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SPECTRAL GRAPHS FOR SELF-SIMILAR QUASI-CONTINUOUS LINEAR CHAINS

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1 Abstract

We construct self-similar functions and linear operators to deduce a self-similar variant of the Laplacian operator and of the D'Alembertian wave operator. The exigence of self-similarity as a symmetry property requires the introduction of non-local particle-particle interactions. We derive a self-similar linear wave operator describing the dynamics of a quasi-continuous linear chain of infinite length with a spatially self-similar distribution of nonlocal inter-particle springs. The self-similarity of the nonlocal harmonic particle-particle interactions results in a dispersion relation of the form of a Weierstrass-Mandelbrot function which exhibits self-similar and fractal features. We also derive a continuum approximation which relates the self-similar Laplacian to fractional integrals and yields in the low-frequency regime a power law frequency-dependence of the oscillator density.

Keywords: Self-similarity, self-similar functions, affine transformations, Weierstrass-Mandelbrot function, fractal functions, fractals, power laws, fractional integrals.

2 Introduction

The development of *Fractal Geometry* by Mandelbrot [1] launched a scientific revolution already in the seventies of the last century whereas the mathematical roots originate much earlier in the 19^{th} century [2]. However it is only recently that problems of fractal and self-similar media have become a subject in analytical mechanics. This is true in statics and dynamics. One important reason for this seems to be the considerable mathematical difficulty even to define physical problems on fractals and this is even more so for the construction of analytical solutions to these problems. Inspired by the exotic electromagnetic properties which fractal gaskets reveal when used as "*fractal antennae*" [5, 8], it had also been found that fractal gaskets exhibit exotic vibrational properties [3] which may open the door for new technological applications. An improved understanding of these properties could raise an enormous new interdisciplinary field for basic research and applications in a wide range of mechanical disciplines including fluid mechanics and the mechanics of granular media and solids. However a "fractal mechanics" has yet to be developed. Some crucial steps have already

*filename: papier˙fractal.tex Corresponding author, Email: michel@lmm.jussieu.fr been performed (see papers [9, 3, 13, 12, 4, 10] and the references therein). In [9] the fractal counterpart of the static harmonic calculus has been described by means of the Sierpinski Gasket by employing a Graph theoretical approach to define the Laplacian on the Sierpinski Gasket. In paper [3] the vibrational spectrum of a Sierpinski gasket was numerically modeled, however no rigorous approach was given. A significant contribution by analyzing Fourier spectra of fractal Sierpinski signals has been given in [4]. Closed form solutions for the dynamic Green's function and the vibrational spectrum of a linear chain with spatially exponential properties is given in a recent paper [11]. A similar fractal type of linear chain as analyzed in the present paper has been considered by Tarasov very recently [13].

In the present paper we utilize elements of lattice dynamics of linear chains together with a methodology to account for self-similarity which is newly developed in this paper. The demonstration is organized as follows: § 3 is devoted to the construction of self-similar functions and operators. By using this approach we construct a self-similar analogue to the Laplace operator to define a self-similar variant of the wave equation for a self-similar linear dynamic system in § 4. We hope the present approach launches some interdisciplinary work and collaborations also in fields concerned with fractal aspects of turbulence and fluid mechanics. It seems there are analogue situations [14] where the present approach could be useful.

A similar linear chain as in the present paper was considered by Tarasov only very recently [13]. However there is a crucial difference between the discrete Tarasov chain and the quasi-continuous chain being subject of our paper: The Tarasov-chain is discrete, i.e. there is a well defined distance between next neighbour particles. In the Tarasov-chain each particle interacts with particles of order N^s where $N \geq 1$ is an integer and $s = 0, 1, 2, ...$ assumes all positive integers including $s = 0$ which corresponds to the next neighbour. The Tarasov nonlocal harmonic interaction exhibits fractal, but not self-similar features.

In contrast we consider here a *quasi-continuous* chain with harmonic *exact self-similar* non-local interparticle interactions. In our chain any particle at space-point x interacts harmonically (spring constants ξ^s, $0 < \xi < 1$) with particles located at $x \pm hN^s$ where $N \in \mathbb{R}$ ($N > 1$) can be also non-integer and $s = -\infty, .., 0, .. + \infty$ is running over all positive and negative integers including zero[1]. In contrast to the Tarasov-chain, the elastic energy (density) introduced in our chain is an exactly self-similar function.

3 Construction of self-similar functions and linear operators

In this paragraph we define the term "self-similarity" with respect to functions and operators. We call a scalar function $\phi(h)$ *exact self-similar* with respect to variable h if the condition

$$\phi(Nh) = \Lambda\phi(h) \tag{1}$$

is satisfied where Λ and N are fixed scalar numbers and $h > 0$ a scalar variable. We call (1) the "affine problem"[2] where N and Λ represent *given* parameters and function $\phi(h)$ an unknown "solution" to these parameters of (1) to be determined. As we will see below for a given N solutions $\phi(h)$ exist only in a certain range of admissible Λ. From this definition of the problem follows that if $\phi(h)$ is a solution of (1) it is also a solution of $\phi(N^s h) = \Lambda^s\phi(h)$ where $s \in \mathbb{Z}$ can assume all positive and negative integers including zero. We emphasize that non-integer s are not admitted. The discrete set of pairs Λ^s, N^s are for all $s \in \mathbb{Z}$ related by a power law with the same power δ, i.e. $\Lambda = N^\delta$ hence $\Lambda^s = (N^s)^\delta$. We can hence define (1) also by replacing Λ and N by any positive or negative integer power Λ^s and N^s.

The affine problem (1) is the eigenvalue problem for a linear operator $\hat{A}_N$ with a certain given fixed parameter N and eigenfunctions $\phi(h)$ to be determined which correspond to an admissible range of eigenvalues $\Lambda = N^\delta$ (or equivalently to an admissible range of exponent $\delta = \ln\Lambda / \ln N$). For a function $f(x, h)$ we denote by $\hat{A}_N(h)f(x, h) =: f(x, Nh)$ when the affine transformation is only performed with respect to variable h.

We assume $\Lambda, N \in \mathbb{R}$ for physical reasons without too much loss of generality to be real and positive. Moreover, the definition (1) does not necessarily need to be restricted to the scalar case. We also can define self-similarity of a vector valued function $\vec{\phi}(\vec{h}) \in \mathbb{R}^n$ in the fully analogous manner, where $\mathbf{N}$ and $\mathbf{\Lambda}$ are Hermitian positive definite $n \times n$ matrices. In this paper however we confine us to the scalar case. For our convenience we define the "affine" operator $\hat{A}_N$ as follows

[1] Owing to this symmetry in s we confine on $N > 1$ without any loss of generality.

[2] where we restrict here to affine transformations $h' = Nh + c$ with $c = 0$.

$$\hat{A}_N \phi(h) =: \phi(Nh) \tag{2}$$

It is easily verified that the affine operator $\hat{A}_N$ is *linear*, that is it fulfills the relation

$$\hat{A}_N \left(c_1\phi_1(h) + c_2\phi_2(h) \right) = c_1\phi_1(Nh) + c_2\phi_2(Nh) \tag{3}$$

and

$$\hat{A}_N^s \phi(h) = \phi(N^s h), \qquad\qquad s = 0 \pm 1, \pm 2, .. \pm \infty \tag{4}$$

From this follows that we can define affine operator functions for any smooth function $g(\tau)$ that can be expanded into a Taylor series as

$$g(\tau) = \sum_{s=0}^{\infty} a_s \tau^s \tag{5}$$

we define an affine operator function in the form

$$g(\xi \hat{A}_N) = \sum_{s=0}^{\infty} a_s \xi^s \hat{A}_N^s \tag{6}$$

where ξ denotes a scalar parameter. The operator function which is defined by (6) acts on a function $\phi(h)$ as follows

$$g(\xi \hat{A}_N)\phi(h) = \sum_{s=0}^{\infty} a_s \xi^s \phi(N^s h) \tag{7}$$

where relation (4) with expansion (6) has been used. The convergence of series (7) has to be verified for a function $\phi(h)$ to be admissible. An explicit representation of the affine operator $\hat{A}_N$ can be obtained when we write $\phi(h) = \phi(e^{\ln h}) = \bar{\phi}(\ln h)$. Hence application of $\hat{A}_N$ on $\phi(h)$ is nothing but a translation in the variable v in $\bar{\phi}(v = \ln h)$. We introduce

$$\hat{A}_N \overline{\phi}(v) = \overline{\phi}(v + \ln N)\,, \qquad \overline{\phi}(v) = \phi(e^v) \tag{8}$$

so that

$$\hat{A}_N \phi(h) = \phi(Nh) = \bar{\phi}(\ln N + \ln h) = e^{\ln N \frac{d}{dv}} \phi(e^v)|_{v=\ln h} \tag{9}$$

where we assume that $\phi(h)$ is a sufficiently smooth function. The exponential operator $e^{\ln N \frac{d}{dv}}$ performs a translation in the variable v by $\ln N$ and is defined by

$$e^{\tau \frac{d}{dv}} = \sum_{s=0}^{\infty} \frac{\tau^s}{s!} \frac{d^s}{dv^s} \qquad \text{with} \qquad e^{\tau \frac{d}{dv}} \bar{\phi}(v)|_{v=v_0} = \bar{\phi}(v_0 + \tau) \tag{10}$$

Hence the affine operator $\hat{A}_N$ can be written explicitly in the form

$$\hat{A}_N(h) = e^{\ln N \frac{\mathrm{d}}{\mathrm{d(ln\,h)}}} \tag{11}$$

This relation is immediately verified in view of

$$\hat{A}_N(h)\phi(h) = e^{\ln N \frac{\mathrm{d}}{\mathrm{d(ln\,h)}}} \phi(e^{\ln h}) = \phi(e^{\ln h + \ln N}) = \phi(Nh) \tag{12}$$

With this machinery we are now able to construct self-similar functions and operators. This will be performed in the next subsection in order to define the wave propagation problem for a self-similar quasi-continuous linear chain (subsequent section 4).

3.1 Construction of self-similar functions

A self-similar function solving problem (1) is formally given by the series

$$\phi(h) = \sum_{s=-\infty}^{\infty} \Lambda^{-s} \hat{A}_N^s f(h) = \sum_{s=-\infty}^{\infty} \Lambda^{-s} f(N^s h) \tag{13}$$

for any function $f(h)$ for which the series (13) is uniformly convergent for all h. We introduce the self-similar operator

$$\hat{T}_N = \sum_{s=-\infty}^{\infty} \Lambda^{-s} \hat{A}_N^s \tag{14}$$

that fulfils formally the condition of self-similarity $\hat{A}_N \hat{T}_N = \Lambda \hat{T}_n$ and hence (13) solves the affine problem (1). In view of the symmetry with respect to inversion of the sign of s in (13) we can restrict ourselves to $N > 1$ without any loss of generality[3]: Let us look for admissible functions $f(t)$ for which (13) is convergent. To this end we have to demand simultaneous convergence of the partial sums over positive and negative s. Let us assume that (where we can confine on $t > 0$)

$$\lim_{t\to 0} f(t) = a_0 t^{\alpha} \tag{15}$$

For $t \to \infty$ we have to demand that $f(t)$ increase is not stronger than a power of t, i.e.

$$\lim_{t\to\infty} f(t) = C t^{\beta} \tag{16}$$

Both exponents α, β are allowed to take positive or negative values and do not need to be integers. A brief consideration of partial sums yields the following requirements for $\Lambda = N^{\delta}$, namely: Summation over $s < 0$ in (13) requires absolute convergence of a geometrical series leading to the condition for its argument $\Lambda N^{-\alpha} < 1$. That is we have to demand $\delta < \alpha$. The partial sum over $s > 0$ requires absolute convergence of a geometrical series leading to the condition for its argument $\Lambda^{-1} N^{\beta} < 1$ which corresponds to $\delta > \beta$. Both conditions are simultaneously met if

$$N^{\beta} < \Lambda = N^{\delta} < N^{\alpha} \tag{17}$$

or equivalently

$$\beta < \delta = \frac{\ln \Lambda}{\ln N} < \alpha \tag{18}$$

Relations (17) and (18) require additionally $\beta < \alpha$, that is only functions $f(t)$ with the behaviour (15) and (16) with $\beta < \alpha$ are *admissible* in (13). These cases include bounded functions $|f(t)| < M$ which correspond to $\beta = 0$. To this category refers for instance to any periodic function.

3.2 A self-similar analogue to the Laplace operator

In the sprit of (13) and (14) we construct an exactly self-similar function from the second difference according to

$$\phi(x, h) = \hat{T}_N(h) \left(u(x + h) + u(x - h) - 2u(x)\right) \tag{19}$$

where $\hat{T}_N(h)$ expresses that the affine operator $\hat{A}_N(h)$ acts only on the dependence on h, that is $\hat{A}_n(h) v(x, h) = v(x, Nh)$. We have with $\xi = \Lambda^{-1}$ the expression

$$\phi(x, h) = \sum_{s=-\infty}^{\infty} \xi^s \left\{u(x + N^s h) + u(x - N^s h) - 2u(x)\right\} \tag{20}$$

[3] We also can exclude the trivial case $N = 1$.

which is a self-similar function with respect to its dependence on h with $\hat{A}_N(h)\phi(x,h) = \phi(x, Nh) = \xi^{-1}\phi(x,h)$ but a regular function with respect to x. The function $\phi(x,h)$ exists if the series (20) is convergent. Let us assume that $u(x)$ is a smooth function with a convergent Taylor series for any h. Then we have with $u(x \pm h) = e^{\pm h \frac{d}{dx}} u(x)$ and $u(x+h) + u(x-h) - 2u(x) = \left(e^{h\frac{d}{dx}} + e^{-h\frac{d}{dx}} - 2\right) u(x)$ which can be written as

$$u(x+h) + u(x-h) - 2u(x) = 4\sinh^2\left(\frac{h}{2}\frac{d}{dx}\right) u(x) = h^2 \frac{d^2}{dx^2} u(x) + orders\ h^{\geq 4} \tag{21}$$

thus $\alpha = 2$ in criteria (15) is met. If we demand $u(x)$ being Fourier transformable we have as necessary condition that

$$\int_{-\infty}^{\infty} |u(x)|\, \mathrm{d}x < \infty \tag{22}$$

exists. This is true if $|u(t)|$ tends to zero as $t \to \pm\infty$ as $|t|^\beta$ where $\beta < -1$. We have then the condition that

$$\beta < \delta = -\frac{\ln \xi}{\ln N} < \alpha = 2 \tag{23}$$

However we will see below that only $\delta > 0$ is *physically admissible*, i.e. compatible with harmonic particle-particle interactions which decrease with increasing particle-particle distance.

The 1D Laplacian Δ_1 is defined by

$$\Delta_1 u(x) = \frac{d^2}{dx^2} u(x) = \lim_{\tau \to 0} \frac{(u(x+\tau) + u(x-\tau) - 2u(x))}{\tau^2} \tag{24}$$

Let us now define a self-similar analogue to the 1D Laplacian. We emphasize that also other definitions could be imagined. However, the definition to follow has a certain "physical" justification as we will see in § 4. In analogy to (24) we put with $\xi = N^{-\delta}$

$$\Delta_{(\delta,N,\tau)} u(x) =: const \lim_{\tau \to 0} \tau^{-\lambda} \phi(x,\tau) \tag{25}$$

$$= const \lim_{\tau \to 0} \tau^{-\lambda} \sum_{s=-\infty}^{\infty} \xi^s \left(u(x + N^s\tau) + u(x - N^s\tau) - 2u(x)\right) \tag{26}$$

where we have introduced a renormalisation-multiplier $\tau^{-\lambda}$ with the power λ to be determined to guarantee the limiting case being finite. The constant factor *const* indicates that there is a certain arbitrariness in this definition and will be chosen conveniently. Let us consider the limit $\tau \to 0$ by the special sequence $\tau_n = N^{-n}h$ with $n \to \infty$ and h being constant. Unlike in the the 1D case (24), the result of this limiting process depends crucially on the choice of the sequence τ_n. Then we have (by putting in (25) $const = h^\lambda$)

$$\Delta_{(\delta,N,h)} u(x) = \lim_{n \to \infty} N^{\lambda n} \xi^n \sum_{s=-\infty}^{\infty} \xi^{s-n} \left(u(x + N^{s-n}h) + u(x - N^{s-n}h) - 2u(x)\right) \tag{27}$$

which assumes by replacing $s - n \to s$ the form

$$\Delta_{(\delta,N,h)} u(x) = \phi(x,h) \lim_{n \to \infty} N^{-(\delta-\lambda)n} \tag{28}$$

which is only finite and nonzero if $\lambda = \delta$. The "Laplacian" can then be defined simply by

$$\Delta_{(\delta,N,h)} u(x) =: \lim_{n \to \infty} N^{\delta n} \phi(x, N^{-n}h) = \phi(x,h) \tag{29}$$

or by using (19) and (21) we can simply write

$$\Delta_{(\delta,N,h)} = 4\hat{T}_N(h) \sinh^2\left(\frac{h}{2}\frac{\partial}{\partial x}\right) = 4\sum_{s=-\infty}^{\infty} N^{-\delta s} \sinh^2\left(\frac{N^s h}{2}\frac{\partial}{\partial x}\right) \tag{30}$$

where $\hat{T}_N(h)$ is the self-similar operator defined in (14). The self-similar analogue of Laplace operator defined by (30) depends on the parameters δ, N, h. We furthermore observe the self-similarity of Laplacian (30), namely

$$\Delta_{(\delta,N,Nh)} = N^\delta \Delta_{(\delta,N,h)} \tag{31}$$

3.3 Continuum approximation - link to fractional integrals

For numerical evaluations it may be convenient to utilize the following continuum approximation of the self-similar Laplacian (30). To this end we put $N = 1 + \epsilon$ and $s\epsilon = v$ where $\epsilon > 0$ is assumed to be "small" so that $\mathrm{d}v \approx \epsilon$ and $N^s = (1+\epsilon)^{\frac{v}{\epsilon}} \approx e^v$. In this approximation $N^s \approx e^v$ becomes a (quasi)-continuous variable when s runs through $s \in \mathbb{Z}$. Then we can write (13) in the form

$$\phi(h) = \sum_{s=-\infty}^{\infty} N^{-s\delta} f(N^s h) \approx \frac{1}{\epsilon}\int_{-\infty}^{\infty} e^{-\delta v} f(he^v)\mathrm{d}v \tag{32}$$

which can be further written with $he^v = \tau$ and $\frac{\mathrm{d}\tau}{\tau} = \mathrm{d}v$ and $\tau(v \to -\infty) = 0$ and $\tau(v \to \infty) = \infty$ as

$$\phi(h) \approx \frac{h^\delta}{\epsilon}\int_0^\infty \frac{f(\tau)}{\tau^{1+\delta}}\,\mathrm{d}\tau \tag{33}$$

In this continuous approximation the function $\phi(h)$ obeys the same scaling behaviour as (13), namely $\phi(h\lambda) = \lambda^\delta \phi(h)$ but in contrast to (13) λ can assume any continuous value. This is due to the fact that (33) is holding for $N = 1+\epsilon$ with sufficiently small $\epsilon > 0$ since in this limiting case there exists for any continuous value λ an integer m such that $N^m = \lambda$. The existence requirement for integral (33) leads to the same requirements for $f(t)$ as in (13), namely inequality (18). Application of the approximate relation (33) on Laplacian (30) yields

$$\Delta_{(\delta,\epsilon,h)}u(x) \approx \frac{h^\delta}{\epsilon}\int_0^\infty \frac{(u(x-\tau)+u(x+\tau)-2u(x))}{\tau^{1+\delta}}\,\mathrm{d}\tau \tag{34}$$

where this integral exists for $\beta < \delta < 2$ and $\beta < -1$ because the required existence of integral (22) and relation (21).

If δ is in the range $\beta < \delta < 1$ we can split (34) into the two integrals

$$\Delta_{(\delta,\epsilon,h)}u(x) \approx \frac{h^\delta}{\epsilon}\int_0^\infty \frac{(u(x+\tau)-u(x))}{\tau^{1+\delta}}\,\mathrm{d}\tau + \frac{h^\delta}{\epsilon}\int_0^\infty \frac{(u(x-\tau)-u(x))}{\tau^{1+\delta}}\,\mathrm{d}\tau \tag{35}$$

By performing two partial integrations and by taking into account the vanishing boundary terms at $\tau = 0$ and $\tau = \infty$ for $\beta < \delta < 1$ we can re-write (34) in the form

$$\Delta_{(\delta,\epsilon,h)}u(x) \approx \frac{h^\delta}{\delta(\delta-1)\epsilon}\int_x^\infty (\tau-x)^{1-\delta}\,\frac{d^2u}{d\tau^2}(\tau)\,\mathrm{d}\tau + \frac{h^\delta}{\delta(\delta-1)\epsilon}\int_{-\infty}^x (x-\tau)^{1-\delta}\,\frac{d^2u}{d\tau^2}(\tau)\,\mathrm{d}\tau \tag{36}$$

We observe here the remarkable fact that this integral is a convolution of the conventional 1D Laplacian $\frac{d^2u}{dx^2}(x)$, namely

$$\Delta_{(\delta,\epsilon,h)}u(x) \approx \int_{-\infty}^\infty g(|x-\tau|)\frac{d^2u}{d\tau^2}(\tau)\mathrm{d}\tau \tag{37}$$

with the kernel

$$g(|x-\tau|) = \frac{h^\delta}{\delta(\delta-1)\epsilon}\,|x-\tau|^{1-\delta} \tag{38}$$

Further illuminating is the possibility to express (36) in terms of *fractional integrals*. To this end we put $D = 2 - \delta > 0$ which is positive in the admissible range of δ. For $0 < \delta < 1$ the quantity D can be identified with the estimated fractal dimension of the fractal spectral graph of the Laplacian [7] as outlined in the next section. The fractional integral is defined by (e.g. [15])

$$\mathcal{D}^D_{a,x}v(x) = \frac{1}{\Gamma(D)}\int_a^x (x-\tau)^{D-1}v(\tau)\mathrm{d}\tau \tag{39}$$

where $\Gamma(D)$ denotes the Γ-function which represents the generalization of the factorial function to non-integer $D > 0$. The Γ-function is defined as

$$\Gamma(D) = \int_0^\infty \tau^{D-1} e^{-\tau} \mathrm{d}\tau \,, \qquad D > 0 \tag{40}$$

For positive integers $D > 0$ the Gamma function reproduces the factorial function $\Gamma(D) = (D-1)!$ with $D = 1, 2, ..\infty$. The Laplacian (36) can be expressed in the form

$$\Delta_{(\delta=2-D,\epsilon,h)} u(x) \approx \frac{h^{2-D}}{\epsilon} \frac{\Gamma(D)}{(D-1)(D-2)} \left(\mathcal{D}^D_{x,\infty} + \mathcal{D}^D_{-\infty,x} \right) \Delta_1 u(x) \tag{41}$$

where $\Delta_1 u(x) = \frac{d^2}{dx^2} u(x)$ denotes the conventional 1D-Laplacian.

Although this approximation holds mathematically for $\beta < \delta < 1$ with $\beta < -1$ it will be demonstrated in the next section that we have to demand for any physical system $\delta > 0$ in Laplacian (30). This is due to that fact that physical inter-particle-interactions have to decay with increasing inter-particle-distance and to diverge when the inter-particle-distance tends to zero. Hence the requirement of convergence of the above integrals together with the demand for the Laplacian to describe a physical system with harmonically interacting particles restricts δ further within the interval

$$0 < \delta < 1 \tag{42}$$

In the next section it will be also shown that interval (42) is also the range where the spectral graphs of the Laplacian reveal fractal features.

4 The physical model

We consider an infinitely long quasi-continuous linear chain of identical particles. Any space-point x corresponds to a "material point" or particle. The mass density of particles is assumed to be spatially homogeneous and equal to one for any space point x. Any particle is associated with one degree of freedom which is represented by the displacement field $u(x,t)$ where x is its spatial (Lagrangian) coordinate and t indicates time. In this sense we consider a quasi continuous spatial distribution of particles. Any particle at space-point x is non-locally connected by harmonic springs of strength ξ^s to particles located at $x \pm N^s h$, where $N > 1$ and $N \in \mathbb{R}$ is not necessarily integer, $h > 0$, and $s = 0, \pm 1, \pm 2, .. \pm \infty$. The requirement of decreasing spring constants with increasing particle-particle distance leads to the requirement that $\xi = N^{-\delta} < 1$ $(N > 1)$ i.e. only chains with $\delta > 0$ are physically admissible. In order to get exact self-similarity we avoid the notion of "next-neighbour particles" in our chain which would be equivalent to the introduction of an internal length scale (the next neighbour distance). To admit particle interactions over arbitrarily close distances $N^s h \to 0$ $(s \to -\infty)$ our chain has to be *quasi-continuous*. This is the principal difference to the *discrete* chain considered recently by Tarasov [13] which is not self-similar.

The Hamiltonian which describes our chain can be written as

$$H = \frac{1}{2} \int_{-\infty}^{\infty} \left(\dot{u}^2(x,t) + \mathcal{V}(x,t,h) \right) \mathrm{d}x \tag{43}$$

In the spirit of (13) the elastic energy density $\mathcal{V}(x,t,h)$ is assumed to be constructed self-similarly, namely[4]

$$\mathcal{V}(x,t,h) = \frac{1}{2} \hat{T}_N(h) \left[(u(x,t) - u(x+h,t))^2 + (u(x,t) - u(x-h,t))^2 \right] \tag{44}$$

where $\hat{T}_N(h)$ is the self-similar operator (14) to arrive at

$$\mathcal{V}(x,t,h) = \frac{1}{2} \sum_{s=-\infty}^{\infty} \xi^s \left[(u(x,t) - u(x+hN^s,t))^2 + (u(x,t) - u(x-hN^s,t))^2 \right] \tag{45}$$

The elastic energy density $\mathcal{V}(x,t,h)$ fulfills the condition of self-similarity with respect to h, namely

$$\hat{A}_N(h) \mathcal{V}(x,t,h) = \mathcal{V}(x,t,Nh) = \xi^{-1} \mathcal{V}(x,t,h) \tag{46}$$

[4]The additional factor 1/2 in the elastic energy avoids double counting.

where $\xi^{-1} = N^{\delta}$.

The criteria of convergence of (45) yields $\alpha = 2$ as for the Laplacian (20). To determine β we have to demand that $u(x,t)$ is Fourier transformable [5] thus we have to have an asymptotic behaviour of $|u(x,t) - u(x \pm \tau, t)|$ as τ^{β} where $\beta < -1$ as $\tau \to \infty$. From this follows $|u(x,t) - u(x \pm \tau, t)|^2$ behaves then as $\tau^{2\beta}$ where $2\beta < -2$. Hence, the elastic energy density (45) converges if

$$2\beta < \delta < \alpha = 2 \tag{47}$$

where $\beta < -1$. In this relation exponent δ ($\xi = N^{-\delta}$) can take values greater or smaller than -2. However the requirement of the convergence of the equation of motion (eq. (51 below) depends on the behaviour of $|u(x,t) - u(x \pm \tau, t)|$ for $\tau \to \infty$. From this follows that

$$\beta < \delta < \alpha = 2 \tag{48}$$

where $\beta < -1$. Relation (48) determines the range of the admissible values of δ in order to achieve convergence. However, we emphasize that physically only chains are admissible with $\delta > 0$ in order to have decreasing inter-particle spring-constants with increasing inter-particle distance. We will see below that the additional requirement $\delta > 0$ works out in a natural way as a consequence of the convergence requirement of the dispersion relation.

If (48) is fulfilled the convergence of the equation of motion (eqs (50), (51)) is guaranteed since relation (47) is also fulfilled (since $\beta < -1$).

The equation of motion is obtained by

$$\frac{\partial^2 u}{\partial t^2} = -\frac{\delta H}{\delta u} \tag{49}$$

(where $\delta./\delta u$ stands for a functional derivative) to arrive at

$$\frac{\partial^2 u}{\partial t^2} = -\sum_{s=-\infty}^{\infty} \xi^s \left(2u(x,t) - u(x + hN^s, t) - u(x - hN^s, t)\right) \tag{50}$$

$$\frac{\partial^2 u}{\partial t^2} = \Delta_{(\delta,N,h)} u(x,t) \tag{51}$$

with the self-similar Laplacian $\Delta_{(\delta,N,h)}$ of equation (30). We can re-write (51) in the compact form of a wave equation

$$\Box_{(\delta,N,h)} u(x,t) = 0 \tag{52}$$

where $\Box_{(\delta,N,h)}$ is the *self-similar analogue of the d'Alembertian wave operator* having the form

$$\Box_{(\delta,N,h)} = \Delta_{(\delta,N,h)} - \frac{\partial^2}{\partial t^2} \tag{53}$$

The d'Alembertian (53) with the Laplacian (30) describes the wave propagation in the self-similar chain (43). It appears be useful and feasible to extended this approach to a general description of wave propagation phenomena in fractal and self-similar material systems.

Now the goal is to determine the spectral graph (dispersion relation), which is constituted by the (negative) eigenvalues of the (semi-)negative definite Laplacian (30). To this end we make use of the fact that the displacement field $u(x,t)$ is Fourier transformable (guaranteed by choosing $\beta < -1$ in (48)) and that the exponentials e^{ikx} are eigenfunctions of the self-similar Laplacian (30). We hence write the Fourier integral

$$u(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{u}(k,t) e^{ikx} \mathrm{d}k \tag{54}$$

to re-write (51) for the Fourier amplitudes $\tilde{u}(k,t)$ in the form

[5] This assumption defines the (function) space of eigenmodes and corresponds to infinite body boundary conditions.

$$\frac{\partial^2 \tilde{u}}{\partial t^2}(k,t) = -\omega^2(k)\, \tilde{u}(k,t) \tag{55}$$

The spectral graph $\omega^2(k)$ (dispersion relation) is obtained by replacing $\frac{\partial}{\partial x}$ by ik in (30) to arrive at

$$\omega^2(k) = 4\hat{T}_N(h)\, \sin^2(\frac{kh}{2}) \tag{56}$$

which yields by applying the self-similar operator $\hat{T}_N(h)$ (eqs (13), (14))

$$\omega^2(kh) = 4 \sum_{s=-\infty}^{\infty} N^{-\delta s} \sin^2(\frac{khN^s}{2}) \tag{57}$$

The spectral graph of (57) is a *Weierstrass-Mandelbrot function* which is a continuous but nowhere differentiable function [1] and fulfills the condition of self-similar symmetry, namely

$$\omega^2(Nkh) = N^{\delta}\, \omega^2(kh) \tag{58}$$

A similar consideration as above shows that convergence of (57) restricts δ to the range

$$0 < \delta < 2 \tag{59}$$

which is a subset of both intervals (47) and (48). Hence *only* exponents δ in the interval (59) are *admissible* in Hamiltonian (43) with the elastic energy density (45) in order to have a "well-posed" problem. Condition (59) includes automatically "*physical admissibility*" which requires $\delta > 0$ ($\xi = N^{-\delta} < 1$) in (45) in order to have spring constants $N^{-\delta s} = (L_s/h)^{-\delta}$ which decrease monotonously and tend versus zero with increasing inter-particle distances $L_s \to \infty$ and diverge for inter-particle distances $L_s \to 0$. It was shown by Hardy [7] that for $\xi N > 1$ and $\xi = N^{-\delta} < 1$ or equivalently for

$$0 < \delta < 1 \tag{60}$$

the Mandelbrot-Weierstrass function of the form (57) is not only self-similar but also a *fractal* curve of (estimated) non-integer fractal (Hausdorff) dimension $D = 2 - \delta > 1$. Figs. 1-3 show spectral graphs for different decreasing values of admissible $0 < \delta < 1$ and increasing fractal dimension. The increase of the fractal dimension from Figs. 1-3 is indicated by the increasingly irregular harsh behaviour of the curves.

To evaluate (57) approximately it is convenient to replace the series by an integral utilizing a similar substitution as in section 3.3 ($\epsilon \approx \ln N$). By doing so we smoothen the Weierstrass-Mandelbrot function (57). It is important to notice that the resulting approximate spectral function is hence differentiable and has not a fractal dimension $D > 1$ in the interval (60). For sufficiently "small" kh we arrive at

$$\omega^2(kh) \approx \frac{(hk)^{\delta}}{\epsilon} C \tag{61}$$

which is only finite if $(kh)^{\delta}$ is in the order of magnitude of ϵ or smaller. This regime which includes the limit $k \to 0$ is hence characterized by a power law behaviour $\omega(k) \approx Const\, k^{\delta/2}$ of the dispersion relation. The constant C introduced in (61) is given by the integral

$$C = 2\int_0^{\infty} \frac{(1-\cos\tau)}{\tau^{1+\delta}} \mathrm{d}\tau \tag{62}$$

This approximation holds for "small" $\epsilon \approx \ln N \neq 0$ ($0 < \epsilon \ll 1$)[6] which corresponds to the limiting case that $N^s = e^v$ is continuous. For this limiting case we thus can obtain the oscillator density $\rho(\omega)$ from

$$\rho(\omega) = \frac{1}{2\pi}\frac{dk}{d\omega} \tag{63}$$

[6] $\epsilon = 0$ has to be excluded since it corresponds to $N = 1$.

which is normalized such that $\rho(\omega)\mathrm{d}\omega$ counts the number of normal oscillators having frequencies within the interval $[\omega, \omega + \mathrm{d}\omega]$. We obtain then

$$\rho(\omega) = \frac{1}{\pi\delta h}\left(\frac{\epsilon}{C}\right)^{\frac{1}{\delta}} \omega^{\frac{2}{\delta}-1} \tag{64}$$

We emphasize that neither the dependence on k of the Weierstrass-Mandelbrot function (57) is represented by a *continuous* k^{δ} dependence nor this function is differentiable with respect to k. Application of (63) is hence justified only to be applied on the approximative representation (61) which holds for sufficiently small ϵ, i.e. when $N = 1 + \epsilon$ is sufficiently close to 1 so that N^s is a quasi-continuous function. However (63) is not generally applicable on (57) for any arbitrary $N > 1$. We can hence consider (64) as the low-frequency regime $\omega \to 0$ of the oscillator density holding *only* in the quasi-continuous case of self-similarity, i.e. when N^s assumes quasi-continuous values by varying $s \in \mathbb{Z}$. This is only true if $N = 1 + \epsilon$ is sufficiently close to 1 $(0 < \epsilon \ll 1)$.

5 Conclusions

We have depicted how self-similar functions and linear operators can be constructed in a simple manner by utilizing a certain category of conventional "admissible" functions. This approach enables us to construct non-local self-similar analogues to the Laplacian and d'Alembert wave operator. The linear self-similar equation of motion describes the propagation of waves in a quasi-continuous linear chain with harmonic non-local self-similar particle-interactions. The complexity which comes into play by the self-similarity of the non-local interactions is completely captured by the spectral graphs which assume the forms of Weierstrass-Mandelbrot functions (57) exhibiting exact self-similarity and for certain parameter combinations (relation (60)) fractal features.

The resulting self-similar wave operator (53) with the Laplacian (30) can be generalized to describe wave propagation in fractal and self-similar structures which are fractal subspaces embedded in Euclidean spaces of 1-3 dimensions. The development of such an approach could be a crucial step towards a better understanding of the dynamics in materials with a scale hierarchies of internal structures which may be fractal and self-similar.

We hope to inspire further work and collaborations in this direction to develop appropriate approaches useful for the modelling of static and dynamic problems in self-similar and fractal structures in a wider interdisciplinary context.

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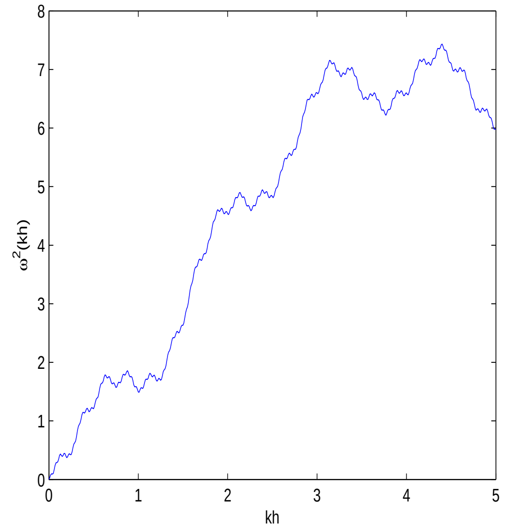

Figure 1: $\delta = \log 4 / \log 5$

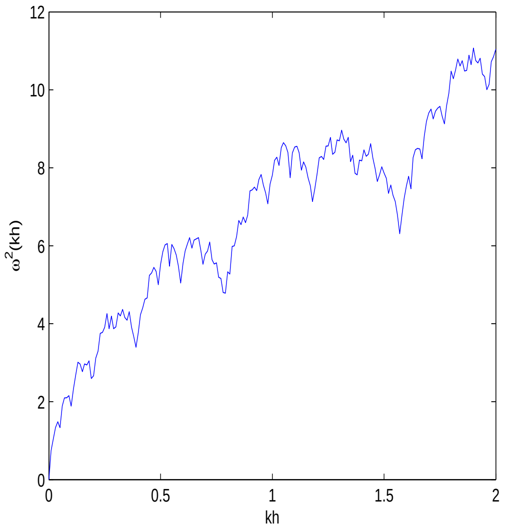

Figure 2: $\delta = 0.5$

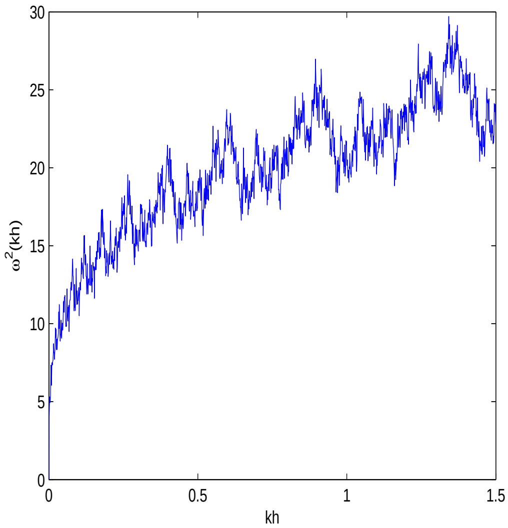

Figure 3: $\delta = 0.25$