Asymptotic Solutions of 1-D Singularly Perturbed Convection-Diffusion Equations with a Turning Point: The Compatible Case

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Abstract

In this article, we consider a convection-diffusion equation with a small diffusion coefficient ϵ. It is a version of a linearized Navier-Stokes equation. Due to the small parameter ϵ multiplied to the highest order of differential operators, the so-called turning point transition layers are displayed where flows in opposite directions collide. For example, a turning point can be observed where the Kuroshio and Kurile Currents meet, from opposite directions, in the North Pacific. Unlike boundary layers, turning point transition layers occur where the convective flows collide and more delicate analysis is necessitated.

Especially, we consider a single turning point with multiple-orders in one-dimensional spaces and provide sharp estimations for the solution with compatible conditions. A main difficulty when solving the problem arises from the fact that the diffusion coefficient is very small in comparison with other terms and it causes a singularity in the solution. We use the asymptotic analysis that is different from typical methods in the singular perturbation problem considered here. The matching technique has been typically used, but this method brings about the difficulty in constructing a globally matched solution. Our method is relatively easy to analyze, and turning point transition layers are systematically and easily constructed.

Contents

1. Introduction

We consider a convection-diffusion equation with the following boundary conditions:

$$\begin{cases} L_\epsilon u^\epsilon = -\epsilon u^\epsilon_{xx} - b u^\epsilon_x = f & \text{in } \Omega = (-1,1) \\ u^\epsilon(-1) = \alpha, \quad u^\epsilon(1) = \beta \end{cases} \tag{1.1}$$

where $0 < \epsilon \ll 1$, $b = b(x)$, $f = f(x)$ are smooth on [-1, 1], and α, β are constants. This type of boundary value problem is considered a one-dimensional linearized Navier-Stokes equation around a constant flow which changes directions at a point, called turning point.

Here, a turning point transition layer is a phenomenon that occurs at the point where the tangential velocity vanishes and changes the sign. Our aim is constructing sharp estimation of the solution when turning point phenomenon arises in the convection-diffusion equation. Particularly, when the viscous effect of a fluid is larger than the inertia effect, flow velocity change within a very narrow region compared with the characteristic length where u or its derivatives become very large. Although these regions shrink to 0 as ϵ goes to 0, they are difficult to analyze. Turbulence phenomenon is one of the most important problems of classical physics. Since turbulence arises from laminar flow via transition as the viscosity is decreased, research on turning points is important in the context of a transition layer.

First, we impose on the convection coefficient b the following conditions:

$$\begin{gathered} b < 0 \quad \text{for } x < 0, \quad b = 0 \quad \text{for } x = 0, \quad b > 0 \quad \text{for } x > 0, \\ b_x(x) \geq c x^{p-1} \quad \forall x \in [-1,1]. \end{gathered} \tag{1.2}$$

where $c \geq \delta > 0$ is a constant. WLOG, we can set $\delta = 1$. Here, we assume that b has a zero with p-order where $p \geq 1$ and p is odd, that is $b(x) = \sum_{k=0}^{\infty} b_{p+k} x^{p+k}$. Note that a turning point with p-order is created at $x = 0$ by the first line of (1.2). Additionally, we write ϵ^{p+1} for ϵ in (1.1) throughout this article for convenience of analysis below. That is, the following is our model problem:

$$\begin{cases} L_\epsilon u^\epsilon = -\epsilon^{p+1} u^\epsilon_{xx} - b u^\epsilon_x = f & \text{in } \Omega = (-1,1) \\ u^\epsilon(-1) = \alpha, \quad u^\epsilon(1) = \beta \end{cases} \tag{1.1b}$$

In this paper, we concentrate on solving a single multiple-order turning point with compatible conditions in a one-dimensional boundary value problem. It is organized as follows.

From Ch. 2 to Ch. 3, we study theoretical background of this problem in detail. First of all, we derive the model equation by linearizing the Navier-Stokes equation. It goes without saying that the Navier-Stokes equation is the most important equation of fluid dynamics. The model equation describes behaviors of fluid in certain physical circumstances. We look into the perturbation theory because it is useful and essential in problems concerning small parameters. Also, we introduce one of classical methodologies of the asymptotic analysis with an example. By studying about this method, the advantages of our method will be revealed well.

From Ch. 4, we consider our model equation. In Ch. 4, we consider the homogeneous problem with inhomogeneous boundary conditions. Our asymptotic analysis for this case gives an asymptotic solution. Then, we can analyze the convergence of the asymptotic solutions and further make estimates in Sobolev spaces. This result can be developed to apply to the inhomogeneous problem with homogeneous boundary conditions in Ch. 5.

Finally, we present future directions of this research.

2. Derivation of the Model Equation

To derive the model equation under consideration in this thesis, we study the Navier-Stokes equation. The model equation is a linearized version of the Navier-Stokes equation, and we justify the model equation from the full Navier-Stokes and Euler equations. The Euler equations are considered the limit problems of the Navier-Stokes equations, i.e., when the viscosity is zero. In chapter 3, we give more details on this viewpoint. In this article, we consider only Newtonian fluid whose stress is linear with respect to strain rate.

2.1 Derivation of the Navier-Stokes Equation

In this section, we derive the Navier-Stokes equation on the rectangular coordinates. As a first stage, we derive the continuity equation from the law of conservation of mass. In fluid mechanical viewpoint, it states that the rate of increase of mass within a fixed volume must equal to the rate of the inflow through the boundaries. That is,

$$\int_V \frac{\partial \rho}{\partial t} dV = -\int_{\partial V} \rho \vec{u} \cdot d\vec{A} \tag{2.1}$$

where ρ is the density, and $\vec{u}$ is the velocity of fluid

Using the divergence theorem, Eq. (2.1) is transformed to:

$$\int_V \left[\frac{\partial \rho}{\partial t} + \nabla \cdot \left(\rho \vec{u} \right) dV \right] = 0. \tag{2.2}$$

Since Eq. (2.2) is valid for an arbitrary volume V, we obtain the continuity equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \left(\rho \vec{u} \right) = 0. \tag{2.3}$$

Next, we consider the motion on infinitesimal fluid element that is of rectangular parallelepiped shape. By Newton's Second Law, the net force on the element must be equal to the mass times acceleration of the element. The latter is equal to the sum of the rate of momentum inflow plus the rate of momentum change in the element. That is,

$$\sum \vec{F} = \int_{\partial V} \rho \vec{u}\left(\vec{u}\cdot\vec{n}\right) dA + \frac{\partial}{\partial t}\int_{V} \rho \vec{u} dV. \tag{2.4}$$

The net force equals the sum of the normal stress, shearing stress, external pressure, and the body forces f, e.g. gravity. For example, at the *i*th - component,

$$\begin{aligned}\sum F_i = &(\tau_{xx}\big|_{x_0+dx} - \tau_{xx}\big|_{x_0})dydz + (\tau_{yx}\big|_{y_0+dy} - \tau_{yx}\big|_{y_0})dxdz + (\tau_{zx}\big|_{z_0+dz} - \tau_{zx}\big|_{z_0})dxdy \\ &-(p\big|_{x_0+dx} - p\big|_{x_0})dydz + f_i dxdydz.\end{aligned} \tag{2.5}$$

Dividing Eq. (2.4) by the infinitesimal volume *dxdydz* and applying Eq. (2.5), we deduce that:

$$\lim_{dx,dy,dz\to 0} \frac{\sum F_i}{dxdydz} = \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} - \frac{\partial p}{\partial x} + f_i. \tag{2.6a}$$

Similarly, for the other components,

$$\begin{aligned}\lim_{dx,dy,dz\to 0} \frac{\sum F_j}{dxdydz} &= \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} - \frac{\partial p}{\partial y} + f_j \\ \lim_{dx,dy,dz\to 0} \frac{\sum F_k}{dxdydz} &= \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} - \frac{\partial p}{\partial z} + f_k.\end{aligned} \tag{2.6b}$$

At the *i*th – component surface of the control volume V_i,

$$\int_{\partial V_i} \rho \vec{u}\left(\vec{u}\cdot\vec{n}\right) dA = (\rho \vec{u} u_i\big|_{x_0+dx} - \rho \vec{u} u_i\big|_{x_0})dydz. \tag{2.7}$$

We then deduce that,

$$\begin{aligned}\lim_{dx,dy,dz\to 0} \frac{\int_{\partial V} \rho \vec{u}\left(\vec{u}\cdot\vec{n}\right) dA}{dxdydz} &= \frac{\partial}{\partial x}\left(\rho \vec{u} u_i\right) + \frac{\partial}{\partial y}\left(\rho \vec{u} u_j\right) + \frac{\partial}{\partial z}\left(\rho \vec{u} u_k\right) \\ &= \vec{u}\nabla\cdot\left(\rho\vec{u}\right) + \rho\left[u_i \frac{\partial}{\partial x}\left(\vec{u}\right) + u_j \frac{\partial}{\partial y}\left(\vec{u}\right) + u_k \frac{\partial}{\partial z}\left(\vec{u}\right)\right].\end{aligned} \tag{2.8a}$$

From (2.3), we find that

$$\lim_{dx,dy,dz\to 0} \frac{\int_{\partial V} \rho \vec{u}\left(\vec{u}\cdot\vec{n}\right) dA}{dxdydz} = -\vec{u}\frac{\partial \rho}{\partial t} + \rho\left[u_i \frac{\partial}{\partial x}\left(\vec{u}\right) + u_j \frac{\partial}{\partial y}\left(\vec{u}\right) + u_k \frac{\partial}{\partial z}\left(\vec{u}\right)\right]. \tag{2.8b}$$

The second term of the right side of (2.4) is found to be

$$\lim_{dx,dy,dz\to 0} \frac{\frac{\partial}{\partial t}\int_V \rho \vec{u} dV}{dxdydz} = \frac{\frac{\partial}{\partial t}\rho \vec{u} dxdydz}{dxdydz} = \rho \frac{\partial}{\partial t}\vec{u} + \vec{u}\frac{\partial}{\partial t}\rho \tag{2.9}$$

From Eq. (2.4), (2.6), (2.8), and (2.9), we obtain the the Navier-Stokes equation:

$$\rho\left(\frac{\partial\vec{u}}{\partial t}+\vec{u}\cdot\nabla\vec{u}\right)=-\nabla p+\nabla\cdot\tau+\vec{f}, \qquad (2.10)$$

where τ is the deviatric stress tensor (τ_{kl}). Here, the left side of (2.10) represents the inertia effect of flow and the term $\nabla\cdot\tau$ does the viscosity effect. Note that (2.10) follows from the laws of mass and momentum conservation.

2.2 The Euler Equation and Asymptotic Behavior

There have been many efforts to find a proper approximation of the Navier-Stokes equation. One of the best approximations is an inviscid flow approximation. When the inviscid condition $\mu=0$, i.e. $\tau_{ij}=0$ as in (2.12) below, is applied to the Navier-Stokes equation, the Euler equation is obtained:

$$\rho\left(\frac{\partial\vec{u}}{\partial t}+\vec{u}\cdot\nabla\vec{u}\right)=-\nabla p+\vec{f}. \qquad (2.11)$$

Historically, the Euler equation is older than the Navier-Stokes equation. An inviscid flow is a flow of which shear stress can be neglected. If velocity gradient is very small, this flow can be also regarded as an inviscid flow. We now consider the following constitutive equation for Newtonian fluid:

$$\tau_{ij}=\mu\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)-\frac{2}{3}\mu(\nabla\cdot\vec{u})\delta_{ij}, \qquad (2.12)$$

in the Cartesian coordinates system, where μ is viscosity coefficient. The Euler equation (2.11) or (2.10) with $\mu=0$ (i.e. $\nabla\cdot\tau=0$) is used a good approximation of the Navier-Stokes equation in many areas in the fluid dynamics.

To understand the convergence of the Euler solutions to the Navier-Stokes solutions, an asymptotic expansion has been used. In this method, the solution is expressed by a power series of a small parameter like viscosity coefficient. By doing this, we can catch the properties of the function, and we can use some finite order of power series up to the required accuracy. This feature is useful in theoretical area as well as in applied area.

However, in our problem, we notice that this methodology is not valid. If the fluid velocity changes steeply in a relatively thin region like boundary layers, this flow has a large deviatric stress tensor even if μ is small. As indicated in (2.12), the Euler equation is then no longer a good approximation of the Navier-Stokes equation in this region. Thus, we need a different method to investigate this problem, called singularly perturbed problem.

2.3 Linearized Navier-Stokes Equation

Here, we deduce our model equation from the Navier-Stokes equation. Firstly, we assume that the model equation satisfies the steady state condition $\frac{\partial \vec{u}}{\partial t} = 0$ and our fluid is incompressible. With these conditions, the Navier-Stokes equation reads

$$\rho \vec{u} \cdot \nabla \vec{u} = -\nabla p + \mu \nabla^2 \vec{u} + \vec{f}, \tag{2.13}$$

where μ is a constant.

Thanks to the incompressibility, i.e. $\nabla \cdot \vec{u} = 0$, $-\nabla p$ is dropped in the weak formulation in the Sobolev space $H_0^1(\Omega)$. The term $\vec{u} \cdot \nabla \vec{u}$ expresses the convective acceleration of the flow, and because this term is non-linear, it is difficult to analyze theoretically and numerically, e.g. if this term is linear, the linear stability theory is available. As an approximation of the non-linear term, we substitute $-b\nabla \vec{u}$ for $\rho \vec{u} \cdot \nabla \vec{u}$, where $b = b(\vec{x})$ is a polynomial, and obtain a linearized equation:

$$-\mu \nabla^2 \vec{u} - b\nabla \vec{u} = \vec{f}. \tag{2.14}$$

Especially, restricting to one-dimensional problem, we obtain our model equation:

$$-\mu \frac{\partial^2 u}{\partial x^2} - b \frac{\partial u}{\partial x} = f, \tag{2.15}$$

where $b = b(x)$.

3. Singular Perturbation Theory

The perturbation theory is to determine the behavior of the solutions $x = x^{\epsilon}$ of some problem depending on a small parameter ϵ as $\epsilon \to 0$. Roughly speaking, its goal is to construct a formal asymptotic solution up to a small error. Once we have constructed such an asymptotic solution, we would like to know whether it converges to exact solution $x = x^{\epsilon}$ as ϵ goes to 0. In case of nonlinear problems, since a small error in nonlinear equations can lead to a big error in the solution, the analysis would be much more involved.

In this chapter, we study singular perturbation theory. Comparing with regularly perturbed problems, we justify our treatment of the singularly perturbed problems. We provide one of representative methods for solving singularly perturbed problems: matched asymptotic expansion method.

3.1 Properties of Singularly Perturbed Problem

Singularly perturbed problems have distinctions with regularly perturbed problems. In regular perturbation problems, the solution's behavior is the same qualitatively whether the parameter ε is zero or non-zero. That is, regular perturbation problems can be solved by the same method used to solve un-perturbed problems. Consider the boundary problem:

$$\begin{cases} -\epsilon u^{\epsilon}_{xx} + u^{\epsilon} = x & \text{in } \Omega = (0,1), \\ u^{\epsilon}(0) = 0, \quad u^{\epsilon}(1) = 1. \end{cases} \tag{3.1}$$

Then, the solution is $u^{\epsilon} = x$ whether $\epsilon = 0$ or not.

On the other hand, singular perturbation problems behave differently from un-perturbation problems. The following example illustrates this situation. Consider the initial value problem:

$$\begin{cases} \epsilon^2 u^{\varepsilon}_{xx} + u^{\varepsilon} = 0 & \text{in } \Omega = (0,1), \\ u^{\varepsilon}(0) = 1, \quad u^{\varepsilon}_{x}(0) = 0. \end{cases} \tag{3.2}$$

Then, the exact solution is $u^{\epsilon} = \cos\frac{x}{\epsilon}$ if $\epsilon \neq 0$, but the solution at $\epsilon = 0$ does not exist.

In the Navier-Stokes equation, the viscosity coefficient is often considered as the perturbation parameter. The solution of singularly perturbed problems often behaves depending on several different length or time scales. Layer problems are a case depending on length. Assume the following situation.

The domain is divided into two sub-domains. On one of these, the solution behaves as regularly perturbed problems do. On the other sub-domain, the solution behaves as singularly perturbed problems do. Then, the latter sub-domain is called a boundary layer, or an interior layer, according to the position of the sub-domain.

To solve singularly perturbed problem, the basic approach is to reduce the target problem to several easier problems and to assemble the results from those featly. Especially, asymptotic analysis expresses a function as a power series of a parameter, then mathematical analysis according to the order of the parameter becomes possible. Such an approach gives obvious intuition to the function, and help to establish the connection with other functions. Moreover, the asymptotic analysis and the physics theory share one of a fundamental property of nature, uncertainty principle, and the power and effect of the asymptotic approach is obvious when we consider active interaction between analytic and numerical methods. Consequently, the asymptotic analysis is a useful tool in perturbation problems.

3.2 Matched Asymptotic Expansion Method

Matched asymptotic analysis is a method for solving problem where the solution depend on rapid variation in a narrow domain. A typical example is layer problems. In order to construct the solution, we derive asymptotic solutions in the each domain where the model equation has different limit behavior. Here, the solution that is similar to regular perturbed solution is called an outer solution, and the corresponding sub-domain is called an outer layer. While the other solution that has singular limiting behavior is called an inner solution, and the corresponding sub-domain is called an inner layer. After all, we form a uniformly valid solution over the entire domain through asymptotic matching of the inner and the outer solution.

Consider the following example:

$$\begin{cases} \epsilon u_{xx} + 2u_x + u = 0 \qquad \text{in } \Omega = (0,1), \\ u(0) = 0, \quad u(1) = 1. \end{cases} \tag{3.5}$$

When $\epsilon = 0$, regular solution $u = Ae^{-\frac{1}{2}x}$. Then, at least one boundary condition is not satisfied, so the solution of this equation should have a boundary layer.

The outer solution is obtained by $\epsilon \to 0$ and the inner solution is obtained by setting $X = x/\epsilon$ and $\epsilon \to 0$. Then,

$$\begin{cases} u_{\text{outer}}(x) = \dfrac{e^{-x/2}}{e^{-1/2}} \\ u_{\text{inner}}(X) = \dfrac{1-e^{-2X}}{e^{-1/2}} \end{cases} \tag{3.6}$$

Note that each solution satisfies only one boundary condition. Here, we can confirm the matching condition:

$$\lim_{x \to 0+} u_{\text{outer}}(x) = \lim_{X \to \infty} u_{\text{inner}}(X). \tag{3.7}$$

Finally, the global solution is obtained by

$$u_{\text{global}} = u_{\text{inner}} + u_{\text{outer}} - u_{\text{overlap}} \tag{3.8}$$

where u_{overlap} has the common asymptotic behavior of the inner and the outer solutions in the matching region. So, $u_{\text{outer}} \sim u_{\text{overlap}}$ and $u_{\text{global}} \sim u_{\text{inner}}$ inside the boundary layer, and $u_{\text{inner}} \sim u_{\text{overlap}}$ and $u_{\text{global}} \sim u_{\text{outer}}$ away from the boundary layer. In this problem, $u_{\text{overlap}} = e^{1/2}$ clearly. Thus, the global solution is

$$u_{\text{global}}(x,\epsilon) = e^{1/2}(e^{-x/2} - e^{-2x/\epsilon}). \tag{3.9}$$

Moreover, matched asymptotic expansion method can be applied to nonlinear problems.

This method is a powerful tool for singularly perturbation problems, but the matching process is not simple in many problems and doesn't provide an appropriate estimate of the error. In our analysis below, we do not use the matching techniques. It is showed that our method is easy to analyze, and gives delicate estimates in Sobolev space.

4. Homogeneous Case with Inhomogeneous Boundary

In this chapter, we investigate the model equation which is homogeneous with inhomogeneous boundary for getting estimates of solution. We study the outer solution and the interior solution. Although the analysis in this chapter is relatively simple, it will be extended to more complex cases in future papers.

4.1 Outer Solution

Suppose $u^{\epsilon} \sim \sum_{j=0}^{\infty} \epsilon^j u_l^j$ in $x < 0$ and $u^{\epsilon} \sim \sum_{j=0}^{\infty} \epsilon^j u_r^j$ in $x > 0$, and substitute these expansions in (1.1b). Then, we obtain the following equation:

$$\begin{cases} -\sum_{j=0}^{\infty} \epsilon^j \left(u_{lxx}^{j-(p+1)} + b u_{lx}^j \right) = f & \text{in } [-1,0), \\ -\sum_{j=0}^{\infty} \epsilon^j \left(u_{rxx}^{j-(p+1)} + b u_{rx}^j \right) = f & \text{in } (0,1]. \end{cases} \tag{4.1}$$

where $u_l^{j-(p+1)} = u_r^{j-(p+1)} = 0$ for $j = 0,1,2,...,p$. Since $f = 0$, by imposing boundary conditions $u_l^0(-1) = \alpha$, $u_r^0(1) = \beta$, and $u_l^j(-1) = u_r^j(1) = 0$ for $j \geq 1$, we find that the outer solutions are $u_l^0 = \alpha,$ $u_r^0 = \beta$ and $u_l^j = 0$, $u_r^j = 0$ for $j \geq 1$.

4.2 Interior Solution

Set $u^{\epsilon} \sim \sum_{j=0}^{\infty} \epsilon^j \theta^j$ with $\theta^j = \theta^j(\bar{x})$, $\bar{x} = x/\epsilon$, $\bar{x} \in (-\infty, \infty)$, and $b(x) = \sum_{j=p}^{\infty} b_j x^j = \sum_{j=p}^{\infty} b_j \epsilon^j \bar{x}^j$, $p \geq 1$. Note that p is the order of zero of $b(x)$. Then, Eq. (1.1b) is represented as follows:

$$-\sum_{j=0}^{\infty} \epsilon^{p+j-1} \theta_{\bar{x}\bar{x}}^j - \sum_{j=p}^{\infty} \sum_{k=0}^{\infty} b_j \bar{x}^j \epsilon^{j+k-1} \theta_{\bar{x}}^k = 0. \tag{4.2}$$

By a suitable substitution of variables,

$$\sum_{j=0}^{\infty} \epsilon^{p+j-1} \left(-\theta_{\bar{x}\bar{x}}^j - \sum_{k=0}^{j} b_{p+j-k} \bar{x}^{p+j-k} \theta_{\bar{x}}^k \right) = 0. \tag{4.3}$$

By identification at each power of ϵ, we find that:

$$-\theta_{\bar{x}\bar{x}}^j - b_p \bar{x}^p \theta_{\bar{x}}^j = \sum_{k=0}^{j-1} b_{p+j-k} \bar{x}^{p+j-k} \theta_{\bar{x}}^k \quad \forall j \geq 0. \tag{4.4}$$

Next, we impose the boundary conditions: $\theta^0(x=-1) = \alpha$, $\theta^0(x=1) = \beta$, and $\theta^j(x=-1) = 0$, $\theta^j(x=1) = 0$ for $j \geq 1$. But, these interior solutions are difficult to handle because the form of the solutions is not simple which will appear below. As a result, we introduce the approximate form $\overline{\theta}^j$ of θ^j satisfying Eq (4.4) with the boundary conditions:

$$\begin{cases} \overline{\theta}^0 \to \alpha \quad as\ \overline{x} \to -\infty, \\ \overline{\theta}^0 \to \beta \quad as\ \ \overline{x} \to \infty, \\ \overline{\theta}^j \to 0 \quad as\ \overline{x} \to \pm\infty,\ j \ge 1. \end{cases} \tag{4.5}$$

From (4.4), we obtain the explicit solutions for $\overline{\theta}^0, \overline{\theta}^1$:

$$\overline{\theta}^0(\overline{x}) = c_0^{-1}\Big[\alpha\int_{\overline{x}}^{\infty} e^{-\frac{b_p}{p+1}s^{p+1}}ds + \beta\int_{-\infty}^{\overline{x}} e^{-\frac{b_p}{p+1}s^{p+1}}ds\Big], \tag{4.6}$$

$$\overline{\theta}^1(\overline{x}) = c_0^{-1}(\alpha-\beta)b_{p+1}\frac{1}{p+2}\int_{-\infty}^{\overline{x}} s^{p+2}e^{-\frac{b_p}{p+1}s^{p+1}}ds \tag{4.7}$$

where $c_0 = \int_{-\infty}^{\infty} exp(-b_p s^{p+1}/(p+1))ds$.

Now, we claim that

$$\overline{\theta}_{\overline{x}}^j = P_{(p+2)j}(\overline{x})exp\Big(-\frac{b_p\overline{x}^{p+1}}{p+1}\Big), \qquad \forall j \ge 0 \tag{4.8}$$

where P_s denotes a polynomial in $\overline{x}$ of degree s with coefficients independent of ϵ but its expression may be different at different occurrences.

For $j = 0$, (4.8) follows from (4.6). Assume that (4.8) is valid for $0 \le j \le n$. For $j = n+1$, applying Eq. (4.8) to Eq. (4.4), we find that

$$\begin{aligned} -\Big\{\overline{\theta}_{\overline{x}}^{n+1}e^{\frac{b_p}{p+1}\overline{x}^{p+1}}\Big\}_{\overline{x}} &= \Big\{\sum_{k=0}^{n} b_{(p+1)+n-k}\overline{x}^{(p+1)+n-k}\overline{\theta}_{\overline{x}}^k\Big\}e^{\frac{b_p}{p+1}\overline{x}^{p+1}} \\ &= \Big\{\sum_{k=0}^{n} b_{(p+1)+n-k}\overline{x}^{(p+1)+n-k}P_{(p+2)k}(\overline{x})\Big\} \\ &= P_{(p+2)n+(p+1)}(\overline{x}). \end{aligned} \tag{4.9}$$

With a suitable constant C_{n+1},

$$\overline{\theta}_{\overline{x}}^{n+1} = \Big(P_{(p+2)(n+1)}(\overline{x}) + C_{n+1}\Big)e^{-\frac{b_p}{p+1}\overline{x}^{p+1}}. \tag{4.10}$$

Hence, we proved (4.8).

Integrating (4.8) with suitable constants D_j, we find that

$$\begin{aligned} \overline{\theta}^j &= \int_{-\infty}^{\overline{x}} \overline{\theta}_{\overline{x}}^j(s)ds + D_j \\ &= \int_{-\infty}^{\overline{x}} P_{(p+2)j}(s)e^{-\frac{b_p}{p+1}s^{p+1}}ds + C_j\int_{-\infty}^{\overline{x}} e^{-\frac{b_p}{p+1}s^{p+1}}ds + D_j. \end{aligned} \tag{4.11}$$

From the boundary condition (4.5), $D_j = 0$ and $C_j c_0 = -\int_{-\infty}^{\infty} P_{(p+2)j}(s) e^{-\frac{b_p}{p+1}s^{p+1}} ds$. Thus, C_j is independent of ϵ and $(P_{(p+2)j}(\overline{x}) + C_j)$ is too.

The explicit solutions (4.11) are used to estimate $\overline{\theta}^j$ in the following section.

4.3 Exponentially Small Error Estimate

Here, exponentially small error means that

$$\| \theta^j - \overline{\theta}^j \|_{H^m(\Omega)} \leq \kappa_{jm} e^{-c/\epsilon}, \quad \forall j, m \geq 0, \tag{4.12}$$

where κ_{jm} and c are constants. In this paper, κ_{jm} are constants depending on j, m but its value is different at different occurrences.

Firstly, we claim the following useful pointwise estimations:

$$\left| \frac{d^m \theta^j}{dx^m} \right| \leq \kappa_{jm} \begin{cases} 1 & \text{for } j = 0 \text{ and } m = 0, \\ \epsilon^{-m} exp\left(-c \frac{|x|}{\epsilon}\right) & \text{for } j \geq 1 \text{ or } m \geq 1, \end{cases} \tag{4.13}$$

and, for $\sigma \in [0,1)$,

$$| \theta^j |_{H^m((-1,-\sigma)\cup(\sigma,1))} \leq \kappa_{jm} \begin{cases} 1 & \text{for } j = 0 \text{ and } m = 0, \\ \epsilon^{-m+1/2} exp\left(-c \frac{\sigma}{\epsilon}\right) & \text{for } j \geq 1 \text{ or } m \geq 1. \end{cases} \tag{4.14}$$

To prove it, differentiate Eq. (4.8), then, for $m \geq 1$,

$$\frac{d^m \overline{\theta}^j}{d\overline{x}^m} = P_{(p+2)j+(m-1)p}(\overline{x}) exp\left(-\frac{b_p \overline{x}^{p+1}}{p+1}\right). \tag{4.15}$$

There is a constant c such that

$$exp\left(-\frac{b_p \overline{x}^{p+1}}{p+1}\right) \leq \kappa(c) exp(-2c |\overline{x}|) \tag{4.16}$$

and

$$| P_{(p+2)j+(m-1)p}(\overline{x}) | \leq \kappa_{jm}(c) exp(c |\overline{x}|). \tag{4.17}$$

So, for $j \geq 0$ and $m \geq 1$,

$$\left| \frac{d^m \overline{\theta}^j}{d\overline{x}^m}(\overline{x}) \right| \leq \kappa_{jm}(c) exp(-c |\overline{x}|), \quad \forall \overline{x} \in (-\infty, \infty). \tag{4.18}$$

Assume (4.12) to obtain equations of θ^j from those of $\overline{\theta}^j$. For $j \geq 1$ and $m = 0$, the estimate (4.13) is clear from Eq. (4.6), Eq. (4.11), and Eq. (4.17). Thus, the estimate (4.13) is proved. Meanwhile, the estimate (4.14) is achieved from (4.13) directly. This claim is used later in this paper.

Next, we explore our main issue of this section. From Eq. (4.11),

$$\bar{\theta}^j(\bar{x}) = \int_{-\infty}^{\bar{x}} P_{(p+2)j}(s) e^{-\frac{b_p}{p+1}s^{p+1}} ds + \bar{\theta}^j(-\infty), \text{ or}$$
$$\bar{\theta}^j(\bar{x}) = \int_{\bar{x}}^{\infty} P_{(p+2)j}(s) e^{-\frac{b_p}{p+1}s^{p+1}} ds + \bar{\theta}^j(\infty). \tag{4.19}$$

Then, by (4.16) and (4.17), we obtain that:

$$\begin{aligned} |(\theta^j - \bar{\theta}^j)(x=-1)| &= |(\bar{\theta}^j(-\infty) - \bar{\theta}^j(-1/\epsilon)| \\ &\le \int_{-\infty}^{-1/\epsilon} |P_{(p+2)j}(s)| e^{-b_p s^{p+1}/(p+1)} ds \\ &\le \kappa_j e^{-c/\epsilon}, \end{aligned} \tag{4.20}$$

and $|(\theta^j - \bar{\theta}^j)(x=1)| \le \kappa_j e^{-c/\epsilon}$. Then, $\| \delta^j \|_{H^m(\Omega)} \le \kappa_{jm} e^{-c/\epsilon}$ is equivalent to (4.12) with the setting:

$$\delta^j(x) = \theta^j - \bar{\theta}^j - [(\theta^j - \bar{\theta}^j)_{x=-1}]\frac{1+x}{2} - [(\theta^j - \bar{\theta}^j)_{x=1}]\frac{1-x}{2}. \tag{4.21}$$

Now, apply Eq. (4.4) to δ^j and multiply by ϵ^{p-1}, then

$$-\epsilon^{p+1}\delta^j_{xx} - b_p x^p \delta^j_x = \sum_{k=0}^{j-1} \epsilon^{-(j-k)} x^{p+j-k} \delta^k_x + \tilde{\delta}^j, \tag{4.22}$$

where $\tilde{\delta}^j = -\frac{1}{2}\sum_{k=0}^{j}\Big[b_{j-k+p} x^{j-k+p} \epsilon^{-(j-k)} \{(\theta^k - \bar{\theta}^k)_{x=-1} - (\theta^k - \bar{\theta}^k)_{x=1}\}\Big], \quad \forall\ j \ge 0.$

Since $\delta^j(-1) = \delta^j(1) = 0$ and $\| \delta^j \|_{H^m(\Omega)} \le \kappa_{jm} e^{-c/\epsilon}$, Lemma 4.1 to δ^j gives recursively that:

$$\| \delta^j \|_{H^m(\Omega)} \le \kappa_{jm} P(\epsilon^{-1}) e^{-c/\epsilon} \le \kappa_{jm} e^{-c/(2\epsilon)}. \tag{4.23}$$

Thus, the difference of θ^j and $\bar{\theta}^j$ is exponentially small.

4.4 Error Analysis

Set $w_{\epsilon n} = u^\epsilon - \theta_{\epsilon n}$, where $\theta_{\epsilon n} = \sum_{j=0}^{n} \epsilon^j \theta^j$. By multiplying (4.4) by ϵ^j and summing over $j = 0, 1, \cdots, n,$

$$-\sum_{j=0}^{n}\Big(\epsilon^{j+2}\theta^j_{xx} + \sum_{k=0}^{j} b_{p+j-k} x^{p+j-k} \epsilon^{-(p-k)+1} \theta^k_x\Big) = 0. \tag{4.24}$$

And,

$$
\begin{aligned}
L_\epsilon \theta_{\epsilon n} &= \sum_{j=0}^{n} \epsilon^j (L_\epsilon \theta^j) \\
&= -\sum_{j=0}^{n} \left(\epsilon^{p+j+1} \theta_{xx}^j + b \epsilon^j \theta_x^j \right) \\
&= -\sum_{j=0}^{n} \left(b \epsilon^j \theta_x^j - \sum_{k=0}^{j} b_{p+j-k} x^{p+j-k} \epsilon^k \theta_x^k \right)
\end{aligned} \tag{4.25}
$$

by replacing $\epsilon^{p+j+1}\theta_{xx}^j$ to $\epsilon^{p-1} \times (4.24)$.

By permuting the summations, we find that:

$$
L_\epsilon \theta_{\epsilon n} = -\sum_{j=0}^{n} \epsilon^j \theta_x^j \left(b - \sum_{k=0}^{n-j} b_{p+k} x^{p+k} \right) \equiv -\sum_{j=0}^{n} \epsilon^j \theta_x^j R^{j,n}(b) \equiv -R_{p,1}^n. \tag{4.26}
$$

Then, $L_\epsilon w_{\epsilon n} = R_{p,1}^n$ in Ω, and $w_{\epsilon n}(-1) = w_{\epsilon n}(1) = 0$.

Now, estimate the L^2-norm of $R_{p,1}^n$ as follows. First, by Taylor expansion,

$$
\begin{aligned}
| R^{j,n}(b) | &= \left| b - \sum_{k=0}^{n-j} b_{p+k} x^{p+k} \right| \\
&\le \kappa_n | x |^{n+(p+1)-j} \\
&= \kappa_n \epsilon^{n+(p+1)-j} | \bar{x} |^{n+(p+1)-j} .
\end{aligned} \tag{4.27}
$$

From (4.13), we find that

$$
| x^{-\frac{p-1}{2}} R_{p,1}^n | \le \kappa_n \epsilon^{n+\frac{p+3}{2}} \sum_{j=0}^{n} | \bar{x} |^{n+\frac{p+3}{2}-j} | \theta_x^j | \le \kappa_n \epsilon^{n+\frac{p+1}{2}} exp\left(-c \frac{|x|}{2\epsilon} \right), \tag{4.28}
$$

thus

$$
| x^{-\frac{p-1}{2}} R_{p,1}^n |_{L^2(\Omega)} \le \kappa_n \epsilon^{n+\frac{p}{2}+1}. \tag{4.29}
$$

Then, we can use the following Lemma:

Lemma 4.1 The following regularity results and a priori estimates of the solutions $u = u^\epsilon$ of (1.1b) with boundary conditions $\alpha = \beta = 0$ hold: if $fx^{-(\frac{p-1}{2})} \in H^{m-2}(\Omega),\ m \ge 2$, then $u \in H^m(\Omega)$, and

$$
\| u \|_{L^2(\Omega, x^{p-1}dx)} \le \kappa | fx^{-(\frac{p-1}{2})} |_{L^2(\Omega)}, \quad \| u \|_{H^1(\Omega)} \le \kappa \epsilon^{-\frac{p+1}{2}} | fx^{-(\frac{p-1}{2})} |_{L^2(\Omega)}, \tag{4.30a}
$$

$$
\| u \|_{H^m(\Omega)} \le \kappa_m \epsilon^{-(p+1)(m-\frac{1}{2})} | fx^{-(\frac{p-1}{2})} |_{L^2(\Omega)} + \kappa_m \sum_{j=2}^{m-1} \epsilon^{-(p+1)(j-1)} | D^{m-j} (fx^{-(\frac{p-1}{2})}) |_{H^{m-j}(\Omega)}, \tag{4.30b}
$$

where $\| u \|_{L^2(\Omega, x^{p-1}dx)} = (\int_\Omega x^{p-1} u^2 dx)^{1/2}$ is x^{p-1}-weighted L^2-norm.

PROOF For $v, w \in H_0^1(\Omega)$, defining the bilinear form:

$$a_\epsilon(v,w) = \epsilon^{p+1}(Dv, Dw)_{L^2(\Omega)} - (bv_x, w)_{L^2(\Omega)} \tag{4.31}$$

we consider the weak formulation of (1.1b):

$$a_\epsilon(u,w) = F(w), \qquad F(w) = (f, w)_{L^2(\Omega)}. \tag{4.32}$$

For $u, w \in H_0^1(\Omega)$, thank to the Poincare inequality, we then find that the bilinear form is coercive,

$$\begin{aligned} a_\epsilon(u,u) &= \epsilon^{p+1} | u |^2_{H^1(\Omega)} + \int_{-1}^{1} (\frac{b_x}{2}) u^2 dx \\ &\geq \epsilon^{p+1} | u |^2_{H^1(\Omega)} + \int_{-1}^{1} (\frac{c}{2})(x^{\frac{p-1}{2}} u)^2 dx \\ &\geq \kappa \epsilon^{p+1} \| u \|^2_{H^1(\Omega)}, \end{aligned} \tag{4.33}$$

where $\kappa > 0$ is a constant, and the bilinear form is bounded,

$$a_\epsilon(u,w) \leq \epsilon^{p+1} \| u \|_{H^1(\Omega)} \| w \|_{H^1(\Omega)} + \max_{x \in \Omega} |b(x)| \| u \|_{H^1(\Omega)} \| w \|_{H^1(\Omega)} \tag{4.34}$$

by the Holder inequality. Invoking the Lax-Milgram theorem, there is a unique function $u \in H_0^1(\Omega)$ satisfying (4.32).

Since $b_x \gg cx^{p-1}$ and p is odd, (4.33) leads

$$\begin{aligned} \epsilon^{p+1} | Du |^2_{L^2(\Omega)} + \frac{1}{2} | x^{\frac{p-1}{2}} u |^2_{L^2(\Omega)} &\leq \epsilon^{p+1} | Du |^2_{L^2(\Omega)} + \int_{-1}^{1} (\frac{c}{2})(x^{\frac{p-1}{2}} u)^2 dx \\ &\leq a_\epsilon(u,u) = (f, u)_{L^2(\Omega)} \\ &\leq | fx^{-(\frac{p-1}{2})} |_{L^2(\Omega)} | x^{\frac{p-1}{2}} u |_{L^2(\Omega)} \\ &\leq | fx^{-(\frac{p-1}{2})} |^2_{L^2(\Omega)} + \frac{1}{4} | x^{\frac{p-1}{2}} u |^2_{L^2(\Omega)} . \end{aligned} \tag{4.35}$$

Here, Young's inequality is used in the last inequality. Then, from (4.35),

$$| x^{\frac{p-1}{2}} u |^2_{L^2(\Omega)} \leq \kappa | fx^{-(\frac{p-1}{2})} |_{L^2(\Omega)}, \tag{4.36}$$

and (4.30a) is derived, and the H^2-estimate is directly derived from (1.1b). By differentiating (1.1b), we inductively find the higher estimates H^m, $m \geq 3$.

Finally, we attain the following theorem from the achieved results.

Theorem 4.1 Let u^ϵ be the solution of (1.1b) with $f=0$. Then, for $m,n\geq 0$, there is a constant κ_n independent of ϵ such that:

$$\begin{aligned} &\| u^\epsilon - \theta_{\epsilon n} \|_{L^2(\Omega, x^{p-1}dx)} \leq \kappa_n \epsilon^{n+p+1}, \\ &\| u^\epsilon - \theta_{\epsilon n} \|_{H^m(\Omega)} \leq \kappa_n \epsilon^{(n+\frac{1}{2})-(p+1)(m-1)} \quad \text{for } m=1,2, \end{aligned} \tag{4.37}$$

where $\theta_{\epsilon n} = \sum_{j=0}^{n} \epsilon^j \theta^j$.

Note that, for $m \geq 3$, estimation such as (4.37) can be successively given by Lemma 4.1.

5. Inhomogeneous Case with Homogeneous Boundary

In this chapter, we consider the model problem (1.1b) with $\alpha=\beta=0$ and arbitrary f. Roughly speaking, the methodology is similar to the above chapter, but we need an additional condition unlike the before case. Since $f\neq 0$ and $b(0)=0$, the limit problem $-bu_x^0 = f$ has inconsistency at $x=0$ if $f(0)\neq 0$. To avoid such problem, we introduce the following condition:

$$\frac{d^i f}{dx^i}(0)=0, \quad i=0,1,\cdots,N. \tag{5.1}$$

By setting a suitable integer N, the solution will be smooth at $x=0$, so this condition called the compatible condition.

5.1 Outer Solution

Since $\alpha=\beta=0$, set $u_l^j(-1)=u_r^j(1)=0$ for $j\geq 0$. Then, from (4.1), we find that

$$u_r^j = u_l^j = 0 \quad \forall j \neq (p+1)k,\ k=0,1,2,\cdots, \tag{5.2a}$$

and

$$u_l^0 = -\int_{-1}^{x} b(s)^{-1} f(s)ds, \quad u_r^0 = -\int_{x}^{1} b(s)^{-1} f(s)ds, \tag{5.2b}$$

and, for $k=1,2,\cdots$,

$$u_l^{(p+1)k} = -\int_{-1}^{x} b(s)^{-1} u_{lxx}^{(p+1)(k-1)} ds, \quad u_r^{(p+1)k} = -\int_{x}^{1} b(s)^{-1} u_{rxx}^{(p+1)(k-1)} ds. \tag{5.3}$$

The following section gives the necessity of the compatible condition (5.1). By assuming a suitable N in (5.1), the outer solutions and their derivatives at $x=0^+$ or $x=0^-$ will be finite.

5.2 Role of Compatible Condition

Let $m \ge 1,$ and $k \ge 0$, and set $u_{lxx}^{-(p+1)} = f$. First, we claim that, for $m \ge 1,\ x \in [-1,0),$

$$|\frac{d^m u_l^{(p+1)k}}{dx^m}(x)| \le \kappa_m \sum_{r=0}^{m-1} |x|^{-p+r-m+1} |\frac{d^{r+2} u_l^{(p+1)(k-1)}}{dx^{r+2}}(x)|. \tag{5.4}$$

Note that, from (4.1),

$$-bu_{lx}^{(p+1)k}(x) = u_{lxx}^{(p+1)(k-1)}(x), \tag{5.5}$$

And, by differentiating (5.5) s-times in x, we observe that

$$-b\frac{d^{s+1} u_l^{(p+1)k}}{dx^{s+1}} = \sum_{r=1}^{s} \binom{s}{r} \frac{d^r b}{dx^r} \frac{d^{s-r+1} u_l^{(p+1)k}}{dx^{s-r+1}} + \frac{d^{s+2} u_l^{(p+1)(k-1)}}{dx^{s+2}}. \tag{5.6}$$

Since $\left|\frac{d^r b}{dx^r}\right| \le \kappa \min\{x^{p-r},1\}$ and $|b|^{-1} = \frac{x^p}{|b|} x^{-p} = \frac{px^{p-1}}{|b_x|} x^{-p} \le \frac{px^{p-1}}{cx^{p-1}} x^{-p} \le \kappa x^{-p}$, (5.4) for $m=1$ is followed from (5.5).

Assume that (5.4) holds for $m \le s$. Then,

$$\begin{aligned}
|\frac{d^{s+1} u_l^{(p+1)k}}{dx^{s+1}}| &\le \kappa_s |b|^{-1} \Big\{\sum_{r=1}^{s} \min\{|x|^{p-r},1\} |\frac{d^{s-r+1} u_l^{(p+1)k}}{dx^{s-r+1}}| + |\frac{d^{s+2} u_l^{(p+1)(k-1)}}{dx^{s+2}}|\Big\} \\
&\le \kappa_s |x|^{-p} \Big\{\sum_{r=1}^{s} \min\{|x|^{p-r},1\} (\sum_{l=0}^{s-r} |x|^{-p+l-(s-r+1)+1} |\frac{d^{l+2} u_l^{(p+1)(k-1)}}{dx^{l+2}}|) + |\frac{d^{s+2} u_l^{(p+1)(k-1)}}{dx^{s+2}}|\Big\} \\
&\le \kappa_s |x|^{-p} \sum_{l=0}^{s} |x|^{l-s} |\frac{d^{l+2} u_l^{(p+1)(k-1)}}{dx^{l+2}}|.
\end{aligned} \tag{5.7}$$

In the last inequality, we use the fact that $\min\{|x|^{l-s}, |x|^{l-s-(p-r)}\} \le |x|^{l-s}$.

Moreover, $|x^{-(\frac{p-1}{2})} \frac{d^m u_l^{(p+1)k}}{dx^m}(x)| \le \kappa_m |x|^{-(\frac{p-1}{2})} \sum_{r=0}^{m-1} |x|^{-p+r-m+1} |\frac{d^{r+2} u_l^{(p+1)(k-1)}}{dx^{r+2}}(x)|.$

Thus, to obtain

$$|x^{-(\frac{p-1}{2})} \frac{d^m u_l^{(p+1)k}}{dx^m}(0^-)| \le \kappa_{k,m}, \tag{5.8}$$

the following conditions are required:

$$\frac{d^s u_l^{(p+1)(k-1)}}{dx^s}(0^-) = 0 \quad \text{for } 2 \le s \le m + \frac{1}{2}(3p-1), \tag{5.9}$$

and

$$|\frac{d^{m+\frac{1}{2}(3p+1)} u_l^{(p+1)(k-1)}}{dx^{m+\frac{1}{2}(3p+1)}}(0^-)| \le \kappa_{k-1,m+\frac{1}{2}(3p+1)}. \tag{5.10}$$

Next, claim that

$$\frac{d^n u_l^{(p+1)k}}{dx^n}(0^-)=0$$
$$\text{if} \quad \frac{d^i f}{dx^i}(0)=0 \quad \text{for} \quad 0 \le i \le n+(k+1)(p+1)-2. \tag{5.11}$$

By using the L'Hospital rule on (5.4) with $m=n$, we find that if

$$\frac{d^i u_l^{(p+1)(k-1)}}{dx^i}(0^-)=0 \quad \text{for} \quad 2 \le i \le n+(p+1), \tag{5.12}$$

then the first equation of (5.11) follows. Successively, (5.12) is derived if

$$\frac{d^i u_l^{(p+1)(k-2)}}{dx^i}(0^-)=0 \quad \text{for} \quad 2 \le i \le n+2(p+1). \tag{5.13}$$

Recursively, we know that (5.12) is derived if

$$\frac{d^{i-2} f}{dx^{i-2}}(0^-)=\frac{d^i u_l^{-(p+1)}}{dx^i}(0^-)=0 \quad \text{for} \quad 2 \le i \le n+(k+1)(p+1). \tag{5.14}$$

Thus, the claim (5.11) holds.

Then, (5.9) is obtained by the compatibility condition (5.1) with $N=m+k(p+1)+\frac{1}{2}(3p-5)$.

And, (5.10) is obtained if

$$\frac{d^s u_l^{(p+1)(k-2)}}{dx^s}(0^-)=0 \quad \text{for} \quad 2 \le s \le m+\frac{1}{2}(5p+1), \tag{5.15}$$

and

$$\left| \frac{d^{m+\frac{1}{2}(5p+3)} u_l^{(p+1)(k-2)}}{dx^{m+\frac{1}{2}(5p+3)}}(0^-) \right| \le \kappa_{k-2,m+\frac{1}{2}(5p+3)}. \tag{5.16}$$

Recursively, (5.10) is derived by the condition (5.1) with $N=m+k(p+1)+\frac{1}{2}(3p-5)$ and the fact that f is smooth.

Finally, we attain the following statement.

If (5.1) holds for $N=m+k(p+1)+\frac{1}{2}(3p-5)$, then

$$\left| x^{-(\frac{p-1}{2})} \frac{d^m u_l^{(p+1)k}}{dx^m}(0^-) \right|, \quad \left| x^{-(\frac{p-1}{2})} \frac{d^m u_r^{(p+1)k}}{dx^m}(0^+) \right| \le \kappa_{km}, \tag{5.17}$$

for some constant $\kappa_{km}>0$.

5.3 Interior Solution

Assume enough compatibility condition, we can guarantee the finiteness of $|u_l^j(0^-)|$, $|u_r^j(0^-)|$, $|u_{lx}^j(0^-)|$, and $|u_{rx}^j(0^-)|$, but $u_l^j(0^-)\neq u_r^j(0^+)$ generally. To resolve this disagreement at $x=0$, we introduce interior solutions $\theta_l^j(\bar{x})$ on $(-\infty,0)$ and $\theta_r^j(\bar{x})$ on $(0,\infty)$, satisfying Eq. (4.4), with the following boundary conditions:

$$\theta_r^j(\bar{x})=u_l^j(0^-),\quad \theta_{r\bar{x}}^j(\bar{x})=\epsilon\theta_{rx}^j(x)=\epsilon u_{lx}^j(0^-)\quad at\ \bar{x}=0, \tag{5.18a}$$

$$\theta_l^j(\bar{x})=u_r^j(0^+),\quad \theta_{l\bar{x}}^j(\bar{x})=\epsilon\theta_{lx}^j(x)=\epsilon u_{rx}^j(0^+)\quad at\ \bar{x}=0. \tag{5.18b}$$

Since these are just initial value problems of ordinary differential equation, we can determine θ_l^j, θ_r^j explicitly. And, for $j=0,1$,

$$\theta_r^0=\epsilon u_{lx}^0(0^-)\int_0^{\bar{x}} exp\Big(-\frac{b_p}{p+1}s^{p+1}\Big)ds+u_l^0(0^-), \tag{5.19a}$$

$$\theta_r^1=-\epsilon u_{lx}^0(0^-)\frac{b_{p+1}}{p+2}\int_0^{\bar{x}} s^{p+2}exp\Big(-\frac{b_p}{p+1}s^{p+1}\Big)ds. \tag{5.19b}$$

And, as $\bar{x}\to\infty$,

$$\theta_r^0\to\epsilon u_{lx}^0(0^-)c_{r,0}+u_l^0(0^-)=:c_{r,\infty}^0(\epsilon), \tag{5.20a}$$

$$\theta_r^1\to-\epsilon u_{lx}^0(0^-)\frac{b_{p+1}}{p+2}c_{r,1}=:c_{r,\infty}^1(\epsilon), \tag{5.20b}$$

where $c_{r,0}=\int_0^\infty exp\Big(-\frac{b_p}{p+1}s^{p+1}\Big)ds,\quad c_{r,1}=\int_0^\infty s^{p+2}exp\Big(-\frac{b_p}{p+1}s^{p+1}\Big)ds$.

Now, we derive the following pointwise estimations like the previous chapter.

If the compatibility condition (5.1) hold with $N=k(p+1)+\frac{3}{2}(p-1)$, $k\geq 0$, there are constants κ_{jm} and c such that, for $x\in[0,1]$, $0\leq j\leq (p+1)k+p$,

$$\Big|\frac{d^m\theta_r^j}{dx^m}\Big|\leq\kappa_{jm}\begin{cases}1 & \text{for } m=0,\\ \epsilon^{-m+1}exp\Big(-c\frac{|x|}{\epsilon}\Big) & \text{for } m\geq 1.\end{cases} \tag{5.21}$$

Furthermore, for $\sigma\in[0,1)$,

$$|\theta_r^j|_{H^m((\sigma,1))}\leq\kappa_{jm}\begin{cases}1 & \text{for } m=0,\\ \epsilon^{-m+3/2}exp\Big(-c\frac{\sigma}{\epsilon}\Big) & \text{for } m\geq 1.\end{cases} \tag{5.22}$$

In addition, there exist constants $c^j_{r,\infty}(\epsilon)$ with $|c^j_{r,\infty}(\epsilon)|\le\kappa_j$ such that for $j\ge 0$,

$$\theta^j_r(\bar{x})\to c^j_{r,\infty}(\epsilon)\quad\text{as}\quad \bar{x}\to\infty. \tag{5.23}$$

First, note that $|u^j_l(0^-)|, |u^j_r(0^+)|, |u^j_{lx}(0^-)|, |u^j_{rx}(0^+)|\le\kappa_j$ for $0\le j\le(p+1)k+p$ from the statement (5.17). Then, by using Eq. (5.18), (5.19) and mathematical induction, we obtain that:

$$\theta^j_{r\bar{x}}=\epsilon P_{(p+2)j}(\bar{x})exp\Big(-\frac{b_p\bar{x}^{p+1}}{p+1}\Big),\qquad 0\le j\le(p+1)k+p. \tag{5.24}$$

Then, using the same method with the proof of estimation in section 4.3, (5.21) and (5.22) are derived.

Next, since $\theta^j_r(\bar{x})$ is bounded from (5.22), take a sequence $p_n\to\infty$ such that $\theta^j_r(p_n)\to c^j_{r,\infty}(\epsilon)$ for $c^j_{r,\infty}(\epsilon)$. Here, $c^j_{r,\infty}(\epsilon)$ is defined similarly to $c^0_{r,\infty}(\epsilon)$ in (5.20). Observe that $\theta^j_r(\bar{x})=\theta^j_r(p_n)+\int_{p_n}^{\bar{x}}\theta^j_{r\bar{x}}(s)ds$, and take $p_n\to\infty$, then, from (4.16), (4.17), and (5.24),

$$|\theta^j_r(\bar{x})-c^j_{r,\infty}(\epsilon)|=\Big|\int_{\bar{x}}^{\infty}\theta^j_{r\bar{x}}(s)ds\Big|\le\kappa\epsilon\Big|\int_{\bar{x}}^{\infty}exp(-cs)ds\Big|\le\kappa\epsilon\, exp(-c\bar{x}). \tag{5.25}$$

Clearly, these results can be valid for the left interior solutions also.

From now on, we construct the global solution of the model problem. Let $\varphi\cup\psi$ mean the function on (-1,1) equal to the restriction of φ on (-1,0) and to the restriction of ψ on (0,1).

Then, $u^j_l\cup\theta^j_r$ and $\theta^j_l\cup u^j_r$ are $C^1([-1,1])$ and $H^2((-1,1))$ clearly. Thus, setting a linear combination $g^j:=(u^j_l\cup\theta^j_r)+(\theta^j_l\cup u^j_r)$ gives $C^1([-1,1])$ and $H^2((-1,1))$ functions.

However, $g^j(x=-1)=\theta^j_l(x=-1)=c^j_{l,\infty}(\epsilon)+e.s.t.$ and $g^j(x=1)=\theta^j_r(x=1)=c^j_{r,\infty}(\epsilon)+e.s.t.$ Here, $e.s.t.$ means exponentially small term. Since there is a difference between the boundary values of g^j and the boundary condition $\alpha=\beta=0$, we introduce interior solutions $\zeta^j(\bar{x})$ that is a interior solution of homogeneous case with the following boundary conditions:

$$\zeta^j=-c^j_{l,\infty}(\epsilon)\ \text{at}\ x=-1,\quad \zeta^j=-c^j_{r,\infty}(\epsilon)\ \text{at}\ x=1\qquad\text{for}\ \ j\ge 0. \tag{5.26}$$

To construct ζ^j, the interior solution $\sum_{k=0}^{\infty}\epsilon^k\bar{\theta}^k(\bar{x};-c^j_{l,\infty}(\epsilon),-c^j_{r,\infty}(\epsilon))$ is required in each order of ϵ. Here, $\bar{\theta}^j(\bar{x};\alpha,\beta)$ denotes a interior solution $\bar{\theta}^j(\bar{x})$ in homogeneous problem with boundary values α,β. Thus,

$$
\begin{aligned}
\zeta_{\epsilon} &= \sum_{j=0}^{\infty} \epsilon^{j} \zeta^{j} \\
&= \overline{\theta}^{0}(\overline{x}; -c_{l,\infty}^{0}(\epsilon), -c_{r,\infty}^{0}(\epsilon)) \\
&+ \epsilon \left[\overline{\theta}^{0}(\overline{x}; -c_{l,\infty}^{1}(\epsilon), -c_{r,\infty}^{1}(\epsilon)) + \overline{\theta}^{1}(\overline{x}; -c_{l,\infty}^{0}(\epsilon), -c_{r,\infty}^{0}(\epsilon)) \right] \\
&+ \epsilon^{2} \left[\overline{\theta}^{0}(\overline{x}; -c_{l,\infty}^{2}(\epsilon), -c_{r,\infty}^{2}(\epsilon)) + \overline{\theta}^{1}(\overline{x}; -c_{l,\infty}^{1}(\epsilon), -c_{r,\infty}^{1}(\epsilon)) + \overline{\theta}^{2}(\overline{x}; -c_{l,\infty}^{0}(\epsilon), -c_{r,\infty}^{0}(\epsilon)) \right] \\
&+ \qquad \cdots \qquad .
\end{aligned}
\tag{5.27}
$$

Then, the solution $\sum_{j=0}^{\infty} \epsilon^{j} (g^{j} + \zeta^{j})$ satisfies the required boundary conditions of the current case, and the results for θ^{j} in chapter 4 can be used to ζ^{j} identically.

5.4 Error Analysis

Now, we investigate an asymptotic error of the global solution constructed in previous section. Let

$$
w_{\epsilon n} = u^{\epsilon} - g_{\epsilon n} - \zeta_{\epsilon n} \tag{5.28a}
$$

where

$$
g_{\epsilon n} = \sum_{j=0}^{(p+1)n} \epsilon^{j} g^{j}, \quad \zeta_{\epsilon n} = \sum_{j=0}^{(p+1)n} \epsilon^{j} \zeta^{j}. \tag{5.28b}
$$

From the outer solutions in section 4.1 and the interior solutions $\theta_{l}^{j}, \theta_{r}^{j}, \zeta^{j}$, after some elementary calculations, we find that

$$
L_{\epsilon} w_{\epsilon n} = R_{p,2}^{n} + R_{p,3}^{n} + R_{p,4}^{n} + e.s.t. \quad \text{in } \Omega, \tag{5.29a}
$$

$$
w_{\epsilon n}(-1) = w_{\epsilon n}(1) = 0 \tag{5.29b}
$$

where

$$
\begin{aligned}
& R_{p,2}^{n} = \epsilon^{(p+1)n+(p+1)} (u_{lxx}^{(p+1)n} \cup u_{rxx}^{(p+1)n}), \; R_{p,3}^{n} = \sum_{j=0}^{(p+1)n} \epsilon^{j} (\theta_{lx}^{j} \cup \theta_{rx}^{j}) R^{j,(p+1)n}(b), \\
& R_{p,4}^{n} = \sum_{j=0}^{(p+1)n} \epsilon^{j} \zeta_{x}^{j} R^{j,(p+1)n}(b).
\end{aligned}
\tag{5.29c}
$$

Here, $R^{j,(p+1)n}$ are as in (4.26). Notice that, from the statement (5.17), $|u_{lxx}^{j}(0^{-})|, |u_{rxx}^{j}(0^{+})| \le \kappa_{j}$ for $0 \le j \le (p+1)n + p$ if we take $N = n(p+1) + \frac{1}{2}(3p-1)$. Then, we can estimate the L^{2}-norms of $x^{-(\frac{p-1}{2})} R_{p,2}^{n}$, $x^{-(\frac{p-1}{2})} R_{p,3}^{n}$, and $x^{-(\frac{p-1}{2})} R_{p,4}^{n}$ as follows. We first easily find that

$$|x^{-(\frac{p-1}{2})}R_{p,2}^n|_{L^2(\Omega)} \le \kappa_n \epsilon^{(p+1)n+(p+1)}. \tag{5.30}$$

Using (4.17), (4.27) and (5.21), we find

$$\begin{aligned}|x^{-(\frac{p-1}{2})}R_{p,3}^n| &\le \kappa_n \epsilon^{(p+1)n+\frac{1}{2}(p+3)} \sum_{j=0}^{(p+1)n} |\bar{x}|^{(p+1)n+\frac{1}{2}(p+3)-j} (|\theta_{rx}^j|_{\chi_{(0,1]}} + |\theta_{lx}^j|_{\chi_{[-1,0)}}) \\ &\le \kappa_n \epsilon^{(p+1)n+\frac{1}{2}(p+3)} exp\Big(-\frac{c|x|}{2\epsilon}\Big),\end{aligned} \tag{5.31}$$

and using (4.27) and (4.13), we obtain

$$\begin{aligned}|x^{-(\frac{p-1}{2})}R_{p,4}^n| &\le \kappa_n \epsilon^{(p+1)n+\frac{1}{2}(p+3)} \sum_{j=0}^{(p+1)n} |\bar{x}|^{(p+1)n+\frac{1}{2}(p+3)-j} |\zeta_x^j| \\ &\le \kappa_n \epsilon^{(p+1)n+\frac{1}{2}(p+1)} exp\Big(-\frac{c|x|}{2\epsilon}\Big).\end{aligned} \tag{5.32}$$

Thus,

$$|x^{-(\frac{p-1}{2})}R_{p,3}^n|_{L^2(\Omega)} \le \kappa \epsilon^{(p+1)n+\frac{1}{2}(p+4)}, \quad |x^{-(\frac{p-1}{2})}R_{p,4}^n|_{L^2(\Omega)} \le \kappa \epsilon^{(p+1)n+\frac{1}{2}(p+2)}. \tag{5.33}$$

Now, From (5.30) and (5.33), the Eq. (5.29a) is bounded by $\kappa_n \epsilon^{(p+1)n+(\frac{p}{2}+1)}$ with respect to L^2-norm. By applying Lemma 4.1 with $u = w_{\epsilon n}$, we attain the following estimation.

Theorem 5.1 Assume that the compatibility conditions (5.1) hold with $N = n(p+1)+\frac{1}{2}(3p-1)$. Let u^ϵ be the solution of (1.1b) with $\alpha = \beta = 0$. Then there exists a constant $\kappa_n > 0$ independent of ϵ such that

$$\begin{aligned}\|u^\epsilon - g_{\epsilon n} - \zeta_{\epsilon n}\|_{L^2(\Omega, x^{p-1}dx)} &\le \kappa_n \epsilon^{(p+1)n+(\frac{p}{2}+1)} \quad \text{for } m=0, \\ \|u^\epsilon - g_{\epsilon n} - \zeta_{\epsilon n}\|_{H^m(\Omega)} &\le \kappa_n \epsilon^{(p+1)n-(m-1)p+\frac{1}{2}} \quad \text{for } m=1,2,\end{aligned} \tag{5.34}$$

Note that, for $m \ge 3$, estimation such as (5.34) can be successively given by Lemma 4.1.

6. Conclusion

In this study, we have presented several estimations for the solution of the convection-diffusion equation with a multiple-order turning point. We have studied the homogeneous case with inhomogeneous boundary conditions, and the inhomogeneous case with the compatible condition in one dimensional space. We have investigated outer and interior solutions to obtain. Unlike typical methods, a priori estimate can be applicable in our works, to give sharp estimations by a relatively simple process.

As the beginning stage of research about turning points, we concentrate on studying in a one dimensional problem with compatible conditions. In the future, research on incompatible cases will be needed to drop the compatible condition. Meanwhile, it can be considered two- or three-dimensional problems. Because a proper application of interior solution makes the numerical solution stable, by using the interior solution provided in this paper, we can make a numerically stable scheme in two- or three-dimensional domain. In those cases, the turning point would become a line or a surface, and we would have to consider the normal directional velocity even if the tangential velocity is zero at the turning point.

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