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Topological degeneracy and vortex manipulation in Kitaev's honeycomb model

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The classification of loop symmetries in Kitaev's honeycomb lattice model provides a natural framework to study the abelian topological degeneracy. We derive a perturbative low-energy effective Hamiltonian, that is valid to all orders of the expansion and for all possible toroidal configurations. Using this form we demonstrate at what order the system's topological degeneracy is lifted by finite size effects and note that in the thermodynamic limit it is robust to all orders. Further, we demonstrate that the loop symmetries themselves correspond to the creation, propagation and annihilation of fermions. Importantly, we note that these fermions, made from pairs of vortices, can be moved with no additional energy cost.

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Recently, Kitaev introduced a spin-1/2 quantum lattice model with abelian and non-abelian topological phases [1]. This model is relevant to on-going research into topologically fault-tolerant quantum information processing [2–4]. The system comprises of two-body interactions and is exactly solvable, which makes it attractive both theoretically [5–16] and experimentally [17–20].

Here, by classifying the loop symmetries of the system according to their homology, we derive a convenient form of the effective Hamiltonian on the torus. The result is valid for all orders of the Brillouin-Wigner perturbative expansion around the fully dimerized point as well as for all toroidal configurations. This allows the system's topological degeneracy to be addressed and shows at what order in the expansion the degeneracy is lifted. In the thermodynamic limit the system's topological degeneracy remains to all orders. In a separate analysis, valid for the full parameter space, we examine the paired-vortex excitations created by applying certain open string operations to the ground state. These vortex-pairs are fermions and can be freely transported. For the manipulation of single vortices we extend the perturbative analysis of [15] to all areas of the phase diagram.

The Hamiltonian for the system can be written as

$$H = -\sum_{\alpha\in\{x,y,z\}} \sum_{i,j} J_\alpha K_{ij}^{\alpha,\alpha} \tag{1}$$

where $K_{ij}^{\alpha,\beta} \equiv \sigma_i^\alpha \otimes \sigma_j^\alpha$ denotes the exchange interaction occurring between the sites i, j connected by a β-link, see FIG. 1. In what follows we will use $K_{ij}^{\alpha} \equiv K_{ij}^{\alpha,\alpha}$ whenever $\alpha = \beta$. Following [1], we consider loops of n non-repeating K operators, $K_{ij}^{\alpha^{(1)}} K_{jk}^{\alpha^{(2)}}, ..., K_{li}^{\alpha^{(n)}}$, where $\alpha^{(m)} \in x, y, z$. Any loop constructed in this way commutes with the Hamiltonian and with all other loops. When the model is mapped to free Majorana fermions coupled to a $\mathbb{Z}_2$ gauge field, these loop operators become Wilson loops [1]. The plaquette operators

$$W_p = \sigma_1^x \sigma_2^y \sigma_3^z \sigma_4^x \sigma_5^y \sigma_6^z, \tag{2}$$

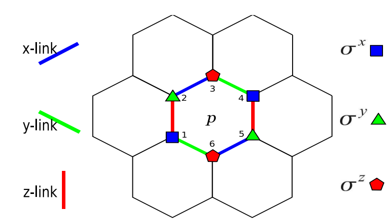

FIG. 1: The honeycomb lattice and the operator W_p.

where the numbers 1 through 6 label lattice sites on single hexagonal plaquette p, see FIG. 1, are the closed loop operators around each of the hexagons of the lattice. Since they commute with the Hamiltonian and with each other we may choose energy eigenvectors $|\, n\rangle$ such that $w_p = \langle n\, |W_p|\, n\rangle = \pm 1$. If $w_p = -1$, one says that the state $|\, n\rangle$ carries a vortex at p.

For a finite system of N spins on a torus there are $N/2$ plaquettes. The product of all plaquette operators is the identity and this is the single nontrivial relation between them. Hence there are only $N/2-1$ independent quantum numbers, $\{w_1, ..., w_{N/2-1}\}$. All homologically trivial loops are products of plaquettes. The relevant homology is $\mathbb{Z}_2$ homology, since loop operators square to the identity. To describe the full symmetry group generated by loop operators, we introduce generators for the nontrivial $\mathbb{Z}_2$ homology classes of the surface that the lattice lives on. At most one generator per homology class is necessary, since all elements of any homology class can be generated from an arbitrary element of that class using the plaquettes. The $\mathbb{Z}_2$ homology group of the torus is $\mathbb{Z}_2 \times \mathbb{Z}_2$, so it is enough to add two homologically nontrivial loops as generators. The third nontrivial class is generated from the product of these two. The full loop symmetry group of the torus is the abelian group with $N/2 + 1$ independent generators of order 2, that is $\mathbb{Z}_2^{N/2+1}$. All closed loop symmetries can be written in the form

$$C_{(k,l)} = G_k F_l(W_1, W_2, \ldots.., W_{N-1}). \tag{3}$$

Here $k \in \{0,1,2,3\}$, $G_0 = I$ and G_1, G_2 and G_3 are arbitrarily chosen symmetries from the three nontrivial homology classes. The F_l, with $l \in \{1,\ldots,2^{N/2-1}\}$, run through all monomials in the W_p.

The loop symmetries play an important role in the perturbation theory of the abelian phase of the model. Following Kitaev we take $J_z \gg J_x, J_y$ and write the Hamiltonian as $H = H_0 + U$, where $H_0 = -J_z \sum_{ij} K^z_{ij}$ is the unperturbed Hamiltonian and $U = -\sum_{\alpha\in\{x,y\}} J_\alpha \sum_{ij} K^\alpha_{ij}$ is the perturbative contribution. H_0 has a $2^{N/2}$-fold degenerate ground state space spanned by ferromagnetic configurations of the dimers on z-links. To understand how this degeneracy behaves under perturbation we analyse the Brillouin-Wigner expansion [21, 22]. The method returns the systems energies E as an implicit non-linear eigenvalue problem and thus, for the actual calculation of coefficients to high orders, can be difficult to apply [23]. However, we will take advantage of the infinite but exact nature of the series by recognizing that on the torus the form of the Hamiltonian is restricted, allowing one term for each element of the group of loop symmetries. This will facilitate a general discussion on the system's topological degeneracy.

Define $\mathcal{P}$ to be the projector onto this space and note that for any exact eigenstate of the full Hamiltonian $|\,\psi\rangle$, its projection $|\,\psi_0\rangle$ onto the subspace satisfies

$$\left[E_0 + \sum_{n=1}^{\infty} H^{(n)}\right] |\,\psi_0\rangle = E|\,\psi_0\rangle = H_{\text{eff}}|\,\psi_0\rangle, \tag{4}$$

where $H^{(n)} = \mathcal{P}U\mathcal{G}^{n-1}\mathcal{P}$, $\mathcal{G} = [1/(E-H_0)]\,(1-\mathcal{P})U$ and E_0 is the ground state energy of H_0. As was mentioned above, (4) is an implicit but perturbatively exact equation for E. The eigenstates, with eigenvalue E, of the effective system and full system are related by the expression $|\,\psi\rangle = (1-\mathcal{G})^{-1}|\,\psi_0\rangle = \sum_{n=0}^{\infty}\mathcal{G}^n|\,\psi_0\rangle$.

Calculating the n^{th} order correction is equivalent to finding the non-zero elements of the matrix $H^{(n)}$. Contributions to $H^{(n)}$ come from the length n products $K^{\alpha^{(1)}}_{ij}\ldots,K^{\alpha^{(n)}}_{lm}$ with $\alpha^{(m)} \in x, y$ that preserve the low-energy subspace. Hence any such contribution comes from an element of the group of loop symmetries from which all factors K^z_{ij} have been removed.

The resulting low-energy effective Hamiltonian can be written in terms of operators acting on the spins of the dimers using the following transformation rules:

$$\begin{array}{lll} \mathcal{P}[\sigma^x\otimes\sigma^y] \;\rightarrow\; +\sigma^y_{\text{e}}, & \mathcal{P}[\sigma^x\otimes\sigma^x] \;\rightarrow\; +\sigma^x_{\text{e}}, & \\ \mathcal{P}[\sigma^y\otimes\sigma^y] \;\rightarrow\; -\sigma^x_{\text{e}}, & \mathcal{P}[\sigma^z\otimes I\;] \;\rightarrow\; +\sigma^z_{\text{e}}, & \\ \mathcal{P}[\sigma^z\otimes\sigma^z] \;\rightarrow\; +I_{\text{e}}, & & \end{array} \tag{5}$$

where the subscript e indicates the effective spin operation. This transformation can be applied directly to the loop symmetries themselves, without removing the z-links first, and does not change the resulting operator on the low-energy subspace. Thus, we need only to examine the effective representations of the projections of the loop symmetries to understand the possible forms of the effective Hamiltonian. The lowest order non-constant contributions come from the plaquette operators $\mathcal{P}[W_p] \rightarrow Q_p = \sigma^y_{e(l)}\sigma^y_{e(r)}\sigma^z_{e(u)}\sigma^z_{e(d)}$, where l, r, u, d denotes the positions (left, right, up and down) of the effective spins, relative to the plaquette p [1]. Expanding to all orders, we have contributions from all loop symmetries, both homologically trivial and non-trivial. To come to an explicit expression for the effective Hamiltonian, we now introduce a particular generating set for the loop symmetry group, constructed from $N/2-1$ plaquettes and the operators $Z \equiv \prod_i \sigma^z_i$,where i represents lattice sites in the horizontal direction of alternating x and y-links and $V \equiv \prod K^{x,y}_{jk}\prod K^{y,x}_{lm}$, where the products take place over vertically arranged x- and y-links. The projections $\mathcal{P}(Z) \rightarrow z$ and $\mathcal{P}(V) \rightarrow y$ act by σ^z_e and σ^y_e on the relevant effective spins, see FIG. 2. In analogy to (3) we can now write the full effective Hamiltonian as

$$H_{\text{eff}} = \sum_{k=0}^{3}\sum_{l=1}^{2^{N/2-2}} d_{k,l}G_k(z,y)F_l(Q_1,Q_2,...,Q_{N/2-2}), \tag{6}$$

where $G_0 = I, G_1 = z, G_2 = y$ and $G_3 = zy$ and the $d_{k,l}$ are constants which depend on J_x, J_y and J_z. This form is strictly valid for when the effective square toroidal lattice has an even number of plaquettes Q_p along both directions. The inside sum only runs to $2^{N/2-2}$ because, as a result of the projection, we now have two non-trivial relations $\prod Q_b = \prod Q_w = 1$, see FIG. 2. The arguments here can be generalised to odd-odd and odd-even lattices examined in [24].

In general, $d_{k,l} \sim O\left(J_x^{n_x}J_y^{n_y}\right)$, where n_x and n_y are the respective number of x-links and y-links used to make $G_k(z,y)F_l(Q_1,Q_2,...,Q_{N/2-2})$. In the thermodynamic limit, and for homologically nontrivial loops ($k>0$), the values of n_x and n_y go to infinity and the limiting form of (6) is similar to the planar Hamiltonian addressed in [12] but with additional topological degrees of freedom.

We can now analyse the topological degeneracy of the abelian phase. The general argument for topological ground state degeneracy depends on the existence of operators T_1 and T_2 that both create particle/anti-particle pairs from the vacuum, bring the particle (or anti-particle) around the torus and then annihilate the pair [2, 25, 26]. These operators should commute with the Hamiltonian but not with each other. Hence T_1 and T_2 operators for the honeycomb system cannot be contained within the group of commuting loop symmetries. However, the low-energy effective representations of the homologically nontrivial loops' generators have the factorizations $z = z_b z_w$ and $y = y_b y_w$, where z_b and y_b act

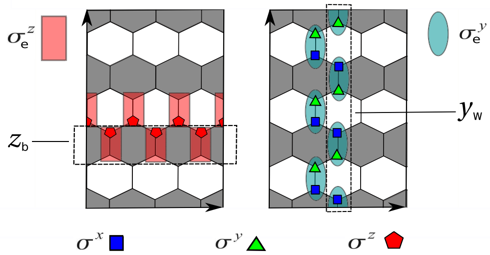

FIG. 2: The Z and V chains with their projections on to the dimerized subspace. The projections may be factorised into products $\mathcal{P}[Z] \to z_b z_w$ and $\mathcal{P}[V] \to y_b y_w$. Each of the individual factors z_b, z_w, y_b, y_w also commute with the homologically trivial components of the effective Hamiltonian but obey the relation $z_j^{-1} y_k^{-1} z_j y_k = e^{i\pi(1-\delta_{jk})} I$. In the text, dimers are referred to as black (white) if they are shared by a black (white) plaquette.

with effective σ^z's and σ^y's respectively on the spins of the 'black' dimers involved in z and y, while z_w and y_w do the same for the 'white' dimers, (see FIG. 2).

These black and white operators correspond to the nontrivial loop operators on the square lattice and dual square lattice of the toric code (cf. [2]) and thus obey the commutation relations $z_j^{-1} y_k^{-1} z_j y_k = e^{i\pi(1-\delta_{jk})} I$. Since these operators commute with the effective plaquette operators Q_p they also commute with all homologically trivial components of H_{eff}. However, they do not commute with all of the homologically non-trivial components. If we define C' as the homologically non-trivial loop with the least number of x- and y-links, then the topological degeneracy is first broken at the order M, where M is the number of x- and y-links in C'.

For the typical system sizes that can be handled by numerical diagonalization and other numerical methods, these finite size terms are appreciable and must be taken into account to produce a good fit to exact numerical results. For example, in all possible $N < 36$ spin toroidal configurations, additional non-constant effects will occur at the 4th order or lower. In larger tori the topological degeneracy of the system can be robust well beyond the 4th order toric code approximation and indeed in the thermodynamic limit it exists to all orders of the perturbation theory. Only in this case are the eigenstates of the effective Hamiltonian still those of the toric code and the ground state of each vortex configuration still exactly 4-fold degenerate. The ground state energies of different vortex configurations, not related by a translation, are generally different and are calculable using the planar form of the effective Hamiltonian [12].

We now concentrate on the full Hamiltonian and consider the physical properties associated with open ended strings of overlapping K^α operators. We first note that $\{\sigma_i^\alpha, W_p\} = 0$ when the site i belongs to an α-link at plaquette p. Hence, the operator σ_j^α changes the vorticity of the two plaquettes sharing this α-link by either creating or annihilating a pair of vortices, or moving a vortex from one plaquette to the other. It follows that the K operators satisfy $[K_{ij}^\alpha, W_p] = 0$ $(\forall i, j)$, $[K_{ij}^{\alpha,\beta}, W_p] = 0$ $(i, j \notin p)$ and $\{K_{ij}^{\alpha,\beta}, W_p\} = 0$ $(i \vee j \in p)$.

Now define a path s on the lattice as some ordered set of $|s|$ neighboring sites connecting the endpoints i and j. A string operator, denoted as S_{ij}^s, of overlapping K^α operators along this path s can be represented as a site ordered product of σ^α and $K^{\alpha,\beta}$ operators. If we assume that a $K^{\alpha,\beta}$ always acts first we see that the total operator can be interpreted as creation of two vortex-pairs and subsequent movement of one of the pairs along the path s. Importantly, we see that σ^α correspond to a rotation of one vortex-pair, whereas $K^{\alpha,\beta}$ moves the pair without a rotation (see FIG. 3). If i and j are neighboring sites and s is a homologically trivial loop then by definition $C_{(k,l)} = S_{ij}^s = \prod_p W_p$, where the product is over all plaquettes enclosed by s. If we treat a vortex-pair as a composite particle then the simplest loop operator $C_{(k,l)} = W_p$ (constructed from single σ^α operators) rotates the composite particle by 2π. The resulting overall phase of $e^{i\pi}$ suggests that the vortex-pairs are fermions for all values of J_α.

Suppose now that the first and last links along the path s are α- and μ-links, respectively, and that the ends of the string S_{ij}^s are given by the operators σ^β or $K^{\beta,\alpha}$ and σ^ν or $K^{\nu,\mu}$. Then $S_{ij}^s H S_{ij}^s = H + 2J_\gamma K_{ia}^\gamma + 2J_\tau K_{jb}^\tau$, where a and b are the sites connected to i and j by the respective γ- and τ-links, $\gamma \neq \alpha \neq \beta$ and $\tau \neq \mu \neq \nu$. One immediately sees that the ground state $|0\rangle$, which is from the vortex-free sector [1, 27], satisfies

$$\langle S_{ij}^s H S_{ij}^s \rangle = E_0 + 2J_\gamma \langle K_{ia}^\gamma \rangle + 2J_\tau \langle K_{jb}^\gamma \rangle \tag{7}$$

where E_0 in this case is the ground state energy of the full Hamiltonian. Note that all quantities on the right hand side of (7) have been calculated explicitly [1, 9]. The crucial point is that the energy of the state $S_{ij}^s|0\rangle$ depends only on the ends of the string and, if we assume a translationally invariant ground state, this energy contribution is the same for links of the same type. This implies that, even when J_x, J_y and J_z are not equal, the vortex-pairs can be propagated freely provided the relative orientation of the pair remains constant.

These fermionic vortex-pairs are distinct from the fermions introduced as redundant degrees of freedom in [1], those obtained by Jordan-Wigner transformation [7, 8, 11] and the vorticity preserving free-fermionic excitations of [12]. In the gapped phase however, the low-energy vortex-pair configurations are related to certain fermionic e-m composites of the toric code [1, 2].

The movement of vortex-pairs is in contrast to the situation encountered when one wants to separate individual vortices. Crucially, this cannot be done using overlapping K^α terms and indeed can only be achieved if we use

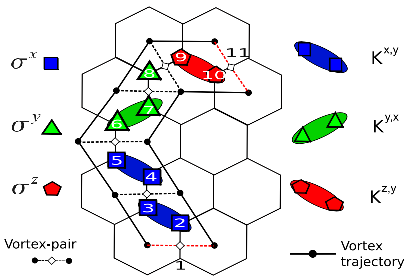

FIG. 3: The operator $K^{x,y}_{2,3}$ is used to create two vortex-pairs from the vacuum with an energy cost of $2J_z\langle K^z_{1,2}+K^z_{3,4}\rangle$. The subsequent operators $K^{x,y}_{4,5}$ and $K^{y,x}_{6,7}$ move one of the pairs in the direction shown. The Pauli operator σ^y_8 rotates this pair and the energy of the system at this time is $E_0+2J_z\langle K^z_{1,2}\rangle+2J_x\langle K^x_{8,9}\rangle$. This new pair is then moved horizontally with no additional energy cost by $K^{z,y}_{9,10}$.

single σ^α's that do not, in general, act on neighbouring sites. To this end we define $D^s_{ij}\equiv\sigma^\alpha_i\sigma^\beta_k....\sigma^\gamma_j$, where it is understood that if i and k are neighboring sites along some link, then $\alpha\neq\beta$. These operators satisfy

$$\langle D^s_{ij}HD^s_{ij}\rangle=E_0+aJ_x\langle K^x\rangle+bJ_y\langle K^y\rangle+cJ_z\langle K^z\rangle, \quad (8)$$

where $a+b+c=|s|$ for some integers a, b and c depending on the path s. The value of (8) scales with the length $|s|$ and implies that there is a string tension for vortex configurations created in this way [28].

These results, valid for all values of the parameters J_α, are in agreement with the perturbative analysis of the gapped phase [15]. There it was shown that while the repeated application of single σ^α excites e or m toric code semions in the low-energy dimerized subspace, it also introduces contributions to the wavefunction from higher energy eigenstates. These high-energy eigenstate contributions also occur when low energy vortex-pairs are excited, but in this case two effective toric code e-m pairs are created in the effective system. However, since (7) implies that the vortex-pairs can be moved freely, there can be no increase in the contribution from these high energy states as these pairs are propagated. This may be useful for the experimental detection of anyons because, in the toric code, the e-m pairs and single e (or m) excitations have mutually anyonic statistics. These points will be addressed in more detail in [29].

In conclusion, we have associated each of the loop symmetries of the full toroidal system with a term in the perturbative expansion. We then demonstrated the order at which the topological degeneracy is broken and noted that, in the thermodynamics limit, it remains to all orders. In a further analysis, valid for all values of the couplings J_α , we showed that the symmetries correspond to propagation of vortex-pairs along closed loops. When treated as composite particles, the vortex-pairs are fermions. We showed that these pairs can be propagated with no additional energy cost but that, in general, single vortices can not.

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