# Cyclic Creep-Damage in Thin-Walled Structures

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# Cyclic creep damage in thin-walled structures

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**Abstract:** Thin-walled structural elements are often subjected to cyclic loadings. This paper presents a material model describing creep behaviour under high-cycle loading conditions ($N \geqslant 5 \times 10^4$–$10^5$). Assuming that the load can be split into two joint acting parts (a static and a superposed, rapidly varying small cyclic component), the asymptotic expansion of two time-scales has been applied to the governing equations of the initial-boundary value creep problem. The system of equations determine two problems. The first is similar to the creep problem by quasi-static loading. The second is the problem of forced vibrations. Both the problems are coupled by constitutive equations. The model is applied to the simulation of the cyclic creep damage behaviour of thin-walled structural elements. The results are discussed for two special numerical examples (a conical shell and a circular plate). The simulations show that the creep and the damage rates as well as the failure time are strongly sensitive to the redistribution of the stress state cycle asymmetry parameter $A_s$. The values of $A_s$ increase during the creep process. For particular cases of the loading frequency, $A_s$ can exceed the critical value. In this case the material model must be extended in order to consider the creep–fatigue damage interaction.

**Keywords:** creep, damage, high-cycle loading, thin-walled structures

## NOTATION

| | |
|---|---|
| $A$ | stress cycle asymmetry parameter |
| $A_s$ | stress state cycle asymmetry parameter |
| $c$ | creep strain |
| $c_{ij}$ | creep strain tensor |
| $E_{ijkl}$ | Hookean tensor |
| $f$ | cyclic frequency of the surface load |
| $k, l, m, n, r$ | exponents in the uniaxial creep damage laws |
| $[\mathbf{K}]$ | stiffness matrix |
| $[\mathbf{M}]$ | mass matrix |
| $n_i$ | outer normal to the body surface |
| $N$ | number of cycles |
| $p_i$ | surface load vector |
| $s_{ij}$ | stress deviator |
| $t, \tau$ | time |
| $t_*$ | rupture time |
| $T$ | cycle period |
| $u_i$ | displacement vector |
| $x_i, z$ | coordinates |
| $\boldsymbol{\delta}$ | generalized displacement vector |
| $\varepsilon_{ij}$ | strain tensor |
| $\sigma$ | stress |
| $\sigma_{eq}$ | equivalent stress in the damage evolution equation |
| $\sigma_{ij}$ | stress tensor |
| $\sigma_{vM}$ | von Mises equivalent stress |
| $\omega$ | damage variable |
| $\Omega$ | angular frequency of the surface load |

### Subscripts and superscripts

| | |
|---|---|
| a | amplitude of the cyclic part |
| cr | creep |
| e | finite element |
| el | elastic |
| eq | equivalent |
| h | hydrostatic |
| $i, j, k, l$ | 1, 2, 3 |
| T | transpose |
| $s$ | axial |
| $\theta$ | tangential |
| 0, 1 | slow and fast components respectively |

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## 1 INTRODUCTION

Creep damage phenomena in materials and structural members (tubes, pressure vessels, chambers, etc.) have been widely studied in recent years. The material model describing the creep behaviour of metals and alloys can be represented as a set of first-order differential equations which contains the constitutive equation for the creep strain rate tensor as a function of the stress tensor, temperature and some internal state variables and appropriate evolution equations [**1**]. The sensitivity of the minimum strain rate (secondary creep) to the stress level can be described by different stress functions (power law, hyperbolic function, etc.) formulated for different mechanisms of creep deformation [**2**]. Primary creep by stationary or varying loading can be characterized by introducing empirical functions (time or strain-hardening functions) or suitable hardening variables with evolution equations. Models which are able to describe creep and creep–plasticity interaction by different kinds of loading have been reviewed in reference [**3**]. The effect of tertiary creep can be described using continuum damage mechanics (CDM) originated by Kachanov [**4**] and Rabotnov [**5**]. The state of the art of CDM has been reported in several review papers, monographs and textbooks (see, for example, references [**6**] to [**12**]).

Here the creep behaviour in the case of fast oscillations (frequencies greater than 1–2 Hz) with amplitudes of the cyclic stress significantly smaller than the value of the constant stress is discussed. Such loading conditions are widespread in engineering applications, where technological or operational conditions (non-stationary flow, combustion, acoustic action, load oscillation, etc.) favour the development of forced vibrations. Creep behaviour observed in such a loading (known as dynamic creep [**13–15**]) can be characterized by an acceleration of creep strain and a decrease in the fracture time in comparison with classical creep by static loading. The creep strain rate becomes additionally dependent on the amplitude of the applied cyclic stress. The available constitutive models representing the creep strain rate by static or slowly varying loading can be used for the dynamic creep behaviour. However, numerical problems can occur in applications to structural analysis. Because of fast oscillations, very small time steps are necessary for stable time integration. Furthermore the influence of the small amplitude of the fast oscillating stress on the strain rate is observable in the global 'slow' time. The numerical effort can be significantly decreased using a suitable time-averaging procedure.

The first investigations were directed at the evaluation of dynamic creep uniaxial models. Most of them are based on experimental data from tension tests with specimens jointly acted upon by static and harmonic varying tension stresses. The approach which has been proposed by several researchers, but most notably by Lasan [**16**], Rabotnov [**5**] and Taira [**13**], substitutes the equivalent stress terms in the corresponding material model for constant loading. Although there was general agreement between the results of experimental studies by the proposed equations, the question about the laws of structural behaviour remained.

The possibility for considering rapidly oscillating loading in multiaxial cases has been discussed in reference [**17**] by applying the asymptotic expansions of two time-scales to the governing mechanical equations of creep. Using a similar procedure a multiaxial creep damage model has been proposed in reference [**18**]. In the uniaxial case this model coincides with the Lasan–Taira–Rabotnov description. Verification of the suggested constitutive equations in the case of complex stress states was reported in reference [**19**] for cyclically loaded thick cylinders, in references [**20**] and [**21**] for plates in bending and in reference [**22**] for notched plates, based on experimental data and results of numerical simulations.

The satisfactory agreement obtained in these investigations permits the numerical simulation of cyclic creep damage behaviour for other types of structural model. Here the cyclic creep behaviour of thin-walled models, discussed previously in reference [**23**] for static loading conditions, is investigated.

## 2 MATHEMATICAL MODEL

The creep equations for bodies, subjected to rapidly oscillating loads, can be formulated using the two time-scales ('fast' and 'slow') method of asymptotic expansions [**17**]. Here the method proposed is extended in order to consider creep damage evolution with application to thin-walled structures.

Assume that the body is determined by the domain $\mathscr{D}$ with the boundary $\mathscr{C} = \mathscr{C}_1 \cup \mathscr{C}_2$. The body is fixed on $\mathscr{C}_1$ and loaded by cyclic forces with constant $p_i^0$ and amplitude $p_i^{\mathrm{a}}$ components, $p_i = p_i^0 + p_i^{\mathrm{a}}\Phi(ft)$ $(i = 1, 2, 3)$ on $\mathscr{C}_2$, where $\Phi(ft)$ is a periodic function with frequency $f = \Omega/2\pi$ and $t$ denotes time.

As follows from experimental observations [**5, 13**] for the case of fast cyclic loading with uniaxial stress $\sigma^0$ and relatively small amplitude $\sigma^{\mathrm{a}}$ the shape of the creep curve is the same as that of the typical static curve. The stress cycle asymmetry parameter is determined as $A = \sigma^{\mathrm{a}}/\sigma^0 < 1$ and the number of cycles to fracture is $N_* = t_* f > 0.5 \times 10^5$. Even such a small cyclic action leads to an appreciable increase in the creep rate, decrease in the fracture time and fracture strain. It is suggested that the maximum uniaxial stress $\sigma^0 + \sigma^{\mathrm{a}}$ is moderate, lying below the yield limit. In the following it will be assumed that the level of loading and the frequency of its cyclic component correspond to the conditions which are necessary for the realization of the phenomenon of dynamic creep in the arbitrary point of the volume.

Under the assumption of small strains and displacements the kinematical equations are

$$\varepsilon_{ij} = \varepsilon_{ij}^{\mathrm{el}} + c_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}) \tag{1}$$

where $\varepsilon_{ij}$, $\varepsilon_{ij}^{\mathrm{el}}$ and $c_{ij}$ denote the components of the strain tensor and its elastic and creep parts respectively, $u_i$ denotes the components of the displacement vector and $(\ldots)_{,j}$ denotes the partial derivative with respect to the coordinate $x_j$. The equilibrium equations under the restriction of negligible volume forces can be written as

$$\sigma_{ij,j} = \rho \ddot{u}_i \tag{2}$$

where $\sigma_{ij}$ are the components of the stress tensor and $\rho$ is the density. The dot means the derivative with respect to the time $t$. Constitutive equations follow from the generalized Hooke's law:

$$\sigma_{ij} = E_{ijkl}(\varepsilon_{kl} - c_{kl}) \tag{3}$$

with $E_{ijkl}$ the components of the tensor of elastic material constants. Appropriate boundary and initial conditions have to be added to equations (1) to (3):

$$u_i\Big|_{\mathscr{C}_1} = \bar{u}_i, \qquad \sigma_{ij}n_j\Big|_{\mathscr{C}_2} = p_i^0 + p_i^a\Phi(ft)$$

$$u_i(0) = \dot{u}_i(0) = 0, \qquad c_{ij}(0) = 0$$

where $\bar{u}_i$ are given boundary displacements. The constitutive model of creep will be formulated following Rabotnov's [5] theory of structural parameters and assuming that the creep strain rate is not influenced by the hydrostatic stress:

$$\dot{c}_{kl} = \frac{3}{2}\frac{\dot{c}_{\mathrm{vM}}}{\sigma_{\mathrm{vM}}} s_{kl}, \qquad \dot{c}_{\mathrm{vM}} = \dot{c}_{\mathrm{vM}}(c_{\mathrm{vM}}, \sigma_{\mathrm{vM}}, \omega) \tag{4}$$

where

$$s_{kl} = \sigma_{kl} - \sigma_{\mathrm{h}}\delta_{kl}, \qquad \sigma_{\mathrm{h}} = \tfrac{1}{3}\sigma_{ij}\delta_{ij}, \qquad \sigma_{\mathrm{vM}} = \sqrt{\tfrac{3}{2}s_{ij}s_{ij}}$$

are the deviator, the hydrostatic stress and the von Mises equivalent stress. Equations (4) are regarded as independent of the kind of stress state. Assuming power-law creep the constitutive and evolution equations can be written as

$$\dot{c}_{\mathrm{vM}} = B\frac{(\sigma_{\mathrm{vM}})^n}{(1-\omega^r)^m}, \qquad \dot{\omega} = D\frac{(\sigma_{\mathrm{eq}})^k}{(1-\omega^r)^l}$$

$$0 \leqslant \omega \leqslant \omega_* \tag{5}$$

where $\omega$ is the phenomenological damage parameter. The constants $B$ and $D$, the exponents $n$, $m$, $k$, $l$ and $r$ and the critical damage value $\omega_*$ are material dependent. They should be identified from uniaxial creep tests.

Now two main assumptions are presented which allow the use of the method of asymptotic expansions for the solution of equations (1) to (5). The first deals with the possibility of asymptotic representation for the principal unknowns by introducing the small parameter $\mu = T/t_* = 1/(ft_*) \ll 1$. As follows from experimental observations [5, 13], for the fast cyclic loading the global creep process occurs there as close to an averaged process, which can be observed in the usual slow time. The second assumption determines that such asymptotic expansions are admissible by using functions which depend on two time-scales: 'slow' and 'fast'.

The asymptotic expansions of the displacement vector $u_i$, the stress tensor $\sigma_{ij}$ and the strain tensor $\varepsilon_{ij}$ $(i, j = 1, 2, 3)$, which are collected in the vector $z = z\{u_i, \sigma_{ij}, \varepsilon_{ij}\}$, can be represented as follows:

$$z(x_1, x_2, x_3, t, \tau) = z^0(x_1, x_2, x_3, t) + \mu z^1(x_1, x_2, x_3, \tau) + \cdots \tag{6}$$

where the functions of oscillatory motion are varied in the time-scale $\tau = t/\mu$, and $t$ and $\tau$ are considered as independent variables. $t$ denotes the 'slow' time variable and $\tau$ is the 'fast' time variable.

The total derivative of the function $z$ can be presented in the following form:

$$\frac{\mathrm{D}z}{\mathrm{D}t} = \left(\frac{\mathrm{d}z}{\mathrm{d}t} + \mu^{-1}\frac{\mathrm{d}z}{\mathrm{d}\tau}\right)_{\tau=t/\mu} \tag{7}$$

The displacements, the stresses and the strains are functions of the coordinates $x_i$ and the time. It is presumed that, after averaging over the period $T$,

$$\langle z^0(x_1, x_2, x_3, t)\rangle = z^0(x_1, x_2, x_3, t)$$

$$\langle z^1(x_1, x_2, x_3, \tau)\rangle = 0$$

where

$$\langle\cdot\rangle = \frac{1}{T}\int_0^T \langle\cdot\rangle\,\mathrm{d}\tau = \int_0^1 \langle\cdot\rangle\,\mathrm{d}\xi$$

is the operator of the averaging and $\xi = \tau/T$. The functions with index 0 and 1 correspond to coefficients of the expansions (6).

Substituting equation (6) into the general system of equations of the creep problem and after applying to the obtained expressions the procedure of averaging, the equations of the initial-boundary value problem will be formulated. The averaging over time produces two sets of equations. The unknowns of the first set are dependent on the 'slow' time variable $t$ only. The unknowns of the second (remaining) set are dependent on the 'fast' time variable $\tau$. The system of equations that determines the global multiaxial creep process in the 'slow' time-scale has the following form:

$$\sigma^0_{ij,j} = 0, \qquad \varepsilon^0_{ij} = \tfrac{1}{2}(u^0_{i,j} + u^0_{j,i}),$$

$$\sigma^0_{ij} = E_{ijkl}(\varepsilon^0_{kl} - c^0_{kl}) \tag{8}$$

with the boundary conditions

$$u_i^0\Big|_{\mathscr{C}_1} = \bar{u}_i, \qquad \sigma_{ij}^0 n_j\Big|_{\mathscr{C}_2} = p_i^0$$

The equations which can be obtained after using the procedure of asymptotic expansions for 'fast' oscillatory motion with time-scale are:

$$\sigma_{ij,j}^1 = \rho u_{i,\xi\xi}^1, \qquad \varepsilon_{ij}^1 = \tfrac{1}{2}(u_{i,j}^1 + u_{j,i}^1),$$
$$\sigma_{ij}^1 = E_{ijkl}\varepsilon_{kl}^1 \tag{9}$$

with the boundary and initial conditions

$$u_i^1\Big|_{\mathscr{C}_1} = \bar{u}_i^1, \qquad \sigma_{ij}^1 n_j\Big|_{\mathscr{C}_2} = \frac{1}{\mu} p_i^1 \Phi(\xi)$$
$$p_i^1 = p_i^{\rm a}, \qquad u_i^1(0) = u_{i,\xi}^1(0) = 0, \qquad c_{ij}^1 \equiv 0$$

The constitutive equations corresponding to cyclic creep should be added to the system of equations (8).

Assuming that the variation in the creep and damage rates during the small cycle period $T$ is negligible in comparison with their averaged values, the averaging of the constitutive and evolution equation in the uniaxial case yields

$$\frac{{\rm d}c^0}{{\rm d}t} = B\langle H\rangle \frac{(\sigma^0)^n}{[1-(\omega^0)^r]^m}$$
$$\frac{{\rm d}\omega^0}{{\rm d}t} = D\langle K\rangle \frac{(\sigma^0)^k}{[1-(\omega^0)^r]^l} \tag{10}$$

with

$$\langle H\rangle = \int_0^1 \left[1 + \mu\frac{\sigma^1(x,\,\xi)}{\sigma^0(x,\,t)}\right]^n {\rm d}\xi$$
$$\langle K\rangle = \int_0^1 \left[1 + \mu\frac{\sigma^1(x,\,\xi)}{\sigma^0(x,\,t)}\right]^k {\rm d}\xi \tag{11}$$

Here $\langle H\rangle$ and $\langle K\rangle$ are slowly varying functions of time $t$. The asymptotically averaged values of the creep strain $c^0 = \langle c\rangle$ and damage parameter $\omega^0 = \langle\omega\rangle$ vary in the slow time-scale and determine the global creep process.

Consider a function $\Phi(f\tau) = \sin(2\pi f\tau)$, so that $\Phi(\xi) = \sin(2\pi\xi)$. The creep deformation cannot be successfully partitioned within the fast time-scale $\xi$. This makes it possible to assume system (9) for the coefficients of fast asymptotic expansions, corresponding to elastic non-stationary motion under the action of a harmonic oscillatory load $p_i^{\rm a}\sin(2\pi\xi)$ on the surface $\mathscr{C}_2$, where $p_i^{\rm a} = p_i^{\rm a}(x_1,\,x_2,\,x_3)$. Taking into account the periodicity of the surface load, the following separation can be performed:

$$\sigma_{ij}^1 = \frac{1}{\mu}\sigma_{ij}^{\rm a}(x_1,\,x_2,\,x_3)\sin(2\pi\xi)$$
$$u_i^1 = \frac{1}{\mu}u_i^{\rm a}(x_1,\,x_2,\,x_3)\sin(2\pi\xi) \tag{12}$$

These variables are determined by solution of the system of equations (9), which presents the forced oscillations problem. Thus

$$\sigma_{ij,j}^{\rm a} + \rho\Omega^2 u_i^{\rm a} = 0, \qquad \sigma_{ij}^{\rm a} = \tfrac{1}{2}E_{ijkl}(u_{k,l}^{\rm a} + u_{l,k}^{\rm a}) \tag{13}$$

with respect to the boundary conditions

$$u_i^{\rm a}\Big|_{\mathscr{C}_1} = \bar{u}_i^{\rm a}, \qquad \sigma_{ij}^{\rm a} n_j\Big|_{\mathscr{C}_2} = p_i^{\rm a}$$

For the constitutive equations, after averaging,

$$\dot{c}_{kl}^0 = \frac{3}{2}\frac{\dot{c}_{\rm vM}^0}{\sigma_{\rm vM}^0}s_{kl}^0, \qquad \dot{c}_{\rm vM}^0 = \dot{c}_{\rm vM}^0(c_{\rm vM}^0,\,\sigma_{\rm vM}^0,\,\omega) \tag{14}$$

where $c_{kl}$ are the components of the creep strain tensor and

$$s_{kl}^0 = \sigma_{kl}^0 - \sigma_{\rm h}^0\delta_{kl}, \qquad \sigma_{\rm h}^0 = \tfrac{1}{3}\sigma_{ij}^0\delta_{ij}$$
$$\sigma_{\rm vM}^0 = \sqrt{\tfrac{3}{2}s_{ij}^0 s_{ij}^0}$$
$$\dot{c}_{\rm vM}^0 = B\langle H\rangle \frac{(\sigma_{\rm vM}^0)^n}{[1-(\omega^0)^r]^m}$$
$$\langle H\rangle = \frac{1}{2\pi}\int_0^1 [1 + A_{\rm s}\sin(2\pi\xi)]^n\,{\rm d}\xi \tag{15}$$

The coefficient $\langle H\rangle$ can be calculated by using the value of the stress state cycle asymmetry parameter $A_{\rm s} = \sigma_{\rm vM}^{\rm a}/\sigma_{\rm vM}^0$, where $\sigma_{\rm vM}^{\rm a} = \sqrt{\tfrac{3}{2}s_{ij}^{\rm a}s_{ij}^{\rm a}}$ is the von Mises stress, $s_{ij}^{\rm a} = \sigma_{ij}^{\rm a} - \sigma_{\rm h}^{\rm a}\delta_{ij}$ is the deviator, and $\sigma_{\rm h}^{\rm a} = \tfrac{1}{3}\sigma_{ij}^{\rm a}\delta_{ij}$ is the mean stress for the oscillating process.

The equivalent stress in the right-hand side of the damage evolution equation (5) has to correspond to the criterion of the durable strength under complex stress state. In the case of quasi-static loading, the Leckie–Hayhurst [24] criterion is mostly used as a damage equivalent stress

$$\sigma_{\rm eq}^0 = \alpha\sigma_{\rm vM}^0 + (1-\alpha)\sigma_1^0 \tag{16}$$

where $\alpha$ is a scalar value ($0 \leqslant \alpha \leqslant 1$) and $\sigma_1^0$ is the maximum principal stress.

 

Experimental observations of uniaxial creep of metals and alloys under cyclic loading show a significant acceleration of creep and decrease in the time to rupture in comparison with pure static loading. The essential quantitative difference between quasi-static and cyclic damage accumulation behaviour can be observed. The uniaxial dynamic creep can be assumed to be frequency independent but strongly influenced by the uniaxial stress cycle asymmetry parameter $A = \sigma^{\mathrm{a}}/\sigma^{0} < A_{\mathrm{cr}}$. The limiting value of $A_{\mathrm{cr}}$ can be obtained from creep tests by cyclic loading (see, for example, references **[13]** and **[25]**). This value corresponds to two different situations: $A < A_{\mathrm{cr}}$, fracture occurs as the result of the cyclic creep damage process; $A > A_{\mathrm{cr}}$, fracture due to creep–fatigue interaction. Here the discussion is limited to the first case. The frequency range of the considered phenomenon is referred to high-cycle loading, when the cycle period is essentially small and the number of cycles is rather high.

Applying the averaging procedure to the damage evolution equation (5) gives

$$\dot{\omega}^0 = D\langle K\rangle \frac{(\sigma^0_{\mathrm{eq}})^k}{[1-(\omega^0)^r]^l}$$

$$\langle K\rangle = \frac{1}{2\pi}\int_0^1 [1 + A_{\mathrm{eq}}\sin(2\pi\xi)]^k \,\mathrm{d}\xi \tag{17}$$

where $A_{\mathrm{eq}} = \sigma^{\mathrm{a}}_{\mathrm{eq}}/\sigma^{0}_{\mathrm{eq}}$, $\sigma^{\mathrm{a}}_{\mathrm{eq}} = \alpha\sigma^{\mathrm{a}}_{i} + (1-\alpha)\sigma^{\mathrm{a}}_{1}$. Thus the damage rate in the case of rapidly oscillating loading becomes additionally dependent on the value of the equivalent stress state asymmetry parameter $A_{\mathrm{eq}}$. Such a dependence has been established by experiments on cyclically loaded steel and aluminium tubular specimens **[26]**.

## 3 NUMERICAL SOLUTION TECHNIQUE

The equations of the initial-boundary value problem can be summarized as follows: for the equivalent static creep problem (slow process),

$$\varepsilon^0_{ij} = \tfrac{1}{2}(u^0_{i,j} + u^0_{j,i}), \qquad u_i\Big|_{\mathscr{C}_1} = \overline{u}_i$$

$$\sigma^0_{ij,j} = 0, \qquad \sigma_{ij}n_j\Big|_{\mathscr{C}_2} = p^0_i$$

$$\sigma^0_{ij} = E_{ijkl}(\varepsilon^0_{kl} - c^0_{kl}), \qquad \dot{c}^0_{kl} = \frac{3}{2}\frac{\dot{c}^0_{\mathrm{vM}}}{\sigma^0_{\mathrm{vM}}} s^0_{kl}$$

$$s^0_{kl} = \sigma^0_{kl} - \sigma^0_{\mathrm{h}}\delta_{kl}, \qquad \sigma^0_{\mathrm{h}} = \tfrac{1}{3}\sigma^0_{ij}\delta_{ij}$$

$$\sigma^0_{\mathrm{vM}} = \sqrt{\tfrac{3}{2}s^0_{ij}s^0_{ij}}$$

$$\dot{c}^0_{\mathrm{vM}} = B\langle H\rangle \frac{(\sigma^0_{\mathrm{vM}})^n}{[1-(\omega^0)^r]^m}$$

$$\langle H\rangle = \frac{1}{2\pi}\int_0^1 [1 + A_{\mathrm{s}}\sin(2\pi\xi)]^n \,\mathrm{d}\xi$$

$$\dot{\omega}^0 = D\langle K\rangle \frac{(\sigma^0_{\mathrm{eq}})^k}{[1-(\omega^0)^r]^l}$$

$$\langle K\rangle = \frac{1}{2\pi}\int_0^1 [1 + A_{\mathrm{eq}}\sin(2\pi\xi)]^k \,\mathrm{d}\xi$$

$$A_{\mathrm{s}} = \frac{\sigma^{\mathrm{a}}_{\mathrm{vM}}}{\sigma^0_{\mathrm{vM}}}, \qquad \sigma^{\mathrm{a}}_{\mathrm{vM}} = \sqrt{\tfrac{3}{2}s^{\mathrm{a}}_{ij}s^{\mathrm{a}}_{ij}}$$

$$s^{\mathrm{a}}_{ij} = \sigma^{\mathrm{a}}_{ij} - \sigma^{\mathrm{a}}_{\mathrm{h}}\delta_{ij}, \qquad \sigma^{\mathrm{a}}_{\mathrm{h}} = \tfrac{1}{3}\sigma^{\mathrm{a}}_{ij}\delta_{ij}$$

$$A_{\mathrm{eq}} = \frac{\sigma^{\mathrm{a}}_{\mathrm{eq}}}{\sigma^0_{\mathrm{eq}}}, \qquad \sigma^0_{\mathrm{eq}} = \alpha\sigma^0_{\mathrm{vM}} + (1-\alpha)\sigma^0_1$$

$$\sigma^{\mathrm{a}}_{\mathrm{eq}} = \alpha\sigma^{\mathrm{a}}_{\mathrm{vM}} + (1-\alpha)\sigma^{\mathrm{a}}_1$$

$$c_{ij}\Big|_{t=0} = 0, \qquad \omega\Big|_{t=0} = 0$$

and for the problem of forced vibrations (fast process),

$$\sigma^{\mathrm{a}}_{ij,j} + \rho\Omega^2 u^{\mathrm{a}}_i = 0$$

$$\sigma^{\mathrm{a}}_{ij} = \tfrac{1}{2}E_{ijkl}(u^{\mathrm{a}}_{k,l} + u^{\mathrm{a}}_{l,k})$$

$$u^{\mathrm{a}}_i\Big|_{\mathscr{C}_1} = \overline{u}^{\mathrm{a}}_i, \; \sigma^{\mathrm{a}}_{ij}n_j\Big|_{\mathscr{C}_2} = p^{\mathrm{a}}_i$$

$$u^{\mathrm{a}}_i\Big|_{\xi=0} = \dot{u}^{\mathrm{a}}_i\Big|_{\xi=0}$$

Equations describing the static creep problem can be solved in the usual way if the functions $\langle H\rangle$ and $\langle K\rangle$ are known. These functions can be determined by calculating fields of amplitude stresses from the solution of the forced-vibrations problem. Both the problems can be solved using the finite element method **[27]**.

As an example the creep damage behaviour of thin shells of revolution under axisymmetrical loading is considered. The surface of revolution produced by rotation of an arbitrary curve is considered as the middle surface of the shell. The finite element in the form of a truncated conical shell was used in the calculations. This element has been discussed in detail in reference **[27]**. The coordinate shell surface is presented by a set of the truncated cones, which are connected by nodal circles. Both regular and irregular finite element grids have been used for discretization of the shell meridian. As usual, the grid condensation was used in the regions of significant stress gradients.

For both the problem of creep damage and the problem of forced vibrations, the following generalized displacements are used as the principal unknowns:

$$\delta^i = \{u^i, w^i, \theta^i\}^{\mathrm{T}}, \qquad 1 < i < N_i \tag{18}$$

where the radial $w^i$ and the axial $u^i$ displacements along the global coordinate axes and the angles of rotation of the normal vectors, $\theta^i$, are included. $N_i$ is the number of nodes. The displacements which are normal to the shell origin are approximated by third-order polynomials. For the tangential displacements a linear dependence is used. The kinematical equations can be presented in the following form:

$$\epsilon_s^0 = e_s^0 + c_s^0, \qquad \epsilon_\theta^0 = e_\theta^0 + c_\theta^0 \tag{19}$$

where $\epsilon_s^0$ and $\epsilon_\theta^0$ are the total strains, $e_s^0$ and $e_\theta^0$ are the elastic strains and $c_s^0$ and $c_\theta^0$ are the creep strains. $s$ and $\theta$ are the indices which correspond to axial and tangential components respectively. Assuming that the creep strains are known at a fixed time step the generalized Hooke's law takes the form

$$\begin{aligned}\sigma_s^0 =& \frac{E}{1-\nu^2}(\varepsilon_s^0 + \nu\varepsilon_\theta^0) + \frac{Ez}{1-\nu^2}(\chi_s^0 + \nu\chi_\theta^0)\\ &- \frac{E}{1-\nu^2}(c_s^0 + \nu c_\theta^0)\\ \sigma_\theta^0 =& \frac{E}{1-\nu^2}(\varepsilon_\theta^0 + \nu\varepsilon_s^0) + \frac{Ez}{1-\nu^2}(\chi_\theta^0 + \nu\chi_s^0)\\ &- \frac{E}{1-\nu^2}(c_\theta^0 + \nu c_s^0)\end{aligned} \tag{20}$$

where $\chi_s^0$ and $\chi_\theta^0$ are the changes in curvature in the $s$ and $\theta$ directions respectively.

Equations (20) can be rewritten in the matrix form

$$\boldsymbol{\sigma}^{\mathrm{e}} = [\mathbf{D}^{\mathrm{e}}]\boldsymbol{\varepsilon} - \boldsymbol{\sigma}^0 \tag{21}$$

where $\boldsymbol{\sigma}^{\mathrm{e}}$ denotes the stress vector and $[\mathbf{D}^{\mathrm{e}}]$ is the Hookean matrix. The vector $\boldsymbol{\sigma}^0$ is fully determined by the creep strains

$$\boldsymbol{\sigma}^0 = [\mathbf{P}_{\mathrm{m}}]c_{\mathrm{m}} \tag{22}$$

The matrix $[\mathbf{P}_{\mathrm{m}}]$ can be obtained from equations (20) and the components of the vector $c_{\mathrm{m}} = \{c_1, c_2, c_3, c_4\}^{\mathrm{T}}$ can be calculated as

$$c_1 = \int_{-h/2}^{h/2} c_s^0\,\mathrm{d}z, \qquad c_2 = \int_{-h/2}^{h/2} c_s^0 z\,\mathrm{d}z$$

$$c_3 = \int_{-h/2}^{h/2} c_\theta^0\,\mathrm{d}z, \qquad c_4 = \int_{-h/2}^{h/2} c_\theta^0 z\,\mathrm{d}z \tag{23}$$

where $h$ is the shell thickness. The cyclic creep laws (14) and (15) with consideration of the constitutive equation for the damage parameter (17) take the form

$$\begin{aligned}\dot{c}_s^0 &= \frac{3}{2}B\langle H\rangle \frac{(\sigma_{\mathrm{vM}}^0)^{n-1}}{[1-(\omega^0)^r]^m}(\sigma_s^0 - \tfrac{1}{2}\sigma_\theta^0)\\ \dot{c}_\theta^0 &= \frac{3}{2}B\langle H\rangle \frac{(\sigma_{\mathrm{vM}}^0)^{n-1}}{[1-(\omega^0)^r]^m}(\sigma_\theta^0 - \tfrac{1}{2}\sigma_s^0)\end{aligned} \tag{24}$$

The system of linear algebraic equations for the principal unknowns has to be solved at each time step and can be written as follows:

$$[\mathbf{K}]\boldsymbol{\delta}^0 = \boldsymbol{F}^0 \tag{25}$$

Here $[\mathbf{K}]$ denotes the global stiffness matrix, $\boldsymbol{\delta}^0$ is the generalized displacement vector and $\boldsymbol{F}^0$ is the vector of nodal forces. The local (element) matrices

$$\begin{aligned}[\mathbf{K}^{\mathrm{e}}] &= \int_{V_{\mathrm{e}}} [\mathbf{B}^{\mathrm{e}}]^{\mathrm{T}}[\mathbf{D}^{\mathrm{e}}][\mathbf{B}^{\mathrm{e}}]\,\mathrm{d}V_{\mathrm{e}}\\ \boldsymbol{F}^{\mathrm{e}} &= \int_{S_{\mathrm{e}}} [\mathbf{N}^{\mathrm{e}}]^{\mathrm{T}}\boldsymbol{p}^{\mathrm{e}}\,\mathrm{d}S_{\mathrm{e}} - \int_{V_{\mathrm{e}}} [\mathbf{B}^{\mathrm{e}}]^{\mathrm{T}}[\mathbf{D}^{\mathrm{e}}]\boldsymbol{c}_{\mathrm{m}}\,\mathrm{d}V_{\mathrm{e}}\end{aligned} \tag{26}$$

are included as submatrices for the discretizing shell model. $[\mathbf{N}]^{\mathrm{e}}$ denotes the matrix of shape functions; $[\mathbf{B}]^{\mathrm{e}}$ is the matrix which realizes the strain–displacement relation in each finite element.

The creep deformation process is addressed by use of a time step algorithm (third-order predictor–corrector method [**28**]). As a result the system (25) has been solved with constant stiffness matrices in every time step. The right-hand side is integrated by the Gauss method for the surface coordinates and the Newton method for the thickness.

The second system of equations (the forced-oscillations problem) is solved by the finite element method as is the first system of equations. By analogy to equation (25) the equations of motion can be presented in a matrix form as follows:

$$[\mathbf{K}]\boldsymbol{\delta}^1 + [\mathbf{M}]\boldsymbol{\delta}^1_{,\xi\xi} = \boldsymbol{F}^1 \tag{27}$$

Here $[\mathbf{M}]$ denotes the mass matrix. The solution of the system (27) in the frequency range which does not include the resonance zone can be found in the following form:

$$\boldsymbol{\delta}^1 = \boldsymbol{\delta}^{\mathrm{a}}\sin(2\pi\xi) \tag{28}$$

With $\delta,^{1}_{\xi\xi} = -\Omega^2 \sin(2\pi\xi)$ the resolving system is

$$([\mathbf{K}] - \Omega^2[\mathbf{M}])\boldsymbol{\delta}^{\mathrm{a}} = \boldsymbol{F}^{\mathrm{a}} \quad (29)$$

where $\boldsymbol{\delta}^{\mathrm{a}}$ and $\boldsymbol{F}^{\mathrm{a}}$ are the vectors of the amplitude values of nodal displacements and forces respectively. The amplitude stress vector $\boldsymbol{\sigma}^{\mathrm{a}} = [\mathbf{D}][\mathbf{B}]\boldsymbol{\delta}^{\mathrm{a}}$ obtained after solution of system (29) is used to determine the values of influence functions $\langle H \rangle$ and $\langle K \rangle$. Further details of the used algorithms have been presented in references **[17]**, **[18]** and **[29]**.

## 4 NUMERICAL RESULTS

### 4.1 Conical shell

The first example is the creep of a thin conical shell simply supported and axisymmetrically loaded by cyclic internal pressure $p = p^0 + p^{\mathrm{a}} \sin(2\pi f \tau)$ uniformly distributed along the shell meridian (Fig. 1a). The shell length is $L = 0.144$ m, the radii of the middle surface are $R_1 = 0.08$ m and $R_2 = 0.12$ m and the shell thickness is $h = 0.001$ m. The static pressure component is assumed to be $p^0 = 3$ MPa and the amplitude of the cyclic load, $p^{\mathrm{a}}$, has been chosen as 0.15 of $p^0$ for different values of the loading frequency $f$. The eigenfrequencies of the shell are calculated first. For the axisymmetrical oscillations the first three eigenfrequencies are $f_1 = 6.45 \times 10^3$ Hz, $f_2 = 6.93 \times 10^3$ Hz and $f_3 = 7.47 \times 10^3$ Hz. The following material constants for the constitutive equations (5) and (17) have been obtained by processing experimental data on tubular specimens made from steel 20H13 at a temperature of 773 K **[30]**: $E = 1.7 \times 10^5$ MPa, $\nu = 0.3$, $B = 3.19 \times 10^{-17}$ MPa$^{-n}$/h, $D = 1.78 \times 10^{-17}$ MPa$^{-n}$/h, $n = k = m = 6.12$, $r = 1$ and $l = 11.513$. The equivalent stress (16) was used in the simulation with $\alpha = 1$.

Figure 2 shows the distributions of the damage parameter obtained after the final time steps for the static and dynamic creep with different frequencies of loading. In the case of dynamic creep with frequencies much less than the first eigenfrequency the calculated damage distribution is approximately the same as for the static case (Fig. 2, curves 1 and 2) but the fracture time in the cyclic case is slightly smaller. With increasing frequency of oscillation the values of the fracture time significantly decrease (Table 1). For frequencies of loading beyond the first eigenfrequency of the shell ($f = 6700$ Hz) the zone of maximum damage moves along the shell meridian (Fig. 2, curve 3). The damage evolution at the points of spatial discretization, where fracture occurs, is illustrated in Fig. 3. It is seen that increasing the loading frequency leads to higher damage rates.

In the case of static pressure, the damage rate, the damage distribution and consequently the fracture time are sensitive to the stress level and stress state expressed by $\sigma^0_{\mathrm{eq}}$ in the damage evolution equation. For cyclic loading conditions the damage rate becomes additionally dependent on the value and distribution of the stress state cycle asymmetry parameter $A_{\mathrm{s}}$. For frequencies of loading below the first eigenfrequency the distribution of $A_{\mathrm{s}}$ is constant in the initial state (Fig. 4, curve 1) because the dynamic stress distribution for the first vibration mode corresponds to the static stress distribution. During the creep process the values of $A_{\mathrm{s}}$ increase due to relaxation of the stress component $\sigma^0_{\mathrm{vM}}$ (Fig. 4, curve 2). For frequencies beyond the first eigenfrequency the distribution of the dynamic stress component $\sigma^{\mathrm{a}}_{\mathrm{vM}}$ corresponds to the second vibration mode, which leads to qualitatively different distributions of $A_{\mathrm{s}}$ and $\omega$ (Fig. 4, curve 3, and Fig. 2, curve 3). The values and distribution of $A_{\mathrm{s}}$, which influence the creep rate and the damage rate in the case of cyclic creep, are strongly sensitive to the ratio of static to cyclic components of the loading and to the loading frequencies.

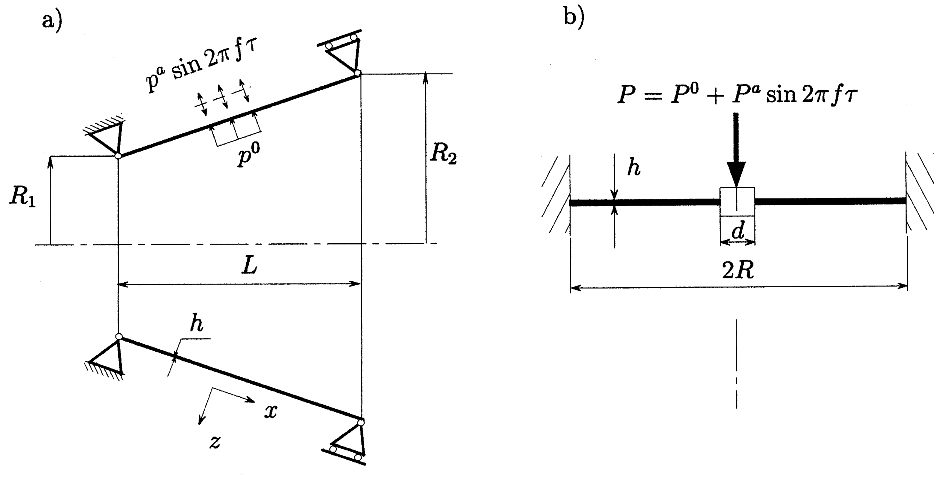


**Fig. 1** Models of the structural members: (a) conical shell; (b) plate with a hole

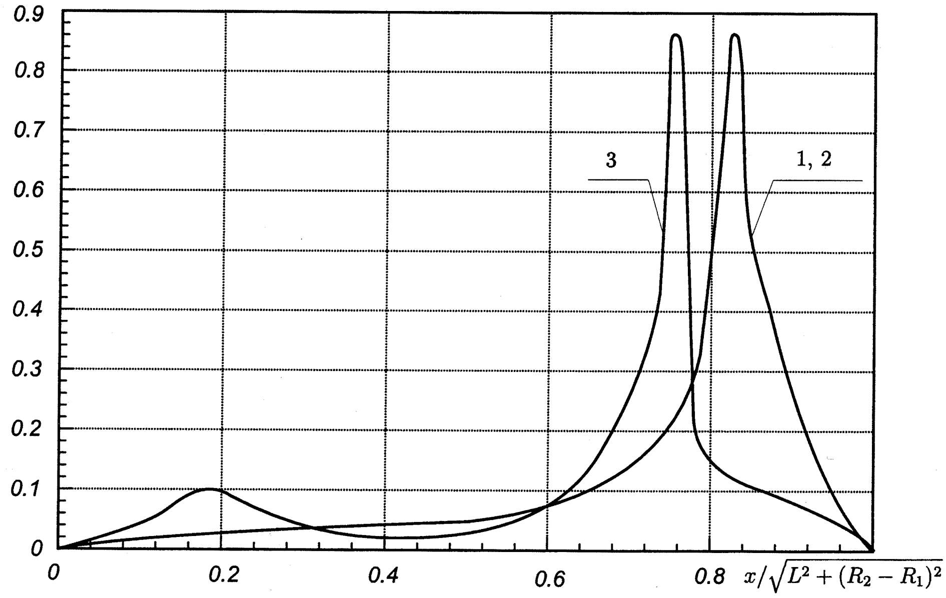


**Fig. 2** Damage parameter versus shell length: curve 1; $t_* = 26.1$ h, $f = 0$ Hz; curve 2, $t_* = 21.8$ h, $f = 25$ Hz; curve 3, $t_* = 15.2$ h, $f = 6700$ Hz

**Table 1** Dependence of fracture time on the frequencies of loading

| $f$ (Hz) | $t_*$ (h) |
|---|---|
| 0 | 26.1 |
| 25 | 21.8 |
| 2000 | 21.2 |
| 3000 | 17.4 |
| 6700 | 15.2 |

### 4.2 Circular plate

The second example is a circular plate, clamped at the external radius (Fig. 1b). For the calculations, $R = 0.05$ m, $h = 0.001$ m and $d = 0.003$ m were given with $R$ as the plate radius, $h$ as the plate thickness and $d$ as the diameter of the central hole. The plate is loaded by a cyclic force on its internal radius in the normal direction. The static force component is assumed to be $P^0 = 100$ N and the amplitude

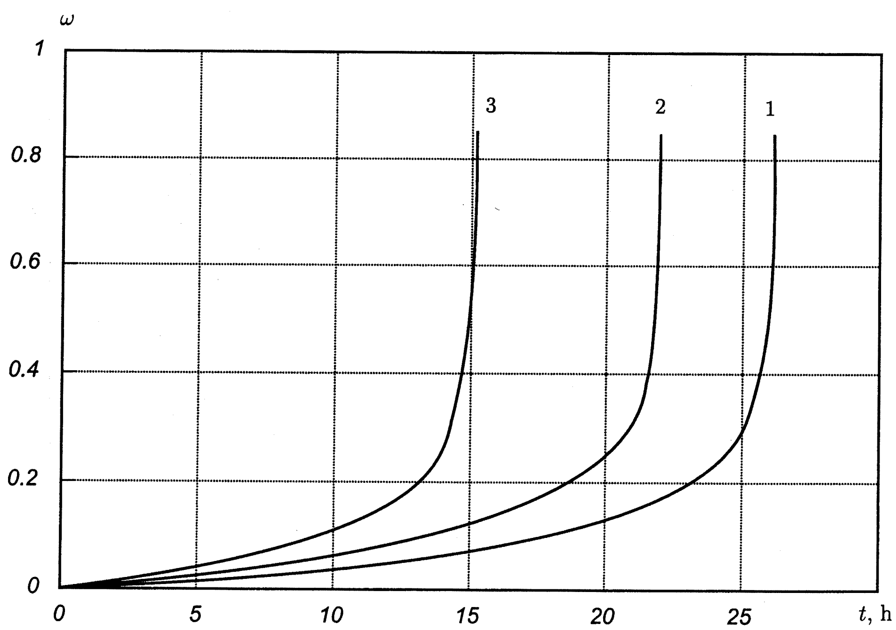


**Fig. 3** Damage parameter versus time: curves 1 to 3 are as in Fig. 2

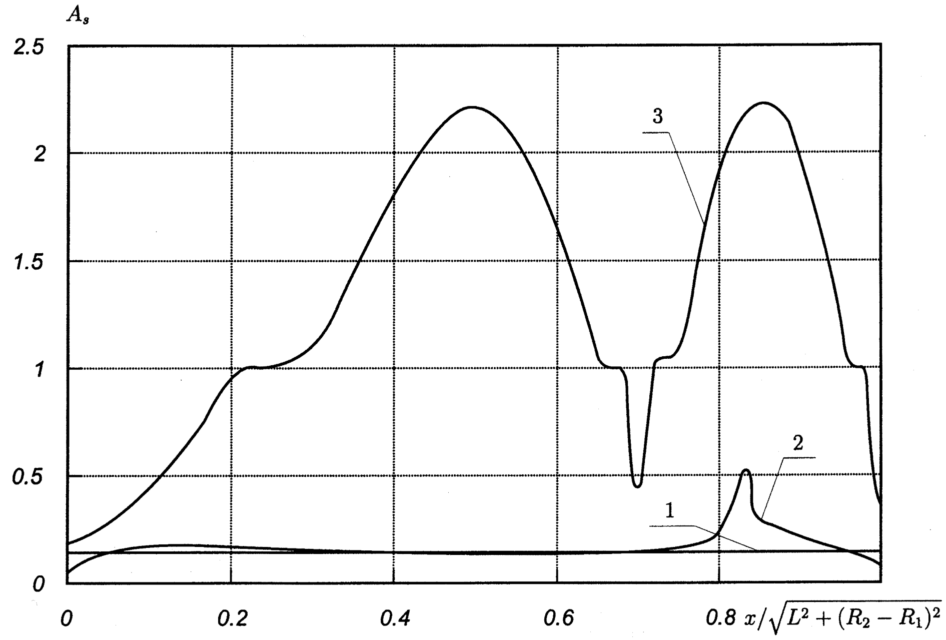


**Fig. 4** Stress state cycle asymmetry parameter versus shell length [outer shell surface ($z = -h/2$)]: curve 1, initial distribution at $t = 0$ h, $f = 25$ Hz; curves 2 and 3 are as in Fig. 2

of cyclic component is $0.2P^0$ for different values of the loading frequency $f$. For axisymmetrical oscillations the first three eigenfrequencies are calculated as $f_1 = 1.732 \times 10^3$ Hz, $f_2 = 4.758 \times 10^3$ Hz and $f_3 = 9.437 \times 10^3$ Hz. The material constants in the constitutive equations (5) and (17) are those for the aluminium alloy $AlCuMg_2$ at 573 K [**23**] as follows: $E = 0.65 \times 10^5$ MPa, $\nu = 0.3$, $B = 0.34 \times 10^{-7}$ MPa$^{-n}$/h, $D = 1.9 \times 10^{-7}$ MPa$^{-n}$/h, $m = n = k = l = 3$, $r = 1.38$ and $\alpha = 1$.

The creep behaviour of the plate in the frequency range 0–3500 Hz has been simulated similarly to the case of the conical shell. According to the numerical results, fracture occurs on the border of the hole due to significant stress concentration. The results of damage accumulation in the neighbourhood of the hole by cyclic loading ($f = 25$ Hz) are presented in Fig. 5. Curves 1, 2 and 3 show the damage growth at different points of the spatial discretization. The fracture time of the plate has been obtained as 3.89 h in the case of pure static loading. In the cyclic cases the fracture time decreases to 3.52 h for the loading frequency

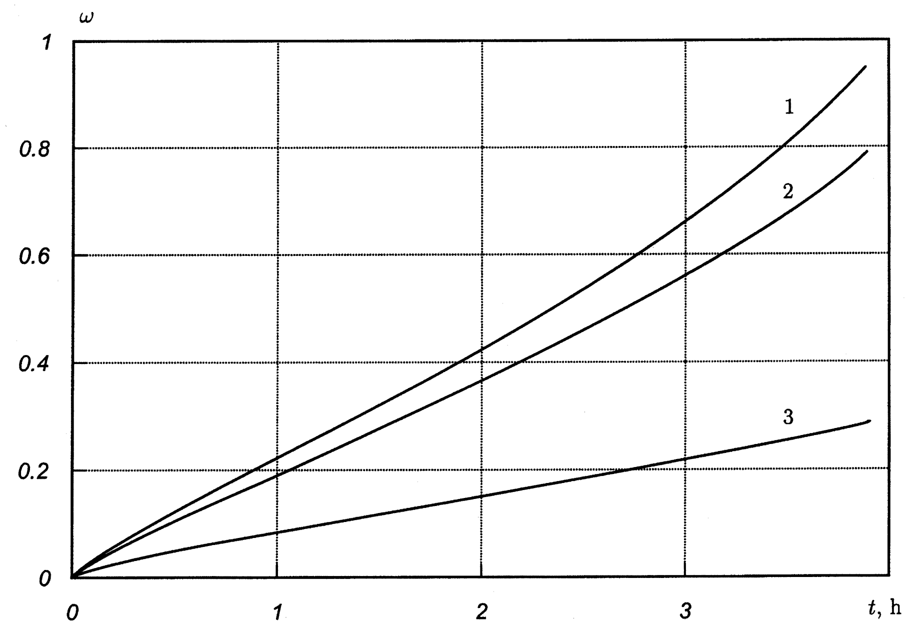


**Fig. 5** Damage parameter versus time at different points in the neighbourhood of the hole: curve 1, $r = 1.5 \times 10^{-3}$ m; curve 2, $r = 1.75 \times 10^{-3}$ m; curve 3, $r = 2 \times 10^{-3}$ m

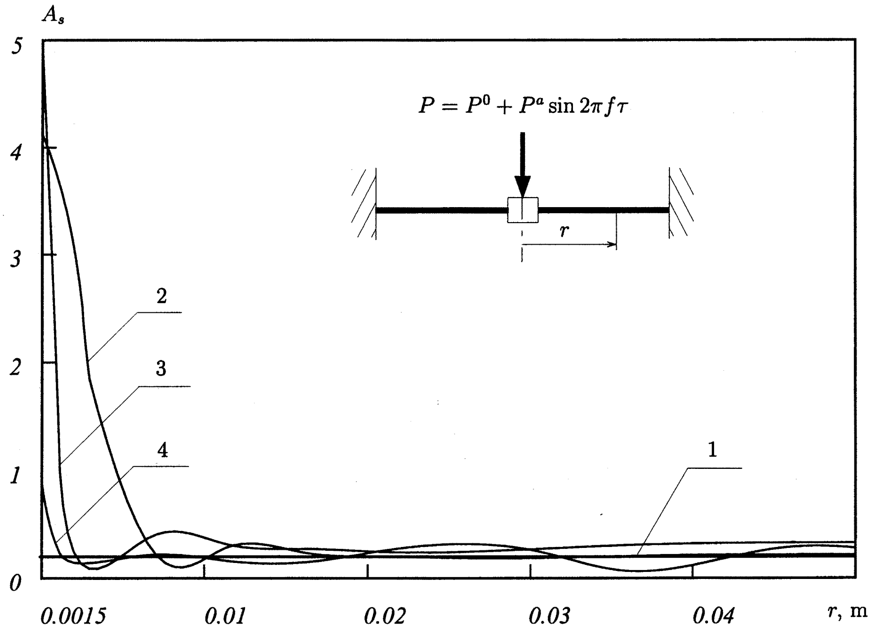


**Fig. 6** Stress state cycle asymmetry parameter versus plate radius: curve 1, initial distribution, $t = 0$ h; curve 2, $t_* = 3.52$ h, $f = 25$ Hz; curve 3, $t_* = 3.17$ h, $f = 500$ Hz; curve 4, $t_* = 3.82$ h, $f = 3500$ Hz

$f = 25$ Hz and to 3.17 h for $f = 500$ Hz. The estimated values of the stress state cycle asymmetry parameter $A_s$ depend on the frequency of forced oscillations. The distribution of $A_s$ along the radius is presented in Fig. 6. In the case when the frequency of the cyclic force component is smaller than the first eigenfrequency, the values of $A_s$ increase up to 4–5; this is particularly observed in the neighbourhood of the hole [Fig. 6, curve 2 ($f = 25$ Hz) and curve 3 ($f = 500$ Hz)]. If the frequency of excitation is greater than $f_1$ [Fig. 6, curve 4 ($f = 3500$ Hz)], the values of $A_s$ are much less than in previous cases, and consequently the time to fracture, $t = 3.82$ h, is close to the static value.

## 5 CONCLUSIONS

The aim of the present paper was the formulation of a numerical concept for the analysis of cyclically loaded thin-walled structures. Under the assumption that the period of the cyclic loading is much smaller than the global period of the whole creep process, the asymptotic expansion method of two time-scales has been applied to the governing mechanical equations of the creep problem. As a result of time averaging, two sets of equations have been obtained. The first corresponds to an 'equivalent' initial-boundary value problem of quasi-static creep and the second to the problem of forced vibrations. Both equation sets are coupled by constitutive and evolution equations of the creep damage process by means of functions of stress state cycle asymmetry parameters. This method leads to the numerical procedure which is similar to that conventionally used for quasi-static creep problems. Based on the numerical examples for the conical shell and circular plate, the results of creep deformation and damage evolution for static and cyclic loading are discussed. The significant influence of the small cyclic component of the external load on fracture time has been illustrated. Further the sensitivity of lifetime predictions in thin-walled structures to the frequencies of the applied cyclic loading is discussed. The investigations are limited to the case of small values of the stress state cycle asymmetry parameter. However, the numerical results obtained for both examples show that the values of $A_s$, which are small in the initial state, can increase during the creep process due to the stress redistribution. If the values of $A_s$ exceed the critical material constant $A_{cr}$, the creep–fatigue damage mechanism must be considered in the material model by means of appropriate internal state variables.

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