On a vector moment problem appearing in the analysis of controllability of neutral type systems

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Abstract

We consider solvability of a vector moment problem by means of its correspondence to controllability problem for a certain delayed system of neutral type. In this way we succeded to determine exsctly the minimal interval on which the moment problem is solvable.

Key words: vector moment problem, neutral type system, spectral assignment.

Studying controllability problem for the following neutral type system

$$\dot{z}(t) = A_{-1}\dot{z}(t-1)+\int_{-1}^{0} A_2(\theta)\dot{z}(t+\theta)\,d\theta+\int_{-1}^{0} A_3(\theta)z(t+\theta)\,d\theta+Bu(t), \quad (1)$$

where A_{-1} is a constant $n \times n$-matrix, $\det A_{-1} \neq 0$, A_2, A_3 are $n \times n$-matrices whose elements belong to $L_2(-1,0)$ and B is a constant $n \times r$-matrix, we reduced it to an equivalent moment problem [3]. We consider the operator model of the neutral type system (1) introduced by Burns and al. in product spaces. The state space is $M_2(-1,0;\mathbb{C}^n) = \mathbb{C}^n \times L_2(-1,0;\mathbb{C}^n)$, shortly M_2, and (1) is rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} y(t) \\ z_t(\cdot) \end{pmatrix} = \mathcal{A}\begin{pmatrix} y(t) \\ z_t(\cdot) \end{pmatrix} + \mathcal{B}u \quad (2)$$

where the operator $\mathcal{A}$ is given by

$$\mathcal{A}\begin{pmatrix} y(t) \\ z_t(\cdot) \end{pmatrix} = \begin{pmatrix} \int_{-1}^{0} A_2(\theta)\dot{z}_t(\theta)\mathrm{d}\theta + \int_{-1}^{0} A_3(\theta)z_t(\theta)\mathrm{d}\theta \\ \mathrm{d}z_t(\theta)/\mathrm{d}\theta \end{pmatrix}$$

with the domain

$$\mathcal{D}(\mathcal{A}) = \left\{(y, z(\cdot)) : z \in H^1(-1,0;\mathbb{C}^n), y = z(0) - A_{-1}z(-1)\right\} \subset M_2.$$

The operator $\mathcal{A}$ is the infinitesimal generator of a C_0-group. The operator $\mathcal{B}$ is defined by $\mathcal{B}u = (Bu, 0)$. The relation between the solutions of the neutral type system (1) and the system (2) is given by the substitutions

$$y(t) = z(t) - A_{-1}z(t-1), \quad z_t(\theta) = z(t+\theta).$$

Complete desription of the spectral properties of operator $\mathcal{A}$ is given in [4]. Consider for simplicity the case when all eigenvalues of A_{-1} are simple with different arguments. Denote these eigenvalues by μ_m, $m = 1, \ldots, n$. We will use the notation $\tilde{\mathcal{A}}$ for $\mathcal{A}$ in the particular case $A_2(\theta) = A_3(\theta) = 0$. Operator $\tilde{\mathcal{A}}$ has a simple eigenvalues

$$\tilde{\lambda}_k^m = \ln|\mu_m| + i(\arg\mu_m + 2\pi k), m = 1, \ldots, n, k \in \mathbb{Z}$$

with corresponding eigenvectors

$$\tilde{\varphi}_k^m = \begin{pmatrix} 0 \\ e^{\tilde{\lambda}_k^m \theta}\Phi_m \end{pmatrix} \in M_2,$$

where $A_{-1}\Phi_m = \mu_m \Phi_m$, $m = 1, \ldots, n$. Besides $\tilde{\mathcal{A}}$ has the eigenvalue $\tilde{\lambda}^0 = 0$ that corresponds to an n-dimensional subspace of generalized eigenvectors: $\tilde{\varphi}_1^0, \ldots, \tilde{\varphi}_n^0$. All eigenvectors $\{\tilde{\varphi}_k^m\} \cup \{\tilde{\varphi}_j^0\}$ constitute a Riesz basis in M_2. In general case if $A_2(\theta)$ and $A_3(\theta)$ are chosen in such a way that all eigenvalues of $\mathcal{A}$, λ_k^m, $m = 1, \ldots, n$, $k \in \mathbb{Z}$, are still simple then they satisfy the condition

$$\sum_{k,m} \left|\lambda_k^m - \tilde{\lambda}_k^m\right|^2 < \infty \tag{3}$$

and the corresponding eigenvectors $\{\varphi_k^m\}$ satisfy

$$\sum_{k,m} \|\varphi_k^m - \tilde{\varphi}_k^m\|^2 < \infty.$$

Eigenvectors $\{\varphi_k^m\}$ together with $\{\varphi_j^0\}$ form a Ries basis in M_2 quadratically close to the spectral basis $\{\tilde{\varphi}\}$ of $\tilde{\mathcal{A}}$.

Let $\{\tilde{\psi}\}$ and $\{\psi\}$ are biorthogonal basises for $\{\tilde{\varphi}\}$ and $\{\varphi\}$ respectively. It is easily checked that $\{\tilde{\psi}\}$ and $\{\psi\}$ are spectral basises of adjoint operators $\tilde{\mathcal{A}}^*$ and $\mathcal{A}^*$,

$$\tilde{\mathcal{A}}^* \tilde{\psi}_k^m = \bar{\tilde{\lambda}}_k^m \tilde{\psi}_k^m, \qquad \mathcal{A}^* \psi_k^m = \bar{\lambda}_k^m \psi_k^m.$$

Besides these basises are also quadraticaly close [2]:

$$\sum_{k,m} \left\|\psi_k^m - \tilde{\psi}_k^m\right\|^2 < \infty.$$

Finally note that as it is shown in [5] vectors $\tilde{\psi}_k^m$, $k \neq 0$, are of the form

$$\tilde{\psi}_k^m = \begin{pmatrix} \frac{1}{k+\frac{1}{2}}\Psi_m \\ * \end{pmatrix} \in M_2 \qquad \psi_k^m = \begin{pmatrix} \frac{1}{k+\frac{1}{2}}\Psi_{mk} \\ * \end{pmatrix}, \tag{4}$$

where Ψ_m, $m = 1, \ldots, n$ are eigenvectors of A_{-1},

$$A_{-1}\Psi_m = \bar{\mu}_m \Psi_m,$$

and

$$\sum_{k,m} \|\Psi_{mk} - \Psi_m\|^2 < \infty, \quad m = 1, \ldots, n. \tag{5}$$

Let now $x \in M_2$. Then

$$x = \sum_{\varphi \in \{\varphi\}} \langle x, \psi \rangle \varphi.$$

A state $x = \begin{pmatrix} y \\ z(\cdot) \end{pmatrix} \in M_2$ is reachable at time T by a control $u(\cdot) \in L_2(0,T;\mathbb{C}^r)$ iff the steering condition

$$x = \begin{pmatrix} y \\ z(\cdot) \end{pmatrix} = \int_0^T \mathrm{e}^{\mathcal{A}t}\mathcal{B}u(t)\mathrm{d}t. \qquad (6)$$

holds. This steering condition may be expanded using the basis $\{\varphi\}$. A state x is reachable iff

$$\sum_{\varphi\in\{\varphi\}} \langle x,\psi\rangle\varphi = \sum_{\varphi\in\{\varphi\}} \int_0^T \langle \mathrm{e}^{\mathcal{A}t}\mathcal{B}u(t),\psi\rangle \mathrm{d}t\varphi,$$

for some $u(\cdot) \in L_2(-1,0;\mathbb{R}^r)$. Then the steering condition (6) can be substituted by the following system of equalities

$$\langle x,\psi\rangle = \int_0^T \langle \mathrm{e}^{\mathcal{A}t}\mathcal{B}u(t),\psi\rangle \mathrm{d}t, \quad \psi\in\{\psi\}. \qquad (7)$$

Let $\{\mathbf{b}_1,\ldots,\mathbf{b}_r\}$ be columns of B and $b_i = \begin{pmatrix} \mathbf{b}_i \\ 0 \end{pmatrix} \in M_2,\ i = 1,\ldots,r$. Then the right hand side of (7) takes the form

$$\int_0^T \langle \mathrm{e}^{\mathcal{A}t}\mathcal{B}u(t),\psi\rangle \mathrm{d}t = \sum_{i=1}^r \int_0^T \langle \mathrm{e}^{\mathcal{A}t}b_i,\psi\rangle u_i(t)\mathrm{d}t.$$

Taking into accout the fact that $\{\psi\}$ is a spectral basis of $\mathcal{A}^*$ we rewrite the steering conditions as

$$s_k^m = \left(k+\frac{1}{2}\right)\langle X,\Psi_k^m\rangle = \int_0^T e^{\lambda_k^m t}\left(b_{k,m}^1 u_1(t) + \ldots + b_{k,m}^r u_r(t)\right)dt, \quad (8)$$

$m = 1,\ldots,n,\ k\in\mathbb{Z}$, where $b_{k,m}^j = \left(k+\frac{1}{2}\right)\langle b_j,\psi_k^m\rangle$. Equations (8) pose a vector moment problem with respect to unknown $u_j(t),\ j = 1,\ldots,r$. Using (4) we observe that

$$\left(k+\frac{1}{2}\right)\langle b_j,\tilde{\psi}_k^m\rangle = \left\langle \begin{pmatrix} \mathbf{b}_j \\ 0 \end{pmatrix}, \begin{pmatrix} \Psi_m \\ * \end{pmatrix} \right\rangle = \langle \mathbf{b}_j,\Psi_m\rangle = b_m^j,$$

So this value does not depend on k. Then from (5) it follows that the coefficients $b_{k,m}^j$ satisfy the condition

$$\sum_{k,m}\left|b_{k,m}^j - b_m^j\right|^2 = \sum_{k,m}|\langle \mathbf{b}_j,\psi_{mk}-\psi_m\rangle|^2 < \infty \quad j = 1,\ldots,r. \qquad (9)$$

In the scalar case $r = 1$, under the conditions

$$b_m^1 = \langle \mathbf{b}_1,\psi_m\rangle \neq 0, \quad m = 1,\ldots,n \qquad (10)$$

(controllability of the pair (A_{-1},B)) and

$$\langle b_1,\psi_k^m\rangle \neq 0, \quad m = 1,\ldots,n; k\in\mathbb{Z} \qquad (11)$$

(approximate controllability of system (2)) the solvability of (8) can be studied by application of the methods given in [1]. In the vector case the problem is essentially more complicated. We showed [3] that the minimal interval of solvability of (2) is related with the first controllability index $n_1(A_{-1},B)$ of the system $\dot{x} = A_{-1}x + Bu$ (see [6]). This result encouraged us to consider the inverse problem: to study moment problems of the form (8) (of course, under some special assumptions) by means of their correspondence to the controllability problems for neutral type systems.

We show that the conditions (3), (9) together with some conditions analogous to (10), (11) are not only necessary but also sufficient in order, for the moment problem (8), to be generated by a neutral type system (1).

Theorem 1. *Moment problem (2) corresponds to the controllability problem for an exact controllable neutral type system of the form (1) if and only if it satisfies the condition*

i) $\sum_{k,m} \left|\lambda_k^m - \tilde{\lambda}_k^m\right|^2 < \infty$, *where* $\tilde{\lambda}_k^m = ln|\mu_m| + i(\arg \mu_m + 2\pi k)$, $m = 1, \ldots, n, k \in \mathbb{Z}$, $\mu_1, \ldots, \mu_r$ *are some nonzero different complex numbers;*

ii) $\sum_{j,k,m} \left|b_{k,m}^j - b_m^j\right|^2 < \infty$, *where* b_m^j $(m = 1, \ldots, n, j = 1, \ldots, r)$ *are some real numbers such that* $\sum_j \left|b_m^j\right| > 0$, $m = 1, \ldots, n$;

iii) $\sum_j \left|b_{k,m}^j\right| > 0$, $m = 1, \ldots, n$, $k \in \mathbb{Z}$.

The $n \times n$ *and* $n \times r$*-matrices* A_{-1} *and* B *being given by*

$$A_{-1} = \operatorname{diag}\{\mu_1, \ldots, \mu_n\}, \qquad B = \{b_m^j\}_{m=1,\ldots,n}^{j=1,\ldots,r}.$$

Our proof of Theorem 1 is based on the following concept. We seek a system (1) in the form

$$\dot{z}(t) = A_{-1}\dot{z}(t-1) + +B\int_{-1}^{0} \left[F_2(\theta)z(t+\theta) + F_3(\theta)\dot{z}(t+\theta)\right] d\theta + Bu(t),$$

where matrices A_{-1}, B are chosen above, F_2, F_3 are $(r \times n)$– indetermined matrices with elements from $L_2(-1, 0)$. In infinite-dimensional form this corresponds to the system

$$\frac{d}{dt}\begin{pmatrix} y(t) \\ z_t(\cdot) \end{pmatrix} = \mathcal{A}\begin{pmatrix} y(t) \\ z_t(\cdot) \end{pmatrix} + \mathcal{B}u = \left(\tilde{\mathcal{A}} + \mathcal{B}\mathcal{P}\right)\begin{pmatrix} y(t) \\ z_t(\cdot) \end{pmatrix} + \mathcal{B}u,$$

where $\mathcal{P} = \begin{pmatrix} \mathbf{p}_1 \\ \vdots \\ \mathbf{p}_r \end{pmatrix}$, and $\mathbf{p}_j$, $j = 1, \ldots, r$ are functionals from space X_{-1}, where

$$X_{-1} = \left\{ \sum_{\tilde{\psi}\in\{\tilde{\psi}\}} d_{\tilde{\psi}}\tilde{\psi}, \sum_{k,m} (d_k^m/k)^2 < \infty \right\},$$

$\{\tilde{\psi}\}$ is a spectral basis of $\tilde{\mathcal{A}}^*$. One needs to choose an operator $\mathcal{P}$ in such a way that

1) $\sigma(\tilde{\mathcal{A}} + \mathcal{B}\mathcal{P}) = \{\lambda_k^m\}$, $k \in \mathbb{Z}$, $m = 1, \ldots, n$;
2) $\langle b^j, \psi_k^m\rangle = b_{km}^j$, ψ_k^m are eigenvalues of $\mathcal{A}^*$.

We reduce this problem to the following one. Denote by $Q(\lambda)$ an $(r \times r)$ matrix-function of the form $Q(\lambda) = \mathcal{P}R(\tilde{\mathcal{A}}, \lambda)\mathcal{B}$, $R(\tilde{\mathcal{A}}, \lambda)$ is resolvent of $\tilde{\mathcal{A}}$. It turns out that all the eigenvalues are the roots of the equation

$$\det(I + Q(\lambda)) = 0$$

and for and $m = 1, \ldots, n$, $k \in \mathbb{Z}$, vector $w^*_{mk} = (b^1_{k,m}, \ldots, b^r_{k,m})$ can be found from the condition:

$$w^*_{mk}(I + Q(\lambda^m_k)) = 0.$$

Finally we give a choice of $\mathcal{P}$ providing the required values of parameters $\{\lambda^m_k\}$, $\{b^j_{k,m}\}$.

Note that Theorem 1 is a direct generalization of pole assignement theorem for neutral type systems with simple control from [5]. Denote by $\mathcal{R}_T$ the set of moment sequences $\{s^m_k\}$ for which (8) is solvable, this means that there exist functions $u_j(t)$, $j = 1, \ldots, r$ from $L_2[0, T]$ such that (8) is satisfied. It follows from general moment problem theory that $\mathcal{R}_T$ is a subset of te space l_2. Reduction to the controllability problem allows to determine the exact values of T for which $\mathcal{R}_T$ may be equal l_2. Combining Theorem 1 and the results of [3] (section 7) we obtain

Theorem 2. *Let the conditions i) – iii) be satisfied. We have:*

1. *If $T > n_1(A_{-1}, B)$ then $\mathcal{R}_T = h_1$.*
2. *If $T = n_1(A_{-1}, B)$ then $\mathcal{R}_T$ is a subspace of h_1 of finite co-dimension in h_1 (it is possible that $\mathcal{R}_T = h_1$).*
3. *If $T < n_1(A_{-1}, B)$ then $\mathcal{R}_T$ is a subspace of h_1 of infinite co-dimension.*

Example. Consider two vector moment problems:

$$\begin{aligned} s^1_k &= \int_0^T e^{(i(\pi+2\pi k)+\varepsilon^1_k)t}(11u_1(t) + u_2(t))dt, \\ s^2_k &= \int_0^T e^{(i2\pi k+\varepsilon^2_k)t}(u_1(t) + u_2(t))dt, \\ s^3_k &= \int_0^T e^{(\log 2+i2\pi k+\varepsilon^3_k)t}(u_1(t) - u_2(t))dt, \\ s^4_k &= \int_0^T e^{(\log 3+i2\pi k+\varepsilon^4_k)t}(u_1(t) - u_2(t))dt, \end{aligned} \tag{12}$$

and

$$\begin{aligned} s^1_k &= \int_0^T e^{(i(\pi+2\pi k)+\varepsilon^1_k)t}(6u_1(t) - u_2(t))dt, \\ s^2_k &= \int_0^T e^{(i2\pi k+\varepsilon^2_k)t}(4u_1(t) - u_2(t))dt, \\ s^3_k &= \int_0^T e^{(\log 2+i2\pi k+\varepsilon^3_k)t}(3u_1(t) - u_2(t))dt, \\ s^4_k &= \int_0^T e^{(\log 3+i2\pi k+\varepsilon^4_k)t}(2u_1(t) - u_2(t))dt, \end{aligned} \tag{13}$$

$k \in \mathbb{Z}$, $\sum_k (\varepsilon^j_k)^2 < \infty$, $j = 1, \ldots, 4$. The both moment problems can be reduced to the controllability problem for systems of the form (1), where

$$A_{-1} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 6 & -5 & -5 & 5 \end{pmatrix},$$

$\{\mu_1, \mu_2, \mu_3, \mu_4\} = \{-1, 1, 2, 3\}$, corresponding eigenvectors of A_{-1}^* are

$$\Psi_1 = \begin{pmatrix} -6 \\ 11 \\ -6 \\ 1 \end{pmatrix}, \Psi_2 = \begin{pmatrix} 6 \\ 1 \\ -4 \\ 1 \end{pmatrix}, \Psi_3 = \begin{pmatrix} 3 \\ -1 \\ -3 \\ 1 \end{pmatrix}, \Psi_4 = \begin{pmatrix} 2 \\ -1 \\ -2 \\ 1 \end{pmatrix},$$

and in the first case $B_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$ and in the second case $B_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$, for a certain choice of $A_2(\theta)$, $A_3(\theta)$. Since

$$n_1(A_{-1}, B_1) = 2, \qquad n_1(A_{-1}, B_2) = 3$$

then the moment problem (12) is solvable for all $\{s_k^j\} \in l_2$ if $T > 2$ while the problem (13) if $T > 3$.

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