2(\mathbf{u} \cdot \gamma)}} \mathbf{b} \right) (s_0) \\ & \text{but } (TR' + \mathbf{u} \cdot \mathbf{b})(s_0) \neq 0. \end{aligned}$$

and

$$\begin{aligned} & f \text{ has the } A_2\text{-singularity at } s_0 \\ \Leftrightarrow \quad & \mathbf{x}_0 = \pm \mathbf{b}(s_0) \text{ and } (TR' + \mathbf{u} \cdot \mathbf{b})(s_0) = 0 \text{ but } \tau(s_0) \neq 0. \end{aligned}$$

■

4.5 Local Diffeomorphic Image of the Spherical Antiorthotomic

Theorem 28 *For the unit speed curve $\gamma = (r_1(s), r_2(s), r_3(s))$ on the unit sphere S^2 and a point $\mathbf{u} \in S^2$ which is not on γ (i.e., $2 - (\mathbf{u} \cdot \mathbf{t})^2 - 2(\mathbf{u} \cdot \gamma) \neq 0$), $k = 1, 2$, if $f(s) = \mathbf{x}_0 \cdot (\gamma(s) - \mathbf{u})$ has the A_k-singularity at $s_0 \in I$ then F is the versal unfolding of f.*

Proof. Since $\mathbf{x} = (x_1, x_2, x_3) \in S^2$, $x_1^2 + x_2^2 + x_3^2 = 1$. x_1, x_2 and x_3 can't be all zero. Without loss of generality, we suppose $x_3 \neq 0$. Let $\mathbf{u} = (u_1, u_2, u_3) \in S^2$. Then by $x_3 = \pm\sqrt{1 - (x_1^2 + x_2^2)}$, we have

$$\begin{aligned} & F(s, \mathbf{x}) = \mathbf{x} \cdot (\gamma - \mathbf{u}) \\ \Rightarrow \quad & \begin{cases} \dfrac{\partial F}{\partial x_1} = r_1(s) - u_1 - \dfrac{x_1}{x_3} [r_3(s) - u_3] \\ \dfrac{\partial F}{\partial x_2} = r_2(s) - u_2 - \dfrac{x_2}{x_3} [r_3(s) - u_3] \end{cases} . \end{aligned}$$

Let $\mathbf{x}_0 = (x_{01}, x_{02}, x_{03}) \in S^2$ and assume $x_{03} \neq 0$, then

$$\begin{aligned} & j^1 \left(\partial F/\partial x_1(s, \mathbf{x}_0)\right)(s_0) \\ = \; & r_1(s_0) - u_1 - \frac{x_{01}}{x_{03}} \left[r_3(s_0) - u_3\right] + r_1'(s_0)s - \frac{x_{01}}{x_{03}} r_3'(s_0)s, \end{aligned}$$

and

$$\begin{aligned} & j^1 \left(\partial F/\partial x_2(s, \mathbf{x}_0)\right)(s_0) \\ = \; & r_2(s_0) - u_2 - \frac{x_{02}}{x_{03}} \left[r_3(s_0) - u_3\right] + r_2'(s_0)s - \frac{x_{02}}{x_{03}} r_3'(s_0)s. \end{aligned}$$

So the $(2-1) \times 2$ matrix $M_1 = (\alpha_{11}, \alpha_{12})$ of coefficients α as in Theorem 22 (6.10 in [3]) is

$$M_1 = \left[r_1(s_0) - u_1 - \frac{x_{01}}{x_{03}} \left(r_3(s_0) - u_3\right),\; r_2(s_0) - u_2 - \frac{x_{02}}{x_{03}} \left(r_3(s_0) - u_3\right) \right];$$

the $(3-1) \times 2$ matrix M_2 of coefficients α as in Theorem 22 (6.10 in [3]) is

$$M_2 = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix} = \begin{bmatrix} r_1(s_0) - u_1 - \frac{x_{01}}{x_{03}} \left(r_3(s_0) - u_3\right) & r_2(s_0) - u_2 - \frac{x_{02}}{x_{03}} \left(r_3(s_0) - u_3\right) \\ r_1'(s_0) - \frac{x_{01}}{x_{03}} r_3'(s_0) & r_2'(s_0) - \frac{x_{02}}{x_{03}} r_3'(s_0) \end{bmatrix}.$$

When f has the A_k-singularity $(k = 1, 2)$ at $s_0 \in I$, $F_{\mathbf{x}_0}(s) = F'_{\mathbf{x}_0}(s) = 0$ and by Proposition 27 $\mathbf{x}_0 = \pm \left(\frac{TR' + \mathbf{u} \cdot \mathbf{b}}{\sqrt{2 - (\mathbf{u} \cdot \mathbf{t})^2 - 2(\mathbf{u} \cdot \boldsymbol{\gamma})}} \mathbf{n} - \frac{R + \mathbf{u} \cdot \mathbf{n}}{\sqrt{2 - (\mathbf{u} \cdot \mathbf{t})^2 - 2(\mathbf{u} \cdot \boldsymbol{\gamma})}} \mathbf{b} \right)(s_0)$.

Now we need to find the condition for M_2 to be nonsingular.

$$\begin{aligned} \det M_2 \;=\; & \left[r_1(s_0) - u_1 - \frac{x_{01}}{x_{03}} (r_3(s_0) - u_3) \right] \left[r_2'(s_0) - \frac{x_{02}}{x_{03}} r_3'(s_0) \right] \\ & - \left[r_2(s_0) - u_2 - \frac{x_{02}}{x_{03}} (r_3(s_0) - u_3) \right] \left[r_1'(s_0) - \frac{x_{01}}{x_{03}} r_3'(s_0) \right] \\ =\; & \frac{1}{x_{03}^2} \left\{ \left[x_{03}(r_1(s_0) - u_1) - x_{01}(r_3(s_0) - u_3) \right] \left[x_{03} r_2'(s_0) - x_{02} r_3'(s_0) \right] \right. \\ & \left. - \left[x_{03}(r_2(s_0) - u_2) - x_{02}(r_3(s_0) - u_3) \right] \left[x_{03} r_1'(s_0) - x_{01} r_3'(s_0) \right] \right\} \\ =\; & \frac{1}{x_{03}^2} \left[x_{03}^2 (r_1(s_0) - u_1) r_2'(s_0) - x_{02} x_{03} (r_1(s_0) - u_1) r_3'(s_0) \right. \end{aligned}$$

$$
\begin{aligned}
& -x_{01}x_{03}(r_3(s_0)-u_{03})r_2'(s_0)+x_{01}x_{02}(r_3(s_0)-u_3)r_3'(s_0) \\
& -x_{03}^2(r_2(s_0)-u_2)r_1'(s_0)+x_{01}x_{03}(r_2(s_0)-u_2)r_3'(s_0) \\
& +x_{02}x_{03}(r_3(s_0)-u_3)r_1'(s_0)-x_{01}x_{02}(r_3(s_0)-u_3)r_3'(s_0)\big] \\
= & \frac{1}{x_{03}^2}\left[x_{03}^2(r_1(s_0)-u_1)r_2'(s_0)-x_{02}x_{03}(r_1(s_0)-u_1)r_3'(s_0)\right. \\
& -x_{01}x_{03}(r_3(s_0)-u_3)r_2'(s_0)-x_{03}^2(r_2(s_0)-u_2)r_1'(s_0) \\
& \left.+x_{01}x_{03}(r_2(s_0)-u_2)r_3'(s_0)+x_{02}x_{03}(r_3(s_0)-u_3)r_1'(s_0)\right] \\
= & \frac{1}{x_{03}}\det(\mathbf{x}_0,\boldsymbol{\gamma}(s_0)-\mathbf{u},\boldsymbol{\gamma}'(s_0)) \\
= & \frac{1}{x_{03}}\det(\mathbf{x}_0,\boldsymbol{\gamma}(s_0)-\mathbf{u},\mathbf{t}(s_0)).
\end{aligned}
$$

$$\det M_2 = 0$$

$\Longleftrightarrow$

$$\mathbf{x}_0, (\boldsymbol{\gamma}(s_0)-\mathbf{u}) \text{ and } \mathbf{t}(s_0) \text{ are linearly dependent}$$

$\overset{\text{by Proposition 27}}{\Longleftrightarrow}$

$$
\begin{aligned}
& (\lambda(\boldsymbol{\gamma}-\mathbf{u})+\mu\mathbf{t})(s_0) \\
= & \left[(TR'+\mathbf{u}\cdot\mathbf{b})\mathbf{n}-(R+\mathbf{u}\cdot\mathbf{n})\mathbf{b}\right](s_0) \text{ for some } \lambda,\mu\in\mathbb{R}
\end{aligned}
\tag{4.12}
$$

$\Longleftrightarrow$

$$
\begin{aligned}
& \left\{\lambda\left(-R\mathbf{n}-TR'\mathbf{b}-[(\mathbf{u}\cdot\mathbf{t})\mathbf{t}+(\mathbf{u}\cdot\mathbf{n})\mathbf{n}+(\mathbf{u}\cdot\mathbf{b})\mathbf{b}]\right)+\mu\mathbf{t}\right\}(s_0) \\
= & \left((TR'+\mathbf{u}\cdot\mathbf{b})\mathbf{n}-(R+\mathbf{u}\cdot\mathbf{n})\mathbf{b}\right)(s_0)
\end{aligned}
$$

$\Longleftrightarrow$

$$
\begin{aligned}
& ([\mu-\lambda(\mathbf{u}\cdot\mathbf{t})]\,\mathbf{t}-[\lambda R+\lambda(\mathbf{u}\cdot\mathbf{n})+TR'+\mathbf{u}\cdot\mathbf{b}]\,\mathbf{n}+ \\
& \quad +[-\lambda TR'-\lambda(\mathbf{u}\cdot\mathbf{b})+R+\mathbf{u}\cdot\mathbf{n}]\,\mathbf{b})\,(s_0) \quad = \mathbf{0}
\end{aligned}
$$

$\Longleftrightarrow$

$$\begin{cases} (\mu - \lambda(\mathbf{u}\cdot\mathbf{t}))\,(s_0) = 0 \\ (\lambda(R + (\mathbf{u}\cdot\mathbf{n})) + TR' + \mathbf{u}\cdot\mathbf{b})\,(s_0) = 0 \\ (-\lambda(TR' + (\mathbf{u}\cdot\mathbf{b})) + R + \mathbf{u}\cdot\mathbf{n})\,(s_0) = 0 \end{cases}, \tag{4.13}$$

which implies $\lambda(R+\mathbf{u}\cdot\mathbf{n})^2 = -\lambda(TR'+\mathbf{u}\cdot\mathbf{b})^2$. So, if $\lambda \neq 0$, then

$$R^2 + 2R(\mathbf{u}\cdot\mathbf{n}) + (\mathbf{u}\cdot\mathbf{n})^2 = -\left[(TR')^2 + 2TR'(\mathbf{u}\cdot\mathbf{b}) + (\mathbf{u}\cdot\mathbf{b})^2\right]$$

$$\Rightarrow \quad 1 + 2R(\mathbf{u}\cdot\mathbf{n}) + 2TR'(\mathbf{u}\cdot\mathbf{b}) + (\mathbf{u}\cdot\mathbf{n})^2 + (\mathbf{u}\cdot\mathbf{b})^2 = 0$$

$$\Rightarrow \quad 2 - (\mathbf{u}\cdot\mathbf{t})^2 - 2(\mathbf{u}\cdot\gamma) = 0.$$

But it is not possible by the assumption in the theorem.

If $\lambda = 0$, then $\mu = 0$ by (4.13) and $[(TR'+\mathbf{u}\cdot\mathbf{b})\mathbf{n} - (R+\mathbf{u}\cdot\mathbf{n})\mathbf{b}]\,(s_0) = \mathbf{0}$ by (4.12). This implies $(TR'+\mathbf{u}\cdot\mathbf{b})(s_0) = (R+\mathbf{u}\cdot\mathbf{n})(s_0) = 0.$ Then we have

$$\mathbf{u} = [(\mathbf{u}\cdot\mathbf{t})\mathbf{t} - R\mathbf{n} - TR'\mathbf{b}]\,(s_0)$$

$$\Rightarrow \quad \mathbf{u} = \gamma(s_0) \text{ by } (\mathbf{u}\cdot\mathbf{t})^2 + (\mathbf{u}\cdot\mathbf{n})^2 + (\mathbf{u}\cdot\mathbf{b})^2 = 1 \text{ and } (TR')^2 + R^2 = 1.$$

This contradicts the assumption.

So when $\mathbf{u}$ is not on γ (i.e., $2 - (\mathbf{u}\cdot\mathbf{t})^2 - 2(\mathbf{u}\cdot\gamma) \neq 0$), M_2 is nonsingular. And M_1 has rank 1. So by Theorem 22 (6.10 in [3]) F is the versal unfolding of f. ■.

So by (4.11), Theorem 22, 28, 24 and Proposition 27 we have:

Theorem 29 *Let γ be a unit speed curve on S^2 and $\mathbf{u}$ be a point on S^2 which is not on γ (i.e., $2 - (\mathbf{u}\cdot\mathbf{t})^2 - 2(\mathbf{u}\cdot\gamma) \neq 0$). Then the spherical antiorthotomic of γ relative to $\mathbf{u}$ is, around the point corresponding to $s = s_0$,*

(a) locally diffeomorphic to a line if $(TR'+\mathbf{u}\cdot\mathbf{b})(s_0) \neq 0$;

(b) locally diffeomorphic to the ordinary cusp C if $(TR'+\mathbf{u}\cdot\mathbf{b})(s_0) = 0$ but $\tau(s_0) \neq 0$.

4.6 Spherical Evolute of the Spherical Orthotomic and the Caustic by Reflection

Suppose a "ray of light" emanating from $\mathbf{u}$ travels along the geodesic (the great circle) on S^2 and is reflected by the curve $\boldsymbol{\gamma}$ at $\boldsymbol{\gamma}(s)$. $\widetilde{\mathbf{u}}$ is the spherical orthotomic as in (4.3). The spherical normal to $\widetilde{\mathbf{u}}$ at s is given as follows:

$$\begin{cases} \mathbf{x}\cdot\mathbf{x}=1, \\ \mathbf{x}\cdot\widetilde{\mathbf{u}}'(s)=0. \end{cases}$$

i.e.,

$$\begin{cases} \mathbf{x}\cdot\mathbf{x}=1, \\ TR'\kappa\mathbf{x}\cdot\{[R(\mathbf{u}\cdot\mathbf{b})-TR'(\mathbf{u}\cdot\mathbf{n})]\,\mathbf{t}-TR'(\mathbf{u}\cdot\mathbf{t})\mathbf{n}+R(\mathbf{u}\cdot\mathbf{t})\mathbf{b}\}=0. \end{cases} \tag{4.14}$$

On the other hand, the reflection $\overline{\mathbf{u}}$ of $\mathbf{u}$ in the plane, where the spherical normal to $\boldsymbol{\gamma}$ at $\boldsymbol{\gamma}(s)$ lies, is given by

$$\begin{aligned} \overline{\mathbf{u}} &= 2\left[(\boldsymbol{\gamma}-\mathbf{u})\cdot\mathbf{t}\right]\mathbf{t}+\mathbf{u} \\ &= -2(\mathbf{u}\cdot\mathbf{t})\mathbf{t}+\mathbf{u}. \end{aligned}$$

The reflected "ray of light" (the great circle) by the curve $\boldsymbol{\gamma}$ at $\boldsymbol{\gamma}(s)$ of the "ray of light" emanating from $\mathbf{u}$ is given by

$$\begin{cases} \mathbf{x}\cdot\mathbf{x}=1, \\ \mathbf{x}\cdot[\boldsymbol{\gamma}(s)\times\overline{\mathbf{u}}(s)]=0. \end{cases} \tag{4.15}$$

Now we compute $\boldsymbol{\gamma}\times\overline{\mathbf{u}}$.

$$\begin{aligned} \boldsymbol{\gamma}\times\overline{\mathbf{u}} &= (-R\mathbf{n}-TR'\mathbf{b})\times[-2(\mathbf{u}\cdot\mathbf{t})\mathbf{t}+\mathbf{u}] \\ &= 2R(\mathbf{u}\cdot\mathbf{t})\mathbf{n}\times\mathbf{t}+2(\mathbf{u}\cdot\mathbf{t})TR'\,\mathbf{b}\times\mathbf{t}-\big\{\big[R(\mathbf{u}\cdot\mathbf{b})-TR'(\mathbf{u}\cdot\mathbf{n})\big]\;\mathbf{t}+ \\ &\quad TR'(\mathbf{u}\cdot\mathbf{t})\mathbf{n}-R(\mathbf{u}\cdot\mathbf{t})\mathbf{b}\big\} \end{aligned}$$

$$= \left[TR'(\mathbf{u}\cdot\mathbf{n}) - R(\mathbf{u}\cdot\mathbf{b})\right]\mathbf{t} + TR'(\mathbf{u}\cdot\mathbf{t})\mathbf{n} - R(\mathbf{u}\cdot\mathbf{t})\mathbf{b}$$

$$= -\left\{\left[R(\mathbf{u}\cdot\mathbf{b}) - TR'(\mathbf{u}\cdot\mathbf{n})\right]\mathbf{t} - TR'(\mathbf{u}\cdot\mathbf{t})\mathbf{n} + R(\mathbf{u}\cdot\mathbf{t})\mathbf{b}\right\}.$$

Hence $\widetilde{\mathbf{u}}' \parallel \boldsymbol{\gamma} \times \overline{\mathbf{u}}$ and then (4.14) and (4.15) define the same great circle. (If $TR' = 0$, we may consider $(R(\mathbf{u}\cdot\mathbf{b}) - TR'(\mathbf{u}\cdot\mathbf{n}))\mathbf{t} - TR'(\mathbf{u}\cdot\mathbf{t})\mathbf{n} + R(\mathbf{u}\cdot\mathbf{t})\mathbf{b}$ as the direction of $\widetilde{\mathbf{u}}'$.)

So the envelope of the family of the spherical normals to $\widetilde{\mathbf{u}}$ is same as the envelope of the family of the reflected "ray of light" of the "ray of light" emanating from $\mathbf{u}$ reflected by the curve $\boldsymbol{\gamma}$, i.e., the spherical evolute of the spherical orthotomic $\widetilde{\mathbf{u}}$ is the same as the caustic by reflection of $\boldsymbol{\gamma}$ relative to $\mathbf{u}$.

Let $F(s,\mathbf{x}) = \mathbf{x}\cdot\{[R(\mathbf{u}\cdot\mathbf{b}) - TR'(\mathbf{u}\cdot\mathbf{n})]\,\mathbf{t} - TR'(\mathbf{u}\cdot\mathbf{t})\mathbf{n} + R(\mathbf{u}\cdot\mathbf{t})\mathbf{b}\}$. Then the spherical evolute of the spherical orthotomic $\widetilde{\mathbf{u}}$ is the envelope of the family $F(s,\mathbf{x}) = 0$, i.e., the discriminant set $\mathfrak{D}_F = \{\mathbf{x} \mid F(s,\mathbf{x}) = \frac{\partial F(s,\mathbf{x})}{\partial s} = 0 \text{ for some } s \in I\}$.

$$F(s,\mathbf{x}) = \frac{\partial F(s,\mathbf{x})}{\partial s} = 0$$

$\Longleftrightarrow$

$$\begin{cases} \mathbf{x}\cdot\mathbf{x} = 1, \\ \mathbf{x}\cdot\{[R(\mathbf{u}\cdot\mathbf{b}) - TR'(\mathbf{u}\cdot\mathbf{n})]\,\mathbf{t} - TR'(\mathbf{u}\cdot\mathbf{t})\mathbf{n} + R(\mathbf{u}\cdot\mathbf{t})\mathbf{b}\} = 0, \\ \mathbf{x}\cdot\{[2TR'\kappa(\mathbf{u}\cdot\mathbf{t})]\,\mathbf{t} + [(\mathbf{u}\cdot\mathbf{b}) - 2TR'\kappa(\mathbf{u}\cdot\mathbf{n})]\,\mathbf{n} + (\mathbf{u}\cdot\mathbf{n})\mathbf{b}\} = 0, \end{cases} \tag{4.16}$$

here

$$\left\{\left[R(\mathbf{u}\cdot\mathbf{b}) - TR'(\mathbf{u}\cdot\mathbf{n})\right]\mathbf{t} - TR'(\mathbf{u}\cdot\mathbf{t})\mathbf{n} + R(\mathbf{u}\cdot\mathbf{t})\mathbf{b}\right\}'$$

$$= \left\{R'(\mathbf{u}\cdot\mathbf{b}) + R(\mathbf{u}\cdot -\tau\mathbf{n}) - (TR')'(\mathbf{u}\cdot\mathbf{n}) - TR'\left[\mathbf{u}\cdot(-\kappa\mathbf{t} + \tau\mathbf{b})\right]\right\}\mathbf{t}$$

$$+ \left[R(\mathbf{u}\cdot\mathbf{b}) - TR'(\mathbf{u}\cdot\mathbf{n})\right](\kappa\mathbf{n}) - \left[(TR')'(\mathbf{u}\cdot\mathbf{t}) + TR'(\mathbf{u}\cdot\kappa\mathbf{n})\right]\mathbf{n}$$

$$-TR'(\mathbf{u}\cdot\mathbf{t})(-\kappa\mathbf{t} + \tau\mathbf{b}) + \left[R'(\mathbf{u}\cdot\mathbf{t}) + R(\mathbf{u}\cdot\kappa\mathbf{n})\right]\mathbf{b} + R(\mathbf{u}\cdot\mathbf{t})(-\tau\mathbf{n})$$

$$\begin{aligned}
&= \left[R'(\mathbf{u}\cdot\mathbf{b}) - (R\tau + (TR')')(\mathbf{u}\cdot\mathbf{n}) + TR'\kappa(\mathbf{u}\cdot\mathbf{t}) - R'(\mathbf{u}\cdot\mathbf{b}) + TR'\kappa(\mathbf{u}\cdot\mathbf{t})\right]\mathbf{t} \\
&\quad + \left\{\kappa\left[R(\mathbf{u}\cdot\mathbf{b}) - TR'(\mathbf{u}\cdot\mathbf{n})\right] - \left[(TR')'(\mathbf{u}\cdot\mathbf{t}) + TR'\kappa(\mathbf{u}\cdot\mathbf{n})\right] - \tau R(\mathbf{u}\cdot\mathbf{t})\right\}\mathbf{n} \\
&\quad + \left[-R'(\mathbf{u}\cdot\mathbf{t}) + R'(\mathbf{u}\cdot\mathbf{t}) + (\mathbf{u}\cdot\mathbf{n})\right]\mathbf{b} \\
&= \left[2TR'\kappa(\mathbf{u}\cdot\mathbf{t})\right]\mathbf{t} + \left[(\mathbf{u}\cdot\mathbf{b}) - 2TR'\kappa(\mathbf{u}\cdot\mathbf{n})\right]\mathbf{n} + (\mathbf{u}\cdot\mathbf{n})\mathbf{b}.
\end{aligned}$$

Suppose $\mathbf{x} = x_1\mathbf{t} + x_2\mathbf{n} + x_3\mathbf{b} \in S^2$ is in (4.16), then we have

$$\begin{cases}
x_1^2 + x_2^2 + x_3^2 = 1 \\
\left[TR'(\mathbf{u}\cdot\mathbf{n}) - R(\mathbf{u}\cdot\mathbf{b})\right]x_1 + TR'(\mathbf{u}\cdot\mathbf{t})x_2 - R(\mathbf{u}\cdot\mathbf{t})x_3 = 0 \\
2TR'\kappa(\mathbf{u}\cdot\mathbf{t})x_1 + \left[(\mathbf{u}\cdot\mathbf{b}) - 2TR'\kappa(\mathbf{u}\cdot\mathbf{n})\right]x_2 + (\mathbf{u}\cdot\mathbf{n})x_3 = 0,
\end{cases}$$

i.e.,

$$\begin{cases}
x_1^2 + x_2^2 + x_3^2 = 1 \\
\left[TR'(\mathbf{u}\cdot\mathbf{n}) - R(\mathbf{u}\cdot\mathbf{b})\right]x_1 + TR'(\mathbf{u}\cdot\mathbf{t})x_2 - R(\mathbf{u}\cdot\mathbf{t})x_3 = 0 \\
2TR'(\mathbf{u}\cdot\mathbf{t})x_1 + \left[R(\mathbf{u}\cdot\mathbf{b}) - 2TR'(\mathbf{u}\cdot\mathbf{n})\right]x_2 + R(\mathbf{u}\cdot\mathbf{n})x_3 = 0.
\end{cases} \tag{4.17}$$

Solving the second and third equations of (4.17), we have

$$\begin{cases}
x_1 = \dfrac{x_3(\mathbf{u}\cdot\mathbf{t})\left[R(\mathbf{u}\cdot\mathbf{b}) - TR'(\mathbf{u}\cdot\mathbf{n})\right]}{3TR'(\mathbf{u}\cdot\mathbf{n})(\mathbf{u}\cdot\mathbf{b}) + 3\kappa(TR')^2(\mathbf{u}\cdot\mathbf{b})^2 - (\mathbf{u}\cdot\mathbf{b})^2\kappa - 2\kappa(TR')^2} \\
x_2 = \dfrac{x_3\left[R(\mathbf{u}\cdot\mathbf{b})(\mathbf{u}\cdot\mathbf{n}) - TR'(\mathbf{u}\cdot\mathbf{n})^2 - 2TR'(\mathbf{u}\cdot\mathbf{t})^2\right]}{3TR'(\mathbf{u}\cdot\mathbf{n})(\mathbf{u}\cdot\mathbf{b}) + 3\kappa(TR')^2(\mathbf{u}\cdot\mathbf{b})^2 - (\mathbf{u}\cdot\mathbf{b})^2\kappa - 2\kappa(TR')^2}.
\end{cases} \tag{4.18}$$

Plugging (4.18) into the first equation of (4.17) and using Maple, we have the three components of the spherical evolute of the spherical orthotomic $\widetilde{\mathbf{u}}$:

$$\begin{cases}
x_1 = \pm\dfrac{(\mathbf{u}\cdot\mathbf{t})(R(\mathbf{u}\cdot\mathbf{b}) - TR'(\mathbf{u}\cdot\mathbf{n}))}{A} \\
x_2 = \pm\dfrac{R(\mathbf{u}\cdot\mathbf{b})(\mathbf{u}\cdot\mathbf{n}) - TR'(\mathbf{u}\cdot\mathbf{n})^2 - 2TR'(\mathbf{u}\cdot\mathbf{t})^2}{A} \\
x_3 = \pm\dfrac{3TR'(\mathbf{u}\cdot\mathbf{n})(\mathbf{u}\cdot\mathbf{b}) + 3\kappa(TR')^2(\mathbf{u}\cdot\mathbf{b})^2 - (\mathbf{u}\cdot\mathbf{b})^2\kappa - 2\kappa(TR')^2}{A},
\end{cases}$$

where

$$A = [12(TR')^3\kappa(\mathbf{u}\cdot\mathbf{n})(\mathbf{u}\cdot\mathbf{b})^3 - 6(TR')^3\kappa(\mathbf{u}\cdot\mathbf{n})(\mathbf{u}\cdot\mathbf{b}) - 4(TR')^2(\mathbf{u}\cdot\mathbf{b})^4$$

$$+3(TR')^2(\mathbf{u}\cdot\mathbf{b})^2 - 6TR'\kappa(\mathbf{u}\cdot\mathbf{n})(\mathbf{u}\cdot\mathbf{b}) + 4\kappa^2(\mathbf{u}\cdot\mathbf{b})^4 - 4(\mathbf{u}\cdot\mathbf{b})^4$$

$$-8(\mathbf{u}\cdot\mathbf{b})^2\kappa^2 + 9(\mathbf{u}\cdot\mathbf{b})^2 + 4\kappa^2 - 4 + 12(TR')^2(\mathbf{u}\cdot\mathbf{n})^2(\mathbf{u}\cdot\mathbf{b})^2$$

$$+4RTR'(\mathbf{u}\cdot\mathbf{n})^3(\mathbf{u}\cdot\mathbf{b}) - 3(TR')^2(\mathbf{u}\cdot\mathbf{n})^2]^{\frac{1}{2}}.$$

Now we can compute the torsion of the spherical evolute of the spherical orthotomic by $\tau = \frac{\mathbf{x}'\times\mathbf{x}''\cdot\mathbf{x}'''}{\|\mathbf{x}'\times\mathbf{x}''\|}$ and then use Theorem 18 and the remark after it to determine when the caustic by reflection locally is diffeomorphic to a line or an ordinary cusp. It does not make too much sense to compute the extremely complex expression for the torsion τ. So we omit the computation of the torsion.

CHAPTER 5

Spherical Conic and its Spherical Orthotomic

5.1 Definition of the Spherical Ellipse and Hyperbola

Definition 30 *A spherical ellipse (hyperbola) is the locus of the points the sum (the absolute value of difference) of whose geodesic distances to two fixed points is constant. These two fixed points are called the foci of the spherical conic.*

For the spherical conic (ellipse or hyperbola) on the sphere, the geodesic distance is given by (1.1).

Now we consider the spherical ellipse (hyperbola) on the sphere with foci $\mathbf{u}_1$ and $\mathbf{u}_2 \in S^2$. Suppose the sum (the absolute value of difference) of the geodesic distances from any point on the spherical conic to the foci is C_1. So the equation of the spherical ellipse is

$$d(\mathbf{x}, \mathbf{u}_1) + d(\mathbf{x}, \mathbf{u}_2) = C_1,$$

and the equation of the spherical hyperbola is

$$|d(\mathbf{x}, \mathbf{u}_1) - d(\mathbf{x}, \mathbf{u}_2)| = C_1, \tag{5.1}$$

where $\mathbf{x} \in S^2$. These imply

$$\begin{aligned} & \left|\arccos\left(1 - \tfrac{1}{2}|\mathbf{x} - \mathbf{u}_1|^2\right) \pm \arccos\left(1 - \tfrac{1}{2}|\mathbf{x} - \mathbf{u}_2|^2\right)\right| = C_1 \\ \Rightarrow \quad & |\arccos(\mathbf{x} \cdot \mathbf{u}_1) \pm \arccos(\mathbf{x} \cdot \mathbf{u}_2)| = C_1 \\ & \text{by } \mathbf{x} \cdot \mathbf{x} = \mathbf{u}_1 \cdot \mathbf{u}_1 = \mathbf{u}_2 \cdot \mathbf{u}_2 = 1. \end{aligned}$$

For the spherical hyperbola we have two equations:

$$\arccos(\mathbf{x}\cdot\mathbf{u}_1)+\arccos(\mathbf{x}\cdot -\mathbf{u}_2)=C_1+\pi, \tag{5.2}$$

$$\arccos(\mathbf{x}\cdot -\mathbf{u}_1)+\arccos(\mathbf{x}\cdot\mathbf{u}_2)=C_1+\pi. \tag{5.3}$$

So (5.2) is the spherical ellipse with the foci $\mathbf{u}_1$ and $-\mathbf{u}_2$ and the sum of the distances $C_1+\pi$, and (5.3) is the spherical ellipse with the foci $-\mathbf{u}_1$ and $\mathbf{u}_2$ and the sum of the distances $C_1+\pi$. In (5.2), if we replace $\mathbf{x}$ by $-\mathbf{x}$ we get (5.3). This implies (5.2) is the antipodal image of (5.3). Thus the spherical hyperbola (5.1) consists of a pair of spherical ellipses (5.2) and (5.3), each of which is just the antipodal image of the other. Furthermore

$$|\arccos(\mathbf{x}\cdot\mathbf{u}_1)\pm\arccos(\mathbf{x}\cdot\mathbf{u}_2)|=C_1$$

$\Longrightarrow$

$$\cos\left(\arccos(\mathbf{x}\cdot\mathbf{u}_1)\pm\arccos(\mathbf{x}\cdot\mathbf{u}_2)\right)=\cos C_1 \tag{5.4}$$

$\Longrightarrow$

$$(\mathbf{x}\cdot\mathbf{u}_1)(\mathbf{x}\cdot\mathbf{u}_2)\mp\sqrt{1-(\mathbf{x}\cdot\mathbf{u}_1)^2}\sqrt{1-(\mathbf{x}\cdot\mathbf{u}_2)^2}=\cos C_1.$$

For $d(\mathbf{x},\mathbf{u}_1)+d(\mathbf{x},\mathbf{u}_2)$, if $\mathbf{x}$ is not on the great circle going through $\mathbf{u}_1$and $\mathbf{u}_2$, we can always find some point $\mathbf{x}'$ on the great circle going through $\mathbf{u}_1$and $\mathbf{u}_2$ such that $d(\mathbf{x}',\mathbf{u}_1)+d(\mathbf{x}',\mathbf{u}_2)>d(\mathbf{x},\mathbf{u}_1)+d(\mathbf{x},\mathbf{u}_2)$. So $d(\mathbf{x},\mathbf{u}_1)+d(\mathbf{x},\mathbf{u}_2)$ achieves its maximum on the great circle going through $\mathbf{u}_1$and $\mathbf{u}_2$. And then $d(\mathbf{x},\mathbf{u}_1)+d(\mathbf{x},\mathbf{u}_2)\le 2\pi-d(\mathbf{u}_1,\mathbf{u}_2)$. We also know $d(\mathbf{x},\mathbf{u}_1)+d(\mathbf{x},\mathbf{u}_2)\ge d(\mathbf{u}_1,\mathbf{u}_2)$. Thus

$$\cos(d(\mathbf{x},\mathbf{u}_1)+d(\mathbf{x},\mathbf{u}_2))\le\cos(d(\mathbf{u}_1,\mathbf{u}_2))=\mathbf{u}_1\cdot\mathbf{u}_2.$$

For $|d(\mathbf{x},\mathbf{u}_1)-d(\mathbf{x},\mathbf{u}_2)|$,

$$\begin{aligned} & |d(\mathbf{x},\mathbf{u}_1)-d(\mathbf{x},\mathbf{u}_2)| \le d(\mathbf{u}_1,\mathbf{u}_2) \\ \Rightarrow\quad & \cos(d(\mathbf{x},\mathbf{u}_1)-d(\mathbf{x},\mathbf{u}_2)) \ge \mathbf{u}_1\cdot\mathbf{u}_2. \end{aligned}$$

In the equation $(\mathbf{x}\cdot\mathbf{u}_1)(\mathbf{x}\cdot\mathbf{u}_2)-\sqrt{1-(\mathbf{x}\cdot\mathbf{u}_1)^2}\sqrt{1-(\mathbf{x}\cdot\mathbf{u}_2)^2}=\cos C_1$ for the spherical ellipse, if we replace $(\mathbf{u}_1,\mathbf{u}_2)$ by $(-\mathbf{u}_1,-\mathbf{u}_2)$, we get the same equation. This happens because we take the cosine in (5.4). Let $C=\cos C_1$, then $(\mathbf{x}\cdot\mathbf{u}_1)(\mathbf{x}\cdot\mathbf{u}_2)-\sqrt{1-(\mathbf{x}\cdot\mathbf{u}_1)^2}\cdot$ $\cdot\sqrt{1-(\mathbf{x}\cdot\mathbf{u}_2)^2}=C$ consists of a pair of spherical ellipses, each of which is just the antipodal image of the other. If we identify a spherical ellipse with its antipodal image we can define a spherical ellipse by the following equation:

$$(\mathbf{x}\cdot\mathbf{u}_1)(\mathbf{x}\cdot\mathbf{u}_2)-\sqrt{1-(\mathbf{x}\cdot\mathbf{u}_1)^2}\sqrt{1-(\mathbf{x}\cdot\mathbf{u}_2)^2}=C, \tag{5.5}$$

where $-1\le C\le \mathbf{u}_1\cdot\mathbf{u}_2$. Actually (5.5) represents the spherical ellipse with the foci $\mathbf{u}_1$ and $\mathbf{u}_2$ and the sum of the distances C_1 and the spherical ellipse with the foci $-\mathbf{u}_1$ and $-\mathbf{u}_2$ and the sum of the distances C_1, where $C=\cos C_1$. If $\mathbf{u}_1=-\mathbf{u}_2$ and $C=-1$ then any $\mathbf{x}\in S^2$ satisfies (5.5). So this case should be excluded from the definition of the spherical ellipse.

And we define a spherical hyperbola by the following equation:

$$(\mathbf{x}\cdot\mathbf{u}_1)(\mathbf{x}\cdot\mathbf{u}_2)+\sqrt{1-(\mathbf{x}\cdot\mathbf{u}_1)^2}\sqrt{1-(\mathbf{x}\cdot\mathbf{u}_2)^2}=C, \tag{5.6}$$

where $\mathbf{u}_1\cdot\mathbf{u}_2\le C\le 1$. If $\mathbf{u}_1=\mathbf{u}_2$ and $C=1$ then any $\mathbf{x}\in S^2$ satisfies (5.6). So this case should be excluded from the definition of the spherical hyperbola. Both (5.5) and (5.6) represent a pair of spherical ellipses, each of which is just the antipodal image of the other. The difference between (5.5) and (5.6) is: for (5.5) the foci $\mathbf{u}_1$ and $\mathbf{u}_2$ lie inside of the same spherical ellipse but for (5.6) the foci $\mathbf{u}_1$ and $\mathbf{u}_2$ lie inside of the different spherical ellipses.

For both the spherical ellipse and the spherical hyperbola we have $((\mathbf{x}\cdot\mathbf{u}_1)(\mathbf{x}\cdot\mathbf{u}_2)-C)^2 = (1-(\mathbf{x}\cdot\mathbf{u}_1)^2)(1-(\mathbf{x}\cdot\mathbf{u}_2)^2)$. We call them the spherical conic. So the spherical conic on the sphere satisfies the equation:

$$(\mathbf{x}\cdot\mathbf{u}_1)^2 + (\mathbf{x}\cdot\mathbf{u}_2)^2 - 2C(\mathbf{x}\cdot\mathbf{u}_1)(\mathbf{x}\cdot\mathbf{u}_2) + C^2 - 1 = 0 \tag{5.7}$$

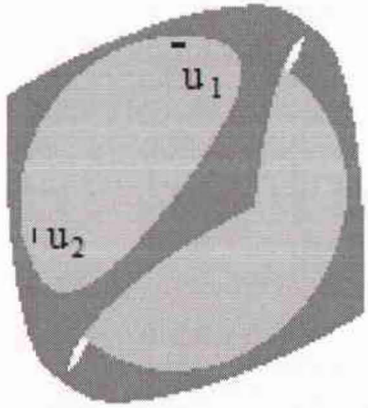


Figure 3: Spherical ellipse (1)

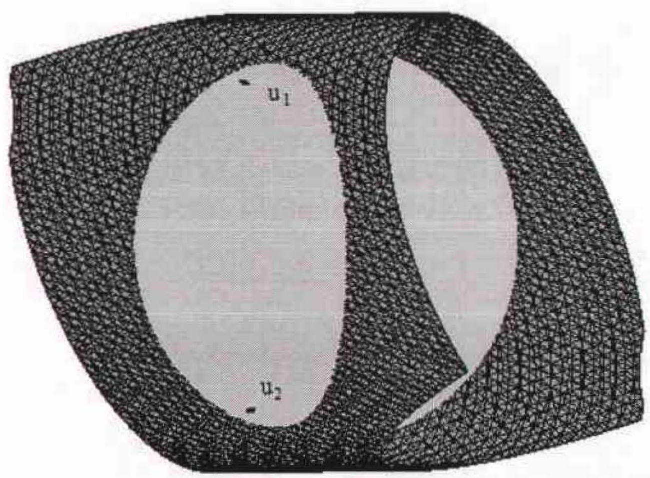


Figure 4: Spherical ellipse (2)

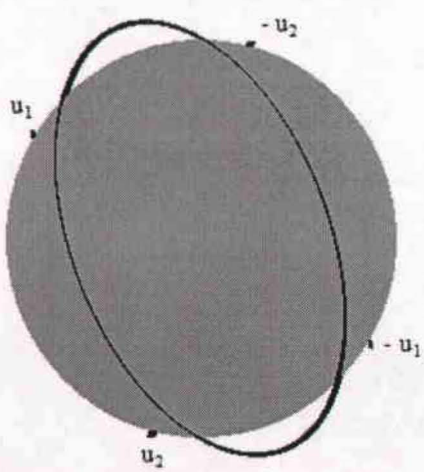


Figure 5: Spherical ellipse (3)

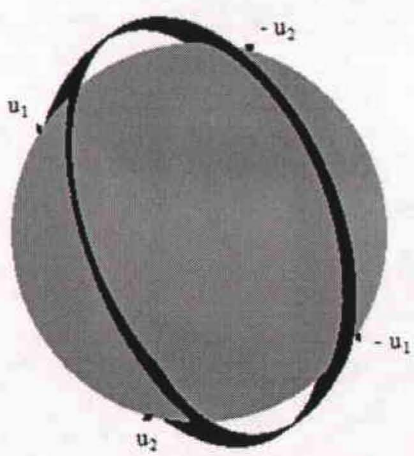


Figure 6: Spherical ellipse (4)

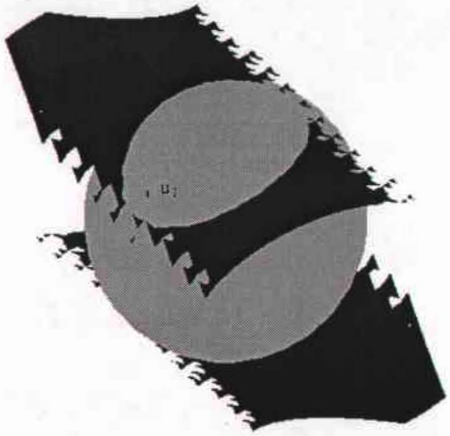


Figure 7: Spherical hyperbola (1)

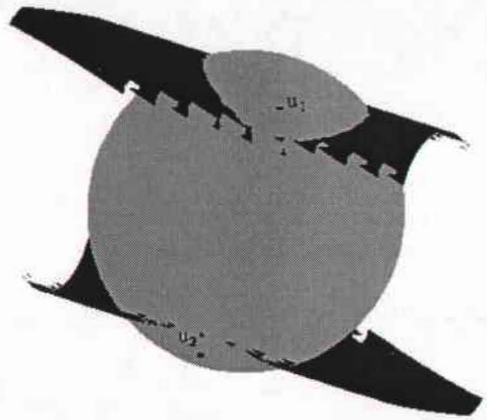


Figure 8: Spherical hyperbola (2)

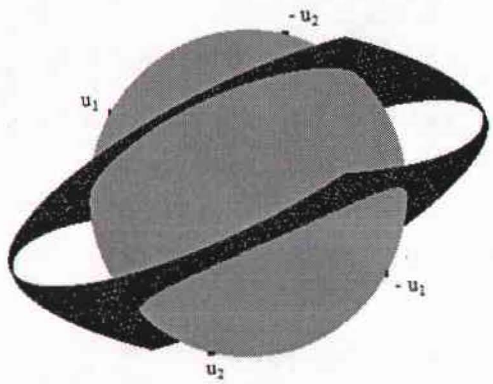


Figure 9: Spherical hyperbola (3)

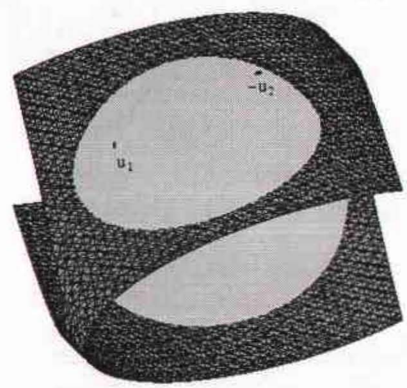


Figure 10: Spherical hyperbola (4)

Figure 3, 4, 5 and 6 show some examples of the spherical ellipses and Figure 7, 8, 9 and 10 show several examples of the spherical hyperbolas.

5.2 Contact between the Spherical Curve and the Spherical Conic

Now we measure the contact between the unit speed curve $\gamma(s)$ and the spherical conic (5.7).

Let

$$f(s) = (\gamma(s) \cdot \mathbf{u}_1)^2 + (\gamma(s) \cdot \mathbf{u}_2)^2 - 2C(\gamma(s) \cdot \mathbf{u}_1)(\gamma(s) \cdot \mathbf{u}_2) + C^2 - 1,$$

then γ has 2-point contact with the spherical conic (5.7) at s iff

$$f(s) = f'(s) = 0.$$

The geodesic circle centered at $\mathbf{u}_2$ is given by

$$\mathbf{x} \cdot \mathbf{u}_2 = C_2, \tag{5.8}$$

where C_2 is the cosine of the geodesic distance between the center $\mathbf{u}_2$ and any point on the circle, i.e., the cosine of the geodesic radius of the circle. Let $g(s) = \widetilde{\mathbf{u}_1}(s) \cdot \mathbf{u}_2 - C_2$, where $\widetilde{\mathbf{u}_1}$ is the spherical orthotomic of γ relative to $\mathbf{u}_1$ (c.f. (4.3)), then the circle (5.8) has 2-point contact with the $\widetilde{\mathbf{u}_1}$ at s iff

$$g(s) = g'(s) = 0.$$

We have the following proposition:

Proposition 31 *Let γ be a unit speed curve on S^2, and $\mathbf{u}_1 \in S^2$ is not on γ or its antipodal image ($\mathbf{u}_1 \notin \pm\gamma$). Let $\widetilde{\mathbf{u}_1}$ be the spherical orthotomic of γ relative to $\mathbf{u}_1$. Suppose $\widetilde{\mathbf{u}_1}$ is smooth at s, i.e., $TR'(s) \neq 0$ by Proposition 19. Then the circle (5.8) going through $\widetilde{\mathbf{u}_1}(s)$ and centered at $\mathbf{u}_2 \in S^2$ has 2-point contact with the $\widetilde{\mathbf{u}_1}$ at s iff its geodesic radius is $d(\mathbf{u}_2, \gamma(s)) + d(\gamma(s), \widetilde{\mathbf{u}_1}(s))$ if $\mathbf{u}_1$ is inside the circle(or $|d(\mathbf{u}_2, \gamma(s)) - d(\gamma(s), \widetilde{\mathbf{u}_1}(s))|$ if $\mathbf{u}_1$ is outside the circle), where $d(\mathbf{u}_2, \gamma(s))$ is geodesic distance between $\mathbf{u}_2$ and $\gamma(s)$, i.e., the arclength of the shorter portion of the great circle connecting $\mathbf{u}_2$ and $\gamma(s)$.*

Proof. In order to prove the geodesic radius is $d(\mathbf{u}_2, \gamma(s)) + d(\gamma(s), \widetilde{\mathbf{u}_1}(s))$

$(|d(\mathbf{u}_2, \gamma(s)) - d(\gamma(s), \widetilde{\mathbf{u}_1}(s))|)$, it is equivalent to prove that the great circle going through $\mathbf{u}_2$ and $\gamma(s)$ coincides with the great circle going through $\gamma(s)$ and $\widetilde{\mathbf{u}_1}$, which is equivalent to prove $(\gamma \times \mathbf{u}_2) \times (\widetilde{\mathbf{u}_1} \times \gamma)(s) = \mathbf{0}$.

Here

$$\begin{aligned}
\widetilde{\mathbf{u}_1} \times \boldsymbol{\gamma} &= \left(\mathbf{u}_1 - \left\{2TR'\left[TR'(\mathbf{u}_1\cdot\mathbf{n}) - R(\mathbf{u}_1\cdot\mathbf{b})\right]\mathbf{n} - 2R\left[TR'(\mathbf{u}_1\cdot\mathbf{n}) - R(\mathbf{u}_1\cdot\mathbf{b})\right]\mathbf{b}\right\}\right) \\
&\quad \times(-R\mathbf{n} - TR'\mathbf{b}) \\
&= R^2\left[-2R(\mathbf{u}_1\cdot\mathbf{b}) + 2TR'(\mathbf{u}_1\cdot\mathbf{n})\right]\mathbf{t} + (TR')^2\left[-2R(\mathbf{u}_1\cdot\mathbf{b}) + 2TR'(\mathbf{u}_1\cdot\mathbf{n})\right]\mathbf{t} \\
&\quad - \left[TR'(\mathbf{u}_1\cdot\mathbf{n}) - R(\mathbf{u}_1\cdot\mathbf{b})\right]\mathbf{t} + TR'(\mathbf{u}_1\cdot\mathbf{t})\mathbf{n} - R(\mathbf{u}_1\cdot\mathbf{t})\mathbf{b} \\
&= \left[-2R^3(\mathbf{u}_1\cdot\mathbf{b}) + 2R^2TR'(\mathbf{u}_1\cdot\mathbf{n}) - 2R(TR')^2(\mathbf{u}_1\cdot\mathbf{b}) + 2(TR')^3(\mathbf{u}_1\cdot\mathbf{n})\right. \\
&\quad \left. -TR'(\mathbf{u}_1\cdot\mathbf{n}) + R(\mathbf{u}_1\cdot\mathbf{b})\right]\mathbf{t} + TR'(\mathbf{u}_1\cdot\mathbf{t})\mathbf{n} - R(\mathbf{u}_1\cdot\mathbf{t})\mathbf{b} \\
&= \left[TR'(\mathbf{u}_1\cdot\mathbf{n}) - R(\mathbf{u}_1\cdot\mathbf{b})\right]\mathbf{t} + TR'(\mathbf{u}_1\cdot\mathbf{t})\mathbf{n} - R(\mathbf{u}_1\cdot\mathbf{t})\mathbf{b}.
\end{aligned}$$

So

$$\begin{aligned}
&(\boldsymbol{\gamma} \times \mathbf{u}_2) \times (\widetilde{\mathbf{u}_1} \times \boldsymbol{\gamma}) \\
&= \left\{\left[TR'(\mathbf{u}_2\cdot\mathbf{n}) - R(\mathbf{u}_2\cdot\mathbf{b})\right]\mathbf{t} - TR'(\mathbf{u}_2\cdot\mathbf{t})\mathbf{n} + R(\mathbf{u}_2\cdot\mathbf{t})\mathbf{b}\right\} \times \\
&\quad \left\{\left[TR'(\mathbf{u}_1\cdot\mathbf{n}) - R(\mathbf{u}_1\cdot\mathbf{b})\right]\mathbf{t} + TR'(\mathbf{u}_1\cdot\mathbf{t})\mathbf{n} - R(\mathbf{u}_1\cdot\mathbf{t})\mathbf{b}\right\} \\
&= TR'(\mathbf{u}_1\cdot\mathbf{t})\left[TR'(\mathbf{u}_2\cdot\mathbf{n}) - R(\mathbf{u}_2\cdot\mathbf{b})\right]\mathbf{b} + R(\mathbf{u}_1\cdot\mathbf{t})\left[TR'(\mathbf{u}_2\cdot\mathbf{n}) - R(\mathbf{u}_2\cdot\mathbf{b})\right]\mathbf{n} \\
&\quad + TR'(\mathbf{u}_2\cdot\mathbf{t})\left[TR'(\mathbf{u}_1\cdot\mathbf{n}) - R(\mathbf{u}_1\cdot\mathbf{b})\right]\mathbf{b} + RTR'(\mathbf{u}_1\cdot\mathbf{t})(\mathbf{u}_2\cdot\mathbf{t})\mathbf{t} \\
&\quad + R(\mathbf{u}_2\cdot\mathbf{t})\left[TR'(\mathbf{u}_1\cdot\mathbf{n}) - R(\mathbf{u}_1\cdot\mathbf{b})\right]\mathbf{n} - RTR'(\mathbf{u}_1\cdot\mathbf{t})(\mathbf{u}_2\cdot\mathbf{t})\mathbf{t} \\
&= TR'\left\{(\mathbf{u}_1\cdot\mathbf{t})\left[TR'(\mathbf{u}_2\cdot\mathbf{n}) - R(\mathbf{u}_2\cdot\mathbf{b})\right] + (\mathbf{u}_2\cdot\mathbf{t})\left[TR'(\mathbf{u}_1\cdot\mathbf{n}) - R(\mathbf{u}_1\cdot\mathbf{b})\right]\right\}\mathbf{b} \\
&\quad + R\left\{(\mathbf{u}_1\cdot\mathbf{t})\left[TR'(\mathbf{u}_2\cdot\mathbf{n}) - R(\mathbf{u}_2\cdot\mathbf{b})\right] + (\mathbf{u}_2\cdot\mathbf{t})\left[TR'(\mathbf{u}_1\cdot\mathbf{n}) - R(\mathbf{u}_1\cdot\mathbf{b})\right]\right\}\mathbf{n}
\end{aligned}$$

Thus

$$\begin{aligned}
&(\boldsymbol{\gamma} \times \mathbf{u}_2) \times (\widetilde{\mathbf{u}_1} \times \boldsymbol{\gamma})(s) = \mathbf{0} \\
\Leftrightarrow \quad &\{(\mathbf{u}_1\cdot\mathbf{t})\left[TR'(\mathbf{u}_2\cdot\mathbf{n}) - R(\mathbf{u}_2\cdot\mathbf{b})\right] + (\mathbf{u}_2\cdot\mathbf{t})\left[TR'(\mathbf{u}_1\cdot\mathbf{n}) - R(\mathbf{u}_1\cdot\mathbf{b})\right]\}(s) = 0.
\end{aligned}$$

On the other hand

$$\begin{aligned} g'(s) &= \widetilde{\mathbf{u_1}}'(s)\cdot \mathbf{u}_2 \\ &= (-2)\left\{TR'\kappa\left[R(\mathbf{u}\cdot\mathbf{b}) - TR'(\mathbf{u}\cdot\mathbf{n})\right]\mathbf{t} - (TR')^2\kappa(\mathbf{u}\cdot\mathbf{t})\mathbf{n} + TR'(\mathbf{u}\cdot\mathbf{t})\mathbf{b}\right\}\cdot\mathbf{u}_2 \\ &= 2TR'\kappa\left\{(\mathbf{u}_1\cdot\mathbf{t})\left[TR'(\mathbf{u}_2\cdot\mathbf{n}) - R(\mathbf{u}_2\cdot\mathbf{b})\right] + (\mathbf{u}_2\cdot\mathbf{t})\left[(TR'(\mathbf{u}_1\cdot\mathbf{n}) - R(\mathbf{u}_1\cdot\mathbf{b})\right]\right\}. \end{aligned}$$

Since $TR'(s) \neq 0$,

$$g'(s) = 0 \;\Leftrightarrow\; (\boldsymbol{\gamma}\times\mathbf{u}_2)\times(\widetilde{\mathbf{u_1}}\times\boldsymbol{\gamma})(s) = \mathbf{0}.$$

■

Here is a proposition about a relationship between the contact of the spherical orthotomic of $\boldsymbol{\gamma}$ relative to $\mathbf{u}_1$ with the circle (5.8) and the contact of the spherical conic (5.7) with $\boldsymbol{\gamma}$:

Proposition 32 *Let $\boldsymbol{\gamma}$ be a unit speed curve on S^2, and $\mathbf{u}_1 \in S^2$ is not on $\boldsymbol{\gamma}$ or its antipodal image ($\mathbf{u}_1 \notin \pm\boldsymbol{\gamma}$). Let $\widetilde{\mathbf{u_1}}$ be the spherical orthotomic of $\boldsymbol{\gamma}$ relative to $\mathbf{u}_1$. Suppose $\widetilde{\mathbf{u_1}}$ is smooth at s, i.e., $TR'(s) \neq 0$ by Proposition 19 and the circle (5.8) going through $\widetilde{\mathbf{u_1}}(s)$ and centered at $\mathbf{u}_2 \in S^2$ has 2-point contact with the $\widetilde{\mathbf{u_1}}$ at s, if and only if the spherical conic (5.7) going through $\boldsymbol{\gamma}(s)$ also has 2-point contact with $\boldsymbol{\gamma}$ at s.*

Proof. *Proof of sufficiency*: By Proposition 31, $g(s) = g'(s) = 0 \Rightarrow d(\mathbf{u}_2, \widetilde{\mathbf{u_1}}(s)) = d(\mathbf{u}_2, \boldsymbol{\gamma}(s)) + d(\boldsymbol{\gamma}(s), \widetilde{\mathbf{u_1}}(s))$ if $\mathbf{u}_1$ is inside the circle and then the spherical conic is a spherical ellipse (or $|d(\mathbf{u}_2, \boldsymbol{\gamma}(s)) - d(\boldsymbol{\gamma}(s), \widetilde{\mathbf{u_1}}(s))|$ if $\mathbf{u}_1$ is outside the circle and then the spherical conic is a spherical hyperbola). By the construction of the spherical orthotomic we know $d(\boldsymbol{\gamma}(s), \mathbf{u}_1) = d(\boldsymbol{\gamma}(s), \widetilde{\mathbf{u_1}}(s))$. So $d(\mathbf{u}_2, \widetilde{\mathbf{u_1}}(s)) = d(\mathbf{u}_2, \boldsymbol{\gamma}(s)) + d(\boldsymbol{\gamma}(s), \widetilde{\mathbf{u_1}}(s)) = d(\mathbf{u}_2, \boldsymbol{\gamma}(s)) + d(\boldsymbol{\gamma}(s), \mathbf{u}_1)$ if the spherical conic is a spherical ellipse(or $|d(\mathbf{u}_2, \boldsymbol{\gamma}(s)) - d(\boldsymbol{\gamma}(s), \widetilde{\mathbf{u_1}}(s))|$ if the spherical conic is a spherical hyperbola), i.e., the geodesic radius of the circle (5.8) going through $\widetilde{\mathbf{u}_1}(s)$ and centered at $\mathbf{u}_2$ is equal to the sum (or the absolute value of difference) of the geodesic distances from any point on the spherical conic (5.7) going through $\boldsymbol{\gamma}(s)$ to the foci $\mathbf{u}_1$ and $\mathbf{u}_2 \in S^2$. Hence C_2 as in (5.8) is equal to C as in (5.7). So by $TR' \neq 0$ we have

$$\begin{aligned}
& g(s) = g'(s) = 0 \\
\Rightarrow\quad & (\mathbf{u}_1\cdot\mathbf{t})(\mathbf{u}_2\cdot\mathbf{t})+(\mathbf{u}_1\cdot\mathbf{n})(\mathbf{u}_2\cdot\mathbf{n})+(\mathbf{u}_1\cdot\mathbf{b})(\mathbf{u}_2\cdot\mathbf{b}) \\
& -2\left[TR'(\mathbf{u}_1\cdot\mathbf{n})-R(\mathbf{u}_1\cdot\mathbf{b})\right]\left[TR'(\mathbf{u}_2\cdot\mathbf{n})-R(\mathbf{u}_2\cdot\mathbf{b})\right]-C=0
\end{aligned}$$

$$\text{and } (\mathbf{u}_1\cdot\mathbf{t})\left[TR'(\mathbf{u}_2\cdot\mathbf{n})-R(\mathbf{u}_2\cdot\mathbf{b})\right]+(\mathbf{u}_2\cdot\mathbf{t})\left[TR'(\mathbf{u}_1\cdot\mathbf{n})-R(\mathbf{u}_1\cdot\mathbf{b})\right]=0.$$

On the other hand

$$\begin{aligned}
f(s) &= (\boldsymbol{\gamma}(s)\cdot\mathbf{u}_1)^2+(\boldsymbol{\gamma}(s)\cdot\mathbf{u}_2)^2-2C(\boldsymbol{\gamma}(s)\cdot\mathbf{u}_1)(\boldsymbol{\gamma}(s)\cdot\mathbf{u}_2)+C^2-1 \\
&= (\boldsymbol{\gamma}(s)\cdot\mathbf{u}_1)^2+(\boldsymbol{\gamma}(s)\cdot\mathbf{u}_2)^2-(\boldsymbol{\gamma}(s)\cdot\mathbf{u}_1)^2(\boldsymbol{\gamma}(s)\cdot\mathbf{u}_2)^2-1+ \\
&\quad [C-(\boldsymbol{\gamma}(s)\cdot\mathbf{u}_1)(\boldsymbol{\gamma}(s)\cdot\mathbf{u}_2)]^2 \\
&= \left[-R(\mathbf{u}_1\cdot\mathbf{n})-TR'(\mathbf{u}_1\cdot\mathbf{b})\right]^2+\left[-R(\mathbf{u}_2\cdot\mathbf{n})-TR'(\mathbf{u}_2\cdot\mathbf{b})\right]^2 \\
&\quad -\left[-R(\mathbf{u}_1\cdot\mathbf{n})-TR'(\mathbf{u}_1\cdot\mathbf{b})\right]^2\left[-R(\mathbf{u}_2\cdot\mathbf{n})-TR'(\mathbf{u}_2\cdot\mathbf{b})\right]^2 \\
&\quad -1+\left\{C-\left[-R(\mathbf{u}_1\cdot\mathbf{n})-TR'(\mathbf{u}_1\cdot\mathbf{b})\right]\left[-R(\mathbf{u}_2\cdot\mathbf{n})-TR'(\mathbf{u}_2\cdot\mathbf{b})\right]\right\}^2 \\
&= \left[R(\mathbf{u}_1\cdot\mathbf{n})+TR'(\mathbf{u}_1\cdot\mathbf{b})\right]^2+\left[R(\mathbf{u}_2\cdot\mathbf{n})+TR'(\mathbf{u}_2\cdot\mathbf{b})\right]^2 \\
&\quad -\left[R(\mathbf{u}_1\cdot\mathbf{n})+TR'(\mathbf{u}_1\cdot\mathbf{b})\right]^2\left[R(\mathbf{u}_2\cdot\mathbf{n})+TR'(\mathbf{u}_2\cdot\mathbf{b})\right]^2 \\
&\quad -1+\{(\mathbf{u}_1\cdot\mathbf{t})(\mathbf{u}_2\cdot\mathbf{t})+(\mathbf{u}_1\cdot\mathbf{n})(\mathbf{u}_2\cdot\mathbf{n})+(\mathbf{u}_1\cdot\mathbf{b})(\mathbf{u}_2\cdot\mathbf{b}) \\
&\quad -2\left[TR'(\mathbf{u}_1\cdot\mathbf{n})-R(\mathbf{u}_1\cdot\mathbf{b})\right]\left[TR'(\mathbf{u}_2\cdot\mathbf{n})-R(\mathbf{u}_2\cdot\mathbf{b})\right] \\
&\quad -\left[R(\mathbf{u}_1\cdot\mathbf{n})+TR'(\mathbf{u}_1\cdot\mathbf{b})\right]\left[R(\mathbf{u}_2\cdot\mathbf{n})+TR'(\mathbf{u}_2\cdot\mathbf{b})\right]\}^2,
\end{aligned}$$

and

$$f'(s) = 2(\gamma(s)\cdot\mathbf{u}_1)(\mathbf{u}_1\cdot\mathbf{t}) + 2(\gamma(s)\cdot\mathbf{u}_2)(\mathbf{u}_2\cdot\mathbf{t}) - 2C(\gamma(s)\cdot\mathbf{u}_2)(\mathbf{u}_1\cdot\mathbf{t}) - 2C(\gamma(s)\cdot\mathbf{u}_1)(\mathbf{u}_2\cdot\mathbf{t}) \tag{5.9}$$

$$= 2\{(\gamma(s)\cdot\mathbf{u}_1)(\mathbf{u}_1\cdot\mathbf{t}) + (\gamma(s)\cdot\mathbf{u}_2)(\mathbf{u}_2\cdot\mathbf{t}) - C[(\gamma(s)\cdot\mathbf{u}_2)(\mathbf{u}_1\cdot\mathbf{t}) + (\gamma(s)\cdot\mathbf{u}_1)(\mathbf{u}_2\cdot\mathbf{t})]\}$$

$$= (-2)\Big(\big[R(\mathbf{u}_1\cdot\mathbf{n}) + TR'(\mathbf{u}_1\cdot\mathbf{b})\big](\mathbf{u}_1\cdot\mathbf{t}) + \big[R(\mathbf{u}_2\cdot\mathbf{n}) + TR'(\mathbf{u}_2\cdot\mathbf{b})\big](\mathbf{u}_2\cdot\mathbf{t}) - \{(\mathbf{u}_1\cdot\mathbf{t})(\mathbf{u}_2\cdot\mathbf{t}) + (\mathbf{u}_1\cdot\mathbf{n})(\mathbf{u}_2\cdot\mathbf{n}) + (\mathbf{u}_1\cdot\mathbf{b})(\mathbf{u}_2\cdot\mathbf{b}) - 2\big[TR'(\mathbf{u}_1\cdot\mathbf{n}) - R(\mathbf{u}_1\cdot\mathbf{b})\big]\big[TR'(\mathbf{u}_2\cdot\mathbf{n}) - R(\mathbf{u}_2\cdot\mathbf{b})\big]\} \times \times \{\big[R(\mathbf{u}_2\cdot\mathbf{n}) + TR'(\mathbf{u}_2\cdot\mathbf{b})\big](\mathbf{u}_1\cdot\mathbf{t}) + \big[R(\mathbf{u}_1\cdot\mathbf{n}) + TR'(\mathbf{u}_1\cdot\mathbf{b})\big](\mathbf{u}_2\cdot\mathbf{t})\}\Big).$$

By

$$\begin{cases} R^2 + (TR')^2 = 1, \\ \mathbf{u}_1\cdot\mathbf{u}_1 = 1, \\ \mathbf{u}_2\cdot\mathbf{u}_2 = 1, \\ (\mathbf{u}_1\cdot\mathbf{t})\,[TR'(\mathbf{u}_2\cdot\mathbf{n}) - R(\mathbf{u}_2\cdot\mathbf{b})] + (\mathbf{u}_2\cdot\mathbf{t})\,[TR'(\mathbf{u}_1\cdot\mathbf{n}) - R(\mathbf{u}_1\cdot\mathbf{b})] = 0 \end{cases}$$

and some computation using Maple we can prove that $f(s) = f'(s) = 0$.

(Maple codes for proving $f(s) = 0$ **:**

simplify($[R(\mathbf{u}_1\cdot\mathbf{n}) + TR'(\mathbf{u}_1\cdot\mathbf{b})]^2 + [R(\mathbf{u}_2\cdot\mathbf{n}) + TR'(\mathbf{u}_2\cdot\mathbf{b})]^2 -$

$[R(\mathbf{u}_1\cdot\mathbf{n}) + TR'(\mathbf{u}_1\cdot\mathbf{b})]^2\,[R(\mathbf{u}_2\cdot\mathbf{n}) + TR'(\mathbf{u}_2\cdot\mathbf{b})]^2 - 1 +$

$\{(\mathbf{u}_1\cdot\mathbf{t})(\mathbf{u}_2\cdot\mathbf{t}) + (\mathbf{u}_1\cdot\mathbf{n})(\mathbf{u}_2\cdot\mathbf{n}) + (\mathbf{u}_1\cdot\mathbf{b})(\mathbf{u}_2\cdot\mathbf{b}) -$

$2\,[TR'(\mathbf{u}_1\cdot\mathbf{n}) - R(\mathbf{u}_1\cdot\mathbf{b})]\,[TR'(\mathbf{u}_2\cdot\mathbf{n}) - R(\mathbf{u}_2\cdot\mathbf{b})]$

$-\,[R(\mathbf{u}_1\cdot\mathbf{n}) + TR'(\mathbf{u}_1\cdot\mathbf{b})]\,[R(\mathbf{u}_2\cdot\mathbf{n}) + TR'(\mathbf{u}_2\cdot\mathbf{b})]\}^2,$

$\{R^2 + (TR')^2 = 1, (\mathbf{u}_1\cdot\mathbf{t})^2 + (\mathbf{u}_1\cdot\mathbf{b})^2 + (\mathbf{u}_1\cdot\mathbf{n})^2 = 1, (\mathbf{u}_2\cdot\mathbf{t})^2 + (\mathbf{u}_2\cdot\mathbf{b})^2 + (\mathbf{u}_2\cdot\mathbf{n})^2 = 1,$

$(\mathbf{u}_1\cdot\mathbf{t})\,[TR'(\mathbf{u}_2\cdot\mathbf{n})-R(\mathbf{u}_2\cdot\mathbf{b})]+(\mathbf{u}_2\cdot\mathbf{t})\,[TR'(\mathbf{u}_1\cdot\mathbf{n})-R(\mathbf{u}_1\cdot\mathbf{b})]=0\});$

Maple codes for proving $f'(s)=0$:

$\mathsf{simplify}((-2)\,([R(\mathbf{u}_1\cdot\mathbf{n})+TR'(\mathbf{u}_1\cdot\mathbf{b})]\,(\mathbf{u}_1\cdot\mathbf{t})+[R(\mathbf{u}_2\cdot\mathbf{n})+TR'(\mathbf{u}_2\cdot\mathbf{b})]\,(\mathbf{u}_2\cdot\mathbf{t})$

$-\{(\mathbf{u}_1\cdot\mathbf{t})(\mathbf{u}_2\cdot\mathbf{t})+(\mathbf{u}_1\cdot\mathbf{n})(\mathbf{u}_2\cdot\mathbf{n})+(\mathbf{u}_1\cdot\mathbf{b})(\mathbf{u}_2\cdot\mathbf{b})$

$-2\,[TR'(\mathbf{u}_1\cdot\mathbf{n})-R(\mathbf{u}_1\cdot\mathbf{b})]\,[TR'(\mathbf{u}_2\cdot\mathbf{n})-R(\mathbf{u}_2\cdot\mathbf{b})]\}\times$

$\times\{[R(\mathbf{u}_2\cdot\mathbf{n})+TR'(\mathbf{u}_2\cdot\mathbf{b})]\,(\mathbf{u}_1\cdot\mathbf{t})+[R(\mathbf{u}_1\cdot\mathbf{n})+TR'(\mathbf{u}_1\cdot\mathbf{b})]\,(\mathbf{u}_2\cdot\mathbf{t})\})\,,$

$\{R^2+(TR')^2=1,(\mathbf{u}_1\cdot\mathbf{t})^2+(\mathbf{u}_1\cdot\mathbf{b})^2+(\mathbf{u}_1\cdot\mathbf{n})^2=1,(\mathbf{u}_2\cdot\mathbf{t})^2+(\mathbf{u}_2\cdot\mathbf{b})^2+(\mathbf{u}_2\cdot\mathbf{n})^2=1,$

$(\mathbf{u}_1\cdot\mathbf{t})\,[TR'(\mathbf{u}_2\cdot\mathbf{n})-R(\mathbf{u}_2\cdot\mathbf{b})]+(\mathbf{u}_2\cdot\mathbf{t})\,[TR'(\mathbf{u}_1\cdot\mathbf{n})-R(\mathbf{u}_1\cdot\mathbf{b})]=0\});$

Proof of necessity: $f(s)=f'(s)=0$ implies the spherical conic (5.7) is tangent to γ at s. Then by the following Theorem 33 and Theorem 34 the great circle going through $\mathbf{u}_2$ and $\gamma(s)$ coincides with the great circle going through $\gamma(s)$ and $\widetilde{\mathbf{u}_1}(s)$. By Proposition 31 we know $g(s)=g'(s)=0$. ■

Remark 10 *It's easy to see that in Proposition 32 the spherical orthotomic relative to* $\mathbf{u}_1$ *of the spherical conic with foci* $\mathbf{u}_1$,$\mathbf{u}_2$, *going through* $\gamma(s)$ *and having* 2*-point contact at* s *with* γ, *is the circle going through* $\widetilde{\mathbf{u}_1}(s)$ *and centered at* $\mathbf{u}_2\in S^2$ *having* 2*-point contact with the* $\widetilde{\mathbf{u}_1}$ *at* s.

5.3 Theorem on the Incident Angle and the Reflection Angle for Spherical Conic

Here we have the theorem about the incident angle and the reflection angle:

Theorem 33 *Let* $\mathbf{x}:I\to S^2$ *be a spherical ellipse defined by*

$$(\mathbf{x}\cdot\mathbf{u}_1)(\mathbf{x}\cdot\mathbf{u}_2)-\sqrt{1-(\mathbf{x}\cdot\mathbf{u}_1)^2}\sqrt{1-(\mathbf{x}\cdot\mathbf{u}_2)^2}=C$$

with the foci $\mathbf{u}_1$ *and* $\mathbf{u}_2 \in S^2$, $-1 \leq C \leq \mathbf{u}_1 \cdot \mathbf{u}_2$. *Let* $\mathbf{x}_0 \in S^2$ *be any point on the ellipse (5.7),* $\mathbf{x}_0 \neq \pm\mathbf{u}_1, \pm\mathbf{u}_2$ *and* $\mathbf{x}_0'$ *be the tangent to* $\mathbf{x}$ *at* $\mathbf{x}_0$. *Consider the great circle going through* $\mathbf{u}_1$ *and* $\mathbf{x}_0$ *as the incident ray and the great circle going through* $\mathbf{x}_0$ *and* $\mathbf{u}_2$ *as the reflection ray. Then the incident angle, made by the tangent* $\mathbf{x}_0'$ *and the great circle going through* $\mathbf{u}_1$ *and* $\mathbf{x}_0$, *is equal to the reflection angle, made by the tangent* $\mathbf{x}_0'$ *and the great circle going through* $\mathbf{x}_0$ *and* $\mathbf{u}_2$.

Proof. The tangent to the incident ray $\mathbf{u}_1\mathbf{x}_0$ is:

$$\begin{aligned}(\mathbf{u}_1 \times \mathbf{x}_0) \times \mathbf{x}_0 &= \mathbf{x}_0 \times (\mathbf{u}_1 \times \mathbf{x}_0) \\ &= (\mathbf{x}_0 \cdot \mathbf{u}_1)\mathbf{x}_0 - \mathbf{u}_1.\end{aligned}$$

$[(\mathbf{x}_0 \cdot \mathbf{u}_1)\mathbf{x}_0 - \mathbf{u}_1]^2 = (\mathbf{x}_0 \cdot \mathbf{u}_1)^2 + 1 - 2(\mathbf{x}_0 \cdot \mathbf{u}_1)^2 = 1 - (\mathbf{x}_0 \cdot \mathbf{u}_1)^2 > 0$ since $\mathbf{x}_0 \neq \pm\mathbf{u}_1$.

The normalized tangent to the incident ray $\mathbf{u}_1\mathbf{x}_0$ is:

$$\frac{1}{\sqrt{1 - (\mathbf{x}_0 \cdot \mathbf{u}_1)^2}} \left[(\mathbf{x}_0 \cdot \mathbf{u}_1)\mathbf{x}_0 - \mathbf{u}_1\right].$$

The tangent to the incident ray $\mathbf{x}_0\mathbf{u}_2$ is:

$$(\mathbf{x}_0 \times \mathbf{u}_2) \times \mathbf{x}_0 = \mathbf{u}_2 - (\mathbf{x}_0 \cdot \mathbf{u}_2)\mathbf{x}_0.$$

The normalized tangent to the incident ray $\mathbf{x}_0\mathbf{u}_2$ is:

$$\frac{1}{\sqrt{1 - (\mathbf{x}_0 \cdot \mathbf{u}_2)^2}} \left[\mathbf{u}_2 - (\mathbf{x}_0 \cdot \mathbf{u}_2)\mathbf{x}_0\right],$$

where $1 - (\mathbf{x}_0 \cdot \mathbf{u}_2)^2 > 0$ since $\mathbf{x}_0 \neq \pm\mathbf{u}_2$. (see Figure 11). Then

the incident angle is equal to the reflection angle

$\iff$

$$\begin{cases} \dfrac{1}{\sqrt{1-(\mathbf{x}_0\cdot\mathbf{u}_1)^2}}\left[(\mathbf{x}_0\cdot\mathbf{u}_1)\mathbf{x}_0-\mathbf{u}_1\right]\cdot\mathbf{x}_0' \\ = \dfrac{1}{\sqrt{1-(\mathbf{x}_0\cdot\mathbf{u}_2)^2}}\left[\mathbf{u}_2-(\mathbf{x}_0\cdot\mathbf{u}_2)\mathbf{x}_0\right]\cdot\mathbf{x}_0' \end{cases}$$

$\iff$

$$-\sqrt{1-(\mathbf{x}_0\cdot\mathbf{u}_2)^2}(\mathbf{x}_0'\cdot\mathbf{u}_1)=\sqrt{1-(\mathbf{x}_0\cdot\mathbf{u}_1)^2}(\mathbf{x}_0'\cdot\mathbf{u}_2). \tag{5.10}$$

Plugging $\mathbf{x}_0$ into (5.7) we have

$$(\mathbf{x}_0\cdot\mathbf{u}_1)^2+(\mathbf{x}_0\cdot\mathbf{u}_2)^2-2C(\mathbf{x}_0\cdot\mathbf{u}_1)(\mathbf{x}_0\cdot\mathbf{u}_2)+C^2-1=0. \tag{5.11}$$

Differentiating (5.7) and evaluating at $\mathbf{x}_0$, we get

$$2(\mathbf{x}_0\cdot\mathbf{u}_1)(\mathbf{x}_0'\cdot\mathbf{u}_1)+2(\mathbf{x}_0\cdot\mathbf{u}_2)(\mathbf{x}_0'\cdot\mathbf{u}_2)-2C((\mathbf{x}_0'\cdot\mathbf{u}_1)(\mathbf{x}_0\cdot\mathbf{u}_2)+(\mathbf{x}_0\cdot\mathbf{u}_1)(\mathbf{x}_0'\cdot\mathbf{u}_2))=0. \tag{5.12}$$

Now we'll prove (5.10) in the following cases.

Case 1: $(\mathbf{x}_0'\cdot\mathbf{u}_1)(\mathbf{x}_0\cdot\mathbf{u}_2)+(\mathbf{x}_0\cdot\mathbf{u}_1)(\mathbf{x}_0'\cdot\mathbf{u}_2)\neq 0$.

Then we have

$$C=\frac{(\mathbf{x}_0\cdot\mathbf{u}_1)(\mathbf{x}_0'\cdot\mathbf{u}_1)+(\mathbf{x}_0\cdot\mathbf{u}_2)(\mathbf{x}_0'\cdot\mathbf{u}_2)}{(\mathbf{x}_0'\cdot\mathbf{u}_1)(\mathbf{x}_0\cdot\mathbf{u}_2)+(\mathbf{x}_0\cdot\mathbf{u}_1)(\mathbf{x}_0'\cdot\mathbf{u}_2)}. \tag{5.13}$$

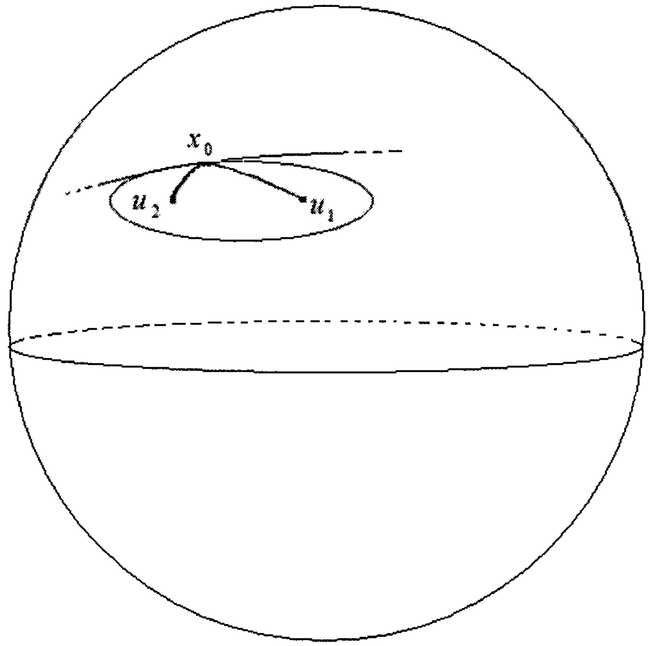


Figure 11: The incident angle is equal to the reflection angle.

Plugging (5.13) into (5.11) yields

$$\frac{\left[(\mathbf{x}_0\cdot\mathbf{u}_1)^2-(\mathbf{x}_0\cdot\mathbf{u}_2)^2\right]\left\{(\mathbf{x}_0'\cdot\mathbf{u}_1)^2\left[1-(\mathbf{x}_0\cdot\mathbf{u}_2)^2\right]-(\mathbf{x}_0'\cdot\mathbf{u}_2)^2\left[1-(\mathbf{x}_0\cdot\mathbf{u}_1)^2\right]\right\}}{(\mathbf{x}_0'\cdot\mathbf{u}_1)(\mathbf{x}_0\cdot\mathbf{u}_2)+(\mathbf{x}_0\cdot\mathbf{u}_1)(\mathbf{x}_0'\cdot\mathbf{u}_2)}=0. \tag{5.14}$$

Then we have the following subcases:

Subcase 1.1: $\mathbf{x}_0\cdot\mathbf{u}_1-\mathbf{x}_0\cdot\mathbf{u}_2=0$.

Then the assumption $(\mathbf{x}_0'\cdot\mathbf{u}_1)(\mathbf{x}_0\cdot\mathbf{u}_2)+(\mathbf{x}_0\cdot\mathbf{u}_1)(\mathbf{x}_0'\cdot\mathbf{u}_2)\neq 0\Rightarrow \mathbf{x}_0\cdot\mathbf{u}_1\neq 0$, $\mathbf{x}_0\cdot\mathbf{u}_2\neq 0$. By (5.13) we have

$$C=\frac{(\mathbf{x}_0\cdot\mathbf{u}_1)(\mathbf{x}_0'\cdot\mathbf{u}_1)+(\mathbf{x}_0\cdot\mathbf{u}_2)(\mathbf{x}_0'\cdot\mathbf{u}_2)}{(\mathbf{x}_0'\cdot\mathbf{u}_1)(\mathbf{x}_0\cdot\mathbf{u}_2)+(\mathbf{x}_0\cdot\mathbf{u}_1)(\mathbf{x}_0'\cdot\mathbf{u}_2)}=\frac{(\mathbf{x}_0\cdot\mathbf{u}_1)((\mathbf{x}_0'\cdot\mathbf{u}_1)+(\mathbf{x}_0'\cdot\mathbf{u}_2))}{(\mathbf{x}_0\cdot\mathbf{u}_1)((\mathbf{x}_0'\cdot\mathbf{u}_1)+(\mathbf{x}_0'\cdot\mathbf{u}_2))}=1.$$

From $\mathbf{x}_0 \cdot \mathbf{u}_1 - \mathbf{x}_0 \cdot \mathbf{u}_2 = 0$, we also know $d(\mathbf{x}_0, \mathbf{u}_1) = d(\mathbf{x}_0, \mathbf{u}_2)$. Then

$$\begin{aligned} & C = 1 \\ \Rightarrow\quad & d(\mathbf{x}_0, \mathbf{u}_1) + d(\mathbf{x}_0, \mathbf{u}_2) = 2\pi \text{ or } 0 \\ \Rightarrow\quad & d(\mathbf{x}_0, \mathbf{u}_1) = d(\mathbf{x}_0, \mathbf{u}_2) = \pi \text{ or } d(\mathbf{x}_0, \mathbf{u}_1) = d(\mathbf{x}_0, \mathbf{u}_2) = 0 \\ \Rightarrow\quad & \mathbf{u}_1 = \mathbf{u}_2 = -\mathbf{x}_0 \text{ or } \mathbf{u}_1 = \mathbf{u}_2 = \mathbf{x}_0, \end{aligned}$$

which contradicts that $\mathbf{x}_0 \neq \pm\mathbf{u}_1$. So this subcase can be excluded. Furthermore,

$$\begin{aligned} & C = 1 \\ \overset{\text{by (5.7)}}{\Rightarrow}\quad & (\mathbf{x} \cdot \mathbf{u}_1)^2 + (\mathbf{x} \cdot \mathbf{u}_2)^2 - 2(\mathbf{x} \cdot \mathbf{u}_1)(\mathbf{x} \cdot \mathbf{u}_2) = 0 \\ \Rightarrow\quad & \mathbf{x} \cdot \mathbf{u}_1 = \mathbf{x} \cdot \mathbf{u}_2 \end{aligned}$$

and then

$$\begin{aligned} & d(\mathbf{x}, \mathbf{u}_1) = d(\mathbf{x}, \mathbf{u}_2) \\ \overset{\text{by } C=1}{\Rightarrow}\quad & d(\mathbf{x}, \mathbf{u}_1) + d(\mathbf{x}, \mathbf{u}_2) = 2\pi \text{ or } 0 \\ \Rightarrow\quad & d(\mathbf{x}, \mathbf{u}_1) = d(\mathbf{x}, \mathbf{u}_2) = \pi \text{ or } 0 \text{ for all } \mathbf{x} \text{ on } (5.7) \\ \Rightarrow\quad & \mathbf{u}_1 = \mathbf{u}_2 \end{aligned}$$

and the spherical ellipse (5.7) degenerates into a single point.

Subcase 1.2: $\mathbf{x}_0 \cdot \mathbf{u}_1 + \mathbf{x}_0 \cdot \mathbf{u}_2 = 0$.

Again from the assumption $(\mathbf{x}_0' \cdot \mathbf{u}_1)(\mathbf{x}_0 \cdot \mathbf{u}_2) + (\mathbf{x}_0 \cdot \mathbf{u}_1)(\mathbf{x}_0' \cdot \mathbf{u}_2) \neq 0$, we have

$$\begin{cases} \mathbf{x}_0 \cdot \mathbf{u}_1 \neq 0, \\ \mathbf{x}_0 \cdot \mathbf{u}_2 \neq 0. \end{cases}$$

By (5.13) we have

$$C = \frac{(\mathbf{x}_0 \cdot \mathbf{u}_1)(\mathbf{x}_0' \cdot \mathbf{u}_1) + (\mathbf{x}_0 \cdot \mathbf{u}_2)(\mathbf{x}_0' \cdot \mathbf{u}_2)}{(\mathbf{x}_0' \cdot \mathbf{u}_1)(\mathbf{x}_0 \cdot \mathbf{u}_2) + (\mathbf{x}_0 \cdot \mathbf{u}_1)(\mathbf{x}_0' \cdot \mathbf{u}_2)} = \frac{(\mathbf{x}_0 \cdot \mathbf{u}_1)((\mathbf{x}_0' \cdot \mathbf{u}_1) - (\mathbf{x}_0' \cdot \mathbf{u}_2))}{(\mathbf{x}_0 \cdot \mathbf{u}_1)(-(\mathbf{x}_0' \cdot \mathbf{u}_1) + (\mathbf{x}_0' \cdot \mathbf{u}_2))} = -1.$$

Then by (5.7) we have

$$(\mathbf{x}\cdot\mathbf{u}_1)^2+(\mathbf{x}\cdot\mathbf{u}_2)^2+2(\mathbf{x}\cdot\mathbf{u}_1)(\mathbf{x}\cdot\mathbf{u}_2)=0 \quad \Rightarrow\ \mathbf{x}\cdot\mathbf{u}_1+\mathbf{x}\cdot\mathbf{u}_2=0$$
$$\Rightarrow\ \mathbf{x}_0'\cdot\mathbf{u}_1+\mathbf{x}_0'\cdot\mathbf{u}_2=0.$$

From $\mathbf{x}_0\cdot\mathbf{u}_1+\mathbf{x}_0\cdot\mathbf{u}_2=0$ we also have $\sqrt{1-(\mathbf{x}_0\cdot\mathbf{u}_1)^2}=\sqrt{1-(\mathbf{x}_0\cdot\mathbf{u}_2)^2}$. So (5.10) follows. Furthermore,

$$\mathbf{x}\cdot\mathbf{u}_1+\mathbf{x}\cdot\mathbf{u}_2=0 \quad \Rightarrow\ \mathbf{x}\cdot(\mathbf{u}_1+\mathbf{u}_2)=0$$
$$\Rightarrow\ \mathbf{x} \text{ is the great circle perpendicular to } \mathbf{u}_1+\mathbf{u}_2.$$

Subcase 1.3: $(\mathbf{x}_0'\cdot\mathbf{u}_1)^2(1-(\mathbf{x}_0\cdot\mathbf{u}_2)^2)-(\mathbf{x}_0'\cdot\mathbf{u}_2)^2(1-(\mathbf{x}_0\cdot\mathbf{u}_1)^2)=0$. In this subcase we have

$$\sqrt{1-(\mathbf{x}_0\cdot\mathbf{u}_2)^2}(\mathbf{x}_0'\cdot\mathbf{u}_1)=\pm\sqrt{1-(\mathbf{x}_0\cdot\mathbf{u}_1)^2}(\mathbf{x}_0'\cdot\mathbf{u}_2).$$

If $(\mathbf{x}_0'\cdot\mathbf{u}_1)=0$ then $\sqrt{1-(\mathbf{x}_0\cdot\mathbf{u}_1)^2}(\mathbf{x}_0'\cdot\mathbf{u}_2)=0$. In this case, $\sqrt{1-(\mathbf{x}_0\cdot\mathbf{u}_2)^2}(\mathbf{x}_0'\cdot\mathbf{u}_1)=-\sqrt{1-(\mathbf{x}_0\cdot\mathbf{u}_1)^2}(\mathbf{x}_0'\cdot\mathbf{u}_2)$ holds. So we assume $\mathbf{x}_0'\cdot\mathbf{u}_1\neq 0$. Suppose that

$$\sqrt{1-(\mathbf{x}_0\cdot\mathbf{u}_2)^2}(\mathbf{x}_0'\cdot\mathbf{u}_1)=\sqrt{1-(\mathbf{x}_0\cdot\mathbf{u}_1)^2}(\mathbf{x}_0'\cdot\mathbf{u}_2). \tag{5.15}$$

From $(\mathbf{x}_0\cdot\mathbf{u}_1)(\mathbf{x}_0\cdot\mathbf{u}_2)-\sqrt{1-(\mathbf{x}_0\cdot\mathbf{u}_1)^2}\sqrt{1-(\mathbf{x}_0\cdot\mathbf{u}_2)^2}=C$ and (5.13) we know

$$\begin{aligned}&\left[(\mathbf{x}_0\cdot\mathbf{u}_1)(\mathbf{x}_0\cdot\mathbf{u}_2)-\sqrt{1-(\mathbf{x}_0\cdot\mathbf{u}_1)^2}\sqrt{1-(\mathbf{x}_0\cdot\mathbf{u}_2)^2}\right][(\mathbf{x}_0'\cdot\mathbf{u}_1)(\mathbf{x}_0\cdot\mathbf{u}_2)+\\&+(\mathbf{x}_0\cdot\mathbf{u}_1)(\mathbf{x}_0'\cdot\mathbf{u}_2)]-[(\mathbf{x}_0\cdot\mathbf{u}_1)(\mathbf{x}_0'\cdot\mathbf{u}_1)+(\mathbf{x}_0\cdot\mathbf{u}_2)(\mathbf{x}_0'\cdot\mathbf{u}_2)]\ =0\end{aligned} \tag{5.16}$$

By (5.15), (5.16) is equal to

$$\begin{aligned}&\left[(\mathbf{x}_0\cdot\mathbf{u}_1)(\mathbf{x}_0\cdot\mathbf{u}_2)-\frac{(\mathbf{x}_0'\cdot\mathbf{u}_2)}{(\mathbf{x}_0'\cdot\mathbf{u}_1)}\sqrt{1-(\mathbf{x}_0\cdot\mathbf{u}_1)^2}\sqrt{1-(\mathbf{x}_0\cdot\mathbf{u}_1)^2}\right][(\mathbf{x}_0'\cdot\mathbf{u}_1)(\mathbf{x}_0\cdot\mathbf{u}_2)+\\&+(\mathbf{x}_0\cdot\mathbf{u}_1)(\mathbf{x}_0'\cdot\mathbf{u}_2)]-\left[(\mathbf{x}_0\cdot\mathbf{u}_1)(\mathbf{x}_0'\cdot\mathbf{u}_1)+(\mathbf{x}_0\cdot\mathbf{u}_2)(\mathbf{x}_0'\cdot\mathbf{u}_2)\right]\\=&\ \frac{2(\mathbf{x}_0'\cdot\mathbf{u}_2)\left[(\mathbf{x}_0\cdot\mathbf{u}_1)^2-1\right]\left[(\mathbf{x}_0'\cdot\mathbf{u}_1)(\mathbf{x}_0\cdot\mathbf{u}_2)+(\mathbf{x}_0\cdot\mathbf{u}_1)(\mathbf{x}_0'\cdot\mathbf{u}_2)\right]}{(\mathbf{x}_0'\cdot\mathbf{u}_1)}\end{aligned}$$

by $(\mathbf{x}_0' \cdot \mathbf{u}_1)^2 \left[1 - (\mathbf{x}_0 \cdot \mathbf{u}_2)^2\right] - (\mathbf{x}_0' \cdot \mathbf{u}_2)^2 \left[1 - (\mathbf{x}_0 \cdot \mathbf{u}_1)^2\right] = 0$ using Maple.

(**Maple code:** simplify($\left[(\mathbf{x}_0 \cdot \mathbf{u}_1)(\mathbf{x}_0 \cdot \mathbf{u}_2) - \frac{(\mathbf{x}_0' \cdot \mathbf{u}_2)}{(\mathbf{x}_0' \cdot \mathbf{u}_1)}(1 - (\mathbf{x}_0 \cdot \mathbf{u}_1)^2)\right] \times$
$\times \left[(\mathbf{x}_0' \cdot \mathbf{u}_1)(\mathbf{x}_0 \cdot \mathbf{u}_2) + (\mathbf{x}_0 \cdot \mathbf{u}_1)(\mathbf{x}_0' \cdot \mathbf{u}_2)\right] - \left[(\mathbf{x}_0 \cdot \mathbf{u}_1)(\mathbf{x}_0' \cdot \mathbf{u}_1) + (\mathbf{x}_0 \cdot \mathbf{u}_2)(\mathbf{x}_0' \cdot \mathbf{u}_2)\right],$
$\{(\mathbf{x}_0' \cdot \mathbf{u}_1)^2 \left[1 - (\mathbf{x}_0 \cdot \mathbf{u}_2)^2\right] = (\mathbf{x}_0' \cdot \mathbf{u}_2)^2 \left[1 - (\mathbf{x}_0 \cdot \mathbf{u}_1)^2\right]\}$);)

So

$$(5.15) \;\Rightarrow\; (\mathbf{x}_0' \cdot \mathbf{u}_2)\left[(\mathbf{x}_0 \cdot \mathbf{u}_1)^2 - 1\right]\left[(\mathbf{x}_0' \cdot \mathbf{u}_1)(\mathbf{x}_0 \cdot \mathbf{u}_2) + (\mathbf{x}_0 \cdot \mathbf{u}_1)(\mathbf{x}_0' \cdot \mathbf{u}_2)\right] = 0.$$

By the assumption $(\mathbf{x}_0' \cdot \mathbf{u}_1)(\mathbf{x}_0 \cdot \mathbf{u}_2) + (\mathbf{x}_0 \cdot \mathbf{u}_1)(\mathbf{x}_0' \cdot \mathbf{u}_2) \neq 0$, we have $(\mathbf{x}_0' \cdot \mathbf{u}_2)\left[(\mathbf{x}_0 \cdot \mathbf{u}_1)^2 - 1\right] = 0$, which implies $(\mathbf{x}_0' \cdot \mathbf{u}_1)^2 \left[1 - (\mathbf{x}_0 \cdot \mathbf{u}_2)^2\right] = 0$ by

$$(\mathbf{x}_0' \cdot \mathbf{u}_1)^2 \left[1 - (\mathbf{x}_0 \cdot \mathbf{u}_2)^2\right] - (\mathbf{x}_0' \cdot \mathbf{u}_2)^2 \left[1 - (\mathbf{x}_0 \cdot \mathbf{u}_1)^2\right] = 0.$$

Thus if $(\mathbf{x}_0' \cdot \mathbf{u}_1)^2 \left[1 - (\mathbf{x}_0 \cdot \mathbf{u}_2)^2\right] \neq 0$, then

$$\begin{aligned} & (\mathbf{x}_0' \cdot \mathbf{u}_1)^2 \left[1 - (\mathbf{x}_0 \cdot \mathbf{u}_2)^2\right] - (\mathbf{x}_0' \cdot \mathbf{u}_2)^2 \left[1 - (\mathbf{x}_0 \cdot \mathbf{u}_1)^2\right] = 0 \\ \Rightarrow\; & \sqrt{1 - (\mathbf{x}_0 \cdot \mathbf{u}_2)^2}(\mathbf{x}_0' \cdot \mathbf{u}_1) = -\sqrt{1 - (\mathbf{x}_0 \cdot \mathbf{u}_1)^2}(\mathbf{x}_0' \cdot \mathbf{u}_2). \end{aligned}$$

$\sqrt{1 - (\mathbf{x}_0 \cdot \mathbf{u}_2)^2}(\mathbf{x}_0' \cdot \mathbf{u}_1) = -\sqrt{1 - (\mathbf{x}_0 \cdot \mathbf{u}_1)^2}(\mathbf{x}_0' \cdot \mathbf{u}_2)$ still holds when $(\mathbf{x}_0' \cdot \mathbf{u}_2)\left[(\mathbf{x}_0 \cdot \mathbf{u}_1)^2 - 1\right] = 0$ since $(\mathbf{x}_0' \cdot \mathbf{u}_1)^2 \left[1 - (\mathbf{x}_0 \cdot \mathbf{u}_2)^2\right] = 0$ by $(\mathbf{x}_0' \cdot \mathbf{u}_1)^2 \left[1 - (\mathbf{x}_0 \cdot \mathbf{u}_2)^2\right] - (\mathbf{x}_0' \cdot \mathbf{u}_2)^2 \left[1 - (\mathbf{x}_0 \cdot \mathbf{u}_1)^2\right] = 0$. So we have
$\sqrt{1 - (\mathbf{x}_0 \cdot \mathbf{u}_2)^2}(\mathbf{x}_0' \cdot \mathbf{u}_1) = -\sqrt{1 - (\mathbf{x}_0 \cdot \mathbf{u}_1)^2}(\mathbf{x}_0' \cdot \mathbf{u}_2)$ in this subcase no matter if $(\mathbf{x}_0' \cdot \mathbf{u}_2)\left[(\mathbf{x}_0 \cdot \mathbf{u}_1)^2 - 1\right] = 0$.

Case 2: $(\mathbf{x}_0' \cdot \mathbf{u}_1)(\mathbf{x}_0 \cdot \mathbf{u}_2) + (\mathbf{x}_0 \cdot \mathbf{u}_1)(\mathbf{x}_0' \cdot \mathbf{u}_2) = 0$.

In this case by (5.12) we have $(\mathbf{x}_0 \cdot \mathbf{u}_1)(\mathbf{x}_0' \cdot \mathbf{u}_1) + (\mathbf{x}_0 \cdot \mathbf{u}_2)(\mathbf{x}_0' \cdot \mathbf{u}_2) = 0$.

Subcase 2.1: $\mathbf{x}_0 \cdot \mathbf{u}_1 \neq 0$.

$$\begin{cases} (\mathbf{x}_0' \cdot \mathbf{u}_1)(\mathbf{x}_0 \cdot \mathbf{u}_2) + (\mathbf{x}_0 \cdot \mathbf{u}_1)(\mathbf{x}_0' \cdot \mathbf{u}_2) = 0 \\ (\mathbf{x}_0 \cdot \mathbf{u}_1)(\mathbf{x}_0' \cdot \mathbf{u}_1) + (\mathbf{x}_0 \cdot \mathbf{u}_2)(\mathbf{x}_0' \cdot \mathbf{u}_2) = 0 \end{cases}$$

$$\Rightarrow \quad \frac{(\mathbf{x}_0' \cdot \mathbf{u}_2)(\mathbf{x}_0 \cdot \mathbf{u}_1 - \mathbf{x}_0 \cdot \mathbf{u}_2)(\mathbf{x}_0 \cdot \mathbf{u}_1 + \mathbf{x}_0 \cdot \mathbf{u}_2)}{(\mathbf{x}_0 \cdot \mathbf{u}_1)} = 0.$$

Subcase 2.1.1: If $\mathbf{x}_0' \cdot \mathbf{u}_2 = 0$ then

$$\begin{aligned} & (\mathbf{x}_0 \cdot \mathbf{u}_1)(\mathbf{x}_0' \cdot \mathbf{u}_1) + (\mathbf{x}_0 \cdot \mathbf{u}_2)(\mathbf{x}_0' \cdot \mathbf{u}_2) = 0 \\ \Rightarrow \quad & (\mathbf{x}_0 \cdot \mathbf{u}_1)(\mathbf{x}_0' \cdot \mathbf{u}_1) = 0 \\ \Rightarrow \quad & (\mathbf{x}_0' \cdot \mathbf{u}_1) = 0 \text{ because } (\mathbf{x}_0 \cdot \mathbf{u}_1) \neq 0. \end{aligned}$$

So $\sqrt{1 - (\mathbf{x}_0 \cdot \mathbf{u}_2)^2}(\mathbf{x}_0' \cdot \mathbf{u}_1) = -\sqrt{1 - (\mathbf{x}_0 \cdot \mathbf{u}_1)^2}(\mathbf{x}_0' \cdot \mathbf{u}_2)$ holds.

Subcase 2.1.2: If $\mathbf{x}_0 \cdot \mathbf{u}_1 - \mathbf{x}_0 \cdot \mathbf{u}_2 = 0$ then

$$(\mathbf{x}_0' \cdot \mathbf{u}_1)(\mathbf{x}_0 \cdot \mathbf{u}_2) + (\mathbf{x}_0 \cdot \mathbf{u}_1)(\mathbf{x}_0' \cdot \mathbf{u}_2) = 0 \Rightarrow \quad (\mathbf{x}_0' \cdot \mathbf{u}_1) + (\mathbf{x}_0' \cdot \mathbf{u}_2) = 0 \text{ since } (\mathbf{x}_0 \cdot \mathbf{u}_1) \neq 0.$$

So $\sqrt{1 - (\mathbf{x}_0 \cdot \mathbf{u}_2)^2}(\mathbf{x}_0' \cdot \mathbf{u}_1) = -\sqrt{1 - (\mathbf{x}_0 \cdot \mathbf{u}_1)^2}(\mathbf{x}_0' \cdot \mathbf{u}_2)$ holds.

Subcase 2.1.3: If $\mathbf{x}_0 \cdot \mathbf{u}_1 + \mathbf{x}_0 \cdot \mathbf{u}_2 = 0$ then

$$\begin{aligned} & \cos(d(\mathbf{x}_0, \mathbf{u}_1)) + \cos(d(\mathbf{x}_0, \mathbf{u}_2)) = 0 \\ \Rightarrow \quad & 2\cos\left(\frac{d(\mathbf{x}_0, \mathbf{u}_1) - d(\mathbf{x}_0, \mathbf{u}_2)}{2}\right)\cos\left(\frac{d(\mathbf{x}_0, \mathbf{u}_1) + d(\mathbf{x}_0, \mathbf{u}_2)}{2}\right) = 0 \end{aligned}$$

$$\Rightarrow \quad \left|\frac{d(\mathbf{x}_0, \mathbf{u}_1) - d(\mathbf{x}_0, \mathbf{u}_2)}{2}\right| = \frac{\pi}{2} \text{ or } \frac{d(\mathbf{x}_0, \mathbf{u}_1) + d(\mathbf{x}_0, \mathbf{u}_2)}{2} = \frac{\pi}{2}$$

$$\text{since } 0 \leq \left|\frac{d(\mathbf{x}_0, \mathbf{u}_1) - d(\mathbf{x}_0, \mathbf{u}_2)}{2}\right| \leq \pi \text{ and } 0 \leq \frac{d(\mathbf{x}_0, \mathbf{u}_1) + d(\mathbf{x}_0, \mathbf{u}_2)}{2} \leq \pi.$$

If $\left|\frac{d(\mathbf{x}_0, \mathbf{u}_1) - d(\mathbf{x}_0, \mathbf{u}_2)}{2}\right| = \frac{\pi}{2}$, then $d(\mathbf{x}_0, \mathbf{u}_1) = \pi$ and $d(\mathbf{x}_0, \mathbf{u}_2) = 0$ or $d(\mathbf{x}_0, \mathbf{u}_1) = 0$ and $d(\mathbf{x}_0, \mathbf{u}_2) = \pi$.

So $\mathbf{x}_0 = \mathbf{u}_2$ or $\mathbf{x}_0 = \mathbf{u}_1$. Contradiction.

If $\frac{d(\mathbf{x}_0,\mathbf{u}_1)+d(\mathbf{x}_0,\mathbf{u}_2)}{2} = \frac{\pi}{2}$, then

$$
\begin{aligned}
& d(\mathbf{x}_0, \mathbf{u}_1) + d(\mathbf{x}_0, \mathbf{u}_2) = \pi \\
\Rightarrow \quad & C = \cos(d(\mathbf{x}_0, \mathbf{u}_1) + d(\mathbf{x}_0, \mathbf{u}_2)) = -1 \\
\overset{(5.11)}{\Rightarrow} \quad & (\mathbf{x} \cdot \mathbf{u}_1)^2 + (\mathbf{x} \cdot \mathbf{u}_2)^2 + 2(\mathbf{x} \cdot \mathbf{u}_1)(\mathbf{x} \cdot \mathbf{u}_2) = 0 \\
\Rightarrow \quad & \mathbf{x} \cdot \mathbf{u}_1 + \mathbf{x} \cdot \mathbf{u}_2 = 0 \\
\Rightarrow \quad & \mathbf{x}'_0 \cdot \mathbf{u}_1 + \mathbf{x}'_0 \cdot \mathbf{u}_2 = 0 \\
\Rightarrow \quad & \sqrt{1 - (\mathbf{x}_0 \cdot \mathbf{u}_2)^2}(\mathbf{x}'_0 \cdot \mathbf{u}_1) = -\sqrt{1 - (\mathbf{x}_0 \cdot \mathbf{u}_1)^2}(\mathbf{x}'_0 \cdot \mathbf{u}_2).
\end{aligned}
$$

We also have $\mathbf{x}'_0 \cdot \mathbf{u}_1 - \mathbf{x}'_0 \cdot \mathbf{u}_2 = 0$ from $(\mathbf{x}_0 \cdot \mathbf{u}_1)(\mathbf{x}'_0 \cdot \mathbf{u}_1) + (\mathbf{x}_0 \cdot \mathbf{u}_2)(\mathbf{x}'_0 \cdot \mathbf{u}_2) = 0$. So in this subcase we actually have $(\mathbf{x}'_0 \cdot \mathbf{u}_1) = (\mathbf{x}'_0 \cdot \mathbf{u}_2) = 0$. Furthermore,

$$
\begin{aligned}
& \mathbf{x} \cdot \mathbf{u}_1 + \mathbf{x} \cdot \mathbf{u}_2 = 0 \\
\Rightarrow \quad & \mathbf{x} \cdot (\mathbf{u}_1 + \mathbf{u}_2) = 0 \\
\Rightarrow \quad & \mathbf{x} \text{ is the great circle perpendicular to } \mathbf{u}_1 + \mathbf{u}_2.
\end{aligned}
$$

Subcase 2.2: $\mathbf{x}_0 \cdot \mathbf{u}_1 = 0$.

$$
\begin{cases}
(\mathbf{x}'_0 \cdot \mathbf{u}_1)(\mathbf{x}_0 \cdot \mathbf{u}_2) + (\mathbf{x}_0 \cdot \mathbf{u}_1)(\mathbf{x}'_0 \cdot \mathbf{u}_2) = 0 \\
(\mathbf{x}_0 \cdot \mathbf{u}_1)(\mathbf{x}'_0 \cdot \mathbf{u}_1) + (\mathbf{x}_0 \cdot \mathbf{u}_2)(\mathbf{x}'_0 \cdot \mathbf{u}_2) = 0
\end{cases}
$$

$$
\Rightarrow \quad (\mathbf{x}'_0 \cdot \mathbf{u}_1)(\mathbf{x}_0 \cdot \mathbf{u}_2) = (\mathbf{x}_0 \cdot \mathbf{u}_2)(\mathbf{x}'_0 \cdot \mathbf{u}_2) = 0.
$$

Subcase 2.2.1: If $\mathbf{x}_0 \cdot \mathbf{u}_2 \neq 0$ then

$$
\begin{aligned}
& \mathbf{x}'_0 \cdot \mathbf{u}_1 = \mathbf{x}'_0 \cdot \mathbf{u}_2 = 0 \\
\Rightarrow \quad & \sqrt{1 - (\mathbf{x}_0 \cdot \mathbf{u}_2)^2}(\mathbf{x}'_0 \cdot \mathbf{u}_1) = -\sqrt{1 - (\mathbf{x}_0 \cdot \mathbf{u}_1)^2}(\mathbf{x}'_0 \cdot \mathbf{u}_2).
\end{aligned}
$$

Subcase 2.2.2: $\mathbf{x}_0 \cdot \mathbf{u}_2 = 0$. Then

$$\begin{aligned} & \mathbf{x}_0 \cdot \mathbf{u}_1 = \mathbf{x}_0 \cdot \mathbf{u}_2 = 0 \\ \Rightarrow \quad & d(\mathbf{x}_0, \mathbf{u}_1) = d(\mathbf{x}_0, \mathbf{u}_2) = \frac{\pi}{2} \\ \Rightarrow \quad & C = \cos(d(\mathbf{x}_0, \mathbf{u}_1) + d(\mathbf{x}_0, \mathbf{u}_2)) = -1, \end{aligned}$$

and then *Subcase 2.1.3* works. ■

Remark 11 *In (5.10), if we replace* $\mathbf{u}_1, \mathbf{u}_2$ *by* $-\mathbf{u}_1, -\mathbf{u}_2$ *(5.10) still holds. This means we have proven the following:*

Given any point $\mathbf{x}_0$ *on either of the spherical ellipse represented by (5.5), then the incident angle, made by the tangent* $\mathbf{x}_0'$ *and the great circle going through* $\mathbf{u}_1$ $(-\mathbf{u}_1)$ *and* $\mathbf{x}_0$*, is equal to the reflection angle, made by the tangent* $\mathbf{x}_0'$ *and the great circle going through* $\mathbf{x}_0$ *and* $\mathbf{u}_2$ $(-\mathbf{u}_2)$*.*

Remark 12 *When* $\mathbf{x}_0' \cdot \mathbf{u}_1 = \mathbf{x}_0' \cdot \mathbf{u}_2 = 0$*, the incident angle and the reflection angle are both* $\pi/2$*, i.e., the incident ray and the reflection ray are the great circle going through* $\mathbf{u}_1$ *and* $\mathbf{u}_2$ *which is perpendicular to* $\mathbf{x}_0'$*.*

As a result of Theorem 33 we have a similar theorem for the spherical hyperbola:

Theorem 34 *Let* $\mathbf{x} : I \to S^2$ *be a spherical hyperbola defined by*

$$(\mathbf{x} \cdot \mathbf{u}_1)(\mathbf{x} \cdot \mathbf{u}_2) + \sqrt{1 - (\mathbf{x} \cdot \mathbf{u}_1)^2}\sqrt{1 - (\mathbf{x} \cdot \mathbf{u}_2)^2} = C$$

with the foci $\mathbf{u}_1$ *and* $\mathbf{u}_2 \in S^2$, $\mathbf{u}_1 \cdot \mathbf{u}_2 \leq C \leq 1$*. Let* $\mathbf{x}_0 \in S^2$ *be any point on the hyperbola (5.7),* $\mathbf{x}_0 \neq \pm\mathbf{u}_1, \pm\mathbf{u}_2$ *and* $\mathbf{x}_0'$ *be the tangent to* $\mathbf{x}$ *at* $\mathbf{x}_0$*. Consider the great circle going through* $\mathbf{u}_1$ *and* $\mathbf{x}_0$ *as the incident ray and the great circle going through* $\mathbf{x}_0$ *and* $\mathbf{u}_2$ *as the reflection ray. Then the incident angle, made by the great circle going through* $\mathbf{u}_1$ *and* $\mathbf{x}_0$

and the tangent $\mathbf{x}_0'$*, is equal to the reflection angle, made by the great circle going through* $\mathbf{x}_0$ *and* $\mathbf{u}_2$ *and the tangent* $\mathbf{x}_0'$*.*

Proof. By (5.2) and (5.3) the spherical hyperbola (5.6) consists of a pair of spherical ellipses with foci $(\mathbf{u}_1, -\mathbf{u}_2)$ and $(-\mathbf{u}_1, \mathbf{u}_2)$. Then it follows from Theorem 33 that the incident angle, made by the great circle going through $\mathbf{u}_1$ and $\mathbf{x}_0$ and the tangent $\mathbf{x}_0'$ is equal to the angle, made by the great circle going through $\mathbf{x}_0$ and $-\mathbf{u}_2$ and the tangent $\mathbf{x}_0'$. The great circle going through $\mathbf{x}_0$ and $-\mathbf{u}_2$ is the same as the great circle going through $\mathbf{x}_0$ and $\mathbf{u}_2$. So the incident angle, made by the great circle going through $\mathbf{u}_1$ and $\mathbf{x}_0$ and the tangent $\mathbf{x}_0'$ is equal to the reflection angle, made by the great circle going through $\mathbf{x}_0$ and $\mathbf{u}_2$ and the tangent $\mathbf{x}_0'$. ■

Remark 13 *In [12] Namikawa states Theorem 33 and 34 without giving the proofs.*

5.4 When is a Spherical Conic a Circle?

Generally the spherical conic (5.7) is not a plane curve. If it is a plane curve, it is a circle. When is the spherical conic (5.7) a circle? Here is a theorem on when the spherical conic (5.7) is a circle:

Theorem 35 *Let* $\mathbf{x} : I \to S^2$ *be a unit speed spherical conic defined by (5.7) with the foci* $\mathbf{u}_1$ *and* $\mathbf{u}_2 \in S^2$*, and* $\mathbf{t}, \mathbf{n}, \mathbf{b}$ *be the Frenet-Serret trihedron. Then (5.7) is a circle if and only if* $\mathbf{u}_1 = \pm\mathbf{u}_2$ *or* $C = \pm 1$*.*

Proof. " $\Leftarrow$ " If $\mathbf{u}_1 = \pm\mathbf{u}_2$ it is obvious that $\mathbf{x}$ is a circle. If $C = 1$ and $\mathbf{u}_1 \neq \mathbf{u}_2$ then $(\mathbf{x} \cdot \mathbf{u}_1)^2 + (\mathbf{x} \cdot \mathbf{u}_2)^2 - 2(\mathbf{x} \cdot \mathbf{u}_1)(\mathbf{x} \cdot \mathbf{u}_2) = 0$ by (5.7). This implies

$(\mathbf{x} \cdot \mathbf{u}_1) - (\mathbf{x} \cdot \mathbf{u}_2) = 0 \Rightarrow \mathbf{x} \cdot (\mathbf{u}_1 - \mathbf{u}_2) = 0 \Rightarrow \mathbf{x}$ is the great circle perpendicular to $\mathbf{u}_1 - \mathbf{u}_2$.

Similarly, if $C = -1$ and $\mathbf{u}_1 \neq -\mathbf{u}_2$ we can prove that $\mathbf{x}$ is the great circle perpendicular to $\mathbf{u}_1 + \mathbf{u}_2$.

" $\Rightarrow$ " : $\mathbf{x}$ is a circle. So the curvature κ is constant.

Differentiating (5.7) we get

$$2(\mathbf{x}\cdot\mathbf{u}_1)(\mathbf{u}_1\cdot\mathbf{t}) + 2(\mathbf{x}\cdot\mathbf{u}_2)(\mathbf{u}_2\cdot\mathbf{t}) - 2C\left((\mathbf{u}_1\cdot\mathbf{t})(\mathbf{x}\cdot\mathbf{u}_2) + (\mathbf{x}\cdot\mathbf{u}_1)(\mathbf{u}_2\cdot\mathbf{t})\right) = 0. \tag{5.17}$$

As in Theorem 33, we consider the following cases:

Case 1: $((\mathbf{u}_1\cdot\mathbf{t})(\mathbf{x}\cdot\mathbf{u}_2) + (\mathbf{x}\cdot\mathbf{u}_1)(\mathbf{u}_2\cdot\mathbf{t}))(s) \neq 0$ for all $s \in I$. By (5.14) in Theorem 33 we have

$$\left(\left[(\mathbf{x}\cdot\mathbf{u}_1)^2 - (\mathbf{x}\cdot\mathbf{u}_2)^2\right]\left\{(\mathbf{u}_1\cdot\mathbf{t})^2\left[1-(\mathbf{x}\cdot\mathbf{u}_2)^2\right] - (\mathbf{u}_2\cdot\mathbf{t})^2\left[1-(\mathbf{x}\cdot\mathbf{u}_1)^2\right]\right\}\right)(s) = 0$$

for all $s \in I$. We consider the following subcases:

Subcase 1.1: $\left((\mathbf{u}_1\cdot\mathbf{t})^2\left[1-(\mathbf{x}\cdot\mathbf{u}_2)^2\right] - (\mathbf{u}_2\cdot\mathbf{t})^2\left[1-(\mathbf{x}\cdot\mathbf{u}_1)^2\right]\right)(s) = 0$ for all $s \in I$. The proof for this subcase will be given at the end of this theorem.

Subcase 1.2: $(\mathbf{x}\cdot\mathbf{u}_1 - \mathbf{x}\cdot\mathbf{u}_2)(s_0) = 0$ for some $s_0 \in I$. This implies $C = 1$ as in *Subcase 1.1* of Theorem 33.

Subcase 1.3: $(\mathbf{x}\cdot\mathbf{u}_1 + \mathbf{x}\cdot\mathbf{u}_2)(s_0) = 0$ for some $s_0 \in I$. This implies $C = -1$ as in *Subcase 1.2* of Theorem 33.

Case 2: $((\mathbf{u}_1\cdot\mathbf{t})(\mathbf{x}\cdot\mathbf{u}_2) + (\mathbf{x}\cdot\mathbf{u}_1)(\mathbf{u}_2\cdot\mathbf{t}))(s_0) = 0$ for some $s_0 \in I$. Thus

$$\begin{cases} ((\mathbf{u}_1\cdot\mathbf{t})(\mathbf{x}\cdot\mathbf{u}_2) + (\mathbf{x}\cdot\mathbf{u}_1)(\mathbf{u}_2\cdot\mathbf{t}))(s_0) = 0, \\ ((\mathbf{x}\cdot\mathbf{u}_1)(\mathbf{u}_1\cdot\mathbf{t}) + (\mathbf{x}\cdot\mathbf{u}_2)(\mathbf{u}_2\cdot\mathbf{t}))(s_0) = 0. \end{cases} \tag{5.18}$$

Subcase 2.1: $(\mathbf{x}\cdot\mathbf{u}_1)(s_0) \neq 0$. Then

$$\left(\frac{(\mathbf{u}_1\cdot\mathbf{t})(\mathbf{x}\cdot\mathbf{u}_1 - \mathbf{x}\cdot\mathbf{u}_2)(\mathbf{x}\cdot\mathbf{u}_1 + \mathbf{x}\cdot\mathbf{u}_2)}{(\mathbf{x}\cdot\mathbf{u}_1)}\right)(s_0) = 0 \tag{5.19}$$

by (5.18). From (5.19) we have the following subcases:

Subcase 2.1.1: $(\mathbf{u}_1 \cdot \mathbf{t})(s_0) = 0$. This implies $(\mathbf{u}_2 \cdot \mathbf{t})(s_0) = 0$ by (5.18) and $(\mathbf{x} \cdot \mathbf{u}_1)(s_0) \neq 0$. Then

$$\left((\mathbf{u}_1 \cdot \mathbf{t})^2 \left[1 - (\mathbf{x} \cdot \mathbf{u}_2)^2\right] - (\mathbf{u}_2 \cdot \mathbf{t})^2 \left[1 - (\mathbf{x} \cdot \mathbf{u}_1)^2\right]\right)(s_0) = 0.$$

Subcase 2.1.2: $(\mathbf{x} \cdot \mathbf{u}_1 - \mathbf{x} \cdot \mathbf{u}_2)(s_0) = 0$. This implies $(\mathbf{u}_1 \cdot \mathbf{t})(s_0) = -(\mathbf{u}_2 \cdot \mathbf{t})(s_0)$ by (5.18) and $(\mathbf{x} \cdot \mathbf{u}_1)(s_0) \neq 0$. Then

$$\left((\mathbf{u}_1 \cdot \mathbf{t})^2 \left[1 - (\mathbf{x} \cdot \mathbf{u}_2)^2\right] - (\mathbf{u}_2 \cdot \mathbf{t})^2 \left[1 - (\mathbf{x} \cdot \mathbf{u}_1)^2\right]\right)(s_0) = 0.$$

Subcase 2.1.3: $(\mathbf{x} \cdot \mathbf{u}_1 + \mathbf{x} \cdot \mathbf{u}_2)(s_0) = 0$. This implies $(\mathbf{u}_1 \cdot \mathbf{t})(s_0) = (\mathbf{u}_2 \cdot \mathbf{t})(s_0)$ by (5.18) and $(\mathbf{x} \cdot \mathbf{u}_1)(s_0) \neq 0$. Then

$$\left((\mathbf{u}_1 \cdot \mathbf{t})^2 \left[1 - (\mathbf{x} \cdot \mathbf{u}_2)^2\right] - (\mathbf{u}_2 \cdot \mathbf{t})^2 \left[1 - (\mathbf{x} \cdot \mathbf{u}_1)^2\right]\right)(s_0) = 0.$$

Subcase 2.2: $(\mathbf{x} \cdot \mathbf{u}_1)(s_0) = 0$. This implies $((\mathbf{u}_1 \cdot \mathbf{t})(\mathbf{x} \cdot \mathbf{u}_2))(s_0) = 0$ and $((\mathbf{x} \cdot \mathbf{u}_2)(\mathbf{u}_2 \cdot \mathbf{t}))(s_0) = 0$. Then we have the following subcases:

Subcase 2.2.1: $(\mathbf{x} \cdot \mathbf{u}_2)(s_0) \neq 0$. This implies $(\mathbf{u}_1 \cdot \mathbf{t})(s_0) = (\mathbf{u}_2 \cdot \mathbf{t})(s_0) = 0$. Then

$$\left((\mathbf{u}_1 \cdot \mathbf{t})^2 \left[1 - (\mathbf{x} \cdot \mathbf{u}_2)^2\right] - (\mathbf{u}_2 \cdot \mathbf{t})^2 \left[1 - (\mathbf{x} \cdot \mathbf{u}_1)^2\right]\right)(s_0) = 0.$$

Subcase 2.2.2: $(\mathbf{x} \cdot \mathbf{u}_2)(s_0) = 0$. Then by (5.7) and $(\mathbf{x} \cdot \mathbf{u}_1)(s_0) = 0$ we get $C = \pm 1$.

Summarizing *Case 2*, whenever $((\mathbf{u}_1 \cdot \mathbf{t})(\mathbf{x} \cdot \mathbf{u}_2) + (\mathbf{x} \cdot \mathbf{u}_1)(\mathbf{u}_2 \cdot \mathbf{t}))(s_0) = 0$ we always have

$$\left((\mathbf{u}_1 \cdot \mathbf{t})^2 \left[1 - (\mathbf{x} \cdot \mathbf{u}_2)^2\right] - (\mathbf{u}_2 \cdot \mathbf{t})^2 \left[1 - (\mathbf{x} \cdot \mathbf{u}_1)^2\right]\right)(s_0) = 0 \text{ or } C = \pm 1.$$

Now we just need to prove the theorem for *Subcase 1.1*:

We can reparametrize $\mathbf{x}$ such that $\mathbf{x}(s)$ is periodic function with the period 2π. We know that for the circle $\mathbf{x}$ we have

$$\mathbf{t}(s + \frac{\pi}{2}) \;=\; \mathbf{n}(s), \quad \mathbf{n}(s + \frac{\pi}{2}) = -\mathbf{t}(s),$$

$$\mathbf{t}(s+\pi) = -\mathbf{t}(s), \quad \mathbf{n}(s+\pi) = -\mathbf{n}(s),$$
$$\mathbf{t}(s+\frac{3\pi}{2}) = -\mathbf{n}(s), \quad \mathbf{n}(s+\frac{3\pi}{2}) = \mathbf{t}(s).$$

Since $\mathbf{x}$ is a circle on S^2, we can write $\mathbf{x} = -R\mathbf{n} - h\mathbf{b}$ by (2.4), where R is the radius of $\mathbf{x}$ and $h = -\mathbf{x}\cdot\mathbf{b}$. R, h and $\mathbf{b}$ are all constant. Then

$$((\mathbf{u}_1\cdot\mathbf{t})^2(1-(\mathbf{x}\cdot\mathbf{u}_2)^2) - (\mathbf{u}_2\cdot\mathbf{t})^2(1-(\mathbf{x}\cdot\mathbf{u}_1)^2)))(s) = 0$$

$\Longrightarrow$

$$\begin{aligned} & (\mathbf{u}_1\cdot\mathbf{t})^2\left(1-\left[R^2(\mathbf{u}_2\cdot\mathbf{n})^2+2Rh(\mathbf{u}_2\cdot\mathbf{n})(\mathbf{u}_2\cdot\mathbf{b})+h^2(\mathbf{u}_2\cdot\mathbf{b})^2\right]\right) \\ = \; & (\mathbf{u}_2\cdot\mathbf{t})^2\left(1-\left[R^2(\mathbf{u}_1\cdot\mathbf{n})^2+2Rh(\mathbf{u}_1\cdot\mathbf{n})(\mathbf{u}_1\cdot\mathbf{b})+h^2(\mathbf{u}_1\cdot\mathbf{b})^2\right]\right), \end{aligned} \tag{5.20}$$

here we omit the dependence on s.

Replacing s by $s+\pi/2$ in (5.20) we have

$$\begin{aligned} & (\mathbf{u}_1\cdot\mathbf{n})^2\left(1-\left[R^2(\mathbf{u}_2\cdot\mathbf{t})^2-2Rh(\mathbf{u}_2\cdot\mathbf{t})(\mathbf{u}_2\cdot\mathbf{b})+h^2(\mathbf{u}_2\cdot\mathbf{b})^2\right]\right) \\ = \; & (\mathbf{u}_2\cdot\mathbf{n})^2\left(1-\left[R^2(\mathbf{u}_1\cdot\mathbf{t})^2-2Rh(\mathbf{u}_1\cdot\mathbf{t})(\mathbf{u}_1\cdot\mathbf{b})+h^2(\mathbf{u}_1\cdot\mathbf{b})^2\right]\right). \end{aligned} \tag{5.21}$$

Replacing s by $s+\pi$ in (5.20) we have

$$\begin{aligned} & (\mathbf{u}_1\cdot\mathbf{t})^2\left(1-\left[R^2(\mathbf{u}_2\cdot\mathbf{n})^2-2Rh(\mathbf{u}_2\cdot\mathbf{n})(\mathbf{u}_2\cdot\mathbf{b})+h^2(\mathbf{u}_2\cdot\mathbf{b})^2\right]\right) \\ = \; & (\mathbf{u}_2\cdot\mathbf{t})^2\left(1-\left[R^2(\mathbf{u}_1\cdot\mathbf{n})^2-2Rh(\mathbf{u}_1\cdot\mathbf{n})(\mathbf{u}_1\cdot\mathbf{b})+h^2(\mathbf{u}_1\cdot\mathbf{b})^2\right]\right). \end{aligned} \tag{5.22}$$

Replacing s by $s+3\pi/2$ in (5.20) we have

$$\begin{aligned} & (\mathbf{u}_1\cdot\mathbf{n})^2\left(1-\left[R^2(\mathbf{u}_2\cdot\mathbf{t})^2+2Rh(\mathbf{u}_2\cdot\mathbf{t})(\mathbf{u}_2\cdot\mathbf{b})+h^2(\mathbf{u}_2\cdot\mathbf{b})^2\right]\right) \\ = \; & (\mathbf{u}_2\cdot\mathbf{n})^2\left(1-\left[R^2(\mathbf{u}_1\cdot\mathbf{t})^2+2Rh(\mathbf{u}_1\cdot\mathbf{t})(\mathbf{u}_1\cdot\mathbf{b})+h^2(\mathbf{u}_1\cdot\mathbf{b})^2\right]\right). \end{aligned} \tag{5.23}$$

(5.20) − (5.22) ⇒

$$(\mathbf{u}_1 \cdot \mathbf{t})^2(\mathbf{u}_2 \cdot \mathbf{n})(\mathbf{u}_2 \cdot \mathbf{b}) = (\mathbf{u}_2 \cdot \mathbf{t})^2(\mathbf{u}_1 \cdot \mathbf{n})(\mathbf{u}_1 \cdot \mathbf{b}). \tag{5.24}$$

(5.21) − (5.23) ⇒

$$(\mathbf{u}_1 \cdot \mathbf{n})^2(\mathbf{u}_2 \cdot \mathbf{t})(\mathbf{u}_2 \cdot \mathbf{b}) = (\mathbf{u}_2 \cdot \mathbf{n})^2(\mathbf{u}_1 \cdot \mathbf{t})(\mathbf{u}_1 \cdot \mathbf{b}). \tag{5.25}$$

(5.20) + (5.22) ⇒

$$(\mathbf{u}_1 \cdot \mathbf{t})^2 \left(1 - \left[R^2(\mathbf{u}_2 \cdot \mathbf{n})^2 + h^2(\mathbf{u}_2 \cdot \mathbf{b})^2\right]\right) = (\mathbf{u}_2 \cdot \mathbf{t})^2 \left(1 - \left[R^2(\mathbf{u}_1 \cdot \mathbf{n})^2 + h^2(\mathbf{u}_1 \cdot \mathbf{b})^2\right]\right). \tag{5.26}$$

(5.21) + (5.23) ⇒

$$(\mathbf{u}_1 \cdot \mathbf{n})^2 \left(1 - \left[R^2(\mathbf{u}_2 \cdot \mathbf{t})^2 + h^2(\mathbf{u}_2 \cdot \mathbf{b})^2\right]\right) = (\mathbf{u}_2 \cdot \mathbf{n})^2 \left(1 - \left[R^2(\mathbf{u}_1 \cdot \mathbf{t})^2 + h^2(\mathbf{u}_1 \cdot \mathbf{b})^2\right]\right). \tag{5.27}$$

(5.26) + (5.27) ⇒

$$\begin{aligned} & (\mathbf{u}_1 \cdot \mathbf{t})^2 + (\mathbf{u}_1 \cdot \mathbf{n})^2 - h^2\left[(\mathbf{u}_1 \cdot \mathbf{t})^2 + (\mathbf{u}_1 \cdot \mathbf{n})^2\right](\mathbf{u}_2 \cdot \mathbf{b})^2 \\ = \; & (\mathbf{u}_2 \cdot \mathbf{t})^2 + (\mathbf{u}_2 \cdot \mathbf{n})^2 - h^2\left[(\mathbf{u}_2 \cdot \mathbf{t})^2 + (\mathbf{u}_2 \cdot \mathbf{n})^2\right](\mathbf{u}_1 \cdot \mathbf{b})^2 \end{aligned}$$

So by $(\mathbf{u}_1 \cdot \mathbf{t})^2 + (\mathbf{u}_1 \cdot \mathbf{n})^2 + (\mathbf{u}_1 \cdot \mathbf{b})^2 = 1$ and $(\mathbf{u}_2 \cdot \mathbf{t})^2 + (\mathbf{u}_2 \cdot \mathbf{n})^2 + (\mathbf{u}_2 \cdot \mathbf{b})^2 = 1$, we have

$$\begin{aligned} & 1 - (\mathbf{u}_1 \cdot \mathbf{b})^2 - h^2\left[1 - (\mathbf{u}_1 \cdot \mathbf{b})^2\right](\mathbf{u}_2 \cdot \mathbf{b})^2 = 1 - (\mathbf{u}_2 \cdot \mathbf{b})^2 - h^2\left[1 - (\mathbf{u}_2 \cdot \mathbf{b})^2\right](\mathbf{u}_1 \cdot \mathbf{b})^2 \\ \Rightarrow \quad & -(\mathbf{u}_1 \cdot \mathbf{b})^2 - h^2(\mathbf{u}_2 \cdot \mathbf{b})^2 = -(\mathbf{u}_2 \cdot \mathbf{b})^2 - h^2(\mathbf{u}_1 \cdot \mathbf{b})^2 \\ \Rightarrow \quad & (1 - h^2)\left[(\mathbf{u}_1 \cdot \mathbf{b})^2 - (\mathbf{u}_2 \cdot \mathbf{b})^2\right] = 0. \end{aligned}$$

Then we have the following subcases:

Subcase 1.1.1: $h^2 = 1 \Rightarrow R^2 = 0 \Rightarrow \mathbf{x} = \pm\mathbf{b}$. This can happen only if $\mathbf{u}_1 = \mathbf{u}_2$ and $d(\mathbf{x}, \mathbf{u}_1) = 0$ or π. If $\mathbf{u}_1 \neq \mathbf{u}_2$, $\mathbf{x}$ must be a proper circle or a portion of the great circle connecting $\mathbf{u}_1$ and $\mathbf{u}_2$ if $C = \mathbf{u}_1 \cdot \mathbf{u}_2$.

Subcase 1.1.2: $(\mathbf{u}_1 \cdot \mathbf{b})^2 - (\mathbf{u}_2 \cdot \mathbf{b})^2 = 0$ and $1 - h^2 \neq 0$.

Subcase 1.1.2.1: $(\mathbf{u}_1 \cdot \mathbf{b}) = (\mathbf{u}_2 \cdot \mathbf{b}) = 0$. Then $\mathbf{u}_1$ and $\mathbf{u}_2$ are on the great circle which is perpendicular to $\mathbf{b}$ and parallel to the circle $\mathbf{x}$. $\mathbf{x}$ is a proper circle since $1 - h^2 \neq 0$. We can find only two points $\mathbf{x}_1$ and $\mathbf{x}_2$ on $\mathbf{x}$ such that $d(\mathbf{x}_i, \mathbf{u}_1) = d(\mathbf{x}_i, \mathbf{u}_2)$, $i = 1, 2$. No other points on $\mathbf{x}$ have such property. So $\mathbf{x}$ can't be a spherical hyperbola. On $\mathbf{x}$ we can find two points $\mathbf{x}_1$ and $\mathbf{x}_2$ such that $d(\mathbf{x}_1, \mathbf{u}_1) + d(\mathbf{x}_1, \mathbf{u}_2)$ is minimal and $d(\mathbf{x}_2, \mathbf{u}_1) + d(\mathbf{x}_2, \mathbf{u}_2)$ is maximal. Obviously, $d(\mathbf{x}_1, \mathbf{u}_1) + d(\mathbf{x}_1, \mathbf{u}_2) \neq d(\mathbf{x}_2, \mathbf{u}_1) + d(\mathbf{x}_2, \mathbf{u}_2)$. So $\mathbf{x}$ can't be a spherical ellipse either and this case is excluded.

Subcase 1.1.2.2: $(\mathbf{u}_1 \cdot \mathbf{b}) = (\mathbf{u}_2 \cdot \mathbf{b})$ but $(\mathbf{u}_1 \cdot \mathbf{b}) \neq 0$.

(5.24) $\Rightarrow$

$$(\mathbf{u}_1 \cdot \mathbf{t})^2 (\mathbf{u}_2 \cdot \mathbf{n}) = (\mathbf{u}_2 \cdot \mathbf{t})^2 (\mathbf{u}_1 \cdot \mathbf{n}), \tag{5.28}$$

(5.25) $\Rightarrow$

$$(\mathbf{u}_1 \cdot \mathbf{n})^2 (\mathbf{u}_2 \cdot \mathbf{t}) = (\mathbf{u}_2 \cdot \mathbf{n})^2 (\mathbf{u}_1 \cdot \mathbf{t}), \tag{5.29}$$

for all $s \in I$.

Subcase 1.1.2.2.1: $((\mathbf{u}_1 \cdot \mathbf{t})(\mathbf{u}_2 \cdot \mathbf{t})(\mathbf{u}_1 \cdot \mathbf{n})(\mathbf{u}_2 \cdot \mathbf{n}))(s) \neq 0$ for all $s \in I$. This implies

$$\begin{aligned} \frac{(\mathbf{u}_2 \cdot \mathbf{t})^2 (\mathbf{u}_1 \cdot \mathbf{n})}{(\mathbf{u}_1 \cdot \mathbf{n})^2 (\mathbf{u}_2 \cdot \mathbf{t})} = \frac{(\mathbf{u}_1 \cdot \mathbf{t})^2 (\mathbf{u}_2 \cdot \mathbf{n})}{(\mathbf{u}_2 \cdot \mathbf{n})^2 (\mathbf{u}_1 \cdot \mathbf{t})} \quad &\Rightarrow (\mathbf{u}_1 \cdot \mathbf{t})(\mathbf{u}_1 \cdot \mathbf{n}) - (\mathbf{u}_2 \cdot \mathbf{t})(\mathbf{u}_2 \cdot \mathbf{n}) = 0 \\ &\Rightarrow \frac{1}{2\kappa} \frac{d}{ds} \left[(\mathbf{u}_1 \cdot \mathbf{t})^2 - (\mathbf{u}_2 \cdot \mathbf{t})^2 \right] = 0 \\ &\Rightarrow (\mathbf{u}_1 \cdot \mathbf{t})^2 - (\mathbf{u}_2 \cdot \mathbf{t})^2 = a, \end{aligned}$$

where a is a constant.

Replacing s by $s+\pi/2$ we have

$$\begin{aligned}(\mathbf{u}_1\cdot\mathbf{n})^2-(\mathbf{u}_2\cdot\mathbf{n})^2=a \quad &\Rightarrow\ (\mathbf{u}_1\cdot\mathbf{t})^2+(\mathbf{u}_1\cdot\mathbf{n})^2-\left[(\mathbf{u}_2\cdot\mathbf{t})^2+(\mathbf{u}_2\cdot\mathbf{n})^2\right]=2a\\ &\Rightarrow\ 1-(\mathbf{u}_1\cdot\mathbf{b})^2-\left[1-(\mathbf{u}_2\cdot\mathbf{b})^2\right]=2a\\ &\Rightarrow\ a=0\\ &\Rightarrow\ (\mathbf{u}_1\cdot\mathbf{t})^2=(\mathbf{u}_2\cdot\mathbf{t})^2 \text{ and } (\mathbf{u}_1\cdot\mathbf{n})^2=(\mathbf{u}_2\cdot\mathbf{n})^2.\end{aligned}$$

By (5.28) and (5.29) we know $(\mathbf{u}_1\cdot\mathbf{t})=(\mathbf{u}_2\cdot\mathbf{t})$, $(\mathbf{u}_1\cdot\mathbf{n})=(\mathbf{u}_2\cdot\mathbf{n})$. Thus $\mathbf{u}_1=\mathbf{u}_2$.

Now we consider the cases where $((\mathbf{u}_1\cdot\mathbf{t})(\mathbf{u}_2\cdot\mathbf{t})(\mathbf{u}_1\cdot\mathbf{n})(\mathbf{u}_2\cdot\mathbf{n}))(s_0)=0$ for some $s_0\in I$.

Subcase 1.1.2.2.2: $(\mathbf{u}_1\cdot\mathbf{t})(s_0)=0$ for some $s_0\in I$. (5.28) or (5.29) $\Rightarrow$ $\left[(\mathbf{u}_2\cdot\mathbf{t})(\mathbf{u}_1\cdot\mathbf{n})\right](s_0)=0$. Then we have the following subcases:

Subcase 1.1.2.2.2.1: $(\mathbf{u}_1\cdot\mathbf{n})(s_0)=0$. Then

$$\mathbf{u}_1=\pm\mathbf{b}.$$

And

$$\begin{aligned}(5.26)\quad &\Rightarrow\quad \left[(\mathbf{u}_2\cdot\mathbf{t})^2(1-h^2)\right](s_0)=0;\\ (5.27)\quad &\Rightarrow\quad \left[(\mathbf{u}_2\cdot\mathbf{n})^2(1-h^2)\right](s_0)=0.\end{aligned}$$

Therefore

$$1-h^2\neq 0\quad\Rightarrow\quad (\mathbf{u}_2\cdot\mathbf{t})(s_0)=(\mathbf{u}_2\cdot\mathbf{n})(s_0)=0.$$

So $\mathbf{u}_2=\pm\mathbf{b}$. By the assumption $(\mathbf{u}_1\cdot\mathbf{b})=(\mathbf{u}_2\cdot\mathbf{b})$ we know $\mathbf{u}_1=\mathbf{u}_2=\pm\mathbf{b}$.

Subcase 1.1.2.2.2.2: $(\mathbf{u}_2\cdot\mathbf{t})(s_0)=0$.

(5.21)

$$\begin{aligned}&\Rightarrow\quad \left((\mathbf{u}_1\cdot\mathbf{n})^2\left[1-h^2(\mathbf{u}_2\cdot\mathbf{b})^2\right]\right)(s_0)=\left((\mathbf{u}_2\cdot\mathbf{n})^2\left[1-h^2(\mathbf{u}_1\cdot\mathbf{b})^2\right]\right)(s_0)\\ &\Rightarrow\quad \left(\left[(\mathbf{u}_1\cdot\mathbf{n})^2-(\mathbf{u}_2\cdot\mathbf{n})^2\right]\left[1-h^2(\mathbf{u}_1\cdot\mathbf{b})^2\right]\right)(s_0)=0\end{aligned}$$

since $(\mathbf{u}_1\cdot\mathbf{b})=(\mathbf{u}_2\cdot\mathbf{b})$.

Then

$\left((\mathbf{u}_1 \cdot \mathbf{n})^2 - (\mathbf{u}_2 \cdot \mathbf{n})^2\right)(s_0) = 0$ since $1-h^2 \neq 0$, $h^2 \leq 1$ and $(\mathbf{u}_1 \cdot \mathbf{b})^2 \leq 1$, i.e., $1-h^2(\mathbf{u}_1 \cdot \mathbf{b})^2 > 0$. And we have $(\mathbf{u}_1 \cdot \mathbf{n})(s_0) = \pm(\mathbf{u}_2 \cdot \mathbf{n})(s_0)$.

If $(\mathbf{u}_1 \cdot \mathbf{n})(s_0) = (\mathbf{u}_2 \cdot \mathbf{n})(s_0)$ we have $(\mathbf{u}_1 \cdot \mathbf{t})(s_0) = 0, (\mathbf{u}_2 \cdot \mathbf{t})(s_0) = 0, (\mathbf{u}_1 \cdot \mathbf{n})(s_0) = (\mathbf{u}_2 \cdot \mathbf{n})(s_0)$ and $(\mathbf{u}_1 \cdot \mathbf{b}) = (\mathbf{u}_2 \cdot \mathbf{b})$. This implies $\mathbf{u}_1 = \mathbf{u}_2$.

If $(\mathbf{u}_1 \cdot \mathbf{n})(s_0) = -(\mathbf{u}_2 \cdot \mathbf{n})(s_0)$ then we have $((\mathbf{u}_1 + \mathbf{u}_2) \cdot \mathbf{n})(s_0) = 0, (\mathbf{u}_1 \cdot \mathbf{t})(s_0) = 0, (\mathbf{u}_2 \cdot \mathbf{t})(s_0) = 0$ and $(\mathbf{u}_1 \cdot \mathbf{b}) = (\mathbf{u}_2 \cdot \mathbf{b})$.

In Figure 12, $\mathbf{u}_1 - \mathbf{u}_2$ is parallel to the circle $\mathbf{x}$ because $(\mathbf{u}_1 \cdot \mathbf{b}) = (\mathbf{u}_2 \cdot \mathbf{b})$. $\mathbf{A}$ and $\mathbf{B}$ are the intersections of the circle $\mathbf{x}$ and the great circle going through $\mathbf{u}_1$ and $\mathbf{u}_2$. $\mathbf{D}$ is the middle point between $\mathbf{A}$ and $\mathbf{B}$ on the circle $\mathbf{x}$. $\mathbf{u}_1 + \mathbf{u}_2$ is perpendicular to the circle $\mathbf{x}$ because $((\mathbf{u}_1+\mathbf{u}_2)\cdot\mathbf{n})(s_0) = 0, (\mathbf{u}_1\cdot\mathbf{t})(s_0) = 0, (\mathbf{u}_2\cdot\mathbf{t})(s_0) = 0$. Then $\mathbf{u}_1+\mathbf{u}_2$ goes through the center of $\mathbf{x}$. Also by $(\mathbf{u}_1 \cdot \mathbf{t})(s_0) = 0, (\mathbf{u}_2 \cdot \mathbf{t})(s_0) = 0$ we may take $\mathbf{A}$ to be $-h\mathbf{b} - R\mathbf{n}(s_0), \mathbf{B}$ to be $-h\mathbf{b} + R\mathbf{n}(s_0)$ and $\mathbf{D}$ to be $-h\mathbf{b} + R\mathbf{t}(s_0)$, noting (2.4) and $\mathbf{t}(s_0 + \frac{3\pi}{2}) = -\ \mathbf{n}(s_0)$.

If $\mathbf{x}$ is a spherical hyperbola then $|d(\mathbf{u}_1, \mathbf{A}) - d(\mathbf{u}_2, \mathbf{A})| = d(\mathbf{u}_1, \mathbf{u}_2)$ and $d(\mathbf{u}_1, \mathbf{D}) - d(\mathbf{u}_2, \mathbf{D}) = 0$ imply $d(\mathbf{u}_1, \mathbf{u}_2) = 0$ and thus $\mathbf{u}_1 = \mathbf{u}_2$.

If $\mathbf{x}$ is a spherical ellipse then $d(\mathbf{A}, \mathbf{B}) = d(\mathbf{u}_1, \mathbf{A}) + d(\mathbf{u}_2, \mathbf{A}) = d(\mathbf{u}_1, \mathbf{D}) + d(\mathbf{u}_2, \mathbf{D}) = 2d(\mathbf{u}_1, \mathbf{D})$. So $\mathbf{u}_1 \cdot \mathbf{D} = -h(\mathbf{u}_1 \cdot \mathbf{b}), \mathbf{u}_2 \cdot \mathbf{D} = -h(\mathbf{u}_2 \cdot \mathbf{b})$ and $\mathbf{A} \cdot \mathbf{B} = h^2 - R^2$ imply

$$\begin{aligned} 2\cos^2\left[\arccos(-h(\mathbf{u}_1 \cdot \mathbf{b}))\right] - 1 = h^2 - R^2 \quad &\Rightarrow \quad 2h^2(\mathbf{u}_1 \cdot \mathbf{b})^2 - 1 = h^2 - R^2 \\ &\Rightarrow \quad h^2((\mathbf{u}_1 \cdot \mathbf{b})^2 - 1) = 0 \text{ since } h^2 + R^2 = 1. \end{aligned}$$

If $(\mathbf{u}_1 \cdot \mathbf{b})^2 - 1 = 0$ then $\mathbf{u}_1 = \mathbf{u}_2 = \pm\mathbf{b}$ since $(\mathbf{u}_1 \cdot \mathbf{b}) = (\mathbf{u}_2 \cdot \mathbf{b})$ and $\mathbf{u}_1 \cdot \mathbf{u}_1 = \mathbf{u}_2 \cdot \mathbf{u}_2 = (\mathbf{b} \cdot \mathbf{b}) = 1$.

If $h = 0$ then $\mathbf{x} = -R\mathbf{n}$. And

$$((\mathbf{u}_1 + \mathbf{u}_2) \cdot \mathbf{n})(s_0) = 0 \quad \Rightarrow \quad ((\mathbf{u}_1 + \mathbf{u}_2) \cdot \mathbf{x})(s_0) = 0$$

$$\Rightarrow \quad \cos(d(\mathbf{u}_1, \mathbf{x}(s_0)) + \cos(d(\mathbf{u}_2, \mathbf{x}(s_0))) = 0$$

$$\Rightarrow \; 2\cos\left(\frac{d(\mathbf{u}_1, \mathbf{x}(s_0)) + d(\mathbf{u}_2, \mathbf{x}(s_0))}{2}\right)\cos\left(\frac{d(\mathbf{u}_1, \mathbf{x}(s_0)) - d(\mathbf{u}_2, \mathbf{x}(s_0))}{2}\right) = 0.$$

So

$$\cos\left(\frac{d(\mathbf{u}_1, \mathbf{x}(s_0)) + d(\mathbf{u}_2, \mathbf{x}(s_0))}{2}\right) = 0$$

$$\Rightarrow \quad d(\mathbf{u}_1, \mathbf{x}(s_0)) + d(\mathbf{u}_2, \mathbf{x}(s_0)) = \pi$$

$$\Rightarrow \quad C = -1.$$

And

$$\cos\left(\frac{d(\mathbf{u}_1, \mathbf{x}(s_0)) - d(\mathbf{u}_2, \mathbf{x}(s_0))}{2}\right) = 0$$

$$\Rightarrow \quad d(\mathbf{u}_1, \mathbf{x}(s_0)) - d(\mathbf{u}_2, \mathbf{x}(s_0)) = \pi$$

$$\Rightarrow \quad d(\mathbf{u}_1, \mathbf{x}(s_0)) = \pi, d(\mathbf{u}_2, \mathbf{x}(s_0)) = 0$$

$$\Rightarrow \quad \mathbf{u}_1 = -\mathbf{x}(s_0), \mathbf{u}_2 = \mathbf{x}(s_0)$$

$$\Rightarrow \quad \mathbf{u}_1 = -\mathbf{u}_2.$$

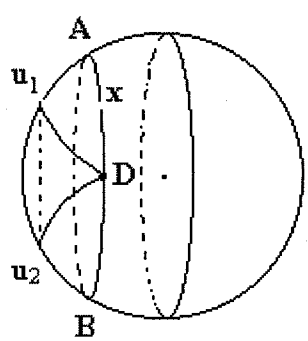


Figure 12: Subcase 1.1.2.2.2.2 $(\mathbf{u}_2 \cdot \mathbf{t})(s_0) = 0$

Subcase 1.1.2.2.3: $(\mathbf{u}_2 \cdot \mathbf{t})(s_0) = 0$ for some $s_0 \in I$. By the symmetry the proof of this subcase is similar to that of *Subcase 1.1.2.2.2*.

Subcase 1.1.2.2.4: $(\mathbf{u}_1 \cdot \mathbf{n})(s_0) = 0$ for some $s_0 \in I$.

$$(5.28) \text{ or } (5.29) \Rightarrow ((\mathbf{u}_1 \cdot \mathbf{t})(\mathbf{u}_2 \cdot \mathbf{n}))(s_0) = 0.$$

Then we have the following subcases:

Subcase 1.1.2.2.4.1: $(\mathbf{u}_1 \cdot \mathbf{t})(s_0) = 0$. This implies $\mathbf{u}_1 = \pm\mathbf{b}$. And

$$\begin{aligned} (5.26) \quad &\Rightarrow \quad \left((\mathbf{u}_2 \cdot \mathbf{t})^2(1 - h^2)\right)(s_0) = 0; \\ (5.27) \quad &\Rightarrow \quad \left((\mathbf{u}_2 \cdot \mathbf{n})^2(1 - h^2)\right)(s_0) = 0. \end{aligned}$$

Therefore

$$1 - h^2 \neq 0 \;\Rightarrow\; (\mathbf{u}_2 \cdot \mathbf{t})(s_0) = (\mathbf{u}_2 \cdot \mathbf{n})(s_0) = 0 \;\Rightarrow\; \mathbf{u}_2 = \pm\mathbf{b}.$$

By the assumption $(\mathbf{u}_1 \cdot \mathbf{b}) = (\mathbf{u}_2 \cdot \mathbf{b})$ we know $\mathbf{u}_1 = \mathbf{u}_2 = \pm\mathbf{b}$.

Subcase 1.1.2.2.4.2: $(\mathbf{u}_2 \cdot \mathbf{n})(s_0) = 0$.

$$\begin{aligned} (5.20) \quad &\Rightarrow \left((\mathbf{u}_1 \cdot \mathbf{t})^2 \left[1 - h^2(\mathbf{u}_2 \cdot \mathbf{b})^2\right]\right)(s_0) = \left((\mathbf{u}_2 \cdot \mathbf{t})^2 \left[1 - h^2(\mathbf{u}_1 \cdot \mathbf{b})^2\right]\right)(s_0) \\ &\Rightarrow \left(\left[(\mathbf{u}_1 \cdot \mathbf{t})^2 - (\mathbf{u}_2 \cdot \mathbf{t})^2\right]\left[1 - h^2(\mathbf{u}_1 \cdot \mathbf{b})^2\right]\right)(s_0) = 0 \\ &\text{since } (\mathbf{u}_1 \cdot \mathbf{b}) = (\mathbf{u}_2 \cdot \mathbf{b}). \end{aligned}$$

It follows $\left((\mathbf{u}_1 \cdot \mathbf{t})^2 - (\mathbf{u}_2 \cdot \mathbf{t})^2\right)(s_0) = 0$ since $1 - h^2(\mathbf{u}_1 \cdot \mathbf{b})^2 > 0$.

If $(\mathbf{u}_1 \cdot \mathbf{t})(s_0) = (\mathbf{u}_2 \cdot \mathbf{t})(s_0)$ then $\mathbf{u}_1 = \mathbf{u}_2$.

If $(\mathbf{u}_1 \cdot \mathbf{t})(s_0) = -(\mathbf{u}_2 \cdot \mathbf{t})(s_0)$ then we have

$$(\mathbf{u}_1 \cdot \mathbf{n})(s_0) = 0,\; (\mathbf{u}_2 \cdot \mathbf{n})(s_0) = 0,\; (\mathbf{u}_1 \cdot \mathbf{b}) = (\mathbf{u}_2 \cdot \mathbf{b}) \text{ and } \left((\mathbf{u}_1 + \mathbf{u}_2) \cdot \mathbf{t}\right)(s_0) = 0,$$

i.e.,

$$\begin{cases} (\mathbf{u}_1 \cdot \mathbf{t})(s_0 + \frac{\pi}{2}) = 0, \\ (\mathbf{u}_2 \cdot \mathbf{t})(s_0 + \frac{\pi}{2}) = 0, \\ (\mathbf{u}_1 \cdot \mathbf{b}) = (\mathbf{u}_2 \cdot \mathbf{b}) \\ ((\mathbf{u}_1 + \mathbf{u}_2) \cdot \mathbf{n})(s_0 + \frac{\pi}{2}) = 0. \end{cases}$$

Then the same argument as in *Subcase 1.1.2.2.2.2* applies here.

Subcase 1.1.2.2.5: $(\mathbf{u}_2 \cdot \mathbf{n})(s_0) = 0$ for some $s_0 \in I$. By the symmetry the proof of this subcase is similar to that of *Subcase 1.1.2.2.4*.

Subcase 1.1.2.3: $(\mathbf{u}_1 \cdot \mathbf{b}) = -(\mathbf{u}_2 \cdot \mathbf{b})$ but $(\mathbf{u}_1 \cdot \mathbf{b}) \neq 0$.

(5.24) $\Rightarrow$

$$(\mathbf{u}_1 \cdot \mathbf{t})^2(\mathbf{u}_2 \cdot \mathbf{n}) = -(\mathbf{u}_2 \cdot \mathbf{t})^2(\mathbf{u}_1 \cdot \mathbf{n}). \tag{5.30}$$

(5.25) $\Rightarrow$

$$(\mathbf{u}_1 \cdot \mathbf{n})^2(\mathbf{u}_2 \cdot \mathbf{t}) = -(\mathbf{u}_2 \cdot \mathbf{n})^2(\mathbf{u}_1 \cdot \mathbf{t}). \tag{5.31}$$

Subcase 1.1.2.3.1: $((\mathbf{u}_1 \cdot \mathbf{t})(\mathbf{u}_2 \cdot \mathbf{t})(\mathbf{u}_1 \cdot \mathbf{n})(\mathbf{u}_2 \cdot \mathbf{n}))(s) \neq 0$ for all $s \in I$. Then we have

$$\begin{aligned} & \frac{(\mathbf{u}_2 \cdot \mathbf{t})^2(\mathbf{u}_1 \cdot \mathbf{n})}{(\mathbf{u}_1 \cdot \mathbf{n})^2(\mathbf{u}_2 \cdot \mathbf{t})} = \frac{-(\mathbf{u}_1 \cdot \mathbf{t})^2(\mathbf{u}_2 \cdot \mathbf{n})}{-(\mathbf{u}_2 \cdot \mathbf{n})^2(\mathbf{u}_1 \cdot \mathbf{t})} \\ \Rightarrow \quad & (\mathbf{u}_1 \cdot \mathbf{t})(\mathbf{u}_1 \cdot \mathbf{n}) - (\mathbf{u}_2 \cdot \mathbf{t})(\mathbf{u}_2 \cdot \mathbf{n}) = 0. \end{aligned}$$

And the proof for *Subcase 1.1.2.2.1* works here.

Now we consider the cases where $((\mathbf{u}_1 \cdot \mathbf{t})(\mathbf{u}_2 \cdot \mathbf{t})(\mathbf{u}_1 \cdot \mathbf{n})(\mathbf{u}_2 \cdot \mathbf{n}))(s_0) = 0$ for some $s_0 \in I$.

Subcase 1.1.2.3.2: $(\mathbf{u}_1 \cdot \mathbf{t})(s_0) = 0$ for some $s_0 \in I$. (5.30) or (5.31) $\Rightarrow$ $((\mathbf{u}_2 \cdot \mathbf{t})(\mathbf{u}_1 \cdot \mathbf{n}))(s_0) = 0$. Then we have the following subcases:

Subcase 1.1.2.3.2.1: $(\mathbf{u}_1 \cdot \mathbf{n})(s_0) = 0$. This implies

$$\mathbf{u}_1 = \pm \mathbf{b}.$$

And

$$(5.26) \Rightarrow \left[(\mathbf{u}_2 \cdot \mathbf{t})^2(1-h^2)\right](s_0) = 0;$$

$$(5.27) \Rightarrow \left((\mathbf{u}_2 \cdot \mathbf{n})^2(1-h^2)\right)(s_0) = 0.$$

Therefore

$$1-h^2 \neq 0 \Rightarrow (\mathbf{u}_2 \cdot \mathbf{t})(s_0) = (\mathbf{u}_2 \cdot \mathbf{n})(s_0) = 0.$$

So $\mathbf{u}_2 = \pm\mathbf{b}$. By the assumption $(\mathbf{u}_1 \cdot \mathbf{b}) = -(\mathbf{u}_2 \cdot \mathbf{b})$ we know $\mathbf{u}_1 = -\mathbf{u}_2 = \pm\mathbf{b}$.

Subcase 1.1.2.3.2.2: $(\mathbf{u}_2 \cdot \mathbf{t})(s_0) = 0$.

$$(5.21) \Rightarrow \quad \left((\mathbf{u}_1 \cdot \mathbf{n})^2\left[1-h^2(\mathbf{u}_2 \cdot \mathbf{b})^2\right]\right)(s_0) = \left((\mathbf{u}_2 \cdot \mathbf{n})^2\left[1-h^2(\mathbf{u}_1 \cdot \mathbf{b})^2\right]\right)(s_0)$$

$$\Rightarrow \left(\left[(\mathbf{u}_1 \cdot \mathbf{n})^2 - (\mathbf{u}_2 \cdot \mathbf{n})^2\right]\left[1-h^2(\mathbf{u}_1 \cdot \mathbf{b})^2\right]\right)(s_0) = 0 \text{ since } (\mathbf{u}_1 \cdot \mathbf{b}) = -(\mathbf{u}_2 \cdot \mathbf{b}).$$

This implies

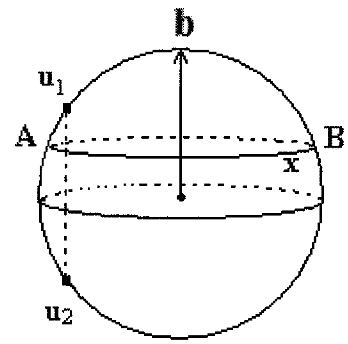
$$\left((\mathbf{u}_1 \cdot \mathbf{n})^2 - (\mathbf{u}_2 \cdot \mathbf{n})^2\right)(s_0) = 0 \text{ since } 1-h^2(\mathbf{u}_1 \cdot \mathbf{b})^2 > 0.$$

If $(\mathbf{u}_1 \cdot \mathbf{n})(s_0) = (\mathbf{u}_2 \cdot \mathbf{n})(s_0)$ then we have $(\mathbf{u}_1 \cdot \mathbf{t})(s_0) = 0$, $(\mathbf{u}_2 \cdot \mathbf{t})(s_0) = 0$, $(\mathbf{u}_1 \cdot \mathbf{b}) = -(\mathbf{u}_2 \cdot \mathbf{b})$ and $\left[(\mathbf{u}_1 - \mathbf{u}_2) \cdot \mathbf{n}\right](s_0) = 0$.

If $\mathbf{x}$ is a spherical ellipse then

$$\begin{aligned} d(\mathbf{u}_1, \mathbf{A}) + d(\mathbf{u}_2, \mathbf{A}) &= d(\mathbf{u}_1, \mathbf{B}) + d(\mathbf{u}_2, \mathbf{B}) \Rightarrow d(\mathbf{u}_1, \mathbf{u}_2) = 2\pi - d(\mathbf{u}_1, \mathbf{u}_2) \\ &\Rightarrow d(\mathbf{u}_1, \mathbf{u}_2) = \pi \Rightarrow \mathbf{u}_1 = -\mathbf{u}_2 \end{aligned}$$

(see Figure 13).



Figurel3: Subcase 1.1.2.3.2.2 $(\mathbf{u}_2 \cdot \mathbf{t})(s_0) = 0$

Suppose $\mathbf{x}$ is a spherical hyperbola.

$$\mathbf{x} = -h\mathbf{b} - R\mathbf{n} \Rightarrow \begin{cases} (\mathbf{u}_1 \cdot \mathbf{x}) = -h(\mathbf{u}_1 \cdot \mathbf{b}) - R(\mathbf{u}_1 \cdot \mathbf{n}) \\ (\mathbf{u}_2 \cdot \mathbf{x}) = -h(\mathbf{u}_2 \cdot \mathbf{b}) - R(\mathbf{u}_2 \cdot \mathbf{n}) = h(\mathbf{u}_1 \cdot \mathbf{b}) - R(\mathbf{u}_1 \cdot \mathbf{n}). \end{cases}$$

By (5.7) we have

$$\begin{aligned} &[-h(\mathbf{u}_1 \cdot \mathbf{b}) - R(\mathbf{u}_1 \cdot \mathbf{n})]^2 + [h(\mathbf{u}_1 \cdot \mathbf{b}) - R(\mathbf{u}_1 \cdot \mathbf{n})]^2 \\ &-2C\,[-h(\mathbf{u}_1 \cdot \mathbf{b}) - R(\mathbf{u}_1 \cdot \mathbf{n})]\,[h(\mathbf{u}_1 \cdot \mathbf{b}) - R(\mathbf{u}_1 \cdot \mathbf{n})] + C^2 - 1 \quad = 0 \end{aligned}$$

where $C = \cos(d(\mathbf{u}_2, \mathbf{A}) - d(\mathbf{u}_1, \mathbf{A})) = \sqrt{1-h^2} = R$. It follows

$$2h^2(\mathbf{u}_1 \cdot \mathbf{b})^2(1+R) - h^2 + 2R^2(R-1)(\mathbf{u}_1 \cdot \mathbf{n})^2 = 0. \tag{5.32}$$

Differentiating (5.32) we get $4R(R-1)(\mathbf{u}_1 \cdot \mathbf{n})(\mathbf{u}_1 \cdot \mathbf{t}) = 0$. If $\mathbf{u}_1$ is perpendicular to the circle $\mathbf{x}$ then $\mathbf{u}_1 = \pm\mathbf{b}$. By $(\mathbf{u}_1 \cdot \mathbf{b}) = -(\mathbf{u}_2 \cdot \mathbf{b})$ we know $\mathbf{u}_1 = -\mathbf{u}_2$. If $\mathbf{u}_1$ is not perpendicular to the circle $\mathbf{x}$ then we can find an s such that $[(\mathbf{u}_1 \cdot \mathbf{n})(\mathbf{u}_1 \cdot \mathbf{t})]\,(s) \neq 0$ and so $R = 1$ because $R = \sqrt{1-h^2} \neq 0$. Hence $C = 1$.

Subcase 1.1.2.3.3: $(\mathbf{u}_2 \cdot \mathbf{t})(s_0) = 0$ for some $s_0 \in I$. By the symmetry the proof of this subcase is similar to that of *Subcase 1.1.2.3.2.*

Subcase 1.1.2.3.4: $(\mathbf{u}_1 \cdot \mathbf{n})(s_0) = 0$ for some $s_0 \in I$.

$$(5.30) \text{ or } (5.31) \quad \Rightarrow \quad [(\mathbf{u}_2 \cdot \mathbf{n})(\mathbf{u}_1 \cdot \mathbf{t})]\,(s_0) = 0.$$

Then we have the following subcases:

Subcase 1.1.2.3.4.1: $(\mathbf{u}_1 \cdot \mathbf{t})(s_0) = 0$. It follows

$$\mathbf{u}_1 = \pm\mathbf{b}.$$

And

$$(5.26) \quad \Rightarrow \quad \left[(\mathbf{u}_2 \cdot \mathbf{t})^2(1 - h^2)\right](s_0) = 0;$$

$$(5.27) \quad \Rightarrow \quad \left[(\mathbf{u}_2 \cdot \mathbf{n})^2(1 - h^2)\right](s_0) = 0.$$

Therefore

$$1 - h^2 \neq 0 \quad \Rightarrow \quad (\mathbf{u}_2 \cdot \mathbf{t})(s_0) = (\mathbf{u}_2 \cdot \mathbf{n})(s_0) = 0 \quad \Rightarrow \quad \mathbf{u}_2 = \pm\mathbf{b}.$$

By the assumption $(\mathbf{u}_1 \cdot \mathbf{b}) = -(\mathbf{u}_2 \cdot \mathbf{b})$ we know $\mathbf{u}_1 = -\mathbf{u}_2 = \pm\mathbf{b}$.

Subcase 1.1.2.3.4.2: $(\mathbf{u}_2 \cdot \mathbf{n})(s_0) = 0$.

$$\begin{aligned} &(5.20) \\ \Rightarrow \quad &\left((\mathbf{u}_1 \cdot \mathbf{t})^2\left[1 - h^2(\mathbf{u}_2 \cdot \mathbf{b})^2\right]\right)(s_0) = \left((\mathbf{u}_2 \cdot \mathbf{t})^2\left[1 - h^2(\mathbf{u}_1 \cdot \mathbf{b})^2\right]\right)(s_0) \\ \Rightarrow \quad &\left(\left[(\mathbf{u}_1 \cdot \mathbf{t})^2 - (\mathbf{u}_2 \cdot \mathbf{t})^2\right]\left[1 - h^2(\mathbf{u}_1 \cdot \mathbf{b})^2\right]\right)(s_0) = 0 \\ &\text{since } (\mathbf{u}_1 \cdot \mathbf{b}) = -(\mathbf{u}_2 \cdot \mathbf{b}). \end{aligned}$$

Then $\left[(\mathbf{u}_1 \cdot \mathbf{t})^2 - (\mathbf{u}_2 \cdot \mathbf{t})^2\right](s_0) = 0$ since $1 - h^2(\mathbf{u}_1 \cdot \mathbf{b})^2 > 0$.

If $(\mathbf{u}_1 \cdot \mathbf{t})(s_0) = -(\mathbf{u}_2 \cdot \mathbf{t})(s_0)$ then $\mathbf{u}_1 = -\mathbf{u}_2$. Suppose $(\mathbf{u}_1 \cdot \mathbf{t})(s_0) = (\mathbf{u}_2 \cdot \mathbf{t})(s_0)$. Then we have $(\mathbf{u}_1 \cdot \mathbf{n})(s_0) = (\mathbf{u}_2 \cdot \mathbf{n})(s_0) = 0$, $[(\mathbf{u}_1 - \mathbf{u}_2) \cdot \mathbf{t}]\,(s_0) = 0$ and $(\mathbf{u}_1 \cdot \mathbf{b}) = -(\mathbf{u}_2 \cdot \mathbf{b})$, i.e.,

$(\mathbf{u}_1 \cdot \mathbf{t})(s_0 + \frac{\pi}{2}) = 0$, $(\mathbf{u}_2 \cdot \mathbf{t})(s_0 + \frac{\pi}{2}) = 0$, $(\mathbf{u}_1 \cdot \mathbf{b}) = -(\mathbf{u}_2 \cdot \mathbf{b})$ and $((\mathbf{u}_1 - \mathbf{u}_2) \cdot \mathbf{n})(s_0 + \frac{\pi}{2}) = 0$. Then the same argument as in *Subcase 1.1.2.3.2.2* applies here.

Subcase 1.1.2.3.5: $(\mathbf{u}_2 \cdot \mathbf{n})(s_0) = 0$ for some $s_0 \in I$. By the symmetry the proof of this subcase is similar to that of *Subcase 1.1.2.3.4.* ■

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Part II

Local Classifications of the Ruled Surfaces of Normals and Binormals of a Space Curve

CHAPTER 6

Introduction

Given a space curve, we can consider the ruled surfaces associated with the tangents, normals, binormals and Darboux vectors of the curve. Among these ruled surfaces the most studied one is the ruled surface of the tangents, i.e., the tangent developable.

Cleave studied the local form of the tangent developable at points of zero torsion on space curves in [4].

Mond gives the normal forms of the tangent developable of a space curve whose torsion vanishes to order k only $(0 \leq k \leq 4)$ in [10] and describes the developable surface of a space curve in the neighborhood of a point t_0 on the curve at which the curvature is nonvanishing and the torsion vanishes to order 4, or at which there is a nondegenerate zero of curvature.

Shcherbak in [15] investigates the hypersurface consisting of all the hyperplanes tangent to a space curve and the curve consisting of the osculating hyperplanes to the space curve in the dual projective space $\mathbb{RP}^n$, and study the local structure of these two objects in terms of the reflection points on the space curve.

Ishikawa in [6] classifies the C^∞-equivalency classes of the envelopes of the osculating hyperplanes to a curve in projective space and in [8] gives a local and topological classification of the tangent developables of space curves. For a survey on the tangent developables of space curves see [7].

Izumiya et al in [9] study the singularity of the rectifying developable, i.e., the ruled surfaces of Darboux vectors of the space curve.

For a unit speed curve it is trivial that its ruled surfaces of normals and binormals are locally smooth.

In this part we'll define the *type* of a smooth curve in terms of the *orthogonal-right equivalence* and then give the local classification of the ruled surfaces of normals and binormals of a general space curve under the left-right action $\mathcal{A}$ according to the types of the curve. For this purpose we prove two lemmas on the relationship of the powers of terms in the Taylor series of an invertible function and its inverse.

CHAPTER 7

Some Preliminaries and the Results

7.1 Definitions of the Ruled Surfaces of Normals and Binormals

Let $\gamma : (\mathbb{R}, 0) \to \mathbb{R}^3, 0$ be a smooth regular curve. Let $s(t)$ be the length function of the curve, then $\widetilde{\gamma}(t) = \gamma(s^{-1}(t))$ is a unit-speed reparametrization, and $\gamma(t) = \widetilde{\gamma}(s(t))$, $s'(t) = \|\gamma'(t)\|$. Let $\widetilde{\kappa}(s)$, $\widetilde{T}(s)$ and $\widetilde{N}(s)$ be the curvature, unit tangent and unit normal of $\widetilde{\gamma}(s)$, then the curvature, tangent and normal of $\gamma(t)$ are defined to be $\kappa(t) = \widetilde{\kappa}(s(t))$, $T(t) = \widetilde{T}(s(t))$ and $N(t) = \widetilde{N}(s(t))$. Differentiating $T(t) = \widetilde{T}(s(t))$ gives

$$T'(t) = s'(t)\widetilde{\kappa}(s(t))\,\widetilde{N}(s(t)) \Rightarrow N(t) = \widetilde{N}(s(t)) = \frac{1}{s'(t)\widetilde{\kappa}(s(t))}T'(t).$$

Differentiating $\gamma(t) = \widetilde{\gamma}(s(t))$ we get

$$\begin{aligned} \gamma'(t) = s'(t)\widetilde{T}(s(t)) \quad &\Rightarrow \quad T(t) = \widetilde{T}(s(t)) = \frac{\gamma'(t)}{s'(t)} \\ &\Rightarrow \quad T'(t) = \frac{s'(t)\gamma''(t) - s''(t)\gamma'(t)}{[s'(t)]^2}. \end{aligned}$$

So

$$\begin{aligned} N(t) &= \widetilde{N}(s(t)) \\ &= \frac{1}{s'(t)\widetilde{\kappa}(s(t))}\frac{s'(t)\gamma''(t) - s''(t)\gamma'(t)}{[s'(t)]^2} \\ &= \frac{\gamma''(t)s'(t) - s''(t)\gamma'(t)}{[s'(t)]^3\,\widetilde{\kappa}(s(t))}. \end{aligned}$$

Thus the direction of the normal of $\gamma(t)$ is $s'(t)\gamma''(t) - s''(t)\gamma'(t)$.

For the regular curve $\gamma(t)$ we define its *ruled surface of normals* to be

$$RS_n(\gamma)(t,u) = \gamma(t) + u\left[s'(t)\gamma''(t) - s''(t)\gamma'(t)\right].$$

The direction of the binormal of $\gamma(t)$ is $\gamma'(t) \times \gamma''(t)$. For the regular curve $\gamma(t)$ we define its *ruled surface of binormals* to be

$$RS_b(\gamma)(t,u) = \gamma(t) + u\left(\gamma'(t) \times \gamma''(t)\right)$$

7.2 Type of a Smooth Curve and $\mathcal{A}$-Equivalence

Definition 36 *Let $\gamma : (\mathbb{R},0) \to \mathbb{R}^3, 0$ be a smooth regular curve. We say that γ is of the finite type $(m, m+r, m+r+s)$ if there exist a diffeomorphism-germ $h : \mathbb{R}, 0 \to \mathbb{R}, 0$ and an orthogonal matrix $M \in \mathbb{R}^3 \times \mathbb{R}^3$ such that $M\gamma(h(t)) = (t^m, at^{m+r} + o(t^{m+r}), bt^{m+r+s} + o(t^{m+r+s}))$, where m, r, s are integers, $m \geq 1, r, s > 0$ and $a, b \neq 0$, $M\gamma$ stands for the product of the matrix M and column vector γ.*

Remark 14 *In this definition, for $M\gamma(h(t)) = (t^m, at^{m+r} + o(t^{m+r}), bt^{m+r+s} + o(t^{m+r+s}))$, if r or $s = 0$, e.g., $s = 0$, then applying the orthogonal matrix*

$$\frac{1}{\sqrt{a^2+b^2}}\begin{bmatrix} 1 & 0 & 0 \\ 0 & a & b \\ 0 & -b & a \end{bmatrix}$$

to $M\gamma(h(t))$ yields $(t^m, \sqrt{a^2+b^2}t^{m+r} + o(t^{m+r}), \frac{1}{\sqrt{a^2+b^2}}(-ab+ab)t^{m+r} + o(t^{m+r}))$. So we can always assume $r, s > 0$.

If $M\gamma(h(t)) = (t^m, at^{m+r} + o(t^{m+r}), bt^{m+r+s} + o(t^{m+r+s}))$ we call γ *orthogonal-right equivalent* to $(t^m, at^{m+r}+o(t^{m+r}), bt^{m+r+s}+o(t^{m+r+s}))$. So γ is of the type $(1, 1+r, 1+r+s)$ with $r, s > 0$ since γ is a regular curve and a permutation matrix is an orthogonal matrix. Given the left-right action of the group $\mathcal{A}$ (for its definition see Definition 37) we'll find the $\mathcal{A}$-type of $RS_n(\gamma)(t,u)$ and $RS_b(\gamma)(t,u)$, i.e., normal forms of $RS_n(\gamma)(t,u)$ and $RS_b(\gamma)(t,u)$

under the left-right action of the group $\mathcal{A}$ according to the types of the curve. Here is the definition of the left-right action of the group $\mathcal{A}$:

Definition 37 *Given two map-germs f and $g : \mathbb{R}^n, a \to \mathbb{R}^p, b$, f and g are said to be $\mathcal{A}$-equivalent if there are diffeomorphism-germs, h_1 of $\mathbb{R}^n, a$ and h_2 of $\mathbb{R}^p, b$ such that $f \circ h_1 = h_2 \circ g$, which is denoted by $f \overset{\mathcal{A}}{\backsim} g$.*

For this part $\mathcal{A}$ is the left-right action on $\mathbb{R}^2 \times \mathbb{R}^3$.

Now we prove that $RS_n(\gamma) \overset{\mathcal{A}}{\backsim} RS_n(\gamma(h))$ and $RS_b(\gamma) \overset{\mathcal{A}}{\backsim} RS_b(\gamma(h))$ for a diffeomorphism-germ h.

Proposition 38 *Let $\gamma : (\mathbb{R}, 0) \to \mathbb{R}^3, 0$ be a smooth regular curve and $h : \mathbb{R}, 0 \to \mathbb{R}, 0$ be a diffeomorphism-germ, then*

(i): $RS_n(\gamma) \overset{\mathcal{A}}{\backsim} RS_n(\gamma(h))$ and

(ii): $RS_b(\gamma) \overset{\mathcal{A}}{\backsim} RS_b(\gamma(h))$.

Proof. (i): Let $s_\gamma(t)$ and $s_{\gamma(h)}(t)$ be the length functions of the curves γ and $\gamma(h)$ respectively.

We have

$$\begin{cases} s'_\gamma(t) = \|\gamma'(t)\|, s''_\gamma(t) = \dfrac{\gamma'(t) \cdot \gamma''(t)}{\|\gamma'(t)\|}, \\ s'_{\gamma(h)}(t) = \| \left[\gamma\left(h(t)\right)\right]' \| = \|\gamma'\left(h(t)\right) h'(t)\|, \\ s''_{\gamma(h)}(t) = \dfrac{\left[\gamma'\left(h(t)\right) h'(t)\right] \cdot \left[\gamma''\left(h(t)\right)\left(h'(t)\right)^2 + \gamma'\left(h(t)\right) h''(t)\right]}{\|\gamma'\left(h(t)\right) h'(t)\|}. \end{cases}$$

So

$$RS_n(\gamma)(t, u) = \gamma(t) + \frac{u}{\|\gamma'(t)\|}\left\{\|\gamma'(t)\|^2 \gamma''(t) - \left[\gamma'(t) \cdot \gamma''(t)\right] \gamma'(t)\right\},$$

and

$$
\begin{aligned}
& RS_n(\gamma(h))(t,u) \\
= \;& \gamma(h(t)) + u\left[s'_{\gamma(h)}(t)\left[\gamma(h(t))\right]'' - s''_{\gamma(h)}(t)\left[\gamma(h(t))\right]'\right] \\
= \;& \gamma(h(t)) + u\left\{\|\gamma'(h(t))h'(t)\|\left[\gamma''(h(t))(h'(t))^2 + \gamma'(h(t))h''(t)\right]\right. \\
& \left. - \frac{(h'(t))^3\left[\gamma'(h(t))\cdot\gamma''(h(t))\right] + h'(t)h''(t)\left[\gamma'(h(t))\cdot\gamma'(h(t))\right]}{\|\gamma'(h(t))h'(t)\|} h'(t)\gamma'(h(t))\right\} \\
= \;& \gamma(h(t)) + \frac{u}{\|\gamma'(h(t))h'(t)\|}\left\{\|\gamma'(h(t))\|^2\left[(h'(t))^4\gamma''(h(t)) + (h'(t))^2h''(t)\gamma'(h(t))\right] + \right. \\
& \left. -(h'(t))^4\left[\gamma'(h(t))\cdot\gamma''(h(t))\right]\gamma'(h(t)) - \|\gamma'(h(t))\|^2(h'(t))^2h''(t)\gamma'(h(t))\right\} \\
= \;& \gamma(h(t)) + \frac{u(h'(t))^3}{\|\gamma'(h(t))\|}\left\{\|\gamma'(h(t))\|^2\gamma''(h(t)) - \left[\gamma'(h(t))\cdot\gamma''(h(t))\right]\gamma'(h(t))\right\}.
\end{aligned}
$$

Let $F(t,u) = (h(t), u(h'(t))^3)$, then $F : \mathbb{R}^2, 0 \to \mathbb{R}^2, 0$ is a diffeomorphism-germ and $RS_n(\gamma)(F(t,u)) = RS_n(\gamma(h))(t,u)$. Thus $RS_n(\gamma) \overset{\mathcal{A}}{\backsim} RS_n(\gamma(h))$.

(ii): Since

$$RS_b(\gamma)(t,u) = \gamma(t) + u\left(\gamma'(t) \times \gamma''(t)\right),$$

and

$$
\begin{aligned}
& RS_b\left(\gamma(h)\right)(t,u) \\
= \;& \gamma\left(h(t)\right) + u\left(h'(t)\gamma'\left(h(t)\right) \times \left[h'(t)\gamma''\left(h(t)\right) + h''(t)\gamma'\left(h(t)\right)\right]\right) \\
= \;& \gamma\left(h(t)\right) + u\left(h'(t)\right)^2\left(\gamma'\left(h(t)\right) \times \gamma''\left(h(t)\right)\right),
\end{aligned}
$$

by using $F(t,u) = \left(h(t), u\left(h'(t)\right)^2\right)$, we have

$$RS_b(\gamma(h))(t,u) = RS_b(\gamma)(F(t,u)).$$

Hence $RS_b(\gamma) \overset{\mathcal{A}}{\backsim} RS_b(\gamma(h))$. ∎

And then we prove that $RS_n(\gamma) \overset{\mathcal{A}}{\backsim} RS_n(M\gamma)$ and $RS_b(\gamma) \overset{\mathcal{A}}{\backsim} RS_b(M\gamma)$ for an orthogonal matrix M.

Proposition 39 *Let $\gamma : (\mathbb{R}, 0) \to \mathbb{R}^3, 0$ be a smooth regular curve and $M \in \mathbb{R}^3 \times \mathbb{R}^3$ be an orthogonal matrix. Then*

(i): $RS_n(\gamma) \overset{\mathcal{A}}{\backsim} RS_n(M\gamma)$ and

(ii): $RS_b(\gamma) \overset{\mathcal{A}}{\backsim} RS_b(M\gamma)$.

Proof. (i): Since M is an orthogonal matrix $\|M\mathbf{x}\| = \|\mathbf{x}\|$ for any $\mathbf{x} \in \mathbb{R}^3$. So

$$s_\gamma(t) = s_{M\gamma}(t), s'_\gamma(t) = s'_{M\gamma}(t), s''_\gamma(t) = s''_{M\gamma}(t).$$

Then

$$\begin{aligned} RS_n(M\gamma)(t,u) &= M\gamma(t) + u\left[s'_{M\gamma}(t)M\gamma''(t) - s''_{M\gamma}(t)M\gamma'(t)\right] \\ &= M\left(\gamma(t) + u\left[s'(t)\gamma''(t) - s''(t)\gamma'(t)\right]\right) \\ &= MRS_n(\gamma)(t,u). \end{aligned}$$

Therefore $RS_n(\gamma) \overset{\mathcal{A}}{\backsim} RS_n(M\gamma)$.

(ii): Since M is an orthogonal matrix $M\gamma$ is just the rotation of γ by some angle in $\mathbb{R}^3$. So

$$\begin{aligned} RS_b(M\gamma)(t,u) &= M\gamma(t) + u\left[M\gamma'(t) \times M\gamma''(t)\right] \\ &= M\left(\gamma(t) + u\left(\gamma'(t) \times \gamma''(t)\right)\right). \end{aligned}$$

Hence $RS_b(\gamma) \overset{\mathcal{A}}{\backsim} RS_b(M\gamma)$. ■

Now we'll consider the relationship between the vanishing orders of the curvature κ of γ, of $\gamma' \times \gamma^{(q)}$ and the type of γ. Obviously $\kappa(0) = 0 \Leftrightarrow \gamma'(0) \times \gamma''(0) = 0$. For the higher order derivatives of the curvature κ we have the following lemma. This lemma is due to Dr. Leslie Wilson.

Lemma 40 *Let $\gamma : (\mathbb{R}, 0) \to \mathbb{R}^3$ be a smooth regular curve, $q \geq 3$. Then the following are equivalent:*

(a) $\gamma'(0)\times \gamma''(0) = \gamma'(0)\times \gamma'''(0) = \cdots = \gamma'(0) \times \gamma^{(q-1)}(0) = 0$ but $\gamma'(0)\times \gamma^{(q)}(0) \neq 0$;

(b) γ is of type $(1, q, q+s)$, for some $s > 0$.

(c) $\kappa(0) = \kappa'(0) = \cdots = \kappa^{(q-3)}(0) = 0$ but $\kappa^{(q-2)}(0) \neq 0$.

Proof. Suppose M is an orthogonal matrix. Then

$$(M\gamma)'(t) \times (M\gamma)^{(r)}(t) = M\left[\gamma'(t) \times \gamma^{(r)}(t)\right]. \tag{7.1}$$

Let $h : \mathbb{R}, 0 \to \mathbb{R}, 0$ be a diffeomorphism-germ. By Faa' de Bruno's formula

$$(g(h))^{(m)} = \sum \frac{m!}{k_1! \cdots k_m! \cdot (1!)^{k_1} \cdots (m!)^{k_m}} \cdot g^{(n)} \cdot (h')^{k_1} \cdots (h^{(m)})^{k_m},$$

where the sum ranges over $n = 1, \ldots, m$ and all nonnegative integers $k_1, \ldots, k_m$ such that $k_1 + \cdots + k_m = n$ and $k_1 + 2k_2 + \cdots + mk_m = m$, and $h^{(i)}$ is the i-th derivative of h (see [14]), it is easy to see that $\gamma'(0) \times \gamma''(0) = \gamma'(0) \times \gamma'''(0) = \cdots = \gamma'(0) \times \gamma^{(q-1)}(0) = 0$ but $\gamma'(0) \times \gamma^{(q)}(0) \neq 0, q \geq 3$ implies that $(\gamma(h))'(0) \times (\gamma(h))''(0) = (\gamma(h))'(0) \times (\gamma(h))'''(0) = \cdots = (\gamma(h))'(0) \times (\gamma(h))^{(q-1)}(0) = 0$ but $(\gamma(h))'(0) \times (\gamma(h))^{(q)}(0) \neq 0$, $q \geq 3$. Applying this argument to the curve $\gamma(h)$ and the diffeomorphism-germ $h^{-1} : \mathbb{R}, 0 \to \mathbb{R}, 0$ and by (7.1), we know that $\gamma'(0)\times \gamma''(0) = \gamma'(0)\times \gamma'''(0) = \cdots = \gamma'(0) \times \gamma^{(q-1)}(0) = 0$ but $\gamma'(0)\times \gamma^{(q)}(0) \neq 0, q \geq 3$ is equivalent to $(M\gamma(h))'(0)\times (M\gamma(h))''(0) = (M\gamma(h))'(0)\times (M\gamma(h))'''(0) = \cdots = (M\gamma(h))'(0) \times (M\gamma(h))^{(q-1)}(0) = 0$ but $(M\gamma(h))'(0) \times (M\gamma(h))^{(q)}(0) \neq 0$, $q \geq 3$.

$(a) \Rightarrow (b)$: We have

$$\gamma'(t) \times \gamma^{(r)}(t) = \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & \gamma_2'(t) & \gamma_3'(t) \\ 0 & \gamma_2^{(r)}(t) & \gamma_3^{(r)}(t) \end{pmatrix} \tag{7.2}$$

$$= (\gamma_2'(t)\gamma_3^{(r)}(t) - \gamma_2^{(r)}(t)\gamma_3'(t), -\gamma_3^{(r)}(t), \gamma_2^{(r)}(t))$$

If γ is orthogonal-right equivalent to $(t^m+o(t^m), at^{m+r}+o(t^{m+r}), bt^{m+r+s}+o(t^{m+r+s})), a \neq 0,\ b \neq 0$, then we must have $m = 1$ and $1 + r = q$ and γ is of type $(1, q, q+s)$, for some $s > 0$.

$(b) \Rightarrow (a)$: It's obvious by the first paragraph above.

$(b) \Rightarrow (c)$: Let $\gamma(t) = (\gamma_1(t), \gamma_2(t), \gamma_3(t))$ be of type $(1, q, q+s),\ q \geq 3,\ s > 0$. Then there exist an orthogonal matrix M and a diffeomorphism-germ $h : \mathbb{R}, 0 \to \mathbb{R}, 0$ such that $M\gamma(h(t)) = (t, b_pt^q + b_{p+1}t^{q+1} + \cdots, c_{p+s}t^{q+s} + c_{p+s+1}t^{q+s+1} + \cdots)$ for some $b_q, b_{q+1}, \cdots, c_{q+s}, c_{q+s+1}, \cdots \in \mathbb{R},\ b_q \neq 0,\ c_{q+s} \neq 0$. We denote $M\gamma(h(t))$ by $\overline{\gamma}(t)$. Let $\overline{\kappa}$ be the curvature of $\overline{\gamma}$. From Section 7 we know the unit binormal

$$B(t) = N(t) \times T(t) = \frac{\gamma'(t)}{s'(t)} \times \frac{\gamma''(t)s'(t) - s''(t)\gamma'(t)}{[s'(t)]^3\,\kappa(t)} = \frac{\gamma'(t) \times \gamma''(t)}{[s'(t)]^3\,\kappa(t)}.$$

So

$$\kappa(t)B(t) = \frac{\gamma'(t) \times \gamma''(t)}{[s'(t)]^3}. \tag{7.3}$$

By $\|B(t)\| = 1$, $s'(t) \neq 0$ and (7.3) we have

$$|\kappa(t)| = \frac{1}{|s'(t)|^3}\|\gamma'(t) \times \gamma''(t)\|. \tag{7.4}$$

By $\|(M\gamma)'(t) \times (M\gamma)''(t)\| = \|M\,[\gamma'(t) \times \gamma''(t)]\,\| = \|\gamma'(t) \times \gamma''(t)\|$, $\|(\gamma(h))'(t) \times (\gamma(h))''(t)\| = |h'(t)|\,\|(\gamma' \times \gamma''(t))(h(t))\|$ and Proposition 2.19 in [3] we know $\kappa(0) = \kappa'(0) =$

$\cdots = \kappa^{(q-3)}(0) = 0$ but $\kappa^{(q-2)}(0) \neq 0$ is equivalent to $\overline{\kappa}(0) = \overline{\kappa}'(0) = \cdots = \overline{\kappa}^{(q-3)}(0) = 0$ but $\overline{\kappa}^{(q-2)}(0) \neq 0$. So we may just assume $\gamma(t) = (t, b_p t^q + b_{p+1} t^{q+1} + \cdots, c_{p+s} t^{q+s} + c_{p+s+1} t^{q+s+1} + \cdots)$ for the purpose of this lemma.

By (7.3) and

$$\gamma'(t) \times \gamma''(t) = t^{q-2}((d_1 e_2 - d_2 e_1) b_q c_{q+s} t^{q+s-1} + O(t^{q+s+1}), -e_2 c_{q+s} t^s + O(t^{s+1}), d_2 b_q + O(t))$$

for $q \geq 2$, we have

$$\kappa(t) = d_2 b_q t^{q-2} u$$

where u is a unit. So $\kappa(0) = \kappa'(0) = \cdots = \kappa^{(q-3)}(0) = 0$ but $\kappa^{(q-2)}(0) \neq 0$ is equivalent to that γ is of type $(1, q, q+s)$, for some $s > 0$. ■

7.3 Main Theorems

We'll prove the following results in the following sections.

Theorem 41 *Let $\gamma : (\mathbb{R}, 0) \to \mathbb{R}^3, 0$ be a smooth regular curve. Let $\mathcal{A}$ be the left-right action on $\mathbb{R}^2 \times \mathbb{R}^3$.*

(i) If γ is of the type $(1,2,s)$, $s \geq 3$, i.e., $\gamma'(0) \times \gamma''(0) \neq 0$, then $RS_n(\gamma)(t,u)$ is $\mathcal{A}$-equivalent to the map $(t,u) \to (t,u,0)$, i.e., an immersion.

(ii) If γ is of the type $(1, q, q+s)$, $q \geq 3, s \geq 1$, or equivalently $\kappa(0) = \kappa'(0) = \cdots = \kappa^{(q-3)}(0) = 0$ but $\kappa^{(q-2)}(0) \neq 0$ then $RS_n(\gamma)(t,u) \overset{\mathcal{A}}{\sim} (t, t^{q-2}u, 0)$;

Theorem 42 *Let $\gamma : (\mathbb{R}, 0) \to \mathbb{R}^3, 0$ be a smooth regular curve. Let $\mathcal{A}$ be the left-right action on $\mathbb{R}^2 \times \mathbb{R}^3$.*

(i) If γ is of the type $(1,2,s)$, $s \geq 3$, i.e., $\gamma'(0) \times \gamma''(0) \neq 0$, then $RS_b(\gamma)(t,u) \overset{\mathcal{A}}{\sim} (t,u,0)$.

(ii) If γ is of the type $(1,q,q+s)$, $s \geq 1, q \geq 3$, or equivalently $\kappa(0) = \kappa'(0) = \cdots = \kappa^{(q-3)}(0) = 0$ but $\kappa^{(q-2)}(0) \neq 0$ then $RS_b(\gamma)(t,u) \overset{\mathcal{A}}{\sim} (t, t^{q-2}u, 0)$.

We first prove 2 lemmas.

7.4 Two Lemmas about Taylor Series

We denote the Taylor expansion of the function f about $t = a$ by $\widehat{f}_a$.

Lemma 43 *Let (t,u) be the coordinates in $\mathbb{R}^2$. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be C^∞ function and $f(0,u) = 0$ for any $u \in \mathbb{R}$. Given a point $(0,b) \in \mathbb{R}^2$, suppose $\frac{\partial f}{\partial t}(0,b) \neq 0$ (so $f(t,u)$ has an inverse function, denoted by $f^{-1}(t,u)$, with respect to t in a neighborhood of $(0,b)$) and the Taylor expansion $\widehat{f}_{(0,b)}$ of f about $(t,u) = (0,b)$ has such a form:*

$$\widehat{f}_{(0,b)} = \sum_{i=1}^{r_0} a_i t^i + \sum_{i \geq r_0+1,\, i - c\deg(p_i(u)) \geq r} t^i p_i(u),$$

where $r, c \geq 1$ are integers and $p_i(u)$ is a polynomial in u, and $r_0 \geq 1$ is the largest integer such that $\deg(a_i) = 0$ for $1 \leq i \leq r_0$ and $\deg(p_{r_0+1}(u)) > 0$. Then the Taylor expansion $\overset{\wedge}{f^{-1}}_{(0,b)}$ of $f^{-1}(t,u)$ about $(t,u) = (0,b)$ has the form:

$$\overset{\wedge}{f^{-1}}_{(0,b)} = \sum_{i=1}^{r_0} b_i t^i + \sum_{i \geq r_0+1,\, i - c\deg(q_i(u)) \geq r} t^i q_i(u), \tag{7.5}$$

where b_i's are some constants and $q_i(u)$'s are some polynomials in u with $i - c\deg(q_i(u)) \geq r$ and r_0 is the largest integer such that $\deg(b_i) = 0$ for $1 \leq i \leq r_0$ and $\deg(q_{r_0+1}(u)) > 0$. Furthermore, if $a_2 = a_3 = \cdots = a_{r_0'} = 0$ for $2 \leq r_0' \leq r_0$, then $b_2 = b_3 = \cdots = b_{r_0'} = 0$.

Proof. For the simplicity we omit the dependence on u and let $g(t) = f^{-1}(t,u)$ and $h(t) = f(t,u)$. So $g(0) = h(0) = 0$. Differentiating $g(h(t)) = t$ and evaluating at $t = 0$ gives

$g'(0) \cdot h'(0) = 1$. Now we'll use Faa' de Bruno's formula to show the Lemma:

$$(g(h))^{(m)} = \sum \frac{m!}{k_1! \cdots k_m! \cdot (1!)^{k_1} \cdots (m!)^{k_m}} \cdot g^{(n)} \cdot (h')^{k_1} \cdots (h^{(m)})^{k_m}, \tag{7.6}$$

where the sum ranges over $n = 1, \ldots, m$ and all nonnegative integers $k_1, \ldots, k_m$ such that $k_1 + \cdots + k_m = n$ and $k_1 + 2k_2 + \cdots + mk_m = m$, and $h^{(i)}$ is the i-th derivative of h. (see [14]) All the derivatives of h are taken at $t = 0$, those of g at $h(0) = 0$.

If $n = m$, then

$$\begin{cases} k_1 + \cdots + k_m = n \\ k_1 + 2k_2 + \cdots + mk_m = m \end{cases} \Rightarrow \quad k_2 + \cdots + (m-1)k_m = 0$$

$$\Rightarrow \begin{cases} k_2 = k_3 = \cdots = k_m = 0 \\ k_1 = m; \end{cases}$$

and if $k_1 = m$, then

$$k_1 + 2k_2 + \cdots + mk_m = m \quad \Rightarrow \begin{cases} k_2 = k_3 = \cdots = k_m = 0 \\ n = m. \end{cases}$$

So $n = m \Leftrightarrow k_1 = m$.

If $n = 1$, then $k_1 = k_2 = \cdots = k_{m-1} = 0$ and $k_m = 1$; if $k_m = 1$, then $k_1 + 2k_2 + \cdots + mk_m = m$ implies $k_1 = k_2 = \cdots = k_{m-1} = 0$ and by $k_1 + \cdots + k_m = n$ we have $n = k_m = 1$. So $k_m = 1 \Leftrightarrow n = 1$.

By Faa' de Bruno's formula (7.6), for $m > 1$ we have

$$\begin{aligned} g^{(m)} &= -\frac{g' \cdot h^{(m)}}{(h')^m} - \left\{ \sum \frac{m!}{k_1! \cdots k_{m-1}! \cdot (1!)^{k_1} \cdots [(m-1)!]^{k_{m-1}}} \right\} \cdot \\ &\quad \cdot \frac{g^{(n)} \cdot (h')^{k_1} \cdots (h^{(m-1)})^{k_{m-1}}}{(h')^m}, \end{aligned} \tag{7.7}$$

where the sum ranges over $n = 2, \ldots, m-1$ and all nonnegative integers $k_1, \ldots, k_{m-1}$ such that $k_1 + \cdots + k_{m-1} = n$ and $k_1 + 2k_2 + \cdots + (m-1)k_{m-1} = m$. Especially, when $m = 2, 3$

we get $g^{(2)} \cdot (h')^2 + g' \cdot h^{(2)} = 0$, and $g^{(3)} \cdot (h')^3 + 3g^{(2)} \cdot h' \cdot h^{(2)} + g' \cdot h^{(3)} = 0$.

Hence $g^{(2)} = -\frac{g' \cdot h^{(2)}}{(h')^2}$, $g^{(3)} = -\frac{3g^{(2)} \cdot h' \cdot h^{(2)} + g' \cdot h^{(3)}}{(h')^3}, \ldots$.
By assumption, $h'(0), h^{(2)}(0), \ldots, h^{(r_0)}(0)$ are constants and $h^{(r_0+1)}(0), h^{(r_0+2)}(0), \ldots$ are all the polynomials in u, with $a_{(r_0+1)j}u^j \neq 0$ in $h^{(r_0+1)}(0) = a_{(r_0+1)j}u^j + \ldots$. So $g'(0)$, $g^{(2)}(0)$ and $g^{(3)}(0)$ are constants. By the formula (7.7) above and induction we know $g'(0), g^{(2)}(0), \ldots$, and $g^{(r_0)}(0)$ are constants and $g^{(m)}(0)$ is a polynomials in u for any m with $\deg(g^{(r_0+1)}(0)) = \deg(h^{(r_0+1)}(0)) > 0$. In the same way one can prove that $h^{(2)}(0) = \cdots = h^{(r_0')}(0) = 0$ implies $g^{(2)}(0) = \cdots = g^{(r_0')}(0) = 0$ for $r_0' \leq r_0$.

Since $f(0,u) = 0$ for any $u \in \mathbb{R}$, $f^{-1}(0,u) = 0$ for any $u \in \mathbb{R}$ and hence by the above the Taylor expansion $\hat{f^{-1}}_{(0,b)}$ of $f^{-1}(t,u)$ about $(t,u) = (0,b)$ has the form:

$$\hat{f^{-1}}_{(0,b)} = \sum_{i=1}^{r_0} b_i t^i + \sum_{i \geq r_0+1} t^i q_i(u), \tag{7.8}$$

where $q_i(u)$ is a polynomial in u and r_0 is the largest integer such that $\deg(b_i) = 0$ for $1 \leq i \leq r_0$ and $\deg(q_{r_0+1}(u)) > 0$ and there is no pure terms $u^j (j > 0)$ in $\hat{f^{-1}}_{(0,b)}$.

Let $i_0 > r_0$ be the smallest positive integers such that there exists a monomial $t^{i_0} q_{i_0}(u)$ in the expansion $\hat{f^{-1}}_{(0,b)}$ with $i_0 - c\deg(q_{i_0}(u)) < r$.

$$\text{Then any } t^i q_i(u) \text{ with } i < i_0 \text{ in the expansion } \hat{f^{-1}}_{(0,b)} \text{ must satisfy} \tag{7.9}$$
$$\text{either } i - c\deg(q_i(u)) \geq r \text{ or } 1 \leq i \leq r_0 \text{ (and } \deg(q_i(u)) = 0).$$

Let's rewrite the Taylor expansion of $f(t,u)$ in the following form:

$$a_1 t + \left[\sum_{i=2}^{r_0} a_i t^i + \sum_{i \geq r_0+1,\, i-\deg(p_i(u)) \geq r} t^i p_i(u) \right].$$

Put $\psi(t,u) = (f^{-1}(t,u),u)$. Then $\psi(0,u) = (0,u)$ for any $u \in \mathbb{R}$. We expand $f(\psi(t,u)) = t$ about $(0,b)$:

$$a_{10}\,\overset{\wedge}{f^{-1}}_{(0,b)} + \left[\sum_{i=2}^{r_0} a_{i0}(\overset{\wedge}{f^{-1}}_{(0,b)})^i + \sum_{i\geq r_0+1,\, i-cj\geq r} a_{ij}(\overset{\wedge}{f^{-1}}_{(0,b)})^i u^j\right] = t. \tag{7.10}$$

Now we consider the term $t^{i_0}q_{i_0}(u)$ in the above Taylor expansion (7.10).

$a_1 = \dfrac{\partial f}{\partial t}(0,b) \neq 0$, so $a_1\ \overset{\wedge}{f^{-1}}_{(0,b)}$ has the term $a_1 t^{i_0}q_{i_0}(u)$. Considering $i_0 > r_0 \geq 1$, so in order to cancel the term $a_1 t^{i_0}q_{i_0}(u)$ in $a_1\ \overset{\wedge}{f^{-1}}_{(0,b)}$we must have the term $-a_1 t^{i_0}q_{i_0}(u)$ in $\sum_{i=2}^{r_0} a_i(\overset{\wedge}{f^{-1}}_{(0,b)})^i + \sum_{i\geq r_0+1,\, i-c\deg(p_i(u))\geq r} (\overset{\wedge}{f^{-1}}_{(0,b)})^i p_i(u)$, in which each exponent $i \geq 2$. Thus $-a_1 t^{i_0}q_{i_0}(u)$ must be

case 1: the sum of the products of some terms like $t^k q_k(u)$ ($-a_1 t^{i_0}q_{i_0}(u)$ is from $(\sum_{i=2}^{r_0} a_i(\overset{\wedge}{f^{-1}}_{(0,b)})^i$) or

case 2: the sum of the products of some $t^k q_k(u)$'s and some $p_i(u)$ with $k < i_0$ ($-a_1 t^{i_0}q_{i_0}(u)$ is from $\sum_{i\geq r_0+1,\, i-c\deg(p_i(u))\geq r} (\overset{\wedge}{f^{-1}}_{(0,b)})^i p_i(u)$) or

case 3: the sum of the sum of the products of some terms like $t^k q_k(u)$ and the sum of the products of some $t^k q_k(u)$'s and some $p_i(u)$ with $k < i_0$ ($-a_1 t^{i_0}q_{i_0}(u)$ is from $\sum_{i=2}^{r_0} a_i(\overset{\wedge}{f^{-1}}_{(0,b)})^i$ and $\sum_{i\geq r_0+1,\, i-c\deg(p_i(u))\geq r} (\overset{\wedge}{f^{-1}}_{(0,b)})^i p_i(u)$).

In case 1, for each factor $t^k q_k(u)$ in the products we have the following 2 subcases:

subcase (i) $i_0 > k > r_0$ and so $k - c\deg(q_k(u)) \geq r$ by (7.9). This contradicts $i_0 - c\deg(q_{i_0}(u)) < r$.

subcase (ii) $\deg(q_k(u)) = 0$ and $k \leq r_0$. The products contain at least one factor as in subcase (i). Then $-a_1 t^{i_0}q_{i_0}(u)$ must satisfy $i_0 - c\deg(q_{i_0}(u)) \geq r$. This contradicts $i_0 - c\deg(q_{i_0}(u)) < r$.

In case 2, suppose the product in the sum $-a_1 t^{i_0} q_{i_0}(u)$ is the product of $t^{k_1} q_{k_1}(u),\ldots,$ $t^{k_d} q_{k_d}(u)$ $(d > 1)$ and $p_d(u)$, where $\deg(q_{k_i}(u)) > 0$ for $1 \le i \le p$ and $\deg(q_{k_i}(u)) = 0$ for $1 + p \le i \le d$ where $1 \le p \le d$. Then $d - c\deg(p_d(u)) \ge r$ by the assumption about $\hat{f}_{(0,b)}$. For each $t^{k_i} q_{k_i}(u)$, $1 \le i \le d$, we have

$k_i - c\deg(q_{k_i}(u)) \ge r$ for $1 \le i \le p$ $(k_i < i_0)$, and $k_i \le r_0$ and $\deg(q_{k_i}(u)) = 0$ for $1 + p \le i \le d$. So $\sum_{m=1}^{d} k_m - c\sum_{m=1}^{d} \deg(q_{k_i}(u)) \ge pr + \sum_{i=p+1}^{d} k_i$ and then

$$\begin{aligned}
&\sum_{m=1}^{d} k_m - c\left[\deg(p_d(u)) + \sum_{m=1}^{d} \deg(q_{k_i}(u))\right] \\
\ge\ & pr + \sum_{i=p+1}^{d} k_i - c\deg(p_d(u)) \\
\ge\ & pr + (d-p) - c\deg(p_d(u)) \\
\ge\ & (r-1)p + d - c\deg(p_d(u)) \\
\ge\ & (r-1)p + r
\end{aligned}$$

Thus $-a_1 t^{i_0} q_{i_0}(u)$ must satisfy $i_0 - \deg(q_{i_0}(u)) \ge r$, This contradicts $i_0 - \deg(q_{i_0}(u)) < r$.

From the case 1 & 2 we know case 3 is also impossible. ■

Remark 15 *1. In* $\widehat{f}_{(0,b)}$, *if* $r_0 = 0$, *i.e., the coefficient of* t *depends on* u, *then* $\overset{\wedge}{f^{-1}}_{(0,b)}$ *may not has the form* $\sum_{i=1}^{r_0} b_i t^i + \sum_{i \ge r_0+1,\, i - c\deg(q_i(u)) \ge r} t^i q_i(u)$, *e.g.,* $\widehat{f}_{(0,b)} = t(1-u)$ *and then* $\overset{\wedge}{f^{-1}}_{(0,b)} = (\sum_{i=0}^{\infty} u^i)t$.

2. If $r = 0$ *then* $\overset{\wedge}{f^{-1}}_{(0,b)}$ *may not has the form* $\sum_{i=1}^{r_0} b_i t^i + \sum_{i \ge r_0+1,\, i - c\deg(q_i(u)) \ge r} t^i q_i(u)$, *e.g., let* $c = 1$ *and* $\widehat{f}_{(0,b)} = t + t^2u^2 + 3t^3u^2$, *then* $\overset{\wedge}{f^{-1}}_{(0,b)} = t - t^2u^2 + (2u^4 - 3u^2)t^3 + (15u^4 - 5u^6)t^4 + O(u^5)$, *where* $\deg(2u^4 - 3u^2) = 4 > 3$ *and* $\deg(15u^4 - 5u^6) = 6 > 4$.

3. If $u \in \mathbb{R}^n$, $n > 1$, *the lemma still holds.*

(Question1: in $\widehat{f}_{(0,b)} = \sum_{i=1}^{r_0} a_{i0}t^i + \sum_{i \geq r_0+1,\, i-cj \geq r} a_{ij}t^iu^j$, if $a_{(r_0+1)j} \neq 0$ for some $j \geq 1$,is it possible that there is a term $b_{i0}t^iu^0$ with $i \geq r_0+1$ in $\hat{f^{-1}}_{(0,b)} = \sum_{i=1}^{r_0} b_{i0}t^i + \sum_{i \geq r_0+1,\, i-cj \geq r} b_{ij}t^iu^j$?

Question2: Graded version of the Lemma:Let $r_2 > r_1 \geq 0$ and $s > 0$ be the integers. If $\widehat{f}_{(0,b)} = \sum_{i=1}^{r_0} a_{i0}t^i + \sum_{i \geq r_0+1,\, i-cj \geq r_2}^{r_0+s} a_{ij}t^iu^j + \sum_{i \geq r_0+s+1,\, i-cj \geq r_1} a_{ij}t^iu^j$ ($a_{10} \neq 0$), can we have $\hat{f^{-1}}_{(0,b)} = \sum_{i=1}^{r_0} b_{i0}t^i + \sum_{i \geq r_0+1,\, i-cj \geq r_2}^{r_0+s} b_{ij}t^iu^j + \sum_{i \geq r_0+s+1,\, i-cj \geq r_1} b_{ij}t^iu^j$?)

Lemma 44 *Let (t,u) be the coordinates in $\mathbb{R}^2$. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be C^∞ function and $f(0,u) = 0$ for any $u \in \mathbb{R}$. Suppose $\frac{\partial f}{\partial t}(0,0) \neq 0$ (so $f(t,u)$ has an inverse function, denoted by $f^{-1}(t,u)$, with respect to t in a neighborhood of $(0,0)$) and the Taylor expansion $\widehat{f}_{(a,0)}$ of f about $(t,u) = (a,0)$ for a near 0 has such a form:*

$$\hat{f}_{(a,0)} = a_{10}t + \sum_{i \geq 1,\, j-ci \geq r} a_{ij}t^iu^j,$$

where $r \geq 0, c \geq 1$ are integers and $a_{ij} \in \mathbb{R}$. Then the Taylor expansion $\hat{f^{-1}}_{(a_{10}a,0)}$ of $f^{-1}(t,u)$ about $(t,u) = (a_{10}a, 0)$ for a near 0 has the form:

$$\hat{f^{-1}}_{(a_{10}a,0)} = a_{10}^{-1}t + \sum_{i \geq 1,\, j-ci \geq r} b_{ij}t^iu^j, \tag{7.11}$$

where $b_{ij} \in \mathbb{R}$.

Proof. Obviously $f^{-1}(0,u) = 0$ for any u and $\frac{\partial f^{-1}}{\partial t}(a_{10}a, 0) = a_{10}^{-1}$.

$$f^{-1}(f(t,u)) = t \Rightarrow f^{-1}(f(t,0)) = t \Rightarrow f^{-1}(a_{10}t, 0) = t \Rightarrow f^{-1}(t,0) = a_{10}^{-1}t.$$

We can write the Taylor expansion $\hat{f^{-1}}_{(a_{10}a,0)}$ of f^{-1} about $(t,u) = (a_{10}a, 0)$ as

$$\hat{f^{-1}}_{(a_{10}a,0)} = a_{10}^{-1}t + \sum_{i \geq 1, j \geq 1} b_{ij}t^iu^j.$$

Put $\psi(t,u) = (f^{-1}(t,u), u)$. Then $\psi(a_{10}a, 0) = (a, 0)$ for any a near 0.

We expand $f(\psi(t,u)) = t$ about $(a_{10}a, 0)$:

$$a_{10}\,\hat{f}^{-1}{}_{(a_{10}\,a,0)} + \sum_{i\geq 1,\, j-ci\geq r} a_{ij}(\hat{f}^{-1}{}_{(a_{10}\,a,0)})^i u^j = t. \tag{7.12}$$

Suppose j_0 and i_0 are the smallest integers ≥ 1 such that there exists a nonzero monomial $t^{i_0}u^{j_0}$ with $j_0 - ci_0 < r$ in the Taylor expansion $\hat{f}^{-1}{}_{(a_{10}\,a,0)}$.

(1): If $i_0 = 1$, in order for (7.12) to hold $\sum_{i\geq 1,\, j-ci\geq r} a_{ij}(\hat{f}^{-1}{}_{(a_{10}\,a,0)})^i u^j$ must contain the term tu^{j_0} (up to the coefficient), which can only be contributed by $a_{1j}\ \hat{f}^{-1}{}_{(a_{10}\,a,0)}\ u^j$ for some j with $j - ci_0 = j - c \geq r$. This contributing term is either tu^j with $j - c \geq r$ or $tu^{j_1}u^j$ with $j - c \geq r$ and $j_1 < j_0$. In both cases, $j + j_1 - c > j - c \geq r$. This implies $j_0 - ci_0 \geq r$. Contradiction.

(2): If $i_0 > 1$, the term $t^{i_0}u^{j_0}$ in $\sum_{i\geq 1,\, j-ci\geq r} a_{ij}(\hat{f}^{-1}{}_{(a_{10}\,a,0)})^i u^j$ is of the form $(\prod_{k=1}^{d}(t^{i_k}u^{j_k}))u^{j'}$ (up to the coefficient) or the sum of terms of such form, where $\sum_{k=1}^{d} i_k = i_0$, $1 \leq i_k \leq i_0$, $j' - cd \geq r$ and $\sum_{k=1}^{d} j_k + j' = j_0$. We have the following 2 subcases:

Subcase(i): If $j' = j_0$,then $j_k = 0$ and $i_k = 1$ for $1 \leq k \leq i_0$ (t is from $a_{10}^{-1}t$ in $\hat{f}^{-1}{}_{(a_{10}\,a,0)}$), and $d = i_0$. Then

$$(\prod_{k=1}^{d}(t^{i_k}u^{j_k}))u^j = t^{\sum_{k=1}^{i_0} 1}u^{j'} = t^{i_0}u^{j'}.$$

And $j' - cd \geq r$ implies $j_0 - ci_0 \geq r$. Contradiction.

Subcase(ii): In the case $j' < j_0$, suppose $j_k > 0$ for $1 \leq i \leq p$ and $j_k = 0$ for $1+p \leq i \leq d$ where $1 \leq p \leq d$. If $j_k > 0$ then $j_k - ci_k \geq r$ because of $j_k < j_0$, $i_k \leq i_0$ and the minimality of j_0. And $j_k = 0$ implies $i_k = 1$. So $j' + \sum_{k=1}^{d} j_k - c\sum_{k=1}^{d} i_k \geq j' + pr - c(d-p)$. Since $j' - cd \geq r$, $-c(d-p) \geq (r - j')(1 - \frac{p}{d}) \geq r - j'$ noting $r - j' \leq 0$ and $1 - \frac{p}{d} \leq 1$. So $j' + pr - c(d-p) \geq j' + pr + r - j' \geq r$. This implies $j_0 - ci_0 \geq r$. Contradiction. ■

Remark 16 *For any $b \in \mathbb{R}$, if $\widehat{f}_{(0,b)}$ has the form $a_{10}t + \sum_{i\geq 1,\, j-ci\geq r} a_{ij}t^i u^j$, then $\overset{\wedge}{f^{-1}}_{(0,b)}$ also has the form $a_{10}^{-1}t + \sum_{i\geq 1,\, j-ci\geq r} b_{ij}t^i u^j$.*

Remark 17 *If $u \in \mathbb{R}^n$, $n > 1$, the lemma still holds.*

CHAPTER 8

Proofs of Main Theorems

8.1 Proof of Theorem 41

The differential of $RS_n(t,u)$ at $(t,u)=(0,0)$ is

$$
\begin{aligned}
& D(RS_n(\gamma)(t,u))|_{(0,0)} \\
= & \left.\begin{bmatrix} \gamma'(t)+u\left[s''(t)\gamma''(t)+s'(t)\gamma'''(t)-s'''(t)\gamma'(t)-s''(t)\gamma'(t)\right] \\ s'(t)\gamma''(t)-s''(t)\gamma'(t) \end{bmatrix}\right|_{(0,0)} \\
= & \begin{bmatrix} \gamma'(0) \\ s'(0)\gamma''(0)-s''(0)\gamma'(0) \end{bmatrix}
\end{aligned}
$$

Case (i): If γ is of the type $(1,2,s)$, $s\geq 3$ then $\gamma'(0)\times\ \gamma''(0)\neq 0$, i.e., $\kappa(0)\neq 0$ by Lemma 40, then $D(RS_n(\gamma)(t,u))$ is nonsingular at $(0,0)$ and $RS_n(\gamma)(t,u)$ is an immersion in a neighborhood of $(0,0)$. So $RS_n(\gamma)(t,u)$ is $\mathcal{A}$-equivalent to the map $(t,u)\to(t,u,0)$.

Case (ii): Suppose γ is of the type $(1,q,q+s)$, $q\geq 3, s\geq 1$, or equivalently $\kappa(0)=\kappa'(0)=\cdots=\kappa^{(q-3)}(0)=0$ but $\kappa^{(q-2)}(0)\neq 0$ by Lemma 40, and suppose γ and $\overline{\gamma}$ are orthogonal-right equivalent to $(t,b_qt^q+b_{q+1}q^{q+1}+O(t^{q+2}),c_{q+1}t^{q+1}+O(t^{q+2})),b_q\neq 0$. By Proposition 38 and Proposition 39 we know $RS_n(\gamma)\overset{\mathcal{A}}{\backsim}RS_n(\overline{\gamma})$. So for the purpose of finding the $\mathcal{A}$-type of $RS_n(\gamma)$ we may assume $\gamma=(t,b_qt^q+b_{q+1}q^{q+1}+O(t^{q+2}),c_{q+1}t^{q+1}+O(t^{q+2})),b_q\neq 0$. We'll prove that the $\mathcal{A}$-type of $RS_n(\gamma)$ does not depend on the values of $b_q,\cdots,c_{q+1},\cdots$. It is only determined by the type $(1,q,q+s)$ of γ. By some computation we have

$$\begin{aligned} RS_n(\gamma)(t,u) &= \gamma(t) + u\left[s'(t)\gamma''(t) - s''(t)\gamma'(t)\right] \\ &= \Big((t + u\left[q^2(1-q)b_q^2 t^{2q-3} + O(t^{2q-2})\right], \\ &\quad b_q t^q + O(t^{q+1}) + ut^{q-2}\left[q(q-1)b_q + O(t)\right], \\ &\quad c_{q+1}t^{q+1} + O(t^{q+2}) + ut^{q-1}\left[q(q+1)c_{q+1} + O(t)\right]\Big) \\ &= (f_{u1}(t), f_{u2}(t), f_{u3}(t)), \end{aligned}$$

where

$$\begin{aligned} f_{u1}(t) &= t + u\left[q^2(1-q)b_q^2 t^{2q-3} + O(t^{2q-2})\right], \\ f_{u2}(t) &= b_q t^q + O(t^{q+1}) + ut^{q-2}\left[q(q-1)b_q + O(t)\right], \\ f_{u3}(t) &= c_{q+1}t^{q+1} + O(t^{q+2}) + ut^{q-1}\left[q(q+1)c_{q+1} + O(t)\right]. \end{aligned}$$

So the Taylor expansion $\widehat{f_{u1}}_0$ of f_{u1} at $t=0$ has the form $\sum_{i=1}^{r_0} a_i t^i + \sum_{i \geq r_0+1,\, i-c\deg(p_i(u)) \geq r} t^i p_i(u)$ for any u as in Lemma *43*, where $r_0 = 2q-4$, $c = 2q-3$, $r = 0$, $a_1 = 1$, $a_i = 0$ for $2 \leq i \leq 2q-4$ and $p_i(u)$ is a multiple of u for $i \geq r_0+1$. So $\frac{\partial f_{u1}}{\partial t}(0) = 1$, then $f_{u1}^{-1}(t)$ exists with respect to t and $\widehat{f_{u1}^{-1}}_0$ also has the form

$$\sum_{i=1}^{r_0} b_i t^i + \sum_{i \geq r_0+1,\, i-c\deg(q_i(u)) \geq r} t^i q_i(u) = t + \sum_{i \geq 2q-3,\, i-(2q-3)\deg(q_i(u)) \geq 0} t^i q_i(u) \qquad (8.1)$$

for any u by Lemma *43*, where $q_i(u)$'s are some polynomials.

Rewrite

$$f_{u2}(f_{u1}^{-1}(t)) = ug_1(t,u) + g_2(t) = ug_{1u}(t) + g_2(t)$$

and

$$f_{u3}(f_{u1}^{-1}(t)) = uh_1(t,u) + h_2(t) = uh_{1u}(t) + h_2(t)$$

for some functions $g_{1u}(t)$, $g_2(t)$, $h_{1u}(t)$ and $h_2(t)$. After the change of the variables $(t,u) \to (f_{u1}^{-1}(t), u)$,

$$RS_n(\gamma)(t,u) = (t, ug_1(t,u) + g_2(t), uh_1(t,u) + h_2(t)).$$

By (7.7) we can easily see the exponent of u in $q_i(u)$ in (8.1) is at least 1 (no constant term in $q_i(u)$) because $p_i(u)$'s in $\widehat{f_{u1\,0}}$ are all the multiples of u. Rewrite $\overset{\wedge}{f_{u1}^{-1}}{}_0$ in the following form:

$$\overset{\wedge}{f_{u1}^{-1}}{}_0 = t + \sum_{i \geq 2q-3,\, i-(2q-3)\deg(q_i(u)) \geq 0} t^i q_i(u) = t + \sum_{i \geq 2q-3,\ j>0,\, i-(2q-3)\,j \geq 0} a_{ij} t^i u^j, \quad (8.2)$$

$a_{ij} \in \mathbb{R}$. Then consider

$$u \left(t + \sum_{i \geq 2q-3,\, i-(2q-3)\,j \geq 0} a_{ij} t^i u^j \right)^k = t^k u \left(1 + \sum_{i \geq 2q-3,\ j>0,\, i-(2q-3)\,j \geq 0} a_{ij} t^{i-1} u^j \right)^k \quad (8.3)$$

for $k \geq q-2 \geq 1$, i.e., ut^k in $f_{u2}(t)$ with t replaced by $\overset{\wedge}{f_{u1}^{-1}}{}_0$. For the exponents i, j of the terms $t^i u^j$ with $j > 0$ and $i \geq (2q-3)j \geq 3$ in (8.3) we have

$$i - (2q-3)j \geq 0 \Rightarrow i - (q-2)j \geq (q-1)j$$

$$\Rightarrow ki - k(q-2)j \geq k(q-1)j \geq 2k \geq k + q - 2 \text{ for } k \geq q-2.$$

So $k(i-1) - (kj+1)(q-2) \geq 0$ for $k \geq q-2$ and (8.3) can be rewritten as

$$t^k (u + \sum_{i \geq 2q-4,\ j>1,\, i-(q-2)\,j \geq 0} b_{ij} t^i u^j) \quad (8.4)$$

for some $b_{ij} \in \mathbb{R}$. The change of the variables $(X, Y, Z) \rightarrow (X, Y - g_2(X), Z - h_2(X)))$ further gives $RS_n(\gamma)(t, u)$ the form $RS_n(\gamma)(t, u) = (t, ug_1(t, u), uh_1(t, u))$, where the Taylor expansions of $ug_{1u}(t)$ about $t = 0$ has the form

$$t^{q-2} \left[q(q-1) b_q^2 u + \sum_{i \geq 2q-4,\ j>1,\, i-(q-2)\,j \geq 0} c_{ij} t^i u^j \right]$$

and similarly the Taylor expansions of $uh_1(t, u)$ about $t = 0$ has the form

$$t^{q-1} \left[q(q+1) c_{q+1} u + \sum_{i \geq 2q-4,\, j>1,\, i-(q-2)\,j \geq 0} d_{ij} t^i u^j \right] \quad (8.5)$$

for any u by the forms of $\hat{f}{}^{-1}_{u1\,0}$, $\hat{f}_{u20}$, $\hat{f}_{u30}$ and (8.4), where $c_{ij}, d_{ij} \in \mathbb{R}$. Factor $ug_{1u}(t) = t^{q-2}g_3(t,u)$. So

$$\frac{\partial g_3(0,0)}{u} = q(q-1)b_q^2 \neq 0,$$

considering that the type $(1, q, q+s)$ of γ guarantees $b_q \neq 0$. The Taylor expansion $\hat{g}_{3(0,b)}$ of $g_3(t,u)$ about $(t,u) = (0,b)$ near $(0,0)$ has the form

$$q(q-1)b_q^2 u + \sum_{i \geq 2q-4,\ j > 1,\, i-(q-2)\,j \geq 0} c_{ij} t^i u^j,$$

noting that the coefficients of t^i in the Taylor expansions of $ug_{1u}(t)$ about $t = 0$ are all the polynomials in u. Then the inverse $g_3^{-1}(t,u)$ of $g_3(t,u)$ with respect to u exists in a neighborhood of $(t,u) = (0,0)$ and the Taylor expansion of $g_3^{-1}(t,u)$ about $(t,u) = (0,b)$ near $(0,0)$ has the form

$$\frac{1}{q(q-1)b_q^2}u + \sum_{i \geq 2q-4,\ j > 1,\, i-(q-2)\,j \geq 0} e_{ij} t^i u^j, e_{ij} \in \mathbb{R} \tag{8.6}$$

by Lemma *44 (*with the roles of t and u switched in Lemma *44),* where that the exponent j of u^j is bigger than 1 is from (7.7). By the change of the variables $(t,u) \to (t, g_3^{-1}(t,u))$, $RS_n(t,u)$ takes the form

$$RS_n(t,u) = (t, t^{q-2}u, g_3^{-1}(t,u)h_1(t, g_3^{-1}(t,u)).$$

Consider $uh_1(t,u)$ with u replaced by $g_3^{-1}(t,u)$ (c.f. (8.5) and (8.6)) and then we have :

$$t^{q-1}\left[q(q+1)c_{q+1}\left(\frac{1}{q(q-1)}u + \sum_{i_1 \geq 2q-4,\ j_1 > 1,\, i_1-(q-2)\,j_1 \geq 0} e_{i_1 j_1} t^{i_1} u^{j_1}\right) + \right. \tag{8.7}$$

$$\left. + \sum_{i_2 \geq 2q-4,\, j_2 > 1,\, i_2-(q-2)\,j_2 \geq 0} d_{i_2 j_2} t^{i_2} \left(\frac{1}{q(q-1)}u + \sum_{i_1 \geq 2q-4,\ j_1 > 1,\, i_1-(q-2)\,j_1 \geq 0} e_{i_1 j_1} t^{i_1} u^{j_1}\right)^{j_2}\right]$$

In the second summation of (8.7), about the exponents of the term $t^i u^j$ we have $i = i_2 + i_1 k, j = j_2 - k + j_1 k$ for some $k, 1 \leq k \leq j_2$, and

$$\begin{aligned} & i_2 + i_1 k - (q-2)(j_2 - k + j_1 k) \\ = \; & i_2 - (q-2)(j_2 - k) + k\,[i_1 - (q-2)j_1] \\ \geq \; & i_2 - (q-2)j_2 + k\,[i_1 - (q-2)j_1] \geq 0 \end{aligned}$$

because of the restrictions $i_1 - (q-2)j_1 \geq 0$ and $i_2 - (q-2)j_2 \geq 0$. In the first summation of (8.7), about the exponents of the term $t^i u^j$ it's easy to see $i - (q-2)\,j \geq 0$. So the Taylor expansion of $h_3(t,u) = g_3^{-1}(t,u)h_1(t, g_3^{-1}(t,u))$ about $(t,u) = (0,b)$ for b near 0 has the form

$$\sum_{i \geq q-1,\, i-(q-2)\,j \geq 0} f_{ij} t^i u^j, f_{ij} \in \mathbb{R}. \tag{8.8}$$

8.2 $\mathcal{C}^k$ Composite Function Property

Here we state a definition about the $\mathcal{C}^k$ composite function property in [2]. For the definition and basic properties of semianalytic, subanalytic, Nash subanalytic and semiproper see [2] and [1].

Let $\Omega \in \mathbb{R}^m$ be a subset and $\varphi : \Omega \to \mathbb{R}^n$ denote a semiproper real analytic mapping. $X = \varphi(\Omega)$. Let $(\varphi^* \mathcal{C}^k(X))\hat{}$ denote the subalgebra of all functions $f \in \mathcal{C}^k(\Omega)$ such that f is "formally a composite with φ"; i.e.,, for each $a \in X$, there is $g \in \mathcal{C}^k(X)$ such that $f - \varphi^*(g)$ is k-flat on $\varphi^{-1}(a)$. A function f is k-flat on Ω iff the k-th Taylor polynomial of f is 0 about each point of Ω.

Put $\mathcal{C}^{(\infty)}(X) = \cap_{k \in \mathbb{N}} \mathcal{C}^k(X)$. For our case we define a map $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ by $\varphi(t,u) =$

$(t, t^{q-2}u)$, $q \geq 3$. Let $A > 0, B > 0$ and $\Omega = \{(t,u) | \, |t| \leq A, |u| \leq B\}$. Then

$$\begin{aligned}\varphi(\Omega) = \; & \{(x,y) \mid -Bx^{q-2} \leq y \leq Bx^{q-2}, 0 \leq x \leq A\} \\ & \cup \{(x,y) \mid Bx^{q-2} \leq y \leq -Bx^{q-2}, -A \leq x \leq 0\} \text{ if } q \text{ is odd;}\end{aligned}$$

and

$$\varphi(\Omega) = \{(x,y) \mid -Bx^{q-2} \leq y \leq Bx^{q-2}, -A \leq x \leq A\} \text{ if } q \text{ is even.}$$

So $\varphi(\Omega)$ is closed (compact) semianalytic. The class of Nash subanalytic sets includes all semianalytic sets. So $\varphi(\Omega)$ is a closed Nash subanalytic set. By Theorem 1.6 in [2] and the remark after it we have

$$\mathcal{C}^{(\infty)}(\varphi(\Omega)) = \mathcal{C}^{\infty}(\varphi(\Omega)).$$

By (8.8) $\hat{h}_{3(0,b)} = \sum_{i \geq q-1, i-(q-2)j \geq 0} f_{ij} t^i u^j$, with $i - (q-2)j \geq 0$ and $i \geq q-1$, so for any $k \in \mathbb{N}$ there is a $g \in \mathcal{C}^k(X)$ such that $h_3 - \varphi^*(g)$ is k-flat on $\varphi^{-1}((0,0)) = \{(0,b) \mid \, |b| \leq B\}$ because each term $f_{ij}t^iu^j$ in $\hat{h}_{3(0,b)}$ satisfies $i - (q-2)j \geq 0$. If $t \neq 0$, φ is a bianalytic map and $h_3 = h_3 \circ \varphi^{-1} \circ \varphi$. So for any $(a,b) \in \varphi(\Omega)$ and for any $k \in \mathbb{N}$ there is $g \in \mathcal{C}^k(X)$ such that $h_3 - \varphi^*(g)$ is k-flat on $\varphi^{-1}((a,b))$. Hence $h_3 \in \mathcal{C}^{(\infty)}(\varphi(\Omega)) = \mathcal{C}^{\infty}(\varphi(\Omega))$. Then there $g \in \mathcal{C}^{\infty}(X)$ such that $h_3 = g \circ \varphi$.

The change of the variables $(X, Y, Z) \to (X, Y, Z - g(X, Y))$ leads to $RS_n(\gamma)(t, u) = (t, t^{q-2}u, 0)$. ∎

8.3 Proof of Theorem 42

Now we consider *the ruled surface of binormals* of $\gamma(t)$. The differential of $RS_b(t,u)$ at $(t,u) = (0,0)$ is

$$D\left(RS_b(\gamma)(t,u)\right)|_{(0,0)} = \left.\begin{bmatrix} \gamma'(t) + u(\gamma'(t) \times \gamma'''(t)) \\ \gamma'(t) \times \gamma''(t) \end{bmatrix}\right|_{(0,0)} = \begin{bmatrix} \gamma'(0) \\ \gamma'(0) \times \gamma''(0) \end{bmatrix}$$

Case 1: If γ is of the type $(1,2,s)$, $s \geq 3$ then $\gamma'(0)\times\ \gamma''(0) \neq 0$, i.e., $\kappa(0) \neq 0$ by Lemma 40, then $\gamma'(0) \neq 0$ and $\gamma'(0) \perp \gamma'(0) \times \gamma''(0)$ implies $D(RS_b(\gamma)(t,u))$ is nonsingular at $(0,0)$. So $RS_b(\gamma)(t,u)$ is an immersion in a neighborhood of $(0,0)$. So $RS_b(\gamma)(t,u)$ is $\mathcal{A}$-equivalent to the map $(t,u) \rightarrow (t,u,0)$.

Case 2: Suppose γ is of the type $(1,q,q+s)$, $q \geq 3, s \geq 1$, or equivalently $\kappa(0) = \kappa'(0) = \cdots = \kappa^{(q-3)}(0) = 0$ but $\kappa^{(q-2)}(0) \neq 0$ by Lemma 40. By Proposition 38 and Proposition 39 we may assume that the curve γ takes the form

$$\gamma(t) = (t, b_q t^q + b_{q+1} q^{q+1} + O(t^{q+2}), c_{q+1} t^{q+1} + O(t^{q+2})), b_q \neq 0,$$

then by some computation we have

$$\begin{aligned} RS_b(\gamma)(t,u) \ &= \gamma(t) + u\,(\gamma'(t) \times \gamma''(t)) \\ &= \big(t + u\left[q(q+1)b_q c_{q+1} t^{2(q-1)} + O(t^{2q-1})\right], \\ &\quad b_q t^q + O(t^{q+1}) + u t^{q-1}\left[-q(q+1)c_{q+1} + O(t)\right], \\ &\quad c_{q+1} t^{q+1} + O(t^{q+2}) + u t^{q-2}\left[q(q-1)b_q + O(t)\right]\big) \\ &= (f_{u1}(t), f_{u2}(t), f_{u3}(t)), \end{aligned}$$

where

$$\begin{aligned} f_{u1}(t) &= t + u\left[q(q+1)b_q c_{q+1} t^{2(q-1)} + O(t^{2q-1})\right], \\ f_{u2}(t) &= b_q t^q + O(t^{q+1}) + u t^{q-1}\left[-q(q+1)c_{q+1} + O(t)\right], \\ f_{u3}(t) &= c_{q+1} t^{q+1} + O(t^{q+2}) + u t^{q-2}\left[q(q-1)b_q + O(t)\right]. \end{aligned}$$

So the Taylor expansion $\widehat{f_{u10}}$ of f_{u1} at $t = 0$ has the form

$$t + \sum_{i \geq 2q-2,\ j>0,\, i-(2q-2)\,j \geq 0} a_{ij} t^i u^j, a_{ij} \in \mathbb{R}.$$

For $f_{u2}(f_{u1}^{-1}(t))$ and $f_{u3}(f_{u1}^{-1}(t))$ we consider

$$u\left(t+\sum_{i\geq 2q-2,\ j>0,\ i-(2q-2)\,j\geq 0} a_{ij}t^iu^j\right)^k = t^ku\left(1+\sum_{i\geq 2q-2,\ j>0,\ i-(2q-2)\,j\geq 0} a_{ij}t^{i-1}u^j\right)^k. \tag{8.9}$$

For $k \geq q-2$, from $i-(2q-2)\,j \geq 0$ we have $i-(q-2)j \geq qj \Rightarrow ki-kj(q-2) \geq kjq \geq 3k$ since $q \geq 3$ and $j \geq 1$ ($j \geq 1$ is from (7.7) as pointed out in the paragraph before (8.2)). So $ki-kj(q-2) \geq 3k > k+q-2 \Rightarrow$

$$k(i-1)-(kj+1)(q-2) \geq 0 \text{ for } k \geq q-2. \tag{8.10}$$

For $k \geq q-1$, from $i-(2q-2)\,j \geq 0$ we have

$$i-(q-1)j \geq (q-1)j$$

$$\Rightarrow \quad ki-kj(q-1) \geq kj(q-1) \geq 2k \text{ since } q \geq 3 \text{ and } j \geq 1$$

($j \geq 1$ is from (7.7) as pointed out in the paragraph before (8.2)). So

$$ki-kj(q-1) \geq 2k > k+q-1$$

$$\Rightarrow \quad k(i-1)-(kj+1)(q-1) \geq 0 \text{ for } k \geq q-1.$$

Then for $k \geq q-2$, (8.9) can be rewritten as

$$t^k\left(u+\sum_{i\geq 2q-3,\ j>1,\ i-(q-2)\,j\geq 0} b_{ij}t^iu^j\right), b_{ij} \in \mathbb{R}.$$

Then for $k \geq q-1$, (8.9) can be rewritten as

$$t^k\left(u+\sum_{i\geq 2q-3,\ j>1,\ i-(q-1)\,j\geq 0} b_{ij}t^iu^j\right), b_{ij} \in \mathbb{R}.$$

Then in the same way as in the case of the ruled surface of normals we can show that by some changes of variables we have

$$RS_b(\gamma)(t,u) \overset{\mathcal{A}}{\sim} \left(t,\ t^{q-1}\left[-q(q+1)c_{q+1}u+\sum_{i\geq 2q-3,\ j>1,\ i-(q-1)\,j\geq 0} c_{ij}t^iu^j\right],\right.$$

$$\left. t^{q-2}\left[q(q-1)b_qu+\sum_{i\geq 2q-3,\ j>1,\ i-(q-2)\,j\geq 0} d_{ij}t^iu^j\right]\right),$$

$c_{ij}, d_{ij} \in \mathbb{R}$. The Taylor expansion of the inverse function of

$$q(q-1)b_q u + \sum_{i \geq 2q-3,\ j>1,\, i-(q-2)\,j \geq 0} d_{ij} t^i u^j$$

with respect to u about $(t,u) = (0,b)$ near $(0,0)$ has the form

$$\frac{1}{q(q-1)b_q} u + \sum_{i \geq 2q-3,\ j>1,\, i-(q-2)\,j \geq 0} e_{ij} t^i u^j, e_{ij} \in \mathbb{R}$$

by *Lemma 44*. Consider $t^{q-1}\left[-q(q+1)c_{q+1}u + \sum_{i \geq 2q-3,\ j>1,\ i-(q-1)\,j \geq 0} c_{ij} t^i u^j\right]$ with u replaced by $\frac{1}{q(q-1)} u + \sum_{i \geq 2q-3,\ j>1,\, i-(q-2)\,j \geq 0} e_{ij} t^i u^j$ and we have:

$$t^{q-1}\left(-q(q+1)c_{q+1}\left[\frac{1}{q(q-1)b_q} u + \sum_{i_1 \geq 2q-3,\ j_1>1,\, i_1-(q-2)\,j_1 \geq 0} e_{i_1 j_1} t^{i_1} u^{j_1}\right] + \right. \tag{8.11}$$

$$\left. + \sum_{i_2 \geq 2q-3,\ j_2>1,\ i_2-(q-1)\,j_2 \geq 0} c_{ij} t^{i_2} \left[\frac{1}{q(q-1)b_q} u + \sum_{i_1 \geq 2q-3,\ j_1>1,\, i_1-(q-2)\,j_1 \geq 0} e_{i_1 j_1} t^{i_1} u^{j_1}\right]^{j_2}\right)$$

In the second summation of (8.11), about the exponents of the term $t^i u^j$ we have $i = i_2 + i_1 k,\ j = j_2 - k + j_1 k$ for some $k,\ 1 \leq k \leq j_2$, and

$$\begin{aligned} & i_2 + i_1 k - (q-2)(j_2 - k + j_1 k) \\ = \ & i_2 - (q-2)(j_2 - k) + k\,[i_1 - (q-2)j_1] \\ \geq \ & i_2 - (q-2)j_2 + k\,[i_1 - (q-2)j_1] \geq 0 \end{aligned} \tag{8.12}$$

because of the restrictions $i_1 - (q-2)j_1 \geq 0$ and $i_2 - (q-1)j_2 \geq i_2 - (q-2)j_2 \geq 0$. In the first summation of (8.11), about the exponents of the term $t^i u^j$ it's easy to see $i - (q-2)\,j \geq 0$. So the Taylor expansion of (8.11) about $(t,u) = (0,b)$ for b near 0 has the form

$$\sum_{i \geq q-1,\, i-(q-2)\,j \geq 0} f_{ij} t^i u^j, f_{ij} \in \mathbb{R}.$$

Then in the same way as in the case of the ruled surface of normals we can show $RS_b(\gamma)(t,u) \overset{\mathcal{A}}{\backsim} (t, \sum_{i \geq q-1,\, i-(q-2)\,j \geq 0} f_{ij} t^i u^j, t^{q-2}u) \overset{\mathcal{A}}{\backsim} (t, 0, t^{q-2}u) \overset{\mathcal{A}}{\backsim} (t, t^{q-2}u, 0)$. ∎

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