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Frédéric BOYER*   &   Dominique PRIMAULT**

Institut de Recherche en Cybernétique de Nantes, UMR
(*Ecole des Mines de Nantes, **Ecole Centrale de Nantes)
1 Rue de la Noë, B.P 92101
44321 Nantes cedex 3, FRANCE
E.mail: Frederic.Boyer@emn.fr

Abstract: This article is devoted to the dynamics of flexible multi-body systems and to their links with a fundamental set of equations discovered by H. Poincaré one hundred years ago [1]. These equations, called "Poincaré-Chetayev equations", are today known to be the foundation of the Lagrangian reduction theory. Starting with the extension of these equations to a Cosserat medium, we show that the two basic sets of equations used in flexible multi-body dynamics. The generalized Newton-Euler model of flexible multi-body systems in the floating frame approach and the partial differential equations of the nonlinear geometrically exact theory in the Galilean approach, are Poincaré-Chetayev equations.

Key words: Flexible multi-body systems, Lie groups, Poincaré-Chetayev equations, Floating frame, Nonlinear beam.

1. Introduction

The dynamic modeling of flexible multi-body systems aroused a great interest during the last few years [2-10]. Two types of systems are susceptible of requiring such a theory: the fast light industrial manipulators and the large space structures. To model these systems, two theories can be used depending on the type of reference of links deformation they use. The first theory, called "floating frame approach"[5-10], is often restricted to the field of linear elasticity since it considers the link deformations as modal perturbations of the overall motions of some moving floating frames. In this approach, the most efficient simulation and control algorithms are based on the model said of "Newton-Euler" [5-9] and opposed to the Lagrangian one [10]. To obtain the generalized Newton-Euler model of a flexible multi-body system, several methods have been proposed. This model appeared for the first time in [5] where the authors used the Newton and Euler laws joined to modal projections. Alternatively, Meirovitch [6] proposed to proceed from Lagrange's equations with quasi coordinates. Invoking the corresponding non-holonomic velocities Bremer [7], used the Hamel's equations. Finally, such a model can be obtained by applying the virtual power principle [8] and invoking the notion of Eulerian and Lagrangian description of motions as classified in [11]. In the floating frame context, the Newton-Euler formulation of the dynamics presents many advantages on the Lagrangian one. First it requires few computational efforts since only the dynamics of single links have to be written and to be joined with a recursive kinematic model of the chain. Second, and for the same reason, the dynamic contributions have a simple physical insight contrary to the Lagrangian approach where they appear via complex symbolic kinetic energies. This advantage is crucial when we search to add some non-linear improvements as the dynamic stiffening effect [12]. Finally thanks to its recursive character, this model can be used in a second time to achieve fast direct [13] and inverse o(n) (with n the number of links) algorithms [14].

The second theory (called Galilean one) was originally developed for the large space structures submitted to finite deformations and small strains [2-4]. It is based on the following basic remark: when the elastic motions are of the same order of magnitude as the rigid ones, it becomes artificial

and too much involved to separate them. Thus in this theory the link deformations references are Galilean and the fields of the links are global. Most of the time, this approach is premised on the Reissner beam theory [15]. In this framework, each constitutive cross section remaining rigid (Cosserat medium [16]), its configuration space can be identified to maps applying the material line of the beam onto $SO(3) \times \mathbb{R}^3$. The numerical procedure used in [2-4] to solve the rotational dynamic weak form is essentially achieved on the group and not, as in a Lagrangian approach, on one of its parameter space. The method is based on a spatial finite element reduction and an implicit one step integration scheme (of the Newmark class). The standard predictor-corrector Newmark scheme must be geometrically revisited since the curvature of $SO(3)$ prevents the standard linear operations of vector spaces. Once this obstacle is surrounded by a pullback operation of the dynamics on the reference configuration, the non-linear initial problem is replaced by a finite succession of linear ones via a Newton procedure. To solve the linear tangent dynamics, an exact intrinsic linearization of the weak form is achieved via an objective derivation. Since no approximation is done before the spatial and time discretizations, this approach is often qualified of "geometrically exact".

The purpose of this paper is to show that these equations, namely:

- the generalised Newton-Euler model of flexible multi-body system in the floating frame,

- the geometrically exact model of a multi-beam system in the Galilean frame,

can be deduced in a straight-forward manner of a set of equations discovered by Poincaré in [1] and refined by chetayev [17,18]. These equations today known as Poincaré-Chetayev [19-21] or Euler-Poincaré equations [22] are a generalization of Lagrange equations on a Lie group not forcibly commutative [23,24]. In flexible multibody dynamics, contrary to standard structural dynamics, the elastic bodies of the mechanism endure not only deformations but also rigid overall motions. It is this particular point which is directly concerned with the result of Poincaré. The article is structured as follows. After introducing our notations (section 2), we extend the Poincaré-Chetayev equations to the case of a continuum following the construction of Cosserat brothers [24] (section 3). Then, doing several remarks (section 4), we finally apply (section 5) these equations to a relevant case of flexible multi-body sytem: an open loop flexible manipulator.

2. Notations

In the subsequent developments we use some basic tools of differential geometry. We invite the reader to consult the appendix 4 of [23] for much detail about this aspect. We now quickly recall the definitions required by the article. In the following, G is a n dimensional Lie group of transformations of $\mathbb{R}^3 \to \mathbb{R}^3$. Its identity element is denoted by e. The motion of a material system whose the configuration space is G is defined by a map $g : t \in \mathbb{R}^+ \to g(t) \in G$. In the action space $\mathbb{R}^3$, it is defined by the set of transformed configurations: $(\Sigma(t) = g(t).\Sigma_o \subset \mathbb{R}^3, t \in \mathbb{R}^+)$ where $\Sigma_o \subset \mathbb{R}^3$ is the reference configuration of the system also named "material space". The Lie algebra of G is denoted by $\mathfrak{g}$ and defined as the tangent space to G at e denoted $T_e G$, endowed with the Lie bracket [,]. Introducing the internal product $\langle \ , \ \rangle$ on G, we define $\mathfrak{g}^*$ as the vector space of 1 forms on G. L_g and R_g denote the left and right translations on G. L_{g*} and R_{g*} are their tangent maps. Dualizing these linear maps through the use on the internal product gives the cotangent maps to L_g and R_g, respectively denoted: L_g^* and R_g^*. Differentiating the group automorphism $Ad_g : h \in G \mapsto R_{g^{-1}}(L_g(h))$ at $h = e$, gives the action map Ad_{g*} of G on $\mathfrak{g}$. Then differentiating Ad_{g*} with respect to g at $g = e$, defines the adjoint map $ad_{(.)*}$ of $\mathfrak{g} \to \mathfrak{g}$. Dualizing $ad_{(.)*}$ defines

the co-adjoint map: $ad_{(.)}^*: g^* \to g^*$. A left (respect. right) invariant field on G is a vector field invariant by L_{g*} (respect. R_{g*}). Once endowed with the Poisson bracket of vector fields denoted $[\ ,\ ]$, the vector space of left (respect. right) invariant vector fields on G realizes another definition of g. On the dual side, the vector space of left (respect. right) invariant 1 forms on G realizes another definition of g^*. From the point of view of the action space, $g \cong (T_e G, [\ ,\ ])$ is identified to the space of infinitesimal transformations applied to the reference configuration Σ_o ("right" or "material" transformations) or to the current configuration $\Sigma(t)$ ("left" or "spatial" transformations). Alternatively the space of left (respect. right) invariant fields realizes another definition of the material (respect. spatial) infinitesimal transformations. To get the dynamic equations of the material system in terms of the infinitesimal material transformations (respect. spatial ones) corresponds to write the dynamics in the "material (respect spatial) setting". The following computations are achieved "in coordinates" as in the original works of Poincaré and Chetayev. The summation convention on repeated indexes is systematically adopted excepted for equations (27-30). Finally, we will use sometimes the conventional notation: $f(x_1, \ldots x_k) = f(x_i)$, $i = 1, \ldots k$; for any function f, vector x and integer k.

3. Poincaré-Chetayev equations of a Cosserat medium

A Cosserat medium is a continuum of micro-solids (for instance the cross sections of a beam, the transverse rigid material lines of a shell or the grains of a micro-polar continuum) [25]. Consequently the spatial configurations of the medium can be described by the action in each point of a sub-manifold D of the medium reference configuration (the reference line for a beam, the reference membrane for a shell), of an n dimensional group G (typically $SE(3)$ or $SO(3)$) onto an elementary rigid micro-solid. Hence, contrary to the finite dimensional case treated by Poincaré [1], the group transformations are not only parameterized by the time, but also by the material coordinates ($X^I, I = 1, \ldots p$, $p \leq 3$) of D. The space of parameters (be careful they are not the group parameters) will be denoted by $P = \mathbb{R}^+ \times D$, where $\mathbb{R}^+$ is the time axis. We shall denote by x an arbitrary distinguished point of P of coordinates $(x^i)_{i=0,\ldots p} = (t, X^I)_{I=1,\ldots p} = (t, X)$. To get the Poincaré-Chetayev equations of a Cosserat medium, we are going to apply the Hamilton principle to a field of Lagrangian of the form:

$$\mathcal{L}: P \to \wedge^p(T^* D) \quad , \quad x \mapsto \mathcal{L}(q(x), \eta_i)\, dV \quad , \quad i = 0, \ldots p. \tag{1}$$

Where $\mathcal{L}$ is the Lagrangian density on D, $T^* D$ is the cotangent bundle to D, $\wedge$ is the exterior product, dV is the volume p form on D, $q(x)$ is the vector of group parameters of the current transformation at X applied on the micro-solid. And $\eta_i(x)$ is the current infinitesimal transformation endured by the micro-solid: at X, in a basis of g, when we shift along the $i^{th.}$ coordinate line of P. In the following, we first concentrate our attention on the material setting since it is the most interesting for solid mechanics. In this setting, $\eta_i(x)$ has two interpretations depending on the definition of g we adopt. If we define it as $(T_e(G), [,])$, the η_i's are defined as:

$$\eta_i(x) = L_{g(x)^{-1}*}(\partial_{x^i} g(x)) = \eta_i^\alpha(x) e_\alpha \quad , \quad i = 0, \ldots p. \tag{2}$$

Where $(e_\alpha)_{\alpha=1,..n}$ is a basis of infinitesimal material transformations (transformations applied on material particles). Alternatively, if we define the Lie algebra as the space of left invariant fields provided with the Poisson bracket, the η_i^α 's are defined by:

$$\eta_i(x) = \partial_{x^i} g(x) = \eta_i^\alpha(x) X_{\alpha,g(x)} \quad , \quad i = 0,..p \quad , \tag{3}$$

where we introduced the basis of left invariant vectors on G: $(X_\alpha : g \mapsto X_{\alpha,g} = L_{g*}(e_\alpha)$, $\alpha = 1,..n; g \in G)$ (the base point on G follows the basis index). In fact, by simple analysis of the variance of (2) and (3), we understand that the set of these vector fields on P realizes a unique field of 1-forms on P with values in the Lie algebra of the group: $\eta : P \to \mathfrak{g} \otimes T^*P$, [24], ($\otimes$ is the tensorial product). For instance, if we adopt $(T_e(G),[,])$ as definition of the Lie algebra, this field of 1-form is defined as:

$$\eta(x) = (L_{g^{-1}*}(\partial_{x^i} g))(x) dx^i = \eta_i^\alpha(x) e_\alpha \otimes dx^i \quad , \tag{4}$$

and gives the infinitesimal transformation to apply on the left of $g(x)$ when we shift from any point x to $x + dx$ in P. Alternatively adopting the left invariant fields as definition of the Lie algebra, this field of 1-forms is:

$$\eta(x) = (\partial_{x^i} g)(x) dx^i = \eta_i^\alpha(x) X_{\alpha,g(x)} \otimes dx^i, \tag{5}$$

and gives the change of $g(x)$ in the field of left invariant basis covering G, when we shift from x to $x + dx$ in P. In order to apply the Hamilton principle to a Lagragian density of the form (1), we have to establish a formula which plays a key role in the variation calculus on a non commutative Lie group. This relation is a consequence of the fact that the variation δ is achieved at fixed time and fixed material parameters. To establish this result, let us first remark that the variation of a $C^\infty(G)$ function f at $g(x)$ is:

$$\delta f(g(x)) = \delta g(x).f = \Omega^\alpha(x) X_{\alpha,g(x)}.f \quad , \tag{6}$$

Where the $\Omega^\alpha(x)$'s are the components of the virtual infinitesimal transformations in the basis of left invariant fields. On the other hand, the derivative of any f, with respect to a parameter τ of a space-time curve $\gamma : \tau \in \mathbb{R} \mapsto \gamma(\tau) \in P$ passing through x is:

$$\frac{df(g(x))}{d\tau} = \frac{dg(x)}{d\tau}.f = (\eta_i^\alpha(x) X_{\alpha,g(x)} \otimes dx^i).(f, \xi_x) = \eta_i^\alpha(x) \xi^i(x) X_{\alpha,g(x)}.f \quad , \tag{7}$$

where $\xi_x = \xi^i(x)\partial_{x^i}$ is the tangent vector to the curve γ at x. Moreover, note that in particular, when $\tau = x^j$, $\xi_x = \partial_{x^j}$, and (7) becomes:

$$\partial_{x^j} f(g(x)) = \partial_{x^j} g(x).f = (\eta_i^\alpha(x) X_{\alpha,g(x)} \otimes dx^i).(f, \partial_{x^j}) = \eta_j^\alpha(x) X_{\alpha,g(x)}.f \tag{8}$$

Hence, the variation process being achieved at fixed time, and fixed material position (label), we have:

$$\frac{d}{d\tau}(\delta f) - \delta(\frac{df}{d\tau}) = 0 \quad , \quad \forall f \in C^\infty(G) \quad , \forall \gamma \text{ passing through } x. \tag{9}$$

Inserting (6) and (7) into (9), we find (g is evaluated at x):

$$\frac{d}{d\tau}(\delta f) - \delta\left(\frac{df}{d\tau}\right) = \eta_i^\alpha \xi^i X_{\alpha,g}.(\Omega^\beta X_{\beta,g}.f) - \Omega^\beta X_{\beta,g}.(\eta_i^\alpha \xi^i X_{\alpha,g}.f) = 0$$

And since this last relation must be verified for any γ, i.e. for any set of ξ^i's, the "fixed parameters condition" can be rewritten:

$$\eta_i^\alpha (X_{\alpha,g}.\Omega^\beta)(X_{\beta,g}.f) + \eta_i^\alpha \Omega^\beta X_{\alpha,g}.(X_{\beta,g}.f)$$

$$-\Omega^\beta (X_{\beta,g}.\eta_i^\alpha)(X_{\alpha,g}.f) - \Omega^\beta \eta_i^\alpha X_{\beta,g}.(X_{\alpha,g}.f) = 0 \quad , \quad \forall f \in C^\infty(G) \tag{10}$$

Considering (6) and (8) we recognize in (10) the terms:

$$\Omega^\beta (X_{\beta,g}.\eta_i^\alpha) = \delta\eta_i^\alpha, \quad \text{and:} \quad \eta_j^\alpha (X_{\alpha,g}.\Omega^\beta) = \partial_{x^j}\Omega^\beta,$$

as so as the Poisson bracket of the left invariant basis vectors:

$$[X_\alpha, X_\beta] = c_{\alpha\beta}^\gamma X_\gamma \quad , \quad \alpha, \beta, \gamma = 1,...n$$

with $c_{\alpha\beta}^\gamma$'s the structure constants of the Lie algebra g. Finally we obtain the following relations:

$$\delta\eta_i^\alpha = \partial_{x^i}\Omega^\alpha + c_{\beta\gamma}^\alpha \eta_i^\beta \Omega^\gamma \quad , \quad i = 0,1,..p; \ \alpha = 1,..n. \tag{11}$$

The equations (11) are a generalization of the one-parameter formulae of [1]. In the same way, the "fixed parameters condition" expressed in the right invariant basis can be written:

$$\delta\mu_i^\alpha = \partial_{x^i}\Omega^\alpha - c_{\beta\gamma}^\alpha \mu_i^\beta \Omega^\gamma \quad , \quad i = 0,1,..p; \ \alpha = 1,..n. \tag{12}$$

Where the structure constants in the right invariant basis $(Z_\alpha : g \mapsto Z_{\alpha,g} = R_{g*}(e_\alpha)$, $\alpha = 1,..n; g \in G)$ are the opposites of those in the left invariant one [22]. Before applying the Hamilton principle, we have to model the fields of external efforts applied on the medium. Geometrically, the resultant of current efforts applied on the micro-solid at X in the configuration $g(x)$, is an element of g^*. Hence, we define the two fields of exterior forms with values in the dual of the Lie algebra:

The field of external efforts applied inside the medium:

$$\bar{F} : \mathbb{R}^+ \times \overset{o}{D} \mapsto g^* \otimes \wedge^p(T^* \overset{o}{D}), \ (t,X) \mapsto \bar{F} = \bar{\mathcal{F}}_\alpha(x, g(x))\omega_{g(x)}^\alpha \otimes dV. \tag{13}$$

The field of current external efforts applied on the boundaries of the medium:

$$\tilde{F} : \mathbb{R}^+ \times \partial D \mapsto g^* \otimes \wedge^{p-1}(T^*\partial D), \quad (t,X) \mapsto \tilde{F} = \tilde{\mathcal{F}}_\alpha(x, g(x))\omega_{g(x)}^\alpha \otimes dS \tag{14}$$

Where $(\omega^\alpha : g \mapsto \omega_g^\alpha, \ \alpha = 1,..n; g \in G)$ is the basis of left invariant 1 forms dual to $(X_\alpha, \alpha = 1,...n)$, and dS is the surface $p-1$ form on ∂D. Now, let us write the modified Hamilton principle:

$$\delta A = \delta\int_{t_1}^{t_2}\int_D \mathcal{L}(\eta_i^\alpha, q^\beta) \, dVdt = \int_{t_1}^{t_2} \delta W_{ext.} dt \quad , \quad \forall \delta g \tag{15}$$

Where A denotes the action of the Lagrangian and where we introduced the virtual work of external efforts applied to the medium:

$$\delta W_{ext.} = \int_D \bar{\mathcal{F}}_\alpha \Omega^\alpha dV + \int_{\partial D} \tilde{\mathcal{F}}_\alpha \Omega^\alpha dS \tag{16}$$

Going over the variation of the action, we have:

$$\delta A = \delta \int_{t_1}^{t_2} \int_D \mathcal{L}(\eta_i^\alpha, q^\beta) dV dt = \int_{t_1}^{t_2} \int_D \left(\frac{\partial \mathcal{L}}{\partial \eta_i^\alpha} \delta \eta_i^\alpha + \frac{\partial \mathcal{L}}{\partial q^\beta} \delta q^\beta \right) dV dt$$

Then taking into account the constraints (11), gives:

$$\delta A = \int_{t_1}^{t_2} \int_D \left(\frac{\partial \mathcal{L}}{\partial \eta_i^\alpha} (\partial_{x^i} \Omega^\alpha + c_{\beta\gamma}^\alpha \eta_i^\beta \Omega^\gamma) + \frac{\partial \mathcal{L}}{\partial q^\beta} \delta q^\beta \right) dV dt \tag{17}$$

But we have:

$$\frac{\partial \mathcal{L}}{\partial \eta_i^\alpha} \partial_{x^i} \Omega^\alpha = \partial_{x^i} \left(\frac{\partial \mathcal{L}}{\partial \eta_i^\alpha} \Omega^\alpha \right) - \partial_{x^i} \left(\frac{\partial \mathcal{L}}{\partial \eta_i^\alpha} \right) \Omega^\alpha \tag{18}$$

Then let us rewrite the first term of the right hand side of (18) as:

$$\int_{t_1}^{t_2} \int_D \partial_{x^i} \left(\frac{\partial \mathcal{L}}{\partial \eta_i^\alpha} \Omega^\alpha \right) dV dt = \int_{t_1}^{t_2} \int_D \partial_t \left(\frac{\partial \mathcal{L}}{\partial \eta_o^\alpha} \Omega^\alpha \right) + \partial_{X^I} \left(\frac{\partial \mathcal{L}}{\partial \eta_I^\alpha} \Omega^\alpha \right) dV dt$$

Integration by part of the time component followed by an application of the Stokes theorem to the spatial terms gives:

$$\int_{t_1}^{t_2} \int_D \partial_i \left(\frac{\partial \mathcal{L}}{\partial \eta_i^\alpha} \Omega^\alpha \right) dV dt = \left[\int_D \left(\frac{\partial \mathcal{L}}{\partial \eta_o^\alpha} \Omega^\alpha \right) dV \right]_{t_1}^{t_2} + \int_{t_1}^{t_2} \int_{\partial D} \left(\frac{\partial \mathcal{L}}{\partial \eta_I^\alpha} \Omega^\alpha \right) N_I \, dS dt$$

Where N_I is the $I^{th.}$ component of the unit normal to ∂D. Moreover the variation $\delta g(x) = \Omega^\alpha X_{\alpha, g(x)}$ vanishing at the end times, we have more simply:

$$\int_{t_1}^{t_2} \int_D \partial_{x^i} \left(\frac{\partial \mathcal{L}}{\partial \eta_i^\alpha} \Omega^\alpha \right) dV dt = \int_{t_1}^{t_2} \int_{\partial D} \left(\frac{\partial \mathcal{L}}{\partial \eta_I^\alpha} \Omega^\alpha \right) N_I \, dS \, dt \tag{19}$$

Taking into account (18) and (19) in (17), we find:

$$\delta A = \int_{t_1}^{t_2} \left(\int_{\partial D} \left(\frac{\partial \mathcal{L}}{\partial \eta_I^\alpha} \Omega^\alpha \right) N_I \, dS + \int_D \left(-\partial_{x^i} \left(\frac{\partial \mathcal{L}}{\partial \eta_i^\alpha} \right) \Omega^\alpha + c_{\beta\alpha}^\gamma \eta_i^\beta \Omega^\alpha \frac{\partial \mathcal{L}}{\partial \eta_i^\gamma} + \frac{\partial \mathcal{L}}{\partial q^\beta} \delta q^\beta \right) dV \right) dt$$

And since from (6):

$$\frac{\partial \mathcal{L}}{\partial q^\beta} \delta q^\beta = \frac{\partial \mathcal{L}}{\partial q^\beta} \Omega^\alpha X_{\alpha, g}(q^\beta), \tag{20}$$

we have :

$$\delta A = \int_{t_1}^{t_2} \left(\int_{\partial D} \left(\frac{\partial \mathcal{L}}{\partial \eta_I^\alpha} \Omega^\alpha \right) N_I \, dS + \int_D \left(-\partial_{x^i} \left(\frac{\partial \mathcal{L}}{\partial \eta_i^\alpha} \right) + c_{\beta\alpha}^\gamma \eta_i^\beta \frac{\partial \mathcal{L}}{\partial \eta_i^\gamma} + \frac{\partial \mathcal{L}}{\partial q^\beta} X_{\alpha,g}(q^\beta) \right) \Omega^\alpha dV \right) dt$$

And the modified Hamilton principle (15) becomes:

$$0 = \int_{t_1}^{t_2} \left(\int_{\partial D} \left(\frac{\partial \mathcal{L}}{\partial \eta_I^\alpha} N_I - \tilde{\mathcal{F}}_\alpha \right) \Omega^\alpha dS + \int_D \left(-\partial_{x^i} \left(\frac{\partial \mathcal{L}}{\partial \eta_i^\alpha} \right) + c_{\beta\alpha}^\gamma \eta_i^\beta \frac{\partial \mathcal{L}}{\partial \eta_i^\gamma} + \frac{\partial \mathcal{L}}{\partial q^\beta} X_{\alpha,g}(q^\beta) - \bar{\mathcal{F}}_\alpha \right) \Omega^\alpha dV \right) dt$$

This equation being verified for any variation, i.e. for any set of independent Ω^α 's, we obtain:

The field equations in $\mathbf{g}^*$ *identified to the space of left invariant 1 forms:*

$$\left(\frac{\partial}{\partial x^i} \left(\frac{\partial \mathcal{L}}{\partial \eta_i^\alpha} \right) - c_{\beta\alpha}^\gamma \eta_i^\beta(x) \frac{\partial \mathcal{L}}{\partial \eta_i^\gamma} - \frac{\partial \mathcal{L}}{\partial q^\beta} X_{\alpha,g(x)}(q^\beta) - \bar{\mathcal{F}}_\alpha(x, g(x)) \right) \omega_{g(x)}^\alpha = 0 \quad , \quad \forall x \in \mathbb{R}^+ \times \overset{o}{D} \qquad (21)$$

Boundary equations in the same space:

$$\left(\frac{\partial \mathcal{L}}{\partial \eta_I^\alpha} N_I(x) - \tilde{\mathcal{F}}_\alpha(x, g(x)) \right) \omega_{g(x)}^\alpha = 0 \quad , \quad \forall x \in \mathbb{R}^+ \times \partial D \qquad (22)$$

Let us notice that for evaluating the term (20), there is no need to introduce a chart of parameters on G. In fact, using the basis of infinitesimal material transformations $(e_\alpha)_{\alpha=1,..n}$ of $(T_e G, [\,,\,])$, we have:

$$\frac{\partial \mathcal{L}}{\partial q^\beta} X_{\alpha,g}(q^\beta) = \frac{d}{d\varepsilon} \Big|_{\varepsilon=0} (\mathcal{L}(\eta(x), L_{g(x)}(\exp(\varepsilon e_\alpha)))) \quad , \quad \alpha = 1,..n. \qquad (23)$$

Where: $\exp : (T_e G, [\,,\,]) \to G$ is the natural map of the Lie algebra into the group [23] and where the Lagrangian density is now a function of the transformations, i.e.: $\mathcal{L} = \mathcal{L}(\eta, g)$. Finally the terms (23) take charge of the symmetry defect of the Lagrangian in the material setting (cf. remark 4 below). Now if we seek for the equations of Poincaré-Chetayev in the spatial setting, the field of 1 form with value in the Lie algebra (5) is replaced by:

$$\mu(x) = (\partial_{x^i} g)(x) dx^i = \mu_i^\alpha(x) Z_{\alpha,g(x)} \otimes dx^i,$$

Applying the same computational process to the spatial Lagrangian density: $\mathcal{L}(\mu, q) dV$ with (12) instead of (11) gives:

$$\left(\frac{\partial}{\partial x^i} \left(\frac{\partial \mathcal{L}}{\partial \mu_i^\alpha} \right) + c_{\beta\alpha}^\lambda \mu_i^\beta(x) \frac{\partial \mathcal{L}}{\partial \mu_i^\lambda} - \frac{\partial \mathcal{L}}{\partial q^\beta} Z_{\alpha,g(x)}(q^\beta) - \bar{\mathcal{E}}_\alpha(x, g(x)) \right) \lambda_{g(x)}^\alpha = 0 \quad , \quad \forall x \in \mathbb{R}^+ \times \overset{o}{D} \qquad (24)$$

$$\left(\frac{\partial \mathcal{L}}{\partial \mu_I^\alpha} N_I(x) - \tilde{\mathcal{E}}_\alpha(x, g(x)) \right) \lambda_{g(x)}^\alpha = 0 \quad , \quad \forall x \in \mathbb{R}^+ \times \partial D \qquad (25)$$

Where $(\lambda^\alpha : g \mapsto \lambda_g^\alpha, \alpha = 1,..n; g \in G)$ is the basis of right invariant 1 forms dual to $(Z_\alpha, \alpha = 1,..n)$, and $\bar{\mathcal{E}}_\alpha$ and $\tilde{\mathcal{E}}_\alpha$ are the components of the form fields of external internal and boundary efforts in such a basis, generally dependent of x and $g(x)$. Finally, the following terms take charge of the symmetry defect of the Lagrangian in the spatial setting:

$$\frac{\partial \mathcal{L}}{\partial q^\beta} Z_{\alpha, g(x)}(q^\beta) = \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} \mathcal{L}(\mu(x), R_{g(x)}(\exp(\varepsilon e_\alpha))) , \quad \alpha = 1,..n \quad . \tag{26}$$

where $\exp(\varepsilon e_\alpha)$ is now applied on the left of $g(x)$ so that $(e_\alpha)_{\alpha=1,..n}$ is a basis of infinitesimal spatial transformations (transformations applied on spatial points). Finally, we recognize in (21) and (24) the components of the co-adjoint map $ad_{(.)}^* : g^* \to g^*$ in the dual of left and right invariant basis respectively [24]. Alternatively, applying to each sight of (21)-(22) and (24)-(25) the co-tangent maps $L_{g(x)}^*$ and $R_{g(x)}^*$ respectively, we obtain the dynamics in the basis $(f^\alpha = L_{g(x)}^*(\omega_{g(x)}^\alpha)$ $= R_{g(x)}^*(\lambda_{g(x)}^\alpha), \alpha = 1,..n)$ of g^* where g is now defined as $(T_e G, [,])$:

$$\sum_{i=0}^p \left(\frac{\partial}{\partial x^i} \left(\frac{\partial \mathcal{L}}{\partial \eta_i} \right) - ad_{\eta_i}^* \left[\frac{\partial \mathcal{L}}{\partial \eta_i} \right] \right) - X_{g(x)}(\mathcal{L}) = \bar{\mathcal{F}} \quad , \quad \forall x \in \mathbb{R}^+ \times \overset{o}{D} \tag{27}$$

$$\sum_{I=1}^p \left(\frac{\partial \mathcal{L}}{\partial \eta_I} N_I \right) = \tilde{\mathcal{F}} \quad , \quad \forall x \in \mathbb{R}^+ \times \partial D \tag{28}$$

Where we introduced the notations: $X_{g(x)}(\mathcal{L}) = \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} \mathcal{L}(\eta(x), L_{g(x)}(\exp(\varepsilon e_\alpha))) f^\alpha$,

$\frac{\partial \mathcal{L}}{\partial \eta_i} = \frac{\partial \mathcal{L}}{\partial \eta_i^\alpha} f^\alpha$, $\frac{\partial \mathcal{L}}{\partial \eta_I} = \frac{\partial \mathcal{L}}{\partial \eta_I^\alpha} f^\alpha$, $\bar{\mathcal{F}} = \bar{\mathcal{F}}_\alpha f^\alpha$, $\tilde{\mathcal{F}} = \tilde{\mathcal{F}}_\alpha f^\alpha$.

In the same way we have in the spatial setting:

$$\sum_{i=0}^p \left(\frac{\partial}{\partial x^i} \left(\frac{\partial \mathcal{L}}{\partial \mu_i} \right) + ad_{\mu_i}^* \left[\frac{\partial \mathcal{L}}{\partial \mu_i} \right] \right) - Z_{g(x)}(\mathcal{L}) = \bar{\mathcal{E}} \quad , \quad \forall x \in \mathbb{R}^+ \times \overset{o}{D} \tag{29}$$

$$\sum_{I=1}^p \left(\frac{\partial \mathcal{L}}{\partial \mu_I} N_I(x) \right) = \tilde{\mathcal{E}} \quad , \quad \forall x \in \mathbb{R}^+ \times \partial D \tag{30}$$

Where: $Z_{g(x)}(\mathcal{L}) = \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} \mathcal{L}(\mu(x), R_{g(x)}(\exp(\varepsilon e_\alpha))) f^\alpha$, $\frac{\partial \mathcal{L}}{\partial \mu_i} = \frac{\partial \mathcal{L}}{\partial \mu_i^\alpha} f^\alpha$, $\frac{\partial \mathcal{L}}{\partial \mu_I} = \frac{\partial \mathcal{L}}{\partial \mu_I^\alpha} f^\alpha$, $\bar{\mathcal{E}} = \bar{\mathcal{E}}_\alpha f^\alpha$, $\tilde{\mathcal{E}} = \tilde{\mathcal{E}}_\alpha f^\alpha$. Finally let us remark that in (27)-(28) the base $(f^\alpha, \alpha = 1,..n)$ is the dual basis to infinitesimal material transformations while in (29)-(30) it is the dual basis to infinitesimal spatial transformations.

4. Remarks

1°) It is worth noting that even if the system is continuous and so, in a certain manner infinite-dimensional, the group used in the above construction is finite but parameterized by the material manifold D (we say that the group is gauged over D, [24]). This is radically different from fluid

mechanics, where the configuration space of an ideal fluid is a group actually infinite dimensional [26]. In fact, the configuration space of our Cosserat medium is the set: $\mathcal{C} := \{g : D \mapsto G\}$.

2°) Let us remark by comparison of (21)-(22) and (27)-(28) (respect. (24)-(25) and (29)-(30)) that the Poincaré-Chetayev equations are in components in the left (respect. right) invariant dual basis the same as those in the dual basis to material (respect. spatial) infinitesimal transformations. Adopting the right and left invariant fields as definition of the Lie Algebra corresponds in fact to the well-known mobile frame method proposed by Cartan [27] to study some problems of integrability. This point of view has also its counterpart in the action space of the group, where we know that the material formulation is in components the same as the spatial one in the mobile frame of the body.

3°) When the parameter space P reduces to the time axis $\mathbb{R}^+$, the partial differential equations (27)-(28) and (29)-(30) degenerate into the classical Poincaré-Chetayev ordinary differential equations [1]:

Finite dimensional Poincaré-Chetayev equations in the material setting:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \eta_o}\right) - ad^*_{\eta_o}\left[\frac{\partial L}{\partial \eta_o}\right] - X_{g(t)}(L) = F \quad , \quad \forall t \in \mathbb{R}^+ \tag{31}$$

Where $X_{g(t)}(L) = \left.\dfrac{d}{d\varepsilon}\right|_{\varepsilon=0} L(\eta_o, L_{g(t)}(\exp(\varepsilon e_\alpha)))f^\alpha$, $L = L(\eta_o, g)$ is the Lagrangian of the system in

the material setting, $\dfrac{\partial L}{\partial \eta_o} = \dfrac{\partial L}{\partial \eta_o}f^\alpha$, and $F = F_\alpha(t, g)f^\alpha$ is the 1 form of external material

efforts applied on the system.

Finite dimensional Poincaré-Chetayev equations in the spatial setting:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \mu_o}\right) + ad^*_{\mu_o}\left[\frac{\partial L}{\partial \mu_o}\right] - Z_{g(t)}(L) = E \quad , \quad \forall t \in \mathbb{R}^+ \tag{32}$$

Where $X_{g(t)}(L) = \left.\dfrac{d}{d\varepsilon}\right|_{\varepsilon=0} L(\mu_o, R_{g(t)}(\exp(\varepsilon e_\alpha)))f^\alpha$, $L = L(\mu_o, g)$ is the Lagrangian of the system

in the spatial setting, $\dfrac{\partial L}{\partial \mu_o} = \dfrac{\partial L}{\partial \mu_o}f^\alpha$, and $E = E_\alpha(t, g)f^\alpha$ is the 1 form of external spatial efforts

applied on the system.

4°) These equations have been extended (and connected to several sets of equations of analytical dynamics) by Rumyantsev [20] in the case where the Lie algebra of invariant fields is replaced by any closed system of infinitesimal linear operators $(X_\alpha, \alpha = 1,..n)$. In this case equations of the Poincaré-Chetayev type still occur, where the structure constants of the Lie algebra are replaced by variable coefficients $c^\lambda_{\beta\alpha}$.

5°) These equations are specially interesting when the Lagrangian and the external forces densities do not depend on the configuration $g(x)$. This case has been widely studied by Arnold [23], and Marsden and his co-workers [22]. It is related to the Lagrangian reduction theory. In this context, if the Lagrangian (and in our case, the external forces) of the system is invariant by the left transformations (for the rigid body) or the right ones (for the incompressible fluid), then once expressed in its Lie algebra, it is configuration independent. The resulting dynamic equations are

equations (27)-(28) or (29)-(30) where: first, the symmetry defect terms (23) (respect. (26)) do not appear any more, and second: the components of forces (13) and (14) (and respectively, their spatial counterparts) are configuration independent. In the modern terminology the dynamics are reduced in the Lie algebra g of the system group symmetry. Therefore they are first order p.d.e. in terms of the velocities η_o or μ_o only (Eulerian formulation). Consequently, they can be integrated with respect to the time in a first place and in a second step, thanks to a reconstruction equation "$\partial_t g(x) = L_{g(x)*}(\eta_o(x))$ or $\partial_t g(x) = R_{g(x)*}(\mu_o(x))$", we recover the motion of the medium.

6°) When G is commutative, we have: $X_\alpha = Z_\alpha$, and: $[X_\alpha, X_\beta] = [Z_\alpha, Z_\beta] = 0$, $\forall \alpha, \beta = 1,..n$. Consequently, the field of local basis $(X_\alpha, \alpha = 1,..n)$ derives from a set $(q^\alpha, \alpha = 1,..n)$ of $C^\infty(G)$ coordinate functions verifying: $X_\alpha = \partial_{q^\alpha}$, $\omega^\alpha = dq^\alpha$, $\alpha = 1,..n$. As a result, we have: $\eta^\alpha = \dot{q}^\alpha$, $\alpha = 1,..n$, and the velocities are integrable (holonomic). In this latter case, the Poincaré-Chetayev equations (31) and (32) degenerate into the unique set of Lagrange equations:

$$\left(\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^\alpha} \right) - \frac{\partial L}{\partial q^\alpha} - Q_\alpha \right) dq^\alpha = 0$$

5. Application to flexible multi-body systems

In order to illustrate the application of Poincaré-Chetayev equations to flexible multi-body systems, we consider in the following the specific case of a flexible manipulator and show how these equations allow to recover: 1°) the geometrically exact equations in the Galilean frame, and 2°) the generalized Newton-Euler equations in the floating frame, of a flexible manipulator. The extension to other multi-body topologies is straightforward.

5.1 Partial differential equations of a flexible manipulator in the Galilean frame

Basic kinematics of a body

In this approach each body of the system is modeled by a nonlinear beam theory due to Reissner [15]. Subsequently we consider the case of a straight beam of initial length L, with a constant cross section. Σ_o is the reference configuration of the beam. Its points are material particles labeled by their position $X = X^I E_I$ in the material frame (O, E_1, E_2, E_3) attached to Σ_o, and such that E_1 is the material axis of the beam. The spatial current position of the X particle is written: $x(X,t) = x^i(t, X^I)e_i$. Where (O, e_1, e_2, e_3) is a spatial frame. The material and spatial frames are practically taken merged. In the Reissner theory the beam cross sections are supposed to be rigid. Hence, they realized the micro-solids of a one dimensional Cosserat medium. Thus, referring to the general construction of section 3, the space of parameters is here: $P = \mathbb{R}^+ \times [0, L]$. Moreover, the current three-dimensional position field of the beam results of the action of the time-parameterized transformation $\varphi_t : \Sigma_o \to \mathbb{R}^3$ defined by:

$\forall X = (X^1, X^2, X^3) \in \Sigma_o:$
$$x(t, X) = \varphi_t(X) = X^1 e_1 + d(t, X^1) + R(t, X^1).(X^2 E_2 + X^3 E_3), \tag{33}$$

where $d(t, X^1)$ is the current translation applied to the gravity center of the X^1 cross section, and $R(t, X^1)$ is the rotation applied to that same cross-section. Moreover the transformation (33) can be rewritten under the homogeneous formalism as:

$$\begin{pmatrix} x(t, X) \\ 1 \end{pmatrix} = \begin{pmatrix} R(t, X^1) & d(t, X^1) + X^1.e_1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} r \\ 1 \end{pmatrix} = g(t, X^1) \cdot \begin{pmatrix} r \\ 1 \end{pmatrix} \tag{34}$$

Where the 4×4 homogeneous matrices g realize $SE(3)$: the group of Euclidean displacements in $\mathbb{R}^3$. Thus the configuration space of the beam is realized by $\mathcal{C} := \{g : [0, L] \mapsto SE(3)\}$ (see remark 1). Moreover the field g acts on the subset of Σ_o: $\mathcal{V} := \{r = X^2 E_2 + X^3 E_3 \ / \ X^I E_I \in \Sigma_o\}$ which plays the role of the "generic" micro-solid. The Lie algebra of $SE(3)$, denoted by $se(3)$, is here identified to the space of twists $\mathbb{R}^6$ endowed with the product "$*$" such that:

$$\forall \eta, \eta' \in \mathbb{R}^6 : \qquad \eta * \eta' = \begin{pmatrix} \Omega \\ V \end{pmatrix} * \begin{pmatrix} \Omega' \\ V' \end{pmatrix} = \begin{pmatrix} \Omega \times \Omega' \\ \Omega \times V' - \Omega' \times V \end{pmatrix} \tag{35}$$

In accordance to the material setting, $se(3)$ is here identified to the infinitesimal material rigid displacements provided with the basis:

$$\left(\begin{pmatrix} E_I \\ 0 \end{pmatrix}_{I=1,2,3}, \begin{pmatrix} 0 \\ E_I \end{pmatrix}_{I=1,2,3} \right) = (e_\alpha)_{\alpha=1,..6} \quad . \tag{36}$$

And the field of 1-forms with values in the Lie algebra (4) is:

$$\eta : \mathbb{R}^+ \times [0, L] \mapsto se(3) \otimes T^*(\mathbb{R}^+ \times [0, L])$$

$$\eta(t, X^1) = \eta_o \otimes dt + \eta_1 \otimes dX^1 = \begin{pmatrix} \Omega(t, X^1) \\ V(t, X^1) \end{pmatrix} \otimes dt + \begin{pmatrix} K(t, X^1) \\ \Gamma(t, X^1) \end{pmatrix} \otimes dX^1 \tag{37}$$

Where η_o and η_1 are the twists associated by the isomorphism between $se(3)$ and $\mathbb{R}^6$, to $g^{-1}.\partial_t g$ and $g^{-1}.\partial_{X^1} g$ respectively. Intuitively, $\eta_o(t, X^1)$ is the material infinitesimal transformation allowing passing from the mobile frame of the X^1 cross-section at t to its mobile frame at $t + dt$. And $\eta_1(t, X^1)$ is the infinitesimal transformation allowing at a fixed time t to pass from the mobile frame of the X^1 cross-section to that of the $X^1 + dX^1$ cross section. The dual space of the Lie algebra, $se(3)^*$, is identified to the space of wrenches isomorphic to $\mathbb{R}^6$, the duality product of twists and wrenches reducing to the duality product in $\mathbb{R}^6$. Before to return to the Reissner theory, let us define the co-adjoint action of any vector $\eta \in se(3)$ on any vector λ of its dual [22]:

$$ad_\eta^*(\lambda) = ad_{\begin{pmatrix} \Omega \\ V \end{pmatrix}}^* \begin{pmatrix} \Lambda \\ W \end{pmatrix} = \begin{pmatrix} \Lambda \times \Omega + W \times V \\ W \times \Omega \end{pmatrix}. \tag{38}$$

Strain measures of a body

Now, let us define the strain measures adopted by the theory of Reissner. Following [2], the two strain fields of the beam are:

1°) The field of material strain vector that we note ε and which is defined by:

$$(t,X^1) \in \mathbb{R}^+ \times [0,L] \mapsto \varepsilon(t,X^1) = R^T . \partial_{X^1} \varphi_t(X^1,0,0) - E_1 = \Gamma(t,X^1) - E_1 \quad , \tag{39}$$

where let us remind that the component of this field along E_1 is a measure of the stretching of the beam whereas the two others are relative to the transverse shearing.

2) The field of material curvature:

$$(t,X^1) \in \mathbb{R}^+ \times [0,L] \mapsto R^T . \partial_{X^1} R = \hat{K}(t,X^1), \tag{40}$$

This field of material tensor has already been introduced in (37) through the field of pseudo-vector K associated to $\hat{K}$ by the natural isomorphism of $so(3) \mapsto \mathbb{R}^3$ (this notation is systematically adopted in the following).

Lagrangian of a body

Now, let us write the velocity field as:
$$\partial_t \varphi_t(X) = \partial_t d + \partial_t R . r$$

As for the kinetic energy, we write it (the reference line of the beam passing through the gravity centers of the cross sections):

$$2T = \int_{\Sigma_o} (\partial_t \varphi_t)^2 . dm = \rho A \int_0^L (\partial_t d)^T . (\partial_t d) + ((\partial_t R).r)^T . ((\partial_t R).r) \ dX^1 ,$$

where A is the cross section area. Then, making appear the vector η_o, we can rewrite the kinetic energy as:

$$T = \frac{1}{2} \int_0^L (\Omega^T, V^T) \begin{pmatrix} \rho J & 0 \\ 0 & \rho A 1_3 \end{pmatrix} . \begin{pmatrix} \Omega \\ V \end{pmatrix} dX^1 = \int_0^L \mathcal{T}(\eta_o) dX^1 , \tag{41}$$

where $\mathcal{T}$ is the kinetic energy density and ρJ is the material inertia tensor of the cross section:

$$\rho J = \rho \iint_A \hat{r}^T . \hat{r} \ dX^2 dX^3 = \rho I_p \ E_1 \otimes E_1 + \rho I_a \ E_2 \otimes E_2 + \rho I_a \ E_3 \otimes E_3$$

with I_a and I_p the polar and axial inertia momentum of the generic cross section. Considering the case of an elastic material undergoing small strains, the strain energy can be approximated by a quadratic potential of the strain measures (39) and (40):

$$U = \frac{1}{2} \int_0^L (K^T, (\Gamma - E_1)^T) \begin{pmatrix} H_r & 0 \\ 0 & H_d \end{pmatrix} . \begin{pmatrix} K \\ \Gamma - E_1 \end{pmatrix} dX^1 = \int_0^L \mathcal{U}(\eta_1) \ dX^1 \tag{42}$$

where $\mathcal{U}$ is the strain energy density and H_d and H_r are the reduced Hooke's tensors of the beam in the material frame:

$$H_d = EA \ E_1 \otimes E_1 + GA \ E_2 \otimes E_2 + GA \ E_3 \otimes E_3, \quad H_r = GI_p \ E_1 \otimes E_1 + EI_a \ E_2 \otimes E_2 + EI_a \ E_3 \otimes E_3 \ ,$$

where E and G are the Young and twist moduli. Finally, the Lagrangian density takes the form:

$$\mathcal{L}(\eta_o, \eta_1) = \mathcal{T}(\eta_o) - \mathcal{U}(\eta_1), \tag{43}$$

and does not depend anymore on the configuration of the beam, i.e. is left invariant. This is not surprisingly since the left invariance of the elastic potential corresponds to the concept of frame indifference (the right invariance being related to the isotropy of elastic properties of the medium). On the other hand, the left invariance of the kinetic energy corresponds to the isotropy of space (the right invariance corresponding to the isotropy of inertial properties of the material).

Poincaré-Chetayev equations of a body

Fields equations

The link is submitted to a field of left invariant external (follower) wrenches:

$$(t, X^1) \mapsto \bar{F} = \bar{\mathscr{F}}(t, X^1) dX^1 = \begin{pmatrix} \bar{m}(t, X^1) \\ \bar{n}(t, X^1) \end{pmatrix} dX^1, \tag{44}$$

and to the left invariant (follower) wrenches at its two ends:

$$t \mapsto \tilde{F}_-(t) = \begin{pmatrix} \tilde{M}_-(t) \\ \tilde{N}_-(t) \end{pmatrix}, \quad \text{at } X^1 = 0 \quad \text{and: } t \mapsto \tilde{F}_+(t) = \begin{pmatrix} \tilde{M}_+(t) \\ \tilde{N}_+(t) \end{pmatrix}, \quad \text{at } X^1 = L \tag{45}$$

Considering equations (27)-(28) with: $x^o = t$ and $x^1 = X^1$, we deduce the Poincaré-Chetayev equations of a free mono-dimensional Cosserat medium with a Lagrangian density of the form (43):

$$\frac{\partial}{\partial t} \left(\frac{\partial \mathscr{T}}{\partial \eta_o} \right) - ad^*_{\eta_o} \left(\frac{\partial \mathscr{T}}{\partial \eta_o} \right) - \frac{\partial}{\partial X^1} \left(\frac{\partial \mathscr{U}}{\partial \eta_1} \right) + ad^*_{\eta_1} \left(\frac{\partial \mathscr{U}}{\partial \eta_1} \right) = \bar{\mathscr{F}} \tag{46}$$

Applying the co-adjoint maps $ad^*_{\eta_o}$ and $ad^*_{\eta_1}$ given by (38) to $\mathscr{T}$ and $\mathscr{U}$ respectively, gives:

$$\begin{pmatrix} \rho J.\partial_t \Omega + \Omega \times (\rho J.\Omega) \\ \rho A (\partial_t V + \Omega \times V) \end{pmatrix} = \begin{pmatrix} \partial_{X^1} M + K \times M + (R^T \partial_{X^1} \phi) \times N + \bar{m} \\ \partial_{X^1} N + K \times N + \bar{n} \end{pmatrix} \quad \forall (t, X^1) \in \mathbb{R}^+ \times]0, L[, \tag{47}$$

where we introduced the position field of the reference line: $\varphi_t(X^1, 0, 0) = \phi(t, X^1)$ and the wrench of the internal forces applied on the X^1 cross section:

$$\begin{pmatrix} H_r.K \\ H_d.\varepsilon \end{pmatrix} = \begin{pmatrix} M \\ N \end{pmatrix} = \left(\left(\frac{\partial \mathscr{L}}{\partial K} \right)^T, \left(\frac{\partial \mathscr{L}}{\partial \Gamma} \right)^T \right)^T, \tag{48}$$

where M is the moment of the field of internal forces applied to the cross section and evaluated in its gravity center, and N is its resultant.

Boundary equations

Applying the general equations (28) with $N_1(0) = -1$ and $N_1(L) = 1$, gives:

$$\frac{\partial \mathscr{L}}{\partial \eta_1}(t, 0) = -\tilde{F}_-(t) \quad , \quad \frac{\partial \mathscr{L}}{\partial \eta_1}(t, L) = \tilde{F}_+(t) \quad , \quad t \in \mathbb{R}^+$$

Finally from (47):

$$\begin{pmatrix} M(t,0) \\ N(t,0) \end{pmatrix} = \begin{pmatrix} M_-(t) \\ N_-(t) \end{pmatrix} \ , \ \begin{pmatrix} M(t,L) \\ N(t,L) \end{pmatrix} = -\begin{pmatrix} M_-(t) \\ N_-(t) \end{pmatrix}, \ t \in \mathbb{R}^+ \qquad (49)$$

The equations (47)-(49) prove to be the partial differential equations of Reissner [15]. These equations can be interpreted as tensorial equations in the material space or alternatively as equations in terms of components in the field of mobile frames $(R(t, X^1).E_1, R(t, X^1).E_2, R(t, X^1).E_3)$. In order to integrate them we have to complete them with the Hooke's law, the reconstruction equations: $\partial_t R = R.\hat{\Omega}$, $\partial_t \phi = R.V$, and the definitions of the strain measures. Before to close this subsection, let us remark that the equations given in [15] are expressed in the spatial setting. These equations can be computed by applying the general Poincaré-Chetayev equations (29)-(30), to a spatial Lagrangian now depending of the configuration. In this case it is necessary to compute the symmetry defect term $Z_{g(x)}(L)$ to get the good result.

Partial differential equations of a flexible manipulator

We now consider the specific case of a flexible manipulator out of gravity. The manipulator is constituted of p links denoted from the basis to the tip: $B_o, B_1,...B_p$. The basis B_o is rigid and fixed, while the other links are modeled by Reissner beams. The links are connected by one d.o.f. rotational joints of axis (from the basis to the tip): $a_1, a_2,...a_p$. The joints are actuated with torques: $\tau_1, \tau_2,...\tau_p$ applied around $a_1, a_2,...a_p$ respectively. Moreover they are assumed to be concentrated at points. All the notations adopted in the model of a single link are kept but indexed with the link number. The vectors a_j's are spatial vectors. We define their material counterparts by: $A_j(t) = R_j^T(t, L_j).a_j(t) = R_{j+1}^T(t,0).a_j(t)$, $j = 1,..p$. From these material vectors, we define the operator $A_j^\perp$ projecting any vector V onto the perpendicular space to A_j:

$$j = 1,..p: \quad A_j^\perp : \mathbb{R}^3 \to \mathbb{R}^2, \quad V \mapsto A_j^\perp.V = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.(V - (A_j.V)A_j)$$

With these notations, the dynamics of the manipulator is realized by the set of equations:

Field equations ($j = 1,..p$):
$$X^1 \in]0, L_j[: \quad \begin{pmatrix} \rho_j \, J_j.\partial_t \Omega_j + \Omega_j \times (\rho_j \, J_j.\Omega_j) \\ \rho_j A_j \, (\partial_t V_j + \Omega_j \times V_j) \end{pmatrix} = \begin{pmatrix} \partial_{X^1} M_j + K_j \times M_j + (R_j^T \partial_{X^1} \phi_j) \times N_j \\ \partial_{X^1} N_j + K_j \times N_j \end{pmatrix} \qquad (50)$$

Reconstruction equations of the links configurations ($j = 1,..p$)

For $X^1 \in [0, L_j]$: $\quad \partial_t R_j(t, X^1) = R_j(t, X^1).\hat{\Omega}_j(t, X^1), \quad \partial_t \phi_j(t, X^1) = R_j(t, X^1).V_j(t, X^1) \qquad (51)$

These equations are initialized in accordance with an arbitrary rigid reference configuration of the manipulator.

Reconstruction equations of the joints rotations

For $j = 1,..p-1$: $\quad {}^j R_{j+1}(t) = R_j^T(t, L_j).R_{j+1}(t, 0) \quad$, and: $\quad {}^0 R_1(t) = R_1(t, 0)$. $\qquad (52)$

Boundary conditions of the links ($j = 1,..p-1$)

$$N_j(t,0) = -\tilde{N}_j(t) \quad , \quad N_j(t,L_j) = -\tilde{N}_{j+1}(t), \quad M_j(t,0) = \tilde{M}_j(t) \quad , \quad M_j(t,L_j) = -\tilde{M}_{j+1}(t)$$

And:

$$N_p(t,0) = -\tilde{N}_p \quad , \quad N_p(t,L_p) = 0 \quad , \quad M_p(t,0) = -\tilde{M}_p(t) \quad , \quad M_p(t,L_p) = 0 \qquad (53)$$

Where, the tip of the manipulator is free, and $\tilde{N}_j$ and $\tilde{M}_j$ are the force and torque applied by B_{j-1} onto B_j. Moreover, $\tilde{N}_j$ and $A_j^{\perp}.\tilde{M}_j$ are vectors of Lagrange multipliers that evolve in order to satisfy the following conditions.

Connection conditions ($j = 1,..p-1$)

$$V_j(t,L_j) = {}^jR_{j+1}(t).V_{j+1}(t,0) \quad , \quad A_j^{\perp}(t).\Omega_j(t,L_j) = A_j^{\perp}(t).({}^jR_{j+1}(t).\Omega_{j+1}(t,0))$$

And:

$$V_1(t,0) = 0 \quad , \quad A_1^{\perp}.\Omega_1(t,0) = 0 \qquad (54)$$

5.2 Equations of a flexible manipulator in the floating frame

Basic kinematics of a link

In the floating frame approach, each link of the manipulator is three dimensional elastic body undergoing small deformations superimposed to finite overall motions. We consider a generic free link of reference configuration $\Sigma_o \subset \mathbb{R}^3$. We attach to Σ_o a material frame (O, E_1, E_2, E_3). The current configuration of the body is denoted $\Sigma(t)$ and is immersed in the geometric space $\mathbb{R}^3$ endowed with a spatial frame (O, e_1, e_2, e_3). The material and spatial frames are taken merged. In the floating frame approach, the current transformation φ_t mapping Σ_o onto $\Sigma(t)$ can be written as the composition of two transformations. The first one is a pure deformation mapping Σ_o onto $\Sigma_o(t)$. It is denoted by φ_t^e. The second one that we will write φ_t^r is a rigid displacement transforming $\Sigma_o(t)$ into $\Sigma(t)$. We thus have the sequence of transformations:

$$\varphi_t = \varphi_t^r \circ \varphi_t^e : \Sigma_o \overset{\varphi_t^e}{\mapsto} \Sigma_o(t) \overset{\varphi_t^r}{\mapsto} \Sigma(t) \quad ,$$

transforming a material point *X* into a spatial one *x* following the law:

$$x(t,X) = \varphi_t(X) = \varphi_t^r(\varphi_t^e(X)) \qquad (55)$$

As previously, the rigid transformation is written as:

$$\varphi_t^r(X') = d_o(t) + R(t).X' \qquad (56)$$

Where X' is a point of $\Sigma_o(t)$ and d_o is the displacement of the reference point of the body. As for the elastic transformation, we can write it:

$$\varphi_t^e(X) = X' = X + d(t,X) , \qquad (57)$$

where *d* is the field of displacement (of material nature) mapping the position of the particle in the reference configuration on its image by the pure deformation. As previously, the set of all rigid

transformations realizes the Lie group $SE(3)$ here acting on the current deformed configuration $\Sigma_o(t)$. Whereas φ_t^e is a point of $Diff(\Sigma_o)$: the space of diffeomorphisms of $\mathbb{R}^3$ into itself, restricted to Σ_o. Moreover, the floating frame coincides with the mobile frame, image of the material frame by the rigid component of the transformation. With the composition of maps (55), the configuration space of the body is the group: $G = SE(3) \times Diff(\Sigma_o)$. For what concerns $Diff(\Sigma_o)$, we replace it by a finite group calling upon the modal reduction:

$$d(t,X) = \sum_{\alpha=1}^{m} \Phi_\alpha(X)\, q^\alpha(t) = \Phi_\alpha(X) q^\alpha(t), \qquad \forall X \in \Sigma_o \ , \tag{58}$$

where the Φ_α's are the natural modes of the body under certain boundary conditions (the modal indexes will be Greek and the indexes of $\mathbb{R}^3$ will be Roman). They are material vectors, i.e.: $\Phi_\alpha = \Phi_\alpha^K.E_K$. Let us note that this modal decomposition supposes first that the body endures small deformations. Under these conditions the group of diffeomorphisms $Diff(\Sigma_o)$ is parameterized by the modal coordinates: $q^\alpha, (\alpha = 1,..m)$ and is geometrically replaced by the linear space $\mathbb{R}^m$, i.e., by a commutative Lie group. Thus, the Lie group: $G = SE(3) \times \mathbb{R}^m$ realizes the configuration space of the elastic body. And any two transformations of that group g and g' are composed as:

$$\forall g, g' \in G, \quad g \circ g' = \begin{pmatrix} \begin{pmatrix} R & d \\ 0 & 1 \end{pmatrix} \\ q \end{pmatrix} \circ \begin{pmatrix} \begin{pmatrix} R' & d' \\ 0 & 1 \end{pmatrix} \\ q' \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} R.R' & R.d'+d \\ 0 & 1 \end{pmatrix} \\ q+q' \end{pmatrix}, \tag{59}$$

Where: $q = (q_1, q_2, ..q_m)^T$ denotes the vector of the modal coordinates. The Lie algebra g of G is realized by $se(3) \times \mathbb{R}^m$, and thanks to the natural isomorphism pairing $se(3)$ and $\mathbb{R}^6$, by the space $\mathbb{R}^{6+m}$ provided with the product "$*$" such that:

$$\forall \eta, \eta' \in \mathbb{R}^{6+m}, \quad \eta * \eta' = \begin{pmatrix} \Omega \\ V \\ \dot{q} \end{pmatrix} * \begin{pmatrix} \Omega' \\ V' \\ \dot{q}' \end{pmatrix} = \begin{pmatrix} \Omega \times \Omega' \\ \Omega \times V' - \Omega' \times V \\ (\dot{q}+\dot{q}')-(\dot{q}'+\dot{q}) \end{pmatrix} = \begin{pmatrix} \Omega \times \Omega' \\ \Omega \times V' - \Omega' \times V \\ 0 \end{pmatrix} \tag{60}$$

where $\dot{q} = (\dot{q}^1, ...\dot{q}^m)^T$ takes the sense of a matrix of modal velocities (we denote the time differentiation by a point). Moreover, in accordance to the material setting, $\mathbb{R}^{6+m}$ is identified to the space of infinitesimal material transformations of basis:

$$(e_\alpha)_{1,..6+m} = \begin{pmatrix} \begin{pmatrix} E_I \\ 0 \\ 0 \end{pmatrix}_{I=1,2,3} , \begin{pmatrix} 0 \\ E_I \\ 0 \end{pmatrix}_{I=1,2,3} , \begin{pmatrix} 0 \\ 0 \\ \partial_{q^\alpha} \end{pmatrix}_{\alpha=1,..m} \end{pmatrix}.$$

And the 1 form with value in the Lie algebra (4) reduces to:

$$\eta(t) = \eta_o(t) \otimes dt = \begin{pmatrix} \Omega \\ V \\ \dot{q} \end{pmatrix} \otimes dt \ ,$$

where η_o is the vector of $\mathbb{R}^{6+m}$ associated by the isomorphism between $se(3)$ and $\mathbb{R}^6$, to $L_{g^{-1}*}(\dot{g})$, i.e. verifying: $\hat{\Omega} = R^T.\dot{R}$ and $V = R^T.\dot{d}_o$. The dual of the algebra, noted $se(3)^* \times \mathbb{R}^m$, is again $\mathbb{R}^{6+m}$ where the six first components are those of a wrench in the material frame, and the last m ones are those of generalized modal forces. As for the duality product, it reduces to the duality product in $\mathbb{R}^{6+m}$. Finally, the co-adjoint action of any vector $\xi \in g$ on any vector λ of its dual is simply:

$$ad_\xi^*(\lambda) = ad^*_{\binom{\Omega}{\substack{V\\\dot{q}}}}\begin{pmatrix} \Lambda \\ W \\ Q \end{pmatrix} = \begin{pmatrix} \Lambda \times \Omega + W \times V \\ W \times \Omega \\ 0 \end{pmatrix}. \tag{61}$$

Lagrangian of one link

Taking into account (56), (57) and (58) in (55), it comes:

$$\varphi_t(X) = d_o(t) + R(t).(X + \Phi_\alpha(X).q^\alpha(t)).$$

And the velocity field is so:

$$\dot{\varphi}_t(X) = \dot{d}_o + \dot{R}.(X + \Phi_\alpha(X).q^\alpha) + R.\Phi_\alpha(X).\dot{q}^\alpha$$

Inserting this expression in the body kinetic energy and making appear the vector η_o of the Lie algebra, we find:

$$T = \frac{1}{2}\int_{\Sigma_o} \dot{\varphi}_t^T \dot{\varphi}_t dm = \frac{1}{2}mV^T.V + \frac{1}{2}\Omega^T.(J.\Omega) + \frac{1}{2}m_{\alpha\beta}\dot{q}^\alpha\dot{q}^\beta + V^T.(\Omega \times ms + a_\alpha\dot{q}^\alpha) + \Omega^T.\beta_\alpha\dot{q}^\alpha$$

Where from the right to the left, we find the rigid kinetic energy of translation, that of rotation, the kinetic energy of the deformation and the coupling terms, and where we defined the material tensors:

$$m_{\alpha\beta} = \int_{\Sigma_o} \Phi_\alpha^T.\Phi_\beta\, dm \ , \quad a_\alpha = \int_{\Sigma_o} \Phi_\alpha\, dm \ , \quad \alpha_\beta = \int_{\Sigma_o} X \times \Phi_\beta\, dm \ , \quad \lambda_{\alpha\beta} = \int_{\Sigma_o} \Phi_\alpha \times \Phi_\beta\, dm$$

$$m = \int_{\Sigma_o} dm \ , \quad ms = \int_{\Sigma_o} X + \Phi_\alpha q^\alpha dm \ , \quad \beta_v = \alpha_v + \lambda_{\gamma v}.q^\gamma$$

$$J = \int_{\Sigma_o} (X + \Phi_\alpha.q^\alpha)^{\wedge T}.(X + \Phi_\beta.q^\beta)\, dm = J_{rr} + (J_{re,\alpha} + J_{er,\alpha}^T)q^\alpha + J_{ee,\alpha\beta}q^\alpha q^\beta$$

The kinetic energy is then completed with the potential strain energy, modeled by a quadratic form of the modal coordinates:

$$2U = K_{\alpha\beta}\, q^\alpha q^\beta = q^T.K.q \ ,$$

where K is the modal stiffness matrix. Finally the Lagrangian of a free link takes the reduced form in $se(3) \times \mathbb{R}^m$:

$$L(\eta_o, q) = \frac{1}{2}(\Omega^T, V^T, \dot{q}^T).\begin{pmatrix} J & m\hat{s} & \beta \\ m\hat{s}^T & m.1 & a \\ \beta^T & a^T & M \end{pmatrix}.\begin{pmatrix} \Omega \\ V \\ \dot{q} \end{pmatrix} - \frac{1}{2}q^T.K.q \ . \tag{62}$$

Where we introduced the matrix of generalized modal inertia:

$$M = mat_{\alpha,\beta=1,..m}(m_{\alpha\beta}) \ ,$$

the matrices of material vectors:

$$\beta = row_{\alpha=1,..m}(\beta_\alpha) , \qquad a = row_{\alpha=1,..m}(a_\alpha).$$

And $m\hat{s}$ is the skew symmetric tensor such that: $m\hat{s}.v = ms \times v, \ \forall v \in \mathbb{R}^3$.

Poincaré-Chetayev equations of a link

Applying the finite dimensional Poincaré-Chetayev equations (31) to a Lagrangian of the form (62) gives:

$$\begin{pmatrix} \dfrac{d}{dt}\left(\dfrac{\partial T}{\partial \Omega}\right) \\[2mm] \dfrac{d}{dt}\left(\dfrac{\partial T}{\partial V}\right) \\[2mm] \dfrac{d}{dt}\left(\dfrac{\partial T}{\partial \dot{q}}\right) \end{pmatrix} - ad^*_{\begin{pmatrix}\Omega\\V\\\dot{q}\end{pmatrix}} \begin{pmatrix} \dfrac{\partial T}{\partial \Omega} \\[2mm] \dfrac{\partial T}{\partial V} \\[2mm] \dfrac{\partial T}{\partial \dot{q}} \end{pmatrix} - X_g(L) = \begin{pmatrix}0\\0\\0\end{pmatrix},$$

Then, noticing that $m\dot{s} = a_\alpha \dot{q}^\alpha = a.\dot{q}$, and $\dot{\beta}.\dot{q} = \dot{\beta}_\alpha \dot{q}^\alpha = 0$, and then invoking the relation:

$$V \times (\Omega \times ms) - \Omega \times (V \times ms) = (V \times \Omega) \times ms = ms \times (\Omega \times V),$$

It comes after few computations:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \eta_o}\right) - ad^*_{\eta_o}\left(\frac{\partial T}{\partial \eta_o}\right) = \begin{pmatrix} J & m\hat{s} & \beta \\ m\hat{s}^T & m.1 & a \\ \beta^T & a^T & M \end{pmatrix} \begin{pmatrix}\dot{\Omega}\\\dot{V}\\\ddot{q}\end{pmatrix} + \begin{pmatrix} \dot{J}.\Omega + \Omega \times (J.\Omega + \beta.\dot{q}) + ms \times (\Omega \times V) \\ \Omega \times (mV + 2a.\dot{q} + \Omega \times ms) \\ \dot{\beta}^T.\Omega \end{pmatrix}.$$

Now, since the Lagrangian (62) depends on the configuration only through the deformations, the symmetry defect term of (31) takes the form:

$$X_g(L) = \left(0,0,\left(\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} L(\eta, q + \varepsilon \ \partial_q)\right)^T\right)^T = (0,0,(\partial_q L)^T) = (0,0,row_{\alpha=1,..m}(\partial_{q_\alpha} L))^T ,$$

with:

$$\frac{\partial L}{\partial q^\alpha} = \frac{1}{2}\Omega^T.(J_{re,\alpha} + J_{er,\alpha}).\Omega + \frac{1}{2}q^\beta.\Omega^T.(J_{ee,\alpha\beta} + J_{ee,\beta\alpha}).\Omega + \dot{q}^\beta \lambda^T_{\alpha\beta}.\Omega + V^T.(\Omega \times a_\alpha) - K_{\alpha\beta}.q^\beta$$

Moreover, remarking that $\lambda_{\gamma\alpha} = -\lambda_{\alpha\gamma}$ and that for any 3×3 matrix A, we have:

$$-\Omega^T.((A^T + A).\Omega) = -(A.\Omega)^T.\Omega - \Omega^T.(A.\Omega) = -2 \ \Omega^T.A.\Omega ,$$

the symmetry defect term finally takes the form:

$$\partial_q L = col_{\alpha=1,..m}(\ \Omega^T.J_{er,\alpha}.\Omega - q^\beta \Omega^T.J_{ee,\alpha\beta}.\Omega + 2\dot{q}^\beta \lambda^T_{\beta\alpha}.\Omega + K_{\alpha\beta}.q^\beta)$$

And the dynamics of the free elastic body in $g = se3 \times \mathbb{R}^m \cong \mathbb{R}^{6+m}$, is:

$$\begin{pmatrix} J & m\hat{s} & \beta \\ m\hat{s}^T & m.1 & a \\ \beta^T & a^T & M \end{pmatrix} . \begin{pmatrix} \dot{\Omega} \\ \gamma \\ \ddot{q} \end{pmatrix} + \begin{pmatrix} \Omega \times (J.\Omega) + 2J_{re,\alpha}.\Omega \dot{q}^\alpha + 2J_{ee,\beta\alpha}.\Omega q^\beta \dot{q}^\alpha \\ 2\Omega \times (a.\dot{q}) + \Omega \times (\Omega \times ms) \\ col_{\alpha=1,..m}(2\dot{q}^\beta \lambda_{\beta\alpha}^T.\Omega - \Omega^T.J_{er,\alpha}.\Omega - q^\beta \Omega^T.J_{ee,\alpha\beta}.\Omega + K_{\alpha\beta}.q^\beta) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} , (63)$$

where we introduced the material acceleration of the reference point: $\gamma = \dot{V} + \Omega \times V$. The equations (63) are today well known of the multiboby systems community working in floating references. They correspond in this context to the "Generalized Newton-Euler model" of an elastic body [5-10]. They can be interpreted as tensorial equations in the material space or as equations in terms of components in the mobile floating frame basis $(R(t).E_1, R(t).E_2, R(t).E_3)$. In order to integrate them we have to complete (63) with the reconstruction equation $\dot{R} = R.\hat{\Omega}$.

Generalized Newton-Euler model of a flexible manipulator

We now consider the same manipulator as that defined in section 4. Each link is now modeled through the floating frame approach. The connection points of the bodies are denoted $O_1, O_2, ..O_p$ from the basis to the tool. The wrenches induced by joints are of the follower type. The floating frames are based at the connection points, and the modal shape functions are cantilevered at the same points. Indexing all the tensors by the index of the bodies, the generalized Newton-Euler model of the flexible manipulator is given by the three sets of equations:

- *Links dynamics*

For $j = 0, .., p$:
$$\begin{pmatrix} \mathbb{J}_{rrj} & J_{rej} \\ J_{erj} & J_{eej} \end{pmatrix} . \begin{Bmatrix} \dot{\mathbb{V}}_j \\ \ddot{q}_j \end{Bmatrix} + \begin{Bmatrix} \mathbb{C}_j \\ c_j \end{Bmatrix} = \begin{Bmatrix} \mathbb{F}_j - {}^j\mathbb{T}_{j+1}.\mathbb{F}_{j+1} \\ -\Phi_j^T.{}^j\mathbb{R}_{j+1}.\mathbb{F}_{j+1} \end{Bmatrix} . \qquad (64)$$

- *Model of velocities*

For $j = 1, .., p$:
$$\mathbb{V}_j = {}^j\mathbb{T}_{j-1}.\mathbb{V}_{j-1} + {}^j\mathbb{R}_{j-1}.\Phi_{j-1}.\dot{q}_{j-1} + \dot{q}_{rj}.\mathbb{A}_j . \qquad (65)$$

- *Model of accelerations*

For $j = 1, .., p$:
$$\dot{\mathbb{V}}_j = {}^j\mathbb{T}_{j-1}.\dot{\mathbb{V}}_{j-1} + {}^j\mathbb{R}_{j-1}.\Phi_{j-1}.\ddot{q}_{j-1} + \mathbb{H}_j . \qquad (66)$$

Where (64) is deduced from (63) by posing $(\dot{\mathbb{V}}_j^T, \ddot{q}_j^T)^T = (\dot{\Omega}_j^T, \gamma_j^T, \ddot{q}_j^T)^T$. $\mathbb{F}_j$ is the wrench applied by the $j^{th.}$ body onto its successor. Φ_j is the $6 \times m$ matrix of displacement and rotational cantilever shape vectors of the $j^{th.}$ link evaluated at O_{j+1}. ${}^j\mathbb{R}_{j-1}$ is the 6×6 matrix of change of floating basis. ${}^j\mathbb{T}_{j-1}$ is the 6×6 matrix carrying the twist of the $(j-1)^{th.}$ body from its floating frame to that of its successor. $\mathbb{A}_j$ is the 6×1 vector of the $j^{th.}$ joint axis. And $\mathbb{H}_j$ is the 6×1 vector of Coriolis and centrifugal accelerations introduced by the $j^{th.}$ joint. Equations (64)-(66) correspond to the "Generalized Newton-Euler model" of flexible manipulators. They have been initially established in [5] and recovered in [6-9]. We invite the reader to consult [12-14] for more details about their use in flexible multi-body dynamics.

6. Concluding remarks

After having extended the equations of Poincaré-Chetayev to the case of a Cosserat medium, we have shown how this result realizes the natural language of the two main sets of equations today

used in flexible manipulators dynamics. As regards as the Galilean approach, the partial differential equations obtained are the foundations of the geometrically exact approach of numerical treatment of flexible multibody systems [2-4]. On the other hand, adopting floating references of the link deformations, the configuration space has been identified to the Cartesian product of $SE(3)$ and of a space of generalized deformation coordinates. From the geometrical point of view, they are a trivialization of the elastic body dynamics on a principal fiber bundle, where the commutative sub-group is the base manifold (here the modal space and more generally 'the shape space'), and the fiber is the non-commutative sub-group (here $SE(3)$) [22]. Let us note at last that these equations have many uses in multi-body dynamics. Between these uses, let us quote that they have permitted to generate o(n) algorithms (where n is the number of links) of the inverse and forward dynamics of a flexible manipulator in relative coordinates [13,14]. A more recent use of these equations in robotics concerns the study of the locomotion of the poly-articulated systems. In this case the elastic manifold of the preceding example is replaced by the joints' manifold of a rigid multi-body system. The control problem that remains then is the following: what are the joint motions that we need to generate (by the control) to allow the system to move in an adequate manner on $SE(3)$. This question is at the basis of an animal locomotion theory [28].

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