The lower central and derived series of the braid groups of the sphere and the punctured sphere.

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The lower central and derived series of the braid groups $B_n(\mathbb{S}^2)$ and $B_m(\mathbb{S}^2 \setminus \{x_1, \ldots, x_n\})$

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Abstract

Our aim in this paper is to determine the lower central and derived series for the braid groups of the sphere and of the finitely-punctured sphere. We are motivated in part by the study of the generalised Fadell-Neuwirth short exact sequence [**GG2, GG4**], but the problem is of interest in its own right.

The braid groups of the 2-sphere $\mathbb{S}^2$ were studied by Fadell, Van Buskirk and Gillette during the 1960's, and are of particular interest due to the fact that they have torsion elements (which were characterised by Murasugi). We first prove that for all $n \in \mathbb{N}$, the lower central series of the n-string braid group $B_n(\mathbb{S}^2)$ is constant from the commutator subgroup onwards. We obtain a presentation of $\Gamma_2(B_n(\mathbb{S}^2))$, from which we observe that $\Gamma_2(B_4(\mathbb{S}^2))$ is a semi-direct product of the quaternion group of order 8 by a free group of rank 2. As for the derived series of $B_n(\mathbb{S}^2)$, we show that for all $n \geq 5$, it is constant from the derived subgroup onwards. The group $B_n(\mathbb{S}^2)$ being finite and soluble for $n \leq 3$, the critical case is $n = 4$ for which the derived subgroup is the semi-direct product obtained above. By proving a general result concerning the structure of the derived subgroup of a semi-direct product, we are able to determine completely the derived series of $B_4(\mathbb{S}^2)$ which from $(B_4(\mathbb{S}^2))^{(4)}$ onwards coincides with that of the free group of rank 2, as well as its successive derived series quotients.

For $n \geq 1$, the class of m-string braid groups $B_m(\mathbb{S}^2 \setminus \{x_1, \ldots, x_n\})$ of the n-punctured sphere includes the usual Artin braid groups B_m (for $n = 1$), those of the annulus, which are Artin groups of type B (for $n = 2$), and affine Artin groups of type $\widetilde{C}$ (for $n = 3$). Motivated by the study of almost periodic solutions of algebraic equations with almost periodic coefficients, Gorin and Lin determined the commutator subgroup of the Artin braid groups. We extend their results, and show that the lower central series of B_m is completely determined for all

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$m \in \mathbb{N}$, and that the derived series is determined for all $m \neq 4$. In the exceptional case $m = 4$, we determine some higher elements of the derived series and its quotients.

When $n \geq 2$, we prove that the lower central series (respectively derived series) of $B_m(\mathbb{S}^2 \setminus \{x_1, \ldots, x_n\})$ is constant from the commutator subgroup onwards for all $m \geq 3$ (respectively $m \geq 5$). The case $m = 1$ is that of the free group of rank $n - 1$. The case $n = 2$ is of particular interest notably when $m = 2$ also. In this case, the commutator subgroup is a free group of infinite rank. We then go on to show that $B_2(\mathbb{S}^2 \setminus \{x_1, x_2\})$ admits various interpretations, as the Baumslag-Solitar group $\mathrm{BS}(2, 2)$, or as a one relator group with non-trivial centre for example. We conclude from this latter fact that $B_2(\mathbb{S}^2 \setminus \{x_1, x_2\})$ is residually nilpotent, and that from the commutator subgroup onwards, its lower central series coincides with that of the free product $\mathbb{Z}_2 * \mathbb{Z}$. Further, its lower central series quotients Γ_i/Γ_{i+1} are direct sums of copies of $\mathbb{Z}_2$, the number of summands being determined explicitly. In the case $m \geq 3$ and $n = 2$, we obtain a presentation of the derived subgroup, from which we deduce its Abelianisation. Finally, in the case $n = 3$, we obtain partial results for the derived series, and we prove that the lower central series quotients Γ_i/Γ_{i+1} are 2-elementary finitely-generated groups.

Preface

1. Generalities and definitions

Let $n \in \mathbb{N}$. The braid groups of the plane $\mathbb{E}^2$, denoted by B_n, and known as *Artin braid groups*, were introduced by E. Artin in 1925 [**A1**], and further studied in [**A2, A3, Ch**]. Artin showed that B_n admits the following well-known presentation: B_n is generated by elements $\sigma_1, \ldots, \sigma_{n-1}$, subject to the classical *Artin relations*:

$$\left.\begin{array}{l} \sigma_i\sigma_j = \sigma_j\sigma_i \text{ if } |i-j| \geq 2 \text{ and } 1 \leq i,j \leq n-1 \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \text{ for all } 1 \leq i \leq n-2. \end{array}\right\} \tag{1}$$

A natural generalisation to braid groups of arbitrary topological spaces was made at the beginning of the 1960's by Fox (using the notion of configuration space) [**FoN**]. In that paper, Fox and Neuwirth proved some basic results about the braid groups of arbitrary manifolds. In particular, if M^r is a connected manifold of dimension $r \geq 3$ then there is no braid theory (as formulated in this paper). The braid groups of compact, connected surfaces have been widely studied; (finite) presentations were obtained in [**Z1, Z2, Bi1, Sc**]. As well as being interesting in their own right, braid groups have played an important rôle in many branches of mathematics, for example in topology, geometry, algebra and dynamical systems, and notably in the study of knots and links [**BZ**], of the mapping class groups [**Bi2, Bi3**], and of configuration spaces [**CG, FH**]. The reader may consult [**Bi2, Han, MK, R**] for some general references on the theory of braid groups.

Let M be a connected manifold of dimension 2 (or *surface*), perhaps with boundary. Further, we shall suppose that M is homeomorphic to a compact 2-manifold with a finite (possibly zero) number of points removed from its interior. We recall two (equivalent) definitions of surface braid groups. The first is that due to Fox. Let $F_n(M)$ denote the n^{th} *configuration space* of M, namely the set of all ordered n-tuples of distinct points of M:

$$F_n(M) = \{(x_1, \ldots, x_n) \mid x_i \in M \text{ and } x_i \neq x_j \text{ if } i \neq j\}.$$

Since $F_n(M)$ is a subspace of the n-fold Cartesian product of M with itself, the topology on M induces a topology on $F_n(M)$. Then we define the *n-string pure (or unpermuted) braid group* $P_n(M)$ of M to be:

$$P_n(M) = \pi_1(F_n(M)).$$

There is a natural action of the symmetric group S_n on $F_n(M)$ by permutation of coordinates, and the resulting orbit space $F_n(M)/S_n$ shall be denoted by $D_n(M)$. The fundamental group $\pi_1(D_n(M))$ is called the *n-string (full) braid group* of M, and shall be denoted by $B_n(M)$. Notice that the projection $F_n(M) \to D_n(M)$ is a regular $n!$-fold covering map. It is well known that B_n is isomorphic to $B_n(\mathbb{D}^2)$ and $P_n \cong P_n(\mathbb{D}^2)$, where $\mathbb{D}^2$ is the closed 2-disc.

The second definition of surface braid groups is geometric. Let $\mathcal{P} = \{p_1, \ldots, p_n\}$ be a set of n distinct points of M. A *geometric braid* of M with basepoint $\mathcal{P}$ is a collection $\beta = (\beta_1, \ldots, \beta_n)$ of n paths $\beta\colon [0,1] \to M$ such that:

(a) for all $i = 1, \ldots, n$, $\beta_i(0) = p_i$ and $\beta_i(1) \in \mathcal{P}$.

(b) for all $i, j = 1, \ldots, n$ and $i \neq j$, and for all $t \in [0,1]$, $\beta_i(t) \neq \beta_j(t)$.

Two geometric braids are said to be *equivalent* if there exists a homotopy between them through geometric braids. The usual concatenation of paths induces a group operation on the set of equivalence classes of geometric braids. This group is isomorphic to $B_n(M)$, and does not depend on the choice of $\mathcal{P}$. The subgroup of *pure* braids, satisfying additionally $\beta_i(1) = p_i$ for all $i = 1, \ldots, n$, is isomorphic to $P_n(M)$. There is a natural surjective homomorphism $B_n(M) \to S_n$ which to a geometric braid β associates the permutation π defined by $\beta_i(1) = p_{\pi(i)}$. The kernel is precisely $P_n(M)$, and we thus obtain the following short exact sequence:

$$1 \to P_n(M) \to B_n(M) \to S_n \to 1.$$

2. The Fadell-Neuwirth short exact sequence

Let $m, n \in \mathbb{N}$ be positive integers such that $m > n$, and consider the projection

$$p\colon\ F_m(M) \to F_n(M)$$
$$(x_1, \ldots, x_n, \ldots, x_m) \mapsto (x_1, \ldots, x_n).$$

In [**FaN**], Fadell and Neuwirth studied the map p, and showed that it is a locally-trivial fibration. The fibre over a point $(x_1, \ldots, x_n)$ of the base space is $F_{m-n}(M \setminus \{x_1, \ldots, x_n\})$ which may be considered to be a subspace of the total space via the map

$$i\colon\ F_{m-n}(M \setminus \{x_1, \ldots, x_n\}) \to F_m(M)$$

defined by

$$i((y_1, \ldots, y_{m-n})) = (x_1, \ldots, x_n, y_1, \ldots, y_{m-n}).$$

Then p induces a group homomorphism $p_*\colon P_m(M) \to P_n(M)$, which representing $P_m(M)$ geometrically as a collection of m strings, corresponds to forgetting the last $(m-n)$ strings. **We adopt the convention throughout this paper, that unless explicitly stated otherwise, all homomorphisms $P_m(M) \to P_n(M)$ in the text will be this one**.

The fibration $p\colon F_m(M) \to F_n(M)$ gives rise to a long exact sequence of homotopy groups of configuration spaces, from which we obtain the *Fadell-Neuwirth pure braid group short exact sequence*:

$$1 \to P_{m-n}(M \setminus \{x_1, \ldots, x_n\}) \xrightarrow{i_*} P_m(M) \xrightarrow{p_*} P_n(M) \to 1, \qquad (2)$$

where i_* is the group homomorphism induced by i, and $n \geq 3$ if M is the 2-sphere $\mathbb{S}^2$ [**Fa, FVB**], $n \geq 2$ if M is the real projective plane $\mathbb{R}P^2$ [**VB**], and $n \geq 1$ otherwise [**FaN**] (in each case, the condition on n implies that $F_n(M)$ is an Eilenberg-MacLane space). This short exact sequence plays a central rôle in the study of surface braid groups. It was used by [**PR**] to study mapping class groups, in the work of [**GMP**] on Vassiliev invariants for braid groups, as well as to obtain presentations for surface pure braid groups [**Bi1, Sc, GG1, GG4**].

An interesting question is that of whether the Fadell-Neuwirth short exact sequence (2) splits. If the above conditions on n are satisfied then the existence of a section for p_* is equivalent to that of a geometric section for p (cf. [**GG3, GG4**]). In [**A2**], Artin showed that if M is the plane then (2) splits for all $n \in \mathbb{N}$. This implies that P_n may be expressed as a repeated semi-direct product of free groups, which enables one to solve the word problem in the pure and full Artin braid groups. The splitting problem has been studied for other surfaces besides the plane. Fadell and Neuwirth gave various sufficient conditions for the existence of a geometric section for p in the general case [**FaN**]. For the sphere, it was known that there exists a section on the geometric level [**FVB**]. If M is the 2-torus then Birman exhibited an explicit algebraic section for (2) for $m = n+1$ and $n \geq 2$ [**Bi1**]. However, for compact orientable surfaces without boundary of genus $g \geq 2$, she posed the question of whether the short exact sequence (2) splits. In [**GG1**], we provided a complete answer to this question:

THEOREM 1 ([**GG1**]). *If M is a compact orientable surface without boundary of genus $g \geq 2$, the short exact sequence (2) splits if and only if $n = 1$.*

3. A generalisation of the Fadell-Neuwirth short exact sequence

As we mentioned above, the Fadell-Neuwirth short exact sequence is a very important tool in the study of pure surface braid groups, but unfortunately it does not generalise directly to the corresponding full braid groups. However, by considering intermediate coverings between $F_n(M)$ and $D_n(M)$, it is possible to extend it to certain subgroups of $B_n(M)$ [**GG2**]. A special case of this construction may be formulated as follows. Let $m, n \in \mathbb{N}$, and let $D_{m,n}(M)$ denote the quotient space of $F_{m+n}(M)$ by the action of the subgroup $S_m \times S_n$ of S_{m+n}. Then we obtain a fibration $D_{m,n}(M) \to D_m(M)$, defined by forgetting the last n coordinates. We set $B_{m,n}(M) = \pi_1(D_{m,n}(M))$, sometimes termed a *'mixed'* braid group. As in the pure braid group case, we obtain a generalisation of the short exact sequence of Fadell and Neuwirth:

$$1 \to B_n(M \setminus \{x_1, \ldots, x_m\}) \to B_{m,n}(M) \stackrel{p_*}{\to} B_m(M) \to 1, \qquad (3)$$

where again we take $m \geq 3$ if $M = \mathbb{S}^2$, $m \geq 2$ if $M = \mathbb{R}P^2$ and $m \geq 1$ otherwise. Once more, unless explicitly stated, all homomorphisms $B_{m,n}(M) \to B_m(M)$ in the text will be this one.

4. The braid groups of the sphere

The braid groups of the sphere and the real projective plane are of particular interest, notably because they have non-trivial centre (which is also the case for the Artin braid groups), and torsion elements. The braid groups of the sphere were studied during the 1960's [**Fa, FVB, VB, GVB**]: let us recall briefly some of their properties.

If $\mathbb{D}^2 \subseteq \mathbb{S}^2$ is a topological disc, there is a group homomorphism $\iota\colon B_n(\mathbb{D}^2) \to B_n(\mathbb{S}^2)$ induced by the inclusion. If $\beta \in B_n(\mathbb{D}^2)$ then we shall denote its image $\iota(\beta)$ simply by β. It is well known that $B_n(\mathbb{S}^2)$ is generated by $\sigma_1, \ldots, \sigma_{n-1}$ which are subject to the following relations:

$$\left.\begin{array}{l} \sigma_i\sigma_j = \sigma_j\sigma_i \text{ if } |i-j| \geq 2 \text{ and } 1 \leq i,j \leq n-1 \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \text{ for all } 1 \leq i \leq n-2, \text{ and} \\ \sigma_1 \cdots \sigma_{n-2}\sigma_{n-1}^2\sigma_{n-2} \cdots \sigma_1 = 1. \end{array}\right\} \qquad (4)$$

In what follows, the third relation will be referred to as the *surface relation* of $B_n(\mathbb{S}^2)$. It follows from this presentation and equation (1) that $B_n(\mathbb{S}^2)$ is a quotient of B_n. The first three sphere braid groups are finite: $B_1(\mathbb{S}^2)$ is trivial, $B_2(\mathbb{S}^2)$ is cyclic of order 2, and $B_3(\mathbb{S}^2)$ is a ZS-metacyclic group (a group whose Sylow subgroups, commutator subgroup and commutator quotient group are all cyclic) of order 12.

If $n \geq 3$, the so-called 'full twist' Δ_n braid of $B_n(\mathbb{S}^2)$, defined by

$$\Delta_n = (\sigma_1 \cdots \sigma_{n-1})^n,$$

generates the centre $Z(B_n(\mathbb{S}^2))$ of $B_n(\mathbb{S}^2)$, and is a torsion element of order 2. Using Seifert fibre space theory, Murasugi characterised the torsion elements of $B_n(\mathbb{S}^2)$: they are all conjugates of powers of the three elements $\sigma_1 \cdots \sigma_{n-2}\sigma_{n-1}$, $\sigma_1 \cdots \sigma_{n-2}\sigma_{n-1}^2$ and $\sigma_1 \cdots \sigma_{n-3}\sigma_{n-2}^2$ which are respectively n^{th}, $(n-1)^{\text{th}}$ and $(n-2)^{\text{th}}$ roots of Δ_n [**Mu**].

In [**GG4**], we studied the short exact sequence (3) in the case $M = \mathbb{S}^2$ of the sphere:

$$1 \to B_n(\mathbb{S}^2 \setminus \{x_1, \ldots, x_m\}) \to B_{m,n}(\mathbb{S}^2) \xrightarrow{p_*} B_m(\mathbb{S}^2) \to 1, \tag{5}$$

and proved the following results:

Theorem 2 ([**GG4**]).

(a) The short exact sequence

$$1 \to B_n(\mathbb{S}^2 \setminus \{x_1, x_2, x_3\}) \to B_{3,n}(\mathbb{S}^2) \xrightarrow{p_*} B_3(\mathbb{S}^2) \to 1$$

splits if and only if $n \equiv 0, 2 \bmod 3$.

(b) Let $m \geq 4$. *If the homomorphism* $p_* \colon B_{m,n}(\mathbb{S}^2) \to B_m(\mathbb{S}^2)$ *admits a section then there exist* $\varepsilon_1, \varepsilon_2 \in \{0, 1\}$ *such that:*

$$n \equiv \varepsilon_1(m-1)(m-2) - \varepsilon_2 m(m-2) \bmod m(m-1)(m-2).$$

An open question is whether the necessary condition in part (b) is also sufficient. If $n \geq 4$ then $B_n(\mathbb{S}^2)$ is infinite, and it follows from the proof of part (a) that $B_n(\mathbb{S}^2)$ contains an isomorphic copy of the finite group $B_3(\mathbb{S}^2)$ of order 12 if and only if $n \not\equiv 1 \bmod 3$. We have recently shown that $B_n(\mathbb{S}^2)$ contains an isomorphic copy of the quaternion group $\mathcal{Q}_8$ of order 8 if and only if n is even [**GG5**]. The realisation of finite subgroups in $B_n(\mathbb{S}^2)$ and $B_n(\mathbb{R}P^2)$ seems an interesting problem which we are pursuing.

5. Braid group series and motivation for their study

If G is a group, then we recall that its *lower central series* $\{\Gamma_i(G)\}_{i\in\mathbb{N}}$ is defined inductively by $\Gamma_1(G) = G$, and $\Gamma_{i+1}(G) = [G, \Gamma_i(G)]$ for all $i \in \mathbb{N}$, and its *derived series* $\{G^{(i)}\}_{i\in\mathbb{N}\cup\{0\}}$ is defined inductively by $G^{(0)} = G$, and $G^{(i)} = [G^{(i-1)}, G^{(i-1)}]$ for all $i \in \mathbb{N}$. One may check easily that $\Gamma_i(G) \supseteq \Gamma_{i+1}(G)$ and $G^{(i-1)} \supseteq G^{(i)}$ for all $i \in \mathbb{N}$, and for all $j \in \mathbb{N}$, $j > i$, $\Gamma_j(G)$ (resp. $G^{(j)}$) is a normal subgroup of $\Gamma_i(G)$ (resp. $G^{(i)}$). Notice that $\Gamma_2(G) = G^{(1)}$ is the *commutator subgroup* of G. The *Abelianisation* of the group G, denoted by G^{Ab} is the quotient $G/\Gamma_2(G)$; the *Abelianisation* of an element $g \in G$ is its $\Gamma_2(G)$-coset in

G^{Ab}. The group G is said to be *perfect* if $G = G^{(1)}$, or equivalently if $G^{\mathrm{Ab}} = \{1\}$. Following P. Hall, for any group-theoretic property $\mathcal{P}$, a group G is said to be *residually* $\mathcal{P}$ if for any (non-trivial) element $x \in G$, there exists a group H with the property $\mathcal{P}$ and a surjective homomorphism $\varphi\colon G \to H$ such that $\varphi(x) \neq 1$. It is well known that a group G is *residually nilpotent* (respectively *residually soluble*) if and only if $\bigcap_{i\geq 1} \Gamma_i(G) = \{1\}$ (respectively $\bigcap_{i\geq 0} G^{(i)} = \{1\}$). If $g, h \in G$ then $[g, h] = ghg^{-1}h^{-1}$ will denote their commutator, and we shall use the symbol $g \rightleftharpoons h$ to mean that g and h commute.

Our main aim in this monograph is to study the lower central and derived series of the braid groups of the sphere and the punctured sphere. This was motivated in part by the study of the problem of the existence of a section for the short exact sequences (2) and (3). To obtain a positive answer, it suffices of course to exhibit an explicit section (although this may be easier said than done!). However, and in spite of the fact that we possess presentations of surface braid groups, in general it is very difficult to prove directly that such an extension does not split. One of the main methods that we used to prove the non-splitting of (2) for $n \geq 2$ and of (5) for $m \geq 4$ was based on the following observation: let $1 \to K \to G \to Q \to 1$ be a split extension of groups, where K is a normal subgroup of G, and let H be a normal subgroup of G contained in K. Then the extension $1 \to K/H \to G/H \to Q \to 1$ is also split. The condition on H is satisfied for example if H is an element of either the lower central series or the derived series of K. In [**GG1**], considering the extension (2) with $n \geq 3$, we showed that it was sufficient to take $H = \Gamma_2(K)$ to prove the non-splitting of the quotiented extension, and hence that of the full extension. In this case, the kernel $K/\Gamma_2(K)$ is Abelian, which simplifies somewhat the calculations in G/H. This was also the case in [**GG4**] for the extension (5) with $m \geq 4$. However, for the extension (2) with $n = 2$, it was necessary to go a stage further in the lower central series, and take $H = \Gamma_3(K)$. From the point of view of the splitting problem, it is thus helpful to know the lower central and derived series of the braid groups occurring in these group extensions. But these series are of course interesting in their own right, and help us to understand better the structure of surface braid groups.

Let us remark that braid groups of the punctured disc were studied in [**Lam**] in relation with the study of knots in handlebodies, and were used by Bigelow to understand the Lawrence-Krammer representation in his proof of the linearity of the Artin braid groups [**Big**]. Furthermore, during our study of the braid groups of the 2- and 3-punctured sphere, we will also come across some of the *Artin and affine Artin groups* (also known as *generalised braid groups*), notably those of types B and $\widetilde{C}$ [**Bri, T**].

The lower central series of groups and their successive quotients Γ_i/Γ_{i+1} are isomorphism invariants, and have been widely studied using commutator calculus, in particular for free groups of finite rank [**Hal, MKS**]. Falk and Randell, and independently Kohno investigated the lower central series of the pure braid group P_n, and were able to conclude that P_n is residually nilpotent [**FR1, Ko**]. Falk and Randell also studied the lower central series of generalised pure braid groups [**FR2, FR3**].

Using the Reidemeister-Schreier rewriting process, Gorin and Lin obtained a presentation of the commutator subgroup of B_n for $n \geq 3$ [**GL**] (see Theorem 36). For $n \geq 5$, they were able to infer that $(B_n)^{(1)} = (B_n)^{(2)}$, and so $(B_n)^{(1)}$ is perfect. From this it follows that $\Gamma_2(B_n) = \Gamma_3(B_n)$, hence B_n is not residually nilpotent. If $n = 3$ then they showed that $(B_3)^{(1)}$ is a free group of rank 2, while if $n = 4$, they proved that $(B_4)^{(1)}$ is a semi-direct product of two free groups of rank 2. By considering the action, one may see that $(B_4)^{(1)} \supsetneqq (B_4)^{(2)}$. The work of Gorin and Lin on these series was motivated by the study of almost periodic solutions of algebraic equations with almost periodic coefficients.

6. Statement of the main results

Chapter 1 is devoted to determining the lower central series of the braid groups of the sphere. In Theorem 3, we show that for all $n \geq 2$, the lower central series is constant from the commutator subgroup onwards. As in the case of the disc, the case $n = 4$ is particularly interesting: $\Gamma_2(B_4(\mathbb{S}^2))$ is a semi-direct product of the quaternion group of order 8 by the free group of rank 2. Here is the main theorem of Chapter 1:

THEOREM 3. *For all $n \geq 2$, the lower central series of $B_n(\mathbb{S}^2)$ is constant from the commutator subgroup onwards: $\Gamma_m(B_n(\mathbb{S}^2)) = \Gamma_2(B_n(\mathbb{S}^2))$ for all $m \geq 2$. The subgroup $\Gamma_2(B_n(\mathbb{S}^2))$ is as follows:*

(a) If $n = 1, 2$ then $\Gamma_2(B_n(\mathbb{S}^2)) = \{1\}$.
(b) If $n = 3$ then $\Gamma_2(B_n(\mathbb{S}^2)) \cong \mathbb{Z}_3$. Thus $B_3(\mathbb{S}^2) \cong \mathbb{Z}_3 \rtimes \mathbb{Z}_4$, the action being the non-trivial one.
(c) If $n = 4$ then $\Gamma_2(B_4(\mathbb{S}^2))$ admits a presentation of the following form:

generators: g_1, g_2, g_3, *where in terms of the usual generators of* $B_4(\mathbb{S}^2)$, $g_1 = \sigma_1^2\sigma_2\sigma_1^{-3}$, $g_2 = \sigma_1^3\sigma_2\sigma_1^{-4}$ *and* $g_3 = \sigma_3\sigma_1^{-1}$.
relations:

$$\begin{gathered} g_3^4 = 1 \\ [g_3^2, g_i] = 1 \text{ for } i = 1, 2 \\ [g_3, g_2g_1] = 1 \\ g_1^{-1}g_3^{-1}g_1 = g_2g_3g_2^{-1} \\ g_1^{-1}g_3^{-1}g_1 = g_1g_3g_1^{-1}g_3. \end{gathered}$$

Furthermore,

$$\Gamma_2(B_4(\mathbb{S}^2)) \cong \mathcal{Q}_8 \rtimes \mathbb{F}_2(a, b),$$

where $\mathcal{Q}_8 = \langle x, y \mid x^2 = y^2,\ xyx^{-1} = y^{-1}\rangle$ *is the quaternion group of order* 8, *and* $\mathbb{F}_2(a, b)$ *is the free group of rank* 2 *on two generators* a *and* b. *The action is given by:*

$$\begin{aligned} \varphi(a)(x) &= y & \varphi(a)(y) &= xy \\ \varphi(b)(x) &= yx & \varphi(b)(y) &= x. \end{aligned}$$

(d) In the cases $n = 5$ *and* $n \geq 6$, *a presentation for* $\Gamma_2(B_n(\mathbb{S}^2))$ *is given in Chapter 4, by Propositions 64 and 67 respectively.*

The lower central series of $B_n(\mathbb{S}^2)$ is thus completely determined. In particular, if $n \geq 3$ then $B_n(\mathbb{S}^2)$ is not residually nilpotent.

In Chapter 2, we study the derived series of $B_n(\mathbb{S}^2)$. As in the case of the disc, $(B_n(\mathbb{S}^2))^{(1)}$ is perfect if $n \geq 5$, in other words, the derived series of $B_n(\mathbb{S}^2)$ is constant from $(B_n(\mathbb{S}^2))^{(1)}$ onwards. The cases $n = 1, 2, 3$ are straightforward, and the groups $B_n(\mathbb{S}^2)$ are finite and soluble. In the case $n = 4$, we make use of the semi-direct product decomposition of $(B_4(\mathbb{S}^2))^{(1)}$ obtained in Theorem 3. Proposition 29 describes the structure of the commutator subgroup of a general semi-direct product, and shall be applied frequently throughout this monograph. This will enable us to show that from $(B_4(\mathbb{S}^2))^{(4)}$ onwards, the derived series of $B_4(\mathbb{S}^2)$ coincides with that of the free group of rank 2. We also determine some of the derived series quotients of $B_4(\mathbb{S}^2)$:

THEOREM 4. *The derived series of* $B_n(\mathbb{S}^2)$ *is as follows.*
(a) If $n = 1, 2$ *then* $(B_n(\mathbb{S}^2))^{(1)} = \{1\}$.
(b) If $n = 3$ *then* $(B_3(\mathbb{S}^2))^{(1)} \cong \mathbb{Z}_3$ *and* $(B_3(\mathbb{S}^2))^{(2)} = \{1\}$.
(c) Suppose that $n = 4$. *Then:*
(i) $(B_4(\mathbb{S}^2))^{(1)} = \Gamma_2(B_4(\mathbb{S}^2))$ *is given by part (c) of Theorem 3; it is isomorphic to the semi-direct product* $\mathcal{Q}_8 \rtimes \mathbb{F}_2$. *Further,* $B_4(\mathbb{S}^2)/(B_4(\mathbb{S}^2))^{(1)}$ *is isomorphic to* $\mathbb{Z}_6$.

(ii) $(B_4(\mathbb{S}^2))^{(2)}$ *is isomorphic to the semi-direct product* $\mathcal{Q}_8 \rtimes \mathbb{F}_2^{(1)}$*, where* $(\mathbb{F}_2)^{(1)}$ *is the commutator subgroup of the free group* $\mathbb{F}_2 = \mathbb{F}_2(a,b)$ *of rank* 2 *on two generators* a, b*. The action of* $(\mathbb{F}_2)^{(1)}$ *on* $\mathcal{Q}_8$ *is the restriction of the action of* $\mathbb{F}_2(a,b)$ *given in part (c) of Theorem 3. Further,*

$$(B_4(\mathbb{S}^2))^{(1)}/(B_4(\mathbb{S}^2))^{(2)} \cong \mathbb{Z}^2, \text{ and } B_4(\mathbb{S}^2)/(B_4(\mathbb{S}^2))^{(2)} \cong \mathbb{Z}^2 \rtimes \mathbb{Z}_6,$$

where the action of the generator $\overline{\sigma}$ *of* $\mathbb{Z}_6$ *on* $\mathbb{Z}^2$ *is given by left multiplication by the matrix* $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 1 \end{smallmatrix}\right)$*.*

(iii) $(B_4(\mathbb{S}^2))^{(3)}$ *is a subgroup of* $P_4(\mathbb{S}^2)$ *isomorphic to the direct product* $\mathbb{Z}_2 \times (\mathbb{F}_2)^{(2)}$*. Further,*

$$(B_4(\mathbb{S}^2))^{(2)}/(B_4(\mathbb{S}^2))^{(3)} \cong (\mathbb{Z}_2 \times \mathbb{Z}_2) \times (\mathbb{F}_2)^{(1)}/(\mathbb{F}_2)^{(2)}.$$

(iv) $(B_4(\mathbb{S}^2))^{(m)} \cong (\mathbb{F}_2)^{(m-1)}$ *for all* $m \geq 4$*. Further,*

$$(B_4(\mathbb{S}^2))^{(3)}/(B_4(\mathbb{S}^2))^{(4)} \cong \mathbb{Z}_2 \times (\mathbb{F}_2)^{(2)}/(\mathbb{F}_2)^{(3)},$$

and for $m \geq 4$*,*

$$(B_4(\mathbb{S}^2))^{(m)}/(B_4(\mathbb{S}^2))^{(m+1)} \cong (\mathbb{F}_2)^{(m-1)}/(\mathbb{F}_2)^{(m)}.$$

(d) If $n \geq 5$ *then* $(B_n(\mathbb{S}^2))^{(2)} = (B_n(\mathbb{S}^2))^{(1)}$*, so* $(B_n(\mathbb{S}^2))^{(1)}$ *is perfect. A presentation of* $(B_n(\mathbb{S}^2))^{(1)}$ *is given in Propositions 64 and 67.*

In particular, the derived series of $B_n(\mathbb{S}^2)$ is thus completely determined (up to knowing the derived series of the free group $\mathbb{F}_2$ of rank 2, see Remark 27).

Chapter 3 deals with the lower central and derived series of braid groups of the punctured sphere $B_m(\mathbb{S}^2 \setminus \{x_1, \ldots, x_n\})$, $n \geq 1$, and is divided into eight sections, according to the respective values of m and n. In Proposition 31 (Section 1), we recall a presentation of these groups obtained in [**GG4**]. In Section 2, we consider the case $n = 1$, and show that $B_m(\mathbb{S}^2 \setminus \{x_1\})$ is isomorphic to $B_n(\mathbb{D}^2)$ (Proposition 34). In Proposition 5, we study the series of $B_n(\mathbb{D}^2)$ in further detail, thus extending the results of Gorin and Lin:

Proposition 5. *Let* $m \geq 1$*. Then:*

(a) For all $s \geq 3$*,* $\Gamma_s(B_m(\mathbb{D}^2)) = \Gamma_2(B_m(\mathbb{D}^2))$*.*

(b) If $m = 1, 2$ *then* $(B_m(\mathbb{D}^2))^{(s)} = \{1\}$ *for all* $s \geq 1$*.*

(c) If $m = 3$ *then the derived series of* $(B_3(\mathbb{D}^2))^{(1)}$ *is that of the free group* $\mathbb{F}_2(u,v)$ *on two generators* u *and* v*, where* $u = \sigma_2\sigma_1^{-1}$ *and* $v = \sigma_1 u \sigma_1^{-1} = \sigma_1\sigma_2\sigma_1^{-2}$*. Further,*

$$B_3(\mathbb{D}^2)\,/(B_3(\mathbb{D}^2))^{(2)} \cong \mathbb{Z}^2 \rtimes \mathbb{Z},$$

where $\mathbb{Z}^2$ is the free Abelian group generated by the respective Abelianisations $\overline{u}$ and $\overline{v}$ of u and v, and the action is given by $\sigma \cdot \overline{u} = \overline{v}$ and $\sigma \cdot \overline{v} = -\overline{u} + \overline{v}$, where σ is a generator of $\mathbb{Z}$.

(d) If $m = 4$ then

$$(B_4(\mathbb{D}^2))^{(1)} \big/ (B_4(\mathbb{D}^2))^{(2)} \cong \mathbb{Z}^2, \text{ and}$$
$$(B_4(\mathbb{D}^2))^{(2)} \cong \mathbb{F}_2(a,b) \rtimes \Gamma_2(\mathbb{F}_2(u,v)),$$

where $a = \sigma_3\sigma_1^{-1}$ and $b = uau^{-1} = \sigma_2\sigma_3\sigma_1^{-1}\sigma_2^{-1}$.

Hence the lower central series (respectively derived series) of $B_m(\mathbb{D}^2)$ is completely determined for all $m \geq 1$ (respectively for all $m \neq 4$; for the case $m = 3$, this is again up to knowing the derived series of $\mathbb{F}_2$).

In the difficult case of the derived series of $B_4(\mathbb{D}^2)$, we then go on to describe some of the higher order terms and the successive derived series quotients:

Proposition 6.
$$\left(B_4(\mathbb{D}^2)\right)^{(2)} \Big/ \left(B_4(\mathbb{D}^2)\right)^{(3)} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times (\Gamma_2(\mathbb{F}_2(u,v)))^{Ab}.$$

Proposition 7. $(B_4(\mathbb{D}^2))^{(3)} \cong \mathbb{F}_5(z_1, \ldots, z_5) \rtimes (\mathbb{F}_2(u,v))^{(2)}$.

The action for this semi-direct product will be described by equations (20) and (21). From this, we may obtain the Abelianisation of $(B_4(\mathbb{D}^2))^{(3)}$:

Proposition 8.
$$\begin{aligned}\left((B_4(\mathbb{D}^2))^{(3)}\right)^{Ab} &= (B_4(\mathbb{D}^2))^{(3)} \big/ (B_4(\mathbb{D}^2))^{(4)} \\ &\cong \mathbb{Z}^3 \times \mathbb{Z}_{18} \times \mathbb{Z}_{18} \times (\mathbb{F}_2(u,v))^{(2)} \big/ (\mathbb{F}_2(u,v))^{(3)}.\end{aligned}$$

This result suggests that the derived series of $B_4(\mathbb{D}^2)$ is highly non trivial. In principle, using the semi-direct product structure of $(B_4(\mathbb{D}^2))^{(3)}$ and Proposition 29, it is possible to discover further terms of the derived series, but in practice, the calculations become very hard. The main results of Section 2 are summed up in Table 1.

In Section 3, we comment briefly on the case $m = 1$ which is that of a free group of rank $n - 1$. From Section 4 of Chapter 3 onwards, we suppose that $n \geq 2$. If $m \geq 3$ (resp. $m \geq 5$) the lower central series (resp. the derived series) of $B_m(\mathbb{S}^2 \setminus \{x_1, \ldots, x_n\})$ is constant from the commutator subgroup onwards. Once more, for the derived series, $m = 4$ represents a challenging case. Nevertheless, we are able to determine some of the derived series quotients. The main theorem of Section 4 is as follows:

values of m	series/group	result	reference
$\forall m \geq 1$	lower central	$\Gamma_3(G) = \Gamma_2(G)$	[**GL**] (see Theorem 36)
$\forall m \geq 5$	derived	$G^{(2)} = G^{(1)}$	
$m = 3$	derived	$G^{(i)} = (\mathbb{F}_2)^{(i-1)}$, $\forall i \geq 1$	
	$\Gamma_2(G)$	$\mathbb{F}_2$	
	$G^{(1)}/G^{(2)}$	$\mathbb{Z}^2$	
	$G/G^{(2)}$	$\mathbb{Z}^2 \rtimes \mathbb{Z}$	Proposition 5
$m = 4$	$\Gamma_2(G)$	$\mathbb{F}_2 \rtimes \mathbb{F}_2$	[**GL**] (see Theorem 36)
	$G^{(1)}/G^{(2)}$	$\mathbb{Z}^2$	
	$G^{(2)}$	$\mathbb{F}_2 \rtimes (\mathbb{F}_2)^{(1)}$	Proposition 5
	$G^{(2)}/G^{(3)}$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times ((\mathbb{F}_2)^{(1)})^{\mathrm{Ab}}$	Proposition 6
	$G^{(3)}$	$\mathbb{F}_5 \rtimes (\mathbb{F}_2)^{(2)}$	Proposition 7
	$G^{(3)}/G^{(4)}$	$\mathbb{Z}^3 \times \mathbb{Z}_{18} \times \mathbb{Z}_{18} \times ((\mathbb{F}_2)^{(2)})^{\mathrm{Ab}}$	Proposition 8

TABLE 1. Summary of results of Section 2, Chapter 3 concerning the lower central and derived series of $G = B_m(\mathbb{D}^2)$. For the semi-direct product actions, one should consult the corresponding reference.

THEOREM 9. *Let $n \geq 2$. Then:*

(a) If $m \geq 3$ then

$$\Gamma_3(B_m(\mathbb{S}^2 \setminus \{x_1, \ldots, x_n\})) = \Gamma_2(B_m(\mathbb{S}^2 \setminus \{x_1, \ldots, x_n\})).$$

(b) If $m \geq 5$ then

$$(B_m(\mathbb{S}^2 \setminus \{x_1, \ldots, x_n\}))^{(2)} = (B_m(\mathbb{S}^2 \setminus \{x_1, \ldots, x_n\}))^{(1)}.$$

(c) If $m = 4$ then

$$B_4(\mathbb{S}^2 \setminus \{x_1, \ldots, x_n\}) \big/ (B_4(\mathbb{S}^2 \setminus \{x_1, \ldots, x_n\}))^{(2)} \cong \big(\mathbb{Z}^2 \rtimes \mathbb{Z}\big) \times \mathbb{Z}^{n-1}$$

where the semi-direct product structure is that of part (c) of Proposition 5, and

$$(B_4(\mathbb{S}^2 \setminus \{x_1, \ldots, x_n\}))^{(1)} \big/ (B_4(\mathbb{S}^2 \setminus \{x_1, \ldots, x_n\}))^{(2)} \cong \mathbb{Z}^2.$$

Alternatively,

$$B_4(\mathbb{S}^2 \setminus \{x_1, \ldots, x_n\})/(B_4(\mathbb{S}^2 \setminus \{x_1, \ldots, x_n\}))^{(2)} \cong \mathbb{Z}^2 \rtimes \mathbb{Z}^n,$$

where $\mathbb{Z}^2 \cong (B_4(\mathbb{S}^2 \setminus \{x_1, \ldots, x_n\}))^{(1)}/(B_4(\mathbb{S}^2 \setminus \{x_1, \ldots, x_n\}))^{(2)}$ is the free Abelian group with basis $\{\overline{u}, \overline{v}\}$, $\mathbb{Z}^n \cong B_4(\mathbb{S}^2 \setminus \{x_1, \ldots, x_n\})^{Ab}$

values of m	series/group	result	reference
$\forall m \geq 3$	lower central	$\Gamma_3 = \Gamma_2$	Theorem 9
$\forall m \geq 5$	derived	$G^{(2)} = G^{(1)}$	
$m = 4$	$G/G^{(2)}$	$(\mathbb{Z}^2 \rtimes \mathbb{Z}) \times \mathbb{Z}^{n-1}$	
		$\mathbb{Z}^2 \rtimes \mathbb{Z}^n$	
	$G^{(1)}/G^{(2)}$	$\mathbb{Z}^2$	

TABLE 2. Summary of results of Section 4, Chapter 3 concerning the lower central and derived series of $G = B_m(\mathbb{S}^2 \setminus \{x_1, \ldots, x_n\})$, $m \geq 3$, $n \geq 2$.

has basis $\{\sigma, \rho_1, \ldots, \rho_{n-1}\}$*, and the action is given by*

$$\sigma \cdot \overline{u} = \overline{v} \qquad \sigma \cdot \overline{v} = -\overline{u} + \overline{v}$$
$$\rho_i \cdot \overline{u} = \overline{u} \qquad \rho_i \cdot \overline{v} = \overline{v}$$

for all $1 \leq i \leq n-1$*.*

So if $n \geq 2$, the lower central and derived series of the braid group $B_m(\mathbb{S}^2 \setminus \{x_1, \ldots, x_n\})$ are completely determined, with the exception of a small number of values of m: for the lower central series, they consist of just $m = 2$, and for the derived series, $m = 2, 3$ and 4.

The case $m \geq 2$ and $n = 2$ is considered in Sections 5, 6 and 7. Applying the results of Proposition 34, one may see that $B_m(\mathbb{S}^2 \setminus \{x_1, x_2\})$ is isomorphic to the m-string braid group $B_m(\mathbb{A})$ of the annulus $\mathbb{A} = [0,1] \times \mathbb{S}^1$, and is thus an Artin group of type B_m. In Proposition 10, Section 5, we prove the following general result concerning the structure of $\Gamma_2(B_m(\mathbb{S}^2 \setminus \{x_1, x_2\}))$:

PROPOSITION 10. *Let* $m \geq 2$*. Then:*

(a) $B_m(\mathbb{S}^2 \setminus \{x_1, x_2\}) \cong \mathbb{F}_m \rtimes B_m(\mathbb{D}^2)$*, where the action* φ *is given by the Artin representation of* $B_m(\mathbb{D}^2)$ *as a subgroup of* $Aut(\mathbb{F}_m)$ *(see equation (27)).*

(b) $\Gamma_2(B_m(\mathbb{S}^2 \setminus \{x_1, x_2\})) \cong Ker(\rho) \rtimes \Gamma_2(B_m(\mathbb{D}^2))$*, where*

$$\rho\colon\ \mathbb{F}_m(A_{2,3}, \ldots, A_{2,m+2}) \to \mathbb{Z}$$

is the augmentation homomorphism, and the action is that induced by φ *(the generators* $A_{i,j}$ *are described in Proposition 31).*

The semi-direct product structure allows us to determine some derived series quotients:

PROPOSITION 11.

$$\left(B_3\left(\mathbb{S}^2\setminus\{x_1,x_2\}\right)\right)^{(1)} \Big/ \left(B_3\left(\mathbb{S}^2\setminus\{x_1,x_2\}\right)\right)^{(2)} \cong \mathbb{Z}^4.$$

PROPOSITION 12.

$$B_3\left(\mathbb{S}^2\setminus\{x_1,x_2\}\right) \Big/ \left(B_3\left(\mathbb{S}^2\setminus\{x_1,x_2\}\right)\right)^{(2)} \cong \mathbb{Z}^4 \rtimes \mathbb{Z}^2,$$

where $\mathbb{Z}^4$ *has a basis* $\left\{\widetilde{\alpha_0},\widetilde{\beta_0},\widetilde{u},\widetilde{v}\right\}$, $\mathbb{Z}^2$ *has a basis* $\{\sigma,\rho_1\}$, *and the action is given by:*

$$\begin{aligned}
&\sigma\cdot\widetilde{u}=\widetilde{v} && \sigma\cdot\widetilde{v}=-\widetilde{u}+\widetilde{v}\\
&\sigma\cdot\widetilde{\alpha_0}=\widetilde{\beta_0} && \sigma\cdot\widetilde{\beta_0}=\widetilde{\beta_0}-\widetilde{\alpha_0}\\
&\rho_1\cdot\widetilde{\alpha_0}=\widetilde{\alpha_0} && \rho_1\cdot\widetilde{\beta_0}=\widetilde{\beta_0}\\
&\rho_1\cdot\widetilde{u}=-\widetilde{\alpha_0}-\widetilde{u}+\widetilde{v} && \rho_1\cdot\widetilde{v}=-\widetilde{\beta_0}-\widetilde{u}.
\end{aligned}$$

We then give an alternative proof of Proposition 11, showing along the way that the commutator subgroup of $B_3(\mathbb{S}^2\setminus\{x_1,x_2\})$ is the semi-direct product of a given infinite rank subgroup of a free group of rank 5 by a free group of rank 2 (see Proposition 42).

In Section 6, we study the lower central series of $B_2(\mathbb{S}^2\setminus\{x_1,x_2\})$ (which is one of the outstanding cases not covered by Theorem 9). Using an exact sequence due to Stallings (see equation (8)), we prove the following:

COROLLARY 13. $\Gamma_2(B_2(\mathbb{S}^2\setminus\{x_1,x_2\}))\cong\Gamma_2(\mathbb{F}_2(a,b))\rtimes\mathbb{Z}$, *where the action of* $\mathbb{Z}$ *on* $\Gamma_2(\mathbb{F}_2(a,b))$ *is given by conjugation by* $b^{-1}a$.

The group $B_2(\mathbb{S}^2\setminus\{x_1,x_2\})$ is particularly fascinating, not least because it may be interpreted in many different ways: as the 2-string braid group $B_2(\mathbb{A})$ of the annulus (and so as the Artin group of type B_2), and as the Baumslag-Solitar group $\mathrm{BS}(2,2)$, for example (see Remarks 49). It is also a one-relator group with non-trivial (infinite cyclic) centre, which applying results of Kim and McCarron [**KMc, McCa**] implies that:

PROPOSITION 14. $B_2(\mathbb{S}^2\setminus\{x_1,x_2\})$ *is residually nilpotent and residually a finite* 2*-group.*

Further, using the fact that the quotient of $B_2(\mathbb{S}^2\setminus\{x_1,x_2\})$ by its centre is isomorphic to the free product $\mathbb{Z}_2*\mathbb{Z}$, we prove that apart from the first term, the lower central series of these two groups coincide, and applying results of Gaglione and Labute [**Ga, Lab**] which describe the lower central series of certain free products of cyclic groups, we are able to determine completely the lower central series (in terms of that of $\mathbb{Z}_2 * \mathbb{Z}$), as well as the successive lower central series quotients of $B_2(\mathbb{S}^2 \setminus \{x_1, x_2\})$ in an explicit manner:

THEOREM 15. *For all* $i \geq 2$, $\Gamma_i(B_2(\mathbb{S}^2 \setminus \{x_1, x_2\})) \cong \Gamma_i(\mathbb{Z}_2 * \mathbb{Z})$, *and:*

$$\begin{aligned}\Gamma_i(B_2(\mathbb{S}^2 \setminus \{x_1, x_2\}))/\Gamma_{i+1}(B_2(\mathbb{S}^2 \setminus \{x_1, x_2\})) &\cong \Gamma_i(\mathbb{Z}_2 * \mathbb{Z})/\Gamma_{i+1}(\mathbb{Z}_2 * \mathbb{Z}) \\ &\cong \underbrace{\mathbb{Z}_2 \oplus \cdots \oplus \mathbb{Z}_2}_{R_i \text{ times}},\end{aligned}$$

where

$$R_i = \sum_{j=0}^{i-2} \left(\sum_{\substack{k \mid i-j \\ k>1}} \mu\left(\frac{i-j}{k}\right) \frac{k\alpha_k}{i-j} \right),$$

μ *is the Möbius function, and*

$$\alpha_k = \frac{1}{k}\left(Tr \begin{pmatrix} 0 & -1 \\ -1 & 1 \end{pmatrix}^k - 1 \right).$$

From this, we may see (Corollary 53) that apart from the first term, the derived series of $B_2(\mathbb{S}^2 \setminus \{x_1, x_2\})$ is that of $\pi(\mathbb{F}_2)$, where $\pi\colon \mathbb{F}_2 = \mathbb{Z} * \mathbb{Z} \to \mathbb{Z}_2 * \mathbb{Z}$ is the homomorphism obtained by taking the first factor modulo 2.

In Section 7, we consider the more general case of the m-string braid group $B_m(\mathbb{S}^2 \setminus \{x_1, x_2\}))$, $m \geq 3$, which we know to be isomorphic to the m-string braid group $B_m(\mathbb{A})$ of the annulus. With this interpretation, Kent and Peifer gave a nice presentation of this group (Proposition 56) from which they were able to conclude that $B_m(\mathbb{A})$ is a semi-direct product of the affine Artin group $\widetilde{A}_{m-1}$ by $\mathbb{Z}$ (Corollary 57) [**KP**]. Applying Proposition 29 once more, we obtain in Proposition 58 a presentation of $\Gamma_2(B_m(\mathbb{S}^2 \setminus \{x_1, x_2\}))$ (which as we shall see, is isomorphic to $\Gamma_2(\widetilde{A}_{m-1})$), from which we may deduce:

COROLLARY 16. *Let* $m \geq 3$. *Then*

$$\left(B_m\left(\mathbb{S}^2 \setminus \{x_1, x_2\}\right)\right)^{(1)} \Big/ \left(B_m\left(\mathbb{S}^2 \setminus \{x_1, x_2\}\right)\right)^{(2)} \cong \begin{cases} \mathbb{Z}^4 & \text{if } m = 3 \\ \mathbb{Z}^2 & \text{if } m = 4 \\ \mathbb{Z} & \text{if } m \geq 5. \end{cases}$$

The main results of Sections 5, 6 and 7 of Chapter 3 are summed up in Table 3.

values of m	series/group	result	reference
$\forall m \geq 2$	$\Gamma_2(G)$	$(\mathbb{F}_m)^{(1)} \rtimes \Gamma_2(B_m(\mathbb{D}^2))$	Proposition 10
$m = 2$	lower central	$\Gamma_i = \Gamma_i(\mathbb{Z}_2 * \mathbb{Z})$, $i \geq 2$	Theorem 15
	lower central quotients $\Gamma_i(G)/\Gamma_{i+1}(G)$	$\Gamma_i(\mathbb{Z}_2 * \mathbb{Z})/\Gamma_{i+1}(\mathbb{Z}_2 * \mathbb{Z})$	
		$\bigoplus_{j=1}^{R_i} \mathbb{Z}_2$	
	$\Gamma_2(G)$	$(\mathbb{F}_2)^{(1)} \rtimes \mathbb{Z}$	Corollary 13
	$\Gamma_2(G)$	$\mathbb{F}_\infty$	Corollary 47
	$\Gamma_2(G)/\Gamma_3(G)$	$\mathbb{Z}_2$	Proposition 54
	$\Gamma_3(G)$	$\mathbb{F}_\infty$	
$m = 3$	$\Gamma_2(G)$	$\mathbb{F}_\infty \rtimes \mathbb{F}_2$	Proposition 42
	$G^{(1)}/G^{(2)}$	$\mathbb{Z}^4$	Proposition 11
	$G/G^{(2)}$	$\mathbb{Z}^4 \rtimes \mathbb{Z}^2$	Proposition 12
$m = 4$	$G^{(1)}/G^{(2)}$	$\mathbb{Z}^2$	Corollary 16
$m \geq 5$	$G^{(1)}/G^{(2)}$	$\mathbb{Z}$	

TABLE 3. Summary of results of Sections 5, 6 and 7 of Chapter 3 concerning the lower central and derived series of $G = B_m(\mathbb{S}^2 \setminus \{x_1, x_2\})$, $m \geq 2$. In each case, $\mathbb{F}_\infty$ is a given free group of countable infinite rank.

In Section 8 of Chapter 3, we consider $B_m(\mathbb{S}^2 \setminus \{x_1, x_2, x_3\})$, $m \geq 2$, which is also one of the outstanding cases for the derived series not covered by Theorem 9. This group is isomorphic to the affine Artin group of type $\widetilde{C}_m$ for which little seems to be known [**All**]. Despite the existence of nice presentations for this group [**BG**], we were not able to describe satisfactorily the commutator subgroup even for $m = 2$. We obtain however some partial results, notably in Proposition 60 the fact that the successive lower central series quotients of $B_2(\mathbb{S}^2 \setminus \{x_1, x_2, x_3\})$ are finite direct sums of $\mathbb{Z}_2$, which generalises part of Theorem 15, as well as for all $i \geq 1$ and $m \geq 2$, $(B_m(\mathbb{S}^2 \setminus \{x_1, x_2, x_3\}))^{(i)}$ is a semi-direct product of some group K_i by $(B_m(\mathbb{D}^2))^{(i)}$ (Proposition 61).

Finally in Chapter 4, we give presentations of the commutator subgroups $\Gamma_2(B_n(\mathbb{S}^2))$ of the sphere braid groups for $n \geq 4$, and in the case $n = 4$, in Proposition 63 we derive the presentation of $\Gamma_2(B_4(\mathbb{S}^2))$ given in Theorem 3(c).

7. Extension to surfaces of higher genus

Since work on this paper started, one of the authors, in collaboration with P. Bellingeri and S. Gervais has undertaken the study of the lower central series of braid groups of orientable surfaces, with and without boundary, of genus $g \geq 1$ [**BGG**]. We remark that some of the techniques appearing in this monograph were used subsequently in that paper. It is worth stating the corresponding results of [**BGG**] which contrast somewhat with those obtained here for the sphere and punctured sphere.

THEOREM 17 ([**BGG**]). *Let M be a compact, connected orientable surface without boundary, of genus $g \geq 1$, and let $m \geq 3$. Then:*

(a) $\Gamma_1(B_m(M))/\Gamma_2(B_m(M)) \cong \mathbb{Z}^{2g} \oplus \mathbb{Z}_2$.
(b) $\Gamma_2(B_m(M))/\Gamma_3(B_m(M)) \cong \mathbb{Z}_{n-1+g}$.
(c) $\Gamma_3(B_m(M)) = \Gamma_4(B_m(M))$. *Moreover,* $\Gamma_3(B_m(M))$ *is perfect for* $m \geq 5$.

This implies that braid groups of compact, connected orientable surfaces without boundary may be distinguished by their lower central series (indeed by the first two lower central quotients).

THEOREM 18 ([**BGG**]). *Let $g \geq 1$, $q \geq 1$ and $m \geq 3$. Let M be a compact, connected orientable surface of genus g with q boundary components. Then:*

(a) $\Gamma_1(B_m(M))/\Gamma_2(B_m(M)) \cong \mathbb{Z}^{2g+q-1} \oplus \mathbb{Z}_2$.
(b) $\Gamma_2(B_m(M))/\Gamma_3(B_m(M)) \cong \mathbb{Z}$.
(c) $\Gamma_3(B_m(M)) = \Gamma_4(B_m(M))$. *Moreover,* $\Gamma_3(B_m(M))$ *is perfect for* $m \geq 5$.

Thus if $m \geq 3$ and if M a compact surface (with or without boundary) of genus $g \geq 1$, since $\Gamma_3(B_m(M)) \neq \{1\}$, $B_m(M)$ is not residually nilpotent. Moreover, we observe similar phenomena to those seen in Theorem 9 for the punctured sphere (stability of the lower central series for $m \geq 3$, perfectness of the $\Gamma_i(B_m(M))$ for $m \geq 5$). However, they occur one stage further, not from the commutator subgroup onwards, but from Γ_3 onwards.

Just as for $B_2(\mathbb{S}^2 \setminus \{x_1, x_2\})$, the 2-string braid groups represent a very difficult and interesting case. In the case of the 2-torus $\mathbb{T}^2$, we prove that its 2-string braid group is residually nilpotent. Further, arguing as in the proof of Theorem 15, we show that apart from the first term, the lower central series of $B_2(\mathbb{T}^2)$ and $\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$ coincide, and by applying Gaglione's results, we may also determine explicitly all of their successive lower central series quotients. More precisely:

THEOREM 19 ([**BGG**]).
(a) $B_2(\mathbb{T}^2)$ *is residually nilpotent.*
(b) For all $i \geq 2$*:*
(i) $\Gamma_i(B_2(\mathbb{T}^2)) \cong \Gamma_i(\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2)$.
(ii) $\Gamma_i(B_2(\mathbb{T}^2))/\Gamma_{i+1}(B_2(\mathbb{T}^2))$ *is isomorphic to the direct sum of* R_i *copies of* $\mathbb{Z}_2$*, where:*

$$R_i = \sum_{j=1}^{i-2} \left(\sum_{\substack{k|i-j \\ k>1}} \mu\left(\frac{i-j}{k}\right) \frac{k\alpha_k}{i-j} \right) \quad \textit{and} \quad k\alpha_k = 2^k + 2(-1)^k.$$

As in the case of the 2-string braid group of the n-punctured sphere, $n \geq 3$, it seems to be very difficult even to describe the commutator subgroup of the 2-string braid groups of orientable surfaces of higher genus.

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CHAPTER 1

The lower central series of $B_n(\mathbb{S}^2)$

The main aim of this chapter is to prove Theorem 3, which describes the lower central series of $B_n(\mathbb{S}^2)$. This will be carried out in Section 2. Before doing so, in Section 1, we state and prove some general results concerning the splitting of the short exact sequence (6) (Proposition 20), as well as homological conditions for the stabilisation of the lower central series of a group (Lemma 23).

1. Generalities

Let $n \in \mathbb{N}$. Let $B_n(\mathbb{S}^2)$ denote the braid group of $\mathbb{S}^2$ on n strings, let $(B_n(\mathbb{S}^2))^{\text{Ab}} = B_n(\mathbb{S}^2)/\Gamma_2(B_n(\mathbb{S}^2))$ denote the Abelianisation of $B_n(\mathbb{S}^2)$, and let $\alpha\colon B_n(\mathbb{S}^2) \to (B_n(\mathbb{S}^2))^{\text{Ab}}$ be the canonical projection. Then we have the following short exact sequence:

$$1 \longrightarrow \Gamma_2(B_n(\mathbb{S}^2)) \longrightarrow B_n(\mathbb{S}^2) \xrightarrow{\alpha} (B_n(\mathbb{S}^2))^{\text{Ab}} \longrightarrow 1. \qquad (6)$$

We first prove the following result which deals with the splitting of this short exact sequence.

Proposition 20. *Let $n \in \mathbb{N}$.*

(a) $(B_n(\mathbb{S}^2))^{Ab} = B_n(\mathbb{S}^2)/\Gamma_2(B_n(\mathbb{S}^2)) \cong \mathbb{Z}_{2(n-1)}$.

(b) The short exact sequence (6) splits if and only if n is odd, where the action on $\Gamma_2(B_n(\mathbb{S}^2))$ by a generator of $\mathbb{Z}_{2(n-1)}$ is given by conjugation by $\sigma_1 \ldots \sigma_{n-2}\sigma_{n-1}^2$.

(c) If n is even then $B_n(\mathbb{S}^2)$ is not isomorphic to the semi-direct product of a subgroup K by $\mathbb{Z}_{2(n-1)}$.

Proof.

(a) This follows easily from the presentation (4) of the group $B_n(\mathbb{S}^2)$. The generators σ_i of $B_n(\mathbb{S}^2)$ are all identified by α to a single generator $\widetilde{\sigma} = \alpha(\sigma_i)$ of $\mathbb{Z}_{2(n-1)}$.

(b) In order to construct a section, we consider the elements of $B_n(\mathbb{S}^2)$ of order $2(n-1)$. According to Murasugi's classification of the torsion elements of $B_n(\mathbb{S}^2)$ [**Mu**], these elements are precisely the conjugates of the elements of the form $(\sigma_1 \cdots \sigma_{n-2}\sigma_{n-1}^2)^r$, where r

and $2(n-1)$ are coprime. Such an element projects to $\widetilde{\sigma}^{rn}$ whose order is $2(n-1)/\gcd(rn, 2(n-1))$. Since

$$\gcd(rn, 2(n-1)) = \gcd(n, 2(n-1)) = \gcd(n, 2),$$

the result follows from equation (6) and part (a).

(c) Let $n \in \mathbb{N}$ be even. We first prove the following lemma:

LEMMA 21. *Let G be a group whose Abelianisation G^{Ab} is Hopfian i.e. G^{Ab} is not isomorphic to any of its proper quotients. Suppose that there exists a group H isomorphic to G^{Ab}, a normal subgroup K of G, and a split short exact sequence $1 \to K \to G \to H \to 1$. Then $G \cong \Gamma_2(G) \rtimes G^{Ab}$.*

PROOF OF LEMMA 21. Let $\alpha\colon G \to G^{\mathrm{Ab}}$ denote Abelianisation, let $\xi\colon G \to H$ denote the homomorphism in the given short exact sequence, and let $s\colon H \to G$ be a section for ξ. Since $H \cong G/K$ is Abelian, it follows from standard properties of the commutator subgroup that $\Gamma_2(G) \subseteq K$. Hence we have the following commutative diagram:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Gamma_2(G) & \hookrightarrow & G & \xrightarrow{\alpha} & G^{\mathrm{Ab}} & \longrightarrow & 1 \\ & & \downarrow & & \| & & & & \\ 1 & \longrightarrow & K & \hookrightarrow & G & \underset{s}{\overset{\xi}{\rightleftarrows}} & H & \longrightarrow & 1, \end{array}$$

This extends to a commutative diagram of short exact sequences by taking $\rho\colon G^{\mathrm{Ab}} \to H$ defined by $\rho(y) = \xi(x)$ for all $y \in G^{\mathrm{Ab}}$, where $x \in G$ is any element satisfying $\alpha(x) = y$. This homomorphism is well defined, and is surjective since ξ and α are. But $G^{\mathrm{Ab}} \cong H$ is Hopfian by hypothesis, which implies that ρ is an isomorphism. Hence $\alpha = \rho^{-1} \circ \xi$, and $s \circ \rho$ is a section for α, which proves the lemma. ☐

By taking $G = B_n(\mathbb{S}^2)$ and $K = \mathbb{Z}_{2(n-1)}$ in the statement of Lemma 21, if $B_n(\mathbb{S}^2)$ were a semi-direct product of K with H then this would contradict part (b). This completes the proof of Proposition 20. ☐

REMARK 22. If n is even, let us consider the natural projection $p\colon \mathbb{Z}_{2(n-1)} \to \mathbb{Z}_{n-1}$. Then we have a short exact sequence:

$$1 \longrightarrow \Gamma_2^*(B_n(\mathbb{S}^2)) \longrightarrow B_n(\mathbb{S}^2) \xrightarrow{\alpha^*} \mathbb{Z}_{(n-1)} \longrightarrow 1.$$

where $\alpha^* = p \circ \alpha$, and $\Gamma_2^*(B_n(\mathbb{S}^2))$ is the kernel of α^*. It is not difficult to see that this short exact sequence splits: a section is given by sending the generator of $\mathbb{Z}_{(n-1)}$ to $(\sigma_1 \ldots \sigma_{n-2}\sigma_{n-1}^2)^{2^r}$, where 2^r is the greatest power of 2 dividing n.

Let G be a group which acts on a group H. Following [**HMR**, p. 67], we may define the commutator subgroup with respect to this action by

$$\Gamma_G(H) = \langle (g \star h)\, kh^{-1}k^{-1} \mid g \in G, h, k \in H \rangle, \tag{7}$$

where $g \star h$ denotes the action of g on h. We say that the action is *perfect* if $\Gamma_G(H) = H$. Note that if H is a normal subgroup of G then $H \supseteq \Gamma_G(H) = [G, H] \supseteq [H, H]$ for the action of conjugation of G on H. In particular, if $G = H$ then $\Gamma_G(H) = \Gamma_2(G)$ for the action of conjugation of G on itself. If this action is perfect then the group G is perfect.

LEMMA 23. *Let G be a group, and let G^{Ab} be its Abelianisation. Let $\delta\colon H_2(G, \mathbb{Z}) \to H_2(G^{Ab}, \mathbb{Z})$ be the homomorphism induced by Abelianisation. Then*

$$\Gamma_2(G)/\Gamma_3(G) \cong \mathit{Coker}(\delta) \cong H_0\left(G^{Ab}, H_1(\Gamma_2(G), \mathbb{Z})\right).$$

In particular:

(a) $\Gamma_2(G) = \Gamma_3(G)$ if and only if δ is surjective.
(b) If $H_2(G^{Ab}, \mathbb{Z})$ is trivial then $\Gamma_n(G) = \Gamma_2(G)$ for all $n \geq 2$.
(c) If either the action (by conjugation) of G on $\Gamma_2(G)$ or the action (by conjugation) of G^{Ab} on $H_1(\Gamma_2(G), \mathbb{Z})$ is perfect then $\Gamma_n(G) = \Gamma_2(G)$ for all $n \geq 2$.

PROOF. Recall that if $1 \to K \to G \to Q \to 1$ is an extension of groups then we have a 6-term exact sequence

$$H_2(G) \to H_2(Q) \to K/[G, K] \to H_1(G) \to H_1(Q) \to 1 \tag{8}$$

due to Stallings [**Bro, McCl, St**]. Applying this to the short exact sequence:

$$1 \to \Gamma_2(G) \to G \to G^{\mathrm{Ab}} \to 1, \tag{9}$$

we obtain:

$$H_2(G, \mathbb{Z}) \xrightarrow{\delta} H_2(G^{\mathrm{Ab}}, \mathbb{Z}) \to \Gamma_2(G)/\Gamma_3(G) \to H_1(G, \mathbb{Z}) \to G^{\mathrm{Ab}} \to 1.$$

But $H_1(G, \mathbb{Z}) \to G^{\mathrm{Ab}}$ is an isomorphism, so this becomes

$$H_2(G, \mathbb{Z}) \xrightarrow{\delta} H_2(G^{\mathrm{Ab}}, \mathbb{Z}) \to \Gamma_2(G)/\Gamma_3(G) \to 1.$$

Hence $\Gamma_2(G)/\Gamma_3(G) \cong \mathrm{Coker}\,(\delta)$ which yields the first isomorphism. To obtain the second, we consider the Lyndon-Hochschild-Serre spectral sequence [**Bro, McCl**] applied to the short exact sequence (9), for which the relevant terms are $E^2_{(2,0)} = H_2(G^{\mathrm{Ab}}, \mathbb{Z})$ and $E^2_{(0,1)} =$

$H_0\left(G^{\mathrm{Ab}}, H_1\left(\Gamma_2(G), \mathbb{Z}\right)\right)$. Since $H_1(G) = H_1(G^{\mathrm{Ab}})$, the differential $d_2\colon E^2_{(2,0)} \to E^2_{(0,1)}$ is surjective, with kernel $E^\infty_{(2,0)}$. From the general definition of the filtration of $H_2(G)$ given by the spectral sequence, we have a surjection $H_2(G) \to E^\infty_{(2,0)}$, and hence the following exact sequence:

$$H_2(G) \to E^\infty_{(2,0)} \hookrightarrow E^2_{(2,0)} \to E^2_{(0,1)} \to 1.$$

Hence $\operatorname{Im}(\delta) = E^\infty_{(2,0)}$, and

$$\operatorname{Coker}(\delta) = E^2_{(2,0)}/\operatorname{Im}(\delta) \cong E^2_{(0,1)} = H_0\left(G^{\mathrm{Ab}}, H_1\left(\Gamma_2(G), \mathbb{Z}\right)\right)$$

as required. From the first isomorphism, one may check that part (a) is satisfied. Part (b) then follows easily.

To prove part (c), if the action by conjugation of G on $\Gamma_2(G)$ is perfect then $\Gamma_G(\Gamma_2(G)) = [G, \Gamma_2(G)] = \Gamma_3(G) = \Gamma_2(G)$ and the result is clear. Now let us consider the action of G on $H_1(\Gamma_2(G)) = (\Gamma_2(G))^{\mathrm{Ab}}$ given by conjugation, defined by $g \cdot \widetilde{h} = \widetilde{ghg^{-1}}$, where $g, h \in G$, and $\widetilde{\ }$ denotes Abelianisation in $\Gamma_2(G)$. If $g \in \Gamma_2(G)$ then the induced action on $(\Gamma_2(G))^{\mathrm{Ab}}$ is trivial, so the original action factors through G^{Ab}, and we obtain an action of G^{Ab} on $(\Gamma_2(G))^{\mathrm{Ab}}$ given by $\widetilde{g} \cdot \widetilde{h} = \widetilde{ghg^{-1}}$ ($\widetilde{g}$ denotes the Abelianisation of g in G). Suppose that this action is perfect, so that $\Gamma_{G^{\mathrm{Ab}}}((\Gamma_2(G))^{\mathrm{Ab}}) = (\Gamma_2(G))^{\mathrm{Ab}}$. Now

$$\Gamma_{G^{\mathrm{Ab}}}((\Gamma_2(G))^{\mathrm{Ab}}) = [G, \Gamma_2(G)]/[\Gamma_2(G), \Gamma_2(G)] = \Gamma_3(G)/[\Gamma_2(G), \Gamma_2(G)],$$

and since $\Gamma_3(G) \subseteq \Gamma_2(G)$, it follows that $\Gamma_3(G) = \Gamma_2(G)$, which implies the result. ☐

Remark 24. The hypothesis of part (b) of the lemma holds for example if G^{Ab} is cyclic. Recall that if G^{Ab} is finitely-generated then this condition is also necessary: if H is a finitely-generated Abelian group satisfying $H_2(H, \mathbb{Z}) = \{0\}$ then H is cyclic.

2. The lower central series of $B_n(\mathbb{S}^2)$

Now we come to the main result of this chapter.

Theorem 3. *For all $n \geq 2$, the lower central series of $B_n(\mathbb{S}^2)$ is constant from the commutator subgroup onwards: $\Gamma_m(B_n(\mathbb{S}^2)) = \Gamma_2(B_n(\mathbb{S}^2))$ for all $m \geq 2$. The subgroup $\Gamma_2(B_n(\mathbb{S}^2))$ is as follows:*

(a) If $n = 1, 2$ then $\Gamma_2(B_n(\mathbb{S}^2)) = \{1\}$.
(b) If $n = 3$ then $\Gamma_2(B_n(\mathbb{S}^2)) \cong \mathbb{Z}_3$. Thus $B_3(\mathbb{S}^2) \cong \mathbb{Z}_3 \rtimes \mathbb{Z}_4$, the action being the non-trivial one.
(c) If $n = 4$ then $\Gamma_2(B_4(\mathbb{S}^2))$ admits a presentation of the following form:

generators: *g_1, g_2, g_3, where in terms of the usual generators of $B_4(\mathbb{S}^2)$, $g_1 = \sigma_1^2\sigma_2\sigma_1^{-3}$, $g_2 = \sigma_1^3\sigma_2\sigma_1^{-4}$ and $g_3 = \sigma_3\sigma_1^{-1}$.*

relations:

$$g_3^4 = 1 \tag{10}$$

$$[g_3^2, g_1] = 1 \tag{11}$$

$$[g_3^2, g_2] = 1 \tag{12}$$

$$[g_3, g_2g_1] = 1 \tag{13}$$

$$g_1^{-1}g_3^{-1}g_1 = g_2g_3g_2^{-1} \tag{14}$$

$$g_1^{-1}g_3^{-1}g_1 = g_1g_3g_1^{-1}g_3. \tag{15}$$

Furthermore,

$$\Gamma_2(B_4(\mathbb{S}^2)) \cong \mathcal{Q}_8 \rtimes \mathbb{F}_2(a,b),$$

where $\mathcal{Q}_8 = \langle x, y \mid x^2 = y^2,\ xyx^{-1} = y^{-1}\rangle$ is the quaternion group of order 8, and $\mathbb{F}_2(a,b)$ is the free group of rank 2 on two generators a and b. The action is given by:

$$\varphi(a)(x) = y \qquad \varphi(a)(y) = xy$$

$$\varphi(b)(x) = yx \qquad \varphi(b)(y) = x.$$

(d) In the cases $n = 5$ and $n \geq 6$, a presentation for $\Gamma_2(B_n(\mathbb{S}^2))$ is given in Chapter 4, by Propositions 64 and 67 respectively.

PROOF. The first part of the theorem, $\Gamma_m(B_n(\mathbb{S}^2)) = \Gamma_2(B_n(\mathbb{S}^2))$ for $m \geq 2$, follows from Lemma 23(b) and Remark 24.

Now let us consider the rest of the theorem.

(a) If $n = 1, 2$ then $B_n(\mathbb{S}^2) \cong \mathbb{Z}_n$, and the result follows easily.

(b) Let $n = 3$. Then $B_3(\mathbb{S}^2)$ is a ZS-metacyclic group (a group whose Sylow subgroups, commutator subgroup and commutator quotient group are all cyclic) of order 12 [**FVB**]. It follows from Proposition 20(a) that $(B_3(\mathbb{S}^2))^{\text{Ab}} \cong \mathbb{Z}_4$, and hence $\Gamma_2(B_3(\mathbb{S}^2)) \cong \mathbb{Z}_3$.

From Proposition 20(b), the short exact sequence (6) splits, so $B_3(\mathbb{S}^2) \cong \mathbb{Z}_3 \rtimes \mathbb{Z}_4$, and the action of the generator $\widetilde{\sigma}$ of $(B_3(\mathbb{S}^2))^{\text{Ab}}$ on the generator ρ of $\mathbb{Z}_3$ is given by $\widetilde{\sigma} \cdot \rho = \rho^{-1}$ i.e. the non-trivial action.

(c) Let $n = 4$. To obtain the given presentation of $\Gamma_2(B_4(\mathbb{S}^2))$, one applies the Reidemeister-Schreier rewriting process to the short exact sequence (6). The calculations are deferred to Proposition 63, see Section 2 of Chapter 4.

Using this presentation, let us prove the second part of (c) of Theorem 3, that $\Gamma_2(B_4(\mathbb{S}^2)) \cong \mathcal{Q}_8 \rtimes \mathbb{F}_2(a,b)$. This will be achieved by the following two propositions.

Proposition 25. *The normal subgroup of $\Gamma_2(B_4(\mathbb{S}^2))$ generated by g_3 is isomorphic to a quotient of the quaternion group $\mathcal{Q}_8$.*

Proof. Let N be the normal subgroup of $\Gamma_2(B_4(\mathbb{S}^2))$ generated by g_3, and let H be the subgroup of $\Gamma_2(B_4(\mathbb{S}^2))$ generated by g_3 and $g_1g_3g_1^{-1}$. Clearly $H \subseteq N$. To prove the converse, it suffices to show that if we conjugate g_3 and $g_1g_3g_1^{-1}$ by $g_1^{\pm 1}$ and $g_2^{\pm 1}$, we obtain elements of H. This is a consequence of the following equalities:

$$\begin{aligned}
g_2g_3g_2^{-1} &= g_1^{-1}g_3^{-1}g_1 \quad \text{by equation (14)}\\
&= g_1g_3g_1^{-1}\cdot g_3 \quad \text{by equation (15)}\\
g_1^2g_3g_1^{-2} &= g_3^{-1}\cdot g_1g_3^{-1}g_1^{-1} \quad \text{by equation (15)}\\
g_2g_1g_3g_1^{-1}g_2^{-1} &= g_3 \quad \text{by equation (13)}\\
g_2^{-1}g_3g_2 &= g_1g_3g_1^{-1} \quad \text{by equation (13)}\\
g_2^{-1}g_1g_3g_1^{-1}g_2 &= g_2^{-1}g_1^{-1}g_3^{-1}g_1g_3^{-1}g_2 \quad \text{by equation (15)}\\
&= g_3\cdot g_2^{-1}g_3^{-1}g_2 \quad \text{by equation (14).}
\end{aligned}$$

Hence $H = N$ is normal in $\Gamma_2(B_4(\mathbb{S}^2))$. Now $g_3^2 = (g_1g_3g_1^{-1})^2$ by equation (11), and $(g_1g_3g_1^{-1}g_3)^2 = (g_1^{-1}g_3^{-1}g_1)^2 = g_3^{-2} = g_3^2$ by equations (15) and (10). By equations (10) and (11) it thus follows that $[g_1g_3g_1^{-1}, g_3] = g_3^2$, and hence $g_1g_3g_1^{-1}\cdot g_3g_1g_3^{-1}g_1^{-1} = g_3^3 = g_3^{-1}$. So $g_1g_3g_1^{-1}$ and g_3 satisfy a set of defining relations of $\mathcal{Q}_8$, and thus H is a quotient of $\mathcal{Q}_8$. ☐

Proposition 26. *With H as defined as in the proof of Proposition 25, $H \cong \mathcal{Q}_8$, and $\Gamma_2(B_4(\mathbb{S}^2)) \cong \mathcal{Q}_8 \rtimes \mathbb{F}_2(a,b)$, the action being given by $\varphi(a)(x) = axa^{-1} = y$, $\varphi(a)(y) = aya^{-1} = xy$, $\varphi(b)(x) = bxb^{-1} = yx$ and $\varphi(b)(y) = byb^{-1} = x$.*

Proof. Let $\mathcal{Q}_8$ be generated by x and y, subject to the relations $x^2 = y^2$ and $xyx^{-1} = y^{-1}$. We remark that if $z \in \mathcal{Q}_8$ and $w \in \mathbb{F}_2(a,b)$ then $wzw^{-1} = \varphi(w)(z)$, and $[z,w] = z\cdot\varphi(w)(z^{-1})$. Consider the map

$$\psi\colon\ \{g_1, g_2, g_3\} \to \mathcal{Q}_8 \rtimes \mathbb{F}_2(a,b)$$

defined as follows: $\psi(g_1) = a$, $\psi(g_2) = b$ and $\psi(g_3) = x$. It is straightforward to check that the images under ψ of relations (10)–(13) hold in $\mathcal{Q}_8 \rtimes \mathbb{F}_2(a,b)$. As for relation (14), the right-hand side yields $bxb^{-1} = \varphi(b)(x) = yx$ from the definition of the action, while the left-hand side yields $a^{-1}x^{-1}a = \varphi(a^{-1})(x^{-1})$. Now $\varphi(a)(xy^{-1}) = x^{-1}$, so $\varphi(a^{-1})(x^{-1}) = xy^{-1} = yx$ in $\mathcal{Q}_8$. So relation (14) is preserved under ψ. Finally, consider relation (15). From the previous relation, the left-hand side yields yx. As for the right-hand side, we obtain $\varphi(a)(x) \cdot x = yx$ also. So ψ extends to a homomorphism, which we also call ψ, from $\Gamma_2(B_4(\mathbb{S}^2))$ into $\mathcal{Q}_8 \rtimes \mathbb{F}_2(a,b)$. Since $\psi(g_1 g_3 g_1^{-1}) = y$, this homomorphism is certainly surjective. Further, since the normal subgroup H of Proposition 25 is generated by g_3 and $g_1 g_3 g_1^{-1}$, it follows that H is mapped surjectively onto $\mathcal{Q}_8$. But H is a quotient of $\mathcal{Q}_8$, and since $\mathcal{Q}_8$ is finite, H is isomorphic to $\mathcal{Q}_8$. This proves the first part of the proposition. The induced map from the quotient $\Gamma_2(B_4(\mathbb{S}^2))$ by H (which is the normal subgroup generated by g_3) into the quotient of $\mathcal{Q}_8 \rtimes \mathbb{F}_2(a,b)$ by $\mathcal{Q}_8$ is a surjective homomorphism from a free group on two generators into a free group on two generators, so is an isomorphism by the Hopfian property of free groups of finite rank. This completes the proof of the proposition, as well as that of part (c) of Theorem 3. □

(d) Now suppose that $n \geq 5$. The presentations are given in Chapter 4, Propositions 64 and 67 respectively. This completes the proof of Theorem 3. □

CHAPTER 2

The derived series of $B_n(\mathbb{S}^2)$

In this chapter, we study the derived series of $B_n(\mathbb{S}^2)$. The aim is to prove the following result, which shows that for all $n \neq 3, 4$, $(B_n(\mathbb{S}^2))^{(1)}$ is perfect. The difficult case is $n = 4$, but using the semi-direct product structure of $(B_4(\mathbb{S}^2))^{(1)}$ obtained in Theorem 3, we shall be able to prove that the derived series of $B_4(\mathbb{S}^2)$ coincides from a certain point with that of the free group of rank 2. Before doing so, we state and prove Proposition 29 which describes the commutator subgroup of a general semi-direct product.

THEOREM 4. *The derived series of $B_n(\mathbb{S}^2)$ is as follows.*

(a) If $n = 1, 2$ then $(B_n(\mathbb{S}^2))^{(1)} = \{1\}$.

(b) If $n = 3$ then $(B_3(\mathbb{S}^2))^{(1)} \cong \mathbb{Z}_3$ and $(B_3(\mathbb{S}^2))^{(2)} = \{1\}$.

(c) Suppose that $n = 4$. Then:

(i) $(B_4(\mathbb{S}^2))^{(1)} = \Gamma_2(B_4(\mathbb{S}^2))$ is given by part (c) of Theorem 3; it is isomorphic to the semi-direct product $\mathcal{Q}_8 \rtimes \mathbb{F}_2$. Further, $B_4(\mathbb{S}^2)/(B_4(\mathbb{S}^2))^{(1)}$ is isomorphic to $\mathbb{Z}_6$.

(ii) $(B_4(\mathbb{S}^2))^{(2)}$ is isomorphic to the semi-direct product $\mathcal{Q}_8 \rtimes (\mathbb{F}_2)^{(1)}$, where $(\mathbb{F}_2)^{(1)}$ is the commutator subgroup of the free group $\mathbb{F}_2(a, b)$ of rank 2 on two generators a, b. The action of $(\mathbb{F}_2)^{(1)}$ on $\mathcal{Q}_8$ is the restriction of the action of $\mathbb{F}_2(a, b)$ given in part (c) of Theorem 3. Further,

$$(B_4(\mathbb{S}^2))^{(1)}/(B_4(\mathbb{S}^2))^{(2)} \cong \mathbb{Z}^2, \text{ and } B_4(\mathbb{S}^2)/(B_4(\mathbb{S}^2))^{(2)} \cong \mathbb{Z}^2 \rtimes \mathbb{Z}_6,$$

where the action of the generator $\overline{\sigma}$ of $\mathbb{Z}_6$ on $\mathbb{Z}^2$ is given by left multiplication by the matrix $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 1 \end{smallmatrix}\right)$.

(iii) $(B_4(\mathbb{S}^2))^{(3)}$ is a subgroup of $P_4(\mathbb{S}^2)$ isomorphic to the direct product $\mathbb{Z}_2 \times (\mathbb{F}_2)^{(2)}$. Further,

$$(B_4(\mathbb{S}^2))^{(2)}/(B_4(\mathbb{S}^2))^{(3)} \cong (\mathbb{Z}_2 \times \mathbb{Z}_2) \times (\mathbb{F}_2)^{(1)}/(\mathbb{F}_2)^{(2)}.$$

(iv) $(B_4(\mathbb{S}^2))^{(m)} \cong (\mathbb{F}_2)^{(m-1)}$ for all $m \geq 4$. Further,

$$(B_4(\mathbb{S}^2))^{(3)}/(B_4(\mathbb{S}^2))^{(4)} \cong \mathbb{Z}_2 \times (\mathbb{F}_2)^{(2)}/(\mathbb{F}_2)^{(3)},$$

and for $m \geq 4$,

$$(B_4(\mathbb{S}^2))^{(m)}/(B_4(\mathbb{S}^2))^{(m+1)} \cong (\mathbb{F}_2)^{(m-1)}/(\mathbb{F}_2)^{(m)}.$$

(d) If $n \geq 5$ then $(B_n(\mathbb{S}^2))^{(2)} = (B_n(\mathbb{S}^2))^{(1)}$, so $(B_n(\mathbb{S}^2))^{(1)}$ is perfect. A presentation of $(B_n(\mathbb{S}^2))^{(1)}$ is given in Propositions 64 and 67.

REMARK 27. In part (c) of Theorem 4 and also in what follows, we shall often refer to the derived series of $\mathbb{F}_2(a,b)$ as well as its quotients. We were not able to track down an explicit reference for them, but one may observe that for $i \geq 1$, $(\mathbb{F}_2(a,b))^{(i)}$ is a free group of infinite rank, and hence $(\mathbb{F}_2(a,b))^{(i)}/(\mathbb{F}_2(a,b))^{(i+1)}$ is a free Abelian group of infinite rank. A basis of $(\mathbb{F}_2(a,b))^{(1)} = \Gamma_2(\mathbb{F}_2(a,b))$ may be obtained as follows: considering the short exact sequence (9) with $G = \mathbb{F}_2(a,b)$, $(\mathbb{F}_2(a,b))^{(1)}$ may be identified with the fundamental group of the Cayley graph of $\mathbb{F}_2(a,b)$. Let $\mathcal{T}$ be a maximal tree in this graph. For each $g \in \mathbb{F}_2(a,b)$, let w_g be the word corresponding to the path in $\mathcal{T}$ between e and g. Then a basis is given by the set of elements of the form $w_g[a,b]w_g^{-1}$, where g runs over $\mathbb{F}_2(a,b)$. For example, the set $\{a^p b^q [a,b] b^{-q} a^{-p}\}_{p,q\in\mathbb{Z}}$ is a basis of $(\mathbb{F}_2(a,b))^{(1)}$. Since $\mathbb{F}_2(a,b)$ is residually nilpotent and $(\mathbb{F}_2(a,b))^{(i-1)} \subseteq \Gamma_i(\mathbb{F}_2(a,b))$, it follows that $\bigcap_{i\geq 0} (\mathbb{F}_2(a,b))^{(i)} = \{1\}$ and $\mathbb{F}_2(a,b)$ is residually soluble.

We obtain easily the following corollary of Theorem 4:

COROLLARY 28. *Let $n \in \mathbb{N}$. Then $B_n(\mathbb{S}^2)$ is residually soluble if and only if $n \leq 4$.*

PROOF OF COROLLARY 28. Recall that a group G is residually soluble if and only if $\bigcap_{i\geq 0} G^{(i)} = \{1\}$. If $n = 1, 2, 3$, this is obvious, and if $n = 4$, the residual solubility of $B_4(\mathbb{S}^2)$ follows from that of $\mathbb{F}_2(a,b)$. For $n \geq 5$, the result also follows easily, since $(B_n(\mathbb{S}^2))^{(1)}$ is non trivial. □

Before proving Theorem 4, let us state and prove the following proposition which describes the commutator subgroup of a semi-direct product. This result will be used frequently throughout the rest of this paper.

PROPOSITION 29. *Let G, H be groups, and let $\varphi\colon G \to Aut(H)$ be an action of G on H. Let $\widehat{H}$ be the subgroup of H generated by the elements of the form $\varphi(g)(h)\cdot h^{-1}$, where $g \in G, h \in H$, and let L be the subgroup of H generated by $\Gamma_2(H)$ and $\widehat{H}$. Then φ induces an action (also denoted by φ) of $\Gamma_2(G)$ on L, and $L \rtimes_\varphi \Gamma_2(G) = \Gamma_2(H \rtimes_\varphi G)$. In particular, $\Gamma_2(H \rtimes_\varphi G)$ is the subgroup generated by $\Gamma_2(H), \Gamma_2(G)$ and $\widehat{H}$.*

REMARK 30. We claim that L is none other than the commutator subgroup $\Gamma_G(H)$ defined by equation (7) with respect to the given action. To see this, recall that $\Gamma_G(H)$ is the subgroup of H generated by the elements of the form $\varphi(g)(h) \cdot kh^{-1}k^{-1}$, where $g \in G$ and $h, k \in H$. Taking $g = e$ (respectively $k = e$), it follows that $\Gamma_G(H) \supseteq \Gamma_2(H)$ (respectively $\Gamma_G(H) \supseteq \widehat{H}$), and hence $L \subseteq \Gamma_G(H)$. Conversely, $\varphi(g)(h) \cdot kh^{-1}k^{-1} = \varphi(g)(h)h^{-1} \cdot hkh^{-1}k^{-1} \in L$, so $\Gamma_G(H) \subseteq L$, and the claim is proved. Note further that if $h' \in H$ then there exists $h'' \in H$ such that $\varphi(g)(h'') = h'$, so

$$h' \left(\varphi(g)(h) \cdot h^{-1}\right) h'^{-1} = \varphi(g)(h''h)(h''h)^{-1} \cdot (\varphi(g)(h'')h''^{-1})^{-1}.$$

It follows that $\widehat{H}$ and L are normal in H. In particular, $\Gamma_G(H)$ is normal in H.

Proof of Proposition 29. From now on, we shall identify each subgroup H_1 of H (respectively each subgroup G_1 of G) with the corresponding subgroup $\{(h, 1) \mid h \in H_1\}$ (respectively $\{(1, g) \mid g \in G_1\}$) of $H \rtimes_\varphi G$ without further comment. The group operation in $H \rtimes_\varphi G$ shall be written as:

$$(h, g) \star (h', g') = (h.\, \varphi(g)(h), gg'), \text{ where } (h, g), (h', g') \in H \rtimes_\varphi G.$$

The subgroup L is normal in H by Remark 30. Let us show that φ induces an action (also denoted by φ) of G on L. Let $g \in G$. Since $\varphi(g)([h_1, h_2]) = [\varphi(g)(h_1), \varphi(g)(h_2)] \in \Gamma_2(H)$ for all $h_1, h_2 \in H$, and

$$\varphi(g)(\varphi(g')(h)h^{-1}) = \varphi(gg')(h)h^{-1}.\, (\varphi(g)(h)h^{-1})^{-1} \in \widehat{H}$$

for all $h \in H$ and $g' \in G$, it follows that $\varphi(g)(L) \subseteq L$. Clearly $\varphi(g)$ is injective. The surjectivity of $\varphi(g)$ (restricted to L) may be deduced from the following observations:

(a) if $i = 1, 2$ and $h_i' \in H$ then there exists $h_i \in H$ such that $\varphi(g)(h_i) = h_i'$, and hence $\varphi(g)([h_1, h_2]) = [h_1', h_2']$.

(b) If $g' \in G$ and $h, h' \in H$ then

$$\varphi(g) \left(\varphi(g^{-1}g')(h)h^{-1}.\, h\left(\varphi(g^{-1})(h^{-1})h\right) h^{-1}\right) = \varphi(g')(h)h^{-1}.$$

Thus φ induces an action (also denoted by φ) of $\Gamma_2(G)$ on L, and $L \rtimes_\varphi \Gamma_2(G)$ is a subgroup of $H \rtimes_\varphi G$.

Clearly any element of $\Gamma_2(H)$ (respectively $\Gamma_2(G)$) may be written as an element of $\Gamma_2(H \rtimes_\varphi G)$. Further, if $g \in G$ and $h \in H$ then

$$[(1, g), (h, 1)] = (\varphi(g)(h), 1) \star (h^{-1}, 1) = (\varphi(g)(h)h^{-1}, 1),$$

and thus every element of $\widehat{H}$ may be written as an element of $\Gamma_2(H \rtimes_\varphi G)$. This proves that $L \rtimes_\varphi \Gamma_2(G) \subseteq \Gamma_2(H \rtimes_\varphi G)$.

To see the converse, notice that the commutator of two elements $(h_1, g_1), (h_2, g_2) \in H \rtimes_\varphi G$ may be written as:

$$[(h_1, g_1), (h_2, g_2)] = \big(h_1 .\, \varphi(g_1)(h_2).\, \varphi(g_1 g_2 g_1^{-1})(h_1^{-1}). \\ \varphi([g_1, g_2])(h_2^{-1}), [g_1, g_2]\big).$$

The second factor belongs clearly to $\Gamma_2(G)$. The first factor is of the form:

$$[h_1, h_2].\, h_2 h_1 h_2^{-1} \left(\varphi(g_1)(h_2) h_2^{-1}\right) h_2 h_1^{-1} h_2^{-1}. \\ h_2 h_1 \left(\varphi(g_1 g_2 g_1^{-1})(h_1^{-1}) h_1\right) h_1^{-1} h_2^{-1}.\, h_2 \left(\varphi([g_1, g_2])(h_2^{-1}) h_2\right) h_2^{-1},$$

which is a product of elements of L. Hence $\Gamma_2(H \rtimes_\varphi G) \subseteq L \rtimes_\varphi \Gamma_2(G)$, and the proposition follows. □

We now prove the main result of this chapter.

Proof of Theorem 4. Cases (a) and (b) follow directly from Theorem 3.

Now consider case (d), i.e. $n \geq 5$. Let $H \subseteq (B_n(\mathbb{S}^2))^{(1)}$ be a normal subgroup of $B_n(\mathbb{S}^2)$ such that $A = (B_n(\mathbb{S}^2))^{(1)}/H$ is Abelian (notice that this condition is satisfied if $H = (B_n(\mathbb{S}^2))^{(2)}$). Let

$$\begin{cases} \pi\colon\ B_n(\mathbb{S}^2) \to B_n(\mathbb{S}^2)/H \\ \qquad\qquad \beta \mapsto \overline{\beta} \end{cases}$$

denote the canonical projection. So the Abelianisation homomorphism $\alpha\colon\ B_n(\mathbb{S}^2) \to (B_n(\mathbb{S}^2))^{\,\mathrm{Ab}}$ of Chapter 1 factors through $B_n(\mathbb{S}^2)/H$ i.e. there exists a (surjective) homomorphism $\widehat{\alpha}\colon\ B_n(\mathbb{S}^2)/H \to (B_n(\mathbb{S}^2))^{\,\mathrm{Ab}}$ satisfying $\alpha = \widehat{\alpha} \circ \pi$. So we have the following short exact sequence:

$$1 \longrightarrow A \longrightarrow B_n(\mathbb{S}^2)/H \xrightarrow{\widehat{\alpha}} (B_n(\mathbb{S}^2))^{\,\mathrm{Ab}} \longrightarrow 1.$$

Now $\overline{\sigma_1}, \ldots, \overline{\sigma_{n-1}}$ generate $B_n(\mathbb{S}^2)/H$, but since $\alpha(\sigma_i) = \alpha(\sigma_1)$ for $1 \leq i \leq n-1$, it follows that $\widehat{\alpha}(\overline{\sigma_i}) = \widehat{\alpha}(\overline{\sigma_1})$, and so there exists $t_i \in A$ such that $\overline{\sigma_i} = t_i \overline{\sigma_1}$.

We now apply π to each of the relations of equation (4) of $B_n(\mathbb{S}^2)$. First suppose that $3 \leq i \leq n-1$. Since σ_i commutes with σ_1, we have that

$$\overline{\sigma_1} \cdot t_i \overline{\sigma_1} = t_i \overline{\sigma_1} \cdot \overline{\sigma_1},$$

and hence t_i commutes with $\overline{\sigma_1}$.

Now let $4 \leq i \leq n-1$. Since σ_i commutes with σ_2, we obtain

$$t_i \overline{\sigma_1} \cdot t_2 \overline{\sigma_1} = t_2 \overline{\sigma_1} \cdot t_i \overline{\sigma_1}.$$

Since A is Abelian, it follows from the previous paragraph that t_2 commutes with $\overline{\sigma_1}$. Applying this to the image of the relation $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$, under π, we see that $t_2 = t_2^2$, and hence $t_2 = 1$.

Next, if $i \geq 2$ then the relation $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ implies that $t_i = t_{i+1}$, and so $t_2 = \ldots = t_{n-1} = 1$. Hence $\overline{\sigma_1} = \overline{\sigma_2} = \ldots = \overline{\sigma_{n-1}}$. Thus $B_n(\mathbb{S}^2)/H$ is cyclic, generated by $\overline{\sigma_1}$, and finite of order not greater than $2(n-1)$, because the surface relation $\sigma_1 \ldots \sigma_{n-2}\sigma_{n-1}^2\sigma_{n-2}\ldots\sigma_1 = 1$ projects to $\overline{\sigma_1}^{2(n-1)} = 1$. Since $\widehat{\alpha}$ is surjective and $(B_n(\mathbb{S}^2))^{\text{Ab}} \cong \mathbb{Z}_{2(n-1)}$, we conclude that $\widehat{\alpha}$ is an isomorphism, so $B_n(\mathbb{S}^2)/H \cong \mathbb{Z}_{2(n-1)}$, and $A = (B_n(\mathbb{S}^2))^{(1)}/H$ is trivial. In particular

$$(B_n(\mathbb{S}^2))^{(2)} = \left[(B_n(\mathbb{S}^2))^{(1)}, (B_n(\mathbb{S}^2))^{(1)}\right] = (B_n(\mathbb{S}^2))^{(1)},$$

in other words, $(B_n(\mathbb{S}^2))^{(1)}$ is perfect.

Now consider case (c), so $n = 4$. Recall that part (i) was proved in Theorem 3 and Proposition 20. To obtain $(B_4(\mathbb{S}^2))^{(2)}$, it suffices to observe that for the action of $\mathbb{F}_2(a,b)$ on $\mathcal{Q}_8$, the subgroup $\widehat{\mathcal{Q}_8}$ defined in Proposition 29 is $\mathcal{Q}_8$ (which is the case, since by Theorem 3(c), $\varphi(b)(x)x^{-1} = y$ and $\varphi(a)(y)y^{-1} = x$). So $(B_4(\mathbb{S}^2))^{(2)}$ is generated by $\mathcal{Q}_8$ and $(\mathbb{F}_2)^{(1)}$, $(B_4(\mathbb{S}^2))^{(2)} \cong \mathcal{Q}_8 \rtimes (\mathbb{F}_2)^{(1)}$, and the action is the restriction of that of $\mathbb{F}_2(a,b)$ on $\mathcal{Q}_8$, which proves the first part of (c)(ii).

To determine $(B_4(\mathbb{S}^2))^{(3)}$, we first have to describe the subgroup $\widehat{\mathcal{Q}_8}$ for the action of $(\mathbb{F}_2)^{(1)}$ on $\mathcal{Q}_8$. By Theorem 3(c), if $B = [a,b] \in (\mathbb{F}_2(a,b))^{(1)}$ then the automorphism $\varphi(B)$ satisifies $\varphi(B)(z) = x^2 \cdot z$ for $z \in \{x,y\}$ (recall that $x^2 = y^2$). Since $(\mathbb{F}_2(a,b))^{(1)}$ is the subgroup of $\mathbb{F}_2(a,b)$ normally generated by B, and the centre $\langle x^2 \rangle$ of $\mathcal{Q}_8$ is invariant under $\operatorname{Aut}(\mathcal{Q}_8)$, it follows that $\widehat{\mathcal{Q}_8} = \langle x^2 \rangle$. So $(B_4(\mathbb{S}^2))^{(3)}$ is isomorphic to the semi-direct product of $\mathbb{Z}_2$ by $(\mathbb{F}_2)^{(2)}$. But the action is trivial, and so the product is direct. This proves the first part of (c)(iii).

For $m \geq 4$, the subgroup $(B_4(\mathbb{S}^2))^{(m)}$ is clear from the description of $(B_4(\mathbb{S}^2))^{(3)}$, and hence we obtain the first part of (c)(iv).

We now analyse various quotients of the form $B_4(\mathbb{S}^2)/(B_4(\mathbb{S}^2))^{(m)}$ and $(B_4(\mathbb{S}^2))^{(m-1)}/(B_4(\mathbb{S}^2))^{(m)}$ for several values of m. For the quotient $B_4(\mathbb{S}^2)/(B_4(\mathbb{S}^2))^{(m)}$, we shall consider the case $m = 2$ (the case $m = 1$ is given by Proposition 20(a)). For $(B_4(\mathbb{S}^2))^{(m-1)}/(B_4(\mathbb{S}^2))^{(m)}$, we consider the cases $m \geq 2$ (the case $m = 1$ was considered in Proposition 20(a)). If $m > 4$, the problem reduces to the corresponding problem for the free group on two generators.

We adopt the notation used above in the case $n \geq 5$, and again we suppose that $H \subseteq (B_n(\mathbb{S}^2))^{(1)}$ is a normal subgroup of $B_n(\mathbb{S}^2)$ such that $A = (B_n(\mathbb{S}^2))^{(1)}/H$ is Abelian. So we have a short exact sequence:

$$1 \longrightarrow A \longrightarrow B_4(\mathbb{S}^2)/H \xrightarrow{\widehat{\alpha}} \underbrace{\left(B_4(\mathbb{S}^2)\right)^{\mathrm{Ab}}}_{\mathbb{Z}_6} \longrightarrow 1.$$

Now $\overline{\sigma_1}, \overline{\sigma_2}, \overline{\sigma_3}$ generate $B_4(\mathbb{S}^2)/H$. As above, for $i = 2, 3$ we set $\overline{\sigma_i} = t_i\overline{\sigma_1}$, where $t_i \in A$, and we apply π to the relations of $B_4(\mathbb{S}^2)$. The fact that σ_1 commutes with σ_3 implies that t_3 commutes with $\overline{\sigma_1}$. The relation $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$ implies that:

$$\overline{\sigma_1}t_2\overline{\sigma_1}^{-1} = t_2 \cdot \overline{\sigma_1}^2 t_2 \overline{\sigma_1}^{-2}. \tag{16}$$

Now consider the relation $\sigma_2\sigma_3\sigma_2 = \sigma_3\sigma_2\sigma_3$. We have that:

$$t_2\overline{\sigma_1} \cdot t_3\overline{\sigma_1} \cdot t_2\overline{\sigma_1} = t_3\overline{\sigma_1} \cdot t_2\overline{\sigma_1} \cdot t_3\overline{\sigma_1},$$

and so

$$t_2\overline{\sigma_1}^2 t_2 = t_3\overline{\sigma_1}t_2\overline{\sigma_1},$$

since A is Abelian and t_3 commutes with $\overline{\sigma_1}$. Thus:

$$t_2\overline{\sigma_1}^2 t_2 = t_3 t_2\overline{\sigma_1}^2 t_2$$

from equation (16), and so $t_3 = 1$. We conclude that $B_4(\mathbb{S}^2)/H$ is generated by $\overline{\sigma_1}$ and $t_2\overline{\sigma_1}$.

Finally, we consider the image of the surface relation under π. Using equation (16), note first that:

$$\begin{aligned}\overline{\sigma_1}^3 t_2\overline{\sigma_1}^{-3} &= \overline{\sigma_1}(t_2^{-1} \cdot \overline{\sigma_1}t_2\overline{\sigma_1}^{-1})\overline{\sigma_1}^{-1} = \overline{\sigma_1}t_2^{-1}\overline{\sigma_1}^{-1} \cdot \overline{\sigma_1}^2 t_2\overline{\sigma_1}^{-2}\\ &= \overline{\sigma_1}t_2^{-1}\overline{\sigma_1}^{-1} \cdot t_2^{-1}\overline{\sigma_1}t_2\overline{\sigma_1}^{-1} = t_2^{-1},\end{aligned} \tag{17}$$

since A is normal and Abelian. Thus $\sigma_1\sigma_2\sigma_3^2\sigma_2\sigma_1 = 1$ implies that:

$$\begin{aligned}1 &= \overline{\sigma_1} \cdot t_2\overline{\sigma_1} \cdot \overline{\sigma_1}^2 \cdot t_2\overline{\sigma_1} \cdot \overline{\sigma_1} = \overline{\sigma_1}t_2\overline{\sigma_1}^{-1} \cdot \overline{\sigma_1}(\overline{\sigma_1}^3 t_2\overline{\sigma_1}^{-3})\overline{\sigma_1}^{-1} \cdot \overline{\sigma_1}^6\\ &= \overline{\sigma_1}t_2\overline{\sigma_1}^{-1} \cdot \overline{\sigma_1}t_2^{-1}\overline{\sigma_1}^{-1} \cdot \overline{\sigma_1}^6 = \overline{\sigma_1}^6\end{aligned}$$

from equation (17).

Recall that $\Gamma_2(B_4(\mathbb{S}^2))$ is the normal subgroup of $B_4(\mathbb{S}^2)$ generated by the commutators of the generators of $B_4(\mathbb{S}^2)$. Hence A is the normal subgroup of $B_4(\mathbb{S}^2)/H$ generated by $[\overline{\sigma_1}, t_2\overline{\sigma_1}] = \overline{\sigma_1}t_2\overline{\sigma_1}^{-1} \cdot t_2^{-1}$. Since A is Abelian and $t_2 \in A$, the action of conjugation on A by t_2 is trivial. From equation (17), the action of $\overline{\sigma_1}^3$ on t_2 yields t_2^{-1}. Further,

$$\overline{\sigma_1}(\overline{\sigma_1}t_2\overline{\sigma_1}^{-1}t_2^{-1})\overline{\sigma_1}^{-1} = \overline{\sigma_1}^2 t_2\overline{\sigma_1}^{-2} \cdot \overline{\sigma_1}t_2^{-1}\overline{\sigma_1}^{-1} = t_2^{-1}$$

from equation (16), and since

$$\overline{\sigma_1}^2(\overline{\sigma_1}t_2\overline{\sigma_1}^{-1}t_2^{-1})\overline{\sigma_1}^{-2} = \overline{\sigma_1}t_2^{-1}\overline{\sigma_1}^{-1},$$

it follows that A is the Abelian group generated by $\overline{\sigma_1}t_2\overline{\sigma_1}^{-1}t_2^{-1}$, t_2 and $\overline{\sigma_1}t_2\overline{\sigma_1}^{-1}$, and thus by t_2 and $\overline{\sigma_1}t_2\overline{\sigma_1}^{-1}$.

Let $\widetilde{\sigma} = \alpha(\sigma_1)$ denote the generator of $(B_4(\mathbb{S}^2))^{\,\mathrm{Ab}}$. Let $M = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$; notice that M is of order 6. We now let $(B_4(\mathbb{S}^2))^{\,\mathrm{Ab}} \cong \mathbb{Z}_6$ act on $\mathbb{Z}^2$ as follows:

$$\widetilde{\sigma} \cdot \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = M \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} X_2 \\ X_2 - X_1 \end{pmatrix},$$

and so we may form the associated semi-direct product $\mathbb{Z}^2 \rtimes \mathbb{Z}_6$. We now consider the following homomorphism:

$$\begin{aligned} \psi : B_4(\mathbb{S}^2) &\to \mathbb{Z}^2 \rtimes \mathbb{Z}_6 \\ \sigma_1, \sigma_3 &\mapsto \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \widetilde{\sigma} \right) \\ \sigma_2 &\mapsto \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \widetilde{\sigma} \right). \end{aligned}$$

We then check that ψ is well defined: clearly $\psi(\sigma_1\sigma_3) = \psi(\sigma_3\sigma_1)$. To see that $\psi(\sigma_1\sigma_2\sigma_1) = \psi(\sigma_2\sigma_1\sigma_2)$ (and that $\psi(\sigma_3\sigma_2\sigma_3) = \psi(\sigma_2\sigma_3\sigma_2)$),

$$\begin{aligned} \psi(\sigma_1)\cdot\psi(\sigma_2)\cdot\psi(\sigma_1) &= \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \widetilde{\sigma} \right) \cdot \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \widetilde{\sigma} \right) \cdot \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \widetilde{\sigma} \right) \\ &= \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \widetilde{\sigma} \right) \cdot \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \widetilde{\sigma}^2 \right) = \left(\begin{pmatrix} 0 \\ -1 \end{pmatrix}, \widetilde{\sigma}^3 \right). \end{aligned}$$

Similarly,

$$\begin{aligned} \psi(\sigma_2)\cdot\psi(\sigma_1)\cdot\psi(\sigma_2) &= \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \widetilde{\sigma} \right) \cdot \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \widetilde{\sigma} \right) \cdot \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \widetilde{\sigma} \right) \\ &= \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \widetilde{\sigma} \right) \cdot \left(\begin{pmatrix} 0 \\ -1 \end{pmatrix}, \widetilde{\sigma}^2 \right) = \left(\begin{pmatrix} 0 \\ -1 \end{pmatrix}, \widetilde{\sigma}^3 \right). \end{aligned}$$

As for the surface relation,

$$\begin{aligned} \psi(\sigma_1\sigma_2\sigma_3^2\sigma_2\sigma_1) &= \left(\begin{pmatrix} 0 \\ -1 \end{pmatrix}, \widetilde{\sigma}^2 \right) \cdot \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \widetilde{\sigma}^2 \right) \cdot \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \widetilde{\sigma}^2 \right) \\ &= \left(\begin{pmatrix} 0 \\ -1 \end{pmatrix}, \widetilde{\sigma}^2 \right) \cdot \left(\begin{pmatrix} -1 \\ -1 \end{pmatrix}, \widetilde{\sigma}^4 \right) \\ &= \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \widetilde{\sigma}^6 \right) = \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, 1 \right) \end{aligned}$$

as required. Since $\psi(\sigma_1) = \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \widetilde{\sigma} \right)$, $\psi(\sigma_2\sigma_1^{-1}) = \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, 1 \right)$ and $\psi([\sigma_1^{-1}, \sigma_2]) = \left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}, 1 \right)$, we see that ψ is surjective.

Now let $H=(B_4(\mathbb{S}^2))^{(2)}$, and let $\delta\colon\ \mathbb{Z}^2 \rtimes \mathbb{Z}_6 \to \mathbb{Z}_6$ denote the projection onto the second factor. Since $\mathbb{Z}_6$ is Abelian, it follows that $\delta(\psi(x))$ is trivial for all $x \in (B_4(\mathbb{S}^2))^{(1)}$, so $\psi(x)$ belongs to the $\mathbb{Z}^2$-factor. Hence $H=\bigl[(B_4(\mathbb{S}^2))^{(1)},(B_4(\mathbb{S}^2))^{(1)}\bigr] \subseteq \operatorname{Ker}(\psi)$, and thus ψ factors through $A=B_4(\mathbb{S}^2)/H$, inducing a (surjective) homomorphism $\widehat{\psi}\colon\ B_4(\mathbb{S}^2)/H \to \mathbb{Z}^2 \rtimes \mathbb{Z}_6$. From the following commutative diagram of short exact sequences,

$$\begin{array}{ccccccccc}
1 & \longrightarrow & A=\Gamma_2(B_4(\mathbb{S}^2))/H & \longrightarrow & B_4(\mathbb{S}^2)/H & \xrightarrow{\widehat{\alpha}} & (B_4(\mathbb{S}^2))^{\,\mathrm{Ab}} & \longrightarrow & 1\\
 & & \Big\downarrow{\scriptstyle \widehat{\psi}\bigr|_A} & & \Big\downarrow{\scriptstyle \widehat{\psi}} & & \Big\| & & \\
1 & \longrightarrow & \mathbb{Z}^2 & \longrightarrow & \mathbb{Z}^2 \rtimes \mathbb{Z}_6 & \xrightarrow{\delta} & \mathbb{Z}_6 & \longrightarrow & 1,
\end{array}$$

the surjectivity of $\widehat{\psi}$ implies that of $\widehat{\psi}\bigr|_A\colon\ A \to \mathbb{Z}^2$. But A is an Abelian group generated by $\{t_2, \overline{\sigma} t_2 \overline{\sigma}^{-1}\}$, so $\widehat{\psi}\bigr|_A$ is an isomorphism, and by the 5-Lemma, $\widehat{\psi}$ is too. Hence:

$$(B_4(\mathbb{S}^2))^{(1)}\big/(B_4(\mathbb{S}^2))^{(2)} \cong \mathbb{Z}^2 \quad \text{and} \quad B_4(\mathbb{S}^2)\big/(B_4(\mathbb{S}^2))^{(2)} \cong \mathbb{Z}^2 \rtimes \mathbb{Z}_6.$$

In fact the first of these two equations may be obtained directly since we know that $(B_4(\mathbb{S}^2))^{(1)} \cong \mathcal{Q}_8 \rtimes \mathbb{F}_2$, and $(B_4(\mathbb{S}^2))^{(2)}$ is isomorphic to the subgroup $\mathcal{Q}_8 \rtimes (\mathbb{F}_2)^{(1)}$ of $\mathcal{Q}_8 \rtimes \mathbb{F}_2$, so $(B_4(\mathbb{S}^2))^{(1)}/(B_4(\mathbb{S}^2))^{(2)} \cong \mathbb{F}_2/(\mathbb{F}_2)^{(1)} \cong \mathbb{Z}^2$. Similarly, $(B_4(\mathbb{S}^2))^{(2)}/(B_4(\mathbb{S}^2))^{(3)} \cong (\mathbb{Z}_2 \times \mathbb{Z}_2) \times (\mathbb{F}_2)^{(1)}/(\mathbb{F}_2)^{(2)}$, $(B_4(\mathbb{S}^2))^{(3)}/(B_4(\mathbb{S}^2))^{(4)} \cong \mathbb{Z}_2 \times (\mathbb{F}_2)^{(2)}/(\mathbb{F}_2)^{(3)}$, and for $m \geq 4$,

$$(B_4(\mathbb{S}^2))^{(m)}\big/(B_4(\mathbb{S}^2))^{(m+1)} \cong (\mathbb{F}_2)^{(m-1)}\big/(\mathbb{F}_2)^{(m)}.$$

This proves the remaining parts of (c), and thus completes the proof of Theorem 4. $\square$

CHAPTER 3

The lower central and derived series of $B_m(\mathbb{S}^2 \setminus \{x_1, \ldots, x_n\})$

In this chapter, the aim is to determine the lower central and derived series of the m-string braid group of the n-punctured sphere $B_m(\mathbb{S}^2 \setminus \{x_1, \ldots, x_n\})$, $n \geq 1$ according to the values of m and n. In Section 1, we begin by giving a presentation of this group. In Section 2, we deal with the case $n = 1$ which corresponds to the Artin braid groups, and extend the results of Gorin and Lin. The case $m = 1$ which is that of the fundamental group of the n-punctured sphere is dealt with in Section 3. From Section 4 onwards, we suppose that $n \geq 2$. In Section 4, we prove Theorem 9, which if $m \geq 3$ (respectively $m \geq 5$) shows that the lower central series (respectively the derived series) of $B_m(\mathbb{S}^2 \setminus \{x_1, \ldots, x_n\})$ is constant from the commutator subgroup onwards. In Sections 5, 6 and 7, we study the case $n = 2$ which corresponds to that of the braid groups of the annulus (which are isomorphic to the Artin groups of type B). The main results of these three sections are Proposition 10, Corollary 13, Proposition 14, Theorem 15 and Corollary 16. In Section 8, we study $B_m(\mathbb{S}^2 \setminus \{x_1, x_2, x_3\})$, $m \geq 2$, which is isomorphic to the affine Artin group of type $\widetilde{C}_m$, and we prove Propositions 60 and 61.

1. A presentation of $B_m(\mathbb{S}^2 \setminus \{x_1, \ldots, x_n\})$, $n \geq 1$

Let $q \in \mathbb{N}$. If $1 \leq i < j \leq q$, let $A_{i,j} = \sigma_{j-1} \cdots \sigma_{i+1} \sigma_i^2 \sigma_{i+1}^{-1} \cdots \sigma_{j-1}^{-1} \in P_q(\mathbb{S}^2)$ which geometrically corresponds to a twist of the j^{th} string about the i^{th} string, with all other strings remaining vertical. It is well known that the $A_{i,j}$ generate $P_q(\mathbb{S}^2)$.

The following presentation of $B_m(\mathbb{S}^2 \setminus \{x_1, \ldots, x_n\})$ was derived in [**GG4**] using standard results concerning presentations of group extensions [**J**] (see also [**Lam, Ma**] for other presentations).

Proposition 31 ([**GG4**]). *Let $m \geq 1$ and $n \geq 1$. The following constitutes a presentation of the group $B_m(\mathbb{S}^2 \setminus \{x_1, \ldots, x_n\})$:*

generators: *$A_{i,j}$, where $1 \leq i \leq n$ and $n+1 \leq j \leq n+m$, and σ_k, $1 \leq k \leq m-1$.*

relations: *for* $1 \leq i, k \leq n$, $n+1 \leq j < l \leq n+m$ *but* $j \leq n+m$ *if* l *is absent, and* $1 \leq r, s \leq m-1$,

$$A_{i,j}A_{k,l}A_{i,j}^{-1} = A_{k,l} \text{ if } k < i$$

$$A_{i,j}A_{i,l}A_{i,j}^{-1} = A_{j,l}^{-1}A_{i,l}A_{j,l}$$

$$A_{i,j}^{-1}A_{i,l}A_{i,j} = A_{i,l}A_{j,l}A_{i,l}A_{j,l}^{-1}A_{i,l}^{-1}$$

$$A_{i,j}A_{k,l}A_{i,j}^{-1} = A_{j,l}^{-1}A_{i,l}^{-1}A_{j,l}A_{i,l}A_{k,l}A_{i,l}^{-1}A_{j,l}^{-1}A_{i,l}A_{j,l} \text{ if } i < k$$

$$A_{i,j}^{-1}A_{k,l}A_{i,j} = A_{i,l}A_{j,l}A_{i,l}^{-1}A_{j,l}^{-1}A_{k,l}A_{j,l}A_{i,l}A_{j,l}^{-1}A_{i,l}^{-1} \text{ if } i < k$$

$$A_{1,n+m} \cdots A_{n,n+m}\sigma_{m-1} \cdots \sigma_2\sigma_1^2\sigma_2 \cdots \sigma_{m-1} = 1$$

$$\sigma_r\sigma_s = \sigma_s\sigma_r \text{ if } |r-s| \geq 2$$

$$\sigma_r\sigma_{r+1}\sigma_r = \sigma_{r+1}\sigma_r\sigma_{r+1}$$

$$\sigma_r A_{i,j}\sigma_r^{-1} = A_{i,j} \text{ if } r \neq j-n-1, j-n$$

$$\sigma_{j-n}A_{i,j}\sigma_{j-n}^{-1} = A_{i,j+1} \text{ if } n+1 \leq j \leq n+m-1.$$

In the above relations, if $n+1 \leq j < l \leq n+m$ *then* $A_{j,l}$ *(which does not appear in the list of generators) should be rewritten as:*

$$A_{j,l} = \sigma_{l-n-1} \ldots \sigma_{j-n+1}\sigma_{j-n}^2\sigma_{j-n+1}^{-1} \cdots \sigma_{l-n-1}^{-1}. \qquad \square$$

REMARKS 32.

(a) Geometrically, we think of the n punctures labelled as points from 1 to n, and the basepoints of the m strings as points labelled from $n+1$ to $n+m$. The generator $A_{i,j}$ corresponds geometrically to a twist of the $(j-n)^{\text{th}}$ string about the i^{th} puncture, with all other strings remaining vertical.

(b) This presentation was derived in [**GG4**] for $m \geq 1$ and $n \geq 3$ (see Proposition 9 of that paper). But it is also correct for $n = 1, 2$. Indeed, to obtain the result, a presentation of $P_m(\mathbb{S}^2 \setminus \{x_1, \ldots, x_n\})$ was derived (Proposition 7 of [**GG4**]) using the fact that there is a split short exact sequence

$$1 \to P_1(\mathbb{S}^2 \setminus \{x_1, \ldots, x_n, x_{n+1}\}) \to P_2(\mathbb{S}^2 \setminus \{x_1, \ldots, x_n\}) \to P_1(\mathbb{S}^2 \setminus \{x_1, \ldots, x_n\}) \to 1,$$

which is the case for all $n \geq 1$ (as $\pi_2(\mathbb{S}^2 \setminus \{x_1, \ldots, x_n\}) = \{1\}$). To prove Proposition 9 of [**GG4**], we then apply standard techniques to the short exact sequence

$$1 \to P_m(\mathbb{S}^2 \setminus \{x_1, \ldots, x_n\}) \to B_m(\mathbb{S}^2 \setminus \{x_1, \ldots, x_n\}) \to S_m \to 1.$$

From this presentation, we may obtain easily the Abelianisation of $B_m(\mathbb{S}^2 \setminus \{x_1, \ldots, x_n\})$:

PROPOSITION 33 ([**GG4**], Proposition 11). *The Abelianisation of* $B_m(\mathbb{S}^2 \setminus \{x_1, \ldots, x_n\})$ *is a free Abelian group of rank* n. □

2. The case $n = 1$: lower central and derived series of Artin's braid groups $B_m(\mathbb{D}^2)$

As we shall see below, the case $n = 1$ corresponds to that of Artin's braid groups. In Theorem 36, we recall Gorin and Lin's results, which we extend in Proposition 5, notably obtaining descriptions of some of the derived series elements and quotients of $B_m(\mathbb{D}^2)$ for $m = 3, 4$. We begin by proving the following proposition which will allow us to identify certain types of braid groups.

PROPOSITION 34.

(a) Let $z_0 \in Int(\mathbb{D}^2)$ *and* $m \geq 2$. *Then* $P_m(\mathbb{D}^2) \cong P_{m-1}(\mathbb{D}^2 \setminus \{z_0\})$.

(b) Let $m \in \mathbb{N}$, *let* $x_0 \in \mathbb{S}^2$, *and let* $Y \subseteq \mathbb{S}^2 \setminus \{x_0\}$ *be a finite set. Then the inclusion* $\mathbb{S}^2 \setminus \{x_0\} \subseteq \mathbb{D}^2$ *induces an isomorphism*

$$B_m\left(\mathbb{S}^2 \setminus (Y \cup \{x_0\})\right) \cong B_m(\mathbb{D}^2 \setminus Y).$$

(c) Let $z_0 \in Int(\mathbb{D}^2)$ *and* $m \geq 1$. *Then* $B_{m,1}(\mathbb{D}^2) \cong B_m(\mathbb{D}^2 \setminus \{z_0\})$.

(d) Let $m \in \mathbb{N}$ *and* $(x_1, \ldots, x_m) \in F_m(Int(\mathbb{D}^2))$. *Then*

$$B_{m,1}(\mathbb{D}^2) \cong \pi_1(\mathbb{D}^2 \setminus \{x_1, \ldots, x_m\}) \rtimes B_m(\mathbb{D}^2).$$

REMARKS 35.

(a) Part (a) of Proposition 34 is a manifestation of the Artin combing operation [**A2, Bi2, Han**]: any geometric pure braid of the disc is equivalent to a pure braid whose first string is vertical.

(b) Taking $Y = \varnothing$ in part (b), and noting that homeomorphic spaces have isomorphic braid groups leads to the well-known isomorphism $B_m \cong B_m(\mathbb{S}^2 \setminus \{x_0\}) \cong B_m(\mathbb{D}^2)$.

(c) Part (a), and parts (c) and (d) describe respectively the pure braid groups and full braid groups of the annulus. Since the latter are isomorphic to the Artin groups of type B [**Cr**], we recover part (2) of Proposition 2.1 of [**CrP**].

(d) In part (d), the action is given by the well-known Artin representation of the Artin braid group as a subgroup of $\mathrm{Aut}(\mathbb{F}_n)$ [**A1, Bi2, Han**], and may be described as follows: let $\sigma_1, \ldots, \sigma_{m-1}$ denote the standard generators of $B_m(\mathbb{D}^2)$, and let $A_1, \ldots, A_m$ denote those of $\pi_1(\mathbb{D}^2 \setminus \{x_1, \ldots, x_m\}, x_{m+1})$. Then:

$$\sigma_i A_j \sigma_i^{-1} = \begin{cases} A_{i+1} & \text{if } j = i \\ A_{i+1}^{-1} A_i A_{i+1} & \text{if } j = i+1 \\ A_j & \text{otherwise.} \end{cases}$$

This was used by Chow [**Ch, Han**] to obtain a presentation of Artin's pure braid group, and may be applied to the study of the Nielsen equivalence problem for fixed points of surface homeomorphisms [**Gu**].

PROOF OF PROPOSITION 34.

(a) Consider the following Fadell-Neuwirth short exact sequence for the disc:

$$1 \longrightarrow P_{m-1}(\mathbb{D}^2 \setminus \{z_0\}) \longrightarrow P_m(\mathbb{D}^2) \longrightarrow P_1(\mathbb{D}^2) \longrightarrow 1.$$

Since $P_1(\mathbb{D}^2)$ is trivial, it follows that the kernel is equal to $P_m(\mathbb{D}^2)$, and the result follows.

(b) Let $m \in \mathbb{N}$ and $(x_1, \ldots, x_m) \in F_m(\mathbb{S}^2 \setminus (Y \cup \{x_0\}))$. Set $X = \{x_1, \ldots, x_m\}$ and $Y' = Y \cup \{x_0\}$. The inclusion $\mathbb{S}^2 \setminus \{x_0\} \subseteq \mathbb{D}^2$ induces an isomorphism of the free groups $\pi_1(\mathbb{S}^2 \setminus (X \cup Y \cup \{x_0\}))$ and $\pi_1(\mathbb{D}^2 \setminus (X \cup Y))$. Consider the following commutative diagram of short exact sequences:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \pi_1(\mathbb{S}^2 \setminus (X \cup Y')) & \longrightarrow & P_{m+1}(\mathbb{S}^2 \setminus Y') & \longrightarrow & P_m(\mathbb{S}^2 \setminus Y') & \longrightarrow & 1 \\
 & & \downarrow \cong & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \pi_1(\mathbb{D}^2 \setminus (X \cup Y)) & \longrightarrow & P_{m+1}(\mathbb{D}^2 \setminus Y) & \longrightarrow & P_m(\mathbb{D}^2 \setminus Y) & \longrightarrow & 1.
\end{array}$$

Applying induction on m and the 5-Lemma, it follows that $P_m(\mathbb{S}^2 \setminus Y') \cong P_{m+1}(\mathbb{D}^2 \setminus Y)$. By commutativity of the following diagram of short exact sequences

$$\begin{array}{ccccccccc}
1 & \longrightarrow & P_m(\mathbb{S}^2 \setminus Y') & \hookrightarrow & B_m(\mathbb{S}^2 \setminus Y') & \longrightarrow & S_m & \longrightarrow & 1 \\
 & & \downarrow \cong & & \downarrow & & \| & & \\
1 & \longrightarrow & P_m(\mathbb{D}^2 \setminus Y) & \hookrightarrow & B_m(\mathbb{D}^2 \setminus Y) & \longrightarrow & S_m & \longrightarrow & 1,
\end{array}$$

and the 5-Lemma, we see that $B_m(\mathbb{S}^2 \setminus Y') \cong B_m(\mathbb{D}^2 \setminus Y)$, which proves part (b).

(c) From the generalised Fadell-Neuwirth short exact sequence, we have that:

$$1 \longrightarrow B_m(\mathbb{D}^2 \setminus \{z_0\}) \longrightarrow B_{m,1}(\mathbb{D}^2) \longrightarrow B_1(\mathbb{D}^2) \longrightarrow 1.$$

The result then follows easily.

(d) Consider the following generalised Fadell-Neuwirth short exact sequence:

$$1 \longrightarrow \pi_1(\mathbb{D}^2 \setminus \{x_1, \ldots, x_m\}) \longrightarrow B_{m,1}(\mathbb{D}^2) \xrightarrow{p_*} B_m(\mathbb{D}^2) \longrightarrow 1.$$

Since p_* admits a section given by the obvious inclusion $B_m(\mathbb{D}^2) \to B_{m,1}(\mathbb{D}^2)$, the result again follows easily. $\square$

So by Remark 35(b), $B_m(\mathbb{S}^2 \setminus \{x_1\})$ and $B_m(\mathbb{D}^2)$ may be identified with Artin's braid group B_m. The series of such groups were previously studied by Gorin and Lin [**GL**]. For all $m \geq 1$, they determined presentations for $\Gamma_2(B_m(\mathbb{D}^2))$, from which they were able to deduce that:

THEOREM 36 ([**GL**]).

(a) The commutator subgroups $\Gamma_2(B_m(\mathbb{D}^2))$ are finitely presented.
(b) $\Gamma_2(B_3(\mathbb{D}^2))$ is a free group $\mathbb{F}_2(u,v)$ on two generators u and v.
(c) $\Gamma_2(B_4(\mathbb{D}^2))$ is a semi-direct product of the free group $\mathbb{F}_2(a,b)$ by $\mathbb{F}_2(u,v)$, the action (denoted by φ) being given by:

$$\left.\begin{array}{ll} \varphi(u)(a) = uau^{-1} = b & \varphi(u)(b) = ubu^{-1} = b^2a^{-1}b \\ \varphi(v)(a) = vav^{-1} = a^{-1}b & \varphi(v)(b) = vbv^{-1} = (a^{-1}b)^3a^{-2}b. \end{array}\right\} \tag{18}$$

(d) For all $m \geq 5$, the derived subgroup $(B_m(\mathbb{D}^2))^{(1)}$ is perfect, i.e. $(B_m(\mathbb{D}^2))^{(s)} = (B_m(\mathbb{D}^2))^{(1)}$ for all $s \geq 2$.

We now go on to extend their results.

PROPOSITION 5. *Let $m \geq 1$. Then:*

(a) For all $s \geq 3$, $\Gamma_s(B_m(\mathbb{D}^2)) = \Gamma_2(B_m(\mathbb{D}^2))$.
(b) If $m = 1, 2$ then $(B_m(\mathbb{D}^2))^{(s)} = \{1\}$ for all $s \geq 1$.
(c) If $m = 3$ then the derived series of $(B_3(\mathbb{D}^2))^{(1)}$ is that of the free group $\mathbb{F}_2(u,v)$ on two generators u and v, where $u = \sigma_2\sigma_1^{-1}$ and $v = \sigma_1 u \sigma_1^{-1} = \sigma_1\sigma_2\sigma_1^{-2}$. Further,

$$B_3(\mathbb{D}^2)/(B_3(\mathbb{D}^2))^{(2)} \cong \mathbb{Z}^2 \rtimes \mathbb{Z},$$

where $\mathbb{Z}^2$ is the free Abelian group generated by the respective Abelianisations $\overline{u}$ and $\overline{v}$ of u and v, and the action is given by $\sigma \cdot \overline{u} = \overline{v}$ and $\sigma \cdot \overline{v} = -\overline{u} + \overline{v}$, where σ is a generator of $\mathbb{Z}$.
(d) If $m = 4$ then

$$(B_4(\mathbb{D}^2))^{(1)}/(B_4(\mathbb{D}^2))^{(2)} \cong \mathbb{Z}^2, \text{ and}$$
$$(B_4(\mathbb{D}^2))^{(2)} \cong \mathbb{F}_2(a,b) \rtimes \Gamma_2(\mathbb{F}_2(u,v)),$$

where $a = \sigma_3\sigma_1^{-1}$ and $b = uau^{-1} = \sigma_2\sigma_3\sigma_1^{-1}\sigma_2^{-1}$.

PROOF OF PROPOSITION 5.

(a) The result follows from Lemma 23, since $(B_m(\mathbb{D}^2))^{\mathrm{Ab}} \cong \mathbb{Z}$, and $H_2(\mathbb{Z}) = \{1\}$.
(b) Clear.

(c) The first part is a direct consequence of Theorem 36(b). For the second part, the short exact sequence

$$1 \to (B_3(\mathbb{D}^2))^{(1)} \to B_3(\mathbb{D}^2) \to \mathbb{Z} \to 1$$

splits, where $(B_3(\mathbb{D}^2))^{(1)} \cong \mathbb{F}_2(u, v)$ and $\mathbb{Z} \cong \langle \sigma \rangle = (B_3(\mathbb{D}^2))^{\mathrm{Ab}}$, and a section is given by sending σ onto σ_1. So

$$B_3(\mathbb{D}^2) \cong (B_3(\mathbb{D}^2))^{(1)} \rtimes \mathbb{Z},$$

where the action is given by $\sigma \cdot u = \sigma_1 \cdot \sigma_2\sigma_1^{-1} \cdot \sigma_1^{-1} = v$ and $\sigma \cdot v = \sigma_1 \cdot \sigma_1\sigma_2\sigma_1^{-2} \cdot \sigma_1^{-1} = \sigma_1\sigma_2^{-1} \cdot \sigma_1\sigma_2\sigma_1^{-2} = u^{-1}v$. Then $\big((B_3(\mathbb{D}^2))^{(1)}\big)^{\mathrm{Ab}} = (B_3(\mathbb{D}^2))^{(1)}/(B_3(\mathbb{D}^2))^{(2)} \cong \mathbb{Z}^2$ is a free Abelian group with basis $\{\widetilde{u}, \widetilde{v}\}$, and so it follows that

$$B_3(\mathbb{D}^2)/(B_3(\mathbb{D}^2))^{(2)} \cong (B_3(\mathbb{D}^2))^{(1)}/(B_3(\mathbb{D}^2))^{(2)} \rtimes \mathbb{Z} \cong \mathbb{Z}^2 \rtimes \mathbb{Z},$$

with action given by $\sigma \cdot \overline{u} = \overline{v}$ and $\sigma \cdot \overline{v} = -\overline{u} + \overline{v}$ as required.

(d) From Theorem 36(c), we know that $(B_4(\mathbb{D}^2))^{(1)}$ is a semi-direct product of the free group $\mathbb{F}_2(a, b)$ by $\mathbb{F}_2(u, v)$, where a, b, u and v are as defined in the statement of the proposition, and the action is given by equation (18). Under Abelianisation of $(B_4(\mathbb{D}^2))^{(1)}$, we see that a and b are sent to the trivial element, and there are no other relations between u and v other than the fact that they commute. So

$$(B_4(\mathbb{D}^2))^{(1)}/(B_4(\mathbb{D}^2))^{(2)} \cong \mathbb{Z}^2.$$

To see that $(B_4(\mathbb{D}^2))^{(2)} \cong \mathbb{F}_2(a, b) \rtimes \Gamma_2(\mathbb{F}_2(u, v))$, we apply Proposition 29. Since $(uau^{-1})a^{-1} = ba^{-1}$ and $(ubu^{-1})b^{-1} = b^2a^{-1}$, it follows that $a, b \in L$, where L is the subgroup generated by $\Gamma_2(\mathbb{F}_2(a, b))$ and the normal subgroup $[\mathbb{F}_2(u, v), \mathbb{F}_2(a, b)]$ of $\mathbb{F}_2(a, b)$ generated by all elements of the form $(ghg^{-1})h^{-1}$, where $g \in \mathbb{F}_2(u, v)$ and $h \in \mathbb{F}_2(a, b)$. So $L = \mathbb{F}_2(a, b)$, and the result follows. □

Hence the lower central series of $B_m(\mathbb{D}^2)$ is determined for all $m \in \mathbb{N}$, in particular, if $m \geq 3$ then $B_m(\mathbb{D}^2)$ is not residually nilpotent; and the derived series of $B_m(\mathbb{D}^2)$ is determined for all $m \neq 4$. In this case, it remains to determine the higher derived subgroups and their quotients. For the next step, by Proposition 29,

$$(B_4(\mathbb{D}^2))^{(3)} = [(B_4(\mathbb{D}^2))^{(2)}, (B_4(\mathbb{D}^2))^{(2)}] \cong K \rtimes (\mathbb{F}_2(u, v))^{(2)},$$

where K is the subgroup of $\mathbb{F}_2(a, b)$ generated by $\Gamma_2(\mathbb{F}_2(a, b))$ and the normal subgroup of $\mathbb{F}_2(a, b)$ generated by the elements of the form $\varphi(g)(h)h^{-1}$, where $g \in (\mathbb{F}_2(u, v))^{(1)}$ and $h \in \mathbb{F}_2(a, b)$.

Let N be the normal subgroup of $\mathbb{F}_2(a, b)$ generated by $[a, b]$, a^2 and b^2. It may be interpreted as the kernel of the homomorphism

$\psi\colon \mathbb{F}_2(a,b) \to \mathbb{Z}_2 \times \mathbb{Z}_2$ which to a word $w = w(a,b)$ associates the exponent sums modulo 2 of w relative to a and b respectively.

In order to determine K we need to investigate the action of $[u,v]$ and its conjugates on $\mathbb{F}_2(a,b)$. One can check that $\varphi(u^{-1})(a) = ab^{-1}a^2$, $\varphi(u^{-1})(b) = a$, $\varphi(v^{-1})(a) = ab^{-1}a^3$ and $\varphi(v^{-1})(b) = ab^{-1}a^4$, then that:

$$\left.\begin{aligned} \varphi([u,v])(a)a^{-1} &= ab^{-2}(ab^{-1})^4a^{-1} \\ \varphi(u[u,v]u^{-1})(a)a^{-1} &= (ab^{-3}(ab^{-2})^4)^2ab^{-2}a^{-1} \\ \varphi([u,v])(b)b^{-1} &= ab^{-2}(ab^{-1})^5b^{-1} \\ \varphi(u[u,v]u^{-1})(b)b^{-1} &= ab^{-3}(ab^{-2})^5b^{-1}. \end{aligned}\right\} \tag{19}$$

Clearly K contains $[a,b]$. Further, a calculation shows that the element

$$(\varphi([u,v])(a)a^{-1})^{-3}(\varphi([u,v])(b)b^{-1})^2b^{-2}$$

belongs to $\Gamma_2(\mathbb{F}_2(a,b))$, and so to K. Hence b^2 belongs to K too. Considering the element

$$(\varphi([u,v])(a)a^{-1})^{-4}(\varphi([u,v])(b)b^{-1})^3,$$

we infer similarly that $a^2 \in K$. Now K is normal in $\mathbb{F}_2(a,b)$ and contains $[a,b]$, a^2 and b^2, so it contains N. We claim that $N = K$. Since ψ factors through Abelianisation, we see that $\Gamma_2(\mathbb{F}_2(a,b)) \subseteq N$. Let $w \in \mathbb{F}_2(u,v)$. If $\eta \in \{a,b\}$ then

$$\psi(\varphi(w)(\eta^2)) = \psi(w\eta^2w^{-1}) = 2\psi(\varphi(w)(\eta)) = (0,0).$$

Also,

$$\psi(\varphi(w)([a,b])) = \psi([waw^{-1}, wbw^{-1}]) = (0,0).$$

This implies that $\psi(w)(N) \subseteq N$, so $\psi(w)$ induces an endomorphism $\widetilde{\varphi(w)}$ of $\mathbb{Z}_2 \times \mathbb{Z}_2$ satisfying $\psi \circ \varphi(w) = \widetilde{\varphi(w)} \circ \psi$. The surjectivity of ψ and $\varphi(w)$ imply that $\widetilde{\varphi(w)}$ is an automorphism. Furthermore, $\widetilde{\varphi(w_1)} \circ \widetilde{\varphi(w_2)} = \widetilde{\varphi(w_1w_2)}$ for all $w_1, w_2 \in \mathbb{F}_2(u,v)$. Using the above relations for $\widetilde{\varphi([u,v])}$, we see that $\widetilde{\varphi([u,v])} = \mathrm{Id}$. Hence for all $w \in \Gamma_2(\mathbb{F}_2(u,v))$, $\widetilde{\varphi(w)} = \mathrm{Id}$, so $\psi(\varphi(w)(h)h^{-1}) = (0,0)$ for all $h \in \mathbb{F}_2(a,b)$. This implies that $K \subseteq N$, which proves the claim.

Finally, we Abelianise $(B_4(\mathbb{D}^2))^{(2)} \cong \mathbb{F}_2(a,b) \rtimes (\mathbb{F}_2(u,v))^{(1)}$. To the commutativity relations between a, b, u and v, one needs to add the relators the Abelianisation of the relators $\varphi(w)(h)h^{-1}$ where $w \in (\mathbb{F}_2(u,v))^{(1)}$ and $h \in \mathbb{F}_2(a,b)$, and in particular of relations (19), from which we obtain $a^2 = b^2 = 1$. But these are the only extra relations: since $\varphi(w)(h)h^{-1} \in \mathrm{Ker}\,(\psi)$, it follows from that form of N that the Abelianised relations are products of powers of a^2 and b^2. We thus obtain:

$$\left.\begin{aligned}
uz_1u^{-1} &= (uau^{-1})^2 = z_2\\
uz_2u^{-1} &= (ubu^{-1})^2 = b^2a^{-2}ab^2a^{-1}[a,b]b^2\\
&= z_2z_1^{-1}z_5z_3z_2^{-1}z_4^{-1}z_2\\
uz_3u^{-1} &= (uabu^{-1})^2 = b^4b^{-1}a^{-1}b^{-1}a^{-1}ab^4a^{-1}[a,b]b^2\\
&= z_2^2z_3^{-1}z_5^2z_3z_2^{-1}z_4^{-1}z_2\\
uz_4u^{-1} &= b^2a^{-2}ab^2a^{-1}a^2b^{-2} = z_2z_1^{-1}z_5z_1z_2^{-1}\\
uz_5u^{-1} &= b^4b^{-1}a^{-1}b^{-1}a^{-1}ab^4a^{-1} = z_2^2z_3^{-1}z_5^2
\end{aligned}\right\} \tag{20}$$

TABLE 4. The action of u on the basis $z_1, \ldots, z_5$ of N

PROPOSITION 6.
$$\big(B_4(\mathbb{D}^2)\big)^{(2)} \Big/ \big(B_4(\mathbb{D}^2)\big)^{(3)} \cong (\mathbb{Z}_2 \times \mathbb{Z}_2) \times (\Gamma_2(\mathbb{F}_2(u,v)))^{Ab}. \qquad \square$$

Using the Reidemeister-Schreier rewriting process [**MKS**], we may obtain a presentation of N. Let $X = \{a, b\}$ be a generating set of $\mathbb{F}_2(a,b)$ and $U = \{1, a, ab, aba^{-1}\}$ be a Schreier transversal. If $g \in \mathbb{F}_2(a,b)$, let $\overline{g} \in U$ denote its coset representative. A basis of N is given by the set of elements of the form $ux(\overline{ux})^{-1}$ where $u \in U$ and $x \in X$ (we remove all occurrences of the trivial element). A simple calculation shows that N is a free group of rank 5 with basis whose elements are given by a^2, $aba^2b^{-1}a^{-1}$, $bab^{-1}a^{-1}$, ab^2a^{-1} and b^2. This may be transformed into the following basis: $z_1 = a^2$, $z_2 = b^2$, $z_3 = (ab)^2$, $z_4 = ba^2b^{-1}$ and $z_5 = ab^2a^{-1}$. The action of $\mathbb{F}_2(u,v)$ on N is given by equations (20) and (21), see Tables 4 and 5 (we have used the relations $[a,b] = ababb^{-1}a^{-2}b^{-1} = z_3z_2^{-1}z_4^{-1}$, and $(ba^{-1})^2 = z_2z_3^{-1}z_5$). Hence:

PROPOSITION 7. $(B_4(\mathbb{D}^2))^{(3)} \cong \mathbb{F}_5(z_1, \ldots, z_5) \rtimes (\mathbb{F}_2(u,v))^{(2)}$, *where the action is that induced by the action of* $\mathbb{F}_2(u,v)$ *on* $\mathbb{F}_5(x_1, \ldots, x_5)$ *given by equations (20) and (21).* $\square$

From this, we may determine the Abelianisation $((B_4(\mathbb{D}^2))^{(3)})^{\mathrm{Ab}}$ of $(B_4(\mathbb{D}^2))^{(3)}$:

PROPOSITION 8.
$$\begin{aligned}\big((B_4(\mathbb{D}^2))^{(3)}\big)^{Ab} &= (B_4(\mathbb{D}^2))^{(3)} \big/ (B_4(\mathbb{D}^2))^{(4)}\\ &\cong \mathbb{Z}^3 \times \mathbb{Z}_{18} \times \mathbb{Z}_{18} \times (\mathbb{F}_2(u,v))^{(2)} \big/ (\mathbb{F}_2(u,v))^{(3)}.\end{aligned}$$

PROOF. The action of $\mathbb{F}_2(u,v)$ on $\mathbb{F}_5(z_1, \ldots, z_5)$ is by conjugation which leaves $\Gamma_2(\mathbb{F}_5(z_1, \ldots, z_5))$ invariant. It thus induces an action

$$\left.\begin{aligned}
vz_1v^{-1} =& (vav^{-1})^2 = (a^{-1}b)^2 = a^{-2}(ab)^2b^{-2}ba^{-2}b^{-1}b^2\\
=& z_1^{-1}z_3z_2^{-1}z_4^{-1}z_2\\
vz_2v^{-1} =& a^{-2}(ab)^2b^{-2}ba^{-2}b^{-1}b^2a^{-2}(ab)^2b^{-2}ba^{-2}b^{-1}b^2\cdot\\
& (b^{-1}a^{-1})^2ab^2a^{-1}b^2(b^{-1}a^{-1})^2ab^2a^{-1}ba^{-2}b^{-1}b^2\\
=& z_1^{-1}z_3z_2^{-1}z_4^{-1}z_2z_1^{-1}z_3z_2^{-1}z_4^{-1}z_2z_3^{-1}z_5z_2z_3^{-1}z_5z_4^{-1}z_2\\
=& (vz_1v^{-1})^2z_3^{-1}z_5z_2z_3^{-1}z_5z_4^{-1}z_2\\
vz_3v^{-1} =& (vabv^{-1})^2 = (a^{-1}b)^4a^{-2}ba^{-1}ba^{-1}ba^{-1}ba^{-1}ba^{-2}b\\
=& (vz_1v^{-1})^2z_1^{-1}(z_2z_3^{-1}z_5)^2z_4^{-1}z_2\\
=& (vz_1v^{-1})^2z_1^{-1}z_2vz_1^{-2}z_2v^{-1}\\
vz_4v^{-1} =& (a^{-1}b)^4b^{-1}a^{-1}b^{-1}a^{-1}ab^2a^{-1}b^2(a^{-1}b)^{-4}\\
=& (vz_1v^{-1})^2z_3^{-1}z_5z_2(vz_1v^{-1})^{-2}\\
vz_5v^{-1} =& (a^{-1}b)^4a^{-2}ba^{-1}ba^{-1}ba^{-1}ba^{-1}\\
=& (vz_1v^{-1})^2z_1^{-1}(z_2z_3^{-1}z_5)^2
\end{aligned}\right\} \qquad (21)$$

TABLE 5. The action of v on the basis $z_1, \ldots, z_5$ of N

of $\mathbb{F}_2(u,v)$ on $\mathbb{F}_5(z_1, \ldots, z_5)^{\mathrm{Ab}} = \mathbb{Z}^5 = \mathbb{Z}^5[Z_1, \ldots, Z_5]$, where for $i = 1, \ldots, 5$, Z_i is the image of z_i under Abelianisation. For $w \in \mathbb{F}_2(u,v)$, let M_w denote the matrix of this action with respect to the basis $(Z_1, \ldots, Z_5)$ of $\mathbb{Z}^5$, and let

$$\Lambda_w = \mathrm{Im}\,(M_w - I_5)\,.$$

By equations (20) and (21),

$$U = M_u = \begin{pmatrix} 0 & -1 & 0 & 0 & 0\\ 1 & 1 & 2 & 0 & 2\\ 0 & 1 & 0 & 0 & -1\\ 0 & -1 & -1 & 0 & 0\\ 0 & 1 & 2 & 1 & 2 \end{pmatrix}, \quad U^{-1} = M_{u^{-1}} = \begin{pmatrix} 1 & 1 & 2 & 2 & 0\\ -1 & 0 & 0 & 0 & 0\\ 1 & 0 & 0 & -1 & 0\\ 1 & 0 & 2 & 2 & 1\\ -1 & 0 & -1 & 0 & 0 \end{pmatrix},$$

and

$$V = M_v = \begin{pmatrix} -1 & -2 & -3 & 0 & -3\\ 0 & 2 & 3 & 1 & 2\\ 1 & 0 & 0 & -1 & 0\\ -1 & -3 & -3 & 0 & -2\\ 0 & 2 & 2 & 1 & 2 \end{pmatrix}, \quad V^{-1} = M_{v^{-1}} = \begin{pmatrix} 2 & 2 & 5 & 2 & 3\\ 0 & -1 & -1 & -1 & 0\\ 0 & 1 & 0 & 0 & -1\\ 2 & 2 & 4 & 2 & 3\\ -1 & -1 & -1 & 0 & 0 \end{pmatrix}.$$

Then

$$C = M_{[u,v]} = \begin{pmatrix} 3 & 3 & 5 & 2 & 3\\ -3 & -3 & -7 & -3 & -4\\ 0 & 0 & 1 & 0 & 0\\ 2 & 3 & 5 & 3 & 3\\ -3 & -4 & -7 & -3 & -3 \end{pmatrix}, \; C^{-1} = M_{[u,v]^{-1}} = \begin{pmatrix} -3 & -3 & -7 & -4 & -3\\ 3 & 3 & 5 & 3 & 2\\ 0 & 0 & 1 & 0 & 0\\ -4 & -3 & -7 & -3 & -3\\ 3 & 2 & 5 & 3 & 3 \end{pmatrix}.$$

Let L be the subgroup of $\mathbb{Z}^5$ generated by the Λ_w, where $w \in (\mathbb{F}_2(u,v))^{(2)}$. The action of $\mathbb{F}_2(u,v)$ on $\mathbb{Z}^5$ restricts to an action of

$(\mathbb{F}_2(u,v))^{(2)}$ on $\mathbb{Z}^5$. Since L is generated by the relators $M_w(Z) - Z$, where $Z \in \mathbb{Z}^5$ and $w \in (\mathbb{F}_2(u,v))^{(2)}$, it follows that

$$\left((B_4(\mathbb{D}^2))^{(3)}\right)^{\mathrm{Ab}} \cong \mathbb{Z}^5/L \times (\mathbb{F}_2(u,v))^{(2)} \,/\, (\mathbb{F}_2(u,v))^{(3)}. \qquad (22)$$

Let $c = [u,v]$ and $a = u[u,v]u^{-1}$. Consider first the special case $w = [a,c]$, and set $\Sigma = \Lambda_{[a,c]}$. We claim that:
(i) $\mathbb{Z}^5/\Sigma \cong \mathbb{Z}^3 \times \mathbb{Z}_{18} \times \mathbb{Z}_{18}$, and
(ii) $L = \Sigma$.
From this, it is obvious that $\mathbb{Z}^5/L \cong \mathbb{Z}^3 \times \mathbb{Z}_{18} \times \mathbb{Z}_{18}$, and so the result follows from equation (22).

To prove claim (i), one may check that

$$M_{[a,c]} = \begin{pmatrix} -701 & -612 & -1314 & -702 & -612 \\ 1548 & 1351 & 2898 & 1548 & 1350 \\ 0 & 0 & 1 & 0 & 0 \\ -702 & -612 & -1314 & -701 & -612 \\ 1548 & 1350 & 2898 & 1548 & 1351 \end{pmatrix}.$$

So Σ is the free Abelian group of rank 2 freely generated by $A_1 = \begin{pmatrix} -702 \\ 1548 \\ 0 \\ -702 \\ 1548 \end{pmatrix}$ and $A_2 = \begin{pmatrix} -612 \\ 1350 \\ 0 \\ -612 \\ 1350 \end{pmatrix}$, and $\mathbb{Z}^5/\Sigma$ has a finite presentation

$$0 \to \Sigma \xrightarrow{T} \mathbb{Z}^5 \to \mathbb{Z}^5/\Sigma \to 0,$$

where T is the $\mathbb{Z}$-module homomorphism represented by the matrix $A = \begin{pmatrix} -702 & -612 \\ 1548 & 1350 \\ 0 & 0 \\ -702 & -612 \\ 1548 & 1350 \end{pmatrix}$ relative to the bases (A_1, A_2) and $(Z_1, \ldots, Z_5)$. Applying elementary row and column operations to A, and taking $P = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ -11 & -5 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 1 \end{pmatrix}$ and $Q = \begin{pmatrix} 1 & 7 \\ -1 & -8 \end{pmatrix}$, we see that $PAQ = \begin{pmatrix} 18 & 0 \\ 0 & 18 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$, which gives the invariant factors of the Smith normal form of A [**AW**]. A new basis $W_1, \ldots, W_5$ of $\mathbb{Z}^5$ is obtained by taking $W_j = \sum_{i=1}^{5} (P^{-1})_{i,j} Z_i$, so $W_1 = -5Z_1 + 11Z_2 - 5Z_4 + 11Z_5$, $W_2 = -Z_1 + 2Z_2 - Z_4 + 2Z_5$, $W_3 = Z_3$, $W_4 = Z_4$ and $W_5 = Z_5$, and from the form of PAQ, it follows that in $\mathbb{Z}^5/\Sigma$, $18W_1 = 18W_2 = 0$, and that W_3, W_4 and W_5 are free generators. Thus $\mathbb{Z}^5/\Sigma \cong \mathbb{Z}^3 \times \mathbb{Z}_{18} \times \mathbb{Z}_{18}$, which proves claim (i).

We now set about proving claim (ii). Since $[a,c] \in (\mathbb{F}_2(u,v))^{(2)}$, it is clear that $\Sigma \subseteq L$. For the converse, it suffices to check that for all $w \in (\mathbb{F}_2(u,v))^{(2)}$, $\Lambda_w = \mathrm{Im}\,(M_w - I_5) \subseteq \langle A_1, A_2 \rangle$.

First note that u and v induce automorphisms of Σ; indeed, one may check that relative to the basis (A_1, A_2), the matrix of u is $\begin{pmatrix} -996 & -869 \\ 1145 & 999 \end{pmatrix}$, and that of v is $\begin{pmatrix} 18955 & 16531 \\ -21731 & -18952 \end{pmatrix}$. So $M_w(\Sigma) = \Sigma$ for all $w \in \mathbb{F}_2(u,v)$.

Further, since for all $w \in \mathbb{F}_2(u,v)$, M_w is an automorphism of $\mathbb{Z}^5$ which leaves Σ invariant, if $y \in \mathbb{F}_2(u,v)$ satisfies $\Lambda_y = \mathrm{Im}\,(M_y - I_5) \subseteq$

Σ then it follows that

$$\begin{aligned}\Lambda_{wyw^{-1}} = \operatorname{Im}(M_{wyw^{-1}} - I_5) &= \operatorname{Im}(M_w(M_y - I_5)M_{w^{-1}})\\ &= \operatorname{Im}(M_w(M_y - I_5)) \subseteq M_w(\Sigma) = \Sigma.\end{aligned}$$

So for our purposes, it will suffice to consider elements of $(\mathbb{F}_2(u,v))^{(2)}$ modulo conjugation by elements of $\mathbb{F}_2(u,v)$. Moreover, if $w_1, w_2 \in \mathbb{F}_2(u,v)$ satisfy $\operatorname{Im}(M_{w_i} - I_5) \subseteq \Sigma$ for $i = 1, 2$, then $\operatorname{Im}(M_{w_1w_2} - I_5) \subseteq \Sigma$. This follows from the fact that for all $x \in \mathbb{Z}^5$,

$$(M_{w_1w_2} - I_5)(x) = M_{w_1}(M_{w_2} - I_5)(x) + (M_{w_1} - I_5)(x),$$

and the invariance of Σ under M_w.

We now give a generating set for $(\mathbb{F}_2(u,v))^{(2)}$. A generating set of $(\mathbb{F}_2(u,v))^{(1)}$ is given by the set of conjugates $wc^{\pm 1}w^{-1}$, where $w \in \mathbb{F}_2(u,v)$ (cf. Remark 27), and so $(\mathbb{F}_2(u,v))^{(2)}$ is generated, up to conjugacy, by the set of commutators of the $wc^{\pm 1}w^{-1}$. So up to conjugacy, $(\mathbb{F}_2(u,v))^{(2)}$ is generated by the set of elements of the form $[c^{\varepsilon_1}, tc^{\varepsilon_2}t^{-1}]$, where $\varepsilon_1, \varepsilon_2 \in \{1, -1\}$. By conjugating by $c^{-1}tc^{\varepsilon_2}t^{-1}$ if necessary, we may suppose that $\varepsilon_1 = 1$. By the remarks of the previous paragraph, it thus suffices to show that $\operatorname{Im}(M_y - I_5) \subseteq \langle A_1, A_2 \rangle$ for elements y of the form $[c, tc^{\varepsilon_2}t^{-1}]$, where $\varepsilon_2 \in \{1, -1\}$. In order to do this, we shall now calculate $M_y - I_5$ explicitly.

Lemma 37. *For all $t \in \mathbb{F}_2(u,v)$, $M_{tc^{\pm 1}t^{-1}}$ is of the form:*

$$A = \begin{pmatrix} 3m & 3n & 3m+3n-1 & 3m-1 & 3n \\ -3p & -3m & -3m-3p-1 & -3p & -3m-1 \\ 0 & 0 & 1 & 0 & 0 \\ 3m-1 & 3n & 3m+3n-1 & 3m & 3n \\ -3p & -3m-1 & -3m-3p-1 & -3p & -3m \end{pmatrix},$$

where $m, n, p \in \mathbb{Z}$ and $np = m^2$.

Remark 38. One may check easily that the inverse of this matrix is:

$$A^{-1} = \begin{pmatrix} -3m & -3n & -3m-3n-1 & -3m-1 & -3n \\ 3p & 3m & 3m+3p-1 & 3p & 3m-1 \\ 0 & 0 & 1 & 0 & 0 \\ -3m-1 & -3n & -3m-3n-1 & -3m & -3n \\ 3p & 3m-1 & 3m+3p-1 & 3p & 3m \end{pmatrix}.$$

So if A satisfies the conditions of Lemma 37 then it is inversible, and A^{-1} also satisfies the conditions. Notice that A may be obtained simply from A^{-1} via the symmetry $(m,n,p) \mapsto (-m,-n,-p)$.

Proof of Lemma 37. We proceed by induction on the length $\ell(t)$ of the word t. If $\ell(t) = 0$ then t is the trivial element, and clearly $C = M_c$ and $C^{-1} = M_{c^{-1}}$ have the given structure. So suppose that t has word length $\ell(t) \geq 0$, and that $M_{tc^{\pm 1}t^{-1}}$ has the given structure. By Remark 38, it suffices to prove the result for $M_{tct^{-1}}$. Setting $A =$

$M_{tct^{-1}}$, a long but straightforward calculation shows that the respective conjugates of $M_{tct^{-1}}$ by M_u, $M_{u^{-1}}$, M_v and $M_{v^{-1}}$ are:

$$\begin{pmatrix} 9p-3m & 3p & 12p-1-3m & 9p-1-3m & 3p \\ 18m-27p-3n & 3m-9p & 21m-36p-1-3n & 18m-27p-3n & 3m-1-9p \\ 0 & 0 & 1 & 0 & 0 \\ 9p-1-3m & 3p & 12p-1-3m & 9p-3m & 3p \\ 18m-27p-3n & 3m-1-9p & 21m-36p-1-3n & 18m-27p-3n & 3m-9p \end{pmatrix},$$

$$\begin{pmatrix} 9n-3m & -18m+3p+27n & -1+36n-21m+3p & 9n-3m-1 & -18m+3p+27n \\ -3n & 3m-9n & -1-12n+3m & -3n & -1-9n+3m \\ 0 & 0 & 1 & 0 & 0 \\ 9n-3m-1 & -18m+3p+27n & -1+36n-21m+3p & 9n-3m & -18m+3p+27n \\ -3n & -1-9n+3m & -1-12n+3m & -3n & 3m-9n \end{pmatrix},$$

$$\begin{pmatrix} \gamma_1 & -30m+75p+3n & -57m+135p+6n-1 & \gamma_1-1 & -30m+75p+3n \\ -48p+24m-3n & -\gamma_1 & -1-108p+51m-6n & -48p+24m-3n & -1-\gamma_1 \\ 0 & 0 & 1 & 0 & 0 \\ \gamma_1-1 & -30m+75p+3n & -57m+135p+6n-1 & \gamma_1 & -30m+75p+3n \\ -48p+24m-3n & -1-\gamma_1 & -1-108p+51m-6n & -48p+24m-3n & -\gamma_1 \end{pmatrix},$$

where $\gamma_1 = -27m + 60p + 3n$, and

$$\begin{pmatrix} \gamma_2 & -120m+75p+48n & -147m+90p-1+60n & -1+\gamma_2 & -120m+75p+48n \\ -3p+6m-3n & -\gamma_2 & -1-18p+33m-15n & -3p+6m-3n & -1-\gamma_2 \\ 0 & 0 & 1 & 0 & 0 \\ -1+\gamma_2 & -120m+75p+48n & -147m+90p-1+60n & \gamma_2 & -120m+75p+48n \\ -3p+6m-3n & -1-\gamma_2 & -1-18p+33m-15n & -3p+6m-3n & -\gamma_2 \end{pmatrix},$$

where $\gamma_2 = -27m + 15p + 12n$. One may then check that each of these matrices has the form of the statement of the lemma. $\square$

We first consider the case $\varepsilon_2 = 1$, so $y = [c, tct^{-1}]$. With the matrix $M_{tct^{-1}} = A$ given by Lemma 37, a long but straightforward calculation shows once more that $M_{[c,tct^{-1}]} - I_5$ is of the form

$$\begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_1+\alpha_2 & \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 & \beta_1+\beta_2 & \beta_1 & \beta_2 \\ 0 & 0 & 0 & 0 & 0 \\ \alpha_1 & \alpha_2 & \alpha_1+\alpha_2 & \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 & \beta_1+\beta_2 & \beta_1 & \beta_2 \end{pmatrix},$$

where

$$\begin{aligned} \alpha_1 =& 1278m^2 + 216m - 1836pm - 126p - 540nm \\ & - 90n + 648p^2 + 450pn \\ \alpha_2 =& 1728nm - 756pn - 540n^2 - 1080m^2 \\ & + 648pm - 72n + 180m - 108p \\ \beta_1 =& - 1512m^2 - 252m + 2160pm + 144p \\ & + 648nm + 108n - 756p^2 - 540pn \\ \beta_2 =& - 2052nm + 882pn + 648n^2 + 1278m^2 \\ & - 756pm - 216m + 126p + 90n. \end{aligned}$$

So $\operatorname{Im}\left(M_{[c,tct^{-1}]} - I_5\right)$ is generated by the first two columns C_1, C_2 of $M_{[c,tct^{-1}]} - I_5$. It is necessary to show that each belongs to $\langle A_1, A_2 \rangle$, in other words, that for $i = 1, 2$, there exist $\tau_i, \mu_i \in \mathbb{Z}$ such that $\tau_i A_1 +$

$\mu_i A_2 = C_i$; these equations are equivalent to $\tau_i \left(\begin{smallmatrix} -702 \\ 1548 \end{smallmatrix}\right) + \mu_i \left(\begin{smallmatrix} -612 \\ 1350 \end{smallmatrix}\right) = \left(\begin{smallmatrix} \alpha_i \\ \beta_i \end{smallmatrix}\right)$, and admit solutions (*a priori* rational) of the form

$$\begin{pmatrix} \tau_i \\ \mu_i \end{pmatrix} = -\frac{1}{324} \begin{pmatrix} 1350 & 612 \\ -1548 & -702 \end{pmatrix} \begin{pmatrix} \alpha_i \\ \beta_i \end{pmatrix}.$$

Substituting for α_i, β_i, we obtain

$$\begin{aligned}
\tau_1 =& -2469m^2 - 424m + 3570pm + 253p + 1026nm + 171n \\
& - 1272p^2 - 855pn \\
\mu_1 =& 2830m^2 + 486m - 4092pm - 290p - 1176nm - 196n \\
& + 1458p^2 + 980pn \\
\tau_2 =& -3324nm + 1484pn + 1026n^2 + 2086m^2 - 1272pm + 130n \\
& - 342m + 212p \\
\mu_2 =& 3810nm - 1701pn - 1176n^2 - 2391m^2 + 1458pm - 149n \\
& + 392m - 243p,
\end{aligned}$$

and these solutions are clearly integers. Hence $C_1, C_2 \in \langle A_1, A_2 \rangle$ as required.

To deal with the case $\varepsilon_2 = -1$, it suffices to invoke the observation of Remark 38 concerning the symmetry between A and A^{-1}. The above analysis holds, and we obtain the same solutions as above, but replacing everywhere m, n and p by $-m, -n$ and $-p$ respectively. This proves claim (ii), and completes the proof of Proposition 8. $\square$

We would now like to go a stage further, and determine $(B_4(\mathbb{D}^2))^{(4)}$ and/or its Abelianisation. By applying Proposition 29 to Proposition 7, $(B_4(\mathbb{D}^2))^{(4)}$ is isomorphic to $M \rtimes (\mathbb{F}_2(u,v))^{(3)}$, where M is the subgroup of $\mathbb{F}_5(z_i) = \mathbb{F}_5(z_1, \ldots, z_5)$ generated by $\Gamma_2(\mathbb{F}_5(z_i))$ and the normal subgroup generated by the elements of the form $\varphi(g)(h)h^{-1}$, where $g \in (\mathbb{F}_2(u,v))^{(2)}$ and $h \in \mathbb{F}_5(z_i)$. However, the complexity of finding a basis of $(\mathbb{F}_2(u,v))^{(2)}$ and calculating the action on $\mathbb{F}_5(z_i)$ makes it extremely difficult to obtain a description of M. In order to get some idea of the situation, we shall turn our attention to studying the semi-direct product $\mathbb{F}_5(z_i) \rtimes \mathbb{F}_2(u,v)$. In any case, the calculations that follow shall be used later in Section 5 in order to study $\Gamma_2(B_3(\mathbb{S}^2 \setminus \{x_1, x_2\}))$.

From relations (20) and (21), we have an action of $\mathbb{F}_2(u,v)$ on $\mathbb{F}_5(z_1, \ldots, z_5)$, and thus a semi-direct product $\mathbb{F}_5(z_1, \ldots, z_5) \rtimes \mathbb{F}_2(u,v)$. Let

$$\varepsilon\colon \mathbb{F}_5(z_1, \ldots, z_5) \rtimes \mathbb{F}_2(u,v) \to (\mathbb{F}_5(z_1, \ldots, z_5) \rtimes \mathbb{F}_2(u,v))^{\mathrm{Ab}} \qquad (23)$$

be Abelianisation. From relations (20) and (21), it follows that

$$(\mathbb{F}_5(z_1,\ldots,z_5) \rtimes \mathbb{F}_2(u,v))^{\mathrm{Ab}} \cong \mathbb{Z} \oplus \mathbb{Z}^2,$$

where $\varepsilon(u) = (0,1,0)$, $\varepsilon(v) = (0,0,1)$, $\varepsilon(z_i) = (1,0,0)$ if $i = 1,2,3$ and $\varepsilon(z_i) = (-1,0,0)$ if $i = 4,5$.

Let $(Z_1,\ldots,Z_5)$, $(W_1,\ldots,W_5)$ be the bases of $\mathbb{Z}^5$ given in the proof of Proposition 8. Let

$$\widetilde{\varepsilon}\colon\ \mathbb{F}_5(z_1,\ldots,z_5) \rtimes (\mathbb{F}_2(u,v))^{(2)} \longrightarrow \big(\mathbb{F}_5(z_1,\ldots,z_5) \rtimes \mathbb{F}_2(u,v)^{(2)}\big)^{\mathrm{Ab}}$$

be the restriction of ε to $\mathbb{F}_5(z_1,\ldots,z_5) \rtimes (\mathbb{F}_2(u,v))^{(2)}$. We identify this latter group with $\mathbb{Z}_{18} \times \mathbb{Z}_{18} \times \mathbb{Z}^3 \times (\mathbb{F}_2(u,v))^{(2)}/(\mathbb{F}_2(u,v))^{(3)}$ via Proposition 8. Since $Z_1 = 2W_1 - 11W_2 - W_4$, $Z_2 = W_1 - 5W_2 - W_5$ and $W_i = Z_i$ for $i = 3,4,5$, as an element of $\mathbb{Z}_{18} \times \mathbb{Z}_{18} \times \mathbb{Z}^3$, we have $\widetilde{\varepsilon}(z_1) = (\overline{2}, -\overline{11}, 0, -1, 0)$, $\widetilde{\varepsilon}(z_2) = (\overline{1}, -\overline{5}, 0, 0, -1)$, $\widetilde{\varepsilon}(z_3) = (\overline{0}, \overline{0}, 1, 0, 0)$, $\widetilde{\varepsilon}(z_4) = (\overline{0}, \overline{0}, 0, 1, 0)$ and $\widetilde{\varepsilon}(z_5) = (\overline{0}, \overline{0}, 0, 0, 1)$. We thus obtain the following commutative diagram of short exact sequences:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & (B_4(\mathbb{D}^2))^{(4)} & \longrightarrow & G & \xrightarrow{\widetilde{\varepsilon}} & \big((B_4(\mathbb{D}^2))^{(3)}\big)^{\mathrm{Ab}} & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \xi} & & \\
1 & \longrightarrow & \mathrm{Ker}\,(\varepsilon) & \longrightarrow & H & \xrightarrow{\varepsilon} & \mathbb{Z} \oplus \mathbb{Z}^2 & \longrightarrow & 1,
\end{array} \tag{24}$$

where we set

$$G = \mathbb{F}_5(z_1,\ldots,z_5) \rtimes (\mathbb{F}_2(u,v))^{(2)} \text{ and } H = \mathbb{F}_5(z_1,\ldots,z_5) \rtimes \mathbb{F}_2(u,v).$$

The first two vertical arrows are inclusions. Identifying $\big((B_4(\mathbb{D}^2))^{(3)}\big)^{\mathrm{Ab}}$ with $\mathbb{Z}_{18} \times \mathbb{Z}_{18} \times \mathbb{Z}^3 \times (\mathbb{F}_2(u,v))^{(2)}/(\mathbb{F}_2(u,v))^{(3)}$ as above, the induced homomorphism ξ of the Abelianisations sends $(\mathbb{F}_2(u,v))^{(2)}/(\mathbb{F}_2(u,v))^{(3)}$ and the Σ-cosets of W_1 and W_2 onto the trivial element, and

$$\xi(W_3) = -\xi(W_4) = -\xi(W_5) = (1,0,0).$$

Let us determine $\mathrm{Ker}\,(\varepsilon)$. Since ε is Abelianisation, it follows from Proposition 29 that

$$\mathrm{Ker}\,(\varepsilon) = \Gamma_2\big(\mathbb{F}_5(x_i) \rtimes \mathbb{F}_2(u,v)\big) = L \rtimes \Gamma_2\left(\mathbb{F}_2(u,v)\right),$$

where L is the subgroup of $\mathbb{F}_5(z_i)$ generated by $\Gamma_2\left(\mathbb{F}_5(z_i)\right)$ and the normal subgroup generated by the elements of the form $\varphi(g)(h)h^{-1}$, where $g \in \mathbb{F}_2(u,v)$ and $h \in \mathbb{F}_5(z_i)$. Let $\rho\colon\ \mathbb{F}_5(z_i) \to \mathbb{Z}$ be the restriction of ε to the first factor, in other words,

$$\rho(z_i) = \begin{cases} 1 & \text{if } i = 1,2,3 \\ -1 & \text{if } i = 4,5. \end{cases} \tag{25}$$

PROPOSITION 39. *$L = Ker(\rho)$.*

$\cdots$	z_1^{-3}	z_1^{-2}	z_1^{-1}		1	z_1	z_1^2	$\cdots$
	1	1	1	z_1	1	1	1	
	$z_1^{-3}z_2z_1^2$	$z_1^{-2}z_2z_1$	$z_1^{-1}z_2$	z_2	$z_2z_1^{-1}$	$z_1z_2z_1^{-2}$	$z_1^2z_2z_1^{-3}$	
	$z_1^{-3}z_3z_1^2$	$z_1^{-2}z_3z_1$	$z_1^{-1}z_3$	z_3	$z_3z_1^{-1}$	$z_1z_3z_1^{-2}$	$z_1^2z_3z_1^{-3}$	
	$z_1^{-3}z_4z_1^4$	$z_1^{-2}z_4z_1^3$	$z_1^{-1}z_4z_1^2$	z_4	z_4z_1	z_1z_4	$z_1^2z_4z_1^{-1}$	
	$z_1^{-3}z_5z_1^4$	$z_1^{-2}z_5z_1^3$	$z_1^{-1}z_5z_1^2$	z_5	z_5z_1	z_1z_5	$z_1^2z_5z_1^{-1}$	

TABLE 6. Determination of a basis of Ker (ρ)

PROOF. We first apply the Reidemeister-Schreier rewriting process in order to obtain a basis of Ker (ρ). Taking $X = \{z_1, \ldots, z_5\}$ as a basis of $\mathbb{F}_5(z_i)$, and $U = \{z_1^i\}_{i\in\mathbb{Z}}$ as a Schreier transversal, one may check that a basis of Ker (ρ) is given by

$$\left\{z_1^{-i}z_jz_1^{i-1}\right\}_{i\in\mathbb{Z},\, j\in\{2,3\}} \bigcup \left\{z_1^{-i}z_jz_1^{i+1}\right\}_{i\in\mathbb{Z},\, j\in\{4,5\}}.$$

These calculations are presented in Table 6.

We may now show that Ker $(\rho) \subseteq L$. Indeed, since L is normal in $\mathbb{F}_5(z_i)$, and all basis elements of Ker (ρ) are conjugates of $z_2z_1^{-1}$, $z_3z_1^{-1}$, z_4z_1 and z_5z_1 by powers of z_1, it suffices to show that these four elements belong to L. This can be done by studying equation (20). First, $z_2z_1^{-1} = \varphi(u)(z_1)z_1^{-1}$, so $z_2z_1^{-1} \in L$. Next,

$$\varphi(u)(z_2)z_2^{-1} = z_2z_1^{-1} \cdot z_5z_3z_2^{-1}z_4^{-1} \in L,$$

so $z_5z_3z_2^{-1}z_4^{-1} \in L$, and $\varphi(u)(z_5)z_5^{-1} = z_2^2z_3^{-1}z_5 \in L$. Thus

$$\varphi(u)(z_3)z_3^{-1} = z_2^2z_3^{-1}z_5 \cdot z_5z_3z_2^{-1}z_4^{-1} \cdot z_2z_3^{-1} \in L,$$

so $z_2z_3^{-1} = z_2z_1^{-1}(z_3z_1^{-1})^{-1} \in L$, and hence $z_3z_1^{-1} \in L$. Since

$$\varphi(u)(z_5)z_5^{-1} = z_2 \cdot z_2z_3^{-1} \cdot z_5z_1 \cdot z_1^{-1}(z_2z_1^{-1})z_1 \cdot z_2^{-1} \in L,$$

and L is normal in $\mathbb{F}_5(z_i)$, it follows that $z_5z_1 \in L$. Finally, since $z_5z_3z_2^{-1}z_4^{-1} \in L$, and

$$z_5z_3z_2^{-1}z_4^{-1} = z_5z_1 \cdot z_1^{-1}(z_3z_1^{-1})z_1 \cdot z_1^{-1}(z_2z_1^{-1})^{-1}z_1 \cdot (z_4z_1)^{-1},$$

we have $z_4z_1 \in L$. This proves that Ker $(\rho) \subseteq L$.

We now prove that $L \subseteq$ Ker (ρ). Clearly $\Gamma_2(\mathbb{F}_5(z_i)) \subset$ Ker (ρ), and since Ker (ρ) is normal in $\mathbb{F}_5(z_i)$, it suffices to prove that all elements of the form $\varphi(g)(h)h^{-1}$, where $g \in \mathbb{F}_2(u, v)$ and $h \in \mathbb{F}_5(z_i)$, belong to Ker (ρ), which we do by double induction. Let ℓ denote the length function defined on elements of free groups. If $\ell(g) = 0$ or $\ell(h) = 0$ then the result is clearly true. If $\ell(g) = \ell(h) = 1$ then one may check directly using equations (20) and (21) that $\varphi(g)(h)h^{-1} \in \operatorname{Ker}(\rho)$ for $g \in \{u, v\}$. Suppose that $g \in \{u^{-1}, v^{-1}\}$. In order to show that $\varphi(g)(h)h^{-1}$ belongs to $\operatorname{Ker}(\rho)$, it suffices to show that its inverse $h(\varphi(g)(h))^{-1}$ belongs to it. Since $\varphi(g^{-1})$ is an automorphism, there exists $h_1 \in \mathbb{F}_5(z_i)$ such that $\varphi(g^{-1})(h_1) = h$. Thus

$$h(\varphi(g)(h))^{-1} = \varphi(g^{-1})(h_1)(\varphi(g) \circ \varphi(g^{-1})(h))^{-1} = \varphi(g^{-1})(h_1)h_1^{-1},$$

and the result follows from the case $g \in \{u, v\}$.

First suppose that $\ell(g) = 1$, and that the result is true for all h of length less than or equal to $n \geq 1$. Let $h' \in \mathbb{F}_5(z_i)$ be such that $\ell(h') = n + 1$. Set $h' = hz$, where $h, z \in \mathbb{F}_5(z_i)$, $\ell(h) = n$ and $\ell(z) = 1$. Then

$$\varphi(g)(h')h'^{-1} = \varphi(g)(h)h^{-1} \cdot h(\varphi(g)(z)z^{-1})h^{-1}.$$

By induction, both terms on the right-hand side belong to $\operatorname{Ker}(\rho)$, and using the fact that $\operatorname{Ker}(\rho)$ is normal in $\mathbb{F}_5(z_i)$, we see that the result holds for all g of length one, and all h.

Now suppose that the result is true for all g of length less than or equal to $n \geq 1$, and all h. Let $g' \in \mathbb{F}_2(u, v)$ be such that $\ell(g') = n + 1$. Set $g' = gy$, where $g, y \in \mathbb{F}_2(u, v)$, $\ell(g) = n$ and $\ell(y) = 1$. Then

$$\varphi(g')(h)h^{-1} = \varphi(g)(\varphi(y)(h))(\varphi(y)(h))^{-1} \cdot \varphi(y)(h)h^{-1}.$$

Since $\varphi(y)(h) \in \mathbb{F}_5(z_i)$, the result follows by induction. This completes the proof of the inclusion $L \subseteq \operatorname{Ker}(\rho)$, and thus that of the proposition. □

3. The lower central and derived series of $B_1(\mathbb{S}^2 \setminus \{x_1, \ldots, x_n\})$

Let $m = 1$ and $n \geq 1$. The group $B_1(\mathbb{S}^2 \setminus \{x_1, \ldots, x_n\})$ is the fundamental group of $\mathbb{S}^2 \setminus \{x_1, \ldots, x_n\}$, and so is a free group on $n-1$ generators. So its lower central and derived series are those of free groups of finite rank. Further details about the lower central series of such groups may be found in [**Hal, MKS**].

4. The lower central and derived series of $B_m(\mathbb{S}^2 \setminus \{x_1, \ldots, x_n\})$ for $m \geq 3$ and $n \geq 2$

In this section, we prove Theorem 9, which tells us that if $n \geq 2$ then for most values of m, the lower central and derived series of $B_m(\mathbb{S}^2 \setminus \{x_1, \ldots, x_n\})$ are constant from the commutator subgroup onwards.

THEOREM 9. *Let $n \geq 2$. Then:*

(a) If $m \geq 3$ then

$$\Gamma_3(B_m(\mathbb{S}^2 \setminus \{x_1, \ldots, x_n\})) = \Gamma_2(B_m(\mathbb{S}^2 \setminus \{x_1, \ldots, x_n\})).$$

(b) If $m \geq 5$ *then*

$$(B_m(\mathbb{S}^2 \setminus \{x_1, \ldots, x_n\}))^{(2)} = (B_m(\mathbb{S}^2 \setminus \{x_1, \ldots, x_n\}))^{(1)}.$$

(c) If $m = 4$ *then*

$$B_4(\mathbb{S}^2 \setminus \{x_1, \ldots, x_n\})/(B_4(\mathbb{S}^2 \setminus \{x_1, \ldots, x_n\}))^{(2)} \cong \big(\mathbb{Z}^2 \rtimes \mathbb{Z}\big) \times \mathbb{Z}^{n-1}$$

where the semi-direct product structure is that of part (c) of Proposition 5, and

$$(B_4(\mathbb{S}^2 \setminus \{x_1, \ldots, x_n\}))^{(1)}/(B_4(\mathbb{S}^2 \setminus \{x_1, \ldots, x_n\}))^{(2)} \cong \mathbb{Z}^2.$$

Alternatively,

$$B_4(\mathbb{S}^2 \setminus \{x_1, \ldots, x_n\})/(B_4(\mathbb{S}^2 \setminus \{x_1, \ldots, x_n\}))^{(2)} \cong \mathbb{Z}^2 \rtimes \mathbb{Z}^n,$$

where $\mathbb{Z}^2 \cong (B_4(\mathbb{S}^2 \setminus \{x_1, \ldots, x_n\}))^{(1)}/(B_4(\mathbb{S}^2 \setminus \{x_1, \ldots, x_n\}))^{(2)}$ *is the free Abelian group with basis* $\{\overline{u}, \overline{v}\}$*,* $\mathbb{Z}^n \cong B_4(\mathbb{S}^2 \setminus \{x_1, \ldots, x_n\})^{Ab}$ *has basis* $\sigma, \rho_1, \ldots, \rho_{n-1}$*, and the action is given by*

$$\sigma \cdot \overline{u} = \overline{v} \qquad \sigma \cdot \overline{v} = -\overline{u} + \overline{v}$$
$$\rho_i \cdot \overline{u} = \overline{u} \qquad \rho_i \cdot \overline{v} = \overline{v}$$

for all $1 \leq i \leq n-1$*.*

REMARKS 40.

(a) For the lower central series of $B_m(\mathbb{S}^2 \setminus \{x_1, \ldots, x_n\})$, the only case not covered by Theorem 9 is $m = 2$ and $n \geq 2$; it will be discussed in Sections 5 and 6.

(b) For the derived series of $B_m(\mathbb{S}^2 \setminus \{x_1, \ldots, x_n\})$, the outstanding cases are $n \geq 2$ and $m = 2$, $m = 3$ and $m = 4$ (see Sections 5, 6, 7 and 8).

PROOF OF THEOREM 9. The idea of much of the proof is similar to that of Theorem 4. Let $m, n \geq 2$. Set $B_{m,n} = B_m(\mathbb{S}^2 \setminus \{x_1, \ldots, x_n\})$. Then we have a short exact sequence

$$1 \to \Gamma_2(B_{m,n}) \to B_{m,n} \xrightarrow{\alpha} (B_{m,n})^{\mathrm{Ab}} \to 1.$$

From Proposition 33, $(B_{m,n})^{\mathrm{Ab}}$ is a free Abelian group of rank n, generated by $\rho_1, \ldots, \rho_n, \sigma$, and subject to a single relation $\rho_1 \cdots \rho_n \sigma^{2(m-1)} = 1$. Taking the generators of $B_{m,n}$ given by Proposition 31, all of the σ_i are identified to σ by α, and for each $1 \leq i \leq n$, all of the $A_{i,j}$, $n+1 \leq j \leq n+m$, are identified to ρ_i.

Let $H \subseteq \Gamma_2(B_{m,n})$ be a normal subgroup of $B_{m,n}$, and let

$$\begin{cases} \pi\colon\ B_{m,n} \to B_{m,n}/H \\ \qquad\quad \beta \mapsto \overline{\beta} \end{cases}$$

denote the canonical projection. Then α factors through $B_{m,n}/H$, and we have a short exact sequence of the form:

$$1 \longrightarrow K \longrightarrow B_{m,n}/H \xrightarrow{\widehat{\alpha}} (B_{m,n})^{\mathrm{Ab}} \longrightarrow 1,$$

where $\alpha = \widehat{\alpha} \circ \pi$ and $K = \Gamma_2(B_{m,n})/H$.

In what follows, we shall impose one of the following two hypotheses:

(i) K is Abelian.
(ii) K is central in $B_{m,n}/H$.

Remarks 41.

(a) If $H = \Gamma_3(B_{m,n})$ then condition (ii) is satisfied.
(b) If $H = (B_{m,n})^{(2)}$ then condition (i) is satisfied.
(c) Clearly condition (ii) implies condition (i).

From Proposition 31, we conclude that $\overline{\sigma_1}, \ldots, \overline{\sigma_{m-1}}$ and the $\overline{A_{i,j}}$, $1 \leq i \leq n$, $n+1 \leq j \leq n+m$, generate $B_{m,n}/H$. Since α identifies the σ_k to σ, we see that $\widehat{\alpha}$ identifies the $\overline{\sigma_k}$ to σ. So for $2 \leq k \leq m-1$, there exists $t_k \in K$ such that $\overline{\sigma_k} = t_k\overline{\sigma_1}$.

Suppose first that condition (ii) is satisfied. Then t_k commutes with $\overline{\sigma_1}$. For $1 \leq l \leq m-2$, we deduce from applying π to Artin's relations $\overline{\sigma_l}\,\overline{\sigma_{l+1}}\,\overline{\sigma_l} = \overline{\sigma_{l+1}}\,\overline{\sigma_l}\,\overline{\sigma_{l+1}}$ that $t_2 = 1$ and $t_l = t_{l+1}$ if $l \geq 2$. Thus $\overline{\sigma_1} = \cdots = \overline{\sigma_{m-1}}$. This argument holds for all $m \geq 2$.

Now suppose instead that condition (i) is satisfied. Let $m \geq 5$. If $3 \leq k \leq m-1$, then since σ_k commutes with σ_1, we have that

$$\overline{\sigma_1} \cdot t_k\overline{\sigma_1} = t_k\overline{\sigma_1} \cdot \overline{\sigma_1}, \tag{26}$$

and hence t_k commutes with $\overline{\sigma_1}$. Now let $4 \leq l \leq m-1$ (such an l exists). Since σ_l commutes with σ_2, we obtain

$$t_l\overline{\sigma_1} \cdot t_2\overline{\sigma_1} = t_2\overline{\sigma_1} \cdot t_l\overline{\sigma_1}.$$

But K is Abelian, and thus it follows that t_2 commutes with $\overline{\sigma_1}$. Applying this to the image under π of the relation $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$, we see that $t_2 = t_2^2$, and hence $t_2 = 1$. Finally, if $l \geq 2$ then the relation $\sigma_l\sigma_{l+1}\sigma_l = \sigma_{l+1}\sigma_l\sigma_{l+1}$ implies that $t_l = t_{l+1}$, and so $t_2 = \cdots = t_{m-1}$. Hence $\overline{\sigma_1} = \overline{\sigma_2} = \cdots = \overline{\sigma_{m-1}}$.

Let us now consider the $A_{i,j}$. In what follows, we suppose that $m \geq 3$ and $\overline{\sigma_1} = \overline{\sigma_2} = \cdots = \overline{\sigma_{m-1}}$ (which as we have just observed, is the case if either condition (ii) holds, or if condition (i) holds and additionally $m \geq 5$). By Proposition 31, $\sigma_{j-n}A_{i,j}\sigma_{j-n}^{-1} = A_{i,j+1}$ where $n+1 \leq j \leq n+m-1$, and if $r \neq j-n-1, j-n$ then $\sigma_r A_{i,j}\sigma_r^{-1} = A_{i,j}$.

Since for all $n+1 \leq j \leq n+m$,

$$A_{i,j} = \sigma_{j-n-1} \cdots \sigma_1 \cdot A_{i,n+1} \sigma_1^{-1} \cdots \sigma_{j-n-1}^{-1},$$

we see by projecting into $B_{m,n}/H$, and using the condition $m \geq 3$ that

$$\overline{A_{i,j}} = \overline{\sigma_{m-1}}^{j-n-1} \overline{A_{i,n+1}}\, \overline{\sigma_{m-1}}^{-(j-n-1)} = \overline{A_{i,n+1}} = \overline{\alpha_i},$$

where $\alpha_i = A_{i,n+1}$. Taking $j = n+2$, it follows that $\overline{\alpha_i}$ commutes with $\overline{\sigma_1}$. Applying this to the first relation of the presentation of $B_{m,n}$ given in Proposition 31, it follows that the $\overline{\alpha_i}$ commute pairwise. Projecting the remaining relations for $B_{m,n}$ into $B_{m,n}/H$ give nothing new, except for the surface relation which yields $\overline{\alpha_1} \cdots \overline{\alpha_n}\, \overline{\sigma_1}^{2(m-1)} = 1$. Hence $B_{m,n}/H$ is an Abelian group generated by $\overline{\alpha_1}, \ldots, \overline{\alpha_n}$ and $\overline{\sigma_1}$, in which the relation $\overline{\alpha_1} \cdots \overline{\alpha_n}\, \overline{\sigma_1}^{2(m-1)} = 1$ is satisfied. Since $\widehat{\alpha}$ is surjective, we conclude by the Hopfian property of free Abelian groups of finite rank that $\widehat{\alpha}$ is an isomorphism, so $B_{m,n}/H \cong \mathbb{Z}^n$, and $H = \Gamma_2(B_{m,n})$.

Taking $m \geq 3$ and $H = \Gamma_3(B_{m,n})$, condition (ii) is satisfied, and we conclude from the above arguments that $\Gamma_3(B_{m,n}) = \Gamma_2(B_{m,n})$. This proves part (a) of the proposition.

Taking $m \geq 5$ and $H = (B_{m,n})^{(2)}$, condition (i) is satisfied, and we conclude similarly that $(B_{m,n})^{(2)} = \Gamma_2(B_{m,n}) = (B_{m,n})^{(1)}$, which proves part (b) of the proposition.

Now let us prove part (c) of the proposition. Let $m = 4$ and $n \geq 2$, and making use of the previous notation, set $H = (B_{4,n})^{(2)}$. Then condition (i) holds, and indeed $(B_{4,n})^{(1)}/H$ is Abelian. As in the case $m \geq 5$ (cf. equation (26)), we see that t_3 commutes with $\overline{\sigma_1}$. From the remaining two Artin relations, we see that $\overline{\sigma_1} t_2 \overline{\sigma_1} = t_2 \overline{\sigma_1}^2 t_2$ and $t_3 \overline{\sigma_1} t_2 \overline{\sigma_1} t_3 = t_2 \overline{\sigma_1} t_3 \overline{\sigma_1} t_2$. But t_3 commutes with both t_2 and $\overline{\sigma_1}$, hence the second equation reduces to $t_3 \overline{\sigma_1} t_2 \overline{\sigma_1} = t_2 \overline{\sigma_1}^2 t_2$. From the first equation, we see that $t_3 = 1$, in other words, σ_1 and σ_3 are identified under π to $\overline{\sigma}$ say, and there is just one Artin relation of the form $\overline{\sigma}\, \overline{\sigma_2}\, \overline{\sigma} = \overline{\sigma_2}\, \overline{\sigma}\, \overline{\sigma_2}$. Further, for $i = 1, \ldots, n$,

$$\begin{aligned}
\overline{A_{i,n+2}} &= \overline{\sigma_1}\, \overline{A_{i,n+1}}\, \overline{\sigma_1}^{-1} = \overline{\sigma_3}\, \overline{A_{i,n+1}}\, \overline{\sigma_3}^{-1} = \overline{A_{i,n+1}},\\
\overline{A_{i,n+3}} &= \overline{\sigma_2}\, \overline{A_{i,n+2}}\, \overline{\sigma_2}^{-1} = \overline{\sigma_2}\, \overline{A_{i,n+1}}\, \overline{\sigma_2}^{-1} = \overline{A_{i,n+1}}, \text{ and}\\
\overline{A_{i,n+4}} &= \overline{\sigma_3}\, \overline{A_{i,n+3}}\, \overline{\sigma_3}^{-1} = \overline{\sigma_3}\, \overline{A_{i,n+1}}\, \overline{\sigma_3}^{-1} = \overline{A_{i,n+1}}.
\end{aligned}$$

So for each $i = 1, \ldots, n$, the $A_{i,j}$ are identified by π to a single element $\overline{\alpha_i}$ which commutes with both $\overline{\sigma}$ and $\overline{\sigma_2}$. So $B_{4,n}/(B_{4,n})^{(2)}$ is generated by $\overline{\sigma}, \overline{\sigma_2}, \overline{\alpha_1}, \ldots, \overline{\alpha_{n-1}}$, subject to the relations

$$\overline{\sigma}\, \overline{\sigma_2}\, \overline{\sigma} = \overline{\sigma_2}\, \overline{\sigma}\, \overline{\sigma_2}, \quad \overline{\sigma}, \overline{\sigma_2} \rightleftharpoons \overline{\alpha_i}, \quad \overline{\alpha_i}, \rightleftharpoons \overline{\alpha_j}, \quad \text{for all } 1 \leq i, j \leq n-1.$$

So there exist homomorphisms

$$f\colon\ B_{4,n} \to B_3(\mathbb{D}^2) \times \mathbb{Z}^{n-1}$$

given by

$$\begin{aligned} f(\sigma_1) &= f(\sigma_3) = \sigma_1 \\ f(\sigma_2) &= \sigma_2 \\ f(A_{i,j}) &= A_i \text{ if } 1 \leq i \leq n-1 \\ f(A_{n,j}) &= A_{n-1}^{-1} \cdots A_1^{-1} \sigma_1^{-1} \sigma_2^{-1} \sigma_1^{-2} \sigma_2^{-1} \sigma_1^{-1}, \end{aligned}$$

where $(A_1, \ldots, A_{n-1})$ is a basis of $\mathbb{Z}^{n-1}$, and

$$\widetilde{\pi}\colon\ B_3(\mathbb{D}^2) \times \mathbb{Z}^{n-1} \to B_{4,n}/(B_{4,n})^{(2)}$$

defined by

$$\begin{aligned} \widetilde{\pi}(\sigma_1) &= \overline{\sigma} \\ \widetilde{\pi}(\sigma_2) &= \overline{\sigma_2} \\ \widetilde{\pi}(A_i) &= \overline{\alpha_i}, \end{aligned}$$

and satisfying $\pi = \widetilde{\pi} \circ f$. Since f is surjective, to prove the first part of (c), by Proposition 5(c), it suffices to show that $\operatorname{Ker}(\widetilde{\pi}) = (B_3(\mathbb{D}^2))^{(2)}$.

First let us show that $(B_3(\mathbb{D}^2))^{(2)} \subseteq \operatorname{Ker}(\widetilde{\pi})$. Let $y \in (B_3(\mathbb{D}^2))^{(2)}$. In particular, y may be written as a word $w(\sigma_1, \sigma_2)$. Considering this word to be an element x of $B_{4,n}$, since $y \in (B_3(\mathbb{D}^2))^{(2)}$, we have that $x \in (B_{4,n})^{(2)}$ and $f(x) = y$. Since $\pi(x) = e$, it follows that $y \in \operatorname{Ker}(\widetilde{\pi})$.

Conversely, let $y \in \operatorname{Ker}(\widetilde{\pi})$. Since f is surjective, there exists $x \in B_{4,n}$ such that $f(x) = y$, and so $x \in (B_{4,n})^{(2)}$. But since $(B_{4,n})^{(1)}$ is the normal subgroup of $B_{4,n}$ generated by the commutators $[\sigma_k, \sigma_l]$, $[\sigma_k, A_{i,j}]$ and $[A_{i,j}, A_{i',j'}]$, where $1 \leq k, l \leq 3$, $1 \leq i, i' \leq n$ and $1 \leq j, j' \leq 4$, f is surjective and $\mathbb{Z}^{n-1}$ is a direct factor of $B_3(\mathbb{D}^2) \times \mathbb{Z}^{n-1}$, it follows that $f\left((B_{4,n})^{(1)}\right) = (B_3(\mathbb{D}^2))^{(1)}$, and thus $f\left((B_{4,n})^{(2)}\right) = (B_3(\mathbb{D}^2))^{(2)}$. In particular, $y \in (B_3(\mathbb{D}^2))^{(2)}$. We thus conclude that $B_3(\mathbb{D}^2)/(B_3(\mathbb{D}^2))^{(2)} \times \mathbb{Z}^{n-1} \cong B_{4,n}/(B_{4,n})^{(2)}$, which proves the first part of (c).

We now move on to the second part of (c). Consider the homomorphism $B_3(\mathbb{D}^2) \to B_{4,n}$ given by $\sigma_i \mapsto \sigma_i$. Since $g\left((B_3(\mathbb{D}^2))^{(i)}\right) \subseteq (B_{4,n})^{(i)}$ for all $i \in \mathbb{N}$, there is an induced homomorphism

$$g\colon\ B_3(\mathbb{D}^2)/(B_3(\mathbb{D}^2))^{(2)} \to B_{4,n}/(B_{4,n})^{(2)},$$

which sends the coset of σ_i onto $\overline{\sigma_i}$, as well as its restriction

$$g_1\colon\ (B_3(\mathbb{D}^2))^{(1)}/(B_3(\mathbb{D}^2))^{(2)} \to (B_{4,n})^{(1)}/(B_{4,n})^{(2)}.$$

Similarly, since $B_4(\mathbb{S}^2 \setminus \{x_1\}) \cong B_4(\mathbb{D}^2)$ by Proposition 34, the surjective homomorphism $B_4(\mathbb{S}^2 \setminus \{x_1, \ldots, x_n\}) \to B_4(\mathbb{S}^2 \setminus \{x_1\})$ given by closing up the $n-1$ punctures $x_2, \ldots, x_n$ induces a surjective homomorphism

$$h\colon\ B_{4,n}/(B_{4,n})^{(2)} \to B_4(\mathbb{D}^2)/(B_4(\mathbb{D}^2))^{(2)},$$

which sends $\overline{\sigma_i}$ onto the coset of σ_i, as well as its restriction

$$h_1\colon\ (B_{4,n})^{(1)}/(B_{4,n})^{(2)} \to B_4(\mathbb{D}^2)^{(1)}/(B_4(\mathbb{D}^2))^{(2)}.$$

Hence we obtain the following commutative diagram:

$$\begin{array}{ccccc}
(B_3(\mathbb{D}^2))^{(1)}/(B_3(\mathbb{D}^2))^{(2)} & \longrightarrow & B_3(\mathbb{D}^2)/(B_3(\mathbb{D}^2))^{(2)} & \longrightarrow & (B_3(\mathbb{D}^2))^{\mathrm{Ab}} \\
{\scriptstyle g_1}\big\downarrow & & {\scriptstyle g}\big\downarrow & & \big\downarrow \\
(B_{4,n})^{(1)}/(B_{4,n})^{(2)} & \longrightarrow & B_{4,n}/(B_{4,n})^{(2)} & \longrightarrow & (B_{4,n})^{\mathrm{Ab}} \\
{\scriptstyle h_1}\big\downarrow & & {\scriptstyle h}\big\downarrow & & \big\downarrow \\
(B_4(\mathbb{D}^2))^{(1)}/(B_4(\mathbb{D}^2))^{(2)} & \longrightarrow & B_4(\mathbb{D}^2)/(B_4(\mathbb{D}^2))^{(2)} & \longrightarrow & (B_4(\mathbb{D}^2))^{\mathrm{Ab}}.
\end{array}$$

Note that the rows are all short exact sequences.

Now consider the first column. From [**GL**] and Proposition 5, we know that $(B_3(\mathbb{D}^2))^{(1)}/(B_3(\mathbb{D}^2))^{(2)}$ and $(B_4(\mathbb{D}^2))^{(1)}/(B_4(\mathbb{D}^2))^{(2)}$ are both free Abelian groups of rank 2, generated by their respective cosets of $u = \sigma_2\sigma_1^{-1}$ and $v = \sigma_1\sigma_2\sigma_1^{-2}$. By definition of g and h, it follows that $h_1 \circ g_1$ is an isomorphism, sending the $(B_3(\mathbb{D}^2))^{(2)}$-coset of u (respectively v) onto the $(B_4(\mathbb{D}^2))^{(2)}$-coset of u (respectively v). Thus to prove that $(B_{4,n})^{(1)}/(B_{4,n})^{(2)} \cong \mathbb{Z}^2$, it suffices to show that g_1 is surjective. To see this, let $x \in (B_{4,n})^{(1)}/(B_{4,n})^{(2)}$. Since $x \in B_{4,n}/(B_{4,n})^{(2)}$, it follows from above that

$$x = w(\overline{\sigma}, \overline{\sigma_2})\ \overline{\alpha_1}^{m_1} \cdots \overline{\alpha_{n-1}}^{m_{n-1}},$$

where $m_i \in \mathbb{Z}$ for $1 \leq i \leq n-1$ and $w(\overline{\sigma}, \overline{\sigma_2})$ is a word in $\overline{\sigma}$ and $\overline{\sigma_2}$. Projecting into $(B_{4,n})^{\mathrm{Ab}}$, since $\overline{\sigma}$ and $\overline{\sigma_2}$ map onto σ, and $\overline{\alpha_i}$ maps onto ρ_i, and furthermore, $\sigma, \rho_1, \ldots \rho_{n-1}$ generate freely $(B_{4,n})^{\mathrm{Ab}}$, we see by exactness that the m_i are all zero, in other words, $x = w(\overline{\sigma}, \overline{\sigma_2})$. Now take $z = w(\sigma_1, \sigma_2) \in B_3(\mathbb{D}^2)/(B_3(\mathbb{D}^2))^{(2)}$, so that $g(z) = x$. Projecting z into $(B_3(\mathbb{D}^2))^{\mathrm{Ab}}$ yields zero by commutativity of the diagram (the homomorphism $(B_3(\mathbb{D}^2))^{\mathrm{Ab}} \to (B_{4,n})^{\mathrm{Ab}}$ is injective), hence $z \in (B_3(\mathbb{D}^2))^{(1)}/(B_3(\mathbb{D}^2))^{(2)}$, and thus g_1 is surjective. So $(B_4(\mathbb{S}^2 \setminus \{x_1, \ldots, x_n\}))^{(1)}/(B_4(\mathbb{S}^2 \setminus \{x_1, \ldots, x_n\}))^{(2)} \cong \mathbb{Z}^2$, which proves the second part of part (c).

Finally, we prove the last part of part (c). Consider the short exact sequence

$$1 \to (B_{4,n})^{(1)}/(B_{4,n})^{(2)} \to B_{4,n}/(B_{4,n})^{(2)} \xrightarrow{\widehat{\alpha}} (B_{4,n})^{\mathrm{Ab}} \to 1.$$

Recall that $(B_{4,n})^{\mathrm{Ab}}$ is a free Abelian group with basis $\{\sigma, \rho_1, \ldots, \rho_{n-1}\}$, and that up to isomorphism, we may identify $B_3(\mathbb{D}^2)/(B_3(\mathbb{D}^2))^{(2)} \times \mathbb{Z}^{n-1}$ with $B_{4,n}/(B_{4,n})^{(2)}$, where the $\mathbb{Z}^{n-1}$-factor has a basis $\{\overline{\alpha_1}, \ldots, \overline{\alpha_{n-1}}\}$ for which $\widehat{\alpha}(\overline{\alpha_i}) = \rho_i$. It follows that $\widehat{\alpha}$ admits a section given by $\sigma \mapsto \overline{\sigma}$ and $\rho_i \mapsto \overline{\alpha_i}$, and hence

$$B_{4,n}/(B_{4,n})^{(2)} \cong (B_{4,n})^{(1)}/(B_{4,n})^{(2)} \rtimes (B_{4,n})^{\mathrm{Ab}}.$$

Taking the basis $\{\overline{u}, \overline{v}\}$ of $(B_{4,n})^{(1)}/(B_{4,n})^{(2)} \cong \mathbb{Z}^2$, the action is given by $\rho_i \cdot \overline{u} = \overline{\alpha_i}\, \overline{u}\, \overline{\alpha_i}^{-1} = \overline{u}$ and $\rho_i \cdot \overline{v} = \overline{\alpha_i}\, \overline{v}\, \overline{\alpha_i}^{-1} = \overline{v}$ since $\overline{\alpha_i}$ commutes with $\overline{\sigma}$ and $\overline{\sigma_2}$, and $\sigma \cdot \overline{u}$ and $\sigma \cdot \overline{v}$ are obtained as in the proof of the second part of Proposition 5(c). This completes the proof of Theorem 9. □

5. The commutator subgroup of $B_m(\mathbb{S}^2 \setminus \{x_1, x_2\})$, $m \geq 2$

Let $m \geq 2$. As we saw in Theorem 9, the lower central series of $B_m(\mathbb{S}^2 \setminus \{x_1, \ldots, x_n\}))$, $n \geq 2$, is constant from the commutator subgroup onwards if $m \geq 3$. In this section, we study the case $n = 2$ in more detail. The special case $m = n = 2$ will also be analysed later in Section 6, and the case $m \geq 3$ and $n = 2$ will also be discussed in Section 7.

From Remarks 35, we know that $B_m(\mathbb{S}^2 \setminus \{x_1, x_2\}))$ is the m-string braid group of the annulus, and so is isomorphic to the Artin group of type B_m. Presentations of these groups were obtained in [**Lam, Ma**], as well as in [**KP**] (we will come back to this presentation in Proposition 56). Annulus braid groups were also studied in [**Cr, PR**].

Let $m \geq 2$. From Proposition 34, it follows from part (b) that $B_m(\mathbb{S}^2 \setminus \{x_1, x_2\}) \cong B_m(\mathbb{D}^2 \setminus \{x_2\})$, and from part (c) that

$$B_m(\mathbb{D}^2 \setminus \{x_2\}, \{x_3, \ldots, x_{m+1}, x_{m+2}\}) \cong B_{m,1}(\mathbb{D}^2).$$

Hence $B_m(\mathbb{S}^2 \setminus \{x_1, x_2\}) \cong B_{m,1}(\mathbb{D}^2)$. But from part (d),

$$\begin{aligned} B_{m,1}(\mathbb{D}^2) &\cong \pi_1(\mathbb{D}^2 \setminus \{x_3, x_4, \ldots, x_{m+2}\}, x_2) \rtimes B_m(\mathbb{D}^2) \\ &\cong \mathbb{F}_m(A_{2,3}, \ldots, A_{2,m+2}) \rtimes B_m(\mathbb{D}^2), \end{aligned}$$

where $B_m(\mathbb{D}^2)$ is taken to be generated by $\sigma_3, \ldots, \sigma_{m+1}$, and the action φ of the σ_i, $3 \leq i \leq m+1$, on the $A_{2,j}$, $3 \leq j \leq m+2$ is that given by the Artin representation:

$$\sigma_i A_{2,j} \sigma_i^{-1} = \begin{cases} A_{2,j+1} & \text{if } j = i \\ A_{2,j}^{-1} A_{2,j-1} A_{2,j} & \text{if } j = i+1 \\ A_{2,j} & \text{otherwise.} \end{cases} \tag{27}$$

From this, we may deduce that:

$$\sigma_i^{-1} A_{2,j} \sigma_i = \begin{cases} A_{2,j-1} & \text{if } j = i+1 \\ A_{2,j} A_{2,j+1} A_{2,j}^{-1} & \text{if } j = i \\ A_{2,j} & \text{otherwise.} \end{cases} \tag{28}$$

PROPOSITION 10. *Let $m \geq 2$. Then:*

(a) $B_m(\mathbb{S}^2 \setminus \{x_1, x_2\}) \cong \mathbb{F}_m \rtimes B_m(\mathbb{D}^2)$, where the action φ is as given in equation (27).

(b) $\Gamma_2(B_m(\mathbb{S}^2 \setminus \{x_1, x_2\})) \cong Ker(\rho) \rtimes \Gamma_2(B_m(\mathbb{D}^2))$, where

$$\rho \colon \mathbb{F}_m(A_{2,3}, \ldots, A_{2,m+2}) \to \mathbb{Z}$$

is the augmentation homomorphism, and the action is that induced by φ.

PROOF. Part (a) was proved above, and in any case is a restatement of the results of Proposition 34. So let us prove part (b). Set $\mathbb{F}_m = \mathbb{F}_m(A_{2,3}, \ldots, A_{2,m+2})$, and let L be the subgroup of $\mathbb{F}_m$ generated by $\Gamma_2(\mathbb{F}_m)$ and the normal subgroup generated by the elements of the form $\varphi(g)(h) \cdot h^{-1}$, where $g \in B_m(\mathbb{D}^2)$ and $h \in \mathbb{F}_m$. By Proposition 29, it suffices to prove that $L = \operatorname{Ker}(\rho)$.

First we show that $L \subseteq \operatorname{Ker}(\rho)$. Since ρ factors through Abelianisation, we have clearly that $\Gamma_2(\mathbb{F}_m) \subseteq \operatorname{Ker}(\rho)$. Further, since $\operatorname{Ker}(\rho)$ is normal in $\mathbb{F}_m$, it suffices to prove that $\varphi(g)(h) \cdot h^{-1} \in \operatorname{Ker}(\rho)$, where $g \in B_m(\mathbb{D}^2)$ and $h \in \mathbb{F}_m$. This is equivalent to showing that $\rho(h) = \rho(\varphi(g)(h)) = \rho(ghg^{-1})$ and may be achieved by double induction as follows. If g and h are both of length 1, in other words if they are generators or inverses of generators of their respective groups then the result holds using equations (27) and (28). Secondly, if g is of length 1 then the result follows for all h by applying induction on the word length of h (relative to the given basis of $\mathbb{F}_m$) and the fact that

$$gh_1h_2g^{-1} = gh_1g^{-1} \cdot gh_2g^{-1}$$

for all $h_1, h_2 \in \mathbb{F}_m$. Finally the result holds for all g and all h by applying induction on the word length of g (relative to the given generators of $B_m(\mathbb{D}^2)$) and the relation

$$g_1g_2h(g_1g_2)^{-1} = g_1h'g_1^{-1},$$

where $g_1, g_2 \in B_m(\mathbb{D}^2)$ and $h' = g_2 h g_2^{-1} \in \mathbb{F}_m$. This proves that $L \subseteq \operatorname{Ker}(\rho)$.

To see that $\operatorname{Ker}(\rho) \subseteq L$, we determine a basis of $\operatorname{Ker}(\rho)$ with the help of the Reidemeister-Schreier rewriting process. Taking $X = \{A_{2,3}, \ldots, A_{2,m+2}\}$ as a basis of $\mathbb{F}_m$ and $U = \left\{A_{2,3}^i\right\}_{i \in \mathbb{Z}}$ to be a Schreier transversal, we see that a basis of $\operatorname{Ker}(\rho)$ is given by the elements of the form $\left\{A_{2,3}^i A_{2,j} A_{2,3}^{-(i+1)}\right\}_{i \in \mathbb{Z},\, j \in \{4,\ldots,m+2\}}$, or in other words, the conjugates of the $A_{2,j}A_{2,3}^{-1}$ by $A_{2,3}^i$. Since L is normal in $\mathbb{F}_m$, it suffices to prove that the $A_{2,j}A_{2,3}^{-1}$ belong to L. This is the case, since for all $3 \leq j \leq m+1$,

$$\begin{aligned} \varphi(\sigma_j)(A_{2,j})A_{2,j}^{-1} &= A_{2,j+1}A_{2,j}^{-1} \in L, \text{ and} \\ A_{2,j+1}A_{2,3}^{-1} &= A_{2,j+1}A_{2,j}^{-1} \cdot A_{2,j}A_{2,j-1}^{-1} \cdots A_{2,4}A_{2,3}^{-1} \in L. \end{aligned}$$

Thus $\operatorname{Ker}(\rho) \subseteq L$, which completes the proof of part (b) of the proposition. □

We now investigate further the case $m = 3$. By Theorem 36, $\Gamma_2(B_3(\mathbb{D}^2))$ is a free group $\mathbb{F}_2(u, v)$ of rank 2, where $u = \sigma_4\sigma_3^{-1}$ and $v = \sigma_3\sigma_4\sigma_3^{-2}$. For $i \in \mathbb{Z}$, we set

$$\begin{aligned} \alpha_i &= A_{2,3}^i A_{2,4} A_{2,3}^{-(i+1)} = A_{2,3}^i \alpha_0 A_{2,3}^{-i} \quad \text{and} \\ \beta_i &= A_{2,3}^i A_{2,5} A_{2,3}^{-(i+1)} = A_{2,3}^i \beta_0 A_{2,3}^{-i}. \end{aligned}$$

Using relations (27) and (28), one may check that

$$\begin{aligned} uA_{2,3}u^{-1} &= A_{2,3}A_{2,5}A_{2,3}^{-1} & vA_{2,3}v^{-1} &= A_{2,4}A_{2,5}A_{2,4}A_{2,5}^{-1}A_{2,4}^{-1} \\ uA_{2,4}u^{-1} &= A_{2,3} & vA_{2,4}v^{-1} &= A_{2,4}A_{2,5}A_{2,4}^{-1} \\ uA_{2,5}u^{-1} &= A_{2,5}^{-1}A_{2,4}A_{2,5} & vA_{2,5}v^{-1} &= A_{2,5}^{-1}A_{2,4}^{-1}A_{2,3}A_{2,4}A_{2,5}, \end{aligned}$$

then that

$$\left.\begin{aligned} u\alpha_0u^{-1} =& uA_{2,4}A_{2,3}^{-1}u^{-1} = (A_{2,3}A_{2,5}A_{2,3}^{-2})^{-1} = \beta_1^{-1} \\ u\beta_0u^{-1} =& uA_{2,5}A_{2,3}^{-1}u^{-1} = (A_{2,3}^{-1}A_{2,5})^{-1}(A_{2,3}^{-1}A_{2,4})(A_{2,5}A_{2,3}^{-1}) \cdot \\ & \qquad (A_{2,3}A_{2,5}A_{2,3}^{-2})^{-1} = \beta_{-1}^{-1}\alpha_{-1}\beta_0\beta_1^{-1} \\ uA_{2,3}^iu^{-1} =& r_i \cdot A_{2,3}^i, \end{aligned}\right\} \quad (29)$$

where $r_i = A_{2,3}A_{2,5}^{i}A_{2,3}^{-(i+1)} \in \operatorname{Ker}(\rho)$, and finally that

$$\left.\begin{aligned}
v\alpha_0 v^{-1} =& vA_{2,4}A_{2,3}^{-1}v^{-1} = (A_{2,4}A_{2,3}^{-1})(A_{2,3}A_{2,5}A_{2,3}^{-2})\cdot\\
& (A_{2,3}^{2}A_{2,5}A_{2,3}^{-3})(A_{2,3}^{2}A_{2,4}A_{2,3}^{-3})^{-1}(A_{2,3}A_{2,5}A_{2,3}^{-2})^{-1}\cdot\\
& (A_{2,4}A_{2,3}^{-1})^{-1} = \alpha_0\beta_1\beta_2\alpha_2^{-1}\beta_1^{-1}\alpha_0^{-1}\\
v\beta_0 v^{-1} =& vA_{2,5}A_{2,3}^{-1}v^{-1} = (A_{2,3}^{-1}A_{2,5})^{-1}(A_{2,3}^{-2}A_{2,4}A_{2,3})^{-1}\cdot\\
& (A_{2,3}^{-1}A_{2,4})(A_{2,5}A_{2,3}^{-1})(A_{2,3}A_{2,4}A_{2,3}^{-2})(A_{2,3}^{2}A_{2,5}A_{2,3}^{-3})\cdot\\
& (A_{2,3}^{2}A_{2,4}A_{2,3}^{-3})^{-1}(A_{2,3}A_{2,5}A_{2,3}^{-2})^{-1}(A_{2,4}A_{2,3}^{-1})^{-1}\\
& = \beta_{-1}^{-1}\alpha_{-2}^{-1}\alpha_{-1}\beta_0\alpha_1\beta_2\alpha_2^{-1}\beta_1^{-1}\alpha_0^{-1}\\
vA_{2,3}^{i}v^{-1} =& s_iA_{2,3}^{-i},
\end{aligned}\right\} \tag{30}$$

where $s_i = A_{2,4}A_{2,5}A_{2,4}^{i}A_{2,5}^{-1}A_{2,4}^{-1}A_{2,3}^{-i} \in \operatorname{Ker}(\rho)$. Up to expressing the r_i and s_i in terms of the α_i and β_i, we thus obtain a complete set of relations for $\operatorname{Ker}(\rho) \rtimes \mathbb{F}_2(u, v)$:

$$\left.\begin{aligned}
u\alpha_i u^{-1} &= r_i\beta_{i+1}^{-1}r_i^{-1}\\
u\beta_i u^{-1} &= r_i\beta_{i-1}^{-1}\alpha_{i-1}\beta_i\beta_{i+1}^{-1}r_i^{-1}\\
v\alpha_i v^{-1} &= s_i\alpha_i\beta_{i+1}\beta_{i+2}\alpha_{i+2}^{-1}\beta_{i+1}^{-1}\alpha_i^{-1}s_i^{-1}\\
v\beta_i v^{-1} &= s_i\beta_{i-1}^{-1}\alpha_{i-2}^{-1}\alpha_{i-1}\beta_i\alpha_{i+1}\beta_{i+2}\alpha_{i+2}^{-1}\beta_{i+1}^{-1}\alpha_i^{-1}s_i^{-1}
\end{aligned}\right\} \tag{31}$$

for all $i \in \mathbb{Z}$.

Setting $\widetilde{\alpha_i}$ (resp. $\widetilde{\beta_i}$) to be the Abelianisation of α_i (resp. β_i), and Abelianising equations (31), we obtain:

$$\left.\begin{aligned}
\widetilde{\alpha_i} &= -\widetilde{\beta_{i+1}}\\
\widetilde{\alpha_i} &= \widetilde{\alpha_{i+3}}\\
\widetilde{\alpha_i} + \widetilde{\alpha_{i+1}} + \widetilde{\alpha_{i+2}} &= 0
\end{aligned}\right\} \tag{32}$$

for all $i \in \mathbb{Z}$. By Proposition 10(b), $(B_3(\mathbb{S}^2 \setminus \{x_1, x_2\}))^{(1)} \big/ (B_3(\mathbb{S}^2 \setminus \{x_1, x_2\}))^{(2)}$ is Abelian, generated by the $\widetilde{\alpha_i}$ and $\widetilde{\beta_i}$, $i \in \mathbb{Z}$, and the Abelianisations of u and v, subject to these relations, and so is a free Abelian group with basis $\widetilde{\alpha_0}, \widetilde{\alpha_1}$ and the Abelianisations of u and v. Hence:

Proposition 11.

$$\left(B_3\left(\mathbb{S}^2 \setminus \{x_1, x_2\}\right)\right)^{(1)} \Big/ \left(B_3\left(\mathbb{S}^2 \setminus \{x_1, x_2\}\right)\right)^{(2)} \cong \mathbb{Z}^4. \qquad \square$$

With the help of Proposition 11, we may obtain the following:

Proposition 12.

$$B_3\left(\mathbb{S}^2 \setminus \{x_1, x_2\}\right) \Big/ \left(B_3\left(\mathbb{S}^2 \setminus \{x_1, x_2\}\right)\right)^{(2)} \cong \mathbb{Z}^4 \rtimes \mathbb{Z}^2,$$

where $\mathbb{Z}^4$ *has a basis* $\left\{\widetilde{\alpha_0}, \widetilde{\beta_0}, \widetilde{u}, \widetilde{v}\right\}$*,* $\mathbb{Z}^2$ *has a basis* $\{\sigma, \rho_1\}$*, and the action is given by:*

$$\sigma \cdot \widetilde{u} = \widetilde{v} \qquad \sigma \cdot \widetilde{v} = -\widetilde{u} + \widetilde{v}$$
$$\sigma \cdot \widetilde{\alpha_0} = \widetilde{\beta_0} \qquad \sigma \cdot \widetilde{\beta_0} = \widetilde{\beta_0} - \widetilde{\alpha_0}$$
$$\rho_1 \cdot \widetilde{\alpha_0} = \widetilde{\alpha_0} \qquad \rho_1 \cdot \widetilde{\beta_0} = \widetilde{\beta_0}$$
$$\rho_1 \cdot \widetilde{u} = -\widetilde{\alpha_0} - \widetilde{u} + \widetilde{v} \qquad \rho_1 \cdot \widetilde{v} = -\widetilde{\beta_0} - \widetilde{u}.$$

PROOF. Consider the following short exact sequence:

$$1 \to (B_{3,2})^{(1)}/(B_{3,2})^{(2)} \to B_{3,2}/(B_{3,2})^{(2)} \xrightarrow{\widehat{\alpha}} {B_3}^{\mathrm{Ab}} \to 1,$$

where $B_{3,2} = B_3(\mathbb{S}^2 \setminus \{x_1, x_2\})$. From Proposition 11, $(B_{3,2})^{(1)}/(B_{3,2})^{(2)}$ is a free Abelian group of rank 4 with basis $\widetilde{\alpha_0}$, $\widetilde{\beta_0}$, $\widetilde{u}$ and $\widetilde{v}$, where $\alpha_0 = A_{2,4}A_{2,3}^{-1}$, $\beta_0 = A_{2,5}A_{2,3}^{-1}$ and $\widetilde{u}$, $\widetilde{v}$ are the respective Abelianisations of $u = \sigma_4\sigma_3^{-1}$ and $v = \sigma_3\sigma_4\sigma_3^{-2}$. Further, $(B_{3,2})^{\mathrm{Ab}}$ is a free Abelian group of rank 2, with basis σ, ρ_1, where $\widehat{\alpha}(\overline{\sigma_3}) = \widehat{\alpha}(\overline{\sigma_4}) = \sigma$, and $\widehat{\alpha}(\overline{A_{1,j}}) = \rho_1$ for $j = 1, 2, 3$. Then $\sigma \mapsto \overline{\sigma_3}$ and $\rho_1 \mapsto \overline{A_{1,5}}$ defines a section for $\widehat{\alpha}$. Let us now determine the associated action. We have already seen that $\sigma \cdot u = v$, and $\sigma \cdot v = u^{-1}v$, so $\sigma \cdot \widetilde{u} = \widetilde{v}$, and $\sigma \cdot \widetilde{v} = -\widetilde{u} + \widetilde{v}$. Further, from equation (27), we have $\sigma \cdot \alpha_0 = \sigma_3\alpha_0\sigma_3^{-1} = A_{2,4}^{-1}A_{2,3} = \alpha_{-1}^{-1}$, and $\sigma \cdot \beta_0 = \sigma_3\alpha_0\sigma_3^{-1} = A_{2,5}A_{2,4}^{-1} = \beta_0\alpha_0^{-1}$, so by equation (32), $\sigma \cdot \widetilde{\alpha_0} = \widetilde{\beta_0}$ and $\sigma \cdot \widetilde{\beta_0} = \widetilde{\beta_0} - \widetilde{\alpha_0}$. As for the action of ρ_1, we have $\rho_1 \cdot \alpha_0 = \alpha_0$, so $\rho_1 \cdot \widetilde{\alpha_0} = \widetilde{\alpha_0}$, and $\rho_1 \cdot \beta_0 = A_{1,5} \cdot A_{2,5}A_{2,3}^{-1} \cdot A_{1,5}^{-1}$. But $A_{1,5} = \sigma_4^{-1}\sigma_3^{-2}\sigma_4^{-1}A_{2,5}^{-1}$, hence

$$\begin{aligned}\rho_1 \cdot \beta_0 &= \sigma_4^{-1}\sigma_3^{-2}\sigma_3^{-1}A_{2,5}\sigma_4\sigma_3^2\sigma_4A_{2,3}^{-1}\\ &= A_{2,3}A_{2,4}A_{2,5}A_{2,4}^{-1}A_{2,3}^{-2} = \alpha_1\beta_2\alpha_2^{-1},\end{aligned}$$

and thus $\rho_1 \cdot \widetilde{\beta_0} = \widetilde{\beta_0}$. Also,

$$\begin{aligned}\rho_1 \cdot u &= \sigma_4^{-1}\sigma_3^{-2}\sigma_3^{-1}A_{2,5}^{-1}uA_{2,5}\sigma_4\sigma_3^2\sigma_4\\ &= A_{2,3}A_{2,4}A_{2,5}^{-1}A_{2,4}^{-1}A_{2,3}^{-1}A_{2,4}\sigma_4^{-1}\sigma_3^{-3}\sigma_4\sigma_3^2\sigma_4.\end{aligned}$$

But

$$\sigma_4^{-1}\sigma_3^{-3}\sigma_4\sigma_3^2\sigma_4 = \sigma_3^{-1}\sigma_4 \cdot \sigma_3\sigma_4\sigma_3^{-2} = u^{-1}v,$$

so $\rho_1 \cdot u = \alpha_1\beta_1^{-1}\alpha_0^{-1}\alpha_{-1}u^{-1}v$, and $\rho_1 \cdot \widetilde{u} = -\widetilde{\alpha_0} - \widetilde{u} + \widetilde{v}$. Finally,

$$\begin{aligned}\rho_1 \cdot v &= A_{1,5}\sigma_3A_{1,5}^{-1} \cdot \rho_1 \cdot u \cdot A_{1,5}\sigma_3^{-1}A_{1,5}^{-1} = \sigma_3 \cdot \alpha_1\beta_1^{-1}\alpha_0^{-1}\alpha_{-1}u^{-1}v \cdot \sigma_3^{-1}\\ &= A_{2,3}A_{2,4}A_{2,5}^{-1}A_{2,4}^{-1}A_{2,3}^{-1}A_{2,4}^{-1}A_{2,3}A_{2,4}v^{-1}u^{-1}v\\ &= \alpha_1\beta_1^{-1}\alpha_0^{-1}\alpha_{-2}^{-1}\alpha_{-1}v^{-1}u^{-1}v,\end{aligned}$$

and so $\rho_1 \cdot \widetilde{v} = -\widetilde{\beta_0} - \widetilde{u}$, which proves the proposition. □

We now give an alternative proof of Proposition 11. Although it is long, we believe the method to be of interest. We will also make use of some results proved in Section 2 to prove (Proposition 42) that $(B_3(\mathbb{S}^2 \setminus \{x_1,x_2\}))^{(1)}$ is a semi-direct product of an infinite-rank subgroup of $\mathbb{F}_5(z_1,\ldots,z_5)$ by $\mathbb{F}_2(u,v)$.

Given $B_3(\mathbb{S}^2 \setminus \{x_1,x_2\})$, from the generalised Fadell-Neuwirth short exact sequence (equation (5)), we obtain

$$1 \to B_3(\mathbb{S}^2 \setminus \{x_1,x_2\}) \xrightarrow{\iota} B_{3,1}(\mathbb{S}^2 \setminus \{x_1\}) \to B_1(\mathbb{S}^2 \setminus \{x_1\}) \to 1.$$

Clearly ι is an isomorphism, and composing by the inclusion $B_{3,1}(\mathbb{S}^2 \setminus \{x_1\}) \hookrightarrow B_4(\mathbb{S}^2 \setminus \{x_1\})$, we obtain an injective homomorphism

$$f\colon\ B_3(\mathbb{S}^2 \setminus \{x_1,x_2\}) \to B_4(\mathbb{S}^2 \setminus \{x_1\}).$$

Further, we have the following commutative diagram of short exact sequences:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & P_3(\mathbb{S}^2 \setminus \{x_1,x_2\}) & \longrightarrow & B_3(\mathbb{S}^2 \setminus \{x_1,x_2\}) & \xrightarrow{\pi} & S_3 & \longrightarrow & 1 \\
 & & {\scriptstyle \varphi\left|_{P_3(\mathbb{S}^2\setminus\{x_1,x_2\})}\right.}\Big\downarrow{\scriptstyle\cong} & & {\scriptstyle\varphi}\Big\downarrow & & \Big\downarrow & & \\
1 & \longrightarrow & P_4(\mathbb{D}^2) & \longrightarrow & B_4(\mathbb{D}^2) & \xrightarrow{\pi} & S_4 & \longrightarrow & 1,
\end{array} \tag{33}$$

where π is the homomorphism which to a braid associates its permutation, and φ is the composition of f and the isomorphism $B_4(\mathbb{S}^2 \setminus \{x_1\}) \cong B_4(\mathbb{D}^2)$ given by Proposition 34(b). The right-hand vertical homomorphism is the natural inclusion of S_3 in S_4. So φ is also injective, and $\varphi(\sigma_i) = \sigma_i$ for $i = 1,2$. The fact that the left-hand homomorphism $\varphi\left|_{P_3(\mathbb{S}^2\setminus\{x_1,x_2\})}\right.$ is an isomorphism follows from Proposition 34(a). From this, we obtain the following commutative diagram:

$$\begin{array}{ccc}
(B_3(\mathbb{S}^2 \setminus \{x_1,x_2\}))^{(1)} & \xrightarrow{\pi} & A_3 \\
{\scriptstyle\varphi}\Big\downarrow & & \Big\downarrow \\
(B_4(\mathbb{D}^2))^{(1)} & \xrightarrow{\pi} & A_4,
\end{array}$$

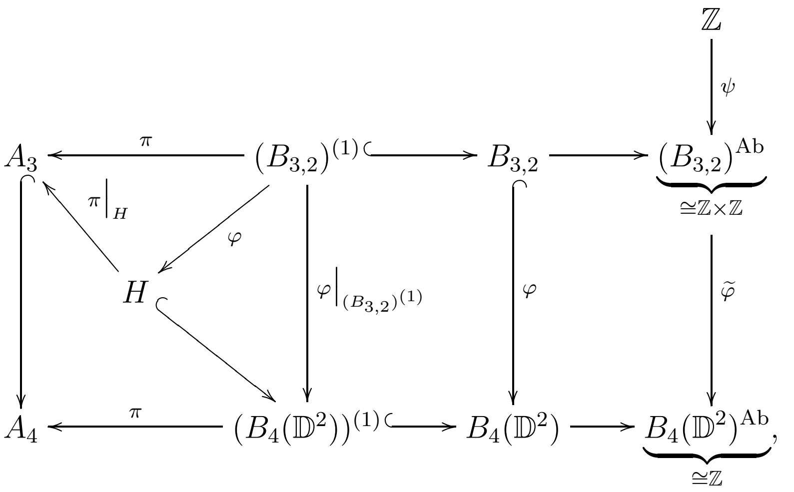
A_n being the alternating subgroup of S_n. By abuse of notation, we use the same symbols for the restriction homomorphisms. If $H = (B_4(\mathbb{D}^2))^{(1)} \cap \pi^{-1}(A_3)$ then $\varphi(B_3(\mathbb{S}^2 \setminus \{x_1,x_2\}))^{(1)} \subseteq H$ by commutativity of the diagram. Since $\pi\left|_{(B_4(\mathbb{D}^2))^{(1)}}\right.$ is surjective onto A_4, it follows that $[(B_4(\mathbb{D}^2))^{(1)} : H] = 4$ (indeed, if $g\colon G_1 \to G_2$ is a surjective group homomorphism, and H_2 is a subgroup of G_2 then $H_1 = g^{-1}(H_2)$ is a subgroup of G_1, and $G_1/H_1 \to G_2/H_2$, $xH_1 \mapsto g(x_1)H_2$ defines a bijection, so $[G_1 : H_1] = [G_2 : H_2]$).

Recall from Theorem 36(c) that $(B_4(\mathbb{D}^2))^{(1)} \cong \mathbb{F}_2(a,b) \rtimes \mathbb{F}_2(u,v)$, where $a = \sigma_3\sigma_1^{-1}$, $b = \sigma_2\sigma_3\sigma_1^{-1}\sigma_2^{-1}$, $u = \sigma_2\sigma_1^{-1}$, $v = \sigma_1\sigma_2\sigma_1^{-2}$, and the action is given by equation (18). The corresponding elements $u = \sigma_2\sigma_1^{-1}$, $v = \sigma_1\sigma_2\sigma_1^{-2}$ of $B_3(\mathbb{S}^2 \setminus \{x_1, x_2\})$ (we use the same symbols to denote these elements) in fact belong to $(B_3(\mathbb{S}^2 \setminus \{x_1, x_2\}))^{(1)}$, generate a free group of rank 2 (by injectivity of φ), and we have

$$\mathbb{F}_2(u,v) \subseteq \varphi((B_3(\mathbb{S}^2 \setminus \{x_1, x_2\}))^{(1)}) \subseteq H. \tag{34}$$

Further, writing the elements of $(B_4(\mathbb{D}^2))^{(1)}$ in the form $\mathbb{F}_2(a,b) \rtimes \mathbb{F}_2(u,v)$, if $(w,z) \in H$ then for all $z' \in \mathbb{F}_2(u,v)$, $z^{-1}z' \in H$ by equation (34), so $(w,z)(e,z^{-1}z') = (w,z') \in H$, and thus $\{w\} \times \mathbb{F}_2(u,v) \subseteq H$. Hence H is of the form $H_1 \rtimes \mathbb{F}_2(u,v)$, where H_1 is an index 4 subgroup of $\mathbb{F}_2(a,b)$. Together with the identity, the elements $\pi(a) = (12)(34)$, $\pi(b) = (13)(24)$ and $\pi(ab) = (14)(23)$ form a set of coset representatives for A_4/A_3, so e, a, b and ab form a set of coset representatives for $\mathbb{F}_2(a,b)/H_1$. If $w = w(a,b) \in \mathbb{F}_2(a,b)$ then $\pi(w) = w(\pi(a), \pi(b))$, and since $\pi(a)$ and $\pi(b)$ generate a group isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$, we see that $\pi(w) \in A_3$ if and only if the exponent sums in w of a and b are both even. In other words, $H_1 = \operatorname{Ker}(\mathbb{F}_2(a,b) \to \mathbb{Z}_2 \times \mathbb{Z}_2)$, where $a \mapsto (1,0)$ and $b \mapsto (0,1)$, is nothing other than the free subgroup N of $\mathbb{F}_2(a,b)$ of rank 5 described on page 21 of Section 2, possessing a basis $z_1 = a^2$, $z_2 = b^2$, $z_3 = (ab)^2$, $z_4 = ba^2b^{-1}$ and $z_5 = ab^2a^{-1}$. Thus $H \cong \mathbb{F}_5(z_1, \ldots, z_5) \rtimes \mathbb{F}_2(u,v)$, where the action is given by equations (20) and (21).

Hence we have a commutative diagram of the form:



(35)

where $B_{3,2} = B_3(\mathbb{S}^2 \setminus \{x_1, x_2\})$, and

$$\widetilde{\varphi}\colon\ B_3(\mathbb{S}^2 \setminus \{x_1, x_2\})^{\mathrm{Ab}} \to B_4(\mathbb{D}^2)^{\mathrm{Ab}}$$

is the homomorphism induced on the Abelianisations. Since $B_4(\mathbb{D}^2)^{\mathrm{Ab}}$ is infinite cyclic, generated by an element $\overline{\sigma}$, say, $B_3(\mathbb{S}^2 \setminus \{x_1, x_2\})^{\mathrm{Ab}}$ is a free Abelian group of rank 2 with basis comprised of σ and A, we have that $\widetilde{\varphi}(\sigma) = \overline{\sigma}$, and $\widetilde{\varphi}(A) = \overline{\sigma}^2$. So $\operatorname{Ker}(\widetilde{\varphi}) \cong \mathbb{Z}$ is the subgroup generated by $(-2, 1)$ relative to the basis (σ, A). Let

$$\psi\colon \mathbb{Z} \to B_3(\mathbb{S}^2 \setminus \{x_1, x_2\})^{\mathrm{Ab}}$$

be defined by $\psi(k) = k(-2, 1)$. Then the final column of the diagram is exact.

Now the idea is the following: given $x \in H$, we may associate an element of $\mathbb{Z}$ using diagram (35). We shall show that this is a homomorphism, ε' say, which satisfies $\varepsilon'(z_i) = 1$ if $i = 1, 2, 3$, $\varepsilon'(z_i) = -1$ if $i = 4, 5$, and $\varepsilon'(u) = \varepsilon'(v) = 0$. From this, in particular, we obtain $\varepsilon' = \eta \circ \varepsilon$, where $\varepsilon\colon \mathbb{F}_5(z_1, \dots, z_5) \rtimes \mathbb{F}_2(u, v) \to \mathbb{Z} \oplus \mathbb{Z}^2$ is the homomorphism defined by equation (23), and $\eta\colon \mathbb{Z} \oplus \mathbb{Z}^2 \to \mathbb{Z}$ is defined by $\eta((1, 0, 0)) = 1$ and $\eta((0, 1, 0)) = \eta((0, 0, 1)) = 0$.

To define ε', we first choose the following correspondence between $B_3(\mathbb{S}^2 \setminus \{x_1, x_2\})$ and $B_4(\mathbb{D}^2)$: the three strings correspond of $B_3(\mathbb{S}^2 \setminus \{x_1, x_2\})$ to the first three strings of $B_4(\mathbb{D}^2)$; the puncture x_1 to the fourth (vertical) string; and the puncture x_2 to the boundary of $\mathbb{D}^2$. In this representation, if $1 \leq i \leq 3$ and $1 \leq j \leq 2$ then $C_{i,j+3}$ will denote the element of $B_3(\mathbb{S}^2 \setminus \{x_1, x_2\})$ represented by a loop based at point i which encircles the j^{th} puncture. Suppose that $z \in B_4(\mathbb{D}^2)$ is such that $\pi(z) \in S_3$. Then we claim that there exists $y_z \in B_3(\mathbb{S}^2 \setminus \{x_1, x_2\})$ such that $\varphi(y_z) = z$; by injectivity of φ, such a y_z is unique. To prove the claim, notice there exists $y' \in B_3(\mathbb{S}^2 \setminus \{x_1, x_2\})$ such that $\pi(z) = \pi(y') = \pi \circ \varphi(y')$ by commutativity of diagram (33). Hence $\varphi(y')^{-1} z \in \operatorname{Ker}(\pi)$. But since the first vertical arrow of that diagram is bijective, there exists $y'' \in P_3(\mathbb{S}^2 \setminus \{x_1, x_2\})$ such that $\varphi(y'') = \varphi(y')^{-1} z$. Hence $z = \varphi(y_z)$, where $y_z = y'y''$, and the claim is proved.

So let $x \in H$. Since $H \subseteq (B_4(\mathbb{D}^2))^{(1)}$, x is sent to 0 in $B_4(\mathbb{D}^2)^{\mathrm{Ab}}$. By the claim, there exists a unique $y_x \in B_3(\mathbb{S}^2 \setminus \{x_1, x_2\})$ such that $\varphi(y_x) = x$, so by commutativity of diagram (35), $\widetilde{y_x} \in \operatorname{Ker}(\widetilde{\varphi})$, where $\widetilde{y_x}$ denotes the Abelianisation of y_x. Thus $\widetilde{y_x} = k(-2, 1)$ relative to the basis (σ, A), where $k \in \mathbb{Z}$, and so $x \mapsto k$ defines a map ε' from H to $\mathbb{Z}$, well defined since y_x is unique, and a homomorphism because φ is. Further, $\psi \circ \varepsilon'(x) = \widetilde{y_x}$. Let us now calculate ε' on the given generating set $\{z_1, \cdots, z_5, u, v\}$ of H. Now $y_u = u$ and $y_v = v$, and since $\mathbb{F}_2(u, v) \subseteq \varphi(B_3(\mathbb{S}^2 \setminus \{x_1, x_2\}))^{(1)}$ by (34), it follows that $\widetilde{y_u} = \widetilde{y_v} = 0$, and so $\varepsilon'(u) = \varepsilon'(v) = 0$. Now consider

$$x_1 = a^2 = (\sigma_3 \sigma_1^{-1})^2 = \sigma_3^2 \sigma_1^{-2}.$$

From the given correspondence between $B_3(\mathbb{S}^2 \setminus \{x_1, x_2\})$ and $B_4(\mathbb{D}^2)$, z_1 may be written as $C_{34}\sigma_1^{-2}$, and so under Abelianisation is sent to $(-2, 1)$. Hence $\varepsilon'(z_1) = 1$. Since z_2 is conjugation of z_1 by σ_2, we obtain similarly that $\varepsilon'(z_2) = 1$. As for z_3,

$$\begin{aligned} z_3 = (ab)^2 &= (\sigma_3\sigma_1^{-1} \cdot \sigma_2\sigma_3\sigma_1^{-1}\sigma_2^{-1})^2 \\ &= \sigma_1^{-1}\sigma_2\sigma_3\sigma_2\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_3\sigma_2\sigma_3\sigma_1^{-1}\sigma_2^{-1} \\ &= \sigma_1^{-1}\sigma_2\sigma_1^{-1}\sigma_3^2\sigma_2\sigma_1^{-1}\sigma_2^{-1} = \sigma_1^{-1}\sigma_2\sigma_1^{-1}C_{34}\sigma_2\sigma_1^{-1}\sigma_2^{-1}, \end{aligned}$$

so $\varepsilon'(z_3) = 1$,

$$\begin{aligned} z_4 = ba^2b^{-1} &= \sigma_2\sigma_3\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-2}\sigma_3^2\sigma_2\sigma_1\sigma_3^{-1}\sigma_2^{-1} \\ &= \sigma_2\sigma_3\sigma_2^{-2}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^2\sigma_2\sigma_1\sigma_3^{-1}\sigma_2^{-1} \\ &= \sigma_3^{-2}\sigma_2\sigma_3\sigma_1^{-1}\sigma_3\sigma_2^2\sigma_3^{-1}\sigma_1\sigma_3^{-1}\sigma_2^{-1} = \sigma_3^{-2}\sigma_2\sigma_1^{-1}\sigma_3^2\sigma_2^2\sigma_1\sigma_3^{-2}\sigma_2^{-1} \\ &= C_{34}^{-1}\sigma_2\sigma_1^{-1}C_{34}\sigma_2^2\sigma_1C_{34}^{-1}\sigma_2^{-1}, \end{aligned}$$

so $\varepsilon'(z_4) = -1$, and

$$\begin{aligned} z_5 = ab^2a^{-1} &= \sigma_1^{-1}\sigma_3\sigma_2\sigma_1^{-2}\sigma_3^2\sigma_2^{-1}\sigma_3^{-1}\sigma_1 \\ &= \sigma_1^{-1}\sigma_3\sigma_2\sigma_1^{-2}\sigma_2^{-1}\sigma_3^{-1}\sigma_2^2\sigma_1 \\ &= \sigma_1^{-2}\sigma_2^{-1}\sigma_3^{-2}\sigma_2\sigma_1\sigma_2^2\sigma_1 = \sigma_1^{-2}\sigma_2^{-1}C_{34}^{-1}\sigma_2\sigma_1\sigma_2^2\sigma_1, \end{aligned}$$

so $\varepsilon'(z_5) = -1$, and thus $\varepsilon' = \eta \circ \varepsilon$ as claimed.

Let $y \in (B_3(\mathbb{S}^2 \setminus \{x_1, x_2\}))^{(1)}$, and let $x = \varphi(y)$. We know that $x \in H$, and $y_x = y$. But $\widetilde{y} = 0$, hence $\varepsilon'(x) = 0$, and $\varphi((B_3(\mathbb{S}^2 \setminus \{x_1, x_2\}))^{(1)}) \subseteq \operatorname{Ker}(\varepsilon')$. Conversely, let $x \in \operatorname{Ker}(\varepsilon')$. Then $\varphi(y_x) = x$, and $\widetilde{y_x} = \psi \circ \varepsilon'(x) = 0$, so $y_x \in (B_3(\mathbb{S}^2 \setminus \{x_1, x_2\}))^{(1)}$. Hence $x \in \varphi((B_3(\mathbb{S}^2 \setminus \{x_1, x_2\}))^{(1)})$, and so $\operatorname{Ker}(\varepsilon') = \varphi((B_3(\mathbb{S}^2 \setminus \{x_1, x_2\}))^{(1)})$. But φ is injective, and thus $\operatorname{Ker}(\varepsilon') \cong (B_3(\mathbb{S}^2 \setminus \{x_1, x_2\}))^{(1)}$. To determine $\operatorname{Ker}(\varepsilon')$, we use the following short exact sequence and the Reidemeister-Schreier rewriting process:

$$1 \to (B_3(\mathbb{S}^2 \setminus \{x_1, x_2\}))^{(1)} \overset{\varphi}{\hookrightarrow} \mathbb{F}_5(z_1, \ldots, z_5) \rtimes \mathbb{F}_2(u, v) \overset{\varepsilon'}{\to} \mathbb{Z} \to 1.$$

The calculations are similar to those given in Section 2 for the kernel of the homomorphism ρ defined by equation (25); the difference is that here $\mathbb{F}_2(u, v) \subseteq \operatorname{Ker}(\varepsilon')$. For all $j \in \mathbb{Z}$, set

$$\alpha_{i,j} = \begin{cases} z_1^j z_i z_1^{-(j+1)} & \text{if } i = 2, 3 \\ z_1^j z_i z_1^{-(j-1)} & \text{if } i = 4, 5. \end{cases}$$

These elements form a basis of $\operatorname{Ker}(\rho)$ (see Table 6 on page 30). To obtain a generating set of $\operatorname{Ker}(\varepsilon')$, we need to add the elements $r_j = z_1^j u z_1^{-j}$ and $s_j = z_1^j v z_1^{-j}$, where $j \in \mathbb{Z}$. The relators of $\operatorname{Ker}(\varepsilon')$ are of the form $z_1^j \mathcal{R} z_1^{-j}$, where $j \in \mathbb{Z}$ and $\mathcal{R}$ runs over the set of relators given by equations (20) and (21). For $i = 1, \ldots, 5$, let us set $t_i = u z_i u^{-1}$ and $w_i = v z_i v^{-1}$. First let $i = 1$. Then for all $j \in \mathbb{Z}$, we have the relator:

$$z_1^j u z_1 u^{-1} t_1^{-1} z_1^{-j} = r_j r_{j+1}^{-1} (z_1^j t_1^{-1} z_1^{-(j+1)})^{-1}.$$

from which it follows that

$$r_{j+1} = (z_1^j t_1 z_1^{-(j+1)})^{-1} r_j. \tag{36}$$

This allows us to delete all of the r_i, $i \in \mathbb{Z} \setminus \{0\}$, from the list of generators. By induction, we obtain:

$$r_j = \begin{cases} (z_1^{j-1} t_1 z_1^{-j})^{-1} (z_1^{j-2} t_1 z_1^{-j+1})^{-1} \cdots (t_1 z_1^{-1})^{-1} u & \text{if } j > 0 \\ (z_1^j t_1 z_1^{-(j+1)})(z_1^{j+1} t_1 z_1^{-(j+2)}) \cdots (z_1^{-1} t_1) u & \text{if } j < 0. \end{cases} \tag{37}$$

In a similar way, we may delete all of the s_j except $s_0 = v$ from the list of generators, and we obtain:

$$s_j = \begin{cases} (z_1^{j-1} w_1 z_1^{-j})^{-1} (z_1^{j-2} w_1 z_1^{-j+1})^{-1} \cdots (w_1 z_1^{-1})^{-1} v & \text{if } j > 0 \\ (z_1^j w_1 z_1^{-(j+1)})(z_1^{j+1} w_1 z_1^{-(j+2)}) \cdots (z_1^{-1} w_1) v & \text{if } j < 0. \end{cases} \tag{38}$$

Notice that in equations (37) and (38), each of the bracketed terms belongs to $\operatorname{Ker}(\rho)$, and hence so do $r_j u^{-1}$ and $s_j v^{-1}$. So they may be expressed in terms of the $\alpha_{i,j}$. Now let $i = 2, 3$ and $j \in \mathbb{Z}$. Then we have a relator

$$z_1^j u z_i u^{-1} t_i^{-1} z_1^{-j} = r_j \alpha_{i,j} r_{j+1}^{-1} z_1^{j+1} t_i^{-1} z_1^{-j},$$

which yields a relation of the form $r_j \alpha_{i,j} r_j^{-1} = z_1^j t_i t_1^{-1} z_1^{-j}$ by equation (36). Using equation (37), we see that the elements $u z_1^j z_i z_1^{-(j+1)} u^{-1}$ may be expressed solely in terms of the $\alpha_{i,j}$. Indeed,

$$u \alpha_{i,j} u^{-1} = (r_j u^{-1})^{-1} z_1^j t_i t_1^{-1} z_1^{-j} (r_j u^{-1}). \tag{39}$$

Similarly,

$$v \alpha_{i,j} v^{-1} = (s_j v^{-1})^{-1} z_1^j w_i w_1^{-1} z_1^{-j} (s_j v^{-1}). \tag{40}$$

Finally, let $i = 4, 5$ and $j \in \mathbb{Z}$. Then we obtain analogously:

$$u \alpha_{i,j} u^{-1} = (r_j u^{-1})^{-1} z_1^j t_i t_1 z_1^{-j} (r_j u^{-1}). \tag{41}$$

Similarly,

$$v \alpha_{i,j} v^{-1} = (s_j v^{-1})^{-1} z_1^j w_i w_1 z_1^{-j} (s_j v^{-1}). \tag{42}$$

This gives a complete set of relations for $\operatorname{Ker}(\varepsilon')$. We conclude that:

Proposition 42. *$(B_3(\mathbb{S}^2 \setminus \{x_1, x_2\}))^{(1)} \cong L \rtimes \mathbb{F}_2(u, v)$, where L is the subgroup of $\mathbb{F}_5(z_1, \ldots, z_5)$ of infinite rank freely generated by $\{\alpha_{i,j}\}_{i \in \mathbb{Z},\, j \in \{2,3,4,5\}}$, the action being given by equations (39), (40), (41), and (42), taking into account equations (37) and (38).* □

From this, we may now determine $(B_3(\mathbb{S}^2 \setminus \{x_1, x_2\}))^{(1)}/(B_3(\mathbb{S}^2 \setminus \{x_1, x_2\}))^{(2)}$ by Abelianising the presentation of $(B_3(\mathbb{S}^2 \setminus \{x_1, x_2\}))^{(1)}$ given by Proposition 42, and thus reprove Proposition 11. First notice that for all $2 \leq i \leq 5$ and all $j, k \in \mathbb{Z}$, $z_1^k \alpha_{i,j} z_1^{-k} = \alpha_{i,j+k}$. If $w \in (B_3(\mathbb{S}^2 \setminus \{x_1, x_2\}))^{(1)}$, let $\widetilde{w} \in (B_3(\mathbb{S}^2 \setminus \{x_1, x_2\}))^{(1)}/(B_3(\mathbb{S}^2 \setminus \{x_1, x_2\}))^{(2)}$ denote its Abelianisation. A simple calculation shows that:

$$\begin{aligned}
\widetilde{t_2 t_1^{-1}} &= \widetilde{\alpha_{2,0}} + \widetilde{\alpha_{5,0}} + \widetilde{\alpha_{3,-1}} - \widetilde{\alpha_{2,-1}} - \widetilde{\alpha_{4,0}} \\
\widetilde{t_3 t_1^{-1}} &= \widetilde{\alpha_{2,1}} - \widetilde{\alpha_{3,1}} + \widetilde{\alpha_{5,1}} + \widetilde{t_2 t_1^{-1}} \\
\widetilde{w_2 w_1^{-1}} &= -\widetilde{\alpha_{4,1}} + \widetilde{\alpha_{2,1}} - \widetilde{\alpha_{3,1}} + \widetilde{\alpha_{5,1}} + \widetilde{\alpha_{5,0}} - \widetilde{\alpha_{4,0}} + \widetilde{\alpha_{2,0}} \\
\widetilde{w_3 w_1^{-1}} &= \widetilde{\alpha_{2,1}} + \widetilde{w_2 w_1^{-1}}.
\end{aligned}$$

Abelianising equations (39) and (40) for $i = 2$ then $i = 3$ yields:

$$\begin{aligned}
\widetilde{\alpha_{5,j}} - \widetilde{\alpha_{4,j}} &= \widetilde{\alpha_{2,j-1}} - \widetilde{\alpha_{3,j-1}} \\
\widetilde{\alpha_{3,j}} - \widetilde{\alpha_{2,j}} &= \widetilde{\alpha_{2,j+1}} - \widetilde{\alpha_{3,j+1}} + \widetilde{\alpha_{5,j+1}} \\
0 &= \widetilde{\alpha_{5,j}} - \widetilde{\alpha_{4,j}} + \widetilde{\alpha_{5,j+1}} - \widetilde{\alpha_{4,j+1}} + \widetilde{\alpha_{2,j+1}} - \widetilde{\alpha_{3,j+1}} \\
\widetilde{\alpha_{2,j+1}} + \widetilde{\alpha_{2,j}} &= \widetilde{\alpha_{3,j}} \qquad (43)
\end{aligned}$$

for all $j \in \mathbb{Z}$. Substituting equation (43) into the three other equations, we obtain:

$$\widetilde{\alpha_{5,j}} - \widetilde{\alpha_{4,j}} = -\widetilde{\alpha_{2,j}} \qquad (44)$$

$$\widetilde{\alpha_{3,j+1}} = \widetilde{\alpha_{5,j+1}} \qquad (45)$$

$$\widetilde{\alpha_{2,j+1}} - \widetilde{\alpha_{2,j}} = \widetilde{\alpha_{4,j+1}}. \qquad (46)$$

Similarly, if $i = 4, 5$,

$$\begin{aligned}
\widetilde{t_4 t_1} &= \widetilde{\alpha_{2,0}} + \widetilde{\alpha_{5,0}} \\
\widetilde{t_5 t_1} &= \widetilde{\alpha_{2,0}} + \widetilde{\alpha_{2,1}} - \widetilde{\alpha_{3,1}} + \widetilde{\alpha_{5,1}} + \widetilde{\alpha_{5,0}} + \widetilde{\alpha_{2,-1}} \\
\widetilde{w_4 w_1} &= \widetilde{\alpha_{3,0}} - \widetilde{\alpha_{4,1}} + \widetilde{\alpha_{2,1}} - \widetilde{\alpha_{3,1}} + \widetilde{\alpha_{5,1}} \\
\widetilde{w_5 w_1} &= \widetilde{\alpha_{3,-1}} - \widetilde{\alpha_{4,0}} + \widetilde{\alpha_{2,0}} - \widetilde{\alpha_{4,1}} + 2\widetilde{\alpha_{2,1}} - \widetilde{\alpha_{3,1}} + \widetilde{\alpha_{5,1}} + \\
&\quad \widetilde{\alpha_{5,0}} + \widetilde{\alpha_{3,-2}} - \widetilde{\alpha_{2,-2}} - \widetilde{\alpha_{4,-1}}.
\end{aligned}$$

Abelianising equations (41) and (42) for $i = 4$ then $i = 5$ and applying the previous equations yields:

$$\begin{aligned}
\widetilde{\alpha_{4,j}} &= \widetilde{\alpha_{2,j}} + \widetilde{\alpha_{5,j}} \text{ which is equivalent to (44)} \\
0 &= \widetilde{\alpha_{2,j-1}} + \widetilde{\alpha_{2,j}} + \widetilde{\alpha_{2,j+1}} && (47) \\
\widetilde{\alpha_{4,j}} &= \widetilde{\alpha_{3,j}} + \widetilde{\alpha_{2,j}} && (48) \\
0 &= \widetilde{\alpha_{2,j-1}} + \widetilde{\alpha_{2,j}} + \widetilde{\alpha_{2,j+1}} \text{ which is the same as (47).}
\end{aligned}$$

By equations (47), (43), (46) and (45) we obtain the solution

$$\begin{aligned}
\widetilde{\alpha_{2,3k}} &= \widetilde{\alpha_{2,0}} = -\widetilde{\alpha_{3,3k+1}} = -\widetilde{\alpha_{5,3k+1}} \\
\widetilde{\alpha_{2,3k+1}} &= \widetilde{\alpha_{2,1}} = -\widetilde{\alpha_{3,3k+2}} = -\widetilde{\alpha_{5,3k+2}} \\
\widetilde{\alpha_{2,3k+2}} &= -(\widetilde{\alpha_{2,0}} + \widetilde{\alpha_{2,1}}) = -\widetilde{\alpha_{3,3k}} = -\widetilde{\alpha_{5,3k}} \\
\widetilde{\alpha_{4,3k}} &= 2\widetilde{\alpha_{2,0}} + \widetilde{\alpha_{2,1}} \\
\widetilde{\alpha_{4,3k+1}} &= -\widetilde{\alpha_{2,0}} + \widetilde{\alpha_{2,1}} \\
\widetilde{\alpha_{4,3k+2}} &= -\widetilde{\alpha_{2,0}} - 2\widetilde{\alpha_{2,1}}
\end{aligned}$$

for all $k \in \mathbb{Z}$, which satisfies the two remaining equations (44) and (48). We conclude that $(B_3(\mathbb{S}^2 \setminus \{x_1, x_2\}))^{(1)}/(B_3(\mathbb{S}^2 \setminus \{x_1, x_2\}))^{(2)}$ is a free Abelian group with basis $\{\widetilde{\alpha_{2,0}}, \widetilde{\alpha_{2,1}}, \widetilde{u}, \widetilde{v}\}$, and this reproves Proposition 11.

We will come back to the special case $m = n = 2$ in the following section.

6. The lower central and derived series of $B_2(\mathbb{S}^2 \setminus \{x_1, x_2\})$

From Section 3 and Theorem 9, the only outstanding case for the lower central series of the punctured sphere is the 2-string braid group. As we shall see in this section, it is particularly challenging. We concentrate here on the case of the two-punctured sphere. The group $B_2(\mathbb{S}^2 \setminus \{x_1, x_2\})$ has many fascinating properties, and as a result, we are able to describe its lower central and derived series in terms of those of the free product $\mathbb{Z}_2 * \mathbb{Z}$. In particular, we prove Corollary 13, Proposition 14 and Theorem 15. As we indicated in the Preface, the techniques used in this section have since been applied in [**BGG**] to study the 2-string braid group of the torus, and similar results were obtained (cf. Theorem 19).

We start by determining $\Gamma_2(P_2(\mathbb{S}^2 \setminus \{x_1, x_2\}))$. The map $F_2(\mathbb{S}^2 \setminus \{x_1, x_2\}) \to F_1(\mathbb{S}^2 \setminus \{x_1, x_2\})$ is a fibration, and the fibre over a point x_3 of the base is of the form $F_1(\mathbb{S}^2 \setminus \{x_1, x_2, x_3\})$. As in equation (2), this gives rise to a short exact sequence of the form:

$$1 \to \pi_1(\mathbb{S}^2 \setminus \{x_1, x_2, x_3\}, x_4) \to P_2(\mathbb{S}^2 \setminus \{x_1, x_2\}, \{x_3, x_4\}) \xrightarrow{p_*} \pi_1(\mathbb{S}^2 \setminus \{x_1, x_2\}, x_3) \to 1.$$

We use the following notation for the generators $\gamma_{i,j}$, $1 \le i, j \le 2$ of $P_2(\mathbb{S}^2 \setminus \{x_1, x_2\}, \{x_3, x_4\})$: the two punctures correspond to the points x_1, x_2; and the two basepoints correspond to x_3, x_4. The generator $\gamma_{i,j}$ is equal to the generator $A_{i,j+2}$ of Proposition 31, and corresponds to a loop based at x_{j+2} which encircles x_i in the positive direction.

Let σ be the standard Artin generator of $B_2(\mathbb{S}^2 \setminus \{x_1, x_2\})$ which geometrically exchanges the points x_3 and x_4. Then a (non-minimal) generating set of $P_2(\mathbb{S}^2 \setminus \{x_1, x_2\})$ is given by the union of the $\gamma_{i,j}$ and σ^2, and a generating set of $B_2(\mathbb{S}^2 \setminus \{x_1, x_2\})$ is given by the union of the $\gamma_{i,j}$ and σ.

The kernel $\pi_1(\mathbb{S}^2 \setminus \{x_1, x_2, x_3\}, x_4)$ of p_* is the free group $\mathbb{F}_2(a, b)$ of rank 2 on a and b, where $a = \gamma_{1,2}$ and $b = \gamma_{2,2}$. The image of p_* is an infinite cyclic group; the homomorphism which sends (one of) its generators to the element $c = \gamma_{2,1}$ of $P_2(\mathbb{S}^2 \setminus \{x_1, x_2\})$ defines a section for p_*. Hence

$$P_2(\mathbb{S}^2 \setminus \{x_1, x_2\}) \cong \mathbb{F}_2(a, b) \rtimes_\varphi \mathbb{Z}, \tag{49}$$

where we identify the second factor $\mathbb{Z}$ with $\langle c \rangle$. The action φ on the kernel is given as follows (this may be checked using the presentation of $P_m(\mathbb{S}^2 \setminus \{x_1, \ldots, x_n\})$ given in [**GG4**]):

$$\begin{cases} \varphi(c)(a) = cac^{-1} = a \\ \varphi(c)(b) = cbc^{-1} = aba^{-1}, \end{cases}$$

which in fact is just conjugation by a.

As well as containing $\Gamma_2(\mathbb{F}_2(a, b))$, by Proposition 29, $\Gamma_2(P_2(\mathbb{S}^2 \setminus \{x_1, x_2\}))$ will also contain elements of the form $[c^j, w]$, where $w \in \mathbb{F}_2(a, b)$ and $j \in \mathbb{Z}$. But from the form of the action φ, $[c^j, w] = [a^j, w]$. Hence

$$\Gamma_2(P_2(\mathbb{S}^2 \setminus \{x_1, x_2\})) = \Gamma_2(\mathbb{F}_2(a, b)), \tag{50}$$

and thus the derived series (with the exception of the first term) of $P_2(\mathbb{S}^2 \setminus \{x_1, x_2\})$ is that of $\mathbb{F}_2(a, b)$.

By Proposition 31, $B_2(\mathbb{S}^2 \setminus \{x_1, x_2\})$ is generated by the $\gamma_{i,j}$, $1 \le i, j \le 2$, and σ, subject to the four relations:

$$\left.\begin{aligned} &\gamma_{1,2}\gamma_{2,2}\sigma^2 = 1 \\ &\gamma_{1,1}\gamma_{2,1}\sigma^2 = 1 \\ &\sigma\gamma_{i,1}\sigma^{-1} = \gamma_{i,2} \text{ for } i = 1, 2. \end{aligned}\right\} \tag{51}$$

Thus:

$$\left.\begin{aligned} c &= \gamma_{2,1} = \gamma_{1,1}^{-1}\sigma^{-2} \\ b &= \gamma_{2,2} = \sigma\gamma_{1,1}^{-1}\sigma^{-3} \\ a &= \gamma_{1,2} = \sigma\gamma_{1,1}\sigma^{-1}. \end{aligned}\right\} \tag{52}$$

In particular, $B_2(\mathbb{S}^2 \setminus \{x_1, x_2\})$ is generated by σ and $\gamma_{1,1}$.

In what follows, we shall sometimes write simply $P_{2,2}$ for $P_2(\mathbb{S}^2 \setminus \{x_1, x_2\})$, and $B_{2,2}$ for $B_2(\mathbb{S}^2 \setminus \{x_1, x_2\})$.

PROPOSITION 43.

(a) The commutator subgroup $[P_{2,2}, P_{2,2}]$ of $P_{2,2}$ is a normal subgroup of $[P_{2,2}, B_{2,2}]$.
(b) The commutator subgroup $[P_{2,2}, B_{2,2}]$ of $P_{2,2}$ and $B_{2,2}$ is a normal subgroup of $P_{2,2}$ and $B_{2,2}$.
(c) The quotient group $[P_{2,2}, B_{2,2}]/[P_{2,2}, P_{2,2}]$ is isomorphic to $\mathbb{Z}$, and is generated by the coset of the element $[\sigma, b] = b^{-1}c$.

PROOF.

(a) This is clear since $P_{2,2} \lhd B_{2,2}$.
(b) The fact that $[P_{2,2}, B_{2,2}]$ is a subgroup of $P_{2,2}$ follows from projection into the symmetric group S_2. Since $P_{2,2} \lhd B_{2,2}$, we see that $[P_{2,2}, B_{2,2}] \lhd B_{2,2}$, and so $[P_{2,2}, B_{2,2}] \lhd P_{2,2}$.
(c) Using equation (52), we see easily that $\sigma^{-2} = ab$, and thus:

$$\left.\begin{aligned} [\sigma, a] &= \sigma^2\sigma^{-2}c^{-1}\sigma^{-2}a^{-1} = c^{-1}aba^{-1} = bc^{-1} = b(b^{-1}c)^{-1}b^{-1} \\ [\sigma, b] &= \sigma^2\gamma_{1,1}^{-1}\sigma^{-4}b^{-1} = b^{-1}a^{-1}ca = b^{-1}c \\ [\sigma, c] &= \sigma\gamma_{1,1}^{-1}\sigma^{-1}\sigma^2\gamma_{1,1} = b(b^{-1}c)^{-1}b^{-1}. \end{aligned}\right\} \tag{53}$$

We know that $[P_{2,2}, B_{2,2}]$ is the normal closure in $B_{2,2}$ of the set of elements of the form $[\rho_1, \rho_2]$ and their inverses, where $\rho_1 \in \{a, b, c\}$ is a generator of $P_{2,2}$, and $\rho_2 \in \{\sigma, a, b, c\}$ is a generator of $B_{2,2}$. If $\rho_2 \in \{a, b, c\}$ then $[\rho_1, \rho_2] \in [P_{2,2}, P_{2,2}]$. So we just need to consider the cosets of the conjugates of elements of the form $[\rho_1, \sigma]$. Consider the following relation:

$$\rho[\rho_1, \sigma]\rho^{-1} = \big[\rho, [\rho_1, \sigma]\big]\,[\rho_1, \sigma]. \tag{54}$$

If $\rho \in P_{2,2}$, then since $[\rho_1, \sigma] \in P_{2,2}$ by (b), it follows that

$$\rho[\rho_1, \sigma]\rho^{-1} \equiv [\rho_1, \sigma] \quad \text{modulo } [P_{2,2}, P_{2,2}].$$

So suppose that $\rho \in B_{2,2} \setminus P_{2,2}$. Then $w = \rho\sigma^{-1} \in P_{2,2}$. By equation (53), we see that

$$\begin{aligned}
&\sigma[\sigma, a]\sigma^{-1} = \sigma bc^{-1}\sigma^{-1} = b^{-1}cb \cdot b^{-1} = b^{-1}[\sigma, a]^{-1}b\\
&\sigma[\sigma, b]\sigma^{-1} = \sigma b^{-1}c\sigma^{-1} = b^{-1}c^{-1}b \cdot b = b^{-1}[\sigma, b]^{-1}b, \quad \text{and}\\
&\sigma[\sigma, c]\sigma^{-1} = \sigma[\sigma, a]\sigma^{-1} = b^{-1}[\sigma, c]^{-1}b.
\end{aligned}$$

In other words, for all $\rho_1 \in \{a, b, c\}$, we have

$$\sigma[\sigma, \rho_1]\sigma^{-1} = b^{-1}[\sigma, \rho_1]^{-1}b.$$

Hence

$$\begin{aligned}
\big[\rho, [\rho_1, \sigma]\big] &= w\sigma[\sigma, \rho_1]\sigma^{-1}w^{-1}[\rho_1, \sigma]^{-1}\\
&= wb^{-1}[\sigma, \rho_1]^{-1}bw^{-1}[\rho_1, \sigma]^{-1} = \big[wb^{-1}, [\sigma, \rho_1]^{-1}\big]\big[\rho_1, \sigma\big]^{-2}.
\end{aligned}$$

Thus by equation (54), we obtain

$$\begin{aligned}
\rho[\rho_1, \sigma]\rho^{-1} &= \big[wb^{-1}, [\sigma, \rho_1]^{-1}\big]\big[\rho_1, \sigma\big]^{-1}\\
&\equiv [\rho_1, \sigma]^{-1} \text{ modulo } [P_{2,2}, P_{2,2}], \text{ since } [\sigma, \rho_1] \in P_{2,2}.
\end{aligned}$$

By equation (53),

$$[\sigma, a] \equiv [\sigma, c] \equiv [\sigma, b]^{-1} \text{ modulo } [P_{2,2}, P_{2,2}],$$

and since $[B_{2,2}, P_{2,2}] = [P_{2,2}, B_{2,2}]$, we conclude that the quotient $[P_{2,2}, B_{2,2}]\big/[P_{2,2}, P_{2,2}]$ is infinite cyclic, and generated by the coset of the element $[\sigma, b] = b^{-1}c$ (using equations (49) and (50), one may check that $b^{-1}c \notin [P_{2,2}, P_{2,2}]$). $\square$

REMARK 44. Let us give an alternative proof of Proposition 43 using Stallings' exact sequence (8). Since $[P_{2,2}, P_{2,2}], [P_{2,2}, B_{2,2}] \trianglelefteq P_{2,2}$ and $[P_{2,2}, P_{2,2}] \subseteq [P_{2,2}, B_{2,2}]$, we see that

$$[P_{2,2}, B_{2,2}]\big/[P_{2,2}, P_{2,2}] \trianglelefteq P_{2,2}\big/[P_{2,2}, P_{2,2}].$$

We thus have the following diagram:

$$\begin{array}{ccccccccc}
 & & & & & & 1 & & \\
 & & & & & & \downarrow & & \\
1 & \to & [P_{2,2}, B_{2,2}]\big/[P_{2,2}, P_{2,2}] & \to & P_{2,2}\big/[P_{2,2}, P_{2,2}] & \to & P_{2,2}\big/[P_{2,2}, B_{2,2}] & \to & 1 \\
 & & & & & & \downarrow & & \\
 & & & & & & B_{2,2}\big/[B_{2,2}, B_{2,2}] & & \\
 & & & & & & \downarrow & & \\
 & & & & & & S_2 & & \\
 & & & & & & \downarrow & & \\
 & & & & & & 1. & &
\end{array}$$

The vertical short exact sequence is that of Stallings applied to the usual exact sequence $1 \to P_{2,2} \to B_{2,2} \to S_2 \to 1$. By Proposition 33, $B_{2,2}\big/[B_{2,2}, B_{2,2}]$ is a free Abelian group of rank 2, with basis $\{\sigma, \gamma_{1,1}\}$ (notationally, here we do not distinguish an element of $B_{2,2}$ and its Abelianisation). The kernel $P_{2,2}\big/[P_{2,2}, B_{2,2}]$ of the projection $B_{2,2}\big/[B_{2,2}, B_{2,2}] \to S_2$ certainly contains the free subgroup of rank 2 with basis $\{\sigma^2, \gamma_{1,1}\}$, and in fact is equal to this subgroup (for otherwise it would contain an element of the form $\sigma^p \gamma_{1,1}^q$, where $p, q \in \mathbb{Z}$ and p is odd, and thus would contain σ, which is clearly not in the kernel). Since $P_{2,2}\big/[P_{2,2}, P_{2,2}]$ is isomorphic to $\mathbb{Z}^3$ (by equation (49)), we see that the kernel $[P_{2,2}, B_{2,2}]\big/[P_{2,2}, P_{2,2}]$ of the horizontal exact sequence is isomorphic to $\mathbb{Z}$. Further, $P_{2,2}\big/[P_{2,2}, P_{2,2}]$ is freely generated by a, b and c. From the relation $\sigma\gamma_{2,1}\sigma^{-1} = \gamma_{2,2}$, we see that $b = [\sigma, c] \cdot c$, and so b and c project to the same element in $P_{2,2}\big/[P_{2,2}, B_{2,2}]$. Hence (the coset of) bc^{-1} is a non-trivial element of $[P_{2,2}, B_{2,2}]\big/[P_{2,2}, P_{2,2}]$, which yields the result.

We thus obtain a short exact sequence of the form:

$$1 \to [P_{2,2}, P_{2,2}] \to [P_{2,2}, B_{2,2}] \to \mathbb{Z} \to 1,$$

for which the homomorphism $s\colon \mathbb{Z} \to [P_{2,2}, B_{2,2}]$ defined by $s(1) = b^{-1}c$ defines a splitting. Since $[P_{2,2}, P_{2,2}]$ is the normal closure in $P_{2,2}$ of the set of elements of the form $[\rho_1, \rho_2]$ and their inverses, where $\rho_1, \rho_2 \in \{a, b, c\}$, and the action in $\mathbb{F}_2(a, b)$ of conjugation by c is just conjugation by a, we see that $[P_{2,2}, P_{2,2}]$ is the normal closure in $\mathbb{F}_2(a, b)$ of the element $[a, b]$, and that:

$$(b^{-1}c)w(b^{-1}c)^{-1} = (b^{-1}a)w(b^{-1}a)^{-1} \quad \text{for all } w \in \mathbb{F}_2(a, b).$$

Hence the action of $b^{-1}c$ on $[P_{2,2}, P_{2,2}]$ is that of conjugation by $b^{-1}a$, and so by the above short exact sequence,

$$\begin{aligned}[P_{2,2}, B_{2,2}] &\cong [P_{2,2}, P_{2,2}] \rtimes_{\psi} \mathbb{Z}\\ &\cong \Gamma_2(\mathbb{F}_2(a,b)) \rtimes_{\psi} \mathbb{Z} \quad \text{by equation (50),}\end{aligned}$$

where the action ψ of $\mathbb{Z}$ on $\Gamma_2(\mathbb{F}_2(a,b))$ is given by conjugation by $b^{-1}a$.

Proposition 45. $[P_{2,2}, B_{2,2}] = [B_{2,2}, B_{2,2}]$.

From this, it follows immediately that:

Corollary 13. $\Gamma_2(B_2(\mathbb{S}^2 \setminus \{x_1, x_2\})) \cong \Gamma_2(\mathbb{F}_2(a,b)) \rtimes_{\psi} \mathbb{Z}$. □

Proof of Proposition 45. Consider the following commutative diagram of short exact sequences (obtained by taking the first two vertical sequences, and the second and third horizontal sequences, and then completing to the whole diagram):

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & [P_{2,2}, B_{2,2}] & \longrightarrow & [B_{2,2}, B_{2,2}] & \longrightarrow & [B_{2,2}, B_{2,2}]\big/[P_{2,2}, B_{2,2}] & \longrightarrow & 1\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & P_{2,2} & \longrightarrow & B_{2,2} & \longrightarrow & \mathbb{Z}_2 & \longrightarrow & 1\\
 & & \downarrow & & \downarrow & & \| & & \\
1 & \longrightarrow & P_{2,2}\big/[P_{2,2}, B_{2,2}] & \longrightarrow & B_{2,2}\big/[B_{2,2}, B_{2,2}] & \longrightarrow & \mathbb{Z}_2 & \longrightarrow & 1.\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 1 & & 1 & & 1 & &
\end{array}$$

As in Remark 44, the third row is Stallings' exact sequence (8) applied to the second row. By exactness of the third vertical sequence, it follows that $[B_{2,2}, B_{2,2}] = [P_{2,2}, B_{2,2}]$. □

We may obtain an alternative description of $\Gamma_2(B_2(\mathbb{S}^2 \setminus \{x_1, x_2\}))$ as a free group of infinite rank. To see this, notice from part (b) of Proposition 34 that $B_m(\mathbb{S}^2 \setminus \{x_1, x_2\}) \cong B_m(\mathbb{D}^2 \setminus \{x_2\})$, and from part (c) that

$$B_m(\mathbb{D}^2 \setminus \{x_2\}, \{x_1, x_3, \ldots, x_{m+1}\}) \cong B_{m,1}(\mathbb{D}^2).$$

Hence $B_2(\mathbb{S}^2 \setminus \{x_1, x_2\}) \cong B_{2,1}(\mathbb{D}^2)$. But from part (d),

$$\begin{aligned}B_{2,1}(\mathbb{D}^2) &\cong \pi_1(\mathbb{D}^2 \setminus \{x_3, x_4\}, x_2) \rtimes B_2(\mathbb{D}^2)\\ &\cong \mathbb{F}_2(\gamma_{2,1}, \gamma_{2,2}) \rtimes_{\varphi} \langle \sigma \rangle, \qquad (55)\end{aligned}$$

where the action, obtained from equations (51), (52) and (53), is given by:

$$\left.\begin{aligned} \varphi(\sigma)(\gamma_{2,1}) &= \gamma_{2,2} \\ \varphi(\sigma)(\gamma_{2,2}) &= \gamma_{2,2}^{-1}\gamma_{2,1}\gamma_{2,2}. \end{aligned}\right\} \tag{56}$$

So if $w = w(\gamma_{2,1}, \gamma_{2,2}) \in \mathbb{F}_2(\gamma_{2,1}, \gamma_{2,2})$ then

$$\varphi(\sigma)(w) = \gamma_{2,2}^{-1} w(\gamma_{2,2}, \gamma_{2,1})\gamma_{2,2}, \tag{57}$$

in other words, the action consists of exchanging $\gamma_{2,1}$ and $\gamma_{2,2}$, then conjugating by $\gamma_{2,2}^{-1}$. Let N denote the normal closure in $\mathbb{F}_2(\gamma_{2,1}, \gamma_{2,2})$ of the elements of the form $\varphi(\sigma^j)(w)\cdot w^{-1}$, where $j \in \mathbb{Z}$ and $w \in \mathbb{F}_2(\gamma_{2,1}, \gamma_{2,2})$, and let L be the subgroup of $\mathbb{F}_2(\gamma_{2,1}, \gamma_{2,2})$ generated by $\Gamma_2(\mathbb{F}_2(\gamma_{2,1}, \gamma_{2,2}))$ and N. Then it follows from Proposition 29 and equation (55) that $\Gamma_2(\gamma_{2,1}(\mathbb{D}^2)) \cong L$.

Proposition 46.

(a) L is the kernel of the homomorphism $\psi\colon \mathbb{F}_2(\gamma_{2,1}, \gamma_{2,2}) \to \mathbb{Z}$, where ψ is augmentation.

(b) L is a free group of infinite rank with basis $\{z_i\}_{i\in\mathbb{Z}}$, where $z_i = \gamma_{2,1}^{i}\gamma_{2,2}\gamma_{2,1}^{-(i+1)}$ for all $i \in \mathbb{Z}$.

Since $L \cong \Gamma_2(B_{2,1}(\mathbb{D}^2)) \cong B_2(\mathbb{S}^2 \setminus \{x_1, x_2\})$, we obtain immediately:

Corollary 47. *$\Gamma_2(B_2(\mathbb{S}^2 \setminus \{x_1, x_2\}))$ is a free group of infinite rank with basis $\{z_i\}_{i\in\mathbb{Z}}$, where $z_i = \gamma_{2,1}^{i}\gamma_{2,2}\gamma_{2,1}^{-(i+1)}$ for all $i \in \mathbb{Z}$.* □

Proof of Proposition 46. First observe that ψ factors through the Abelianisation of $\mathbb{F}_2(\gamma_{2,1}, \gamma_{2,2})$, and so $\Gamma_2(\mathbb{F}_2(\gamma_{2,1}, \gamma_{2,2})) \subseteq \operatorname{Ker}(\psi)$. Secondly, from equation (56), σ commutes with $cb = \gamma_{2,1}\gamma_{2,2}$, and it follows from equation (57) that

$$\sigma^m w(\gamma_{2,1}, \gamma_{2,2})\sigma^{-m} =$$
$$\begin{cases} (\gamma_{2,1}\gamma_{2,2})^{-m/2} w(\gamma_{2,1}, \gamma_{2,2})(\gamma_{2,1}\gamma_{2,2})^{m/2} & \text{if } m \text{ is even} \\ (\gamma_{2,1}\gamma_{2,2})^{-(m-1)/2}\gamma_{2,2}^{-1} w(\gamma_{2,2}, \gamma_{2,1})\gamma_{2,2}(\gamma_{2,1}\gamma_{2,2})^{(m-1)/2} & \text{if } m \text{ is odd.} \end{cases}$$

So for all $j \in \mathbb{Z}$, $\psi(\varphi(\sigma^j)(w)w^{-1}) = \psi([\sigma^j, w]) = 0$. Since the same is true for products and conjugates in $\mathbb{F}_2(\gamma_{2,1}, \gamma_{2,2})$, we see that $N \subseteq \operatorname{Ker}(\psi)$, and thus $L \subseteq \operatorname{Ker}(\psi)$. Now let us show that $\operatorname{Ker}(\psi) \subseteq L$. To see this, we first apply the Reidemeister-Schreier rewriting process in order to obtain a basis of $\operatorname{Ker}(\psi)$ (which is a free group since it is a subgroup of $\mathbb{F}_2(\gamma_{2,1}, \gamma_{2,2})$). Taking $\{\gamma_{2,1}, \gamma_{2,2}\}$ as the basis of $\mathbb{F}_2(\gamma_{2,1}, \gamma_{2,2})$ and $\{\gamma_{2,1}^{i}\}_{i\in\mathbb{Z}}$ as a Schreier transversal, the process yields

$\left\{\gamma_{2,1}^i \gamma_{2,2} \gamma_{2,1}^{-(i+1)}\right\}_{i \in \mathbb{Z}}$ as a basis. But for all $i \in \mathbb{Z}$,

$$\gamma_{2,1}^i \gamma_{2,2} \gamma_{2,1}^{-(i+1)} = \gamma_{2,1}^i \gamma_{2,2} \gamma_{2,1}^{-1} \gamma_{2,1}^{-i} = \gamma_{2,1}^i \varphi(\sigma)(\gamma_{2,1}) \gamma_{2,1}^{-1} \gamma_{2,1}^{-i},$$

which belongs to L by definition. This proves that $\operatorname{Ker}(\psi) = L$, and that L is a free group of infinite rank with the given basis as required. □

REMARK 48. Since $\Gamma_2(\mathbb{F}_2(\gamma_{2,1}, \gamma_{2,2}))$ is the normal closure of the commutator $[\gamma_{2,1}, \gamma_{2,2}]$ in $\mathbb{F}_2(\gamma_{2,1}, \gamma_{2,2})$, and

$$\begin{aligned}[\gamma_{2,1}, \gamma_{2,2}] &= (\gamma_{2,1} \cdot \gamma_{2,2} \gamma_{2,1}^{-1} \cdot \gamma_{2,1}^{-1})(\gamma_{2,2} \gamma_{2,1}^{-1})^{-1} \\ &= (\gamma_{2,1} \cdot \varphi(\sigma)(\gamma_{2,1}) \gamma_{2,1}^{-1} \cdot \gamma_{2,1}^{-1})(\varphi(\sigma)(\gamma_{2,1}) \gamma_{2,1}^{-1})^{-1},\end{aligned}$$

it follows that $\Gamma_2(\mathbb{F}_2(\gamma_{2,1}, \gamma_{2,2}))$ is contained in N, and so $L = N = \operatorname{Ker}(\psi)$.

REMARKS 49. In fact, the group $B_2(\mathbb{S}^2 \setminus \{x_1, x_2\})$ is of particular interest since it may be interpreted in several different ways.

(a) As well as being isomorphic to $B_{2,1}(\mathbb{D}^2)$, it is also isomorphic to the 2-string braid group of the annulus.

(b) One may reduce the presentation given by equation (51) to the following:

$$B_2(\mathbb{S}^2 \setminus \{x_1, x_2\}) = \langle \sigma, \gamma_{2,2} \mid (\sigma\gamma_{2,2})^2 = (\gamma_{2,2}\sigma)^2 \rangle, \tag{58}$$

which is nothing other than the Artin group of type B_2 [**Cr, T**].

(c) The above presentation shows that $B_2(\mathbb{S}^2 \setminus \{x_1, x_2\})$ is a one-relator group. Interpreting it as the 2-string braid group of the annulus, it follows from [**PR**] that it has infinite cyclic centre generated by the full twist of $B_3(\mathbb{D}^2)$, which written in terms of our generators, is of the form $(\sigma\gamma_{2,2})^2$. Further, the relation may be written as $[\sigma, (\sigma\gamma_{2,2})^2] = 1$. In particular, $B_2(\mathbb{S}^2 \setminus \{x_1, x_2\})$ is a one-relator group with non-trivial centre.

(d) Setting $D = \sigma\gamma_{2,2}$, from above, we obtain the presentation

$$\langle \sigma, D \mid [\sigma, D^2] = 1 \rangle. \tag{59}$$

So $B_2(\mathbb{S}^2 \setminus \{x_1, x_2\})$ is isomorphic to the Baumslag-Solitar group $\mathrm{BS}(2,2)$ [**BS**].

(e) Following [**FG**], using the presentation (59), consider the homomorphism of $B_2(\mathbb{S}^2 \setminus \{x_1, x_2\})$ onto $\mathbb{Z}[D] = \langle D \rangle$ given by taking the exponent sum of D. It follows from the Reidemeister-Schreier rewriting process that the kernel is a free group $\mathbb{F}_2(\sigma, D\sigma D^{-1})$ of rank two, and thus that

$$B_2(\mathbb{S}^2 \setminus \{x_1, x_2\}) \cong \mathbb{F}_2(\sigma, D\sigma D^{-1}) \rtimes \mathbb{Z}[D],$$

where the action is given by $D\cdot(\sigma) = D\sigma D^{-1}$, and $D\cdot(D\sigma D^{-1}) = \sigma$. In other words, the action exchanges the two basis elements of the kernel (and not just up to conjugation as in equation (57)), and so is an involution. From this, it follows that $B_2(\mathbb{S}^2 \setminus \{x_1, x_2\})$ is an HNN-extension of the free group $\mathbb{F}_2(\sigma, D\sigma D^{-1})$ with stable letter D.

(f) Still following [**FG**] and using the presentation (59), consider the homomorphism of $B_2(\mathbb{S}^2 \setminus \{x_1, x_2\})$ onto $\mathbb{Z}[\sigma] = \langle \sigma \rangle$ given by taking the exponent sum of σ. Applying the Reidemeister-Schreier rewriting process, one sees that that the kernel is generated by an infinite number of generators $x_i = \sigma_i D \sigma_i^{-1}$, $i \in \mathbb{Z}$, subject to the relations $x_i^2 = x_0^2 = D^2$ for all $i \in \mathbb{Z}$.

Applying [**KMc, McCa**] to Remarks 49(c) above, we see immediately that:

PROPOSITION 14. *$B_2(\mathbb{S}^2 \setminus \{x_1, x_2\})$ is residually nilpotent and residually a finite 2-group.* □

Using the algorithm given in [**CFL**], one may determine the quotient groups of the lower central series of $B_2(\mathbb{S}^2 \setminus \{x_1, x_2\})$. But these quotients may also be obtained explicitly using the results of [**Ga, Lab**]:

THEOREM 15. *For all $i \geq 2$, $\Gamma_i(B_2(\mathbb{S}^2 \setminus \{x_1, x_2\})) \cong \Gamma_i(\mathbb{Z}_2 * \mathbb{Z})$, and:*

$$\begin{aligned}\Gamma_i(B_2(\mathbb{S}^2 \setminus \{x_1, x_2\}))/\Gamma_{i+1}(B_2(\mathbb{S}^2 \setminus \{x_1, x_2\})) &\cong \Gamma_i(\mathbb{Z}_2 * \mathbb{Z})/\Gamma_{i+1}(\mathbb{Z}_2 * \mathbb{Z}) \\ &\cong \underbrace{\mathbb{Z}_2 \oplus \cdots \oplus \mathbb{Z}_2}_{R_i \text{ times}},\end{aligned}$$

where

$$R_i = \sum_{j=0}^{i-2} \left(\sum_{\substack{k \mid i-j \\ k>1}} \mu\left(\frac{i-j}{k}\right) \frac{k\alpha_k}{i-j} \right),$$

μ is the Möbius function, and

$$\alpha_k = \frac{1}{k}\left(Tr \begin{pmatrix} 0 & -1 \\ -1 & 1 \end{pmatrix}^k - 1 \right).$$

REMARKS 50.

(a) One may check that $R_{i+1} = R_i + \sum_{\substack{k|i+1\\k>1}} \mu\left(\frac{i+1}{k}\right) \frac{k\alpha_k}{i+1}$, and that

$$\mathrm{Tr}\begin{pmatrix} 0 & -1 \\ -1 & 1 \end{pmatrix}^k = \left(\frac{1-\sqrt{5}}{2}\right)^k + \left(\frac{1+\sqrt{5}}{2}\right)^k.$$

(b) By induction, one obtains $\begin{pmatrix} 0 & -1 \\ -1 & 1 \end{pmatrix}^k = \begin{pmatrix} f_{k-1} & -f_k \\ -f_k & f_{k+1} \end{pmatrix}$, where $(f_k)_{k\geq 0}$ is the classical Fibonacci sequence defined by $f_0 = 0$, $f_1 = 1$, and $f_{k+2} = f_{k+1} + f_k$ for all $k \geq 0$.

(c) A simple calculation shows that $R_2 = 1$, $R_3 = 2$, $R_4 = 3$, $R_5 = 5$ and $R_6 = 7$.

The following lemma and corollary will be used in the proof of Theorem 15.

LEMMA 51. *Let G be a finitely-generated group.*

(a) Suppose that there exists $i \geq 2$ such that $\Gamma_i(G)/\Gamma_{i+1}(G)$ is a torsion group. Then for all $j \geq i$, $\Gamma_j(G)/\Gamma_{j+1}(G)$ is a torsion group.

(b) Suppose that there exists $i \geq 2$ and $n \in \mathbb{N}$ such that $x^n = 1$ for all $x \in \Gamma_i(G)/\Gamma_{i+1}(G)$. Then for all $j \geq i$, $y^n = 1$ for all $y \in \Gamma_j(G)/\Gamma_{j+1}(G)$.

PROOF OF LEMMA 51. Let X be a finite set of generators of G. From [**MKS**], we recall that for all $i \geq 2$, $\Gamma_i(G)/\Gamma_{i+1}(G)$ is a finitely-generated Abelian group, generated by the cosets of the simple i-fold commutators of elements of X. We prove part (a) by induction on j: suppose that $\Gamma_j(G)/\Gamma_{j+1}(G)$ is a torsion group for some $j \geq 2$. Now let $y \in \Gamma_{j+1}(G)/\Gamma_{j+2}(G)$. Then there exist simple j-fold commutators $x_1, \ldots, x_k \in \Gamma_j(G)$, $z_1, \ldots, z_k \in G$ and $\varepsilon_1, \ldots, \varepsilon_k \in \{\pm 1\}$ such that y is equal to the $\Gamma_{j+2}(G)$-coset of $[x_1, z_1]^{\varepsilon_1} \cdots [x_k, z_k]^{\varepsilon_k}$. By hypothesis, there exist $m_1, \ldots, m_k \in \mathbb{N}$ such that $x_i^{m_i} \in \Gamma_{j+1}(G)$ for $i = 1, \ldots, k$. Set $m = \mathrm{lcm}(m_1, \ldots, m_k)$. Then modulo $\Gamma_{j+2}(G)$,

$$y^m \equiv ([x_1, z_1]^{\varepsilon_1} \cdots [x_k, z_k]^{\varepsilon_k})^m \equiv [x_1^m, z_1]^{\varepsilon_1} \cdots [x_k^m, z_k]^{\varepsilon_k} \equiv 1,$$

since each of the commutators $[x_i^m, z_i]^{\varepsilon_i}$ belongs to $\Gamma_{j+2}(G)$. This proves part (a). Part (b) follows similarly, taking $m_1 = \cdots = m_k = n$ in the above proof. □

COROLLARY 52. *The lower central series quotients of $\mathbb{Z}_2 * \mathbb{Z}$ are isomorphic to the direct sum of a finite number of copies of $\mathbb{Z}_2$.*

PROOF OF COROLLARY 52. Let x, y generate $\mathbb{Z}_2$ and $\mathbb{Z}$ respectively. Then $\Gamma_2(\mathbb{Z}_2 * \mathbb{Z})/\Gamma_3(\mathbb{Z}_2 * \mathbb{Z})$ is a cyclic group generated by the coset of $[x, y]$. But modulo $\Gamma_3(\mathbb{Z}_2 * \mathbb{Z})$, $[x, y]^2 \equiv [x^2, y] \equiv 1$, and the result follows from Lemma 51 and using the fact that the lower central series quotients of $\mathbb{Z}_2 * \mathbb{Z}$ are finitely-generated Abelian groups. □

Proof of Theorem 15. Consider the presentation (59) of the group $B_2(\mathbb{S}^2 \setminus \{x_1, x_2\})$. Let $\mathbb{Z}_2 * \mathbb{Z} = \langle \overline{D}, \overline{\sigma} \mid \overline{D}^2 = 1 \rangle$. Since the centre of $B_2(\mathbb{S}^2 \setminus \{x_1, x_2\})$ is generated by D^2, we obtain the following central extension:

$$1 \to \langle D^2 \rangle \to B_2(\mathbb{S}^2 \setminus \{x_1, x_2\}) \xrightarrow{\psi} \mathbb{Z}_2 * \mathbb{Z} \to 1,$$

where $\psi(D) = \overline{D}$ and $\psi(\sigma) = \overline{\sigma}$. Since ψ is surjective, for $i \geq 2$, it induces a surjection $\psi_i \colon \Gamma_i(B_2(\mathbb{S}^2 \setminus \{x_1, x_2\})) \to \Gamma_i(\mathbb{Z}_2 * \mathbb{Z})$. Using the fact that $(B_2(\mathbb{S}^2 \setminus \{x_1, x_2\}))^{\mathrm{Ab}} = \langle D, \sigma \rangle \cong \mathbb{Z}^2$, this gives rise to the following commutative diagram of short exact sequences:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \Gamma_2(B_2(\mathbb{S}^2 \setminus \{x_1, x_2\})) & \longrightarrow & B_2(\mathbb{S}^2 \setminus \{x_1, x_2\}) & \xrightarrow{\mathrm{Ab}} & \mathbb{Z} \oplus \mathbb{Z} & \longrightarrow & 1 \\
 & & \Big\downarrow{\scriptstyle \psi_2} & & \Big\downarrow{\scriptstyle \psi} & & \Big\downarrow & & \\
1 & \longrightarrow & \Gamma_2(\mathbb{Z}_2 * \mathbb{Z}) & \longrightarrow & \mathbb{Z}_2 * \mathbb{Z} & \xrightarrow{\mathrm{Ab}} & \mathbb{Z}_2 \oplus \mathbb{Z} & \longrightarrow & 1,
\end{array}$$

where Ab denotes Abelianisation. Now ψ_2 is injective, since if $x \in \operatorname{Ker}(\psi_2)$ then $x \in \operatorname{Ker}(\psi)$, so there exists $k \in \mathbb{Z}$ such that $x = D^{2k}$. But since $x \in \Gamma_2(B_2(\mathbb{S}^2 \setminus \{x_1, x_2\}))$, its Abelianisation is trivial, so $k = 0$. Hence ψ_2 is an isomorphism. But for $i \geq 2$, since ψ_{i+1} is the restriction of ψ_2 to $\Gamma_{i+1}(B_2(\mathbb{S}^2 \setminus \{x_1, x_2\}))$ onto $\Gamma_{i+1}(\mathbb{Z}_2 * \mathbb{Z})$, it follows that ψ_i is an isomorphism for all $i \geq 2$, and that

$$\Gamma_i(B_2(\mathbb{S}^2 \setminus \{x_1, x_2\}))/\Gamma_{i+1}(B_2(\mathbb{S}^2 \setminus \{x_1, x_2\})) \cong \Gamma_i(\mathbb{Z}_2 * \mathbb{Z})/\Gamma_{i+1}(\mathbb{Z}_2 * \mathbb{Z}).$$

This proves the first part of the theorem.

We now calculate the successive lower central series quotients $\Gamma_i(\mathbb{Z}_2 * \mathbb{Z})/\Gamma_{i+1}(\mathbb{Z}_2 * \mathbb{Z})$. This may be done by applying the results of [**Ga, Lab**]; we follow those of [**Ga**]. From Corollary 52, for each $i \geq 2$, $\Gamma_i(\mathbb{Z}_2 * \mathbb{Z})/\Gamma_{i+1}(\mathbb{Z}_2 * \mathbb{Z})$ is the direct sum of a finite number, denoted by R_i in [**Ga**], of copies of $\mathbb{Z}_2$.

To determine R_i, one may first check that in Theorem 2.2 of [**Ga**], $U_\infty(x) = 0$ and $R_k^\infty = 0$ for all $k \geq 2$ (R_k^∞ represents the rank of the free abelian factor of $\Gamma_k(\mathbb{Z}_2 * \mathbb{Z})/\Gamma_{k+1}(\mathbb{Z}_2 * \mathbb{Z})$). Secondly, referring to the notation of Section 3 of that paper, we see that $y = x$, $z = \frac{x}{1-x}$,

$U(x) = \frac{x^2}{1-x}$, and

$$\begin{aligned}\frac{d}{dx}\left(\ln(1-U(x))\right) &= \frac{x(x-2)}{(x-1)(x^2+x-1)}\\ &= \frac{-1}{x-1} + \frac{1}{x-\lambda_+} + \frac{1}{x-\lambda_-},\end{aligned}$$

where $\lambda_\pm = \frac{-1\pm\sqrt{5}}{2}$ are the roots of x^2+x-1. So from equation (3.22) of [**Ga**], we observe that for $k \geq 2$,

$$\alpha_k = \frac{1}{k}\left(\operatorname{Tr} M^k - 1\right),$$

where $M = \begin{pmatrix} 0 & -1 \\ -1 & 1 \end{pmatrix}$, and $\operatorname{Tr} M^k = (-1)^k \left(\lambda_+^k + \lambda_-^k\right)$. The second part of the theorem then follows from Theorem 3.4 of [**Ga**]. □

We may thus describe the derived series of $B_2(\mathbb{S}^2 \setminus \{x_1, x_2\})$ in terms of that of the free group of rank 2:

COROLLARY 53. *For all $i \in \mathbb{N}$,*

$$(B_2(\mathbb{S}^2 \setminus \{x_1, x_2\}))^{(i)} \cong \pi((\mathbb{Z} * \mathbb{Z})^{(i)}),$$

where $\pi\colon \mathbb{Z} * \mathbb{Z} \to \mathbb{Z}_2 * \mathbb{Z}$ is the homomorphism obtained by taking the first factor modulo 2.

PROOF. Let G_1, G_2 be two groups. If $\pi\colon G_1 \to G_2$ is a surjective homomorphism, then the restriction $\pi|_{(G_1)^{(1)}}\colon (G_1)^{(1)} \to (G_2)^{(1)}$, and by induction on i, so is the restriction $\pi|_{(G_1)^{(i)}}\colon (G_1)^{(n)} \to (G_2)^{(i)}$. Taking $G_1 = \mathbb{Z} * \mathbb{Z}$ and $G_2 = \mathbb{Z}_2 * \mathbb{Z}$, it follows that

$$(\mathbb{Z}_2 * \mathbb{Z})^{(i)} = \pi((\mathbb{Z} * \mathbb{Z})^{(i)}).$$

But

$$(B_2(\mathbb{S}^2 \setminus \{x_1, x_2\}))^{(i)} \cong (\mathbb{Z}_2 * \mathbb{Z})^{(i)}$$

by Theorem 15 which proves the corollary. □

We now determine explicitly $\Gamma_3(B_2(\mathbb{S}^2 \setminus \{x_1, x_2\}))$.

PROPOSITION 54. *Let $\rho_2\colon \Gamma_2(B_2(\mathbb{S}^2 \setminus \{x_1, x_2\})) \to \mathbb{Z}_2$ be the homomorphism defined by $\rho_2(z_n) = 1$ for all $n \in \mathbb{Z}$, where $\{z_n\}_{n\in\mathbb{Z}}$ is the basis given by Corollary 47. Then $\Gamma_3(B_2(\mathbb{S}^2 \setminus \{x_1, x_2\})) = Ker(\rho_2)$. In particular, $\Gamma_3(B_2(\mathbb{S}^2 \setminus \{x_1, x_2\}))$ is a free group of infinite rank with a basis given by $\left\{z_n z_0^{-1}\right\}_{n\in\mathbb{Z}\setminus\{0\}} \bigcup \{z_m^2\}_{m\in\mathbb{Z}}$, and*

$$\Gamma_2(B_2(\mathbb{S}^2 \setminus \{x_1, x_2\}))/\Gamma_3(B_2(\mathbb{S}^2 \setminus \{x_1, x_2\})) \cong \mathbb{Z}_2.$$

REMARK 55. Since $R_2 = 1$, this agrees with the result of Theorem 15 in the case $i = 2$.

PROOF. We start by calculating the action under conjugation of the generators $\gamma_{2,1}, \gamma_{2,2}$ and σ of $B_2(\mathbb{S}^2 \setminus \{x_1, x_2\})$ on the generators z_n of $\Gamma_2(B_2(\mathbb{S}^2 \setminus \{x_1, x_2\}))$. Clearly $\gamma_{2,1} z_n \gamma_{2,1}^{-1} = z_{n+1}$ and $\gamma_{2,2} z_n \gamma_{2,2}^{-1} = z_0 z_{n+1} z_0^{-1}$. Further, it follows from equation (57) that

$$\sigma z_n \sigma^{-1} = \gamma_{2,2}^{n-1} \gamma_{2,1} \gamma_{2,2}^{-n},$$

which rewriting in terms of the z_i yields:

$$\sigma z_n \sigma^{-1} = \begin{cases} z_0 z_1 \cdots z_{n-2} z_{n-1}^{-1} z_{n-2}^{-1} \cdots z_1^{-1} z_0^{-1} & \text{if } n > 0 \\ z_{-1}^{-1} \cdots z_{-|n|}^{-1} z_{-(|n|+1)}^{-1} z_{-|n|} \cdots z_{-1} & \text{if } n \leq 0. \end{cases}$$

Let us apply the Reidemeister-Schreier rewriting process to the basis $\{z_n\}_{n \in \mathbb{Z}}$ of $\Gamma_2(B_2(\mathbb{S}^2 \setminus \{x_1, x_2\}))$, taking the Schreier transversal $\{1, z_0\}$ for ρ_2. This yields a basis $\left\{z_n z_0^{-1}\right\}_{n \in \mathbb{Z} \setminus \{0\}} \bigcup \{z_0 z_m\}_{m \in \mathbb{Z}}$ of $\operatorname{Ker}(\rho_2)$, or equivalently a basis $\left\{z_n z_0^{-1}\right\}_{n \in \mathbb{Z} \setminus \{0\}} \bigcup \{z_m^2\}_{m \in \mathbb{Z}}$. Since

$$z_n z_0^{-1} = \begin{cases} (z_n z_{n-1}^{-1})(z_{n-1} z_{n-2}^{-1}) \cdots (z_1 z_0^{-1}) & \text{for all } n > 0 \\ (z_{n+1} z_n^{-1})^{-1} (z_{n+2} z_{n+1}^{-1})^{-1} \cdots (z_0 z_{-1}^{-1})^{-1} & \text{for all } n < 0, \end{cases}$$

and $z_{i+1} z_i^{-1} = [a, z_i] \in \Gamma_3(B_2(\mathbb{S}^2 \setminus \{x_1, x_2\}))$ for all $i \in \mathbb{Z}$, we see that $z_n z_0^{-1} \in \Gamma_3(B_2(\mathbb{S}^2 \setminus \{x_1, x_2\}))$ for all $n \neq 0$. Finally, if $m \in \mathbb{Z}$ then $z_m z_0 = (z_m z_0^{-1}) z_0^2$. But $[\sigma, z_1] = z_0^{-1} z_1^{-1}$, so $z_0^2 = (z_1 z_0^{-1})^{-1} [\sigma, z_0]^{-1} \in \Gamma_3(B_2(\mathbb{S}^2 \setminus \{x_1, x_2\}))$. Thus $\operatorname{Ker}(\rho_2) \subseteq \Gamma_3(B_2(\mathbb{S}^2 \setminus \{x_1, x_2\}))$.

To prove the converse, observe first that $\Gamma_3(B_2(\mathbb{S}^2 \setminus \{x_1, x_2\}))$ is the normal closure in $B_2(\mathbb{S}^2 \setminus \{x_1, x_2\})$ of the commutators $[\gamma_{2,1}, z_n]$, $[\gamma_{2,2}, z_n]$ and $[\sigma, z_n]$, where $n \in \mathbb{Z}$. It follows easily from the above expressions that these elements belong to $\operatorname{Ker}(\rho_2)$. Further, conjugation by each of $\gamma_{2,1}$, $\gamma_{2,2}$ and σ induce automorphisms of $\Gamma_2(B_2(\mathbb{S}^2 \setminus \{x_1, x_2\}))$, each of which leaves $\operatorname{Ker}(\rho_2)$ invariant, and so induces an automorphism of $\mathbb{Z}_2$, which is in fact the identity in all three cases. Hence for all $n \in \mathbb{Z}$, all conjugates of $[\gamma_{2,1}, z_n]$, $[\gamma_{2,2}, z_n]$ and $[\sigma, z_n]$ by elements of $B_2(\mathbb{S}^2 \setminus \{x_1, x_2\})$ belong to $\operatorname{Ker}(\rho_2)$, and so $\Gamma_3(B_2(\mathbb{S}^2 \setminus \{x_1, x_2\})) \subseteq \operatorname{Ker}(\rho_2)$. We conclude that $\Gamma_3(B_2(\mathbb{S}^2 \setminus \{x_1, x_2\})) \subseteq \operatorname{Ker}(\rho_2)$, and $\Gamma_2(B_2(\mathbb{S}^2 \setminus \{x_1, x_2\}))/\Gamma_3(B_2(\mathbb{S}^2 \setminus \{x_1, x_2\})) \cong \mathbb{Z}_2$. □

7. The commutator subgroup of $B_m(\mathbb{S}^2 \setminus \{x_1, x_2\})$, $m \geq 3$

As we already observed in Remarks 35, $B_m(\mathbb{S}^2 \setminus \{x_1, x_2\})$ may be identified with the m-string braid group of the annulus. The case $m = 2$ having already been studied in Section 6, let us now suppose that $m \geq 3$. In this case, we know from Theorem 9 that the lower central series is constant from the commutator subgroup onwards. The following presentation of $B_m(\mathbb{S}^2 \setminus \{x_1, x_2\})$ was obtained by Kent and Peifer:

PROPOSITION 56 ([**KP**]). *If $m \geq 3$ then $B_m(\mathbb{S}^2 \setminus \{x_1, x_2\})$ admits a presentation of the following form:*

generators: *$\sigma_0, \sigma_1, \ldots, \sigma_{m-1}$ and τ.*
relations:

$$\sigma_i \sigma_j = \sigma_j \sigma_i \textit{ if } |i-j| \neq 1, m-1 \textit{ and } 0 \leq i, j \leq m-1 \tag{60}$$

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \textit{ for } 0 \leq i \leq m-1, \textit{ and} \tag{61}$$

$$\tau^{-1} \sigma_i \tau = \sigma_{i+1} \textit{ for } 0 \leq i \leq m-1. \tag{62}$$

The indices should be taken modulo m.

The m points should be thought of as being arranged around the centre of the annulus. The generator σ_0 corresponds to a positive half-twist between the m^{th} and 1^{st} point, while τ is represented geometrically by a rigid rotation of the annulus about the centre by an angle $2\pi/n$. It follows from this presentation that:

COROLLARY 57 ([**KP**]). *If $m \geq 3$ then $B_m(\mathbb{S}^2 \setminus \{x_1, x_2\})$ is isomorphic to the semi-direct product of the affine Artin group $\widetilde{A}_{m-1}$ (generated by $\sigma_0, \sigma_1, \ldots, \sigma_{m-1}$, and subject to relations (60) and (61)) by the infinite cyclic group generated by τ, the action being that of conjugation given by relation (62).*

Then we have the following result:

PROPOSITION 58.

(a) If $m \geq 3$ then $\Gamma_2(B_m(\mathbb{S}^2 \setminus \{x_1, x_2\}))$ is generated by the elements $p_k = \sigma_1^k \sigma_2 \sigma_1^{-(k+1)}$, $r_k = \sigma_1^k \sigma_0 \sigma_1^{-(k+1)}$, for all $k \in \mathbb{Z}$, and $q_i = \sigma_i \sigma_1^{-1}$ for $3 \leq i \leq m-1$.
(b) If $m = 3$, then $\Gamma_2(B_3(\mathbb{S}^2 \setminus \{x_1, x_2\}))$ is defined by the following relations:

$$p_{k+1} p_{k+2}^{-1} p_k^{-1} = 1 \tag{63}$$

$$r_{k+1} r_{k+2}^{-1} r_k^{-1} = 1 \tag{64}$$

$$r_k p_{k+1} r_{k+2} p_{k+2}^{-1} r_{k+1}^{-1} p_k^{-1} = 1, \tag{65}$$

where $k \in \mathbb{Z}$.

(c) If $m \geq 4$ then $\Gamma_2(B_m(\mathbb{S}^2 \setminus \{x_1, x_2\}))$ is defined by the following relations:

$$p_{k+1}p_{k+2}^{-1}p_k^{-1} = 1 \tag{66}$$

$$r_{k+1}r_{k+2}^{-1}r_k^{-1} = 1 \tag{67}$$

$$p_k q_3 p_{k+2} q_3^{-1} p_{k+1}^{-1} q_3^{-1} = 1 \tag{68}$$

$$p_k q_i p_{k+1}^{-1} q_i^{-1} = 1 \textit{ for all } 4 \leq i \leq m-1 \tag{69}$$

$$q_i q_j q_i^{-1} q_j^{-1} = 1 \textit{ for all } 3 \leq i < j-1 \leq m-2 \tag{70}$$

$$q_i q_{i+1} q_i = q_{i+1} q_i q_{i+1} \textit{ for } 3 \leq i \leq m-2 \tag{71}$$

$$r_k p_{k+1} r_{k+1}^{-1} p_k^{-1} = 1 \tag{72}$$

$$r_k q_i r_{k+1}^{-1} q_i^{-1} = 1 \textit{ for all } 3 \leq i \leq m-2 \tag{73}$$

$$r_k q_{m-1} r_{k+2} q_{m-1}^{-1} r_{k+1}^{-1} q_{m-1}^{-1} = 1, \tag{74}$$

where $k \in \mathbb{Z}$.

We may thus deduce the first derived series quotient of the group $\Gamma_2(B_m(\mathbb{S}^2 \setminus \{x_1, x_2\}))$:

Corollary 16. *Let $m \geq 3$. Then*

$$\left(B_m\left(\mathbb{S}^2 \setminus \{x_1, x_2\}\right)\right)^{(1)} \Big/ \left(B_m\left(\mathbb{S}^2 \setminus \{x_1, x_2\}\right)\right)^{(2)} \cong \begin{cases} \mathbb{Z}^4 & \textit{if } m = 3 \\ \mathbb{Z}^2 & \textit{if } m = 4 \\ \mathbb{Z} & \textit{if } m \geq 5. \end{cases}$$

Proof of Proposition 58. We start by applying Proposition 29 to the result of Corollary 57, namely that

$$B_m(\mathbb{S}^2 \setminus \{x_1, x_2\}) \cong \widetilde{A}_{m-1} \rtimes \langle \tau \rangle.$$

If $w = \sigma_{i_1}^{\varepsilon_{i_1}} \cdots \sigma_{i_k}^{\varepsilon_{i_k}} \in \widetilde{A}_{m-1}$, it follows from the action, given by equation (62), that for all $l \in \mathbb{Z}$,

$$\tau^{-l} w \tau^l \cdot w^{-1} = \sigma_{i_1+l}^{\varepsilon_{i_1}} \cdots \sigma_{i_k+l}^{\varepsilon_{i_k}} \cdot \sigma_{i_k}^{-\varepsilon_{i_k}} \cdots \sigma_{i_1}^{-\varepsilon_{i_1}},$$

where the indices should be taken modulo m. Hence $\tau^{-l} w \tau^l \cdot w^{-1} \in \Gamma_2(\widetilde{A}_{m-1})$, and it follows from Proposition 29 that

$$\Gamma_2(B_m(\mathbb{S}^2 \setminus \{x_1, x_2\})) \cong \Gamma_2(\widetilde{A}_{m-1}).$$

A presentation of $\Gamma_2(\widetilde{A}_{m-1})$ may be obtained by observing that $(\widetilde{A}_{m-1})^{\mathrm{Ab}} \cong \mathbb{Z}$, and by applying the Reidemeister-Schreier rewriting process to the generating set $\{\sigma_0, \sigma_1, \ldots, \sigma_{m-1}\}$ of $\widetilde{A}_{m-1}$ and the Schreier transversal $\{\sigma_1^k\}_{k \in \mathbb{Z}}$. The generators and relations not containing σ_0 define a group isomorphic to $B_m(\mathbb{D}^2)$, and using [**GL**], we obtain all of the generators and relations of Proposition 58 not containing r_k. The generator σ_0 of $\widetilde{A}_{m-1}$ gives rise to generators $r_k = \sigma_1^k \sigma_0 \sigma_1^{-(k+1)}$ of $\Gamma_2(\widetilde{A}_{m-1})$, where $k \in \mathbb{Z}$. The relation (61) with $j = 0$ and $i = 1$ yields relations of the form $r_{k+1} r_{k+2}^{-1} r_k^{-1} = 1$, $k \in \mathbb{Z}$, in $\Gamma_2(\widetilde{A}_{m-1})$. If $m = 3$ then we obtain relations (65) in $\Gamma_2(\widetilde{A}_2)$ from relation (61) with $j = 0$ and $i = 2$, and so we deduce the presentation given in part (b). If $m \geq 4$, taking $j = 0$ in relations (60) with $i = 2$ (resp. $3 \leq i \leq m-2$) yields relations (72) (resp. (73)) in $\Gamma_2(\widetilde{A}_{m-1})$. Finally we obtain relations (74) in $\Gamma_2(\widetilde{A}_{m-1})$ by taking $j = 0$ and $i = m-1$ in relation (61), and this gives the presentation of part (c). □

Proof of Corollary 16. It suffices to Abelianise the presentations of Proposition 58, in other words, we add the commutation relations of all of the generators to the given presentations. First let $m = 3$. Equation (65) becomes trivial using equations (63) and (64). Further, it follows from equations (63) (resp. (64)) that all of the p_k (resp. r_k) may be expressed uniquely in terms of p_0 and p_1 (resp. r_0 and r_1), and hence $(B_3(\mathbb{S}^2 \setminus \{x_1, x_2\}))^{\mathrm{Ab}}$ is a free Abelian group of rank 4 with basis $\{p_0, p_1, r_0, r_1\}$

Let $m = 4$. By equation (66), it follows from equation (68) that q_3 Abelianises to the trivial element, and then equation (67) implies that equation (74) becomes trivial. By equation (72), $p_k = r_k$ for all $k \in \mathbb{Z}$. As above, all of the p_k (resp. r_k) may be expressed uniquely in terms of p_0 and p_1 (resp. r_0 and r_1), and thus $(B_4(\mathbb{S}^2 \setminus \{x_1, x_2\}))^{\mathrm{Ab}}$ is a free Abelian group of rank 4 with basis $\{p_0, p_1\}$.

Finally, if $m \geq 5$, by equation (71) we obtain additionally that all of the q_i Abelianise to the trivial element. By equation (69) (resp. (73)), $p_k = p_{k+1}$ (resp. $r_k = r_{k+1}$). Thus $(B_m(\mathbb{S}^2 \setminus \{x_1, x_2\}))^{\mathrm{Ab}}$ is a infinite cyclic group generated by p_0. □

8. The series of $B_m(\mathbb{S}^2 \setminus \{x_1, x_2, x_3\})$

The situation seems to be more difficult in the case of the braid group of the 3-punctured sphere. As we remark below, if $m \geq 2$, $B_m(\mathbb{S}^2 \setminus \{x_1, x_2, x_3\})$ is isomorphic to the affine Artin group of type $\widetilde{C}_m$ for which little seems to be known [**All, ChP**]. We have not even been able to describe the commutator subgroup. We may however obtain some partial results, notably in Proposition 60 the fact that the successive lower central series quotients of $B_2(\mathbb{S}^2 \setminus \{x_1, x_2, x_3\})$ are direct sums of $\mathbb{Z}_2$, which generalises part of Theorem 15.

We begin by considering the case $m = 2$.

PROPOSITION 59 ([**BG**]). *The following constitutes a presentation of the group $B_2(\mathbb{S}^2 \setminus \{x_1, x_2, x_3\})$:*

generators: *σ, ρ_1 and ρ_2.*

relations:

$$(\sigma\rho_1)^2 = (\rho_1\sigma)^2 \tag{75}$$

$$(\sigma\rho_2)^2 = (\rho_2\sigma)^2 \tag{76}$$

$$\rho_1\rho_2 = \rho_2\rho_1.$$

Geometrically, $B_2(\mathbb{S}^2 \setminus \{x_1, x_2, x_3\})$ may be considered as the 2-string braid group of the twice-punctured disc, which in turn may be considered as a subgroup of $B_4(\mathbb{D}^2)$ whose first and fourth strings are vertical. Then with the usual notation, $\rho_1 = A_{1,2}$, $\rho_2 = A_{3,4}$, and σ is the positive half-twist of the second and third strings.

Let G_1 be the group generated by σ and ρ_1 subject to the relation (75), and let G_2 be the group generated by σ and ρ_2 subject to the relation (76). It follows from the above proposition and Remark 49(d) that $B_2(\mathbb{S}^2 \setminus \{x_1, x_2, x_3\})$ may be considered as the amalgamated product $G_1 *_{\langle\sigma\rangle} G_2$ of two copies of the Baumslag-Solitar group $\mathrm{BS}(2,2)$, subject to the additional relation $[\rho_1, \rho_2] = 1$. We wonder if it would be possible to obtain determine the commutator subgroup via this amalgamated product.

The following gives a generalisation to $B_2(\mathbb{S}^2 \setminus \{x_1, x_2, x_3\})$ of part of Theorem 15.

PROPOSITION 60. *For all $i \geq 2$, the lower central series quotient $\Gamma_i(B_2(\mathbb{S}^2 \setminus \{x_1, x_2, x_3\})) / \Gamma_{i+1}(B_2(\mathbb{S}^2 \setminus \{x_1, x_2, x_3\}))$ is isomorphic to the direct sum of a finite number of copies of $\mathbb{Z}_2$.*

PROOF. As in the proof of Lemma 51, since $B_2(\mathbb{S}^2 \setminus \{x_1, x_2, x_3\})$ is finitely generated, it follows that the lower central quotient $\Gamma_i(B_2(\mathbb{S}^2 \setminus \{x_1, x_2, x_3\})) / \Gamma_{i+1}(B_2(\mathbb{S}^2 \setminus \{x_1, x_2, x_3\}))$ is a finitely-generated Abelian group. By part (b) of Lemma 51, it suffices to prove the result in the case $i = 2$, which we do using the presentation of Proposition 59. We know that $\Gamma_2(B_2(\mathbb{S}^2 \setminus \{x_1, x_2, x_3\})) / \Gamma_3(B_2(\mathbb{S}^2 \setminus \{x_1, x_2, x_3\}))$ is generated by the Γ_3-cosets of the commutators of the form $[x, y]$, where $x, y \in \{\sigma, \rho_1, \rho_2\}$, and thus of the commutators $[\sigma, \rho_i]$ for $i = 1, 2$. But $[\sigma, \rho_i] = [\rho_i^{-1}\sigma^{-1}]^{-1}$ by relations (75) and (76). So modulo Γ_3, $[\sigma, \rho_i]$ is congruent to $[\sigma, \rho_i]^{-1}$, in other words, $[\sigma, \rho_i]^2$ is trivial modulo Γ_3, which proves the result. ☐

As was pointed out in [**All, BG**], $B_2(\mathbb{S}^2 \setminus \{x_1, x_2, x_3\})$ is isomorphic to the affine Artin braid group $\widetilde{C}_2$. More generally, for $m \geq 2$, $B_m(\mathbb{S}^2 \setminus$

$\{x_1, x_2, x_3\})$ is isomorphic to $\widetilde{C}_m$ and by [**BG**] has a presentation of the form:

generators: ρ_1, ρ_m and σ_i, $1 \le i \le m-1$.
relations:

$$\sigma_i\sigma_j = \sigma_j\sigma_i \text{ if } |i-j| \ge 2 \text{ and } 1 \le i,j \le m-1$$
$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \text{ for all } 1 \le i \le m-2$$
$$\rho_1 \rightleftharpoons \rho_m$$
$$\rho_1 \rightleftharpoons \sigma_i \text{ for all } 2 \le i \le m-1$$
$$\rho_m \rightleftharpoons \sigma_i \text{ for all } 1 \le i \le m-2$$
$$(\sigma_1\rho_1)^2 = (\rho_1\sigma_1)^2$$
$$(\sigma_{m-1}\rho_m)^2 = (\rho_m\sigma_{m-1})^2.$$

The following result yields information about the derived series quotients of $B_m(\mathbb{S}^2 \setminus \{x_1, x_2, x_3\})$.

PROPOSITION 61. *Let $m \ge 2$. Then $B_m(\mathbb{S}^2 \setminus \{x_1, x_2, x_3\})$ is a semi-direct product of a group K_0 by $B_m(\mathbb{D}^2)$. In particular, for all $i \ge 1$ $(B_m(\mathbb{S}^2 \setminus \{x_1, x_2, x_3\}))^{(i)}$ is a semi-direct of a group K_i by $(B_m(\mathbb{D}^2))^{(i)}$.*

PROOF. Consider the homomorphism of $B_m(\mathbb{S}^2 \setminus \{x_1, x_2, x_3\})$ to $B_m(\mathbb{D}^2)$ which sends ρ_1 and ρ_m onto the trivial element. From the above presentation, it is clearly surjective, and it admits an obvious section. So if K_0 denotes the kernel then $B_m(\mathbb{S}^2 \setminus \{x_1, x_2, x_3\}) \cong K_0 \rtimes B_m(\mathbb{D}^2)$. The second part is obtained by induction on i, using Proposition 29. $\square$

CHAPTER 4

Presentations for $\Gamma_2(B_n(\mathbb{S}^2))$, $n \geq 4$

In this chapter, we give various presentations of $\Gamma_2(B_n(\mathbb{S}^2))$, $n \geq 4$. In Section 1, we begin by giving a general presentation obtained using the Reidemeister-Schreier rewriting process. In Section 2, we consider the case $n = 4$, and derive the presentation given in Theorem 3(c). In Section 3, we restate the presentation given by Proposition 62 for the case $n = 5$, and for $n \geq 6$, we refine the presentation to obtain Proposition 67.

1. A general presentation of $\Gamma_2(B_n(\mathbb{S}^2))$ for $n \geq 4$

PROPOSITION 62. *Let $n \geq 4$. The following constitutes a presentation of the group $\Gamma_2(B_n(\mathbb{S}^2))$:*

generators:

$$w = \sigma_1^{2n-2}$$
$$u_1 = \sigma_2\sigma_1^{-1}, u_2 = \sigma_1\sigma_2\sigma_1^{-2}, \ldots, u_{2n-2} = \sigma_1^{2n-3}\sigma_2\sigma_1^{-(2n-2)}$$
$$v_1 = \sigma_3\sigma_1^{-1}, \ldots, v_{n-3} = \sigma_{n-1}\sigma_1^{-1}.$$

relations:

$$v_iv_j = v_jv_i \text{ if } |i-j| \geq 2 \text{ and } 1 \leq i,j \leq n-3 \tag{77}$$

$$v_iv_{i+1}v_i = v_{i+1}v_iv_{i+1} \text{ for all } 1 \leq i \leq n-4 \tag{78}$$

$$w \rightleftharpoons v_i \tag{79}$$

$$u_iv_ju_{i+1}^{-1}v_j^{-1} = 1 \text{ for } j \geq 2 \text{ and } i = 1,\ldots,2n-3 \tag{80}$$

$$u_{2n-2}v_jwu_1^{-1}w^{-1}v_j^{-1} = 1 \text{ for } 2 \leq j \leq n-3 \tag{81}$$

$$u_iv_1u_{i+2}v_1^{-1}u_{i+1}^{-1}v_1^{-1} = 1 \text{ for } i = 1,\ldots,2n-4 \tag{82}$$

$$u_{2n-3}v_1wu_1w^{-1}v_1^{-1}u_{2n-2}^{-1}v_1^{-1} = 1 \tag{83}$$

$$u_{2n-2}v_1wu_2v_1^{-1}u_1^{-1}w^{-1}v_1^{-1} = 1 \tag{84}$$

$$u_{i+1}u_{i+2}^{-1}u_i^{-1} = 1 \text{ for all } i = 1,\ldots,2n-4 \tag{85}$$

$$u_{2n-2}wu_1^{-1}w^{-1}u_{2n-3}^{-1} = 1 \tag{86}$$

$$wu_1u_2^{-1}w^{-1}u_{2n-2}^{-1} = 1 \tag{87}$$

$$u_2(v_1\cdots v_{n-4}v_{n-3}^2v_{n-4}\cdots v_1)u_{2n-3}w = 1 \tag{88}$$

$$u_3(v_1\cdots v_{n-4}v_{n-3}^2v_{n-4}\cdots v_1)u_{2n-2}w = 1 \tag{89}$$

$$u_i(v_1\cdots v_{n-4}v_{n-3}^2v_{n-4}\cdots v_1)wu_{i-3} = 1 \text{ for } i = 4,\ldots,2n-2 \tag{90}$$

$$u_1(v_1\cdots v_{n-4}v_{n-3}^2v_{n-4}\cdots v_1)u_{2n-4}w = 1. \tag{91}$$

In what follows, we shall denote by equation (m_i) the equation (m) of the above system for the parameter value i.

PROOF. Taking the standard presentation (4) of $B_n(\mathbb{S}^2)$, and the set $\{1, \sigma_1, \sigma_1^2, \ldots, \sigma_1^{2n-3}\}$ as a Schreier tranversal, we apply the Reidemeister-Schreier rewriting process to the following short exact sequence:

$$1 \longrightarrow \Gamma_2(B_n(\mathbb{S}^2)) \longrightarrow B_n(\mathbb{S}^2) \longrightarrow (B_n(\mathbb{S}^2))^{\text{Ab}} \longrightarrow 1.$$

As generators of $\Gamma_2(B_n(\mathbb{S}^2))$, we obtain $w = \sigma_1^{2n-2}$, $\sigma_1^j\sigma_i\sigma_1^{-(j+1)}$ and $\sigma_1^{2n-3}\sigma_i$, where $2 \leq i \leq n-1$ and $0 \leq j \leq 2n-4$. We replace the latter by $\sigma_1^{2n-3}\sigma_i \cdot w^{-1} = \sigma_1^{2n-3}\sigma_i\sigma_1^{-(2n-2)}$. Now turning to the relations, if $i \geq 3$ then for $j = 0,\ldots,2n-4$, the relator $\sigma_1\sigma_i\sigma_1^{-1}\sigma_i^{-1}$ of $B_n(\mathbb{S}^2)$ gives rise to relators

$$\sigma_1^j\sigma_1\sigma_i\sigma_1^{-1}\sigma_i^{-1}\sigma_1^{-j} = \sigma_1^{j+1}\sigma_i\sigma_1^{-(j+2)} \cdot \sigma_1^{j+1}\sigma_i^{-1}\sigma_1^{-j}$$

of $\Gamma_2(B_n(\mathbb{S}^2))$, so

$$\sigma_1^{j+1}\sigma_i\sigma_1^{-(j+2)} = \sigma_1^j\sigma_i\sigma_1^{-(j+1)} = \sigma_i\sigma_1^{-1} = v_{i-2}.$$

If $j = 2n-3$ then we have a relator of the form

$$\sigma_1^{2n-3}\sigma_1\sigma_i\sigma_1^{-1}\sigma_i^{-1}\sigma_1^{-(2n-3)} = \sigma_1^{2n-2} \cdot \sigma_i\sigma_1^{-1} \cdot \sigma_1^{2n-2}\sigma_1^{-(2n-2)}\sigma_i^{-1}\sigma_1^{-(2n-3)},$$

and thus v_{i-2} commutes with w, which gives relation (79). If $1 \leq i, j \leq n-3$ and $|i-j| \geq 2$ then the relator $\sigma_{i+2}\sigma_{j+2}\sigma_{i+2}^{-1}\sigma_{j+2}^{-1}$ gives rise to the single relator $v_i v_j v_i^{-1} v_j^{-1}$, while if $1 \leq i \leq n-4$, the relator $\sigma_{i+2}\sigma_{i+3}\sigma_{i+2}\sigma_{i+3}^{-1}\sigma_{i+2}^{-1}\sigma_{i+3}^{-1}$ yields the single relator $v_i v_{i+1} v_i = v_{i+1} v_i v_{i+1}$, thus we obtain equations (77) and (78).

Now for $i = 1, \ldots, u_{2n-2}$, let $u_i = \sigma_1^{i-1}\sigma_2\sigma_1^{-i}$. From the relator $\sigma_1^{j-1}\sigma_2\sigma_1\sigma_2\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-(j-1)}$, we obtain the relators $u_j u_{j+2} u_{j+1}^{-1}$ if $j = 1, \ldots, 2n-4$, $u_{2n-3} w u_1 w^{-1} u_{2n-2}^{-1}$ if $j = 2n-3$, and $u_{2n-2} w u_2 u_1^{-1} w^{-1}$ if $j = 2n-2$, which gives respectively equations (85), (86) and (87).

If $2 \leq i \leq n-3$ then the relator $\sigma_1^{j-1}\sigma_{i+2}\sigma_2\sigma_{i+2}^{-1}\sigma_2^{-1}\sigma_1^{-(j-1)}$ yields relators $v_i u_{j+1} v_i^{-1} u_j^{-1}$ if $j = 1, \ldots, 2n-3$ and $v_i w u_1 w^{-1} v_i^{-1} u_{2n-2}^{-1}$ if $j = 2n-2$, and so we recover equations (80) and (81).

From the relator $\sigma_1^{j-1}\sigma_3\sigma_2\sigma_3\sigma_2^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-(j-1)}$, we obtain the relators $v_1 u_{j+1} v_1 u_{j+2}^{-1} v_1^{-1} u_j^{-1}$ if $j = 1, \ldots, 2n-4$, $v_1 u_{2n-2} v_1 w u_1^{-1} w^{-1} v_1^{-1} u_{2n-3}^{-1}$ if $j = 2n-3$, and $v_1 w u_1 v_1 u_2^{-1} w^{-1} v_1^{-1} u_{2n-2}^{-1}$ if $j = 2n-2$, which gives equations (82), (83) and (84).

Finally,

$$\begin{aligned}\sigma_1\sigma_2\cdots\sigma_{n-2}\sigma_{n-1}^2\sigma_{n-2}\cdots\sigma_2\sigma_1 &= \sigma_1\sigma_2\sigma_1^{-2}\cdot\sigma_1^2\sigma_3\sigma_1^{-3}\cdots\\ &\cdots\sigma_1^{n-3}\sigma_{n-2}\sigma_1^{-(n-2)}\cdot\sigma_1^{n-2}\sigma_{n-1}\sigma_1^{-(n-1)}\cdot\sigma_1^{n-1}\sigma_{n-1}\sigma_1^{-n}\cdot\\ &\sigma_1^{n}\sigma_{n-2}\sigma_1^{-(n+1)}\cdot\sigma_1^{2n-4}\sigma_2\sigma_1^{-(2n-3)}\cdot\sigma_1^{2n-2},\end{aligned}$$

and conjugating by σ_1^{j-1}, we obtain relators

$$\begin{cases} u_2(v_1\cdots v_{n-4}v_{n-3}^2v_{n-4}\cdots v_1)u_{2n-3}w & \text{if } j=1\\ u_3(v_1\cdots v_{n-4}v_{n-3}^2v_{n-4}\cdots v_1)u_{2n-2}w & \text{if } j=2\\ u_{j+1}(v_1\cdots v_{n-4}v_{n-3}^2v_{n-4}\cdots v_1)wu_{j-2} & \text{if } j=3,\ldots,2n-3\\ wu_1(v_1\cdots v_{n-4}v_{n-3}^2v_{n-4}\cdots v_1)u_{2n-4} & \text{if } j=2n-2.\end{cases}$$

This yields the remaining equations (88), (89), (90) and (91). □

We now simplify somewhat the presentation of $\Gamma_2(B_n(\mathbb{S}^2))$ given by Proposition 62. From equations (80) and (85), for $i = 1, 2$ we obtain the following equations:

$$u_1 v_j = v_j u_2 \quad \text{for all} \quad j \geq 2 \tag{80_1}$$

$$u_2 v_j = v_j u_1^{-1} u_2 \quad \text{for all} \quad j \geq 2. \tag{80_2}$$

This allows us to eliminate equation (81) as follows. For all $j \geq 2$, we have:

$$\begin{aligned} v_j w u_1 w^{-1} v_j^{-1} &= w v_j u_1^{-1} v_j^{-1} w^{-1} \quad \text{by equation (79)} \\ &= w v_j u_2 v_j^{-1} u_2^{-1} w^{-1} \quad \text{by equation } (80_2) \\ &= w u_1 u_2^{-1} w^{-1} \quad \text{by equation } (80_1) \\ &= u_{2n-2} \quad \text{by equation (87)}, \end{aligned}$$

and this is equivalent to equation (81), which we thus delete from the list of relations.

Suppose that for some $2 \leq i \leq 2n-4$, we have equations (80_{i-1}) and (80_i). We now show that they imply (80_{i+1}). For all $j \geq 2$, we have:

$$\begin{aligned} u_{i+1} v_j u_{i+2}^{-1} v_j^{-1} &= u_{i-1}^{-1} u_i v_j u_{i+1}^{-1} u_i v_j^{-1} \quad \text{by equation (85)} \\ &= u_{i-1}^{-1} v_j u_i v_j^{-1} \quad \text{by equation } (80_i) \\ &= 1 \quad \text{by equation } (80_{i-1}), \end{aligned}$$

which yields equation (80_{i+1}). So we may successively delete equations (80_{2n-3}), (80_{2n-4}), …, (80_3) from the list of relations.

We now show that we may delete all but one of the surface relations (88)–(91). First suppose that we have equation (88). Now

$$\begin{aligned} u_{2n-2} w u_3 &= u_{2n-3} w u_1 u_3 \quad \text{by equation (86)} \\ &= u_{2n-3} w u_2 \quad \text{by equation } (85_1). \end{aligned}$$

This implies equation (89) which we delete from the list of relations.

Now suppose that we have equation (90_{i+1}) for some $5 \leq i \leq 2n-2$. Let us write $A = v_1 \cdots v_{n-4} v_{n-3}^2 v_{n-4} \cdots v_1$. Then $w u_{i-2} u_{i+1} = A^{-1}$. So

$$\begin{aligned} w u_{i-3} u_i &= w u_{i-2} u_{i+1} \quad \text{by equation } (85_1) \\ &= A^{-1} \quad \text{by above.} \end{aligned}$$

This yields equation (90_i), and so we may delete successively equations (90_4), …, (90_{2n-3}).

Now suppose that we have (91), so $A u_{2n-4} w u_1 = 1$. Then

$$\begin{aligned} A w u_{2n-5} u_{2n-2} &= A w u_{2n-4} w u_1 w^{-1} \quad \text{by equations (85) and (86)} \\ &= w (A u_{2n-4} w u_1) w^{-1} \quad \text{by equation (79)} \\ &= 1 \quad \text{by above.} \end{aligned}$$

This implies equation (90_{2n-2}) which we delete from the list of relations.

Finally, suppose that we have equation (88). Then

$$\begin{aligned} Au_{2n-4}wu_1 &= Au_{2n-3}u_{2n-2}^{-1}wu_1 \quad \text{by equation } (85_{2n-4}) \\ &= Au_{2n-3}wu_2 \quad \text{by equation (87)} \\ &= 1 \quad \text{by above.} \end{aligned}$$

This yields equation (91) which we delete from the list. It thus follows that we may delete all but one of the surface relations; let us keep equation (88).

Summing up, we may thus delete relations (81), (80_i) for $i = 3, \ldots, 2n-3$ and (89)–(91) from the presentation of $\Gamma_2(B_n(\mathbb{S}^2))$ given by Proposition 62.

2. The derived subgroup of $B_4(\mathbb{S}^2)$

The aim of this section is to use Proposition 62 to derive the presentation of $\Gamma_2(B_4(\mathbb{S}^2))$ given in Theorem 3(c), from which we were able to see that $\Gamma_2(B_4(\mathbb{S}^2)) \cong \mathbb{F}_2 \rtimes \mathcal{Q}_8$.

We first remark that in this case, the relations (77), (78), (80) and (81) do not exist. Further, from relations (85), we may obtain the following:

$$\begin{aligned} u_2 &= u_3u_4^{-1} & u_1 &= u_3u_4^{-1}u_3^{-1} \\ u_5 &= u_3^{-1}u_4 & u_6 &= u_4^{-1}u_3^{-1}u_4, \end{aligned}$$

which we take to be definitions of u_1, u_2, u_5 and u_6, so we delete equation (85) from the list of relations. From equation (88), we see that

$$w = u_4^{-1}u_3v_1^{-2}u_4u_3^{-1}.$$

We conclude that $\Gamma_2(B_4(\mathbb{S}^2))$ is generated by u_3, u_4 and v_1.

Let us return momentarily to the situation of the previous section. Before deleting all but one of the surface relations, we shall derive some other useful relations.

Consider the surface relations (88)–(91). From relations equation (88) and (90_5) (resp. (91) and (90_4)), it follows that u_5 (resp. u_4) commutes with v_1^2. But these two equations are equivalent to the relations

$$u_3 \rightleftharpoons v_1^2, \quad \text{and} \tag{92}$$

$$u_4 \rightleftharpoons v_1^2. \tag{93}$$

Further, equations (91) and (93) imply equation (90_4), and equations (88), (92) and (93) imply equation (90_5), so we replace equations (90_4) and (90_5) by equations (92) and (93).

As in Section 1, we can then delete equations (90_6) and (91) from the list of relations, which becomes: (79), (82), (83), (84), (86) and (87). We now analyse these relations in further detail.

From equation (79) and the definition of w, we see that $v_1 \rightleftharpoons u_4^{-1}u_3u_4u_3^{-1}$. Up to conjugacy, equation (82_1) may be written as follows:

$$\begin{aligned}1 &= u_3v_1^{-1}u_3^{-1}u_3u_4u_3^{-1}v_1^{-1}u_3u_4^{-1}u_3^{-1}v_1\\ &= u_3v_1^{-1}u_3^{-1}u_4u_4^{-1}u_3u_4u_3^{-1}v_1^{-1}u_3u_4^{-1}u_3^{-1}v_1 = u_3v_1^{-1}u_3^{-1}u_4v_1^{-1}u_4^{-1}v_1,\end{aligned}$$

and hence we may replace equation (82_1) by:

$$u_3v_1u_3^{-1} = u_4v_1^{-1}u_4^{-1}v_1. \tag{94}$$

Up to conjugacy, equation (82_2) may be written:

$$u_3^{-1}v_1u_3 = u_4^{-1}v_1u_4v_1^{-1}. \tag{95}$$

By equations (94) and (92), the left-hand side of equation (82_3) may be written:

$$u_3v_1u_3^{-1}u_4v_1^{-1}u_4^{-1}v_1^{-1} = u_3v_1^2u_3^{-1}v_1^{-2} = 1,$$

so relation (82_3) is automatically satisfied, and we thus delete it from the list.

Using the fact that $v_1 \rightleftharpoons u_4^{-1}u_3u_4u_3^{-1}$, equation ($82_4$) may be written:

$$\begin{aligned}1 &= u_4v_1u_4^{-1}u_3^{-1}u_4v_1^{-1}u_4^{-1}u_3v_1^{-1} = u_4v_1u_3^{-1}u_3u_4^{-1}u_3^{-1}u_4v_1^{-1}u_4^{-1}u_3v_1^{-1}\\ &= u_4v_1u_3^{-1}v_1^{-1}u_3u_4^{-1}v_1^{-1},\end{aligned}$$

and from this, we obtain equation (95), using the fact that v_1^2 commutes with u_4. So we delete equation (82_4) from the list.

We now consider equation (79). Using equations (94) and (95), we obtain:

$$\begin{aligned}1 &= u_4^{-1}u_3u_4u_3^{-1}v_1u_3u_4^{-1}u_3^{-1}u_4v_1^{-1} = u_4^{-1}u_3v_1u_4v_1^{-1}u_4^{-1}u_3^{-1}u_4v_1^{-1}\\ &= u_4^{-1}u_3v_1u_4v_1^{-1}u_4^{-1}v_1u_3^{-1}v_1^{-1}u_4 = u_4^{-1}u_3v_1u_3v_1u_3^{-2}v_1^{-1}u_4,\end{aligned}$$

which up to conjugacy, and using the fact that v_1^2 commutes with u_3 yields:

$$u_3^{-2}v_1^{-1}u_3v_1^{-1}u_3v_1^{-1} \cdot v_1^4 = 1. \tag{96}$$

We replace equation (79) by this relation.

From equations (92) and (93), the left-hand side of equations (86) and (87) collapse, and so we delete them from the list.

After immediate cancellations, equation (84) becomes:

$$\begin{aligned} 1 &= u_4^{-1}u_3^{-1}u_4v_1u_4^{-1}u_3v_1^{-1}u_4v_1^{-1} \\ &= u_4^{-1}u_3^{-1}u_4v_1u_4^{-1}v_1^{-1}v_1u_3v_1^{-1}u_4v_1u_4^{-1}u_4v_1^{-2} \\ &= u_4^{-1}u_3^{-1}u_3v_1u_3^{-1}v_1u_3u_3v_1^{-1}u_3^{-1}u_4v_1^{-2}, \end{aligned}$$

which up to conjugacy and inversion yields equation (96). So we delete equation (84) from the list.

After immediate cancellations, the left-hand side of equation (83) becomes:

$$\begin{aligned} u_3^{-1}u_4v_1u_4^{-1}u_3u_4^{-1}u_3^{-1}u_4v_1^{-1}u_4^{-1} = u_3u_4v_1^{-1} &= u_3^{-1}u_4v_1u_4^{-1}v_1^{-1}u_3v_1^{-1} \\ &= u_3^{-1}u_3v_1u_3^{-1}u_3v_1^{-1} = 1, \end{aligned}$$

using the fact that $v_1 \rightleftharpoons u_4^{-1}u_3u_4u_3^{-1}$, and applying equations (94) and (93). So we delete equation (83) from the list.

We are thus left with relations (92), (93), (94), (95) and (96). We now multiply together equations (94) and (95). The product of the left-hand sides, by equation (96), is given by:

$$u_3v_1u_3^{-2}v_1u_3 = v_1,$$

while by equations (92), (93), (94), (95) and (96), the product of the right-hand sides is given by:

$$\begin{aligned} u_4v_1^{-1}u_4^{-1}v_1u_4^{-1}v_1u_4v_1^{-1} &= u_4v_1^{-1}u_4^{-1}v_1u_4^{-1}v_1^{-1}u_4v_1 \\ &= v_1^{-1}v_1u_4v_1^{-1}u_4^{-1}u_3^{-1}v_1^{-1}u_3v_1 \\ &= v_1^{-1}v_1^{-1}u_4v_1u_4^{-1}u_3^{-1}v_1^{-1}u_3v_1 \\ &= v_1^{-1}u_3v_1^{-1}u_3^{-2}v_1^{-1}u_3v_1 = v_1^{-3}. \end{aligned}$$

From these two equations, we conclude that:

$$v_1^4 = 1, \tag{97}$$

and so equation (96) becomes:

$$u_3^{-2}v_1^{-1}u_3v_1^{-1}u_3v_1^{-1} = 1. \tag{98}$$

The list of relations now becomes: (92), (93), (97), (98), (94) and (95). We may rewrite the corresponding presentation as follows:

PROPOSITION 63. *The following constitutes a presentation of the group $\Gamma_2(B_4(\mathbb{S}^2))$:*

generators: g_1, g_2, g_3, *where in terms of the usual generators of $B_4(\mathbb{S}^2)$, $g_1 = u_3 = \sigma_1^2\sigma_2\sigma_1^{-3}$, $g_2 = u_4 = \sigma_1^3\sigma_2\sigma_1^{-4}$ and $g_3 = v_1 = \sigma_3\sigma_1^{-1}$.*

relations:

$$\begin{aligned} g_3^4 &= 1 \\ g_3^2 &\rightleftharpoons g_1 \\ g_3^2 &\rightleftharpoons g_2 \\ g_3 &\rightleftharpoons g_2 g_1 \\ g_2^{-1} g_1^{-1} g_3^{-1} g_1 g_2 g_3^{-1} &= 1 \\ g_1^{-2} g_3^{-1} g_1 g_3^{-1} g_1 g_3^{-1} &= 1. \end{aligned}$$

PROOF. Rewriting u_3, u_4 and v_1 in terms of the g_i, we obtain directly the first three and the last of the given relations. As for the fourth and fifth relations, we obtain respectively:

$$\begin{aligned} g_3 g_2 g_1 g_3^{-1} g_1^{-1} g_2^{-1} &= v_1 u_4 u_3^{-1} v_1^{-1} u_3^{-1} u_4^{-1} \\ &= u_4 (u_4^{-1} v_1 u_4 v_1^{-1} v_1 u_3^{-1} v_1^{-1} u_3^{-1}) u_4^{-1} \\ &= u_4 u_3 (u_3^{-2} v_1^{-1} u_3 v_1^{-1} u_3 v_1^{-1} v_1^4) u_3^{-1} u_4^{-1} = 1 \end{aligned}$$

by equations (95), (97) and (98), and

$$\begin{aligned} g_2^{-1} g_1^{-1} g_3^{-1} g_1 g_2 g_3^{-1} &= u_4^{-1} u_3^{-1} v_1^{-1} u_3 u_4 v_1^{-1} \\ &= v_1 u_4^{-1} (u_4 v_1^{-1} u_4^{-1} v_1 v_1^{-1} u_3^{-1} v_1^{-1} u_3) u_4 v_1^{-1} \\ &= v_1 u_4^{-1} u_3 (v_1 u_3^{-1} v_1^{-1} u_3^{-1} v_1^{-1} u_3^2) u_3^{-1} u_4 v_1^{-1} = 1 \end{aligned}$$

by equations (94), (97) and (98). Thus the presentation we derived with generators u_3, u_4 and v_1 implies the system given by Proposition 63. Conversely, given this system, we have

$$u_3 v_1 u_3^{-1} = u_3^{-1} v_1^{-1} u_3 v_1^{-1} = u_4 v_1^{-1} u_4^{-1} v_1,$$

which is equation (94), and

$$u_3^{-1} v_1 u_3 = u_3^{-1} v_1^{-1} u_3 v_1^2 = u_3 v_1 u_3^{-1} v_1 v_1^2 = u_4^{-1} v_1 u_4 v_1^3 = u_4^{-1} v_1 u_4 v_1^{-1},$$

which is equation (95). Hence the system given by Proposition 63 is equivalent to our presentation with generators u_3, u_4 and v_1, and so in particular is a presentation of $\Gamma_2(B_4(\mathbb{S}^2))$. □

3. The derived subgroup of $B_5(\mathbb{S}^2)$

For the case $n = 5$, we obtain the following presentation directly from Proposition 62:

PROPOSITION 64. *The following constitutes a presentation of the group* $\Gamma_2(B_5(\mathbb{S}^2))$*:*

generators:

$$w = \sigma_1^8$$
$$u_1 = \sigma_2\sigma_1^{-1}, u_2 = \sigma_1\sigma_2\sigma_1^{-2}, \ldots, u_8 = \sigma_1^7\sigma_2\sigma_1^{-8}$$
$$v_1 = \sigma_3\sigma_1^{-1}, v_2 = \sigma_4\sigma_1^{-1}.$$

relations:

$$v_1v_2v_1 = v_2v_1v_2$$
$$w \rightleftharpoons v_i \textit{ for } i = 1, 2$$
$$u_1v_2 = v_2u_2$$
$$u_2v_2 = v_2u_1^{-1}u_2$$
$$u_iv_1u_{i+2}v_1^{-1}u_{i+1}^{-1}v_1^{-1} = 1 \textit{ for } i = 1, \ldots, 6$$
$$u_7v_1wu_1w^{-1}v_1^{-1}u_8^{-1}v_1^{-1} = 1$$
$$u_8v_1wu_2v_1^{-1}u_1^{-1}w^{-1}v_1^{-1} = 1$$
$$u_{i+1}u_{i+2}^{-1}u_i^{-1} = 1 \textit{ for } i = 1, \ldots, 6$$
$$u_8wu_1^{-1}w^{-1}u_7^{-1} = 1$$
$$wu_1u_2^{-1}w^{-1}u_8^{-1} = 1$$
$$u_2(v_1v_2^2v_1)u_7w = 1. \qquad \square$$

4. The derived subgroup of $B_n(\mathbb{S}^2)$ for $n \geq 6$

We now suppose that $n \geq 6$. Then the generator v_3 exists.

Suppose that equation (82_i) holds for some $1 \leq i \leq 2n - 5$. Let us take $j \geq 3$. We eliminate equation (82_{i+1}) as follows: applying successively equations (80) and (77), we obtain:

$$\begin{aligned} u_{i+1}v_1u_{i+3}v_1^{-1}u_{i+2}^{-1}v_1^{-1} &= v_j^{-1}u_iv_jv_1v_j^{-1}u_{i+2}v_jv_1^{-1}v_j^{-1}u_{i+1}^{-1}v_jv_1^{-1} \\ &= v_j^{-1}(u_iv_1u_{i+2}v_1^{-1}u_{i+1}^{-1}v_1^{-1})v_j \\ &= 1 \text{ by equation } (82_i). \end{aligned}$$

It thus follows that we may delete successively equations (82_{2n-4}), …, (82_2) from the list of relations.

Suppose that equation (83) holds. Applying the idea of the previous paragraph, we eliminate equation (84):

$$\begin{aligned} u_{2n-2}v_1wu_2v_1^{-1}u_1^{-1}w^{-1}v_1^{-1} =& v_j^{-1}u_{2n-3}v_jv_1wv_j^{-1}u_1v_jv_1^{-1}w^{-1}v_j^{-1}\cdot \\ & u_{2n-2}^{-1}v_jv_1^{-1} \\ =& v_j^{-1}(u_{2n-3}v_1wu_1v_1^{-1}w^{-1}u_{2n-2}^{-1}v_1^{-1})v_j = 1. \end{aligned}$$

Let us suppose that equation (82_1) holds. Then so does equation (82_{2n-4}). We eliminate equation (83) as follows.

$$\begin{aligned} u_{2n-3}v_1wu_1w^{-1}v_1^{-1}u_{2n-2}^{-1}v_1^{-1} =& v_j^{-1}u_{2n-4}v_jv_1v_j^{-1}u_{2n-2}v_jv_1^{-1}v_j^{-1}\cdot \\ & u_{2n-3}^{-1}v_jv_1^{-1} \\ =& v_j^{-1}(u_{2n-4}v_1u_{2n-2}v_1^{-1}u_{2n-3}^{-1}v_1^{-1})v_j = 1. \end{aligned}$$

PROPOSITION 65. *Let $n \geq 6$. The following constitutes a presentation of the group $\Gamma_2(B_n(\mathbb{S}^2))$:*

generators:

$$w = \sigma_1^{2n-2}$$

$$u_1 = \sigma_2\sigma_1^{-1}, u_2 = \sigma_1\sigma_2\sigma_1^{-2}, \ldots, u_{2n-2} = \sigma_1^{2n-3}\sigma_2\sigma_1^{-(2n-2)}$$

$$v_1 = \sigma_3\sigma_1^{-1}, \ldots, v_{n-3} = \sigma_{n-1}\sigma_1^{-1}.$$

relations:

$$v_iv_j = v_jv_i \text{ if } |i-j| \geq 2 \tag{99}$$

$$v_iv_{i+1}v_i = v_{i+1}v_iv_{i+1} \text{ for all } 1 \leq i < j \leq n-4 \tag{100}$$

$$w \rightleftharpoons v_i \tag{101}$$

$$u_1v_j = v_ju_2, \text{ where } j \geq 2 \tag{102}$$

$$u_2v_j = v_ju_1^{-1}u_2, \text{ where } j \geq 2 \tag{103}$$

$$u_1v_1u_1^{-1}u_2v_1^{-1}u_2^{-1}v_1^{-1} = 1 \tag{104}$$

$$u_{i+1}u_{i+2}^{-1}u_i^{-1} = 1 \text{ for all } i = 1, \ldots, 2n-4 \tag{105}$$

$$u_{2n-2}wu_1^{-1}w^{-1}u_{2n-3}^{-1} = 1 \tag{106}$$

$$wu_1u_2^{-1}w^{-1}u_{2n-2}^{-1} = 1 \tag{107}$$

$$u_2(v_1\cdots v_{n-4}v_{n-3}^2v_{n-4}\cdots v_1)u_{2n-3}w = 1. \tag{108}$$

□

This presentation may be refined further. Set

$$A = v_1\cdots v_{n-4}v_{n-3}^2v_{n-4}\cdots v_1 \text{ and } y = u_2^{-1}u_1u_2u_1^{-1}.$$

Applying equations (106) and (107) to equation (108), we have:

$$1 = u_2Au_{2n-3}w = u_2Au_{2n-2}wu_1^{-1} = u_2Awu_1u_2^{-1}u_1^{-1},$$

so

$$w = A^{-1}y.$$

Since A commutes with w by equation (101), we see that A commutes with y. Equations (106) and (107) are then equivalent to:

$$u_{2n-3} = A^{-1}u_2^{-1}y^{-1}A \tag{109}$$

$$u_{2n-2} = A^{-1}u_2^{-1}u_1y^{-1}A. \tag{110}$$

Let $i \geq 2$. One may check using relations (99) and (100) that A commutes with v_i. Relation (101) is then equivalent to v_i commutes with y. But this is implied by equations (102) and (103). Indeed, from these two relations we see that $v_iu_2v_i^{-1} = u_1$ and $v_iu_1v_i^{-1} = u_1u_2^{-1}$, and then one may check directly that v_i commutes with $u_2^{-1}u_1u_2u_1^{-1}$. This implies that we may delete equations (101_i) for $2 \leq i \leq n-3$.

From equation (105), we may calculate $u_3, \ldots u_{2n-4}, u_{2n-3}$ and u_{2n-2} in terms of u_1 and u_2. Since all but the last two of these elements do not appear anywhere in the rest of the presentation, we may delete relations (105_i) for $i = 1, \ldots, 2n-4$, provided that we keep (as definitions) the expressions for u_{2n-3} and u_{2n-2} in terms of u_1 and u_2. Let us calculate the general term u_i in terms of u_1 and u_2.

For $i \in \mathbb{N}$, we define v_i as follows:

$$v_i = \begin{cases} u_1u_2^{-1} & \text{if } i \equiv 0 \bmod 6 \\ u_1 & \text{if } i \equiv 1 \bmod 6 \\ u_2 & \text{if } i \equiv 2 \bmod 6 \\ u_1^{-1}u_2 & \text{if } i \equiv 3 \bmod 6 \\ u_1^{-1} & \text{if } i \equiv 4 \bmod 6 \\ u_2^{-1} & \text{if } i \equiv 5 \bmod 6. \end{cases}$$

LEMMA 66. *Let $i \in \mathbb{N}$, and let $k \geq 0$ and $0 \leq l \leq 5$ be such that $i = 6k + l + 1$. Then:*

$$u_i = \begin{cases} y^kv_iy^{-k} & \textit{if } l = 0, 1, 2 \\ y^ku_2^{-1}u_1v_iu_1^{-1}u_2y^{-k} & \textit{if } l = 3, 4, 5. \end{cases}$$

PROOF. The proof is by induction on i, one considers the six possible cases depending on the value of i mod 6. □

We can then determine equations (109) and (110) in the three possible cases. We let $k \geq 0$ and $0 \leq l \leq 5$ be such that $2n - 2 = 6k + l + 1$.

(a) <u>$2n - 2 \equiv 0 \bmod 6$ $(l = 5)$:</u>

$$y^ku_2^{-1}y^{-k} = A^{-1}u_2^{-1}A$$
$$y^ku_2^{-1}u_1y^{-k} = A^{-1}u_2^{-1}u_1A.$$

Hence:

$$u_1, u_2 \rightleftharpoons Ay^k.$$

(b) $\underline{2n-2 \equiv 2 \bmod 6\ (l=1)}$:

$$y^k u_1 y^{-k} = A^{-1} u_2^{-1} y^{-1} A$$
$$y^k u_2 y^{-k} = A^{-1} u_2^{-1} u_1 y^{-1} A.$$

(c) $\underline{2n-2 \equiv 4 \bmod 6\ (l=3)}$:

$$y^k u_1^{-1} y^{-k} = A^{-1} u_1^{-1} u_2 A$$
$$y^k u_2 y^{-k} = A^{-1} u_2^{-1} u_1^{-1} u_2 A.$$

PROPOSITION 67. *Let $n \geq 6$. The following constitutes a presentation of the group $\Gamma_2(B_n(\mathbb{S}^2))$:*

generators:

$$u_1 = \sigma_2\sigma_1^{-1}, u_2 = \sigma_1\sigma_2\sigma_1^{-2}$$
$$v_1 = \sigma_3\sigma_1^{-1}, \ldots, v_{n-3} = \sigma_{n-1}\sigma_1^{-1}.$$

relations:

$$v_i v_j = v_j v_i \text{ if } |i-j| \geq 2$$
$$v_i v_{i+1} v_i = v_{i+1} v_i v_{i+1} \text{ for all } 1 \leq i < j \leq n-4$$
$$y \rightleftharpoons v_1$$
$$v_j u_2 v_j^{-1} = u_1, \quad \text{where } j \geq 2$$
$$v_j u_1 v_j^{-1} = u_1 u_2^{-1}, \quad \text{where } j \geq 2$$
$$u_1 v_1 u_1^{-1} u_2 v_1^{-1} u_2^{-1} v_1^{-1} = 1,$$

plus the two corresponding relations from (a), (b) and (c) of the previous paragraph, where

$$y = u_2^{-1} u_1 u_2 u_1^{-1} \text{ and } A = v_1 \cdots v_{n-4} v_{n-3}^2 v_{n-4} \cdots v_1. \qquad \square$$

REMARK 68. From this presentation, one could also delete, for example, the generator u_2.

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