Idempotent version of the Fréchet contingency array problem

Chams Lahlou, Laurent Truffet

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C. Lahlou [a,*], L. Truffet [b]

[a] *Ecole des Mines de Nantes - IRCCyN UMR-CNRS 6597*

4, rue A. Kastler, La Chantrerie. BP 20722. Nantes 44307 Cedex 3 France.

[b] *Ecole des Mines de Nantes*

4, rue A. Kastler, La Chantrerie. BP 20722. Nantes 44307 Cedex 3 France.

Abstract

In this paper we study the idempotent version of the so-called Fréchet correlation array problem. The problem is studied using an algebraic approach. The major result is that there exists a unique upper bound and several lower bounds. The formula for the upper bound is given. An algorithm is proposed to compute one lower bound. Another algorithm is provided to compute all lower bounds, but the number of lower bounds may be a very large number. Note that all these results are only based on the distributive lattice property of the idempotent algebraic structure.

Key words: Algebra, non-additive measures, ordered structure.

* Corresponding author.
Email addresses: `chams.lahlou@emn.fr` (C. Lahlou),
`laurent.truffet@emn.fr` (L. Truffet).

Preprint submitted to Fuzzy Sets and Systems *8 March 2009*

1 Introduction

It is well-known for a long time that dependence concepts play an important role in Probability and Statistics. Many practical applications concern e.g. the management of (insurances, economics, financial,..) risks, performance evaluation of Discrete Event Systems such as manufacturing systems, networks, etc. based on stochastic models (see e.g. [32], [23], [19] and references therein). Pioneering works on the subject are the ones of Fréchet ([14], [15]) who proposed a way to study dependence in Probability and Statistics. Independently, this problem also received attention from Bonferroni [5] and Hoeffding [18]. The Fréchet problem in Probability and Statistics is a particular case of the following problem called in this paper *the abstract Fréchet problem* which is defined hereafter. The main idea of such a work is to obtain optimistic and pessimistic bounds only as functions of given marginals. It means that no dependence model is required. For more details on the importance of such an approach and its applications the reader is referred to Rüschendorf [30] and references therein, Williamson and Downs [35], and Regan *et al.* [29].

1.1 The abstract Fréchet problem

Let us consider the 6-tuple $\mathsf{S} = (\mathbb{S}, \oplus, \odot, \mathbb{0}, \mathbb{1}; \preceq)$ where $\mathbb{S}$ is a set equipped with an addition $\oplus : \mathbb{S} \times \mathbb{S} \to \mathbb{S}$, a multiplication $\odot : \mathbb{S} \times \mathbb{S} \to \mathbb{S}$ and a partial order $\preceq$. The element $\mathbb{0}$ (resp. $\mathbb{1}$) denotes the neutral element for $\oplus$ (resp. $\odot$). As we will see in the sequel, the definition of the abstract Fréchet problem is only based on the addition. However, the role of the multiplication will be explained in Remark 3.1. In particular, the multiplication allows us to study the geometric properties of the set of solutions of the Fréchet problem (see subsection 4.3).

Let $(X, \mathcal{A})$ be a measurable space (i.e. $\mathcal{A}$ is a σ-algebra of parts of the set X). A set function $f : \mathcal{A} \to \mathbb{S}$ is called an $(\mathcal{A}, \mathsf{S})$-*measure* if it obeys the following properties:

$(M1)$. $f(\emptyset) = \mathbb{0}$,

$(M2)$. $f(A \cup B) = f(A) \oplus f(B)$, for any pair $A, B \in \mathcal{A}$ of disjoint sets, i.e. such that $A \cap B = \emptyset$.

Let us denote the set of all $(\mathcal{A}, \mathsf{S})$-*measures* on $\mathcal{A}$ by $\mathtt{M}(\mathcal{A}, \mathsf{S})$.

Note that in Hamm [17] they are called *pseudo-additive measures*. Moreover, if $f \in \mathtt{M}(\mathcal{A}, \mathsf{S})$ satisfies:

(P). $f(X) = \mathbb{1}$,

f is called a $(\mathcal{A}, \mathsf{S})$-*probability*.

Assume that the partial order $\preceq$ defined on $\mathbb{S}$ is such that it is possible to define a partial order $\overset{\mathrm{D}}{\preceq}$ on the set $\mathtt{M}(\mathcal{A} \otimes \mathcal{A}, \mathsf{S})$ where $\mathcal{A} \otimes \mathcal{A}$ denotes the smallest σ-algebra containing the elements of $\mathcal{A} \times \mathcal{A}$.

Let us define the function $\pi_i : X \times X \to X$, $(x_1, x_2) \mapsto x_i$ as the projection on the ith coordinate of $X \times X$ such that for all $A \in \mathcal{A}$: $\pi_i^{-1}(A) \overset{\mathsf{def}}{=} \{x \in X \times X \mid \pi_i(x) \in A\} \in \mathcal{A} \otimes \mathcal{A}$, $i = 1, 2$. Let P and Q be any elements of the set $\mathtt{M}(\mathcal{A}, \mathsf{S})$. P and Q are called *marginals*. Let us denote $\mathtt{F}(P, Q, \mathsf{S})$ the set of all elements H of $\mathtt{M}(\mathcal{A} \otimes \mathcal{A}, \mathsf{S})$ verifying:

(F1). $H \circ \pi_1^{-1} = P$,

(F2). $H \circ \pi_2^{-1} = Q$.

An *extremal element* of $\mathtt{F}(P, Q, \mathsf{S})$ w.r.t $\overset{\mathrm{D}}{\preceq}$ is an element H^* of $\mathtt{F}(P, Q, \mathsf{S})$ such that for any $H \in \mathtt{M}(\mathcal{A} \otimes \mathcal{A}, \mathsf{S})$, $H \neq H^*$, we have: if $H \overset{\mathrm{D}}{\preceq} H^*$ then $H \notin \mathtt{F}(P, Q, \mathsf{S})$ (*minimal element*); if $H^* \overset{\mathrm{D}}{\preceq} H$ then $H \notin \mathtt{F}(P, Q, \mathsf{S})$ (*maximal element*).

Then, the abstract Fréchet problem consists in finding (if exist) extremal (i.e. maximal and minimal) elements with respect to (w.r.t.) the partial order $\overset{\mathrm{D}}{\preceq}$ of the set $\mathtt{F}(P, Q, \mathsf{S})$.

It is easy to see that if $P(X) \neq Q(X)$ then $\mathtt{F}(P, Q, \mathsf{S}) = \emptyset$. Thus, in the sequel we will assume that:

$$P(X) = Q(X). \tag{1}$$

In this paper we study the discrete case, that is the case where the measurable space is the discrete (topological) space:

$$F_n \overset{\mathsf{def}}{=} (X = \{1, \ldots, n\} = I_n, \mathcal{A} = \mathbf{2}^{\{1,\ldots,n\}} = \mathcal{I}_n) \tag{2}$$

where $\mathbf{2}^{(\cdot)}$ denotes the set of all subsets of the set $(\cdot)$.

Conditions (F1) and (F2) in the discrete case are equivalent to:

$$\forall i \in I_n, \bigoplus_{k \in I_n} h(i, k) = p(i) \tag{3a}$$

and

$$\forall j \in I_n, \bigoplus_{l \in I_n} h(l, j) = q(j) \tag{3b}$$

We adopt the following notation convention. Let Ω be either the finite set I_n or $I_n \times I_n$ endowed with the order $\preccurlyeq$ which is defined as follows: if $\Omega = I_n$ it denotes $\leq$, the natural order on the set of real numbers $\mathbb{R}$; if $\Omega = I_n \times I_n$ it denotes the componentwise ordering, that is: $\forall x_1, x_2, y_1, y_2 \in I_n$, $(x_1, x_2) \preccurlyeq (y_1, y_2) \overset{\mathsf{def}}{\Leftrightarrow} x_i \leq y_i, i = 1, 2$.

For any $(\mathbf{2}^\Omega, \mathsf{S})$-measure M, the symbol m (resp. $\overline{M}$) will denote the density (resp. the distribution) of M defined by: $\forall \omega \in \Omega \; m(\omega) = M(\{\omega\})$ (resp. $\overline{M}(\omega) = \oplus_{\{\omega' | \omega' \preccurlyeq \omega\}} M(\{\omega'\})$).

We are interested in finding (if exist) extremal elements of $\mathtt{F}(P, Q, \mathsf{S})$ w.r.t. the particular partial order $\overset{\mathrm{D}}{\preceq}$ on $\mathtt{M}(\mathcal{I}_n \otimes \mathcal{I}_n, \mathsf{S})$ denoted $\overset{\mathrm{D}}{\preceq}_1$ and defined as follows:

$$\forall H, H', \; H \overset{\mathrm{D}}{\preceq}_1 H' \overset{\mathsf{def}}{\Leftrightarrow} \forall i, j \in I_n, \; \overline{H}(i,j) \preceq \overline{H}'(i,j). \tag{4}$$

For example, the statistics case [15, Section I] corresponds to the measurable space $\mathcal{F}_n$ and the semiring $\mathsf{Q}_+ = (\mathbb{Q}_+, +, \times, 0, 1; \leq)$, where $\mathbb{Q}_+$ denotes the set of nonnegative rational numbers, $+$, $\times$ and $\leq$ are the usual addition, multiplication, and the natural order on $\mathbb{R}$, respectively. The probability case [15, Section II] is a particular case of the positive measure case which corresponds to $\mathcal{F}_n$ and the semiring $\mathsf{R}_+ = (\mathbb{R}_+, +, \times, 0, 1; \leq)$.

1.2 Motivations for the study of the idempotence case

In this paper we consider the case where the measurable space is $\mathcal{F}_n$ and $\mathsf{S} = (\mathbb{S}, \oplus, \odot, \mathbb{0}, \mathbb{1}; \preceq)$ is an idempotent semiring, that is a semiring whose $\oplus$ is idempotent (i.e. $\forall s, \; s \oplus s = s$). Such measures appear in many fields of research such as fuzzy theory (see e.g. [10]), large deviation theory (see e.g. [34], [27]), fractal theory (see e.g. [13]), optimization theory/dynamic programming (see e.g. [21]), non linear difference equations (see e.g. [25]), decision/game theory (see e.g. [2]). $(\mathcal{A}, \mathsf{S})$-measures can also be considered as particular cases of Choquet capacities [6]. In the literature $(\mathcal{A}, \mathsf{S})$-measures, as defined in this paper, are closely related to other kinds of measures: maxitive measures [31], decomposable measures [26], null-additive measures [24], possibility measures

([22, and references therein], [11]), measures based on triangular norms or t-norms (see e.g. [12]). For other vocabulary the reader is also referred to Puhalskii [27, Appendix B]. The concept of independence and conditioning are well-known in the context of idempotency and/or fuzzyness. For instance this transfer of probabilistic axioms to optimization/control theory has been successfully applied on dynamic programming and optimization (or decision) processes (see e.g. [7], [8], [9], [1], [28] and references therein) and for particular classes of uncertain dynamical systems (see e.g. [17]). Last but not least it is proved in [33] that the Fréchet array problem is max-plus linear which means that it is linear when addition is max and multiplication is $+$. Thus, it seems natural to go into deeper investigations in dependence problems in the idempotent case.

1.3 Organization of the paper

The paper is organized in order to be self-contained. In Section 2 basic results on Fréchet correlation array problem and basic results on order and idempotent algebra are recalled. In Section 3 we define the Fréchet problem over an idempotent semifield. In Section 4 we present the main results dealing with the problem of Section 3. In subsection 4.1, Theorem 4.1 we prove the existence and the uniqueness of the upper bounding problem. This proof is based on the distributive lattice order property of an idempotent semifield. In subsection 4.2 we mention that it may exist a large number of minimal solutions in the idempotent case (see Examples 4.1 and 4.2). We also provide two algorithms: Algorithm 1 computes one minimal solution and Algorithm 2 computes all minimal solutions. This part of the problem needs further work.

Finally, in subsection 4.3, Theorem 4.2 we show that the set of solutions to the idempotent Fréchet problem is an idempotent convex set. Section 5 concludes this work.

2 Preliminaries

2.1 The Fréchet array problem

In this subsection we consider the so-called Hoeffding-Fréchet problem which corresponds to the abstract Fréchet problem with the measurable space $\mathcal{F}_n = (I_n, \mathcal{I}_n)$ defined by (2) and the naturally ordered semiring $\mathsf{R}_+ = (\mathbb{R}_+, +, \times, 0, 1; \leq)$.

Let us consider the partial order $\overset{\mathrm{D}}{\leq}_1$ on the distributions associated to the elements of $\mathtt{M}(\mathcal{I}_n \otimes \mathcal{I}_n, \mathsf{R}_+)$ defined by (4) (where $\preceq$ is replaced with $\leq$). Let us remark that the R_+-valued measures defined on the discrete topology are completely characterized because of the relationship between the density function and the distribution function. This relation is recalled in the following result.

Result 2.1 *Let P be a $(\mathcal{I}_n, \mathsf{R}_+)$-measure and let H be a $(\mathcal{I}_n \otimes \mathcal{I}_n, \mathsf{R}_+)$-measure.*

A . *If $\overline{H}(i,j)$, for all $i, j = 1, \ldots, n$, are known and verify the condition:*

$$(\mathbf{M}).\ \overline{H}(i,j) + \overline{H}(i-1,j-1) - \overline{H}(i-1,j) - \overline{H}(i,j-1) \geq 0,$$

for all $i, j = 1, \ldots, n$, where by convention $\overline{H}(0,\ \cdot\ ) = \overline{H}(\ \cdot\ ,0) = 0$.
Then, the density h of H is defined, for all $i, j = 1, \ldots, n$, by:

$$h(i,j) = \overline{H}(i,j) + \overline{H}(i-1,j-1) - \overline{H}(i-1,j) - \overline{H}(i,j-1) \tag{5}$$

B . *Conversely, if* $h(l,k) \geq 0$ *are given for all* $k,l = 1,\ldots,n$ *then:*

$$\overline{H}(i,j) = \sum_{l=1}^{i}\sum_{k=1}^{j} h(l,k), i,j = 1,\ldots,n, \tag{6}$$

define the distribution function of the measure H*.*

C . *If* $\overline{P}(i)$*,* $i \in I_n$ *are given numbers which verify the monotonicity condition:*

$$(\mathbf{m}).\ \overline{P}(i) - \overline{P}(i-1) \geq 0,$$

for all $i,\ldots,n$ *with the convention:* $\overline{P}(0) = 0$*.*

Then, the density p *of* P *is defined by:*

$$p(i) = \overline{P}(i) - \overline{P}(i-1),$$

$i = 1,\ldots,n$*.*

D . *Conversely, if* $p(i) \geq 0$*,* $i = 1,\ldots,n$*, are given then:*

$$\overline{P}(i) = \sum_{k=1}^{i} p(k),\ i = 1,\ldots,n,$$

define the distribution function of the measure P*.*

We are now in position to restate Fréchet's result using our settings.

Result 2.2 ([15]) *Let* P *and* Q *be two given marginals on* $\mathcal{F}_n$*. Then, the subset* $\mathtt{F}(P,Q,\mathsf{S})$ *of* $\mathtt{M}(\mathcal{I}_n \otimes \mathcal{I}_n, \mathsf{R}_+)$*:*

i) is not empty if P *and* Q *verify condition (1), i.e.* $P(I_n) = Q(I_n) = \theta$*,* $\theta \in \mathbb{R}_+$*.*

ii) And under i) it has a unique maximal element $F_{\max}$ *and a unique minimal element* $F_{\min}$*, with respect to* $\overset{\mathrm{D}}{\leq}_1$*, which verify condition (*$\mathbf{M}$*, Result 2.1)* *and are characterized by their distribution functions from* $I_n \times I_n$ *into* $\mathbb{R}_+$ *respectively defined by:*

$$(i,j) \mapsto \overline{F}_{\max}(i,j) = \min(\overline{P}(i), \overline{Q}(j)), \tag{7a}$$

and

$$(i,j) \mapsto \overline{F}_{\min}(i,j) = \max(0, \overline{P}(i) + \overline{Q}(j) - \theta), \tag{7b}$$

2.2 Ordered sets

Let $(\mathcal{X}, \preceq)$ be a poset. $(\mathcal{X}, \preceq)$ is a sup-semilattice (resp. inf-semilattice) if any set $\{x_1, x_2\} \subset \mathcal{X}$ has a supremum $\bigvee\{x_1, x_2\}$ (an infimum $\bigwedge\{x_1, x_2\}$). $(\mathcal{X}, \preceq)$ is a lattice iff $(\mathcal{X}, \preceq)$ is a sup- and inf-semilattice. $(\mathcal{X}, \preceq)$ is a complete sup-semilattice (resp. inf-semilattice) if any set $A \subset \mathcal{X}$ has a supremum $\bigvee A$ (an infimum $\bigwedge A$). $(\mathcal{X}, \preceq)$ is a complete lattice iff $(\mathcal{X}, \preceq)$ is a complete sup- and inf-semilattice. A lattice is distributive if $\wedge$ and $\vee$ are left distributive w.r.t one another, i.e. $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$, $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$, and also right distributive, i.e. $(b \wedge c) \vee a = (b \vee a) \wedge (c \vee a)$, $(b \vee c) \wedge a = (a \wedge b) \vee (a \wedge c)$.

Proposition 2.1 *Let* $(\mathcal{X}, \preceq)$ *be a lattice. Then,*

$$a \preceq b \Leftrightarrow b = a \vee b \Leftrightarrow a \wedge b = a.$$

Proof. By definition of $\vee$ and $\wedge$ we have: $b = a \vee b \Rightarrow a \preceq b$ and $a \wedge b = a \Rightarrow a \preceq b$. Assume that $a \preceq b$. Because $\preceq$ is reflexive we have $a \preceq a$ which implies by definition of $\wedge$ that: $a \preceq a \wedge b$. Noticing that $a \wedge b \preceq a$ and $\preceq$ is antisymmetric we conclude that: $a = a \wedge b$. Similarly we prove that $a \preceq b \Rightarrow b = a \vee b$ on the inequality $b \preceq b$. □

2.3 Idempotent algebra

Let us define the fundamental (idempotent) algebraic structures used in this paper.

Definition 2.1 (Basic structures) .

0. Semigroup. A semigroup is a set $\mathbb{S}$ *endowed with an associative operation* $\oplus : \mathbb{S} \times \mathbb{S} \to \mathbb{S}$.

1. Monoid. A monoid is a set $\mathsf{M} = (\mathbb{M}, \oplus, \mathbb{0})$ *which is a semigroup with a neutral element* $\mathbb{0}$*. Moreover, if* $\oplus$ *is commutative then* M *is a* commutative monoid.

2. Group. A group is a monoid $\mathsf{G} = (\mathbb{G}, \odot, \mathbb{1})$ *such that all elements are invertible, i.e. for any element* a *, there exists a unique element* $c = a^{-1}$ *such that* $a \odot c = c \odot a = \mathbb{1}$.

3. Semiring. A semiring is a set $\mathsf{S} = (\mathbb{S}, \oplus, \odot, \mathbb{0}, \mathbb{1})$ *with* $\mathbb{0} \neq \mathbb{1}$ *such that* $(\mathbb{S}, \oplus, \mathbb{0})$ *is a commutative monoid,* $\odot : \mathbb{S} \times \mathbb{S} \to \mathbb{S}$ *is associative and its neutral element is* $\mathbb{1}$*,* $\odot$ *has* $\mathbb{0}$ *as absorbing element,* $\odot$ *distributes over* $\oplus$.

4. Semifield. A semifield is a set $\mathsf{K} = (\mathbb{k}, \oplus, \odot, \mathbb{0}, \mathbb{1})$ *such that* $(\mathbb{k}, \oplus, \odot, \mathbb{0}, \mathbb{1})$ *is a semiring and* $(\mathbb{k} \setminus \{\mathbb{0}\}, \odot, \mathbb{1})$ *is a group.*

Semigroup, Monoid, group, semiring, semifield are said to be idempotent when $\oplus$ is idempotent (i.e. $\forall a$, $a \oplus a = a$).

Let $\mathsf{S} = (\mathbb{S}, \oplus)$ be a commutative idempotent semigroup. We define the natural (or standard) partial order $\preceq$ as follows:

$$x \preceq y \overset{\mathsf{def}}{\Leftrightarrow} \exists z \; y = x \oplus z, \; \Leftrightarrow \; y = x \oplus y \text{ (Because } \oplus \text{ is idempotent).} \qquad (8)$$

The notation $x \succeq y$ means $y \preceq x$. The relation $x \prec y$ means that $x \preceq y$ and $x \neq y$. From now on, $\preceq$ will denote the partial order defined by (8).

Remark 2.1 *Let us note that if* S *is a semigroup the binary relation* $\preceq$ *is only transitive. If* S *is a monoid the binary relation is a preorder (i.e., reflexive and transitive).*

By definition of $\preceq$ and because $\oplus$ is idempotent and commutative, we easily see that $\oplus$ is monotone, i.e.:

$$a \preceq b \Rightarrow \forall c, \; a \oplus c \preceq b \oplus c \qquad (9)$$

By definition of $\preceq$ and because $\oplus$ is idempotent and commutative, we have: $a \preceq a \oplus b$ and $b \preceq a \oplus b$. Conversely, assume that there exists c such that $a \preceq c$ and $b \preceq c$. Then, because $\oplus$ is monotone (see (9)) and idempotent we have $a \oplus b \preceq c \oplus c = c$. Thus, for all a, b the supremum of a and b, $a \vee b$ exists and is $a \oplus b$. This well-known result is recalled in the next Proposition.

Proposition 2.2 *The class of all idempotent commutative semigroups coincides with the class of all sup-semilattices.*

From this Proposition we immediately deduce that the class of idempotent commutative monoids with neutral element $\mathbb{0}$ coincides with the class of supsemilatices having the bottom element $\bot = \mathbb{0}$.

An idempotent commutative semigroup $(\mathbb{S}, \oplus; \preceq)$ is a complete ordered set iff $\forall A \subseteq \mathbb{S}$, $\oplus A \overset{\mathsf{def}}{=} \oplus_{a \in A} a$ exists in $\mathbb{S}$. An idempotent commutative semigroup $(\mathbb{S}, \oplus; \preceq)$ such that $\forall x, y$, $x \wedge y$ exists is called a *lattice semigroup.*

An idempotent semiring $\mathsf{S} = (\mathbb{S}, \oplus, \odot, \mathbb{0}, \mathbb{1}; \preceq)$ is complete if $(\mathbb{S}, \oplus; \preceq)$ is a complete ordered set and $\forall B \subseteq \mathbb{S}$, $\forall c \in \mathbb{S}$: $(\oplus B) \odot c = \oplus_{b \in B} b \odot c$, $c \odot (\oplus B) = \oplus_{b \in B} c \odot b$. One also remarks that any distributive lattice with a bottom element $\bot$ and a top element $\top$ (resp. a complete distributive lattice) is an idempotent semiring (resp. an idempotent complete semiring).

Proposition 2.3 *Let* $\mathsf{S} = (\mathbb{S}, \oplus, \odot, \mathbb{0}, \mathbb{1}; \preceq)$ *be an idempotent semiring. Then,* $\odot$ *is monotone, i.e.:*

$$\begin{cases} a \preceq b \\ c \preceq d \end{cases} \Rightarrow a \odot c \preceq b \odot d. \tag{10}$$

Proof. Assume that $a \preceq b$ and $c \preceq d$. Then, $b = a \oplus b$ and $d = c \oplus d$. By distributivity of $\odot$ over $\oplus$, and because $\oplus$ is commutative one has:

$$\begin{aligned} b \odot d &= (b \oplus a) \odot (d \oplus c) \\ &= a \odot c \oplus \underbrace{a \odot d \oplus b \odot c \oplus b \odot d}_{z}. \end{aligned}$$

Thus, by definition of $\preceq$ (8) the result is proved. □

Let us mention the following useful order properties of idempotent semifields.

Proposition 2.4 *Let* $(\mathbb{k}, \oplus, \odot, \mathbb{0}, \mathbb{1}; \preceq)$ *be an idempotent semifield equipped with the natural (partial) order* $\preceq$ *defined by (8).*

(i). $(\mathbb{k}, \oplus; \preceq)$ *is a lattice semigroup such that* $\oplus = \vee$.

(ii). The lattice $(\mathbb{k}, \preceq)$ *is distributive, i.e. for all* $a, b, c \in \mathbb{S}$*:*

$$a \oplus (b \wedge c) = (a \oplus b) \wedge (a \oplus c) \tag{11a}$$

and

$$a \wedge (b \oplus c) = (a \wedge b) \oplus (a \wedge c). \tag{11b}$$

(iii). $\odot$ *distributes over* $\wedge$*,i.e.:*

$$\forall a, b, c \in \mathbb{S}, \quad \begin{aligned} a \odot (b \wedge c) &= (a \odot b) \wedge (a \odot c), \\ (b \wedge c) \odot a &= (b \odot a) \wedge (c \odot a). \end{aligned} \tag{12}$$

Proof. To prove (i), we just have to remark that by monotonicity of $\odot$ (see (10)):

$$\wedge\{a, b\} = \begin{cases} (a^{-1} \oplus b^{-1})^{-1} & \text{if } \{a, b\} \subset \mathbb{S} \setminus \{\mathbb{0}\} \\ \mathbb{0} & \text{otherwise.} \end{cases}$$

Indeed, assume that $a, b \neq \mathbb{0}$. Because $\oplus = \vee$: $a^{-1} \oplus b^{-1} \geq a^{-1}$ and $a^{-1} \oplus b^{-1} \geq b^{-1}$. Thus, because $\odot$ is monotone: $(a^{-1} \oplus b^{-1})^{-1} \preceq a$ and $(a^{-1} \oplus b^{-1})^{-1} \preceq b$. Hence, by definition of $\wedge$: $(a^{-1} \oplus b^{-1})^{-1} \preceq a \wedge b$. Let $c \preceq a, c \preceq b$. Then $c^{-1} \geq a^{-1}$, $c^{-1} \geq b^{-1}$, and therefore $c^{-1} \geq a^{-1} \oplus b^{-1}$, hence $c \preceq (a^{-1} \oplus b^{-1})^{-1}$. The equality holds because $\preceq$ is antisymmetric. The case $a = \mathbb{0}$ or $b = \mathbb{0}$ is trivial.

For the proof of (ii) the reader is referred to [4, Chap. 12]. The result (iii) can be found in e.g. [3, p. 168]. □

3 Definition of the Idempotent Fréchet problem

Let us consider the measurable space $\mathcal{F}_n$. The basic algebraic structure we consider in this paper is a naturally ordered idempotent semifield $\mathsf{K} = (\Bbbk, \oplus, \odot, \mathbb{0}, \mathbb{1}; \preceq)$.

Remark 3.1 *The choice of this algebraic structure is motivated as follows. Our paper is an algebraic oriented paper and the idempotent semifield hypothesis is a very important one. It includes idempotent semirings such as* $\mathsf{R}_{max} = ([-\infty, +\infty], \oplus = \max, \odot = +, \mathbb{0} = -\infty, \mathbb{1} = 0)$ *and* $\mathsf{R}_{min} = ([-\infty, +\infty], \oplus = \min, \odot = +, \mathbb{0} = +\infty, \mathbb{1} = 0)$ *which play an important role in e.g. optimization theory.*

It allows us to point out that only the distributive lattice property of such a structure is needed to prove one of our main result dealing with the maximal element of the Fréchet problem (see Theorem 4.1 and Remark 4.1). Thus, our result holds for other algebraic structures which have a distributive lattice property such as some incline algebras and fuzzy algebras which play an important role in many fields (see e.g. [20], [3] and references therein).

Finally, this algebraic structure allows us to study the set of all elements which are solution to the idempotent Fréchet problem we define below.

Let us begin by the following fundamental remark.

Remark 3.2 *Let us recall that a* $(\mathcal{I}_n, \mathsf{K})$*-measure* P *(resp. a* $(\mathcal{I}_n \otimes \mathcal{I}_n, \mathsf{K})$*-measure* H*) is completely characterized by its discrete density function, i.e. the application* $p : I_n \to \Bbbk,\ i \mapsto p(i) \overset{def}{=} P(\{i\})$ *(resp.* $h : I_n \times I_n \to \Bbbk,\ (i,j) \mapsto h(i,j) \overset{def}{=} H(\{i\}\times\{j\})$*) but not always by its distribution function. For example* *let us consider the idempotent semifield* $\mathsf{R}_{max} = ([-\infty, +\infty], \oplus = \max, \odot = +, \mathbb{0} = -\infty, \mathbb{1} = 0; \preceq)$. *Note that in this case* $\preceq$ *defined by (8) coincides with* $\leq$ *the natural order on* $\mathbb{R}$. *Let* P *be a* $(\mathcal{I}_3, \mathsf{R}_{max})$*-measure whose density is the constant function* $p = -5$. *Then, its distribution function is also the constant function* $i \mapsto -5$. *But if we take* P' *whose density* p' *is defined by* $p'(i) = -5 - i + 1$, $i \in I_3$ *then* P' *has also the same distribution function as* P.

Because of this remark only (B and D Result 2.1) are still valid. In order to find (if exist) extremal elements of the set $\mathtt{F}(P, Q, \mathsf{S})$ we define the partial order $\overset{\mathrm{D}}{\preceq}_2$ on $\mathtt{M}(\mathcal{I}_n \otimes \mathcal{I}_n, \mathsf{K})$ based on the comparison of density functions as follows.

$$\forall H, H',\ H \overset{\mathrm{D}}{\preceq}_2 H' \overset{\mathsf{def}}{\Leftrightarrow} \forall i, j \in I_n,\ h(i,j) \preceq h'(i,j) \tag{13}$$

Because $\oplus$ is non-decreasing the partial order $\overset{\mathrm{D}}{\preceq}_2$ is stronger than the partial order on distribution functions $\overset{\mathrm{D}}{\preceq}_1$. It means that the extremal solutions (if exist) of the Fréchet problem with partial order $\overset{\mathrm{D}}{\preceq}_2$ are also extremal solutions of the same Fréchet problem (i.e., the same algebraic structure) with partial order $\overset{\mathrm{D}}{\preceq}_1$.

4 Main results

We assume that the conditions described in Section 3 are satisfied and that condition (1) is satisfied, i.e.

$$\oplus_i p(i) = \sigma = \oplus_j q(j).$$

Theorem 4.1 *The set* $\mathtt{F}(P, Q, \mathsf{K})$ *has a unique maximal element* $H_{\max}$ *w.r.t the partial order* $\overset{\mathrm{D}}{\preceq}_2$ *completely characterized by its density function from* $I_n \times I_n \to \mathbb{k}$ *defined by:*

$$(i, j) \mapsto h_{\max}(i, j) = p(i) \wedge q(j). \tag{14}$$

Proof. We have to study solutions (if exist) of the system of equation (3a)-(3b), that is the solution of:

$$(I).\ \forall i, j \in I_n,\ \bigoplus_{k \in I_n} h(i, k) = p(i),\ \bigoplus_{l \in I_n} h(l, j) = q(j)$$

By definition of the partial order $\preceq$ (see (8)) and because $\oplus$ is commutative and associative:

$$(I) \ \Rightarrow \ (II).\ \forall i, j \in I_n,\ h(i, j) \preceq p(i), \text{and } h(i, j) \preceq q(j).$$

Note that $(\mathbb{k}, \preceq)$ is a lattice (see (i), Proposition 2.4) thus it is an inf-semilattice and hence:

$$(II) \ \Leftrightarrow \ (III).\ \forall i, j \in I_n,\ h(i, j) \preceq p(i) \wedge q(j).$$

Because $(\mathbb{k}, \preceq)$ is a distributive lattice (see (ii), Proposition 2.4) we have: $\oplus_k (p(i) \wedge q(k)) = p(i) \wedge (\oplus_k q(k)) = p(i) \wedge \sigma$. Then, because $p(i) \preceq \sigma$ and thanks to Proposition 2.1 one concludes that: $\oplus_k (p(i) \wedge q(k)) = p(i), \forall i$. Similarly, we prove that $\oplus_l (p(l) \wedge q(j)) = q(j), \forall j$ and that: $\oplus_{k,l} p(l) \wedge q(k) = \sigma$. Thus we have proved that the measure $H_{\max}$ with density $(i, j) \mapsto h_{\max}(i, j) \overset{\mathsf{def}}{=} p(i) \wedge q(j)$ is the maximum element of the set $\mathtt{F}(P, Q, \mathsf{K})$ w.r.t $\overset{\mathrm{D}}{\preceq}_2$. □

Remark 4.1 *In the previous proof we only use the fact that a semifield is a distributive lattice.*

Remark 4.2 *For the classical Fréchet problem (i.e., when the semiring is* R_+*) the bound* $h_{\max}$ *is still valid because* $(\mathbb{R}_+, +; \leq)$ *is a naturally ordered inf-semilattice. But, measure* $H_{\max}$*, whose density is* $h_{\max}$*, is not an element of* $\mathtt{F}(P, Q, \mathsf{R}_+)$ *in general.*

4.2 Study of minimal solutions

The main result of this subsection is that there is not always a unique minimal solution for the idempotent Fréchet problem w.r.t $\overset{\mathrm{D}}{\preceq}_2$.

Example 4.1 *Let us consider the idempotent semifield* $\mathsf{R}_{max} = ([-\infty, +\infty], \oplus = \max, \odot = +, \mathbb{0} = -\infty, \mathbb{1} = 0; \preceq)$*, recalling that* $\preceq$ *defined by (8) coincides with* $\leq$ *in this case. Let us take the* $(\mathcal{I}_2, \mathsf{R}_{max})$*-measures* P *and* Q *characterized by their density vector* $p = (\mathbb{1}, -5)^T$ *and* $q = (-2, \mathbb{1})$*, respectively. The minimal solutions of the Fréchet problem (3a)-(3b) are:*

$$\begin{pmatrix} -2 & \mathbb{1} \\ \\ \mathbb{0} & -5 \end{pmatrix} \quad \textit{and} \quad \begin{pmatrix} -2 & \mathbb{1} \\ \\ -5 & \mathbb{0} \end{pmatrix}$$

Moreover, as demonstrated in the following example there may exist a large number of minimal solutions.

Example 4.2 *Let* $n \geq 2$ *be an integer and let us consider the* $(\mathcal{I}_n, \mathsf{R}_{max})$*-measures* P *and* Q *characterized by their density vector* $p = (-1, -2, \ldots, -(n-1), \mathbb{1})^T$ *and* $q = (\mathbb{1}, \mathbb{1}, \ldots, \mathbb{1})$*, respectively. The minimal solutions* $H_{\min} = [h_{\min}(i, j)]$ *of the Fréchet problem (3a)-(3b) are such that :*

(i). for all $i \in I_{n-1}$*, there exists a unique* $k_i \in I_n$ *such that* $h_{\min}(i, j) =$

$$\begin{cases} p(i) & \textit{if } j = k_i \\ \mathbb{0} & \textit{otherwise.} \end{cases}$$

(ii). $h_{\min}(n, j) = \mathbb{1}$ *for* $j \in I_n$.

Since there are n possibilities to satisfy condition (i)*, for a given* $i \in I_{n-1}$*, the number of minimal solutions is* n^{n-1}.

Finding one minimal solution of the Fréchet problem (3a)-(3b) can be done easily by starting from the maximal solution $H_{\max}$, and trying to set $h_{\max}(i, j)$ to $\mathbb{0}$, as long as it is possible. The following algorithm follows this scheme:

Algorithm 1 Computing one minimal solution of the Fréchet problem

Input: density vectors p and q.

Output: a minimal solution $H_{\min}$.

Set $H_{\min}$ to the maximal solution $H_{\max}$;

$E := I_n \times I_n$;

While $E \neq \emptyset$ **do**

  Take any (i, j) in E ;

  $h_{\min}(i, j) := \mathbb{0}$;

  If $H_{\min}$ does not satisfy (3a)-(3b) **then** $h_{\min}(i, j) := h_{\max}(i, j)$;

  $E := E \setminus \{(i, j)\}$;

end

This algorithm provides a minimal solution: otherwise there exists at least one $h_{\min}(i, j)$ that can be set to $\mathbb{0}$ without violating conditions (3a)-(3b), which contradicts the fact that the algorithm tries to set every $h_{\min}(i, j)$ to $\mathbb{0}$. The $\mathcal{O}(n^3)$ time complexity of the algorithm can be improved by storing and updating the number $r(i)$ of elements in a row of $H_{\max}$ equal to $p(i)$ (respectively the number $r(i)$ of elements in a column of $H_{\max}$ equal to $q(i)$). Hence, $h(i,j)$ is set to $\mathbb{O}$ only if condition "$(r(i) > 1$ or $h(i,j) \neq p(i))$ and $(c(i) > 1$ and $h(i,j) \neq q(j))$" is satisfied. Since checking this condition and updating $r(i)$ and $c(i)$ take time $\mathcal{O}(1)$, the complexity of Algorithm 4.2 is lowered to $\mathcal{O}(n^2)$.

The next step is to compute all minimal solutions, which can be done by algorithm 4.2:

Algorithm 2 Computing all minimal solutions of the Fréchet problem

Input: density vectors p and q.

Output: set S of all minimal solutions.

Compute the maximal solution $H_{\max}$;

$E := I_n \times I_n$;

$S := \emptyset$;

MinimalSolution$(p, q, H_{\max}, E, S)$;

Proposition 4.1 *Algorithm 4.2 computes all minimal solutions of the Fréchet array problem (3a)-(3b), i.e. it computes all possible sets* $E \subseteq I_n \times I_n$ *such that* $h_{\min}(i,j) = h_{\max}(i,j)$ *if* $(i,j) \in E$*, and* $h_{\min}(i,j) = \mathbb{O}$ *otherwise.*

Proof. First we prove that the solutions found are minimal. Suppose the algorithm returns one non-minimal solution H. In this case there is at least one element (i,j) such that $h(i,j) = h_{\max}(i,j)$ and H is still a solution if $h(i,j)$ is set to $\mathbb{O}$. Therefore there exist k and l such that $h(i,l) = p(i)$ and $h(k,j) = q(j)$. If (i,j) has been treated by the algorithm before (i,l) and (k,j), then $h(i,l)$ or $h(k,j)$ (or both) should have been set to $\mathbb{O}$: so suppose, without loss of generality, that (i,l) has been first considered. Since $h(i,l) = p(i)$ and $h(k,j) = q(j)$, procedure *MinimalSolution* should have set $h(i,j)$ to $\mathbb{O}$ when

Procedure 1 MinimalSolution(p,q,H,E,S)

```
/* Step 1 */
if E = ∅ then
    S := S ∪ {H} ;
else
    Take any (i, j) in E ;
/* Step 2: generate solutions such that h(i, j) = 𝟘 */
    H' := H ;
    h'(i, j) := 𝟘 ;
    if H' satisfies (3a)-(3b) then
        MinimalSolution(p, q, H', E \ {(i, j)}, S) ;
    end
/* Step 3: generate solutions such that h(i, j) = h_max(i, j) */
    H' := H ;
    L := {(i, k) ∈ E| h(i, k) = h(i, j) and k ≠ j} ;
    C := {(l, j) ∈ E| h(l, j) = h(i, j) and l ≠ i} ;
    foreach (k, l) ∈ L ∪ C do
        h'(k, l) := 𝟘
        if H' does not satisfy (3a)-(3b) then h'(k, l) := h(k, l) ;
    end
    MinimalSolution(p, q, H', E \ ({(i, j)} ∪ L ∪ C), S) ;
end
```

trying to set elements of row i to $\mathbb{0}$ (Step 2), which contradicts our assumption.

Now we prove that all minimal solutions are found. Assume there is a minimal solution H' not generated by the algorithm, and let H be a minimal solution found by the algorithm. Since $H \neq H'$, there exists (i,j) such that $h(i,j) \neq$

$h'(i,j)$. We consider two cases:

- $h'(i,j) = \mathbb{0}$: since H' is a solution, there exist k and l such that $h'(i,l) = h_{\max}(i,l) = p(i)$ and $h'(k,j) = h_{\max}(k,j) = q(j)$. Since H is a minimal solution such that $h(i,j) = h_{\max}(i,j)$, it is not possible to have $\mathbb{0} \prec h(i,l)$ and $\mathbb{0} \prec h(k,j)$ at the same time. Suppose, without loss of generality, that only $h(i,l)$ is equal to $\mathbb{0}$, or, if $h(k,j) = h(i,l) = \mathbb{0}$, that $h(i,l)$ has been set to $\mathbb{0}$ before $h(k,j)$. After setting $h(i,l)$ to $\mathbb{0}$ (Step 1), procedure *MinimalSolution* has set $h(i,l)$ to $h_{\max}(i,l) = p(i)$ and has tried to set elements of row i to $\mathbb{0}$ (Step 2). Because $h(k,j) = h_{\max}(k,j) = q(j)$, the algorithm should have set $h(i,j)$ to $\mathbb{0}$, which contradicts our assumption.
- $h'(i,j) = h_{\max}(i,j)$: since $h(i,j) = \mathbb{0}$, there exist k and l such that $h(i,l) = h_{\max}(i,l) = p(i)$ and $h(k,j) = h_{\max}(k,j) = q(j)$. Since H' is a minimal solution such that $h'(i,j) = h_{\max}(i,j)$, it is not possible to have $\mathbb{0} \prec h'(i,l)$ and $\mathbb{0} \prec h'(k,j)$ at the same time. Therefore, $h(i,l) = \mathbb{0}$ or $h(k,j) = \mathbb{0}$, and the previous case applies.

□

Obviously, this algorithm does not have a polynomial time complexity since the number of minimal solutions can be exponential (see example 4.2).

4.3 Study of other elements

In this subsection we characterize the set $\mathtt{F}(P,Q,\mathsf{K})$ recalling that K is a naturally ordered idempotent semifield. Let us begin by the following definition.

Definition 4.1 *A subset $\mathcal{X}$ of K is an idempotent convex set if $\forall u, v \in \mathcal{X}$,*

$\forall \alpha, \beta \in \mathsf{K}$ *such that* $\alpha \oplus \beta = \mathbb{1}$*:* $\alpha \odot u \oplus \beta \odot v \in \mathcal{X}$.

We give next the main result of this subsection.

Theorem 4.2 *The set* $\mathtt{F}(P, Q, \mathsf{K})$ *is an idempotent convex set.*

Proof. Let H_1 and H_2 be two elements of $\mathtt{F}(P, Q, \mathsf{K})$. Let $H = \alpha \odot H_1 \oplus \beta \odot H_2 = [\alpha \odot h_1(i,j) \oplus \beta \odot h_2(i,j)]$, $\forall \alpha, \beta \in \mathsf{K}$ such that $\alpha \oplus \beta = \mathbb{1}$.

For all $i \in I_n$,

$$\begin{aligned}
\oplus_{k \in I_n} h(i,k) &= \oplus_{k \in I_n} (\alpha \odot h_1(i,k) \oplus \beta \odot h_2(i,k) \\
&= (\oplus_{k \in I_n} \alpha \odot h_1(i,k)) \oplus (\oplus_{k \in I_n} \beta \odot h_2(i,k)) \text{ ($\oplus$ is commutative and associative)} \\
&= \alpha \odot (\oplus_{k \in I_n} h_1(i,k)) \oplus \beta \odot (\oplus_{k \in I_n} h_2(i,k)) \text{ (by distributivity)} \\
&= \alpha \odot p(i) \oplus \beta \odot p(i) \\
&= (\alpha \oplus \beta) \odot p(i) \text{ (by distributivity)} \\
&= p(i).
\end{aligned}$$

We have proved that relation (3a) is verified by H. A similar proof is used to show that H satisfies (3b). Thus, $H \in \mathtt{F}(P, Q, \mathsf{K})$ and the result is proved. □

Noticing that the Minkowski theorem holds for max-plus convex Sets [16], the previous result suggests to investigate the topological properties (e.g. compactness) of the set $H \in \mathtt{F}(P, Q, \mathsf{K})$.

5 Conclusion

In this paper we have studied an idempotent (or fuzzy) version of the Fréchet array problem. In the case of an idempotent semifield the set of all solutions is an idempotent convex set (see subsection 4.3 and Theorem 4.2).

There exists a unique upper bound to this problem. The proof is valid not only for an idempotent semifield but also for a distributive lattice. Such a structure naturally appears in the context of fuzzyness.

The lower bounding problem is more complex. There exist several (maybe many) lower bounds of a given Fréchet array problem. As a further work we need to count exactly their numbers and try to find other algorithms more efficient than the one we have proposed in this paper.

Acknowledgements

The authors would like to thank the anonymous reviewers and the editors for their comments which improved the quality of the paper.

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