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# Exact Computation of Emergy Based on a Reformulation of the Rules of Emergy Algebra

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## Abstract

The emergy algebra is based on four rules which use is sometimes confusing or reserved only to the experts of the domain. The emergy computation does not obey conservation logic (i.e. emergy computation does not obey Kirchoff-like circuit law). In this paper the authors propose to reformulate the emergy rules into four axioms which provide an exact algorithm to compute emergy within a system of interconnected processes at steady state modeled by an oriented graph named emergy graph.

Because emergy algebra follows a logic of memorization the evaluation principles deal with paths in emergy graph. The underlying algebraic structure is the the set of non-negative reals equipped with the maximum (max), the addition (+) and the multiplication ( $\cdot$ ). The maximum is associated with the co-product problem. The addition is linked with the split problem or more generally with the independence of two emergy flows. And the

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5 multiplication is related to the logic of memorization. The axioms describe  
6 how to use the different operators  $\max$ ,  $+$  and  $\cdot$  to combine flows without  
7 any confusion or ambiguity. The method is tested on five benchmark energy  
8 examples.  
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12 *Keywords:* Track summing method, path, memory algebra, energy  
13 algebra, exact energy computation  
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## 16 17 **1. Introduction** 18

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20 According to Odum [1] the energy is defined as the total solar equivalent  
21 energy/exergy of one form that was used up directly or indirectly in the  
22 work of making a product or a service. In Emergy point of view the com-  
23 parison of interconnected processes/components can be based on the same  
24 fundamentals and provide reliable sustainability development dimensionless  
25 numbers. The idea of the energy is based on the maximum power principle  
26 stated by the biologist Lotka [2].  
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29 However, as mentioned in e.g. Hau and Bakshi [3] even if the idea of  
30 energy is attractive only Odum and a small circle of co-workers have devel-  
31 oped the notion of energy and energy analysis since the 1980's. Even if  
32 there are attractive features it is mentioned in Hau and Bakshi [3, Section 1  
33 and subsection 3.2] that energy analysis received many criticisms. Most of  
34 these criticisms could be applied to other popular methods which try to an-  
35alyze in the same framework environmental and industrial/human systems.  
36 As mentioned in Hau and Bakshi [3, Section 5] energy analysis of large and  
37 complex systems is one of the main challenges of energy approach. A system  
38 is large when it possesses a large number of components. A system is com-  
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5 plex when there are splits and co-products within the same system. Roughly  
6 speaking, an emergy system (see the precise definition in Section 2) is rep-  
7 resented by an oriented graph. Each node represents a process/component.  
8 The emergy circulates on the branches of the graph (or diagram associated  
9 with the system) and is assigned at the nodes of the system. Because emergy  
10 can be considered as the memory of all solar used during a process (see e.g.  
11 Bastianoni et al. [4]) the notion of pathway from a source is the fundamen-  
12 tal notion to manipulate for emergy analysis. A pathway from a source of  
13 emergy (e.g.: sun, wind, fuel, ...) on the graph represents the sequel of assig-  
14 nations of the emergy source. According to Odum [1, Chap. 6, p. 90] in a  
15 split branching a pathway of the emergy system divides into several branches  
16 of the same kind e.g. as in hydraulic systems. In a co-product branching, the  
17 flow in each branch is of a different kind e.g. as in combined heat and power  
18 plants (described in e.g. Horlock [5]). The complexity comes from the fact  
19 that the flow circulating on a branch is in fact a combination of splits and  
20 co-products coming upstream this branch. And the emergy upstream flows  
21 cannot be counted more than once.  
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37 The way to combine the emergy upstream flows is described and explained  
38 in e.g. [1, Chap. 6]. It is summarized in e.g. Sciubba and Ulgiati [6, pp.  
39 1965-1966] as follows under the name *emergy algebra*.  
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44 R1 : When only one product is obtained from a process (i.e. a process  
45 with only one output), all source-emergy is assigned to it.  
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48 Concerning processes with more than one output we have.  
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51 R2 : When a flow (of emergy) splits the total emergy splits accordingly,  
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4 based on the exergy/energy flowing through each pathway.  
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7 R3 : When two or more co-products are generated in a process, the total  
8 source-energy is assigned to each of them.  
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12 Finally, a fourth rule describes how energy is assigned within a system of  
13 interconnected processes.  
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17 R4 : Energy cannot be counted twice within a system.  
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20 R4.1 : Energy in feedbacks cannot be double counted.  
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22 R4.2 : Co-products, when reunited, cannot be summed. Only the  
23 energy of the largest co-product flow is accounted for.  
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27 The general method of energy analysis consists in propagating these rules  
28 from energy sources to the outputs of the system of interconnected processes.  
29 Difficulty occurs for large and complex systems. Moreover, the use of these  
30 rules are not easy and seems to be confusing. E.g. concerning the application  
31 of rule R4.2 it is clearly noticed in Lazzaretto [7, p.2201]: "As observed by one  
32 of the reviewers the rule counting the largest energy value [arriving at a node]  
33 is a rather "crude way" of avoiding double counting". This approximation  
34 is made in e.g. Li et al. [8, (2) p. 415] when authors studied the output  
35 energy at node  $G$  of the energy graph (see their figures 8 and 9).  
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44 To bypass these difficulties several numerical methods have been pro-  
45 posed. Most of them are approximation methods based on linear algebra  
46 (It means that they do not use the operator maximum). Some of them are  
47 based on pre-analysis of the system which is not well-suited for an energy  
48 computation. For more details on such approaches see e.g. Li et al. [8,  
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5 subsection 1.3 and references therein]. Few simulation solutions have been  
6 proposed (see Odum and Peterson [9], Maud [10] and references therein). All  
7 these solutions have no mathematical framework and it is difficult to validate  
8 their results. To the best knowledge of the authors only two mathematical  
9 framework have been proposed in the literature. The first is Giannantoni  
10 [11] who proposed another approach based on (non)-linear differential equa-  
11 tions and on a variant of fractional derivatives concept. The second is the  
12 approach of Bastianoni et al. [4] based on (commutative) free monoids.

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19 Contributions of the paper are as follows.

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22 To respect the logic of memorization of the energy algebra a new path-  
23 oriented method is proposed. A path-oriented method is a method which  
24 manipulates paths in a graph. In this paper the proposed method is based  
25 on the *Track summing method* developed by Tennenbaum [12]. The Track  
26 summing method is a path-oriented method which is exact and has been  
27 implemented for energy systems with splits and without co-product. More  
28 precisely authors start from the expression given in Tennenbaum [12, p. viii]  
29 for acyclic source requirements and extend the Track summing method to  
30 interconnected systems with splits and co-products.

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39 It is noticed that the Tennenbaum's Track summing method can be di-  
40 vided into two different parts. The first part is a path-finding problem. The  
41 second part is a computational problem. The path-finding problem can be  
42 solved by method based on a slight modification of methods to enumerate  
43 elementary paths in a graph which have been developed by e.g. Kaufmann  
44 and Malgrange [13], Kaufmann [14], Benzaken [15], Backhouse and Carré  
45 [16], also mentioned in e.g. Gondran and Minoux [17]. And this is clearly  
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Figure 1: List of Emergy Symbols

not the purpose of this paper. It is the subject of a companion paper which is in preparation.

Thus, assuming that all emergy paths ending by a given arc of the emergy graph are known the major contributions are the following ones.

- The paper describes how to compute the exact value of emergy flowing on this arc (see the recursive algorithm subsection 3.1).
- To proceed an axiomatic basis is proposed as a reformulation of the rules R1-R4 to avoid confusing applications of the rules and decide whether or not emergy flows are independent. As an example the abovementioned problem with co-product (i.e the application of rule R4.2) noticed in e.g. Lazzaretto [7, p.2201] is solved (see the illustrative example of subsection 4.4).

## 2. Example and important definitions

The way by which emergy circulates in a multicomponent system is modeled by an oriented graph. The graph has input nodes called sources, intermediate nodes and output (or final) nodes. Each node is represented by an integer (i.e. an element of  $\mathbb{N}$ ).

The drawing conventions for the emergy graph are depicted in Figure 1. A source is represented by the symbol Fig 1.A, an intermediate node on the emergy graph is represented by Fig 1.B, an output node is represented by

Figure 2: Emergy graph  $G_1$

Fig 1.C. Splits are modeled by Fig 1.D and co-products are modeled by Fig 1.E.

Let us consider a system with its associated emergy graph (or diagram)  $G_1$  described by the Figure 2.

According to convention notations of Figure 1 the set of sources is  $\{1, 2\}$ . For numerical application authors assume that the emergy of 1 is 400 seJ and the emergy of node 2 is 100 seJ. The set of intermediate nodes is  $\{3, 4, 5, 6\}$  and the set of output nodes is  $\{7\}$ . Finally, the set of all nodes is  $\{1, 2, 3, 4, 5, 6, 7\}$ .

The set of the arcs is:

$$\mathcal{A}_1 = \{[1; 3], [2; 4], [3; 4], [3; 5], [4; 6], [5; 6], [6; 5], [6; 7]\}.$$

The weight (i.e. the fraction of emergy which is assumed to be given in this paper) of the arcs  $[1; 3]$ ,  $[2; 4]$ ,  $[3; 4]$ ,  $[3; 5]$ ,  $[4; 6]$ ,  $[5; 6]$  is 1. The weight of the arc  $[6; 7]$  is  $4/5$  and the weight of the arc  $[6; 5]$  is  $1/5$ .

There is a split at node 6 and a co-product at node 3.

All this information is encoded using the following 8-tuple:

$$G_1 = (\mathcal{L}_1^s, \mathcal{L}_1^i, \mathcal{L}_1^o, \mathcal{A}_1, \mathbf{R}_{G_1}, \mathbf{\Omega}_{G_1}, \epsilon_{G_1}, \mathcal{E}_1) \quad (1)$$

Where:

- $\mathcal{L}_1^s = \{1, 2\}, \mathcal{L}_1^i = \{3, 4, 5, 6\}, \mathcal{L}_1^o = \{7\}$ .



- The relations between the arcs are stored in the array  $\mathbf{R}_{G_1}$ :

	[1; 3]	[2; 4]	[3; 4]	[3; 5]	[4; 6]	[5; 6]	[6; 5]	[6; 7]
[1; 3]	id	$\perp$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
[2; 4]	$\perp$	id	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
[3; 4]	$\emptyset$	$\emptyset$	id	$\parallel$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
[3; 5]	$\emptyset$	$\emptyset$	$\parallel$	id	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
[4; 6]	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	id	$\emptyset$	$\emptyset$	$\emptyset$
[5; 6]	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	id	$\emptyset$	$\emptyset$
[6; 5]	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	id	$\perp$
[6; 7]	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\perp$	id

The relation **id** denotes the identity relation (i.e. equality of the arcs). The relation  $\emptyset$  means that there is no relation between the arcs. The relation  $\parallel$  means that there is a co-product. In the example there is a co-product at node 3, thus we have:  $[3; 4] \parallel [3; 5]$  (or equivalently  $\mathbf{R}_{G_1}([3; 4], [3; 5]) = \parallel$ ) and  $[3; 5] \parallel [3; 4]$ . To indicate that flows which circulate on arcs are independent we use the symbol relation  $\perp$ . There are two cases of independence. The first case is for a split. In the example there is a split at node 6, thus  $[6; 5] \perp [6; 7]$  and  $[6; 7] \perp [6; 5]$  (or equivalently  $\mathbf{R}_{G_1}([6; 7], [6; 5]) = \perp$ ). The second case is for the sources. In the example, the arcs  $[1; 3]$  and  $[2; 4]$  satisfy this condition, thus  $[1; 3] \perp [2; 4]$  and  $[2; 4] \perp [1; 3]$ .

- The matrix of the weights of the graph,  $\mathbf{\Omega}_{G_1}$ , is as follows:

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$$\Omega_{G_1} =$$

$\curvearrowright$	1	2	3	4	5	6	7
1	0	0	1	0	0	0	0
2	0	0	0	1	0	0	0
3	0	0	0	1	1	0	0
4	0	0	0	0	0	1	0
5	0	0	0	0	0	1	0
6	0	0	0	0	1/5	0	4/5
7	0	0	0	0	0	0	0

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- The vector of assigned energy sources is:

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$$\epsilon_{G_1} = (400, 100, 0, 0, 0, 0, 0).$$

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On this graph we define different notion of paths. A *path*  $\pi$  has the form  $\pi = \underline{1}$  (unit path, i.e. a path with no arc) or e.g.  $\pi = [3; 4][4; 6][6; 5][5; 6]$  which is a path from first node 3 to last node 6 in  $G_1$ . A *path from a source* is a path such that its first node is a source. E.g.  $[1; 3][3; 4][4; 6][6; 5][5; 6][6; 5]$  is a path from a source (1) to node 5. A *simple path* is a path such that all its nodes are different. E.g.  $[4; 6][6; 5]$  is a simple path from node 4 to node 5. A *simple path from a source* is a simple path such that its first node is a source. E.g.  $[2; 4][4; 6][6; 5]$  is a simple path from a source (2) to node 5. Finally, an *energy path* of  $n$  ( $n \geq 1$ ) arcs is a path such that the path with the  $n - 1$  first arcs is a simple path from a source. E.g.  $[2; 4][4; 6][6; 5]$ ,  $[2; 4][4; 6][6; 5][5; 6]$  are energy paths. But the path  $[2; 4][4; 6][6; 5][5; 6][6; 7]$  is not an energy path because the path  $[2; 4][4; 6][6; 5][5; 6]$  is not a simple path from a source.

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- The set of all energy paths of  $G_1$  is assumed to be given in this paper

is as follows:

$$\mathcal{E}_1 = \left\{ \begin{array}{l} [1; 3], [1; 3][3; 4], [1; 3][3; 5], [1; 3][3; 5][5; 6][6; 5], \\ [1; 3][3; 4][4; 6][6; 5], \\ [1; 3][3; 4][4; 6], [1; 3][3; 5][5; 6], [1; 3][3; 4][4; 6][6; 5][5; 6], \\ [1; 3][3; 4][4; 6][6; 7], [1; 3][3; 5][5; 6][6; 7], [2; 4], [2; 4][4; 6][6; 5], \\ [2; 4][4; 6], [2; 4][4; 6][6; 5][5; 6], [2; 4][4; 6][6; 7] \end{array} \right\}.$$

Recall that the computation of energy paths is a further work.

Concatenation.

We define the concatenation of paths by analogy with the concatenation of letters to form words. For example the concatenation of the path  $[1; 3][3; 4][4; 6][6; 5]$  with the path  $[5; 6][6; 5]$  gives the path  $[1; 3][3; 4][4; 6][6; 5][5; 6][6; 5]$ . For pure mathematical reasons we add that the path  $\underline{1}$  concatenated with any other path  $\pi$  gives  $\pi$  (i.e.  $\underline{1}\pi = \pi\underline{1} = \pi$ ). That is why  $\underline{1}$  is called the unit path.

If  $\mathcal{U}$  denotes a set of paths. Then for any path  $\pi$  the set

$$\pi\mathcal{U}$$

denotes the union of the paths obtained by the concatenation of  $\pi$  with the paths of  $\mathcal{U}$ . Following the logic of memorization of the energy algebra the path  $\pi$  can be interpreted as the *past* of the paths of  $\pi\mathcal{U}$ .

For example,

$$[1; 3]\{[3; 4][4; 6][6; 7], [3; 5][5; 6][6; 7]\} = \{[1; 3][3; 4][4; 6][6; 7], [1; 3][3; 5][5; 6][6; 7]\}$$

### 3. Energy evaluation principles

Let us recall that energy algebra obeys a logic of memorization which implies that the definition of energy is based on paths in energy graph.

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5 The general principle is that at a node of the energy graph only energy  
6 flows arriving at this node with the same past (or upstream flow) can be  
7 combined using the maximum, addition and multiplication operators. So,  
8 let us consider the set of nonnegative reals  $\mathbb{R}_+$  equipped with the operations  
9  $\max$  (i.e. maximum),  $+$  (the addition) and  $\cdot$  (the multiplication). The  $\max$   
10 is associated with the co-product  $\parallel$ . The addition is associated with the  
11 independent relation  $\perp$ . And the multiplication is related to the logic of  
12 memorization of the energy.  
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20 Based on previous preliminaries in this section, remarks in the summary  
21 and the introduction of the paper we propose the following definition for  
22 energy.  
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27 **Definition 3.1 (Emergy).** *Let us consider the energy graph*

$$28 \quad G = (\mathcal{L}^s, \mathcal{L}^i, \mathcal{L}^o, \mathcal{A}, \mathbf{R}_G, \mathbf{\Omega}_G, \epsilon_G, \mathcal{E}),$$

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32 where  $\mathcal{E}$  is assumed to be known in this paper. Then, the energy flowing on  
33 arc  $[l; l']$  with  $l, l' \in \mathcal{L}$  is the function denoted  $\mathbf{Em}(\mathcal{E}([l; l']))$ , where  $\mathcal{E}([l; l']) \subseteq$   
34  $\mathcal{E}$  denotes the set of all emergy paths ending by the arc  $[l; l']$ , which satisfies  
35 the following axioms which replace the rules R1-R4 of emergy algebra:  
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41 (a.1) .  $\forall \pi, \forall k \geq 1, \forall a_1, \dots, a_k \in \mathcal{A}$  s.t.  $a_1 \top a_2 \cdots \top a_k$  with  $\top \in \{\mathbf{id}, \perp, \parallel\}$ ,

$$42 \quad \forall \mathcal{U}_1, \dots, \mathcal{U}_k \subseteq \mathcal{E}([l; l']):$$

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46 (a.1.1) . In the case of one output (i.e.  $\top = \mathbf{id}$ ) all emergy having the  
47 same past  $\pi$  is assigned to this output, that is:

$$48 \quad \mathbf{Em}(\cup_{i=1}^k \pi a_i \mathcal{U}_i) = \mathbf{Em}(\pi a_1 (\cup_{i=1}^k \mathcal{U}_i)).$$

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5 (a.1.2) . If the arcs  $a_i$  are independent then the total energy flowing  
6 on arc  $[l; l']$ ,  $\text{Em}(\cup_{i=1}^k \pi a_i \mathcal{U}_i)$  is equal to the sum of the energies  
7 flowing on arc  $[l; l']$  of the system if there was only one arc  $a_i$   
8 after the past  $\pi$ ,  $\text{Em}(\pi a_i \mathcal{U}_i)$ ,  $i = 1, \dots, k$ , when reunited, i.e.:

$$\text{Em}(\cup_{i=1}^k \pi a_i \mathcal{U}_i) = \sum_{i=1}^k \text{Em}(\pi a_i \mathcal{U}_i), \text{ if } \top = \perp.$$

(See the explanation in Appendix A).

16 (a.1.3) . If there is co-product just after  $\pi$  then the total energy flowing on  
17 arc  $[l; l']$ ,  $\text{Em}(\cup_{i=1}^k \pi a_i \mathcal{U}_i)$ , is equal to the maximum of the energies  
18 flowing on arc  $[l; l']$  of the system if there was only one arc  $a_i$  after  
19 the past  $\pi$ ,  $\text{Em}(\pi a_i \mathcal{U}_i)$ ,  $i = 1, \dots, k$ , when reunited, i.e.:

$$\text{Em}(\cup_{i=1}^k \pi a_i \mathcal{U}_i) = \max_{i=1}^k \text{Em}(\pi a_i \mathcal{U}_i), \text{ if } \top = \parallel.$$

(See the explanation in Appendix B).

28 (a.2) . For all path  $\pi$ , for all  $\mathcal{U} \subseteq \mathcal{E}$ ,  $\text{Em}(\pi \mathcal{U}) = \text{Em}(\pi) \cdot \text{Em}(\mathcal{U})$ . It means  
29 that the computation of the energy of energy paths with the same past  
30  $\pi$  can be divided into the computation of the past  $\pi$  and the computation  
31 of the downstream part of the energy paths.

37 (a.3) . For all path  $[l_1; l_2] \dots [l_{k-1}; l_k]$ ,

$$\text{Em}([l_1; l_2] \dots [l_{k-1}; l_k]) = \begin{cases} \prod_{i=1}^{k-1} \Omega_G(l_i, l_{i+1}) & \text{if } l_1 \notin \mathcal{L}^s \\ \epsilon_G(l_1) \cdot \prod_{i=1}^{k-1} \Omega_G(l_i, l_{i+1}) & \text{if } l_1 \in \mathcal{L}^s \end{cases}$$

42 In the case where  $l_1 \in \mathcal{L}^s$   $\text{Em}([l_1; l_2] \dots [l_{k-1}; l_k])$  coincides with the  
43 energy flowing on the path  $[l_1; l_2] \dots [l_{k-1}; l_k]$  which is obtained as the  
44 fraction  $\prod_{i=1}^{k-1} \Omega_G(l_i, l_{i+1})$  of the energy of source  $l_1$ ,  $\epsilon_G(l_1)$ .

48 And for pure mathematical consideration we add:

51 (a.4) . For all path  $\pi$ :  $\text{Em}(\{\pi\}) = \text{Em}(\pi)$ .

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5 We call (a.1)-(a.2) the *tree property*.

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7 The energy rules R1-R4 do not make clearly the difference between the  
8 qualitative analysis of the energy (i.e. the enumeration problem of the  
9 energy paths) and the quantitative analysis of the energy (which is the  
10 focus of the paper). Nevertheless,  
11  
12

- 13  
14 • the rule R1 has been expressed as a particular case of axioms (a.1.1)  
15 and (a.1.2, with  $\forall i = 1, \dots, k: a_i \in [\mathcal{L}^s; l], \mathcal{U}_i = \{[l; l']\}$  for some  
16  $l, l' \in \mathcal{L} \setminus \mathcal{L}^s$ ), and the axioms (a.2)-(a.4). This rule is illustrated in e.g.  
17 [18, Fig. 6.b p. 225]. However, let us remark that this rule seems not  
18 to be always written the same way in the literature (see e.g. [6] –also  
19 used in the Introduction of this paper–, [8], [19], [7], [20]).  
20  
21

22 The basic case of  $n$  sources and one product, usually written under  
23 energy tables, is completely treated in subsection 4.1 as an application  
24 of axioms (a.1.2) and (a.2)-(a.4).  
25  
26

- 27 • The rule R2 concerning splits has been expressed by axioms (a.1.2) and  
28 (a.2)-(a.4).  
29  
30
- 31 • The rule R3 is expressed as a particular case of the axiom (a.1.3) with  
32  $\mathcal{U}_i = \{\underline{1}\}, i = 1, \dots, k$  and the application of (a.2)-(a.4).  
33  
34
- 35 • The rule R4 concerning the double counting problem is expressed by the  
36 application of the axioms (a.1.3) and (a.2)-(a.4) and the computation  
37 of the energy paths  $\mathcal{E}$ .  
38  
39

### 40 41 42 43 44 45 46 47 48 *3.1. Algorithm for energy computation*

49 In this Section we present a recursive algorithm to compute  $\text{Em}(\mathcal{E}([l; l']))$   
50 which is as follows:  
51  
52

Figure 3: Energy graph  $G_0$  with  $n$  sources and one output

- Enter energy graph  $G = (\mathcal{L}^s, \mathcal{L}^i, \mathcal{L}^o, \mathcal{A}, \mathbf{R}_G, \mathbf{\Omega}_G, \epsilon_G, \mathcal{E})$ ,  $l$  and  $l'$
- $\mathcal{X} := \mathcal{E}([l; l'])$ , where  $\mathcal{E}([l; l'])$  is assumed to be known/given.

While  $\mathcal{X} \neq \emptyset$  Do

- 1 Factorize  $\mathcal{X}$  according to (a.1) using the same notations
- 2 Apply (a.1.1) if  $\top = \text{id}$  or (a.1.2) if  $\top = \perp$  or (a.1.3) if  $\top = \parallel$
- 3 Apply (a.2) to each  $\pi a_i \mathcal{U}_i$ ,  $i = 1, \dots, k$  if  $\top \in \{\perp, \parallel\}$  or Apply (a.2) to  $\pi a_1 (\cup_{i=1}^k \mathcal{U}_i)$  if  $\top = \text{id}$
- 4 Evaluate by axiom (a.3) and store  $\text{Em}(\pi a_i)$ ,  $i = 1, \dots, k$
- 5  $\mathcal{X} := \mathcal{X} \setminus (\cup_{i=1}^k \{\pi a_i\})$

EndWhile

- Return  $\text{Em}(\mathcal{E}([l; l']))$

## 4. Numerical examples

### 4.1. $n$ sources, one product

Let us consider the energy graph  $G_0$  with  $n$  sources and one output arc as depicted in Figure 3.

We have:  $\mathcal{L}_0^s = \{1, 2, \dots, n\}$ ,  $\mathcal{L}_0^i = \{n + 1\}$  and  $\mathcal{L}_0^o = \{n + 2\}$ .

$$\mathcal{A}_0 = \{[1; n + 1], [2; n + 1], \dots, [n; n + 1], [n + 1; n + 2]\}$$

The relations between the arcs are stored in the array  $\mathbf{R}_{G_0}$ :

	$[1; n+1]$	$[2; n+1]$	$\dots$	$[n; n+1]$	$[n+1; n+2]$
$[1; n+1]$	id	$\perp$	$\dots$	$\perp$	$\emptyset$
$[2; n+1]$	$\perp$	id	$\perp$	$\dots$	$\emptyset$
$\vdots$	$\vdots$	$\ddots$	$\ddots$	$\ddots$	$\vdots$
$[n; n+1]$	$\perp$	$\perp$	$\dots$	id	$\emptyset$
$[n+1; n+2]$	$\emptyset$	$\emptyset$	$\dots$	$\emptyset$	id

The matrix of the weights of the graph  $G_0$ ,  $\mathbf{\Omega}_{G_0}$ , is as follows:

$\curvearrowright$	1	2	$\dots$	$n$	$n+1$	$n+2$
1	0	0	$\dots$	0	1	0
2	0	0	$\dots$	0	1	0
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\ddots$	$\vdots$	$\vdots$
$n$	0	0	$\dots$	0	1	0
$n+1$	0	0	$\dots$	0	0	1
$n+2$	0	0	$\dots$	0	0	0

Finally, the vector of assigned energy sources is:

$$\boldsymbol{\epsilon}_{G_0} = (\epsilon(1), \epsilon(2), \dots, \epsilon(n), 0, 0).$$

The set of all energy paths denoted  $\mathcal{E}_0$  is:

$$\mathcal{E}_0 = \{[1; n+1], [2; n+1], \dots, [n; n+1], [1; n+1][n+1; n+2], \dots, [n; n+1][n+1; n+2]\}.$$

Let us compute the energy flowing on arc  $\text{Em}(\mathcal{E}_0([n+1; n+2]))$ . The direct application of the rule  $R1$  gives that:

$$\text{Em}(\mathcal{E}_0([n+1; n+2])) = \epsilon(1) + \dots + \epsilon(n).$$



Now, let us compute  $\text{Em}(\mathcal{E}_0([n+1; n+2]))$  using our axiomatic basis.

First, let us assumed that the set of emergy paths ending by the arc  $[n+1; n+2]$  is given. Thus:

$$\mathcal{E}_0([n+1; n+2]) = \{[1; n+1][n+1; n+2], [2; n+1][n+1; n+2], \dots, [n; n+1][n+1; n+2]\}.$$

Rewrite  $\mathcal{E}_0([n+1; n+2])$  as:

$$\mathcal{E}_0([n+1; n+2]) = \cup_{i=1}^n a_i \mathcal{U}_i,$$

with  $a_i = [i, n+1]$ ,  $\mathcal{U}_i = \{[n+1; n+2]\}$ ,  $\forall i = 1, \dots, n$ . And  $a_1 \perp a_2 \perp \dots \perp a_n$ . Thus, applying (a.1.2) with  $\pi = \underline{1}$  it comes:

$$\text{Em}(\mathcal{E}_0([n+1; n+2])) = \sum_{i=1}^n \text{Em}([i; n+1]\{[n+1; n+2]\}).$$

For all  $i = 1, \dots, n$  we apply (a.2), then:

$$\text{Em}([i; n+1]\{[n+1; n+2]\}) = \text{Em}([i; n+1]) \cdot \text{Em}(\{[n+1; n+2]\})$$

By (a.3) because  $i$  is a source:

$$\text{Em}([i; n+1]) = \epsilon(i) \cdot \Omega_{G_0}(i, n+1).$$

Apply (a.4) we have:

$$\text{Em}(\{[n+1; n+2]\}) = \text{Em}([n+1; n+2]),$$

and by (a.3) noticing that  $n+1$  is not a source:

$$\text{Em}([n+1; n+2]) = \Omega_{G_0}(n+1, n+2).$$

Thus,  $\forall i = 1, \dots, n$ , by applying (a.2)-(a.4) we have:

$$\text{Em}([i; n+1]\{[n+1; n+2]\}) = \epsilon(i) \cdot \Omega_{G_0}(i, n+1) \cdot \Omega_{G_0}(n+1, n+2).$$

Finally we have:

$$\begin{aligned}
\text{Em}(\mathcal{E}_0([n+1; n+2])) &= \sum_{i=1}^n \epsilon(i) \cdot \Omega_{G_0}(i, n+1) \cdot \Omega_{G_0}(n+1, n+2) \\
&= \sum_{i=1}^n \epsilon(i) \cdot 1 \cdot 1 \\
&= \sum_{i=1}^n \epsilon(i).
\end{aligned}$$

And the result obtained by application of the rule  $R1$  is retrieved.

#### 4.2. Tennenbaum-like example

Let us consider the energy graph  $G_2$  corresponding to the Figure 4.

Figure 4: Energy graph  $G_2$  Tennenbaum-like net

We have:

$$\mathcal{L}_2^s = \{1, 2\}, \mathcal{L}_2^i = \{3, 4\}, \mathcal{L}_2^o = \{5\}.$$

$$\mathcal{A}_2 = \{[1; 3], [2; 4], [3; 4], [4; 3], [4; 5]\}.$$

The relations between the arcs are stored in the array  $\mathbf{R}_{G_2}$ :

	[1; 3]	[2; 4]	[3; 4]	[4; 3]	[4; 5]
[1; 3]	id	$\perp$	$\emptyset$	$\emptyset$	$\emptyset$
[2; 4]	$\perp$	id	$\emptyset$	$\emptyset$	$\emptyset$
[3; 4]	$\emptyset$	$\emptyset$	id	$\emptyset$	$\emptyset$
[4; 3]	$\emptyset$	$\emptyset$	$\emptyset$	id	$\perp$
[4; 5]	$\emptyset$	$\emptyset$	$\emptyset$	$\perp$	id

The matrix of the weights of the graph  $G_2$ ,  $\Omega_{G_2}$ , is as follows:

$$\Omega_{G_2} = \begin{array}{c|ccccc} \curvearrowright & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & 0 & 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 & 1 & 0 \\ 3 & 0 & 0 & 0 & 1 & 0 \\ 4 & 0 & 0 & 3/5 & 0 & 2/5 \\ 5 & 0 & 0 & 0 & 0 & 0 \end{array}$$

And the vector of assigned energy is:

$$\epsilon_{G_2} = (\epsilon(1), \epsilon(2), 0, 0, 0).$$

The set of all energy paths denoted  $\mathcal{E}_2$  is:

$$\begin{aligned} \mathcal{E}_2 = \{ & [1; 3], [1; 3][3; 4][4; 3], [1; 3][3; 4], [1; 3][3; 4][4; 5], \\ & [2; 4][4; 3], [2; 4], [2; 4][4; 3][3; 4], [2; 4][4; 5] \} \end{aligned}$$

For example let us give the close formula for energy circulating on arc  $[4; 5]$ .

The set of all energy paths ending by the arc  $[4; 5]$  is:

$$\mathcal{E}_2([4; 5]) = \{[1; 3][3; 4][4; 5], [2; 4][4; 5]\},$$

and we compute the energy  $\text{Em}(\mathcal{E}_2([4; 5]))$  as follows.

- 1 and 2 are energy sources, thus by definition of  $\perp$  we have  $[1; 3] \perp [2; 4]$  (i.e.  $\mathbf{R}_{G_2}([1; 3], [2; 4]) = \perp$ ).
- Rewrite  $\text{Em}(\mathcal{E}_2([4; 5]))$  as:

$$\text{Em}(\mathcal{E}_2([4; 5])) = \text{Em}([1; 3]\{[3; 4][4; 5]\} \cup [2; 4]\{[4; 5]\}),$$

with  $[1; 3] \perp [2; 4]$ .

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4  
5 • Apply (a.1.2) with  $\pi = \underline{1}$ ,  $k = 2$ ,  $a_1 = [1; 3]$ ,  $\mathcal{U}_1 = \{[3; 4][4; 5]\}$ ,  $a_2 =$   
6  $[2; 4]$  and  $\mathcal{U}_2 = \{[4; 5]\}$ . Then,  
7  
8

$$\text{Em}(\mathcal{E}_2([4; 5])) = \text{Em}([1; 3]\{[3; 4][4; 5]\}) + \text{Em}([2; 4]\{[4; 5]\})$$

- 9  
10  
11  
12 • Compute  $\text{Em}([2; 4]\{[4; 5]\})$  as follows:  
13  
14

$$\begin{aligned} \text{Em}([2; 4]\{[4; 5]\}) &= \text{Em}([2; 4]) \cdot \text{Em}(\{[4; 5]\}) \\ &\text{by (a.2, } \pi = [2; 4], \mathcal{U} = \{[4; 5]\}) \\ &= \epsilon(2) \cdot \Omega_{G_2}(2, 4) \cdot \text{Em}(\{[4; 5]\}) \\ &\text{by (a.3, } l = 2, l' = 4) \\ &= \epsilon(2) \cdot \Omega_{G_2}(2, 4) \cdot \text{Em}([4; 5]) \\ &\text{by (a.4)} \\ &= \epsilon(2) \cdot \Omega_{G_2}(2, 4) \cdot \Omega_{G_2}(4, 5) \\ &\text{by (a.3, noticing that } 4, 5 \notin \mathcal{L}_2^s). \end{aligned}$$

- 15  
16  
17  
18  
19  
20  
21  
22 • Compute  $\text{Em}([1; 3]\{[3; 4][4; 5]\})$  as follows.  
23  
24  
25

$$\begin{aligned} \text{Em}([1; 3]\{[3; 4][4; 5]\}) &= \text{Em}([1; 3]) \cdot \text{Em}(\{[3; 4][4; 5]\}) \\ &\text{by (a.2)} \\ &= \text{Em}([1; 3]) \cdot \text{Em}([3; 4][4; 5]) \\ &\text{by (a.4)} \end{aligned}$$

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41  
42 Then, applying (a.3) to  $\text{Em}([1; 3])$ ,  $\text{Em}([3; 4][4; 5])$ , we have:

$$\begin{aligned} \text{Em}([1; 3]) &= \epsilon(1) \cdot \Omega_{G_2}(1, 3) \\ \text{Em}([3; 4][4; 5]) &= \Omega_{G_2}(3, 4) \cdot \Omega_{G_2}(4, 5). \end{aligned}$$

43  
44  
45  
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48  
49 Finally, we have:

$$\text{Em}(\mathcal{E}_2([4; 5])) = \epsilon(1) \cdot \Omega_{G_2}(1, 3) \cdot \Omega_{G_2}(3, 4) \cdot \Omega_{G_2}(4, 5) + \epsilon(2) \cdot \Omega_{G_2}(2, 4) \cdot \Omega_{G_2}(4, 5)$$

1  
2  
3  
4  
5 **Numerical application.**

6  $\epsilon(1) = 400 \text{ seJ}$  and  $\epsilon(2) = 100 \text{ seJ}$ .

7 Thus,

8  
9  
10  
11 
$$\text{Em}(\mathcal{E}_2([4; 5])) = 400 \cdot 1 \cdot 1 \cdot \frac{2}{5} + 100 \cdot 1 \cdot 1 \cdot \frac{2}{5} = 160 + 40 = 200 \text{ seJ},$$

12 which is the value obtained at the output of the graph [18, Fig 8.b p. 226].

13  
14  
15  
16  
17 **Remark 4.1.** *The emergy computed corresponds to the entry (4, 5) of the*  
18 *matrix FRM in the Tennenbaum's program (see Tennenbaum [12, pp. 122-*  
19 *126]).*

20  
21  
22  
23  
24 *4.3. Example of Section 2 continued*

25 Let us recall that the set of all emergy paths  $\mathcal{E}_1$  is:

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28  
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31 
$$\mathcal{E}_1 = \left\{ \begin{array}{l} [1; 3], [1; 3][3; 4], [1; 3][3; 5], [1; 3][3; 5][5; 6][6; 5], \\ [1; 3][3; 4][4; 6][6; 5], \\ [1; 3][3; 4][4; 6], [1; 3][3; 5][5; 6], [1; 3][3; 4][4; 6][6; 5][5; 6], \\ [1; 3][3; 4][4; 6][6; 7], [1; 3][3; 5][5; 6][6; 7], [2; 4], [2; 4][4; 6][6; 5], \\ [2; 4][4; 6], [2; 4][4; 6][6; 5][5; 6], [2; 4][4; 6][6; 7] \end{array} \right\}$$

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41 As an illustrative example, let us compute the emergy flowing on the arc [6; 5],

42 i.e.  $\text{Em}(\mathcal{E}([6; 5]))$  with  $\mathcal{E}_1([6; 5]) = \{[1; 3][3; 5][5; 6][6; 5], [1; 3][3; 4][4; 6][6; 5], [2; 4][4; 6][6; 5]\}$ .

43  
44  
45 Because  $1, 2 \in \mathcal{L}_1^s$  we have:  $[1; 3] \perp [2; 4]$ , by definition of  $\perp$ . Thus, we  
46 express  $\mathcal{E}_1([6; 5])$  as follows:

47  
48  
49  
50 
$$\mathcal{E}_1([6; 5]) = [1; 3]\mathcal{U}_1 \cup [2; 4]\mathcal{U}_2,$$

with  $\mathcal{U}_1 = \{[3; 5][5; 6][6; 5], [3; 4][4; 6][6; 5]\}$  and  $\mathcal{U}_2 = \{[4; 6][6; 5]\}$ . And we obtain:

$$\begin{aligned} \text{Em}(\mathcal{E}_1([6; 5])) &= \text{Em}([1; 3]\mathcal{U}_1 \cup [2; 4]\mathcal{U}_2) \\ &= \text{Em}([1; 3]\mathcal{U}_1) + \text{Em}([2; 4]\mathcal{U}_2) \quad \text{by (a.1.2)}. \end{aligned}$$

By an easy computation we have:

$$\begin{aligned} \text{Em}([2; 4]\mathcal{U}_2) &= \text{Em}([2; 4]\{[4; 6][6; 5]\}) \\ &= \text{Em}([2; 4]) \cdot \text{Em}(\{[4; 6][6; 5]\}), \text{ by (a.2)} \\ &= \text{Em}([2; 4]) \cdot \text{Em}([4; 6][6; 5]), \text{ by (a.4)} \\ &= \epsilon(2) \cdot \Omega_{G_1}(2, 4) \cdot \Omega_{G_1}(4, 6) \cdot \Omega_{G_1}(6, 5), \text{ by (a.3)}. \end{aligned}$$

Let us detail the computation of  $\text{Em}([1; 3]\mathcal{U}_1)$ . It comes:

$$\begin{aligned} \text{Em}([1; 3]\mathcal{U}_1) &= \text{Em}([1; 3]) \cdot \text{Em}(\mathcal{U}_1) \quad \text{by (a.2)} \\ &= \epsilon(1) \cdot \Omega_{G_1}(1, 3) \cdot \text{Em}(\mathcal{U}_1) \quad \text{by (a.3)}. \end{aligned}$$

Now, we just have to compute  $\text{Em}(\mathcal{U}_1)$ . We remark that:

$$\mathcal{U}_1 = [3; 4]\{[4; 6][6; 5]\} \cup [3; 5]\{[5; 6][6; 5]\},$$

with  $[3; 4] \parallel [3; 5]$  because there is a co-product at node 3. Then, by applying (a.1.3) we have:

$$\text{Em}(\mathcal{U}_1) = \max(\Omega_{G_1}(3, 4) \cdot \text{Em}(\{[4; 6][6; 5]\}), \Omega_{G_1}(3, 5) \cdot \text{Em}(\{[5; 6][6; 5]\})).$$

Using (a.2) and (a.3) we have:

$$\text{Em}([3; 4]\{[4; 6][6; 5]\}) = \Omega_{G_1}(3, 4) \cdot \Omega_{G_1}(4, 6) \cdot \Omega_{G_1}(6, 5)$$

and

$$\text{Em}([3; 5]\{[5; 6][6; 5]\}) = \Omega_{G_1}(3, 5) \cdot \Omega_{G_1}(5, 6) \cdot \Omega_{G_1}(6, 5).$$

Finally, we obtain:

$$\begin{aligned} \text{Em}(\mathcal{E}_1([6; 5])) &= \epsilon(2) \cdot \Omega_{G_1}(2, 4) \cdot \Omega_{G_1}(4, 6) \cdot \Omega_{G_1}(6, 5) \\ &+ \epsilon(1) \cdot \Omega_{G_1}(1, 3) \cdot \max(\Omega_{G_1}(3, 4) \cdot \Omega_{G_1}(4, 6) \cdot \Omega_{G_1}(6, 5), \\ &\quad \Omega_{G_1}(3, 5) \cdot \Omega_{G_1}(5, 6) \cdot \Omega_{G_1}(6, 5)). \end{aligned}$$

#### Numerical application.

$$\epsilon(1) = 400 \text{ seJ}, \quad \epsilon(2) = 100 \text{ seJ}.$$

Thus,

$$\text{Em}(\mathcal{E}_1([6; 5])) = 100 \cdot 1 \cdot 1 \cdot \frac{1}{5} + 400 \cdot 1 \cdot \max(1 \cdot 1 \cdot \frac{1}{5}, 1 \cdot 1 \cdot \frac{1}{5}) = 100 \text{ seJ}.$$

#### 4.4. Emergy graph with splits and one co-product

Let us consider the emergy graph  $G_3$  of Figure 5 borrowed from Li et al. [8, Fig 8 and 9]. There are splits at nodes 3, 5, 6, 7 and 10, and a co-product at node 4. The set of sources is  $\mathcal{L}_3^s = \{1, 2\}$ , the set of internal nodes is  $\mathcal{L}_3^i = \{3, 4, 5, 6, 7, 8, 9, 10\}$  and the set of the output nodes is  $\mathcal{L}_3^o = \{11, 12, 13, 14\}$ . Because 1 and 2 are sources we have:  $[1; 3] \perp [2; 10]$ . Because 3, 5, 6, 7 and 10 are splitted we have:  $[3; 4] \perp [3; 5]$ ,  $[6; 8] \perp [6; 9]$ ,  $[7; 9] \perp [7; 10]$  and  $[10; 4] \perp [10; 11]$ . Because of the co-product at node 4 we have:  $[4; 6] \parallel [4; 7]$ .

Figure 5: Net with splits and one co-product at node 4

Let us give the main steps of the computation of the emergy circulating on the arc  $[9; 13]$  denoted  $\text{Em}(\mathcal{E}_3([9; 13]))$ , recalling that  $\mathcal{E}_3([9; 13])$  is the set of all emergy paths ending by  $[9; 13]$ .

We assume that the computation of the set  $\mathcal{E}_3([9; 13])$  has already been made and:

$$\mathcal{E}_3([9; 13]) = \left\{ \begin{array}{l} [1; 3][3; 5][5; 7][7; 10][10; 4][4; 6][6; 9][9; 13], \\ [1; 3][3; 5][5; 7][7; 9][9; 13], \\ [1; 3][3; 4][4; 6][6; 9][9; 13], [1; 3][3; 4][4; 7][7; 9][9; 13], \\ [2; 10][10; 4][4; 6][6; 9][9; 13], [2; 10][10; 4][4; 7][7; 9][9; 13] \end{array} \right\}.$$

Because 1 and 2 are sources:  $[1; 3] \perp [2; 10]$ , by definition of  $\perp$ . Then,  $\mathcal{E}_3([9; 13])$  is decomposed as follows:

$$\mathcal{E}_3([9; 13]) = [1; 3]\mathcal{U}_1 \cup [2; 10]\mathcal{U}_2,$$

and applying (a.1.2) we have:

$$\text{Em}(\mathcal{E}_3([9; 13])) = \text{Em}([1; 3]\mathcal{U}_1) + \text{Em}([2; 10]\mathcal{U}_2)$$

with:

$$\mathcal{U}_1 = \left\{ \begin{array}{l} [3; 5][5; 7][7; 10][10; 4][4; 6][6; 9][9; 13], [3; 5][5; 7][7; 9][9; 13], \\ [3; 4][4; 6][6; 9][9; 13], [3; 4][4; 7][7; 9][9; 13] \end{array} \right\} \text{ and}$$

$$\mathcal{U}_2 = \{[10; 4][4; 6][6; 9][9; 13], [10; 4][4; 7][7; 9][9; 13]\}.$$

Applying (a.2) and (a.3) to  $\text{Em}([1; 3]\mathcal{U}_1)$  we have:

$$\text{Em}([1; 3]\mathcal{U}_1) = \epsilon(1) \cdot \Omega_{G_3}(1, 3) \cdot \text{Em}(\mathcal{U}_1)$$

and

$$\text{Em}([2; 10]\mathcal{U}_2) = \epsilon(2) \cdot \Omega_{G_3}(2, 10) \cdot \text{Em}(\mathcal{U}_2).$$

Computation of  $\text{Em}(\mathcal{U}_1)$ . There is a split at node 3 thus  $[3; 4] \perp [3; 5]$  and  $\mathcal{U}_1$  is decomposed as follows:

$$\mathcal{U}_1 = [3; 4]\mathcal{U}_{11} \cup [3; 5]\mathcal{U}_{12}.$$



Thus, applying (a.1.2), (a.2) and (a.3) we have:

$$\text{Em}(\mathcal{U}_1) = \Omega_{G_3}(3, 4) \cdot \text{Em}(\mathcal{U}_{11}) + \Omega_{G_3}(3, 5) \cdot \text{Em}(\mathcal{U}_{12}),$$

with:

$$\mathcal{U}_{11} = \{[4; 6][6; 9][9; 13], [4; 7][7; 9][9; 13]\} \text{ and}$$

$$\mathcal{U}_{12} = \{[5; 7][7; 10][10; 4][4; 6][6; 9][9; 13], [5; 7][7; 9][9; 13]\}.$$

There is a co-product at node 4 with  $[4; 6] \parallel [4; 7]$ , thus  $\mathcal{U}_{11}$  is decomposed as follows:

$$\mathcal{U}_{11} = [4; 6]\mathcal{U}_{111} \cup [4; 7]\mathcal{U}_{112},$$

Hence, using (a.1.3), (a.2) and (a.3) we have:

$$\text{Em}(\mathcal{U}_{11}) = \max(\Omega_{G_3}(4, 6) \cdot \text{Em}(\mathcal{U}_{111}), \Omega_{G_3}(4, 7) \cdot \text{Em}(\mathcal{U}_{112})),$$

with  $\mathcal{U}_{111} = \{[6; 9][9; 13]\}$  and  $\mathcal{U}_{112} = \{[7; 9][9; 13]\}$ .

Noticing that  $[7; 9] \perp [7; 10]$ ,  $\mathcal{U}_{12}$  is decomposed as follows:

$$\mathcal{U}_{12} = [5; 7][7; 10]\mathcal{U}_{121} \cup [5; 7][7; 9]\mathcal{U}_{122},$$

by applying (a.1.2), (a.2) and (a.3) we have:

$$\text{Em}(\mathcal{U}_{12}) = \Omega_{G_3}(5, 7) \cdot \Omega_{G_3}(7, 10) \cdot \text{Em}(\mathcal{U}_{121}) + \Omega_{G_3}(5, 7) \cdot \Omega_{G_3}(7, 9) \cdot \text{Em}(\mathcal{U}_{122}),$$

with:  $\mathcal{U}_{121} = \{[10; 4][4; 6][6; 9][9; 13]\}$  and  $\mathcal{U}_{122} = \{[9; 13]\}$ .

By applying (a.4), (a.2) and (a.3) we have:

$$\text{Em}(\mathcal{U}_{121}) = \Omega_{G_3}(10, 4) \cdot \Omega_{G_3}(4, 6) \cdot \Omega_{G_3}(6, 9) \cdot \Omega_{G_3}(9, 13)$$

and

$$\text{Em}(\mathcal{U}_{122}) = \Omega_{G_3}(9, 13).$$

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2  
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4  
5 Computation of  $\text{Em}(\mathcal{U}_2)$ .

6 Noticing that  $[4; 6] \parallel [4; 7]$ ,  $\mathcal{U}_2$  is decomposed as follows:

$$7 \quad \mathcal{U}_2 = [10; 4][4; 6]\mathcal{U}_{21} \cup [10; 4][4; 7]\mathcal{U}_{22},$$

8  
9  
10  
11  
12 by applying (a.1.3), (a.2) and (a.3) we have:

$$13 \quad \text{Em}(\mathcal{U}_2) = \max(\Omega_{G_3}(10, 4) \cdot \Omega_{G_3}(4, 6) \cdot \text{Em}(\mathcal{U}_{21}), \Omega_{G_3}(10, 4) \cdot \Omega_{G_3}(4, 7) \cdot \text{Em}(\mathcal{U}_{22})),$$

14  
15  
16  
17  
18 with  $\mathcal{U}_{21} = \{[6; 9][9; 13]\}$  and  $\mathcal{U}_{22} = \{[7; 9][9; 13]\}$ .

19  
20  
21 By applying (a.4), (a.2) and (a.3) we have:

$$22 \quad \text{Em}(\mathcal{U}_{21}) = \Omega_{G_3}(6, 9) \cdot \Omega_{G_3}(9, 13)$$

23  
24  
25  
26 and

$$27 \quad \text{Em}(\mathcal{U}_{22}) = \Omega_{G_3}(7, 9) \cdot \Omega_{G_3}(9, 13).$$

28  
29  
30 Finally, the following close formula for  $\text{Em}(\mathcal{E}_3([9; 13]))$  is obtained:

$$31 \quad \begin{aligned} & \text{Em}(\mathcal{E}_3([9; 13])) = \epsilon(1) \cdot \Omega_{G_3}(1, 3) \cdot ( \\ & \Omega_{G_3}(3, 5) \cdot \Omega_{G_3}(5, 7) \cdot ( \\ & \Omega_{G_3}(7, 10) \cdot \Omega_{G_3}(10, 4) \cdot \Omega_{G_3}(4, 6) \cdot \Omega_{G_3}(6, 9) \cdot \Omega_{G_3}(9, 13) + \Omega_{G_3}(7, 9)\Omega_{G_3}(9, 13)) \\ & + \Omega_{G_3}(3, 4) \cdot \max(\Omega_{G_3}(4, 6) \cdot \Omega_{G_3}(6, 9) \cdot \Omega_{G_3}(9, 13), \Omega_{G_3}(4, 7) \cdot \Omega_{G_3}(7, 9) \cdot \Omega_{G_3}(9, 13)) \\ & + \epsilon(2) \cdot \Omega_{G_3}(2, 10) \cdot \Omega_{G_3}(10, 4) \cdot \\ & \max(\Omega_{G_3}(4, 6) \cdot \Omega_{G_3}(6, 9) \cdot \Omega_{G_3}(9, 13), \Omega_{G_3}(4, 7) \cdot \Omega_{G_3}(7, 9) \cdot \Omega_{G_3}(9, 13)). \end{aligned}$$

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33  
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40  
41  
42  
43  
44  
45 **Numerical application.**

46  
47  $\epsilon(1) = 1000 \text{ seJ}$ ,  $\epsilon(2) = 500 \text{ seJ}$ .

48  
49  $\Omega_{G_3}(1, 3) = \Omega_{G_3}(2, 10) = \Omega_{G_3}(4, 6) = \Omega_{G_3}(4, 7) = \Omega_{G_3}(9, 13) = 1$ .

$$\Omega_{G_3}(3, 4) = 5/8, \Omega_{G_3}(3, 5) = 3/8, \Omega_{G_3}(5, 7) = 4/5, \Omega_{G_3}(6, 9) = 1/5,$$

$$\Omega_{G_3}(7, 9) = 2/3, \Omega_{G_3}(7, 10) = \Omega_{G_3}(10, 4) = 1/3.$$

$$\begin{aligned} \text{Em}(\mathcal{E}_3([9; 13])) &= 1000 \cdot 1 \cdot \left(\frac{3}{8} \cdot \frac{4}{5} \cdot \left(\frac{1}{3} \cdot \frac{1}{3} \cdot 1 \cdot \frac{1}{5} \cdot 1 + \frac{2}{3} \cdot 1\right) + \frac{5}{8} \cdot \max\left(1 \cdot \frac{1}{5} \cdot 1, \frac{2}{3} \cdot 1\right)\right) \\ &\quad + 500 \cdot 1 \cdot \frac{1}{3} \cdot \max\left(1 \cdot \frac{1}{5} \cdot 1, 1 \cdot \frac{2}{3} \cdot 1\right) \\ &= \frac{6610}{9} (\approx 734.44) \text{ seJ}. \end{aligned}$$

**Remark 4.2.** *Let us remark that our formula avoid double counting of emergy flows with the same past. In Li et al. [8, p. 415, (2)] authors propose to compute the emergy flowing on arc [9; 13] as the maximum of emergy flowing on arc [6; 9] and the emergy flowing on arc [7; 9]. From a numerical point of view this leads to the value of 727.77 (which is different than ours). But the problem is that emergy flowing on arc [6; 9] has not exactly the same past than the emergy flowing on arc [7; 9]. In fact, the formula used in Li et al. [8] to compute emergy flowing on arc [9; 13] is:*

$$\begin{aligned} &\max(\epsilon(1) \cdot \Omega_{G_3}(1, 3) \cdot \Omega_{G_3}(3, 5) \cdot \Omega_{G_3}(5, 7) \cdot \Omega_{G_3}(7, 9) \\ &\quad + \epsilon(1) \cdot \Omega_{G_3}(1, 3) \cdot \Omega_{G_3}(3, 4) \cdot \Omega_{G_3}(4, 7) \cdot \Omega_{G_3}(7, 9) \\ &\quad + \epsilon(2) \cdot \Omega_{G_3}(2, 10) \cdot \Omega_{G_3}(10, 4) \cdot \Omega_{G_3}(4, 7) \cdot \Omega_{G_3}(7, 9), \\ &\quad \epsilon(1) \cdot \Omega_{G_3}(1, 3) \cdot \Omega_{G_3}(3, 5) \cdot \Omega_{G_3}(5, 7) \cdot \Omega_{G_3}(7, 10) \cdot \Omega_{G_3}(10, 4) \cdot \Omega_{G_3}(4, 6) \cdot \Omega_{G_3}(6, 9) \\ &\quad + \epsilon(1) \cdot \Omega_{G_3}(1, 3) \cdot \Omega_{G_3}(3, 4) \cdot \Omega_{G_3}(4, 6) \cdot \Omega_{G_3}(6, 9) \\ &\quad + \epsilon(2) \cdot \Omega_{G_3}(2, 10) \cdot \Omega_{G_3}(10, 4) \cdot \Omega_{G_3}(4, 6) \cdot \Omega_{G_3}(6, 9)). \end{aligned}$$

*This example illustrates the remark of one of the reviewers in Lazzaretto [7, p.2201]: "As observed by one of the reviewers the rule counting the largest emergy value [arriving at a node] is a rather "crude way" of avoiding double counting".*

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4  
5 4.5. *Odum example*  
6

7 Let us consider the graph  $G_4$  of Figure 6 borrowed from Odum [1, p. 100].  
8  
9 And let us compute emergy flowing on arc  $d$  and emergy flowing on arc  $e$ .  
10

11  
12  
13  
14 Figure 6: Graph  $G_4$  Odum diagram  
15

16 Emergy flowing on arc  $d$ .

17 The set of the emergy paths endind by  $d$  is:  
18

19  
20  
21 
$$\mathcal{E}_4(d) = \{samd, sbcd, samegncd, sbcejlmcd, sbceghklmd, flmd, flmegncd\}$$
  
22  
23

24 Noticing that  $s \perp f$ ,  $\mathcal{E}_4(d)$  is decomposed as follows:  
25

26  
27 
$$\mathcal{E}_4(d) = s\mathcal{U}_1 \cup f\mathcal{U}_2,$$
  
28

29 and applying (a.1.2) we have:  
30

31  
32 
$$\text{Em}(\mathcal{E}_4(d)) = \text{Em}(s\mathcal{U}_1) + \text{Em}(f\mathcal{U}_2)$$
  
33

34 with:  $\mathcal{U}_1 = \{amd, bcd, amegncd, bcejlmcd, bceghklmd\}$  and  $\mathcal{U}_2 = \{lmd, lmegncd\}$ .  
35  
36

37 Applying (a.2) and (a.3) we have:  
38

39  
40 
$$\text{Em}(s\mathcal{U}_1) = \epsilon(1) \cdot \Omega_{G_4}(s) \cdot \text{Em}(\mathcal{U}_1)$$
  
41  
42

43 and  
44

45 
$$\text{Em}(f\mathcal{U}_2) = \epsilon(2) \cdot \Omega_{G_4}(f) \cdot \text{Em}(\mathcal{U}_2).$$
  
46

47 Computation of  $\text{Em}(\mathcal{U}_1)$ . We remark that  $a \perp b$  thus  $\mathcal{U}_1$  is decomposed as  
48  
49 follows:  
50

51 
$$\mathcal{U}_1 = a\mathcal{U}_{11} \cup b\mathcal{U}_{12}.$$
  
52  
53

Thus applying (a.1.2), (a.2) and (a.3) we have:

$$\text{Em}(\mathcal{U}_1) = \Omega_{G_4}(a) \cdot \text{Em}(\mathcal{U}_{11}) + \Omega_{G_4}(b) \cdot \text{Em}(\mathcal{U}_{12}).$$

with:  $\mathcal{U}_{11} = \{md, megncd\}$  and  $\mathcal{U}_{12} = \{cd, cejlm d, ceghklmd\}$ .

Noticing that  $d \parallel e$ ,  $\mathcal{U}_{11}$  is decomposed as follows:

$$\mathcal{U}_{11} = md\mathcal{U}_{111} \cup me\mathcal{U}_{112}.$$

Hence, using (a.1.3), (a.2) and (a.3) we have:

$$\text{Em}(\mathcal{U}_{11}) = \max(\Omega_{G_4}(m) \cdot \Omega_{G_4}(d) \cdot \text{Em}(\mathcal{U}_{111}), \Omega_{G_4}(m) \cdot \Omega_{G_4}(e) \cdot \text{Em}(\mathcal{U}_{112})),$$

with:  $\mathcal{U}_{111} = \{\underline{1}\}$  and  $\mathcal{U}_{112} = \{gncd\}$ .

Noticing that  $\{\underline{1}\} = \underline{1}\{\underline{1}\}$  we have by (a.2) and (a.4):  $\text{Em}(\underline{1}) = \text{Em}(\underline{1}) \cdot \text{Em}(\underline{1})$  and because  $\text{Em}(\cdot)$  is strictly positive we have:  $\text{Em}(\underline{1}) = 1 = \text{Em}(\mathcal{U}_{111})$ .

By (a.4) and (a.3) we have:  $\text{Em}(\mathcal{U}_{112}) = \Omega_{G_4}(g) \cdot \Omega_{G_4}(n) \cdot \Omega_{G_4}(c) \cdot \Omega_{G_4}(d)$ .

Because  $d \parallel e$ ,  $\mathcal{U}_{12}$  is decomposed as follows:

$$\mathcal{U}_{12} = cd\mathcal{U}_{121} \cup ce\mathcal{U}_{122}$$

Hence, using (a.1.3), (a.2) and (a.3) we have:

$$\text{Em}(\mathcal{U}_{12}) = \max(\Omega_{G_4}(c) \cdot \Omega_{G_4}(d) \cdot \text{Em}(\mathcal{U}_{121}), \max(\Omega_{G_4}(c) \cdot \Omega_{G_4}(e) \cdot \text{Em}(\mathcal{U}_{122}))$$

with:  $\mathcal{U}_{121} = \{\underline{1}\}$  (thus  $\text{Em}(\mathcal{U}_{121}) = 1$ ) and  $\mathcal{U}_{122} = \{jlm d, ghklmd\}$ .

Noticing that  $j \perp g$ ,  $\mathcal{U}_{122}$  is decomposed as follows:

$$\mathcal{U}_{122} = j\{lm d\} \cup g\{ghklmd\}$$

Using (a.1.2), (a.2), (a.3) and (a.4) we have:

$$\text{Em}(\mathcal{U}_{122}) = \Omega_{G_4}(j) \cdot \Omega_{G_4}(l) \cdot \Omega_{G_4}(m) \cdot \Omega_{G_4}(d) + \Omega_{G_4}(g) \cdot \Omega_{G_4}(h) \cdot \Omega_{G_4}(k) \cdot \Omega_{G_4}(l) \cdot \Omega_{G_4}(m) \cdot \Omega_{G_4}(d).$$

1  
2  
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4  
5 Computation of  $\text{Em}(\mathcal{U}_2)$ .

6 Noticing that  $d \parallel e$ ,  $\mathcal{U}_2$  is decomposed as follows:

$$7 \quad \mathcal{U}_2 = lmd\mathcal{U}_{21} \cup lme\mathcal{U}_{22}$$

8  
9 Hence, using (a.1.3), (a.2) and (a.3) we have:

$$10 \quad \text{Em}(\mathcal{U}_2) = \max(\Omega_{G_4}(l) \cdot \Omega_{G_4}(m) \cdot \Omega_{G_4}(d) \cdot \text{Em}(\mathcal{U}_{21}), \Omega_{G_4}(l) \cdot \Omega_{G_4}(m) \cdot \Omega_{G_4}(e) \cdot \text{Em}(\mathcal{U}_{22}))$$

11  
12 with:  $\mathcal{U}_{21} = \{\underline{1}\}$  and  $\mathcal{U}_{22} = \{gncd\}$ .

13  
14 As previously,  $\text{Em}(\mathcal{U}_{21}) = 1$  and by (a.4) and (a.3) we have:  $\text{Em}(\mathcal{U}_{22}) =$

$$15 \quad \Omega_{G_4}(g) \cdot \Omega_{G_4}(n) \cdot \Omega_{G_4}(c) \cdot \Omega_{G_4}(d).$$

16  
17  
18 Finally,

$$19 \quad \begin{aligned} 20 \quad \text{Em}(\mathcal{E}_4(d)) &= \epsilon(1) \cdot \Omega_{G_4}(s) \cdot (\Omega_{G_4}(a) \\ 21 \quad &\cdot \max(\Omega_{G_4}(m) \cdot \Omega_{G_4}(d) \cdot 1, \Omega_{G_4}(m) \cdot \Omega_{G_4}(e) \cdot \Omega_{G_4}(g) \cdot \Omega_{G_4}(n) \cdot \Omega_{G_4}(c) \cdot \Omega_{G_4}(d)) \\ 22 \quad &+ \Omega_{G_4}(b) \\ 23 \quad &\cdot \max(\Omega_{G_4}(c) \cdot \Omega_{G_4}(d) \cdot 1, \Omega_{G_4}(c) \cdot \Omega_{G_4}(e) \cdot (\Omega_{G_4}(j) \cdot \Omega_{G_4}(l) \cdot \Omega_{G_4}(m) \cdot \Omega_{G_4}(d) \\ 24 \quad &+ \Omega_{G_4}(g) \cdot \Omega_{G_4}(h) \cdot \Omega_{G_4}(k) \cdot \Omega_{G_4}(l) \cdot \Omega_{G_4}(m) \cdot \Omega_{G_4}(d))) \\ 25 \quad &+ \epsilon(2) \cdot \Omega_{G_4}(f) \cdot \max(\Omega_{G_4}(l) \cdot \Omega_{G_4}(m) \cdot \Omega_{G_4}(d) \cdot 1, \\ 26 \quad &\Omega_{G_4}(l) \cdot \Omega_{G_4}(m) \cdot \Omega_{G_4}(e) \cdot \Omega_{G_4}(g) \cdot \Omega_{G_4}(n) \cdot \Omega_{G_4}(c) \cdot \Omega_{G_4}(d)). \end{aligned}$$

27  
28 Energy flowing on arc  $e$ .

29  
30 The set of energy paths ending by  $e$  is:

$$31 \quad \mathcal{E}_4(e) = \{same, sbce, flme\}$$

32  
33 By decomposing  $\mathcal{E}_4(e)$  as follows:

$$34 \quad \mathcal{E}_4(e) = s(a\{me\} \cup sb\{ce\}) \cup f\{lme\},$$

and using (a.1.2), (a.2)-(a.4) we obtain:

$$\begin{aligned} \text{Em}(\mathcal{E}_4(e)) &= \epsilon(1) \cdot \Omega_{G_4}(s) \cdot (\Omega_{G_4}(a) \cdot \Omega_{G_4}(m) \cdot \Omega_{G_4}(e) + \Omega_{G_4}(b) \cdot \Omega_{G_4}(c) \cdot \Omega_{G_4}(e)) \\ &+ \epsilon(2) \cdot \Omega_{G_4}(f) \cdot \Omega_{G_4}(l) \cdot \Omega_{G_4}(m) \cdot \Omega_{G_4}(e). \end{aligned}$$

**Numerical application.**

$\epsilon(1) = 10,000$  seJ and  $\epsilon(2) = 20,000$  seJ.

$$\begin{aligned} \text{Em}(\mathcal{E}_4(d)) &= 10,000 \cdot 1 \cdot \left( \frac{3}{10} \cdot \max(1 \cdot 1 \cdot 1, 1 \cdot 1 \cdot \frac{1}{2} \cdot 1 \cdot 1 \cdot 1) \right. \\ &+ \frac{7}{10} \cdot \max(1 \cdot 1 \cdot 1, 1 \cdot 1 \cdot (\frac{1}{2} \cdot 1 \cdot 1 \cdot 1 + \frac{1}{2} \cdot 1 \cdot \frac{1}{2} \cdot 1 \cdot 1 \cdot 1)) \\ &+ 20,000 \cdot 1 \cdot \max(1 \cdot 1 \cdot 1 \cdot 1, 1 \cdot 1 \cdot 1 \cdot \frac{1}{2} \cdot 1 \cdot 1 \cdot 1) \\ &= 30,000 \text{ seJ}. \end{aligned}$$

$$\begin{aligned} \text{Em}(\mathcal{E}_4(e)) &= 10,000 \cdot 1 \cdot \left( \frac{3}{10} \cdot 1 \cdot 1 + \frac{7}{10} \cdot 1 \cdot 1 \right) + 20,000 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \\ &= 30,000 \text{ seJ}. \end{aligned}$$

**Remark 4.3.** *We can compute every emergy flowing on each arc of the graph. As a further example the numerical expression of the emergy flowing on arc  $m$  is:*

$$\begin{aligned} \text{Em}(\mathcal{E}_4(m)) &= 10,000 \cdot 1 \cdot \left( \frac{3}{10} \cdot 1 + \frac{7}{10} \cdot 1 \cdot 1 \left( \frac{1}{2} \cdot 1 \cdot 1 + \frac{1}{2} \cdot 1 \cdot \frac{1}{2} \cdot 1 \cdot 1 \right) \right) \\ &+ 20,000 \cdot 1 \cdot 1 \\ &= 28,250 \text{ seJ}, \end{aligned}$$

with

$$\mathcal{E}_4(m) = \{sam, sbcejlm, sbceghklm, flm\}.$$

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5 **5. Conclusion**  
6

7       In this paper the Tennenbaum's Track Summing method has been ex-  
8 tended to the case of emergy networks with both splits and co-products. To  
9 obtain this extension the emergy rules R1-R4 (see the Introduction) have  
10 been reformulated into the axiomatic basis (a.1)-(a.4).  
11

12       The data processing implementation of this axiomatic basis is carried out  
13 by a recursive method (see Section 3.1). And it does not require an expert  
14 on emergy algebra. Even if we cannot formally prove that our axiomatic  
15 basis is logically equivalent to the rules R1-R4 apply on emergy flows with  
16 the same past (in the sense of graph theory) this method has been tested on  
17 benchmark emergy examples and gives the same results.  
18

19       Last but not least. Let us note that our method is not only a computa-  
20 tional method. It also provides a rigorous framework based on an axiomatic  
21 basis to do the emergy evaluation of an emergy graph.  
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37

### 38 **Appendix A. Explanation of axiom (a.1.2)**

39  
40 Let us consider the emergy graph of Figure A.7 such that  $\epsilon(1) = 300 \text{ seJ}$ .  
41  
42 Let us compute  $\text{Em}(\mathcal{E}([6; 5]))$ . We have  $\mathcal{E}([6; 5]) = \{[1; 3][3; 4][4; 6][6; 5], [1; 3][3; 5][5; 6][6; 5]\}$ .  
43  
44 Because there is a split at node 3:  $[3; 4] \perp [3; 5]$ , thus the set  $\mathcal{E}([6; 5])$  is decomposed as follows:  
45  
46  
47

$$48 \mathcal{E}([6; 5]) = [1; 3][3; 4]\mathcal{U}_1 \cup [1; 3][3; 5]\mathcal{U}_2$$

49 with:  $\mathcal{U}_1 = \{[4; 6][6; 5]\}$  and  $\mathcal{U}_2 = \{[5; 6][6; 5]\}$ .  
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Figure A.7: Emergy graph with split

The graph of Figure A.8 explains how to compute the emergy flowing on arc [6; 5] of the system where there is only the arc [3; 4] after the past [1; 3], that is  $\text{Em}([1; 3][3; 4]\mathcal{U}_1)$ .

Figure A.8: First pathway from 1 to 5

The graph of Figure A.9 explains how to compute the emergy flowing on arc [6; 5] of the system where there is only the arc [3; 5] after the past [1; 3], that is  $\text{Em}([1; 3][3; 5]\mathcal{U}_2)$ .

Figure A.9: Second pathway from 1 to 5

Finally, when reunited the graph of Figure A.10 explains how to compute the whole emergy flowing on arc [6; 5] and illustrates the formula:

$$\text{Em}([1; 3][3; 4]\mathcal{U}_1 \cup [1; 3][3; 5]\mathcal{U}_2) = \text{Em}([1; 3][3; 4]\mathcal{U}_1) + \text{Em}([1; 3][3; 5]\mathcal{U}_2).$$

In the general case we have:

$$\text{Em}(\cup_{i=1}^k \pi a_i \mathcal{U}_i) = \sum_{i=1}^k \text{Em}(\pi a_i \mathcal{U}_i),$$

and the addition is well associated with independent relation  $\perp$ .

Figure A.10: Total emergy flowing on arc [6; 5]

## Appendix B. Explanation of axiom (a.1.3)

Let us consider the emergy graph of Figure B.11 such that  $\epsilon(1) = 500 \text{ seJ}$ .  
Let us compute  $\text{Em}(\mathcal{E}([6; 5]))$ . We have  $\mathcal{E}([6; 5]) = \{[1; 3][3; 4][4; 6][6; 5], [1; 3][3; 5][5; 6][6; 5]\}$ .  
Because there is a co-product at node 3:  $[3; 4] \parallel [3; 5]$ , thus the set  $\mathcal{E}([6; 5])$   
is decomposed as follows:

$$\mathcal{E}([6; 5]) = [1; 3][3; 4]\mathcal{U}_1 \cup [1; 3][3; 5]\mathcal{U}_2$$

with:  $\mathcal{U}_1 = \{[4; 6][6; 5]\}$  and  $\mathcal{U}_2 = \{[5; 6][6; 5]\}$ .

Figure B.11: Emergy graph with co-product

The graph of Figure B.12 explains how to compute the emergy flowing on  
arc [6; 5] of the system where there is only the arc [3; 4] after the past [1; 3],  
that is  $\text{Em}([1; 3][3; 4]\mathcal{U}_1)$ .

Figure B.12: Emergy on the first pathway from 1 to 5

The graph of Figure B.13 explains how to compute the emergy flowing on  
arc [6; 5] of the system where there is only the arc [3; 5] after the past [1; 3],  
that is  $\text{Em}([1; 3][3; 5]\mathcal{U}_2)$ .

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7 Figure B.13: Emergy on the second pathway from 1 to 5  
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10 Finally, when reunited the graph of Figure B.14 explains how to compute  
11 the whole emergy flowing on arc [6; 5] and illustrates the formula:  
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$$14 \quad \text{Em}([1; 3][3; 4]\mathcal{U}_1 \cup [1; 3][3; 5]\mathcal{U}_2) = \max(\text{Em}([1; 3][3; 4]\mathcal{U}_1), \text{Em}([1; 3][3; 5]\mathcal{U}_2)).$$

15  
16  
17 In the general case we have:

$$18 \quad \text{Em}(\cup_{i=1}^k \pi a_i \mathcal{U}_i) = \max_{i=1}^k \text{Em}(\pi a_i \mathcal{U}_i),$$

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21 and the maximum is well associated with the co-product ||.  
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29 Figure B.14: Total emergy flowing on arc [6; 5]  
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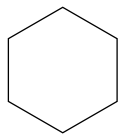
Figure 1



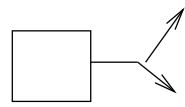
(A)



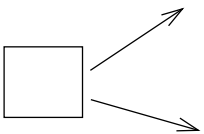
(B)



(C)



(D)



(E)

Figure 2

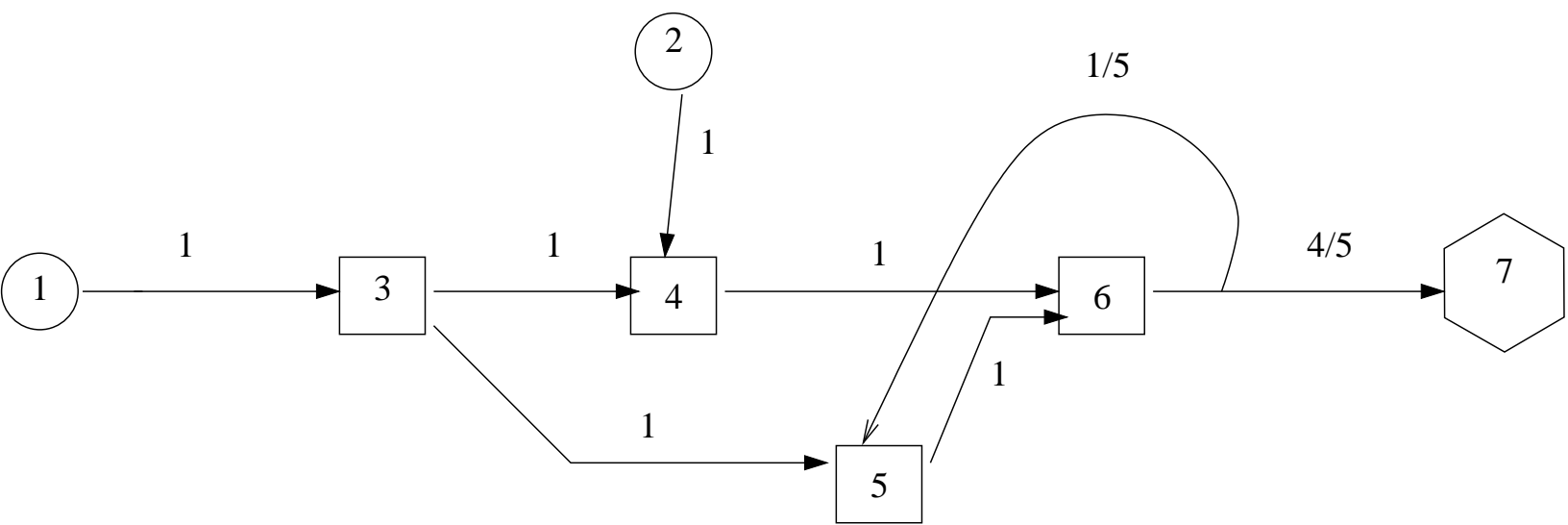


Figure 3

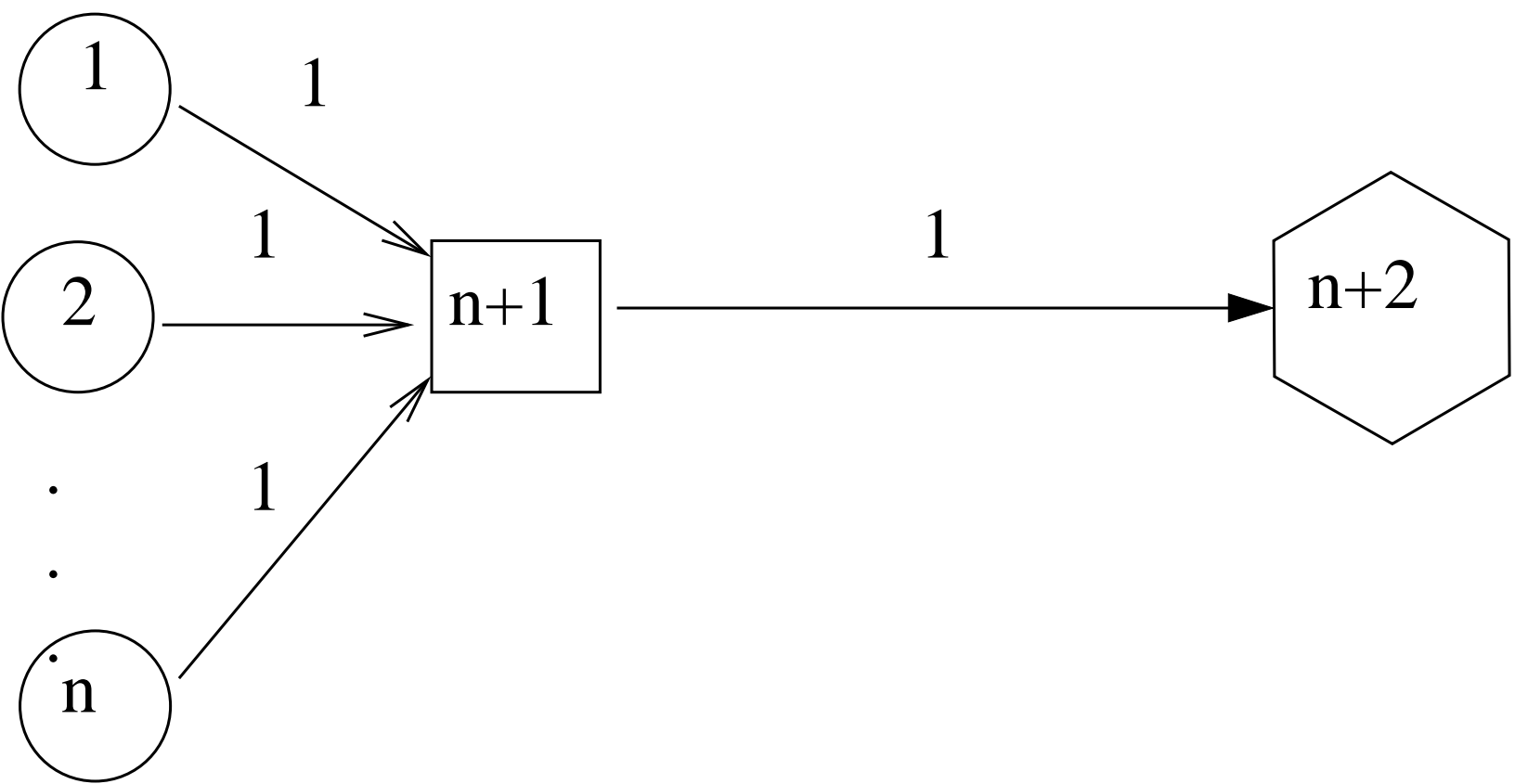




Figure 4

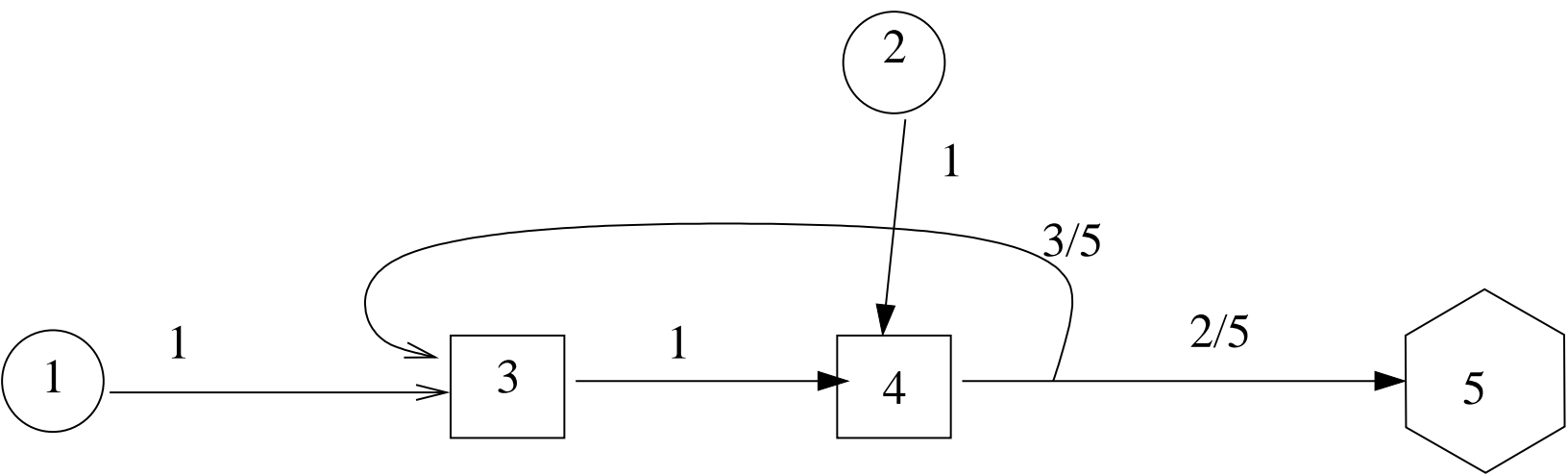


Figure 5

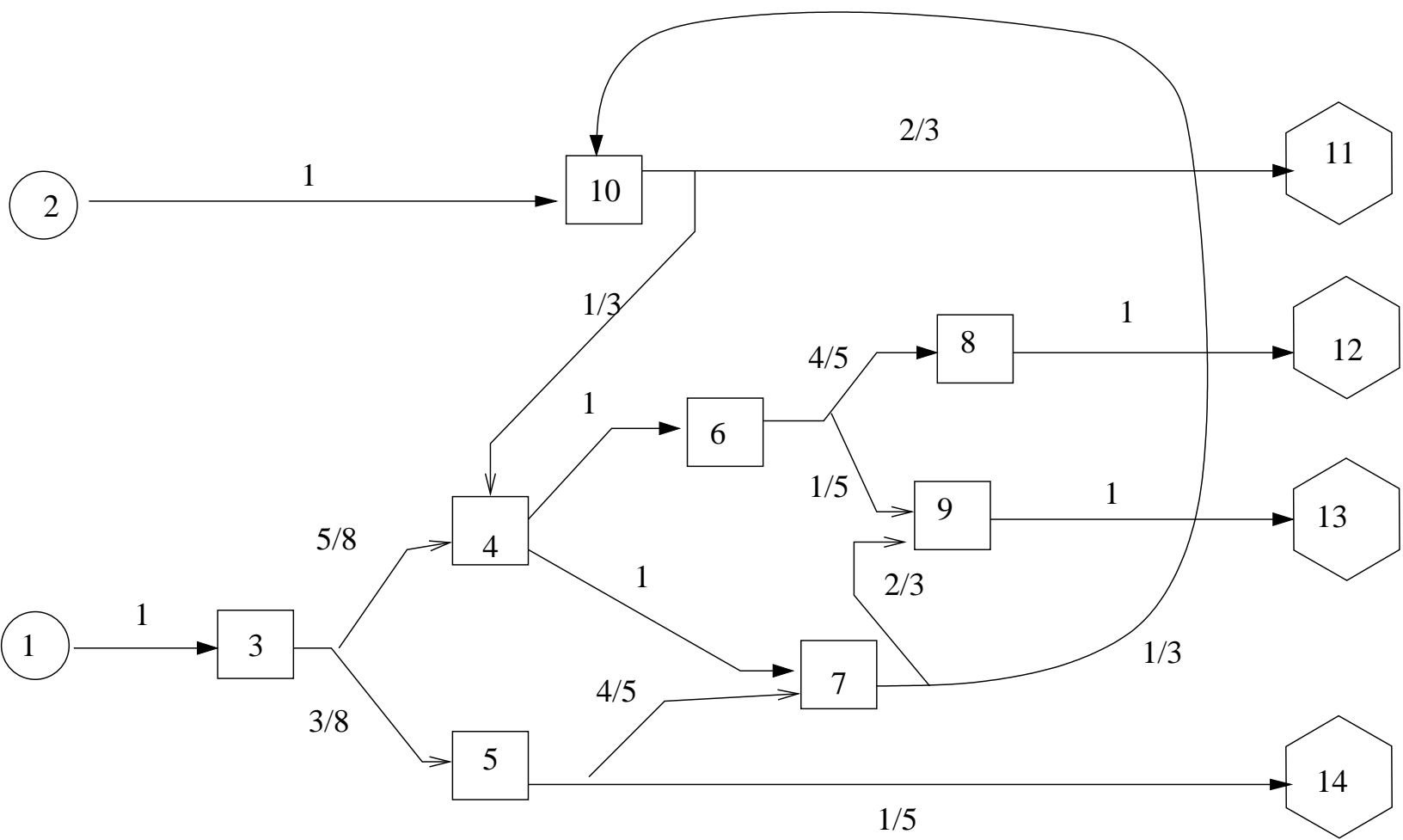


Figure 6

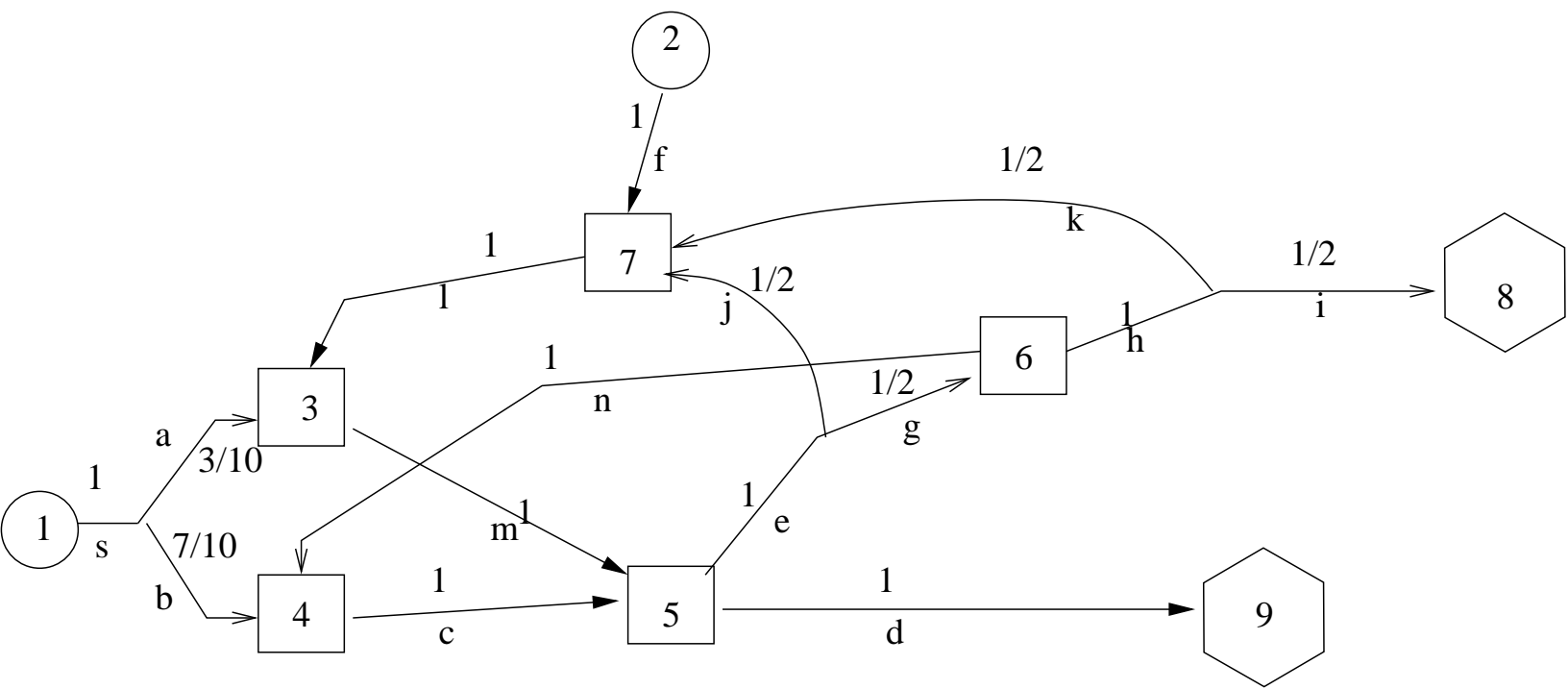


Figure 7

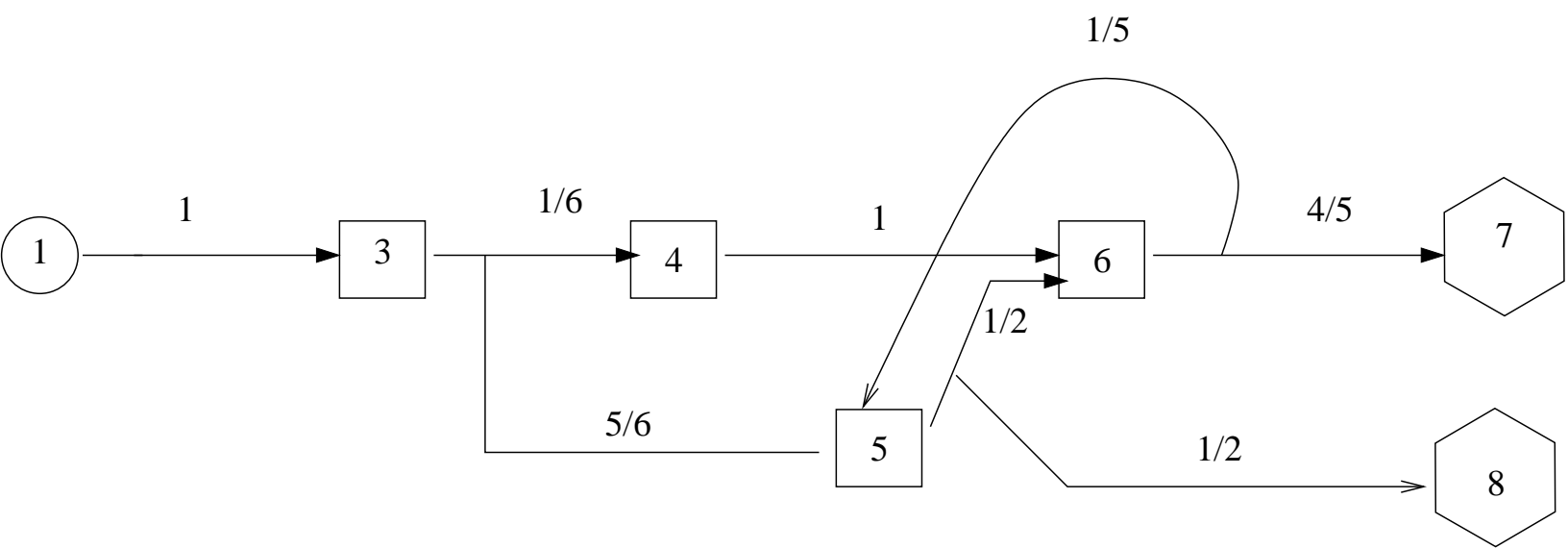


Figure 8

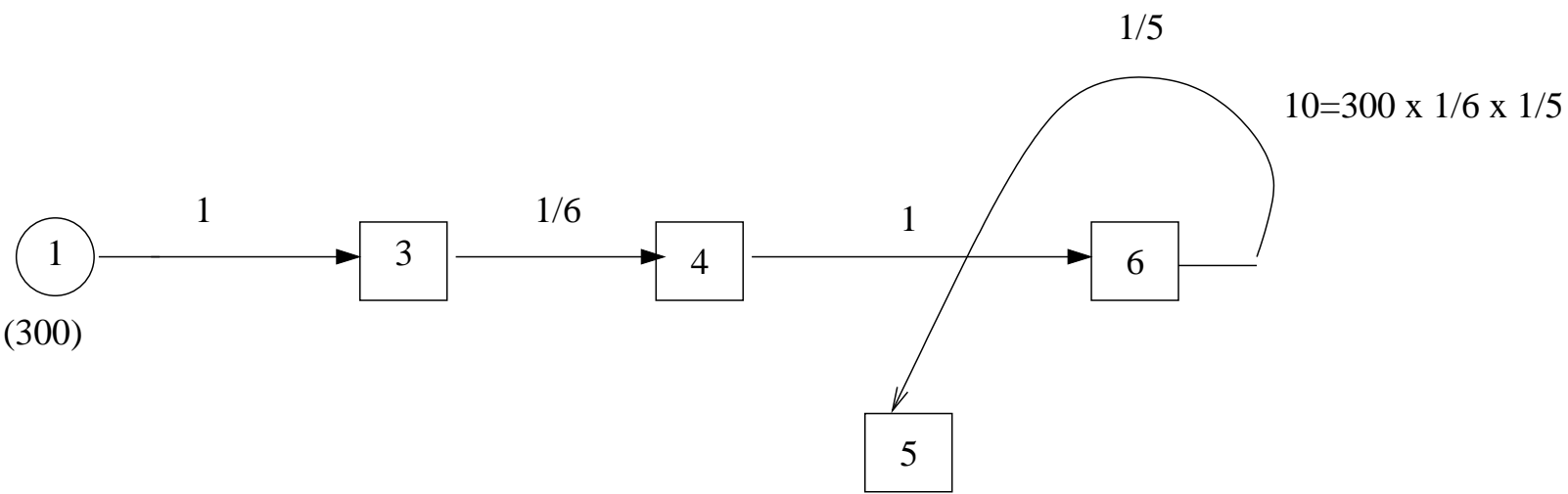


Figure 9

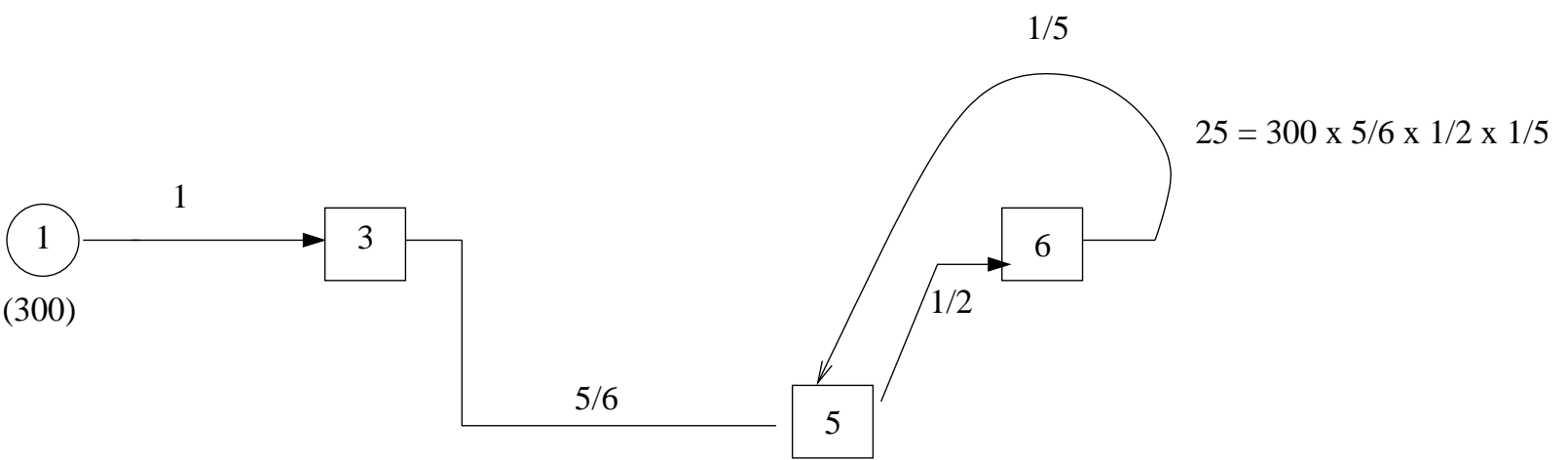


Figure 10

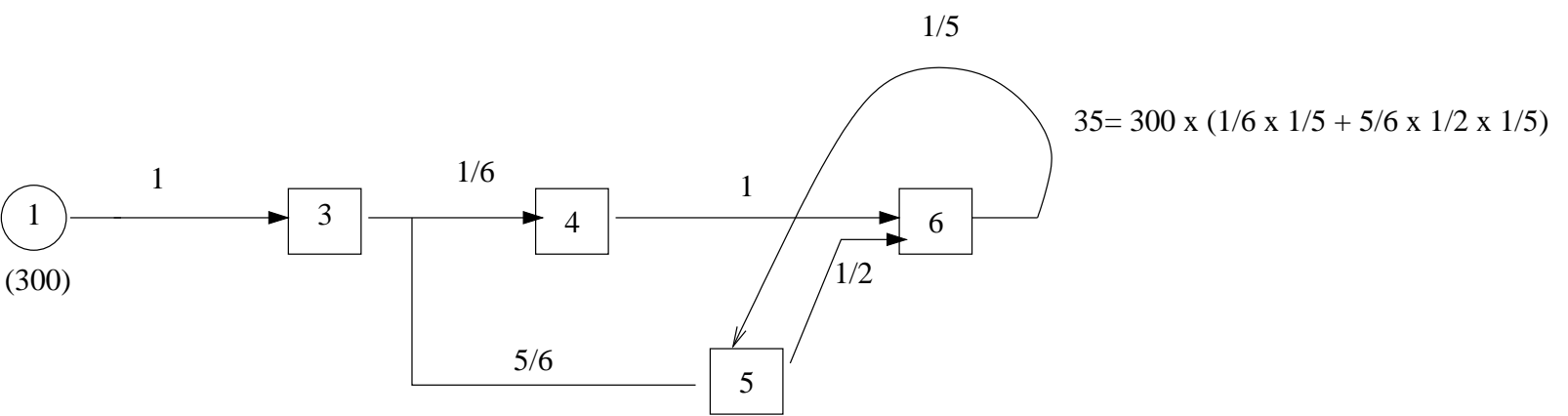


Figure 11

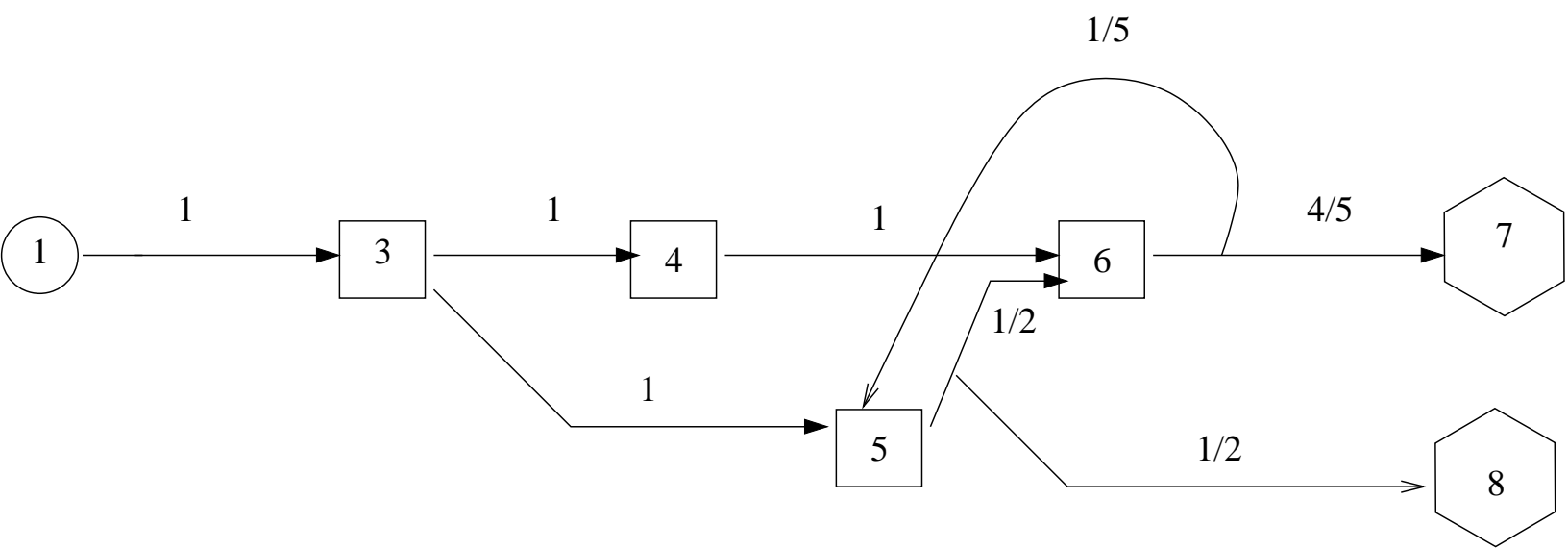




Figure 12

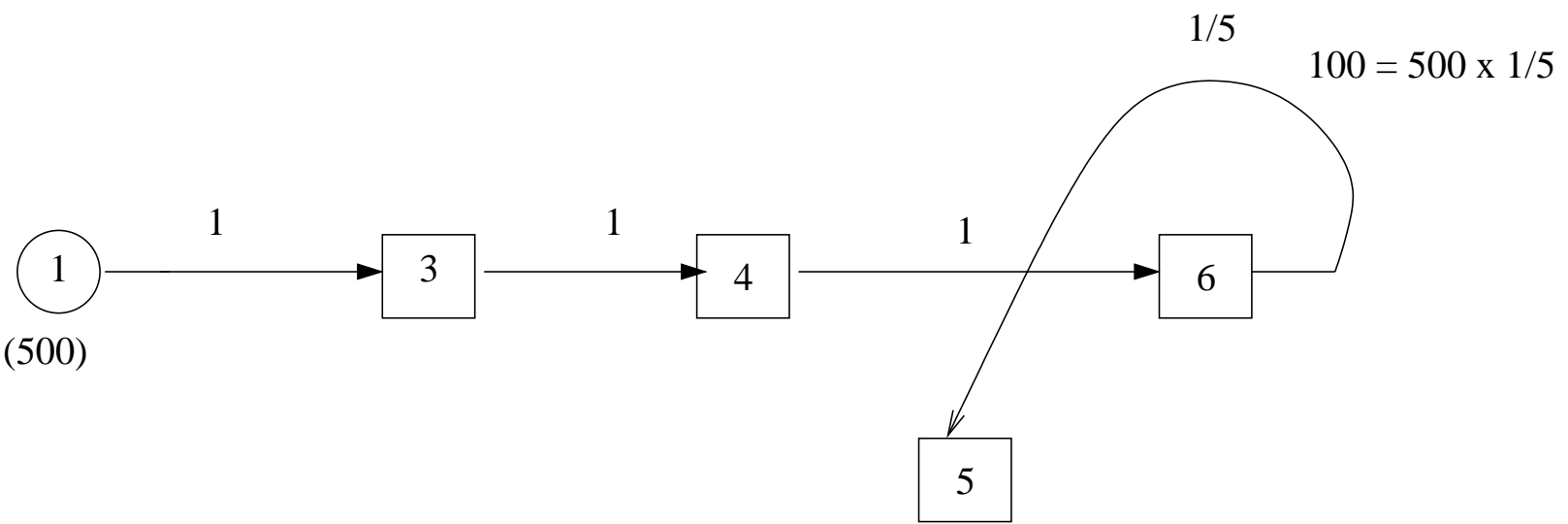


Figure 13

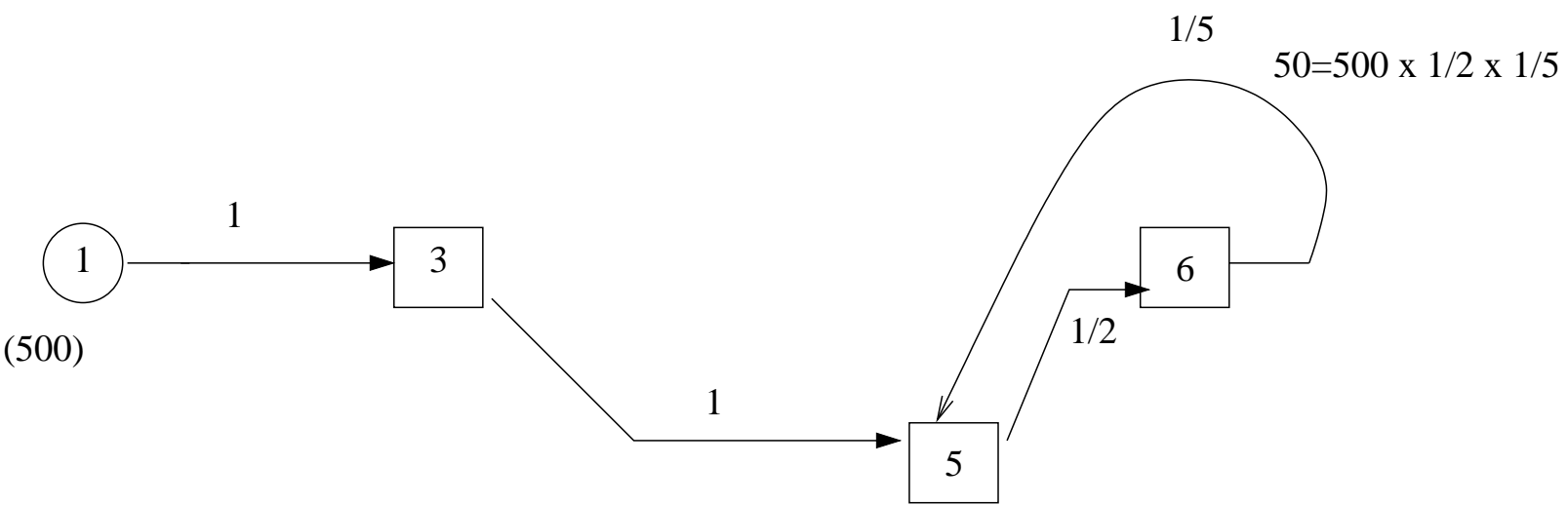
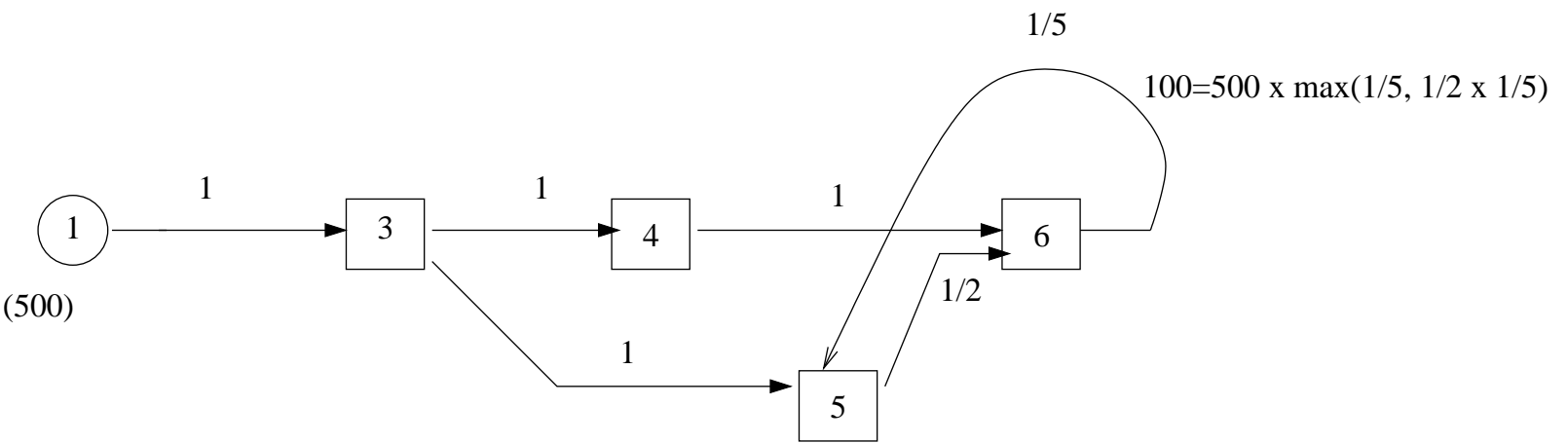


Figure 14



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